Exercise 1

a) In exercise 1 of Homework 3 we found that the characteristic function of X_t is equal to

$$\hat{P}_{X_t}(z) = \exp\left[t\left(iz\gamma - \frac{1}{2}\sigma^2 z^2 + \lambda\left(e^{iz\alpha - \frac{1}{2}\beta^2 z^2} - 1\right)\right)\right].$$

We call the payoff $\varphi(S_T) = f(X_T + x_0)$ where $S_t = S_0 e^{X_t}$ and $x_0 := \log S_0$.

Since $\varphi(s) = \mathbb{1}_{\{s > a\}} \Rightarrow f(x + x_0) = \varphi(S_0 e^x) = \mathbb{1}_{\{S_0 e^x > a\}} = \mathbb{1}_{\{x > \log\left(\frac{a}{S_0}\right)\}}$.

$$\Downarrow$$

$$f(x) = f(x - x_0 + x_0) = \mathbb{1}_{\{x - \cancel{x_0} > \log a - \cancel{\log S_0}\}} = \mathbb{1}_{\{x > \log a\}}$$

First of all, we notice that $f(x) \notin L^1(\mathbb{R})$, since

$$\int_{\mathbb{R}} \mathbb{1}_{\{x > \log(a)\}}\, dx = \int_{\log(a)}^{+\infty} dx = +\infty$$ and doesn't converge.

Then, we define

$$f^\eta(x) = e^{\eta x} f(x) = e^{\eta x}\mathbb{1}_{\{x > \log(a)\}}$$ for a

suitable η s.t. (A1) - (A3) hold.

(A1) $$\int_{\mathbb{R}} e^{\eta x}\mathbb{1}_{\{x > \log a\}}\, dx = \int_{\log a}^{+\infty} e^{\eta x}\, dx = \frac{1}{\eta}\left[e^{\eta x}\right]_{\log a}^{+\infty} = (*)$$

and this converges if $\eta < 0$. In particular,

$$(*) = \frac{1}{\eta}\left[-a^\eta\right] \qquad (1)$$

(A2) $\mathbb{E}\left[e^{-\eta X_T}\right] = \hat{P}_{X_T}(i\eta)$

$$= \exp\left[T\left(-\eta\gamma + \frac{1}{2}\sigma^2\eta^2 + \lambda\left(e^{-\eta\alpha + \frac{1}{2}\beta^2\eta^2} - 1\right)\right)\right]$$

which is finite.

(A3) is equivalent to say that $\hat{g}_\eta \in L^1(\mathbb{R})$. Now,

$$\rightarrow \hat{g}_\eta(u) = \int_{\mathbb{R}} e^{iuy} g_\eta(y)\,dy = \int_{\mathbb{R}} e^{iuy} e^{\eta y} \mathbb{1}_{\{y > \log a\}}\,dy$$

$$= \int_{\log a}^{+\infty} e^{(iu+\eta)y}\,dy = \frac{1}{\eta + iu}\left[e^{(iu+\eta)y}\right]_{\log a}^{+\infty}$$

and this is convergent if $\eta < 0$ and it's equal to $\frac{1}{\eta + iu}\left[-a^{ui+\eta}\right]$

So at the end it is sufficient to take $\eta < 0$ to apply the theorem.

Then taking $x_0 = \log(S_0)$ we can apply Theorem 22 and we have that

$$\mathbb{E}\left[\Psi(S_T)\right] = \mathbb{E}\left[\Psi(S_0 e^{X_T})\right] = \mathbb{E}\left[f(X_T + x_0)\right]$$

$$= \frac{e^{-\eta x_0}}{2\pi} \int_{\mathbb{R}} e^{iux_0}\, \hat{P}_{X_T}(u + i\eta)\, \overline{\hat{g}_\eta(u)}\,du$$

$$= \boxed{\frac{1}{2\pi} \int_{\mathbb{R}} \frac{1}{-\eta + iu} \left(\frac{a}{S_0}\right)^{\eta - iu} \hat{P}_{X_T}(u + i\eta)\,du}$$

where I used that

$$\overline{-\frac{1}{\eta + iu} \cdot a^{\eta + ui}} = \overline{\frac{-\eta + iu}{\eta^2 + u^2}\, a^{\eta + ui}} = \frac{-\eta - iu}{\eta^2 + u^2}\, a^{\eta - iu} = \frac{1}{-\eta + iu}\, a^{\eta - iu}.$$

b) The cdf is given by $F_{X_T}(x) = \mathbb{E}\left[\mathbb{1}_{\{X_T \leq x\}}\right]$.

In point a) I found a formula to compute $\mathbb{E}\left[\mathbb{1}_{\{X_T > \log(\frac{a}{S_0})\}}\right]$.
So for each x I can take a s.t.

$\log\left(\frac{a}{S_0}\right) = x$ and then obtaining $\mathbb{E}\left[\mathbb{1}_{\{X_T > x\}}\right]$.

Now, I put $F_{X_T}(x) = 1 - \mathbb{E}\left[\mathbb{1}_{\{X_T > x\}}\right]$.

c)

I want to estimate $\| P(a) - I_n(P(\cdot))(a)\|_\infty$ using Theorem 10 of lecture 10

Taking $P^1 = [a_{min}, a_{max}] \subset \mathbb{R}$ and $\rho \in (1, +\infty)$, if (A1) - (A5) hold,
then we obtain that

$$\| P(a) - I_n(P(\cdot))(a)\|_\infty \leq C \rho^{-n}$$

for a certain costant C. Moreover we obtain an exponential rate of convergence.
Let us show that (A1) - (A5) hold. Take $f^a(x) := \mathbb{1}_{\{x > \log(a)\}}$.

(A1) $x \mapsto e^{\eta x} f^a(x) \in L^1(\mathbb{R})$: this property follows by (1) for every $a \in P^1$ (already shown for property (A1) of point a)).

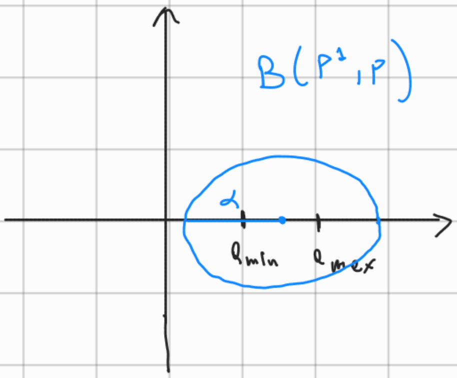
(A2) $a \mapsto \overline{\hat{f}^a_\eta(z)}$ analytic in $B(P^1, \rho)$ $\forall z \in \mathbb{R}$:
fix $z \in \mathbb{R}$.

Since $\overline{\hat{f}^a_\eta(z)} = \frac{1}{-\eta + iz}(a)^{\eta - iz}$, it is enough to show that

the function $a \mapsto a^{\eta - iz}$ is analytic (then $\overline{\hat{f}^a_\eta(z)}$ is analytic since multiply by a complex number is analytic).

Now, $a^{\eta - iz} = \exp\left((\eta - iz) \log a \right)$ where exp and log are the complex exponential and the complex logarithm.

To ensure the well-posedness and the analyticity of $\log a$ we need that $a \neq 0$ and to have this it is enough to take the semi-major of the Bernstein ellipse as $\alpha < \frac{a_{max} + a_{min}}{2}$ (ellipse doesn't touch 0).



(A3) $\mathbb{E}\left[e^{-\eta X_T} \right] < +\infty$: follows from (A2) of point a).

(A4) Since in this case the integrand is a function of a and $\widehat{P^{-\eta}_{X_T}}(z)$ does not depend on a, this condition is already satisfied.

(A5) $\exists\, h \in L^1(\mathbb{R})$ s.t. $\left\| \widehat{g^{a}_{\eta}}(z)\, \widehat{P^{-\eta}_{X_T}}(z) \right\|_{\infty} \leq h(z) \quad \forall z \in \mathbb{R}$:

$$\max_a \left| \widehat{g^{a}_{\eta}}(z)\, \widehat{P^{-\eta}_{X_T}}(z) \right| = \max_a \left| \frac{1}{-\eta + iz} \left(a \right)^{\eta - iz} \hat{P}_{X_T}(z + i\eta) \right|$$

$$\leq \max_a \left| \frac{(a)^{\eta}\, e^{-iz \log(a)}}{-\eta + iz} \right| \left| \hat{P}_{X_T}(z + i\eta) \right|$$

$\left| e^{-iz \log(a)} \right| = 1 \;\; \forall z$

$\eta < 0$

$a^{\eta} \leq (a_{min})^{\eta}$

$$= \left| \frac{(a_{min})^{\eta}}{-\eta + iz} \right| \left| \hat{P}_{X_T}(z + i\eta) \right| = \frac{(a_{min})^{\eta}}{\sqrt{\eta^2 + z^2}} \left| \hat{P}_{X_T}(z + i\eta) \right| \leq \frac{(a_{min})^{\eta}}{\eta} \left| \hat{P}_{X_T}(z + i\eta) \right|$$

So, it remains to show that $\left|\hat{P}_{X_T}(z+i\eta)\right|$ is integrable.

If we show that $\left|\hat{P}_{X_T}(z)\right|$ is integrable (with z real number), then the integrability of $\left|\hat{P}_{X_T}(z+i\eta)\right|$ follows similarly considering the moodolus instead of the absolute value.

Now,

$$\left|\hat{P}_{X_T}(z)\right| = \exp\left[T\left(iz\gamma - \frac{1}{2}\sigma^2 z^2 + \lambda\left(\underbrace{e^{iz\alpha - \frac{1}{2}\beta^2 z^2}}_{\xrightarrow{|z|\to+\infty} 0} - 1\right)\right)\right].$$

$$\Rightarrow \quad \hat{P}_{X_T}(z) \underset{|z|\to+\infty}{\sim} \exp\left(-\frac{T}{2}\sigma^2 z^2 - \lambda\right) \quad \text{which is integrable on } \mathbb{R}.$$

Take Home Exam 5: exercise 1 d) e)

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In this file there are the Matlab codes for *Exercises d) e)*.

Exercise d)

Here it is the code for the solution of part d).

```
function [P] = D_Price_Merton_mod(T,S,lambda,alpha,sigma,beta,L,a,eta)

% P= Price of a digital with Maturity T using the characteristic ...
%function of the price and Fourier pricing rule in the Merton model

% S= Initial price
% lambda,alpha,sigma,beta = parameters Merton
% L = truncation bound for the integral
% a = barriers
% eta=damping factor

%Merton characteristic function

%gamma
gamma=-0.5*sigma^2-lambda*(exp(alpha+0.5*beta^2)-1);

% characteristic function for Jks
Ch_Jk = @(z) exp(1i.*z*alpha-0.5*beta^2*z.^2);
% characteristic function for X
Ch_X = @(z) exp(T*(1i*z*gamma-0.5*sigma^2*z.^2+lambda.*(Ch_Jk(z)-1)));

% Digital Fourier transform
 F_d=@(u) ((a/S).^(-1i*u+eta))./(-eta+1i*u);
%Integrand
integrand=@(u) Ch_X(u+1i*eta).*F_d(u);
% Pricing formula
P=1/(2*pi)*integral(integrand,-L, L);% Price

end
```

Exercise e)

In this section, I firstly present the script for part e).

```

S=1; lambda=0.4; sigma=0.15; alpha=-0.5; beta=0.4; T=0.5;
amin=0.7; amax=1.3; L=50; eta=-1;

a_grid=linspace(amin,amax,100);

nr_Cheb = 2:30;
errors = zeros(size(nr_Cheb));

P_D=zeros(length(a_grid),1);
tic
for j=1:length(a_grid)
    P_D(j)=D_Price_Merton_mod(T,S,lambda,alpha,sigma,beta,L,a_grid(j),eta);
end
time1=toc;

tic;
for j=1:size(nr_Cheb,2)
    P_Cheb=ChebInterpol(@(a) D_Price_Merton_mod(T,S,lambda,alpha,sigma,beta,...
        L,a,eta), a_grid, nr_Cheb(j), amin, amax);
    errors(j)=max(P_Cheb-P_D);
end
time=toc;
%average time
time=toc/size(nr_Cheb,2);

% plot log errors
figure
semilogy(nr_Cheb,exp(-nr_Cheb),'k--')
hold on, grid on
semilogy(nr_Cheb,abs(errors),'or-')
legend('O(exp(- n))')
ylabel('Absolute interp errors')
xlabel('Interpolation order')
```

The following figure shows the results obtained.

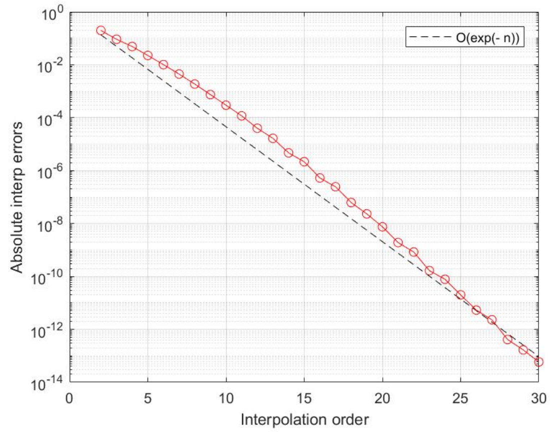

Figure 1: Order of convergence in a semilogy plot.

As we can see from the previous plot, the order of convergence is exponential. Moreover the time occured to run the function obtained in part d) is 0.18s, while for the Chebyschev interpolation is 0.02s. We can observe that both methods are fast, but between them the second method is faster. Moreover, we can notice that the second procedure is also more efficient since, fixed an order n, it computes the price only on the nodes of interpolation intead of each $a \in a\_grid$.