Exercise 2

a) $dX_t = \kappa(\theta - X_t)\,dt + \lambda\,dW_t$ and $v_t + \mathcal{L}v = 0$.

So, by assumption we have
$$\begin{cases} v_t + \kappa(\theta - x)v_x + \frac{\lambda^2}{2} v_{xx} = 0 \\ v(t,x) = \exp\left(\varphi(T-t,\nu) + \psi(T-t,\nu)\,x\right) \end{cases}$$

$$\Rightarrow \quad v_t = \left(\varphi_t(T-t,\nu) + \psi_t(T-t,\nu)\,x\right) v$$
$$v_x = \psi(T-t,\nu)\,v$$
$$v_{xx} = \psi^2(T-t,\nu)\,v$$

Since $v(t,x) \neq 0$ $\forall (t,x)$

$$\Rightarrow \left(\varphi_t(T-t,\nu) + \psi_t(T-t,\nu)\,x\right)\cancel{v} + \kappa(\theta - x)\,\psi(T-t,\nu)\,\cancel{v} + \frac{\lambda^2}{2}\psi^2(T-t,\nu)\,\cancel{v} = 0$$

$$\Rightarrow \begin{cases} 1 \quad \kappa\theta\,\psi(T-t,\nu) + \frac{\lambda^2}{2}\psi^2(T-t,\nu) + \varphi_t(T-t,\nu) = 0 \\ 2 \quad -\kappa\,\psi(T-t,\nu) + \psi_t(T-t,\nu) = 0 \end{cases}$$

b) 2 $\quad \psi_t(T-t,\nu) = \kappa\,\psi(T-t,\nu)$

$$\Rightarrow \psi(T-t,\nu) = \alpha\, e^{-\kappa(T-t)}$$, where α is $\alpha(\nu)$.

1 $\quad \alpha\kappa\theta\, e^{-\kappa(T-t)} + \frac{\lambda^2}{2}\alpha^2 e^{-2\kappa(T-t)} + \varphi_t(T-t,\nu) = 0$

$$\Rightarrow \varphi_t(T-t,\nu) = -\alpha\kappa\theta\, e^{-\kappa(T-t)} - \frac{\lambda^2}{2}\alpha^2 e^{-2\kappa(T-t)}$$

$$\Rightarrow \varphi(T-t,\nu) = -\alpha\theta\, e^{-\kappa(T-t)} - \frac{\lambda^2}{4}\frac{\alpha^2}{\kappa} e^{-2\kappa(T-t)} - \beta$$, where β is $\beta(\nu)$

$$\Rightarrow v(t,x) = \exp\left(-\alpha\theta\, e^{-\kappa(T-t)} - \frac{\lambda^2}{4}\frac{\alpha^2}{\kappa} e^{-2\kappa(T-t)} - \beta + \alpha\, e^{-\kappa(T-t)}\, x\right)$$

Using the terminal condition, we obtain that

$$e^{i\nu x} = v(T,x) = \exp\left(-\alpha\theta - \frac{\lambda^2}{4}\frac{\alpha^2}{\kappa} - \beta + \alpha x\right),$$

which means

$$\begin{cases} \alpha\theta + \dfrac{\lambda^2\alpha^2}{4\kappa} + \beta = 0 \\ \alpha = i\nu \end{cases} \Rightarrow -i\nu\theta + \frac{\lambda^2\nu^2}{4\kappa} = +\beta,$$

Hence,

$$\psi(T-t,\nu) = i\nu\, e^{-\kappa(T-t)}$$

$$\varphi(T-t,\nu) = -i\nu\theta\, e^{-\kappa(T-t)} + \frac{\lambda^2\nu^2}{4\kappa} e^{-2\kappa(T-t)} + i\nu\theta - \frac{\lambda^2\nu^2}{4\kappa}$$

$$= -\frac{\lambda^2\nu^2\left(1-e^{-2\kappa(T-t)}\right)}{4\kappa} + i\nu\theta\left(1-e^{-\kappa(T-t)}\right)$$

And so I obtained that the characteristic function of the SDE solution for $\begin{cases} dX_t = \kappa(\theta - X_t)dt + \lambda dW_t \\ X_t = x \end{cases}$ is

$$\mathbb{E}\left[e^{i\nu X_T} \mid X_t = x\right] = v(t,x) = \exp\left(i\nu\left(e^{-\kappa(T-t)}x + \theta\left(1-e^{-\kappa(T-t)}\right)\right) - \frac{\nu^2\lambda^2}{4\kappa}\left(1-e^{-2\kappa(T-t)}\right)\right).$$

If now I consider the initial condition $X_0 = x$, I have

$$\mathbb{E}\left[e^{i\nu X_t} \mid X_0 = x\right] = v(0,x) \overset{\substack{T=t \\ t=0}}{=} \exp\left(-\frac{\lambda^2\nu^2\left(1-e^{-2\kappa t}\right)}{4\kappa} + i\nu\theta\left(1-e^{-\kappa t}\right) + x\, i\nu\, e^{-\kappa t}\right). \quad (1)$$

c) I can notice know that the expression obtained in (1) is equal to the characteristic function of a normal variable. Indeed,

$$(1) = \exp\left(iv\left(\theta(1-e^{-\kappa t}) + x e^{-\kappa t}\right) - \frac{v^2}{2}\left(\frac{\lambda^2(1-e^{-2\kappa t})}{2\kappa}\right)\right)$$

and this is the characteristic function of a normal variable with mean and variance equal to

$$\boxed{\mu = \theta + e^{-\kappa t}(x-\theta)} \qquad \boxed{\sigma^2 = \frac{\lambda^2(1-e^{-2\kappa t})}{2\kappa}} \quad . \qquad (2)$$

Since the characteristic function determines the law of a r.v. we obtain that $X_t \sim N(\mu, \sigma^2)$.

d) Consider the process

$$X_t = \theta + (x-\theta)e^{-\kappa t} + \lambda e^{-\kappa t}\int_0^t e^{\kappa s}\, dW_s \,, \qquad (3)$$

we can show that it is a solution. Indeed,

set $f(t,x) = \theta + (x-\theta)e^{-\kappa t} + \lambda e^{-\kappa t} x$ (of class C^2).

$$\frac{\partial f}{\partial t}(t,x) = \kappa(\theta - x_0)e^{-\kappa t} - \lambda\kappa\, e^{-\kappa t} x$$

$$\frac{\partial f}{\partial x}(t,x) = \lambda e^{-\kappa t}$$

$$\frac{\partial^2 f}{\partial x^2}(t,x) = 0$$

We apply Itô's formula to $f(t, Y_t)$ when $Y_t = \int_0^t e^{\kappa s} dW_s$ (i.e. $dY_t = e^{\kappa t} dW_t$)

$$X_t = f(t, Y_t) = x + \int_0^t \lambda e^{-\kappa s} dY_s + \int_0^t \kappa(\theta - x) e^{-\kappa s} - \lambda\kappa e^{-\kappa s} Y_s \, ds$$

$$= X_0 + \int_0^t \underbrace{\kappa(\theta - x) e^{-\kappa s} - \lambda\kappa e^{-\kappa s} \int_0^s e^{\kappa r} dW_r}_{\kappa(\theta - X_s)} \, ds + \int_0^t \lambda \, dW_s$$

$$= x + \int_0^t \kappa(\theta - X_s) \, ds + \int_0^t \lambda \, dW_t$$

$\Rightarrow$ $dX_t = \kappa(\theta - X_t) dt + \lambda dW_t$ and solve the Cauchy problem with $X_0 = x$.

It is consistent with the results in the previous part because the variable

$$X_t = \theta + (x - \theta) e^{-\kappa t} + \lambda e^{-\kappa t} \int_0^t e^{\kappa s} dW_s$$

is gaussian, since the last stochastic integral is a stochastic integral of a deterministic function.
Moreover, due to the fact that $\mathbb{E}\left[\int_0^t e^{\kappa s} dW_s\right] = 0$, it turns out that

- $\mathbb{E}[X_t] = \theta + (x - \theta) e^{-\kappa t} = \mu$ (see (2)).

As we saw in Take Home 1 ex. 2 (part 1),

$$\mathbb{E}[X_t^2] = \left(\theta + (x - \theta) e^{-\kappa t}\right)^2 + \frac{1}{2\kappa} \lambda^2 e^{-2\kappa t} \left(e^{2\kappa t} - 1\right)$$

Now,

$$Var(X_t) = \mathbb{E}[X_t^2] - \mathbb{E}[X_t]^2$$

$$= \frac{\ell^2}{2\kappa}\left(1 - e^{-2\kappa t}\right) = \sigma^2 \qquad (\text{see } (2))$$