Computational Finance
FIN-472
Take-Home Exam 1

September 30, 2022

- **Please upload your solutions on moodle by Friday 14.10.2022 at 8h15.**

- **You should upload one PDF file containing the solutions to the theoretical exercises as well as one PDF file with the Matlab code for Exercises** $4.a$ **and** $4.b$ **and the plot for Exercise** $4.c$.

- **In addition, for Exercises** $4.a$, $4.b$ **and** $4.c$ **do not forget to upload in moodle the Matlab codes. You should submit only one Matlab file for each part.**

- **For the theoretical exercises you can either scan your handwritten solutions or write them using a math typesetting software such as LaTeX.**

Exercise 1: Black Scholes model and law of one price (8 points /40)

a) *Black Scholes model:* In the Black Scholes model, the dynamics of the price process $(S_t)_{0 \in [0,T]}$ with respect to the real-world measure $\mathbb{P}$ are given by

$$dS_t = S_t(\mu \, dt + \sigma \, dW_t), \tag{1}$$

for parameters $\sigma > 0$ and μ being constant and for a standard Brownian motion W. *Show that the process*

$$S_t = S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W_t\right)$$

is a solution of (1) and compute $\mathbb{E}[S_T]$.

b) *Law of one price:* Using the law of one price, answer the following questions:

(a) Suppose that a European type derivative's payoff at maturity T is $|S_T - K|$. Explain why the value of such an option is equal to the sum of the values of a Call Option and a Put Option with strike K and expiration date T.

(b) A discount certificate on the underlying S with cap K is a European derivative with payoff equal to $\min\{S_T, K\}$ at maturity T. Explain why the price of this certificate is equal to S_0 minus the price of a Call Option with strike K and expiration date T.

Exercise 2: OU process (12 points /40)

An Ornstein-Uhlenbeck process is defined as solution of the stochastic differential equation

$$dX_t = \kappa(\theta - X_t)\,dt + \lambda dW_t, \quad \text{for } 0 \le t \le T \tag{2}$$

for real parameters $\kappa > 0, \lambda > 0, \theta$ and a standard Brownian motion W. Moreover, denote by x_0 the starting value of the process.

a) *Solve* (2).

b) Using the solution found in a) *show* that

$$\lim_{t \to \infty} \mathbb{E}[X_t] = \theta.$$

c) In this exercise we compute the quantity $\mathbb{E}[X_T^2]$ in two different ways.

- *Compute* $\mathbb{E}[X_T^2]$ by using the solution found in a) and knowing that, if a function f is square-integrable on $[0, t]$, then $\int_0^t f(s)dW_s$ is normally distributed with mean 0 and variance $\int_0^t f^2(s)ds$.

- Define

$$v(t, x) = \mathbb{E}[(X_T^{t,x})^2]$$

where $X_s^{t,x}$ is the solution of

$$\begin{cases} dX_s = \kappa(\theta - X_s)\,ds + \lambda dW_s, & \text{for } t \le s \le T \\ X_t = x. \end{cases}$$

Derive the following partial differential equation (PDE) satisfied by v:

$$\begin{cases} v_t + \mathcal{G}v = 0 \\ v(T, x) = x^2, \end{cases} \tag{3}$$

where $\mathcal{G}$ denotes the generator of the process defined by

$$\mathcal{G}v = \kappa(\theta - x)v_x + \frac{\lambda^2}{2}v_{xx}.$$

Solve the PDE (3) and *derive* the value of $\mathbb{E}[X_T^2]$.

Hint: In order to solve (3), *use the Ansatz* $v(t, x) = a(T-t) + b(T-t)x + c(T-t)x^2$, *derive the equations for* a, b *and* c *(these are functions of* $T - t$*) and solve them.*

Exercise 3: Heston model (8 points /40)

In the Heston model, the dynamics of the price process $(S_t)_{0 \le t \le T}$ with respect to a risk-neutral measure $\mathbb{Q}$ are given by

$$dS_t = rS_t dt + S_t \rho \sqrt{V_t}dW_t + S_t \sqrt{1 - \rho^2}\sqrt{V_t}dW_t^\perp,$$

where the spot variance V_t (square of the volatility) is modeled by

$$dV_t = \kappa(\theta - V_t)dt + \nu\sqrt{V_t}dW_t.$$

Here, W and $W^\perp$ are independent standard Brownian motions, r is a risk-free interest rate, ρ is a correlation parameter and κ, θ, ν are additional model parameters.

Let $U(t, S, V)$ be the price at time t of a European option with payoff function Ψ, when $S_t = S$ and $V_t = V$, i.e. (by risk-neutral pricing rule)

$$U(t, S, V) = e^{-r(T-t)} \mathbb{E}^{\mathbb{Q}}[\Psi(S_T^{t,S,V})].$$

Derive the pricing PDE satisfied by $U(t, S, V)$.

Exercise 4: Up-and-Out option (12 points /40)

Let $(S_t)_{0 \le t \le T}$ be a price process. The up-and-out call barrier option with discrete monitoring at two monitoring dates is a path-dependent option whose payoff function at maturity time T is given by

$$\Psi(S) := (S_T - K)^+ \ \mathbb{1}_{\{S_T < b\}} \mathbb{1}_{\{S_{T/2} < b\}} = \begin{cases} (S_T - K)^+ & \text{if } S_t < b, \text{ for } t = \frac{T}{2}, T, \\ 0 & \text{if } S_t \ge b, \text{ for } t = \frac{T}{2} \text{ or } t = T, \end{cases} \tag{4}$$

where K denotes the strike value and $b \in \mathbb{R}$ is the value of the barrier.

a) Consider a multi-period binomial model over the time interval $[0, T]$ with parameters r, d, u, N, with N even, and denote the initial stock price S_0 by s.
   Write a Matlab function `BinomialpriceBarrierUODM.m` with input parameters

   $$r, d, u, N, T, s, K, b$$

   which computes the binomial price at time $t = 0$ of the up-and-out call barrier option with discrete monitoring at two monitoring dates defined in (4).

   Hint: Adapt the iteration formula introduced in slide 31 (Week 2) in order to price this path-dependent option.

b) Consider now the Black Scholes model where the $\mathbb{Q}$-dynamics of the price process $(S_t)_{0 \le t \le T}$ are given by
   $$dS_t = r S_t \, dt + \sigma S_t dW_t, \tag{5}$$
   for a risk-free interest rate r and volatility σ. Denote the initial price S_0 by s.
   Let
   $$\{t_k := \frac{Tk}{N_{time}} | k = 0, \cdots, N_{time}\}.$$
   be a discretization of $[0, T]$ and let

   $$\{(S_{t_k}^i)_{k=0, \cdots, N_{time}} | i = 1, \cdots, N_{sim}\}$$

   be a set of N_{sim} simulated paths generated by using the discretization scheme

   $$S_{t_{k+1}} - S_{t_k} = r S_{t_k} \Delta t + \sigma S_{t_k} \sqrt{\Delta t} Z, \tag{6}$$

   where $\Delta t = T/N_{time}$ and $Z \approx N(0, 1)$. Then, the Monte Carlo (MC) price at time $t = 0$ of an option maturing at time T with payoff function Ψ is given by

   $$P_{MC} := \frac{e^{-rT}}{N_{sim}} \sum_{i=1}^{N_{sim}} \Psi(S^i).$$

3

Write a Matlab function `MCpriceBarrierUODM.m` with input parameters

$$r, \sigma, N_{time}, N_{sim}, T, s, K, b$$

that computes the MC price P_{MC} of the up-and-out call barrier option with discrete monitoring at two monitoring dates.

c) *Compare* the binomial prices computed in part a) with the MC price computed in b), for $N \to \infty$. In particular, consider the following set of parameters

$$s = 1, \quad r = 0.1, \quad T = 0.5, \quad K = 0.9, \quad \sigma = 0.1, \quad b = 1.3$$

and *plot*

- The constant MC price P_{MC} computed in b) with $N_{time} = 100$ and $N_{sim} = 10^6$,
- The binomial price obtained for $N = 2, 4, \cdots, 200$.

against the values of N. What do you observe?

Remark: For this part, please submit on the moodle the Matlab script/function that generates the needed plot.