Exercise 1

$$dX_t = \kappa(\theta - X_t)\,dt + \sigma X_t\,dW_t$$

a) X_t is a polynomial diffusion iff $b \in Pol_1(\mathbb{R})$, $a \in Pol_2(\mathbb{R})$

In this case,

$$b(x) = \kappa(\theta - x) \in Pol_1(\mathbb{R})$$
$$a(x) = \sigma^2 x^2 \in Pol_2(\mathbb{R}).$$

And so X_t is a polynomial diffusion. Moreover, by Itô's formula if $f \in C^2$

$$df(X_t) = f'(X_t)\,dX_t + \frac{1}{2} f''(X_t)\,d\langle X\rangle_t$$

$$= f'(X_t)\left[\kappa(\theta - X_t)\,dt + \sigma X_t\,dW_t\right] + \frac{1}{2} f''(X_t)\,\sigma^2 X_t^2\,dt$$

$$= \underbrace{\left[\kappa f'(X_t)(\theta - X_t) + \frac{1}{2} f''(X_t)\sigma^2 X_t^2\right]}_{\mathcal{G}f(X_t)} dt + \sigma f'(X_t) X_t\,dW_t$$

Hence,
$$\mathcal{G}f(x) = \kappa f'(x)(\theta - x) + \frac{1}{2} f''(x)\sigma^2 x^2$$

and we can notice again that $\mathcal{G}\,Pol_n(\mathbb{R}) \subseteq Pol_n(\mathbb{R})$ $\forall n \in \mathbb{N}$.

b) If $v(t,x) = \mathbb{E}\left[\exp(iuX_T) \mid X_t = x\right]$ and the conditions of F-K theorem are satisfied v solves

$$\begin{cases} v_t + \mathcal{G}v = 0 \\ v(T,x) = \exp(iux) \end{cases} \longrightarrow \begin{cases} v_t + \kappa v_x(\theta - x) + \frac{1}{2} v_{xx}\sigma^2 x^2 = 0 \\ v(T,x) = \exp(iux) \end{cases}$$

First of all, if we could write $v(t,x) = \exp\left(\phi(T-t) + \psi(T-t)x\right)$, it would be an affine model, but in class we said that only if $q(x) \in Pol_1(\mathbb{R})$ we have an affine model. In this case $a(x) \in Pol_2(\mathbb{R})$.
More precisely,

if $v(t,x) = \exp\left(\phi(T-t) + \psi(T-t)x\right)$ with $\phi(0) = 0$ and $\psi(0) = iu$

then,

$$V_t = \left(\phi_t(T-t) + \psi_t(T-t)x\right) v$$
$$V_x = \psi(T-t)\, v$$
$$V_{xx} = \psi^2(T-t)\, v$$

$$\Rightarrow \quad \phi_t(T-t) + \psi_t(T-t)x + k\psi(T-t)(\theta - x) + \frac{1}{2}\psi^2(T-t)\sigma^2 x^2 = 0$$

$$\underset{\sigma \neq 0}{\Rightarrow} \begin{cases} \phi_t(T-t) + k\psi(T-t)\theta = 0 \\ \psi_t(T-t) - k\psi(T-t) = 0 \\ \psi_t^2(T-t) = 0 \end{cases} \underset{k \neq 0}{\Rightarrow} \begin{cases} \phi_t(T-t) = 0 \\ \boxed{\psi(T-t) = 0} \\ \psi_t(T-t) = 0 \end{cases}$$

But the condition $\psi(T-t) = 0$ is incompatible with $\psi(0) = iu$.

c) Take $L_t = e^{(k+\frac{\sigma^2}{2})t - \sigma W_t} X_t$

By Leibnitz rule

$$dL_t = \underbrace{X_t\, d\left(e^{(k+\sigma^2/2)t - \sigma W_t}\right)}_{①} + e^{(k+\frac{\sigma^2}{2})t - \sigma W_t} \underbrace{dX_t}_{\text{SDE given}} + \underbrace{dX_t\, d\left(e^{(k+\sigma^2/2)t - \sigma W_t}\right)}_{②}$$

① Apply Ito to $f(t, W_t) = e^{(\kappa+\sigma^2/2)t - \sigma W_t}$

$$df(t,X_t) = -\sigma e^{(\kappa+\sigma^2/2)t-\sigma W_t}\, dW_t + \left[\left(\kappa+\frac{\sigma^2}{2}\right) e^{(\kappa+\sigma^2/2)t-\sigma W_t} + \frac{1}{2}\sigma^2 e^{(\kappa+\sigma^2/2)t-\sigma W_t}\right] dt$$

$$= -\sigma e^{(\kappa+\sigma^2/2)t-\sigma W_t}\, dW_t + \left(\kappa+\sigma^2\right) e^{(\kappa+\sigma^2/2)t-\sigma W_t}\, dt$$

② $$dX_t\, d\left(e^{(\kappa+\sigma^2/2)t-\sigma W_t}\right) = \left(\kappa(\theta - X_t)\,dt + \sigma X_t\, dW_t\right)\left(-\sigma e^{(\kappa+\sigma^2/2)t-\sigma W_t}\, dW_t + (\kappa+\sigma^2)\, e^{(\kappa+\sigma^2/2)t-\sigma W_t}\, dt\right)$$

Follows since the covariation between a BV process and a process that admits a quadratic variation is equal to 0 →

$$= -\sigma^2 X_t\, e^{(\kappa+\sigma^2/2)t-\sigma W_t}\, dt$$

Hence, we have that

$$dL_t = X_t\, d\left(e^{(\kappa+\sigma^2/2)t-\sigma W_t}\right) + e^{-\sigma W_t}\, dX_t + dX_t\, d\left(e^{(\kappa+\sigma^2/2)t-\sigma W_t}\right)$$

$$= \cancel{-\sigma e^{(\kappa+\sigma^2/2)t-\sigma W_t} X_t\, dW_t} + \left(\cancel{\kappa} + \cancel{\sigma^2}\right) e^{(\kappa+\sigma^2/2)t-\sigma W_t} X_t\, dt$$

$$+ \kappa\left(\theta \cancel{- X_t}\right) e^{(\kappa+\sigma^2/2)t-\sigma W_t}\, dt + \cancel{\sigma X_t\, e^{(\kappa+\sigma^2/2)t-\sigma W_t}}\, dW_t$$

$$\cancel{-\sigma^2 X_t\, e^{(\kappa+\sigma^2/2)-\sigma W_t}}\, dt$$

$$\Rightarrow dL_t = \kappa\theta\, e^{(\kappa+\sigma^2/2)t-\sigma W_t}\, dt \Rightarrow \underbrace{L_t = L_0 + \kappa\theta \int_0^t e^{(\kappa+\sigma^2/2)s-\sigma W_s}\, ds}$$

Now, since $L_0 = X_0$ and $L_t = e^{(\kappa+\sigma^2/2)t-\sigma W_t} X_t$

$$\boxed{X_t = e^{-(\kappa+\sigma^2/2)t+\sigma W_t}\left[X_0 + \kappa\theta\int_0^t e^{(\kappa+\sigma^2/2)s-\sigma W_s}\, ds\right]}$$

d) $\mathcal{G}f(x) = \kappa f'(x)(\theta - x) + \frac{1}{2} f''(x)\sigma^2 x^2$

I can apply $\mathcal{G}$ to monomials:

$$\begin{aligned}
\mathcal{G}x^n &= n\kappa(\theta - x)x^{n-1} + \frac{1}{2}n(n-1)x^{n-2}\sigma^2 x^2 \\
&= n\kappa\theta x^{n-1} - n\kappa x^n + \frac{1}{2}n(n-1)\sigma^2 x^n \\
&= [n\kappa\theta]x^{n-1} + \left[\frac{1}{2}n(n-1)\sigma^2 - n\kappa\right]x^n
\end{aligned}$$

$\Rightarrow$

$$G_N = \begin{pmatrix}
0 & \kappa\theta & 0 & \cdots & 0 \\
0 & -\kappa & 2\kappa\theta & \ddots & \vdots \\
\vdots & 0 & \sigma^2 - 2\kappa & \ddots & \vdots \\
\vdots & \vdots & 0 & \ddots & \vdots \\
\vdots & \vdots & \vdots & \ddots & N\kappa\theta \\
0 & 0 & 0 & \cdots & \frac{1}{2}N(N-1)\sigma^2 - N\kappa
\end{pmatrix} \in \mathbb{R}^{(N+1)\times(N+1)}$$

e) $$\mathbb{E}[p(X_T) \mid X_t] = H_N(X_t)\, e^{(T-t)G_N}\vec{p}$$

If $N=1$, $p(x) = x$, then $H_1(x) = (1, x)$ and

$$G_1 = \begin{pmatrix} 0 & \kappa\theta \\ 0 & -\kappa \end{pmatrix} \Rightarrow e^{TG_1} = \begin{pmatrix} 1 & (-e^{-\kappa T} + 1)\theta \\ 0 & e^{-\kappa T} \end{pmatrix}$$

Moreover $\vec{p} = (0, 1)^T$.

$$\mathbb{E}[X_T \mid X_t] = (1, X_t)\, e^{(T-t)G_1}\vec{p} = (1, X_t)\begin{pmatrix} 1 & \theta\left(1 - e^{-\kappa(T-t)}\right) \\ 0 & e^{-\kappa(T-t)} \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= (1, x_t) \begin{pmatrix} \theta\left(1 - e^{-\kappa(T-t)}\right) \\ e^{-\kappa(T-t)} \end{pmatrix} = \theta\left(1 - e^{-\kappa(T-t)}\right) + X_t\, e^{-\kappa(T-t)}$$

If I take $t=0$ we have $\mathbb{E}[X_T \mid X_0] = \boxed{\theta\left(1 - e^{-\kappa T}\right) + X_0 e^{-\kappa T}}$
which is coherent with c), since

$$X_t = e^{-(\kappa+\sigma^2/2)t + \sigma W_t} \left(X_0 + \kappa\theta \int_0^t e^{(\kappa+\sigma^2/2)s - \sigma W_s}\, ds \right)$$

$$\Rightarrow \mathbb{E}[X_T] = X_0\, e^{-(\kappa+\sigma^2/2)T}\, \mathbb{E}\left[e^{\sigma W_T}\right] + \kappa\theta \int_0^t e^{(\kappa+\sigma^2/2)(s-T)}\, \mathbb{E}\Big[e^{\sigma \overbrace{(W_T - W_s)}^{\sim N(0,\, t-s)}}\Big]\, ds$$

$$= X_0\, e^{-(\kappa+\cancel{\sigma^2/2})T}\, e^{\cancel{\sigma^2 T/2}} + \cancel{\kappa}\theta\, e^{-(\kappa+\cancel{\sigma^2/2})T} \underbrace{\int_0^T e^{(\kappa+\sigma^2/2)s}\, e^{(T-s)\sigma^2/2}\, ds}_{e^{\sigma^2 T/2} \int_0^T e^{(\kappa + \cancel{\sigma^2/2} - \cancel{\sigma^2/2})s}\, ds \;=\; \cancel{e^{\sigma^2 T/2}} \cdot \frac{1}{\cancel{\kappa}}\left(e^{\kappa T} - 1\right)}$$

$$= \boxed{X_0\, e^{-\kappa T} + \theta\left(1 - e^{-\kappa T}\right)}$$

Take Home Exam 3: exercise 1 f) g) h)

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In this file there are the Matlab codes for *Exercises f) g) h)*.

Exercise f)

```
%parameters
kappa = 0.5; theta = 0.4; sigma = 0.2; X_0 = 1; T = 0.5;
Mom=zeros(1,4);
for i= 1:4
    G_n = GenGar(kappa, theta, sigma, i);
    p=zeros(i+1,1);
    p(i+1)=1;
    Mom(i) = MomCIR(G_n, X_0, T, p);
end
Mom

function [ G_n ] = GenGar(kappa, theta, sigma, n)
% The function constructs the matrix G_n as in Exercise 2, HW5.
% Input: model parameters kappa, theta, sigma
% maximal polynomial degree n
% Output: Matrix G_n
%
% Specify the two non-zero diagonals
d2=(1:n)*(kappa)*(theta);
d1=(0:n)*(-kappa) + 0.5* sigma^2 * (0:n).*(-1:n-1);
% Construct the matrix G_n
G_n = diag(d1, 0) + diag(d2, 1);
end

function [ Mom ] = MomCIR( G_n, X_0, T, p )
% The function applies the moment formula in Exercise 2, HW 5
% Input: Matrix representation G_n of the generator G
% Initial asset price X 0
% Maturity time T
% Coordinate vector p
% Output: Corresponding moment

% Evaluate basis vector in X_0
```

```
n=size(G_n,1);
H_n=X_0.^(0:n-1);
Mom=H_n*expm(G_n*T)*p;
end
```

The four moments computed by the *moment formula for polynomial diffusions* in the previous script are in the following table.

$\mathbb{E}[X_T]$	$\mathbb{E}[X_T^2]$	$\mathbb{E}[X_T^3]$	$\mathbb{E}[X_T^4]$
0.8673	0.7658	0.6885	0.6302

Table 1: Moments

Exercise g)

```
%moments computed before
Mom=[0.8673 0.7658 0.6885 0.6302];

%parameters
kappa = 0.5; theta = 0.4; sigma = 0.2; x0 = 1; T = 0.5;
% We compare the moments with moments computed via MC
Nsim=10^6;
Ntime=100;
x=zeros(Nsim,1);
for i = 1:Nsim
    [t,X] = SimSDEgar_vec(x0, kappa, theta, sigma, T, Ntime);
    x(i)=X(Ntime+1);
end
M1_MC=mean(x);
M2_MC=mean(x.^2);
M3_MC=mean(x.^3);
M4_MC=mean(x.^4);
%comparison
disp('Comparison with MC')
abs(Mom(1)-M1_MC)
abs(Mom(2)-M2_MC)
abs(Mom(3)-M3_MC)
abs(Mom(4)-M4_MC)


function [t,X] = SimSDEgar_vec(x0, kappa, theta, sigma, T, Ntime)
% We simulate the SDE for the GAR model by Euler discretization

% Time steps
delta=T/Ntime;
t=delta.*(0:T);
```

```
%Simulate N times random variable Z
Z= randn(Ntime,1);

%Simulate the path
X=zeros(Ntime+1,1);
X(1)=x0;
for i= 1:Ntime
    X(i+1)=X(i)+ kappa*(theta-X(i))*delta+sigma*(X(i))*sqrt(delta)*Z(i);
end
end
```

The comparisons between the results of the previous part with the moments computed via Monte Carlo simulation are in the following table.

$\mathbb{E}[X_T]$	$\mathbb{E}[X_T^2]$	$\mathbb{E}[X_T^3]$	$\mathbb{E}[X_T^4]$
$2.0504e-04$	$3.0703e-04$	$3.5709e-04$	$3.8814e-04$

Table 2: Comparison with MC moments

It is possible to notice that the distance between the two results are of order 10^{-4}, that means that they are comparable.

Exercise h)

In order to calculate the 4-order "approximation" of the density of X_T, I used as density $w(x)$ the Gaussian density that matches the first two moments calculated in f). This means that if we take $\mu = Mom(1)$ and since we know also the second moment $Mom(2)$, the density $w(x)$ is the Gaussian with mean μ and variance $\sigma^2 = Mom(2) - \mu^2$. Moreover, I take as orthonarmal basis of L^2_w the following

$$\tilde{H}_n(x) = \frac{1}{\sqrt{n!}}\tilde{\mathcal{H}}\left(\frac{x-\mu}{\sigma}\right),$$

where $\tilde{\mathcal{H}}(x)$ are the standard "probabilists" Hermite polynomials. Finally, to compute the approximation of the density $q(x)$ of X_T I used the formula

$$q^{(N)}(x) = \left(\sum_{n=0}^{N} \ell_n \tilde{H}_n(x)\right) w(x), \qquad N = 1, 2, 3, 4.$$

In order to compute the ℓ_n coefficients, I used the fact that $\ell_n = \mathbb{E}[\tilde{H}_n(X_T)]$. Now, the first five Hermite polynomials are

$$\mathcal{H}_0(x) = 1,\ \ \mathcal{H}_1(x) = x,\ \ \mathcal{H}_2(x) = x^2 - 1,\ \ \mathcal{H}_3(x) = x^3 - 3x,\ \ \mathcal{H}_4(x) = x^4 - 6x^2 + 3.$$

This implies that

$$H_0(x) = 1,\ H_1(x) = x,\ H_2(x) = \frac{1}{\sqrt{2}}\left(x^2 - 1\right),\ H_3(x) = \frac{1}{\sqrt{6}}\left(x^3 - 3x\right),\ H_4(x) = \frac{1}{\sqrt{4!}}\left(x^4 - 6x^2 + 3\right).$$

And so we can obtain the coefficients with the following quantities that depend on the the moments that we have computed in f).

$$\ell_0 = 1,\ \ \ell_1 = \mathbb{E}[X_T],\ \ \ell_2 = \frac{1}{\sqrt{2}}\left(\mathbb{E}[X_T^2] - 1\right),\ \ \ell_3 = \frac{1}{\sqrt{6}}\left(\mathbb{E}[X_T^3] - 3\mathbb{E}[X_T]\right),\ \ \ell_4 = \frac{1}{\sqrt{4!}}\left(\mathbb{E}[X_T^4] - 6\mathbb{E}[X_T^2] + 3\right).$$

Therefore, we are ready to compute the “approximations” of the density with the following code.

```
%moments computed before
Mom=[0.8673 0.7658 0.6885 0.6302];

%mean and standard deviation for density w(x)
mu=Mom(1);
sigma=sqrt(Mom(2)-mu^2);
x = [0:.01:2];

%computations of l_n
l=zeros(5,1);
l(1)=1;
l(2)=Mom(1);
l(3)=(Mom(2)-1)/sqrt(2);
l(4)=(Mom(3)-3*Mom(1))/sqrt(6);
l(5)=(Mom(4)-6*Mom(2)+3)/sqrt(factorial(4));

%approximated density for N=1,2,3,4
for i = 1:4
    y=zeros(1,length(x));
    for j = 1:i+1
        n=j-1;
        y=y+l(j)*(2^(-0.5*n)*hermiteH(n, (x-mu)/ (sigma*sqrt(2))) )./(sqrt(factorial(n)));
    end
    y=normpdf(x,mu,sigma).*y;
    plot(x,y)
    hold on
end
legend('q^{(1)}(x)','q^{(2)}(x)','q^{(3)}(x)','q^{(4)}(x)')
```

The plot of the densities obtained with the previous script is the following.

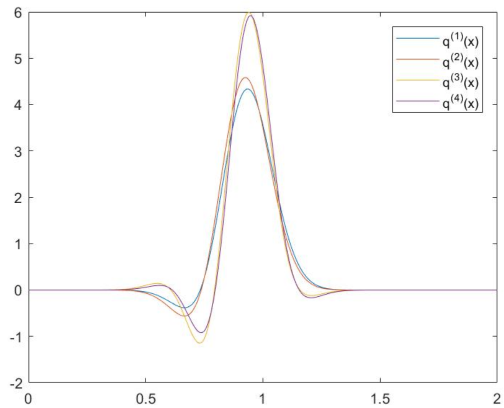

Figure 1: Density approximation for orders $1, 2, 3, 4$.

As we can see frome the previous figure, the density approximations are negative in a small interval and so, in order to obtain more coherent density functions, we could take the maximum between the values of densities and 0. The following figure gives us the result.

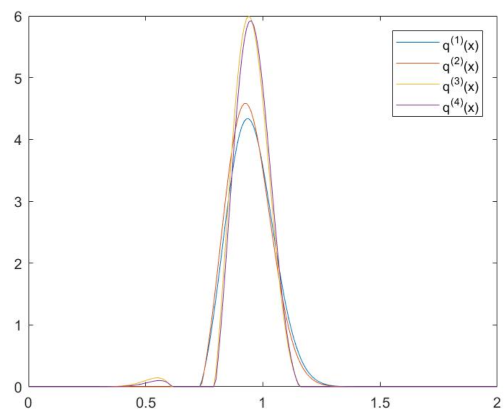

Figure 2: Nonnegative density approximation for orders $1, 2, 3, 4$.

Finally, I computed the coefficients ℓ_n as I said before by hand. Anyway it is possible to find them also with the following code with the moment formula, where in the vector $\vec{p}$ there are the coefficients of H_n.

```
%calcolo dei coefficienti l_n
l=zeros(5,1);
for i=1:5
    H_n=X_0.^(0:i-1);
    G_n = GenGar(kappa, theta, sigma, i-1);
    syms x
```

```
    l(i)=H_n*expm(G_n*T)*(double(fliplr(coeffs(2^(-0.5*(i-1)) * hermiteH(
        i-1, x / sqrt(2)), 'All')))./(sqrt(factorial(i-1))))';
end
```