Computational Finance
FIN-472
Take-Home Exam 5

December 2, 2022

- **Please hand in your solutions online before Friday 16.12.2022.**
- For Exercises *d*) and *e*) do not forget to upload in moodle the Matlab codes.
- Additionally, print out the Matlab code for Exercises *d*) and *e*), and the plots for Exercise *e*). These have to be handed in together with the other solutions on Friday 16.12.2022.

In this exercise we consider the European digital option, whose payoff function is given by

$$\Psi : \mathbb{R}_+ \to \mathbb{R},$$
$$\Psi(s) = \mathbb{1}_{\{s>a\}},$$

for a certain real value $a > 0$. Moreover, we consider the Merton model in which the log-price process $(X_t)_{0\leq t\leq T}$ on the time horizon $[0, T]$ for $T > 0$ is defined as

$$X_t = \gamma t + \sigma W_t + \sum_{k=1}^{N_t} J_k, \tag{1}$$

for some real parameters σ, γ and where W_t is the standard Brownian motion. N_t is a Poisson distributed random variable with parameter λt for $\lambda > 0$ and $J_k \overset{i.i.d}{\sim} \mathcal{N}(\alpha, \beta^2)$, for real parameters α and β. The price process S_t is finally defined as $S_t = S_0 e^{X_t}$, for some initial asset price S_0. Also, we assume vanishing interest rate, i.e. $r = 0$, and we impose the condition $\gamma = -\frac{1}{2}\sigma^2 - \lambda(e^{\alpha+\frac{1}{2}\beta^2} - 1)$ in order to obtain a risk neutral formulation of the model.

a) *Derive* the Fourier pricing formula for the digital option in Merton's model, applying Theorem 22, slide 45, Lecture 3. In particular, find appropriate g and η (damping factor) for which you show that the conditions (A1)-(A3) hold.

   Remark: You can use the characteristic function derived in Exercise 1 of the Homework 3. Also, you can use without proof that this characteristic function extends to complex arguments.

b) *Explain* how you can compute the cumulative distribution function $F_{X_T}(x)$ of X_T, using the formula derived in a).

c) Suppose now that we fix $T > 0$, all the model parameters $S_0, \lambda, \sigma, \alpha, \beta$ and a certain $\eta < 0$. Consider the function

$$P : [a_{\min}, a_{\max}] \to \mathbb{R},$$
$$P(a) := \mathbb{E}[\mathbb{1}_{\{S_T > a\}}]$$

written as Fourier pricing formula, as you derived in a), for some fixed interval $[a_{\min}, a_{\max}]$ ($0 < a_{\min} < a_{\max}$). Consider the Chebyshev interpolation of order n of $P(a)$, denoted by $I_n(P(\cdot))(a)$. *Derive* an appropriate error bound and convergence rate, for $n \to \infty$.

d) *Implement* a Matlab function that computes the price of the European digital option in Merton's model. In particular,

- implement the Fourier pricing formula derived in a);
- use an adequate numerical integration function among the ones provided by Matlab for computing the integral;
- the input should consist of the model and payoff parameters, T, η and the truncation limit L for the integration routine.

e) Consider the Matlab function `ChebInterpol` implemented in Exercise 2 of Homework 10, that computes the Chebyshev interpolation of order n of an input function $f : [a, b] \mapsto \mathbb{R}$ in an arbitrary point $x \in [a, b]$. Moreover, consider the restriction of P (defined in part c)) on the fixed interval $[a_{\min}, a_{\max}]$ and fix following parameters

$$S_0 = 1, \;\; \lambda = 0.4 \;\; \sigma = 0.15, \;\; \alpha = -0.5, \;\; \beta = 0.4,$$
$$T = 0.5, \;\; a_{\min} = 0.7, \;\; a_{\max} = 1.3, \;\; L = 50, \;\; \eta = -1.$$

Then, for all $a_i \in \texttt{linspace}(a_{\min}, a_{\max}, 100)$ ($i = 1, \cdots, 100$) compute the price of the corresponding digital options

- using the function you implemented in part d), and
- using the function `ChebInterpol` with appropriate input parameters,

for interpolation orders $n = \{2, 3, \cdots, 30\}$. For each n, plot the maximal absolute error over the whole computed prices, as done in Exercise 2d) of Homework 10. What do you observe

- generally?
- in terms of efficiency gain?
- in terms of run time gain?