Computational Finance
FIN-472
Take-Home Exam 3
Polynomial expansion methods

October 28, 2022

- **Please upload your solutions on moodle by Friday 11.11.2022 at 8h15.**
- **Please do not forget to upload in Moodle your Matlab codes. You should submit Matlab files for parts f),g),h), for a total of 3 Matlab files.**
- **You should also upload one PDF file containing the solutions to all parts– including the Matlab codes for parts f),g),h) and the plot for part h).**

Consider a GARCH model $(X_t)_{0\leq t\leq T}$ of the form

$$dX_t = \kappa(\theta - X_t)\,dt + \sigma X_t\,dW_t,$$

where κ, θ and σ are model parameters.

a) **(1/20)** Prove that X_t is a polynomial diffusion and write its infinitesimal generator $\mathcal{G}$.

b) **(2/20)** For $u \in \mathbb{R}$, define v as

$$v(t,x) = \mathbb{E}[\exp(iuX_T)|X_t = x].$$

If the conditions of the Feyman-Kac theorem are satisfied, v solves the equation

$$v_t + \mathcal{G}v = 0, \quad v(T,x) = \exp(iux).$$

Can we write the solution of the PDE as

$$v(t,x) = \exp(\phi(T-t) + \psi(T-t)x)$$

for some function ϕ and ψ with $\phi(0) = 0$ and $\psi(0) = iu$?

c) **(2/20)** Solve explicitly the differential equation for X_t.

Hint: Apply Itô to $L_t := e^{\left(\kappa+\frac{\sigma^2}{2}\right)t-\sigma W_t}X_t$.

d) **(1/20)** For $N \in \mathbb{N}$, write the matrix representation G_N of the infinitesimal generator restricted to $\mathrm{Pol}_N(\mathbb{R})$, with respect to the monomial basis given by

$$H_N(x) = (1, x, x^2, \cdots, x^N).$$

e) **(2/20)** Use the moment formula for polynomial diffusions to calculate the first moment $\mathbb{E}[X_T]$. Check that the obtained result is coherent with what one gets from the explicit formula derived in part c).

f) **(4/20)** Consider the set of parameters

$$\kappa = 0.5, \quad \theta = 0.4, \quad \sigma = 0.2, \quad X_0 = 1, \quad T = 0.5.$$

Use the moment formula for polynomial diffusions to find the first 4 moments

$$\mathbb{E}[X_T], \quad \mathbb{E}[X_T^2], \quad \mathbb{E}[X_T^3], \quad \mathbb{E}[X_T^4].$$

g) **(4/20)** Compare the results of the previous part with the moments computed via a Monte Carlo simulation, where you use a forward Euler scheme to simulate $N_{sim} = 10^6$ realizations of X_T and use $N_{time} = 100$ for the time discretization.

h) **(4/20)** Using the moments calculated in part f), calculate the 4-order "approximation" of the density of X_T with a Gaussian that matches the first two moments. Plot the density approximation for orders $1, 2, 3, 4$.