Exercise 1

a) $$S_t = S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t\right)$$

Take $f(t,x) = \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma x\right)$. Then

- $\frac{\partial f}{\partial t}(t,x) = \left(\mu - \frac{\sigma^2}{2}\right) \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma x\right)$
- $\frac{\partial f}{\partial x}(t,x) = \sigma \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma x\right)$
- $\frac{\partial f}{\partial x^2}(t,x) = \sigma^2 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma x\right)$

Now, we can apply Itô's formula to $f(t, W_t)$ and

$$S_t = S_0 f(t, W_t)$$

$$= S_0 \underbrace{f(0, W_0)}_{=1} + \int_0^t \sigma S_0 \exp\left(\left(\mu - \sigma^2/2\right)s + \sigma W_s\right) dW_s$$

$$+ \int_0^t \left(\mu - \frac{\sigma^2}{2}\right) S_0 \exp\left(\left(\mu - \sigma^2/2\right)s + \sigma W_s\right) + \frac{1}{2}\sigma^2 S_0 \exp\left(\left(\mu - \sigma^2/2\right)s + \sigma W_s\right) ds$$

$$= S_0 + \int_0^t \sigma S_s \, dW_s + \int_0^t \left(\mu - \cancel{\frac{\sigma^2}{2}} + \cancel{\frac{\sigma^2}{2}}\right) S_s \, ds$$

$$= S_0 + \int_0^t \mu S_s \, ds + \int_0^t \sigma S_s \, dW_s .$$

This means that: $dS_t = S_t\left(\mu \, dt + \sigma \, dW_t\right)$.

Moreover $\mathbb{E}[S_T] = S_0 \exp\left(\left(\mu - \sigma^2/2\right)T\right) \mathbb{E}\left[\exp(\sigma W_T)\right]$

due to gaussian MGF and $W_T \sim N(0,T)$ $\rightarrow$ $= S_0 \exp\left(\left(\mu - \sigma^2/2\right)T\right) \exp\left(\sigma^2 T/2\right)$.

b) (a) We denote the value of such an option with $V_t(\tilde{P}_1)$ with $0 \leq t \leq T$.

We know that $V_T(\tilde{P}_1) = |S_T - K|$.

Moreover, if we call C_t the value of a Call option and P_t the value of a Put Option with strike K and expiration date T, we call $V_t(\tilde{P}_2) = P_t + C_t$ the value of this portfolio.
Now, we know that

$$V_T(\tilde{P}_2) = P_T + C_T = (S_T - K)_+ + (K - S_T)_+$$

$$= \max\{S_T - K, 0\} + \max\{K - S_T, 0\}$$

$$= \begin{cases} S_T - K & \text{if } S_T - K \geq 0 \\ K - S_T & \text{if } S_T - K < 0 \end{cases} = |S_T - K|$$

Hence, $V_T(\tilde{P}_1) = |S_T - K| = V_T(\tilde{P}_2)$ and by the law of one price we obtain that

$$V_t(\tilde{P}_1) = V_t(\tilde{P}_2) \quad \forall\ t \leq T.$$

(b) We can consider the following two portfolios:

- P_1: long one share, short one call $\Rightarrow V_t(P_1) = S_t - C_t$
- P_2: long one our discount certificate.

At maturity we have

$$V_T(P_2) = \min\{S_T, K\} = \begin{cases} S_T & \text{if } S_T \leq K \\ K & \text{if } S_T > K \end{cases}$$

On the other hand,

$$V_T(P_1) = S_T - \max\{S_T - K, 0\} = \begin{cases} S_T - 0 & \text{if } S_T \leq K \\ S_T - S_T + K & \text{if } S_T > K \end{cases}$$

$$= \begin{cases} S_T & \text{if } S_T \leq K \\ K & \text{if } S_T > K \end{cases}$$

But then, $V_T(P_1) = V_T(P_2)$ and for the law of one price

$$V_t(P_1) = V_t(P_2) \quad \forall t .$$

In particular, this means that the price of our discount certificate at $t=0$ is equal to $S_0 - C_0$.

Exercise 2

$$\begin{cases} dX_t = \kappa(\theta - X_t)\,dt + \lambda\, dW_t \\ X_0 = x_0 \end{cases}$$

a) We can take the process

$$X_t = \theta + (x_0 - \theta)e^{-\kappa t} + \lambda e^{-\kappa t}\int_0^t e^{\kappa s}\, dW_s \qquad (1)$$

and we can show that it is a solution. Indeed,
set $f(t, x) = \theta + (x_0 - \theta)e^{-\kappa t} + \lambda e^{-\kappa t} x$. Then,

$$\frac{\partial f}{\partial t}(t, x) = \kappa(\theta - x_0)e^{-\kappa t} - \lambda\kappa\, e^{-\kappa t} x$$

$$\frac{\partial f}{\partial x}(t, x) = \lambda e^{-\kappa t}$$

$$\frac{\partial^2 f}{\partial x^2}(t, x) = 0$$

We apply Itô's formula to $f(t, Y_t)$ when $Y_t = \int_0^t e^{\kappa s}\, dW_s$
(i.e. $dY_t = e^{\kappa t}\, dW_t$)

$$X_t = f(t, Y_t) = x_0 + \int_0^t \lambda e^{-\kappa s}\, dY_s + \int_0^t \kappa(\theta - x_0)e^{-\kappa s} - \lambda\kappa\, e^{-\kappa s} Y_s\, ds$$

$$= x_0 + \int_0^t \underbrace{\kappa(\theta - x_0)e^{-\kappa s} - \lambda\kappa\, e^{-\kappa s}\int_0^s e^{\kappa r}\, dW_r}_{\kappa(\theta - X_s)}\, ds + \int_0^t \lambda\, dW_s$$

$$= x_0 + \int_0^t \kappa(\theta - X_s)\, ds + \int_0^t \lambda\, dW_t$$

$\Rightarrow$ $dX_t = \kappa(\theta - X_t)\,dt + \lambda\, dW_t$ and solve the Cauchy problem with $X_0 = x_0$.

Remark: It is possible to find the formula in (1) following the idea of what we do in ODE. In particular we can solve firstly the "homogenous" equation and then adding the "particular solution".

b) Now, I know that $$X_t = \theta + (x_0 - \theta) e^{-\kappa t} + \lambda e^{-\kappa t} \int_0^t e^{\kappa s} dW_s .$$

So,
$$\mathbb{E}[X_t] = \theta + (x_0 - \theta) e^{-\kappa t} \qquad \text{since } \mathbb{E}\left[\int_0^t e^{\kappa s} dW_s\right] = 0.$$

Therefore,
$$\lim_{t \to +\infty} \mathbb{E}[X_t] = \lim_{t \to +\infty} \left(\theta + (x_0 - \theta) \underbrace{e^{-\kappa t}}_{\downarrow t\to+\infty \atop 0}\right) = \theta .$$

c) • $$\mathbb{E}[X_T^2] = \mathbb{E}\left[\left(\theta + (x_0 - \theta) e^{-\kappa T} + \lambda e^{-\kappa T} \int_0^T e^{\kappa s} dW_s\right)^2\right]$$

$$= \left(\theta + (x_0 - \theta) e^{-\kappa T}\right)^2 + \lambda^2 e^{-2\kappa T} \underbrace{\mathbb{E}\left[\left(\int_0^T e^{\kappa s} dW_s\right)^2\right]}_{Var(Y_T)} + 2\lambda e^{-\kappa T}\left(\theta + (x_0 - \theta) e^{-\kappa T}\right) \underbrace{\mathbb{E}\left[\int_0^T e^{\kappa s} dW_s\right]}_{=0}$$

$$= \left(\theta + (x_0 - \theta) e^{-\kappa T}\right)^2 + \lambda^2 e^{-2\kappa T} \int_0^T e^{2\kappa s} ds$$

$$= \left(\theta + (x_0 - \theta) e^{-\kappa T}\right)^2 + \frac{1}{2\kappa} \lambda^2 e^{-2\kappa T} \left(e^{2\kappa T} - 1\right) \qquad (2)$$

• We can apply Feynman-Kac Theorem to the PDE problem
$$\begin{cases} v_t + \mathcal{G}v = 0 \\ v(T, x) = x^2 \end{cases}$$
and we have the stochastic representation of v such that

$$v(t,x) = \mathbb{E}\left[\left(X_T^{t,x}\right)^2\right], \qquad \text{since } c \equiv 0.$$
of the theorem

To apply the theorem it's enough to show that the operator $\mathcal{G}$ coincides with

$$\mathcal{G} f(t,x) = b(t,x)\frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2(t,x)\frac{\partial^2 f}{\partial x}$$

where $b(t,x)$ and $\sigma(t,x)$ are taken from the SDE.

Indeed, from the SDE

$$b(t,x) = \kappa(\theta - x) \qquad \sigma(t,x) = \lambda .$$

Then,

$$\mathcal{G} v(t,x) = \kappa(\theta - x) v_x + \frac{1}{2}\lambda^2 v_{xx} \quad \text{as we wanted.}$$

Now, we want to solve the PDE $v_t + \kappa(\theta - x)v_x + \frac{\lambda^2}{2} v_{xx} = 0$.

We can take $v(t,x) = a(T-t) + b(T-t)x + c(T-t)x^2$

$$v_t(t,x) = -\frac{\partial a}{\partial t}(T-t) - \frac{\partial b}{\partial t}(T-t)x - \frac{\partial c}{\partial t}(T-t)x^2$$

$$v_x(t,x) = b(T-t) + 2c(T-t)x$$

$$v_{xx}(t,x) = 2c(T-t)$$

$$\Rightarrow -\frac{\partial a}{\partial t}(T-t) - \frac{\partial b}{\partial t}(T-t)x - \frac{\partial c}{\partial t}(T-t)x^2 + \kappa(\theta-x)\left[b(T-t) + 2c(T-t)x\right] + \lambda^2 c(T-t) = 0$$

$$\Leftrightarrow \begin{cases} \text{(1)} \quad \lambda^2 c(T-t) - \frac{\partial a}{\partial t}(T-t) + \kappa\theta b(T-t) = 0 \\ \text{(2)} \quad -\frac{\partial b}{\partial t}(T-t) - \kappa b(T-t) + 2\kappa\theta c(T-t) = 0 \\ \text{(3)} \quad -\frac{\partial c}{\partial t}(T-t) - 2\kappa c(T-t) = 0 \end{cases}$$

with conditions $c(0) = 1$, $a(0) = b(0) = 0$

So, (3)

$$\frac{\partial c}{\partial t}(T-t) = -2\kappa c(T-t) \Rightarrow c(T-t) = \alpha e^{-2\kappa(T-t)}$$

$$\Rightarrow c(T-t) = e^{-2\kappa(T-t)}$$

$$\Rightarrow \quad \begin{cases} (1) & \dfrac{\partial a}{\partial t}(T-t) = \kappa\theta b(T-t) + \lambda^2 e^{-2\kappa(T-t)} \\ (2) & \dfrac{\partial b}{\partial t}(T-t) + \kappa b(T-t) = 2\kappa\theta e^{-2\kappa(T-t)} \end{cases}$$

(2) $b(T-t) = \alpha e^{-\kappa(T-t)} - 2\theta e^{-2\kappa(T-t)}$ and

$$0 = b(0) = \alpha - 2\theta \Rightarrow \alpha = 2\theta$$

$$\Rightarrow b(T-t) = 2\theta\left(e^{-\kappa(T-t)} - e^{-2\kappa(T-t)}\right)$$

(1) $$\frac{\partial a}{\partial t}(T-t) = 2\kappa\theta^2\left(e^{-\kappa(T-t)} - e^{-2\kappa(T-t)}\right) + \lambda^2 e^{-2\kappa(T-t)}$$

$$= 2\kappa\theta^2 e^{-\kappa(T-t)} + \left(\lambda^2 - 2\kappa\theta^2\right) e^{-2\kappa(T-t)}$$

$$\Rightarrow a(T-t) = -2\theta^2 e^{-\kappa(T-t)} + \left(\theta^2 - \frac{\lambda^2}{2\kappa}\right) e^{-2\kappa(T-t)} + \beta$$

and $$0 = a(0) = -2\theta^2 + \theta^2 - \frac{\lambda^2}{2\kappa} + \beta$$

$$\Rightarrow \beta = \theta^2 + \frac{\lambda^2}{2\kappa}$$

$$\Rightarrow a(T-t) = -2\theta^2 e^{-\kappa(T-t)} + \left(\theta^2 - \frac{\lambda^2}{2\kappa}\right) e^{-2\kappa(T-t)} + \theta^2 + \frac{\lambda^2}{2\kappa}$$

$$= \theta^2\left(1 - 2e^{-\kappa(T-t)} + e^{-2\kappa(T-t)}\right) + \frac{\lambda^2}{2\kappa}\left(1 - e^{-2\kappa(T-t)}\right)$$

At the end we obtained that the function

$$v(t,x) = a(T-t) + b(T-t)x + c(T-t)x^2$$ with

$a(T-t)$, $b(T-t)$, $c(T-t)$ just computed solving the PDE.

Now, to compute $\mathbb{E}[X_T^2]$, I just put $t=0$ and $x=x_0$.
Then,

$$\mathbb{E}[X_T^2] = v(0,x_0) = a(T) + b(T)x_0 + c(T)x_0^2 =$$

$$= \theta^2\left(1-2e^{-\kappa T}+e^{-2\kappa T}\right) + \frac{\lambda^2}{2\kappa}\left(1-e^{-2\kappa T}\right) + 2\theta\left(e^{-\kappa T}-e^{-2\kappa T}\right)x_0 + e^{-2\kappa T}x_0^2 .$$

We can check if it is equal to (2).

$$(2) = \left(\theta + (x_0-\theta)e^{-\kappa T}\right)^2 + \frac{1}{2\kappa}\lambda^2 e^{-2\kappa T}\left(e^{2\kappa T}-1\right)$$

$$= \theta^2 + (x_0-\theta)^2 e^{-2\kappa T} + 2\theta(x_0-\theta)e^{-\kappa T} + \frac{\lambda^2}{2\kappa}\left(1-e^{-2\kappa T}\right)$$

$$= \theta^2\left(1+e^{-2\kappa T}-2e^{-\kappa T}\right) + 2\theta\left(-x_0e^{-2\kappa T}+x_0e^{-\kappa T}\right)$$

$$+ x_0^2e^{-2\kappa T} + \frac{\lambda^2}{2\kappa}\left(1-e^{-2\kappa T}\right)$$

$\square$

Exercise 3

$$U(t,S,V) = = e^{-r(T-t)}\,\mathbb{E}^{\mathbb{Q}}\left[\Psi\left(S_T^{t,S,V}\right)\right] \qquad (1)$$

According to Feynman–Kac representation, the solution of the following PDE

$$\begin{cases} U_t + \mathcal{L}U + cU = 0 \\ U(T,S,V) = \Psi\left(S_T^{S,V}\right) \end{cases}$$

is given by $U(t,S,V) = \mathbb{E}\left[\Psi\left(S_T^{t,S,V}\right)\exp\left(\int_t^T c\left(s, S_s^{t,S,V}\right)ds\right)\right]$ (2)

Now, since we want (1) = (2) we can take $c \equiv -r$ and we have that

$$(2) = e^{-r(T-t)}\,\mathbb{E}\left[\Psi\left(S_T^{t,S,V}\right)\right].$$

So, we need to find the operator $\mathcal{L}$ for our model. $[d=2, m=2]$

We know that

$$\mathcal{L}f(t,x_1,x_2) = \sum_{i=1}^{2} b_i(t,x)\frac{\partial f}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^{m}(\sigma\sigma^T)_{i,j}(t,x)\frac{\partial^2 f}{\partial x_i \partial x_j}$$

In our case we have the SDE:

$$\begin{cases} dS_s = r S_s\,ds + S_s\rho\sqrt{V_s}\,dW_s + S_s\sqrt{1-\rho^2}\sqrt{V_s}\,dW_s^{\perp} \\ dV_s = \kappa(\theta - V_s)\,ds + \nu\sqrt{V_s}\,dW_s \end{cases}$$

with solution $S_s^{t,S,V}$ for $t \le s \le T$ and initial conditions $(S_t, V_t) = (S, V)$.

We have:

- a 2-dim BM given by $\underline{W_s} = (W_s, W_s^{\perp})$
- $b(s, S_s, V_s) = \left(r S_s, \kappa(\theta - V_s)\right)$
- $\sigma(s, S_s, V_s) = \begin{pmatrix} S_s\rho\sqrt{V_s} & S_s\sqrt{1-\rho^2}\sqrt{V_s} \\ \nu\sqrt{V_s} & 0 \end{pmatrix}.$

Then,

$$\boxed{\mathcal{L}U(t,S,V) = rS\,U_S + \kappa(\theta - V)\,U_V + \frac{1}{2}\left(S^2 V\,U_{SS} + 2\rho\nu S V\,U_{SV} + \nu^2 V\,U_{VV}\right)}$$

This means that the PDE satisfied by $U(t,S,V)$ is
$[v(t,x_1,x_2)]$

$$
\begin{cases}
V_t + \frac{1}{2}\left(x_1^2 x_2 V_{x_1x_1} + \nu^2 x_2 V_{x_2x_2} + 2\rho\nu x_1 x_2 V_{x_1x_2} \right) + r x_1 V_{x_1} + \kappa(\theta - x_2) V_{x_2} - rV = 0 \\
v(T, x_1, x_2) = \Psi(x_1, x_2)
\end{cases}
$$