Deterministic and Stochastic Wireless Network Games

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1 Introduction

In this seminar we address the general problem of power control in communications on wireless networks, following what the authors *Zhou, Bambos and Glynn* have done in [3], published in 2018.

1.1 General problem

Power control on wireless networks is a broad field with important application that have been studying for more than twenty years. The aim of this research area is to develop a good distributed power control in order to achieve certain performance on wireless communication network, with sufficient quality of service. This follows directly on the first simple distributed power control algorithm, proposed in [1] by Foschini and Miljanic, where they showed a way to reach minimum power in certain conditions of interference and noise.

A later important approach was considering the wireless networks as a noncooperative multiplayer game. Such game-theoretical formulation induces a distributed power control scheme (best response update) where at each iteration the player chooses the power to maximize its utility. An interesting feature in this approach is the convergence to the Nash equilibrium.

Moreover, it is possible to realize simplier models assuming the entire external environment (as the channel noise) to be static over time, which provides, tipically, a good approximation. However, more generally, it is possible to consider a stochastic and time-varying environment which can provide a more solid model.

1.2 *Zhou, Bambos, Glynn*'s contributions

In our seminar, we present a game-theoretic formulation of power control in wireless communications with some innovations. In the first part, we consider the classic static case, but we do not impose exogenous power bounds on the feasible transmission power. This gives an interesting point of view especially when the transimission power is large. On the other hand, in the second part we allow the environment to be stochastic and time-varying.

In particular, in the deterministic case we present a new version of the fixed point theorem that is necessary for the unbonded assumption in order to estabilish the existence and the uniquness of the Nash equilibrium. In addition, we investigate convergence exhibiting two different algorithms, the synchronous and asynchronous best response dynamics. In the stochastic case, we allow the entire environment to change randomly over time and we investigate stochastic stability, searching for a unique stationary distribution.

2 Deterministic Wireless Networks

We consider a game-theoretical formulation of communications over wireless network. In this section, we take the environment fixed over time in order to analyze a simplier starting point which will be generalized in Section 3.

2.1 Game Model

Consider a network of N communications links, each of which is composed of a transmitter and a corrisponding receiver. We denote with a vector $P = (P_1, \dots, P_N)$ the **power vector** for transmission, where each component P_i is the power used by link i. We assume P to be in $\mathbb{R}_+^N$ [1].

First of all, we introduce a commonly used measure of link service quality that is indicated by **SINR**.

Definition 2.1. Let G_{ij} be the channel gain from the transmitter j to receiver i and let η_i be the overall noise. Given a power vector P, link's *SINR* is the quantity

$$R_i(P) := \frac{G_{ii}P_i}{\sum_{j \neq i} G_{ij}P_j + \eta_i}.$$

Remark 2.1.

1. The overall noise η_i includes both the intrinsic noise (e.g. thermal noise) and the extraneous noise induced by interferers outside of the system.

2. It is possible to represent l'environment of a wireless network with the pair (G, η), where $G \in \mathbb{R}_+^{N \times N}$ is the gain matrix with $G_{ii} > 0$ and η is the overall noise vector.

In this model for wireless network, each link acts as a rational agent and want to minimize its own cost. This cost is composed by two parts: first, a cost associated with power and, second, a cost associated with the quality of service received, i.e. its SINR. In particular, calling $r_i(\cdot)$ the link i's cost of power and f_i the function that maps a given SINR to a cost for link i, we can define the total cost of link i as the quantity

$$C_i := r_i(P) + f_i\left(\frac{G_{ii}P_i}{\sum_{j \neq i} G_{ij}P_j + \eta_i}\right). \tag{2.1}$$

Note that we can take $r_i(\cdot)$ as any convex and increasing function and that $f_i : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ can be interpreted as the inverse utility derived from the quality service. Furthermore, because of its meaning, it is reasonable to make the following structural assumptions.

[1] $\mathbb{R}_+$ denotes nonnegative real numbers.

Assumption 2.1.

1. *f_i strictly deacreasing and strictly convex;*
2. *$lim_{x\to+\infty} f_i(x) = 0$ and $lim_{x\to 0^+} f_i(x) = +\infty$* [2]
3. *f_i continuously differentiable.*

Remark 2.2. These assumptions imply that f_i' is continous on $(0, +\infty)$ and it is strictly negative $\forall x \in (0, +\infty)$. Moreover, $\lim_{x\to+\infty} f_i'(x) = 0$ and $\lim_{x\to 0^+} f_i'(x) = -\infty$. Possible examples of such functions are:

- $f_i(x) = \frac{a}{x^p}$ $\forall a, p > 0$;
- $f_i(x) = b\left(e^{\frac{a}{x^p}} - 1\right)$ $\forall b, a, p > 0$;
- $f_i(x) = \frac{c}{\log^q(x+1)}$ $\forall c, q > 0$;

or any convex combinations of those functions.

This setup naturally induces a N-player noncooperative game and so we can proceed analyzing the concept of Nash equilibrium and best response function.

Definition 2.2. P is a (pure strategy) **Nash equilibrium** if, for each i

$$C_i(P_1, \dots, P_{i-1}, P_i, \dots, P_N) \le C_i(P_1, \dots, P_{i-1}, P_i', \dots, P_N) \quad \forall P_i' \in \mathbb{R}_+$$

In particular, P is a Nash equilibrium if and only if no transmitter i has any incentive to change its power P_i. Under this formulation, one should ask if there exists a Nash equilibrium, if it is unique, if there exist some update schemes that converge to a Nash equilibrium. These are the questions that motivated the authors and previous works. In this paper, following what is done in [3], we do not restrict the action space (where lies the power level P) to be bounded. Indeed, when the action space is a compact set, such existence of NE is a directly consequence of Brouwer's fixed-point theorem, or more generally, of Kakutani's fixed-point theorem. In this case we must develop a new version of fixed-point theorem, as a variant of Tarski's one, which operates in the absence of the endogenous bound.

Firstly we want to discuss of Nash equilibrium from another point of view, i.e. via best response functions.

Definition 2.3. The **best response function** $g_i : \mathbb{R}_+^N \longrightarrow \mathbb{R}_+$ of link i is

$$g_i(P) = \arg\min_{P_i' \in \mathbb{R}_+} C_i(P_1, \dots, P_{i-1}, P_i', \dots, P_N) \tag{2.2}$$

[2]We can assume $f_i(0) = \infty$ to have a continous function to all $\mathbb{R}_+$.

In other words, it gives the minimum cost of power P_i, assuming that all the other links fix their power according to the vector P. As a consequence, we can consider $g : \mathbb{R}_+^N \longrightarrow \mathbb{R}_+^N$ defined as $g(P) := (g_1(P), \ldots, g_N(P))$ and P is a Nash equilibrium if and only if P is a fixed point of g, i.e. $g(P) = P$.

In our setting, since C_i is strictly convex (sum of a convex function and a strictly convex function), for a given P, there exists a unique minimizer $P_i^* \in \mathbb{R}_+$ that minimizes the cost according to definition (2.2). Hence, the best response function is well defined and it is possible to find the minimizer $P_i^* = g_i(P)$ via vanishing the derivative of the cost. In this way we find the condition

$$- f_i'\left(\frac{G_{ii}P_i^*}{\sum_{j\neq i} G_{ij}P_j + \eta_i}\right)\left(\frac{G_{ii}}{\sum_{j\neq i} G_{ij}P_j + \eta_i}\right) = r_i'(P_i^*). \tag{2.3}$$

Now, we are ready to prove an important Lemma which gives us some peculiar properties of the our best response function that are useful hereafter.

Lemma 2.1. *Suppose that for each $i = 1, \ldots, N$ the map $y \mapsto f_i'(y)y$ is increasing in $y \geq 0$. Then*

1. $\forall P, \hat{P} \in \mathbb{R}_+^N, \quad P \leq \hat{P} \Rightarrow g(P) \leq g(\hat{P})$.
2. *For any given P and for any $\alpha > 1$, $\alpha g(P) > g(\alpha P)$.*
3. *For any $P \in \mathbb{R}_+^N$ with all positive components, there exists $\alpha_0 > 0$ such that $\alpha P > g(\alpha P)$ $\forall \alpha \geq \alpha_0$.*

Proof.

1. Assume for contradiction that $g_i(P) > g_i(\hat{P})$ for some i. Denote $g_i(P) = P_i^*$ and $g_i(\hat{P}) = \hat{P}_i^*$. Since $r_i(\cdot)$ is convex, it follows that $r_i'(\hat{P}_i^*) \leq r_i'(P_i^*)$. Starting from equation (2.3), multiplying and dividing the LHS respectively by P_i^* and $\hat{P}_i^*$, we have

$$-f_i'\left(\frac{G_{ii}P_i^*}{\sum_{j\neq i} G_{ij}P_j + \eta_i}\right)\left(\frac{G_{ii}P_i^*}{\sum_{j\neq i} G_{ij}P_j + \eta_i}\right)\frac{1}{P_i^*} = r_i'(P_i^*),$$
$$-f_i'\left(\frac{G_{ii}\hat{P}_i^*}{\sum_{j\neq i} G_{ij}\hat{P}_j + \eta_i}\right)\left(\frac{G_{ii}\hat{P}_i^*}{\sum_{j\neq i} G_{ij}\hat{P}_j + \eta_i}\right)\frac{1}{\hat{P}_i^*} = r_i'(\hat{P}_i^*).$$

Since $P \leq \hat{P}$ and $P_i^* > \hat{P}_i^*$, it follows that

$$\frac{G_{ii}P_i^*}{\sum_{j\neq i} G_{ij}P_j + \eta_i} > \frac{G_{ii}\hat{P}_i^*}{\sum_{j\neq i} G_{ij}\hat{P}_j + \eta_i}.$$

Moreover, since $f_i'(y)y$ is positive and strictly decreasing in y, we have

$$\begin{aligned}
r_i'(P_i^*) &= -f_i'\left(\frac{G_{ii}P_i^*}{\sum_{j\neq i} G_{ij}P_j + \eta_i}\right)\left(\frac{G_{ii}P_i^*}{\sum_{j\neq i} G_{ij}P_j + \eta_i}\right)\frac{1}{P_i^*} \\
&< -f_i'\left(\frac{G_{ii}\hat{P}_i^*}{\sum_{j\neq i} G_{ij}\hat{P}_j + \eta_i}\right)\left(\frac{G_{ii}\hat{P}_i^*}{\sum_{j\neq i} G_{ij}\hat{P}_j + \eta_i}\right)\frac{1}{\hat{P}_i^*} = r_i'(\hat{P}_i^*),
\end{aligned}$$

which contradicts $r_i'(\hat{P}_i^*) \leq r_i'(P_i^*)$.

2. Set any i and define $\bar{P}_i = g_i(P)$, $\hat{P}_i = g_i(\alpha P)$. Since $\alpha > 1$, we have $\alpha P > P$ and so $\hat{P}_i > \bar{P}_i$, due to previous Statement. As before, by convexity of $r_i(\cdot)$, we have that $r_i'(\hat{P}_i) \geq r_i'(\bar{P}_i)$. The optimality conditions give us

$$-f_i'\left(\frac{G_{ii}\bar{P}_i}{\sum_{j\neq i} G_{ij}P_j + \eta_i}\right)\left(\frac{G_{ii}}{\sum_{j\neq i} G_{ij}P_j + \eta_i}\right) = r_i'(\bar{P}_i), \tag{2.4}$$

$$-f_i'\left(\frac{G_{ii}\hat{P}_i}{\alpha\sum_{j\neq i} G_{ij}P_j + \eta_i}\right)\left(\frac{G_{ii}}{\alpha\sum_{j\neq i} G_{ij}P_j + \eta_i}\right) = r_i'(\hat{P}_i). \tag{2.5}$$

Assume, now, by contradiction that $g_i(\alpha P) \geq \alpha g_i(P)$, i.e. $\hat{P}_i \geq \alpha\bar{P}_i$; then we can write $\hat{P}_i = \beta\bar{P}_i$ for some $\beta \geq \alpha > 1$. Plugging this in the equation (2.5) it turns out that

$$-f_i'\left(\frac{G_{ii}\bar{P}_i}{\frac{\alpha}{\beta}\sum_{j\neq i} G_{ij}P_j + \frac{\eta_i}{\beta}}\right)\left(\frac{G_{ii}}{\alpha\sum_{j\neq i} G_{ij}P_j + \eta_i}\right) = r_i'(\hat{P}_i).$$

Now, since $\frac{\alpha}{\beta} \leq 1$ and $\beta > 1$,

$$\left(\frac{G_{ii}\bar{P}_i}{\frac{\alpha}{\beta}\sum_{j\neq i} G_{ij}P_j + \frac{\eta_i}{\beta}}\right) > \left(\frac{G_{ii}\bar{P}_i}{\sum_{j\neq i} G_{ij}P_j + \eta_i}\right),$$

and, beacuse $-f_i'$ is a strictly decreasing function, it follows that

$$-f_i'\left(\frac{G_{ii}\bar{P}_i}{\frac{\alpha}{\beta}\sum_{j\neq i} G_{ij}P_j + \frac{\eta_i}{\beta}}\right) < -f_i'\left(\frac{G_{ii}\bar{P}_i}{\sum_{j\neq i} G_{ij}P_j + \eta_i}\right).$$

Then, using also the inequality $\frac{G_{ii}}{\alpha\sum_{j\neq i} G_{ij}P_j+\eta_i} < \frac{G_{ii}}{\sum_{j\neq i} G_{ij}P_j+\eta_i}$, we have that

$$\begin{aligned}
r_i'(\hat{P}_i) &= -f_i'\left(\frac{G_{ii}\bar{P}_i}{\frac{\alpha}{\beta}\sum_{j\neq i} G_{ij}P_j + \frac{\eta_i}{\beta}}\right)\left(\frac{G_{ii}}{\alpha\sum_{j\neq i} G_{ij}P_j + \eta_i}\right) \\
&< -f_i'\left(\frac{G_{ii}\bar{P}_i}{\sum_{j\neq i} G_{ij}P_j + \eta_i}\right)\left(\frac{G_{ii}}{\sum_{j\neq i} G_{ij}P_j + \eta_i}\right) = r_i'(\bar{P}_i),
\end{aligned}$$

which is a contradiction and so $\alpha g_i(P) > g_i(\alpha P)$. Since it is true for any i we have thesis.

3. We divide this proof into two steps. For the first step, we prove the statement assuming that r_i is linear, i.e. if the cost $C_i(P) = c_i P_i + f_i\left(\frac{G_{ii}P_i}{\sum_{j\neq i} G_{ij}P_j+\eta_i}\right)$, for some $c_i > 0$. Since $-f_i'(x)$ is strictly decreasing, its inverse exists (and well defined in $(0, +\infty)$) and is also strictly decreasing[3]. Let define h_i this inverse function. Due to properties of f_i', it follows that $\lim_{x\to\infty} h_i(x) = 0$ and $\lim_{x\to 0^+} h_i(x) = \infty$. Consequently, for any $x \in (0, +\infty)$ we have $\lim_{\alpha\to\infty} h_i(\alpha x) = 0$, which implies that $h_i(\alpha x) = \frac{o(\alpha)}{\alpha}$ as $\alpha \to \infty$ (x fixed).

   Fix, now, an arbitrary $P \in \mathbb{R}_+^N$ with all positive components. For $\alpha > 1$ and for any i we can rewrite the optimality condition as

$$
\begin{aligned}
g_i(\alpha P) &= \frac{\alpha \sum_{j\neq i} G_{ij}P_j + \eta_i}{G_{ii}}\ h_i\left(\frac{c_i}{G_{ii}}\left(\alpha \sum_{j\neq i} G_{ij}P_j + \eta_i\right)\right) \\
&< \frac{\alpha \sum_{j\neq i} G_{ij}P_j + \eta_i}{G_{ii}}\ h_i\left(\frac{c_i}{G_{ii}}\alpha \sum_{j\neq i} G_{ij}P_j\right) \\
&= \frac{\alpha \sum_{j\neq i} G_{ij}P_j + \eta_i}{G_{ii}}\ \frac{o(\alpha)}{\alpha} = o(\alpha).
\end{aligned}
$$

   Hence, for α large enough, $\alpha P_i > g_i(\alpha P_i)$ for any i. Set α_0 to be this α. We have that $\alpha_0 P > g(\alpha_0 P)$, and consequently for any $\alpha > \alpha_0$,

$$
\alpha P = \frac{\alpha}{\alpha_0}\alpha_0 P > \frac{\alpha}{\alpha_0} g(\alpha_0 P) > g\left(\frac{\alpha}{\alpha_0}\alpha_0 P\right) = g\left(\alpha P\right),
$$

   where we used the second statement of the lemma since $\frac{\alpha}{\alpha_0} > 1$.

   Now, we can prove the general case. Fix an arbitrary $P \in \mathbb{R}_+^N$ with all positive components and set $P^* = g(P)$. By the optimality condition, for any i we have that

$$
-f_i'\left(\frac{G_{ii}P_i^*}{\sum_{j\neq i} G_{ij}P_j + \eta_i}\right)\left(\frac{G_{ii}}{\sum_{j\neq i} G_{ij}P_j + \eta_i}\right) = r_i'(P_i^*) =: c_i. \tag{2.6}
$$

   Now, fix $\alpha > 1$ and plug αP into previous equation in replacement of P to obtain the optimal condition for $g(\alpha P)$. We have

$$
-f_i'\left(\frac{G_{ii}P_i^*}{\alpha \sum_{j\neq i} G_{ij}P_j + \eta_i}\right)\left(\frac{G_{ii}}{\alpha \sum_{j\neq i} G_{ij}P_j + \eta_i}\right) = c_i.
$$

   Since the previous equality must hold for any α, we need that P_i^* increase by the same argument. However, since r_i is convex, if P_i^* increases, also $r_i'(P_i^*)$

[3] It follows simply because given a function $f(x)$ strictly decreasing and surjective, it is bijective and so the inverse function exists. Moreover, taken $c < d \in Im(f)$, there exist $a, b \in dom(f)$ s.t. $f(a) = c$, $f(b) = d$, and if by contradiction $a = f^{-1}(c) \leq f^{-1}(d) = b$, by strictly monotonicity, $c = f(a) \geq f(b) = d$, which is a contradiction.

increases and at a certain point it will exceed c_i. Therefore, because $-f_i'(y)y$ is positive srictly decreasing in y, we need that P_i^* increases less compared with $r_i(P_i) = c_i P_i$ for the c_i defined in equation (2.6). This implies that the value $g(\alpha P)$ under a general convex cost r_i will be smaller than the value of $g(\alpha P)$ under the corresponding linear cost. Then, due to first step we have the thesis also in the general case.

$\square$

Remark 2.3. Since $f_i'(\cdot) < 0$ is increasing, $-f_i'(\cdot) > 0$ is decreasing. So the assumption $-f_i'(y)y$ decreasing means that $-f_i'(y)$ must decrease faster than y increases. Moreover, note that all the examples made in Remark 2.2 satisfy this assumption.

2.2 Nash Equilibrium Characterization

In this section, we want to estabilish the existence of the Nash equilibrium in the absence of endogenous bounds on the power levels. We firstly introduce the classical Tarski-Kantorovitch fixed-point theorem and then we present a new variant of that theorem that works around unboundedness issue, since all fixed-point theorems (Brouwer's, Kakutani's and classical Tarski's) cannot be directly applied here.

2.2.1 Fixed-point Theorems

A huge amount of different kind of fixed-point theorems exists in the literature. In our setting, we are interested in ones that operate on partially ordered sets, as the Tarski-Kantorovitch theorem.

Definition 2.4. Let $(P, \leq)$ be a partially ordered set and $G : P \to P$ be a map.

1. A subset $S \subset P$ is called a *chain* if S is totally ordered under $\leq$.
2. G is called a *poset-mapping continous map* if, for each countable chain $\{c_i\}$ having a supremum, $G(\sup\{c_i\}) = \sup\{G\{c_i\}\}$.

Remark 2.4. If the poset P is also a metric space (such as $\mathbb{R}_+^n$) with the usual continuity concept, a continuous map and a poset-mapping continuous map are not the same. Indeed, a poset-mapping continuous map is necessarily monotonic (increasing). This follows easily since if we take $x \leq y \in P$, then $y = \sup\{x, y\}$; therefore $G(y) = \sup\{G(x), G(y)\}$, which means that $G(x) \leq G(y)$.
In our setting, the best response function g is continous and monotonic and this is sufficient to prove that g is a poset-mapping continuous map. Indeed, take any sequence $\{p^i\} \subset \mathbb{R}_+^N$, then by monotonicity it turns out that $g(\sup\{p^i\}) \geq g(p^i)\ \forall i$, that implies that $g(\sup\{p^i\}) \geq \sup\{g(p^i)\}$. On the other hand, by continuity we have that

$g(\sup\{p^i\}) \leq \sup\{g(p^i)\}$[4] . Therefore, $g(\sup\{p^i\}) = \sup\{g(p^i)\}$, which means that g is a poset-mapping continous map.

We are now ready to state the classical Tarski-Kantorovitch fixed-point theorem.

Theorem 2.1 (Tarski-Kantorovitch fixed-point theorem)**.** *Let $(P, \leq)$ be a partially ordered set and $G : P \to P$ be a poset-mapping continous map. Suppose that:*
1. *$\exists p_1 \in P$ s.t. $p_1 \leq G(p_1)$.*
2. *every countable chain cointained in $\{x \in P \mid x \geq p_1\}$ has a supremum.*

Then G has a fixed point.

The second assumptions of the previous theorem does not hold in our unbounded setting for the domain of P. And so we can state and proof the new variant of fixed-point theorem.

Theorem 2.2. *Let $(P, \leq)$ be a partially ordered set and $G : P \to P$ be a poset-mapping continous map. Suppose that:*
1. *$\exists p_1 \in P$ s.t. $p_1 \leq G(p_1)$.*
2. *every bounded countable chain cointained in $\{x \in P \mid x \geq p_1\}$ has a supremum.*
3. *$\exists p_2 > p_1 \in P,$ s.t. $p_2 \geq G(p_2)$.*

Then G has a fixed point.

Proof. Starting from p_1, consider the sequence $\{G^n(p_1)\}_{n\in\mathbb{N}}$ where G^n means to apply n times the function G. As noted in Remark 2.4, G is monotonic and so $\forall n \in \mathbb{N}$ $G^{n-1}(p_1) \leq G^n(p_1)$. Hence, $\{G^n(p_1)\}_{n\in\mathbb{N}}$ is a chain (increasing).
Moreover, due to condition 3, there exists $p_2 > p_1 \in P$ such that $p_2 \geq G(p_2)$. But, thanks to monotonicity, it turns out that

$$p_2 \geq G^n(p_2) \geq G^n(p_1) \quad \forall n \in \mathbb{N}.$$

From the last inequalities it follows that the chain $\{G^n(p_1)\}_{n\in\mathbb{N}}$ is bounded and thanks to condition 2 it has a supremum, i.e. let $p^* := \sup\left(\{G^n(p_1)\}_{n\in\mathbb{N}}\right)$.
Now, by the definition of poset-mapping continuity of G, we have

$$\begin{aligned} G(p^*) &= G(\sup\{G^n(p_1)\}_{n\in\mathbb{N}}) \\ &= \sup\left(G\left(\{G^n(p_1)\}_{n\in\mathbb{N}}\right)\right) = \sup\{G^n(p_1)\}_{n\in\mathbb{N}} = p^*. \end{aligned}$$

□

Remark 2.5. One could apply the same iteration to p_2 instead of p_1 using the latter as a lower bound to find another fixed point as the limit of $G^n(p_2)$. A priori, applied in our setting to find Nash equilibria, we could have two different ones. Actually, we'll see the uniqueness.

[4]Take $\{p^{i_k}\}$ a subsequence that converges to the $\sup\{p^i\}$, then by continuity $g(\{p^{i_k}\}) \longrightarrow g(\sup\{p^i\})$, and so $g(\sup\{p^i\}) \leq \sup\{g(p^{i_k})\} \leq \sup\{g(p^i)\}$.

Remark 2.6. Even if Theorem 2.2 is a variant of Tarski's theorem that works in unbounded setting, there is still a notion of "boundness" in the assumpions. In particular, we need the p_2 of the third condition as a sort of "upper bound" in order to prevent the iterations from going to infinity.

2.2.2 Existence and Uniqueness

We are now ready to characterize the existence and uniqueness of the Nash equilibrium using last Theorem and the properties of the best response function explored in Lemma 2.1.

Theorem 2.3. *There exists a unique Nash equilibrium.*

Proof. For existence, we want to apply Theorem 2.2 and so it is necessary to prove that the best response function g satisfies the three assumptions. Firstly, in Remark 2.4 we noted that g is a poset-mapping continuous map. Now, take any $P \in \mathbb{R}_+^N$ with all positive components, by Lemma 2.1 there exists α sufficiently large such that $\alpha P > g(\alpha P)$. Setting $p_1 = 0$ and $p_2 = \alpha P$, we have that $p_1 < p_2$, $p_1 \leq g(p_1)$ and $p_2 \geq g(p_2)$. Then, since the poset under discussion is $\mathbb{R}_+^n$, we have that every bounded subset has at least an upper bound, which means that any bounded countable chain has a supremum. Therefore, we can apply Theorem 2.2 and we obtain the existence of the Nash equilibrium as a fixed-point of the best response function.

For the uniqueness, suppose we have two different Nash equilibria $x, y \in \mathbb{R}_+^n$ with $x \neq y$ all positive components. Let i be such that $\frac{y_i}{x_i} = \max_j \frac{y_j}{x_j}$ and call $\beta = \frac{y_i}{x_i}$; wlog (without loss of generality) we can assume that $\beta > 1$. Now, we have that $\forall j \neq i \; y_j \leq \beta x_j$ and $y_i = \beta x_i$, which means that $y \leq \beta x$. Hence, by monotonicity

$$g_i(\beta x) \geq g_i(y) = y_i. \tag{2.7}$$

On the other hand, by Lemma 2.1

$$g_i(\beta x) < \beta g_i(x) = \beta x_i. \tag{2.8}$$

Therefore, we have that $y_i < \beta x_i$ that contradicts the fact that $y_i = \beta x_i$. So, the Nash equilibrium is unique. □

Remark 2.7. Proving the uniqueness we exploited again the unboundness of our setting. Indeed, if one impose bounds on maximum power, then $g's$ domain would be bounded. But, when we consider the quantity βx with $\beta > 1$, it could lie outside of the bounded feasible power set. Consequently, a priori, one could not apply g to such βx and so we cannot assume boundness.

Example 2.1. Consider the two-link case in which

$$f_i(x) = \frac{1}{x} \qquad i = 1, 2 \qquad\qquad r_1(x) = cx \quad \textit{with } 0 < c < 1,$$
$$r_2(x) = x.$$

Fix $\eta_1 = \eta_2 = 0$ and let the gain matrix be $\begin{bmatrix} G & G' \\ G' & G \end{bmatrix}$, where $G > G'$. In this case, by equation (2.1) we have

$$C_1(P) = \frac{G'P_2}{GP_1} + cP_1 \qquad C_2(P) = \frac{G'P_1}{GP_2} + P_2.$$

Now, imposing condition (2.3) it is possible to find the unique Nash equilibrium as $P^{Nash} = \left(P_1^{Nash}, P_2^{Nash}\right)$ where P_1^{Nash} and P_2^{Nash} satisfy

$$\frac{G'P_2^{Nash}}{G\left(P_1^{Nash}\right)^2} = c, \qquad \frac{G'P_1^{Nash}}{G\left(P_2^{Nash}\right)^2} = 1.$$

If we put together the two previous equations, we obtain that

$$\frac{\left(P_2^{Nash}\right)^3}{\left(P_1^{Nash}\right)^3} = c, \qquad P_2^{Nash} = \frac{G'}{Gc^{\frac{1}{3}}}.$$

2.2.3 Convergence

The previous analysis on Nash equilibria through the best response function suggests us a natural way to reach the unique Nash equilibrium. Here, we want to investigate two different algorithms, one synchronous and the other asynchronous.

Synchronous Best Response Dynamics For the synchronous case, at each iteration, each link chooses the best power that minimizes its total cost, assuming that all the other links will choose their best powers from the previous iteration. This can be outlined in the following scheme.

Algorithm 2.1: Synchronous Best Response Dynamics for Power Update

---

```
1 Each link i arbitrarily chooses an initial power P_i^0 ∈ ℝ_+
2 for k = 0, 1, 2, ... do
3     for i = 1, ..., N do
4         P_i^{k+1} = g_i(P^k);
```

Now, we show that, under this scheme, the iterations P^k (k-th power vector iterate) will converge to the unique Nash equilibrium irrespective of the initial power P^0. Indeed, it holds the following result.

Theorem 2.4. *Under Algorithm 2.1, for any initial power P_0, P^k converges to the unique Nash equilibrium.*

Proof. Firstly, we can compactly set $P^k = g^k(P^0)$. By Lemma 2.1, for any initial power vector P^0, there exists a sufficiently large $\alpha > 0$ such that

$$0 \leq P^0 \leq \alpha P^0, \tag{2.9}$$

with $g(0) > 0$ and $\alpha P^0 > g(\alpha P^0)$. Now, if we apply the function g k times to the three sides of the inequality (2.9) we have that $g^k(0) \leq g^k(P^0) \leq g^k(\alpha P^0)$. But, if we consider 0 and αP^0 as the p_1 and p_2, like we did in the existence part of Theorem 2.3, due to what we said in Remark 2.5 we obtain that both $g^k(0)$ and $g^k(\alpha P^0)$ converge. In particular, they converge to fixed-points of g, and so to the same unique Nash equilibrium. This implies that also $g^k(P^0)$ converges to the Nash equilibrium, as we wanted to show. □

Asynchronous Best Response Dynamics An alternative scheme could be made if, at each iteration, not every link necessarily updates its power. This can be outlined in the following scheme.

Algorithm 2.2: Asynchronous Best Response Dynamics for Power Update

Each link i arbitrarily chooses an initial power $P_i^0 \in \mathbb{R}_+$

for $k = 0, 1, 2, \ldots$ **do**

  Let $\mathcal{N}^k \subset \{1, 2, \ldots, N\}$ be a (possibly empty) set of updating links at k

  for $i \in \mathcal{N}^k$ **do**

    $P_i^{k+1} = g_i(P^k)$;

To investigate the convergence of the previous algorithm we, firstly, introduce some notation.

Definition 2.5. Let $\mathcal{N} \subset \{1, 2, \ldots, N\}$ and $\{\mathcal{N}^k\}_{k=1}^{\infty}$ be a sequence of sets contained in $\{1, 2, \ldots, N\}$ $\forall k$. Define

- the *partial best response function* $g^{\mathcal{N}} : \mathbb{R}_+^N \longrightarrow \mathbb{R}_+^N$ the function

$$g_i^{\mathcal{N}}(P) := \begin{cases} g_i(P) & i \in \mathcal{N} \\ P_i & i \notin \mathcal{N} \end{cases} \tag{2.10}$$

- the set of *complete cycle times* $\{t_j\}_{j=0}^{\infty}$ the quantity

$$t_j := \begin{cases} 0 & j = 0 \\ \inf\{k \mid k > t_{j-1} \ \forall\ 1 \le i \le N \ \ \exists\ t_{j-1} < l \le k \ s.t.\ i \in \mathcal{N}^l\} & j \ge 1 \end{cases} \tag{2.11}$$

Intuitively, $g^{\mathcal{N}}$ represent the general iteration of the asynchronous best besponse algorithm, while t_j is the first iteration by when every link has update its power at least j times.

Remark 2.8. Note that, by definition, monotonicity still holds under partial best response updates, i.e. if $P \ge \tilde{P}$, then $g_i^{\mathcal{N}}(P) \ge g_i^{\mathcal{N}}(\tilde{P})\ \forall \mathcal{N}$.

As we did before, we present a convergence result even for the second Algorithm.

Theorem 2.5. *Under Algorithm 2.2, if for each link i $|\{k : i \in \mathcal{N}^k\}| = \infty$, then P^k converges to the unique Nash equilibrium for any initial power vector P^0.*

Proof. Let $\{\mathcal{N}^l\}_{l=1}^{\infty}$ be the sequence of updating sets given in Algorithm 2.2. Firstly, we show that $P \le g(P) \ \Rightarrow\ g^n(P) \le g^{\{\mathcal{N}^l\}_{l=1}^{t_n}}(P)\ \forall n \ge 1$. We proceed by induction.

- The base case consists in proving that $g(P) \le g^{\{\mathcal{N}^l\}_{l=1}^{t_1}}(P)$. By monotonicity, we have
$$P \le g^{\{\mathcal{N}^l\}_{l=1}^{1}}(P) \le g^{\{\mathcal{N}^l\}_{l=1}^{2}}(P) \le \ldots \le g^{\{\mathcal{N}^l\}_{l=1}^{t_1}}(P). \tag{2.12}$$
Now, for each $i \in \{1, 2, \ldots, N\}$, let $\mathcal{N}^{k_i}$ be the first set that contains i. By definition of t_1, we have $k_i \le t_1$. Hence, since $i \in \mathcal{N}^{k_i}$ and due to previous inequality (2.12), it turns out that
$$g_i^{\{\mathcal{N}^l\}_{l=1}^{t_1}}(P) \ge g^{\{\mathcal{N}^l\}_{l=1}^{k_i}}(P) \ge g_i^{\mathcal{N}^{k_i}}\left(g_i^{\{\mathcal{N}^l\}_{l=1}^{k_i-1}}(P)\right) \ge g_i^{\mathcal{N}^{k_i}}(P) = g_i(P),$$
where we used the convention $g_i^{\{\mathcal{N}^l\}_{l=1}^{k_i-1}}(P) = P$ if $k_i - 1 = 0$.

- For the inductive step, assume that $g^n(P) \le g^{\{\mathcal{N}^l\}_{l=1}^{t_n}}(P)$, then we have
$$\begin{aligned} g^{n+1}(P) = g(g^n(P)) \le g\left(g^{\{\mathcal{N}^l\}_{l=1}^{t_n}}(P)\right) &\le g^{\{\mathcal{N}^l\}_{l=t_n+1}^{t_{n+1}}}\left(g^{\{\mathcal{N}^l\}_{l=1}^{t_n}}(P)\right) \\ &= g^{\{\mathcal{N}^l\}_{l=1}^{t_{n+1}}}(P). \end{aligned}$$

By a similar argument, it is possible to prove that if $P \ge g(P)$, then $g^n(P) \ge g^{\{\mathcal{N}^l\}_{l=1}^{t_n}}(P)$, for each $n \ge 1$.
Finally, thanks to Lemma 2.1, pick $\alpha > 0$ such that $0 \le P^0 \le \alpha P^0$ with $0 < g(0)$ and

$\alpha P^0 > g(\alpha P^0)$. Since $g^{\{\mathcal{N}^l\}_{l=1}^n}(0)$ is an increasing sequence and $g^{\{\mathcal{N}^l\}_{l=1}^n}(0) \le g^n(0)$, it has a limit which verifies

$$\lim_{n\to\infty} g^{\{\mathcal{N}^l\}_{l=1}^n}(0) \le \lim_{n\to\infty} g^n(0) \ =: p^*,$$

where p^* is the unique Nash equilibrium. On the other hand, since for any $n \ge 1$ $g^n(0) \le g^{\{\mathcal{N}^l\}_{l=1}^{t_n}}(0)$ and by hypothesis $t_n \to \infty$, we have

$$\begin{aligned}\lim_{n\to\infty} g^{\{\mathcal{N}^l\}_{l=1}^n}(0) &= \lim_{n\to\infty} g^{\{\mathcal{N}^l\}_{l=1}^{t_n}}(0)\\ &\ge \lim_{n\to\infty} g^n(0) = p^*.\end{aligned}$$

Therefore, $\lim_{n\to\infty} g^{\{\mathcal{N}^l\}_{l=1}^n}(0) = p^*$. Similarly, one can prove that $\lim_{n\to\infty} g^{\{\mathcal{N}^l\}_{l=1}^n}(\alpha P) = p^*$ and together these two results imply the thesis, i.e.

$$\lim_{n\to\infty} g^{\{\mathcal{N}^l\}_{l=1}^n}(P^0) = p^*.$$

□

2.2.4 Price of Anarchy

In game theory, it is often used an interesting notion konwn as the **Price of Anarchy** (PoA) to measure a sort of efficiency of a Nash equilibrium. Indeed it is defined as the ratio between the cost achieved by the worst-case Nash equilibrium and the cost achieved by the social optimal solution, which is the optimal system cost (namely, the solution of an appropriate social objective) that tipically is represented by the sum of all the individual costs. The PoA is by definition at least 1 and the smaller the value, the more efficient the Nash equilibrium is. The concept of the PoA was firstly proposed by Koutsoupias and Papadimitriou in [2] and it is usually used in game theory and computer science theory, even in more general forms.

We are interested in verify if there exists an upper bound for the PoA (or find sufficient conditions to have a bound), since in general it may not exist. Indeed the following example shows that there cannot be a bound on PoA for the general case.

Example 2.2. Consider the two-link case of the example 2.1. Let P^* be the optimal solution of the social objective function $C(P) = C_1(P)+C_2(P) = \frac{G'P_2}{GP_1}+cP_1+\frac{G'P_1}{GP_2}+P_2$. Since $c < 1$, it follows that

$$\begin{aligned}C(P^*) \le C((1,1)) &= C_1((1,1)) + C_2((1,1))\\ &= \frac{G'}{G} + c + \frac{G'}{G} + 1 = 2\frac{G'}{G} + c + 1\\ &< 2\left(\frac{G'}{G} + 1\right).\end{aligned}$$

While, evaluating the social objective $C(P)$ at P^{Nash}, since it satisfies $\frac{\left(P_2^{Nash}\right)^3}{\left(P_1^{Nash}\right)^3} = c$ and $P_2^{Nash} = \frac{G'}{Gc^{\frac{1}{3}}}$,it turns out that

$$\begin{aligned} C(P^{Nash}) &= C_1(P^{Nash}) + C_2(P^{Nash}) \\ &> C_2(P^{Nash}) = \frac{G' \frac{P_2^{Nash}}{c^{\frac{1}{3}}}}{G \frac{G'}{Gc^{\frac{1}{3}}}} + P_2^{Nash} = 2\frac{G'}{Gc^{\frac{1}{3}}}. \end{aligned}$$

Consequently, since $G > G'$ the PoA is equal to

$$\frac{C(P^{Nash})}{C(P^*)} = \frac{2\frac{G'}{Gc^{\frac{1}{3}}}}{2\left(\frac{G'}{G}+1\right)} > \frac{1}{c^{\frac{1}{3}}\left(1+\frac{G}{G'}\right)} > \frac{1}{2\frac{G}{G'}c^{\frac{1}{3}}}.$$

So, we can take a sequence of triples (G, G', c) such that the last term $\frac{1}{2\frac{G}{G'}c^{\frac{1}{3}}} \longrightarrow \infty$, which means that the PoA will diverge to infinity and cannot be bound.

For our analysis, we can proceed considering the special case of fully homogeneous wireless network, i.e. with $f_i = \frac{1}{x}$ and $r_i(x) = cx$ for any link i. In this case each player shares the same characteristics of other players. In this special case, we obtain the following remarkable result on the PoA estimate.

Theorem 2.6. *Consider a fully homogenous power control game in which every player shares $f_i(x) = \frac{1}{x}$, $r_i(x) = cx$ and the wireless network is fully symmetric, i.e. $G_{ii} = G$, $G_{ij} = G_{ji} = G'$ $\forall i, j$ and $\eta_i = \eta$ $\forall i$. Then the PoA is at most 2 irrespective of number of players, wireless environment parametres and the cost per unit power. Namely, $\forall N \geq 1,\ \forall c,\ \forall \eta,\ G > 0,\ G' \geq 0$, it holds*

$$\frac{C(P^{Nash})}{C(P^*)} \leq 2.$$

Proof. We can divide this proof into three steps.

- Firs of all, we characterize the social optimal solution P^*.
  The social cost, as we have already said, is given by the sum of all the individual cost, i.e.
  $$C(P) = \sum_{i=1}^{N} \left\{ \frac{\sum_{j \neq i} G' P_j + \eta}{G P_i} + c P_i \right\}.$$
  We show that there must be a unique minimum of the form $P^* = (p, p, \ldots, p)$. To see this, assume on the contrary that $P^* = (P_1, P_2, \ldots, P_N)$ with not all components equal. We note that, by symmetry, the cost function $C(P)$ is invariant

under permutation. Indeed, if $J : \mathbb{R}_+^N \longrightarrow \mathbb{R}_+^N$ is a general permutation which operates on each component by $J_i : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$,

$$C(J(P)) = \sum_{i=1}^{N} \left\{ \frac{\sum_{j\neq i} G' J_j(P_j) + \eta}{G J_i(P_i)} + c J_i(P_i) \right\} = \sum_{i=1}^{N} \left\{ \frac{\sum_{j\neq i} G' P_j + \eta}{G P_i} + c P_i \right\} = C(P).$$

Therefore, in particular, all the N cyclic permutations of P^*, i.e. $(P_1, P_2, \ldots, P_N)$, $(P_2, P_3, \ldots, P_N, P_1)$, $(P_3, P_4, \ldots, P_1, P_2)$, $\ldots$, $(P_N, P_1, P_2, \ldots, P_{N-1})$ are minima of the cost function $C(P)$. Consider, now, the average of those N cycle permutations $\tilde{P} = \left(\frac{\sum_{i=1}^N P_i}{N}, \frac{\sum_{i=1}^N P_i}{N}, \ldots, \frac{\sum_{i=1}^N P_i}{N} \right)$. Evaluating $C(\cdot)$ at $\tilde{P}$ and due to some algebra, we obtain

$$\begin{aligned} C(\tilde{P}) &= N \left\{ \frac{\sum_{j\neq i} G' \frac{\sum_{i=1}^N P_i}{N} + \eta}{G \frac{\sum_{i=1}^N P_i}{N}} + c \frac{\sum_{i=1}^N P_i}{N} \right\} \\ &= N(N-1)\frac{G'}{G} + \frac{\eta}{G} \frac{N^2}{\sum_{i=1}^N P_i} + c \sum_{i=1}^N P_i. \end{aligned}$$

Now, we can rewrite the cost function as

$$\begin{aligned} C(P) &= \frac{G'}{G} \sum_{i=1}^N \sum_{j\neq i} \frac{P_j}{P_i} + \frac{\eta}{G} \sum_{i=1}^N \frac{1}{P_i} + c \sum_{i=1}^N P_i \\ &= \frac{1}{2} \frac{G'}{G} \sum_{i=1}^N \sum_{j\neq i} \left(\frac{P_j}{P_i} + \frac{P_i}{P_j} \right) + \frac{\eta}{G} \sum_{i=1}^N \frac{1}{P_i} + c \sum_{i=1}^N P_i. \end{aligned} \tag{2.13}$$

But, $\frac{P_j}{P_i} + \frac{P_i}{P_j} \geq 2$ and there exists a pair of indexes i, j such that it is > 2, since P_i's are not all the same. Hence,

$$\begin{aligned} (2.13) &> N(N-1)\frac{G'}{G} + \frac{\eta}{G} \sum_{i=1}^N \frac{1}{P_i} + c \sum_{i=1}^N P_i \\ &> N(N-1)\frac{G'}{G} + \frac{\eta}{G} \frac{N^2}{\sum_{i=1}^N P_i} + c \sum_{i=1}^N P_i = C(\tilde{P}), \end{aligned}$$

where the last inequality follows from the classical arithmetic-mean-harmonic-mean inequality (strictly again since P_i's are not all the same). Indeed,

$$\sum_{i=1}^N \frac{1}{P_i} = N \frac{1}{\frac{N}{\sum_{i=1}^N \frac{1}{P_i}}} > N \frac{1}{\frac{\sum_{i=1}^N P_i}{N}} = \frac{N^2}{\sum_{i=1}^N P_i}.$$

So, we proved that $C(P) > C(\tilde{P})$ but P is minimum and it is a contradiction. Therefore there is a unique minimum $P^* = (p, p, \ldots, p)$. To find such p we need to set the gradient of $C(\cdot)$ equal to 0 and we can find that $p = \sqrt{\frac{\eta}{cG}}$ and consequently, $P^* = \left(\sqrt{\frac{\eta}{cG}}, \sqrt{\frac{\eta}{cG}}, \dots, \sqrt{\frac{\eta}{cG}}\right)$.

- For the second step, we can proceed characterizing the Nash equilibrium. Taken $g(\cdot)$ the best response function, the Nash equilibrium satisfies $g(P^{Nash}) = P^{Nash}$. Imposing optimal condition (2.3) for each i, we have that

$$\frac{\sum_{j \neq i} G' P_j + \eta}{G P_i^2} = c.$$

Thanks to Theorem 2.3 we know that there exists a unique Nash equilibrium. By symmetry, the Nash equilibrium must have all components equals. Indeed, as we did before, if $\bar{P}$ is a NE, then also an its permutation satisfies the previous condition and so it would be another NE (against the uniqueness). Hence, $P^{Nash} = (p^{Nash}, p^{Nash}, \dots, p^{Nash})$ where p^{Nash} satisfies

$$\frac{\sum_{j \neq i} G' p^{Nash} + \eta}{G (p^{Nash})^2} = c.$$

Now, solving the second order equation (and excluding the negative root) we find that

$$p^{Nash} = \frac{(N-1)G' + \sqrt{((N-1)G')^2 + 4Gc\eta}}{2Gc}.$$

- Finally, using what we obtained in previous steps, we are ready to realize the upper bound for the PoA.

$$\begin{aligned}
\frac{C(P^{Nash})}{C(P^*)} &= \frac{\sum_{i=1}^N \left\{ \frac{\sum_{j \neq i} G' p^{Nash} + \eta}{G p^{Nash}} + c p^{Nash} \right\}}{\sum_{i=1}^N \left\{ \frac{\sum_{j \neq i} G' p + \eta}{G p} + c p \right\}} \\
&= \frac{(N-1)\frac{G'}{G} + \frac{\eta}{G p^{Nash}} + c p^{Nash}}{(N-1)\frac{G'}{G} + \frac{\eta}{Gp} + cp} \\
&= \frac{(N-1)\frac{G'}{G} + \frac{2c\eta}{(N-1)G' + \sqrt{((N-1)G')^2 + 4Gc\eta}} + \frac{(N-1)G' + \sqrt{((N-1)G')^2 + 4Gc\eta}}{2G}}{(N-1)\frac{G'}{G} + 2\sqrt{\frac{\eta c}{G}}} \\
&\leq \frac{(N-1)\frac{G'}{G} + \frac{2c\eta}{\sqrt{4Gc\eta}} + \frac{(N-1)G' + \sqrt{((N-1)G')^2} + \sqrt{4Gc\eta}}{2G}}{(N-1)\frac{G'}{G} + 2\sqrt{\frac{\eta c}{G}}} \\
&= \frac{2(N-1)\frac{G'}{G} + 2\sqrt{\frac{c\eta}{G}}}{(N-1)\frac{G'}{G} + 2\sqrt{\frac{c\eta}{G}}} \leq 2.
\end{aligned}$$

where for the inequality we used the fact that $\sqrt{x+y} \leq \sqrt{x}+\sqrt{y}$ for any $x, y \geq 0$.

$\square$

3 Stochastic Wireless Networks

In this section we consider a wireless communications setting in which the environment (gain matrix and generalized noise) is random. Firstly, we focus on a worst-case stability characterization of the best response update under such random environment.

3.1 Worst-Case Stability Characterization

First of all, we define our setting.

Definition 3.1. We denote the **environment** for link i (interferences and noise on receiver i) as $\theta_i = \left(\{G_{ij}\}_{j=1}^N, \eta_i\right)$. We denote all the environments in a single matrix as $\theta = (G, \eta) \in \mathbb{R}_+^{N\times(N+1)}$. Finally, we denote with $\theta^k = (G^k, \eta^k) \in \mathbb{R}_+^{N\times(N+1)}$ the joint environment in the k-th iteration following the power control update under Algorithm 2.1.

Now, we note that there is a natural partial ordering on the set of all environments. We can say that, for each individual link i, the environment is more "friendly" to it if G_{ii} is larger and all other G_{ij} and the noise η_i are smaller. Namely, a more "friendly" environment allows link i to transmit using less power while achieving the same SINR. More precisely, we have the following definition.

Definition 3.2. Let $\theta_i = \left(\{G_{ij}\}_{j=1}^N, \eta_i\right)$, $\tilde{\theta}_i = \left(\{\tilde{G}_{ij}\}_{j=1}^N, \tilde{\eta}_i\right) \in \mathbb{R}_+^{N+1}$. We define the partial ordering $\leq$ on $\mathbb{R}_+^{N+1}$ as

$$\theta_i \leq \tilde{\theta}_i \;\Leftrightarrow G_{ii} \geq \tilde{G}_{ii},\; G_{ij} \leq \tilde{G}_{ij}\; \forall j \neq i,\; \eta_i \leq \tilde{\eta}_i.$$

Moreover, let $\theta, \tilde{\theta} \in \mathbb{R}_+^{N\times(N+1)}$ be the joint environments. We say that $\theta \leq \tilde{\theta}$ if and only if $\theta_i \leq \tilde{\theta}_i$ for any i.

Now, we can consider the best response function as a bivariate function $g(P, \theta)$, since we have an explicit dependence on the environment. The properties laid down in Lemma 2.1 are still valid for the first variable of the best response function. Similarly, it is possible to prove a monotonic property for the second variable, which implies that $g(P, \theta)$ is bimonotonic.

Lemma 3.1. *For any fixed $P \in \mathbb{R}_+^N$, $\theta \leq \tilde{\theta} \Rightarrow g(P, \theta) \leq g(P, \tilde{\theta})$.*

Furthermore, because of its meaning, it is reasonable to make the following structural assumptions.

Assumption 3.1.

1. $\forall k,\ \forall i,\ 0 < \underline{G}_{ii} \leq G_{ii}^k \leq \overline{G}_{ii}$;

2. $\forall k,\ \forall i \neq j,\ 0 \leq \underline{G}_{ij} \leq G_{ij}^k \leq \overline{G}_{ij}$;

3. $\forall k,\ \forall i\ 0 < \underline{\eta}_i \leq \eta_i^k \leq \overline{\eta}_i$.

Namely, we assume that the gain matrix and the generalized noise are bounded. Moreover, set

$$\underline{\theta}_i = (0, \dots, 0, \overline{G}_{ii}, 0, \dots, 0, \underline{\eta}_i)$$
$$\overline{\theta}_i = (\overline{G}_{i1}, \dots, \overline{G}_{ii-1}, \underline{G}_{ii}, \overline{G}_{ii+1}, \dots, \overline{G}_{iN}, \overline{\eta}_i),$$

the previous assumption imply the environment boundedness

$$\underline{\theta}_i \leq \theta_i^k \leq \overline{\theta}_i,$$

or equivalently, $\underline{\theta} \leq \theta^k \leq \overline{\theta}$ for any k. We denote by $\mathcal{U}$ the set of all such environments, i.e. $\mathcal{U} = \left\{\theta \mid \underline{\theta} \leq \theta \leq \overline{\theta}\right\}$.

We are, now, ready to prove a worst-case characterization of the behaviour of the best response function under a random environment.

Theorem 3.1. *Given a constant $\epsilon \in \mathbb{R}_+^{N\times(N+1)}$, let $P^e(\epsilon)$ be the equilibrium power vector under Algorithm 2.1 when the joint environment is constant: $\theta^k = \epsilon$, for any k. Then, if $\theta^k \in \mathcal{U}\ \forall k$, for any initial power vector $P^0 \in \mathbb{R}_+^N$, we have*

$$P^e(\underline{\theta}) \leq \liminf_{k\to\infty} P^k \leq \limsup_{k\to\infty} P^k \leq P^e(\overline{\theta}),$$

where P_i^k is given in Algorithm 2.1. Furthermore, if $P^e(\underline{\theta}) \leq P \leq P^e(\overline{\theta})$, then for any $\theta \geq \underline{\theta}$ we have $P^e(\underline{\theta}) \leq g(P, \theta) \leq P^e(\overline{\theta})$.

Proof. Fix $P^0 \in \mathbb{R}_+^N$. By Lemma 2.1, for any $Q > 0$, we can take an $\alpha > 0$ such that $\alpha Q \geq P^0 \geq 0$. Set $\overline{p}^0 = \alpha Q$ and $\underline{P}^0 = 0$. Let $P^k = h(P^0, \theta^1, \dots, \theta^k)$ be the k-iterate in Algorithm 2.1 with initial power equal to P^0 and the realizations of the joint environments being $\theta^1, \dots, \theta^k$. Set $\overline{P}^k = h(\overline{p}^0, \overline{\theta}, \overline{\theta}, \dots, \overline{\theta})$ and $\underline{P}^k = h(\underline{P}^0, \underline{\theta}, \underline{\theta}, \dots, \underline{\theta})$. Since $\underline{P}^0 \leq P_i^0 \leq \overline{P}^0$, by monotonicity on the first variable, we have

$$h(\underline{P}^0, \theta^1, \dots, \theta^k) \leq h(P^0, \theta^1, \dots, \theta^k) \leq h(\overline{P}^0, \theta^1, \dots, \theta^k).$$

On the other hand, by monotonicity on the environment variables, we have

$$h(\underline{P}^0, \underline{\theta}, \underline{\theta}, \dots, \underline{\theta}) \leq h(\underline{P}^0, \theta^1, \dots, \theta^k),$$

$$h(\overline{P}^0, \overline{\theta}, \overline{\theta}, \dots, \overline{\theta}) \geq h(\overline{P}^0, \theta^1, \dots, \theta^k).$$

Hence, it turns out that $\underline{P}^k \leq P^k \leq \overline{P}^k \ \forall k$.
Furthermore, $\overline{P}^k$ decreases and $\lim_{k\to\infty} \overline{P}^k = P^e(\overline{\theta})$, while $\underline{P}^k$ increases and $\lim_{k\to\infty} \underline{P}^k = P^e(\underline{\theta})$. So, we have

$$\overline{P}^k \geq P^j \quad \forall j \geq k, \tag{3.14}$$

$$\underline{P}^k \leq P^j \quad \forall j \geq k. \tag{3.15}$$

Now, define $U^k := \sup_{j\geq k} P^j$ and $L^k := \inf_{j\geq k} P^j$. Inequalities (3.14) and (3.15) imply that $U^k \leq \overline{P}^k$ and $L^k \geq \underline{P}^k$. Therefore,

$$P^e(\underline{\theta}) \leq \liminf_{k\to\infty} P^k \leq \limsup_{k\to\infty} P^k \leq P^e(\overline{\theta}).$$

For the second part of the theorem, again by bimonotonicity, if $P^e(\underline{\theta}) \leq P \leq P^e(\overline{\theta})$, then for any θ with $\underline{\theta} \leq \overline{\theta}$, we have

$$P^e(\underline{\theta}) = g(P^e(\underline{\theta}), \underline{\theta}) \leq g(P, \theta) \leq g(P^e(\overline{\theta}), \overline{\theta}) = P^e(\overline{\theta}).$$

□

3.2 Stochastic Stability: Intuition

In this section, we want to characterize the probabilistic behaviour of the power vector iterate P^k when the environments $\{\Theta^k\}_{k=0}^{\infty}$ follow some stochastic process. We assume that the environments $\{\Theta^k\}$ are *iid* (indipendent and identically distributed) across time with a continuous density function $f(\theta)$ supported on $\mathcal{U}$. On this setting, the power iterates $\{P^k\}_{k=0}^{\infty}$ form a general Markov chain. Our aim is to investigate the probabilistic behaviour of the process answering to some important questions as: *Does there exist a stationary distribution for the Markov Chain? If there exists a stationary distribution, is it unique? Will P^k converge to that unique stationary distribution (if it exists unique) irrespective of the initial condition?*

It is quite important to solve these problems because, a priori, once randomness enters the network, there might be more than one Nash equilibria (unlike the deterministic case where we proved uniqueness), one for each network environment. Moreover, for any practical stochastic system, it is essential having a unique stationary probability measure and convergence to that because they characterize the stochastic stability of the system. In particular, we will see that there exists a unique stationary distribution and we want to give now some intuition on what it happens.

Firstly, Theorem 3.1 tells us that, irrespective of the initial condition, after finitely many iterations, the power vector lies in a N-dimensional hyper-rectangle that we can denote with $\mathcal{H} = \prod_{i=1}^{N}[P_i^e(\underline{\theta}), P_i^e(\overline{\theta})]$. This means that wlog we can assume that the process starts in $\mathcal{H}$. Moreover, once the power vector enters in $\mathcal{H}$ remains in $\mathcal{H}$

again thanks to Theorem 3.1. Secondly, since the best response function $g(P,\theta)$ is bimonotonic, there is a one-to-one correspondence between $\theta \in \mathcal{U}$ and the equilibrium $P_i^e(\theta)$, which is the equilibrium generated if the disturbance is fixed equal to θ. Finally, another consequence of bimonotonicity is that if the current state is in $\mathcal{H}$, the possible locations of the current state is in another smaller hyper-rectangle which is contained in the $\mathcal{H}$. In particular, there will be an area (the small hyper-rectangle around the current state P^k) that the next power iterate can reach with positive probability.

3.3 Stationary Distribution

To formally answer to those previous questions, we need to present some general concepts and terminologies about Markov chains.

Definition 3.3. Let $\mathfrak{S}$ be a σ-algebra of subsets of S, let $(S, \mathfrak{S})$ be a measurable space and take $A \in \mathfrak{S}$. Let $\{X^n\}_{n=0}^{\infty}$ be a $(S, \mathfrak{S})$-valued Markov chain with transition kernel $K(s, A)$[5].

- A σ-finite measure π on $(S, \mathfrak{S})$ is called an **invariant measure** if for any $A \in \mathfrak{S}$ $\pi(A) = \int_S K(s, A)\pi(ds)$. An invariant measure π which is also a probability measure is called a **stationary probability measure**.
- $\{X^n\}_{n=0}^{\infty}$ is called **ϕ-irreducible** if there exists a non trvial measure ϕ on $(S, \mathfrak{S})$ such that
$$\phi(A) > 0 \;\Rightarrow\; P_s(\tau_A < \infty) > 0 \;\; \forall s \in S,$$
where $\tau_A = \min\{n \geq 1 \mid X^n \in A\}$ is the first return time and $P_s(\tau_A < \infty)$ denotes the probability of the first return time being finite, given the initial condition $X^0 = s$.
- A set A is called **Harris recurrent** if $P_s(\sum_{n=1}^{\infty} \mathbb{1}_{\{X^n \in A\}} = \infty) = 1$, for any $s \in S$.
- $\{X^n\}_{n=0}^{\infty}$ is called **Harris recurrent** if it is ϕ-irreducible and $\phi(A) > 0 \Rightarrow A$ is a Harris recurrent, for any $A \in \mathfrak{S}$.
- $A \in \mathfrak{S}$ is called **v_m-small** (or, simplier, small set) if there exists a positive integer m and a non trivial measure v_m on $(S, \mathfrak{S})$ such that
$$K^m(s, B) \geq v_m(B) \quad \forall s \in A \; \forall B \in \mathfrak{S}.$$

[5]Intuitively, $K(s, A)$ gives the probability that the next state lies in the set A starting from s. Formally, a transition kernel is a map $K : S \times \mathfrak{S} \longrightarrow [0, 1]$ satisfies the properties:

- for any $s \in S$, $A \mapsto K(s, A)$ is a proability measure on $(S, \mathfrak{S})$.
- for any $A \in \mathfrak{S}$, $s \mapsto K(s, A)$ is a measurable function on $(S, \mathfrak{S})$.

In our case, the transition kernel is time-invariant and we can denote with $K^m(s, A)$ the m-step of the transition kernel.

- Let $\{X^n\}_{n=0}^{\infty}$ be ϕ-irreducible. $\{X^n\}_{n=0}^{\infty}$ is called **strongly aperiodic** if there exists a v_1-small set A with $v_1(A) > 0$.

- $\{X^n\}_{n=0}^{\infty}$ is called **positive Harris** if it is Harris recurrent and there exists a small set A such that
$$\sup_{a\in A} \mathbb{E}_a[\tau_A] < \infty,$$
which means that the return time to A is uniformly bounded when you start in A.

Now, we are ready to state an important result that we can apply to our setting.

Theorem 3.2. *Let $\{X^n\}_{n=0}^{\infty}$ be a time-homogenous Markov chain on $(S, \mathfrak{S})$ which is positive Harris. Therefore*

- *$\{X^n\}_{n=0}^{\infty}$ has a unique stationary probability measure $\pi(\cdot)$.*

- *If $\{X^n\}_{n=0}^{\infty}$ is strongly aperiodic, then the chain converges to the stationary probability measure in total variation distance, namely for any $s \in S$*
$$\lim_{n\to\infty} \|K^n(s, \cdot) - \pi(\cdot)\|_{TV} = 0,$$
where the n-step transition kernel gives the probability measure after n steps when starting at s.

In our current setting, $\{P^k\}_{k=0}^{\infty}$ is a time-homogenous Markov chain on the space $\mathbb{R}_+^N$, with the standard Borel σ-algebra. We have the following result.

Lemma 3.2. *The Markov chain $\{P^k\}_{k=0}^{\infty}$ has the following properties:*

1. *It is ϕ-irreducible for some (nontrivial) ϕ.*
2. *It is Harris recurrent.*
3. *It is positive Harris chain.*
4. *It is strongly aperiodic.*

Proof.

1. As we said above, we can assume without loss of generality that the initial state P^0 lies in the hyper-rectangle $\mathcal{H} = \prod_{i=1}^N [P_i^e(\underline{\theta}), P_i^e(\overline{\theta})]$. Take any $P \in \mathcal{H}$; due to bimonotonicity of the best response function, there exists a unique $\theta \in \mathcal{U}$ such that $g(P, \theta) = P$. Moreover, such $P = P^e(\theta) = \lim_{k\to\infty} g^k(P^0, \theta)$ for any P^0. Since the function g is continous in both variables, it follows that for any r-neighborhood of P, denoted by $\mathcal{N}_P(r) = \left\{\tilde{P} \in \mathcal{H} \mid \left\|P - \tilde{P}\right\|_1 < r\right\}$, there exists

a $\gamma > 0$ small enough and a positive integer T large enough (they depend on P^0) such that

$$k \leq T \text{ and } \left\|\theta - \theta^k\right\|_1 < \gamma \;\Rightarrow\; P^T \in \mathcal{N}_P(r).$$

Since the density function $f(\theta)$ is continuous and supported on $\mathcal{U}$, it follows that $P^T \in \mathcal{N}_P(r)$ with probability at least $(f_{min}\gamma^N)^T$, where $f_{min} := \min_{\theta \in \mathcal{U}} f(\theta)$. Now, take the minimum power vector $P^e(\underline{\theta})$ and consider the neighborhood $\mathcal{N}_{P^e(\underline{\theta})}(r)$. For any $P^0 \in \mathcal{H}$,

$$P^1 \in \mathcal{H}_{P^0} := \{P \mid g(P^0, \underline{\theta}) \leq P \leq g(P^0, \overline{\theta})\} \subset \mathcal{H}.$$

Consequently, calling $\lambda(\cdot)$ the Lesbegue measure, $A \in \mathfrak{S},\ A \subset \mathcal{H}_{P^0},\ \lambda(A) > 0$, we have that $K(P^0, A) \geq f_{min}\lambda(A) > 0$. Then, by continuity of g, set $\hat{r}$ such that

$$\mathfrak{F} := \bigcap_{P^0 \in \mathcal{N}_{P^e(\underline{\theta})}(\hat{r})} \mathcal{H}_{P^0} \neq \emptyset.$$

Note that it is a hyper-rectangle with positive Lesbegue measure. Then, starting from $\mathcal{N}_{P^e(\underline{\theta})}(r)$, the Markov chain will reach with positive probability any measurable subset of $\mathfrak{F}$ that has a positive Lesbegue measure. This means that starting at $P^0 \in \mathcal{H}$, the Markov chain will enter in $\mathfrak{F}$ with positive probability in $T+1$ steps. Finally, we can take as (nontrivial) measure

$$\phi(A) := \frac{\lambda(A \cap \mathfrak{F})}{\lambda(\mathfrak{F})} \quad \forall A \in \mathfrak{S}.$$

2. Fix $A \in \mathfrak{F}$ with $\phi(A) > 0$. We want to prove that A is Harris recurrent. For the previous discussion, we know that starting at $P^0 = P^e(\overline{\theta})$, there exist γ, T such that $P^{T+1} \in A$ with probability at least $(f_{min}\gamma)^T f_{min}\lambda(A)$. Equivalently, setting $\epsilon = (f_{min}\gamma)^T f_{min}$, we have $K^{T+1}(P^e(\overline{\theta}), A) \geq \epsilon\lambda(A)$. Moreover, starting at any $P \in \mathcal{H}$ we have that following the same realization of the environment $\theta^0, \ldots, \theta^k$, $P^{k+1}_{P^0=P} \leq P^{k+1}_{P^0=P^e(\overline{\theta})}$. Hence,

$$P^{k+1}_{P^0=P} \in \mathcal{N}_{P^e(\underline{\theta})}(\hat{r}) \;\Rightarrow P^{k+1}_{P^0=P^e(\overline{\theta})} \in \mathcal{N}_{P^e(\underline{\theta})}(\hat{r}).$$

Consequently, $K^{T+1}(P^0, A) \geq \epsilon\lambda(A)$ for any $P^0 \in \mathcal{H}$. Then, with probability at least $\epsilon\lambda(A) > 0$, A is visited in $T+1$ steps irrespective of the starting point. By Borel Cantelli Lemma, A is visited infinitely often with probability one.

3. As before, set $\epsilon = (f_{min}\gamma)^T f_{min}$. Take $m = T+1$ and $v_{T+1}(B) = \epsilon\lambda(B \cap \mathfrak{F})$, for any $B \in \mathfrak{S}$; hence, $\mathfrak{F}$ is a v_{T+1}-small set. Indeed, for any $P^0 \in \mathfrak{F}$, $\forall B \in \mathfrak{S}$,

$K^{T+1}(P^0, B) \geq K^{T+1}(P^0, B \cap \mathfrak{F}) \geq \epsilon\lambda(B \cap \mathfrak{F}) = v_{T+1}(B)$. Therefore,

$$\begin{aligned} \sup_{P^0 \in \mathfrak{F}} \mathbb{E}_{P^0}[\tau_{\mathfrak{F}}] &\leq \sup_{P^0 \in \mathcal{H}} \mathbb{E}_{P^0}[\tau_{\mathfrak{F}}] \\ &\leq \sum_{i=1}^{\infty} \epsilon\lambda(\mathfrak{F})(1 - \epsilon\lambda(\mathfrak{F}))^{i-1} i(T+1) \\ &= \frac{T+1}{\epsilon\lambda(\mathfrak{F})} < \infty. \end{aligned}$$

4. Again by continuity, as for the first statement, pick an $r > 0$ such that

$$C := \mathcal{N}_{P^e(\underline{\theta})}(r) \ \cap \bigcap_{P^0 \in \mathcal{N}_{P^e(\underline{\theta})}(\hat{r})} \mathcal{H}_{P^0} \neq \emptyset.$$

   Then, we have $K(c, C) > 0 \ \forall c \in C$. Take $m = 1$ and the measure $v_1(\cdot)$ to be $v_1(A) = f_{min}\lambda(A \cap C) \ \forall A \in \mathfrak{S}$. It follows that C is a v_1-small set because

$$\forall B \in \mathfrak{S}, \ \forall P^0 \in \mathcal{H} \quad K(P^0, B) \geq K(P^0, B \cap C) \geq f_{min}\lambda(B \cap C) = v_1(B).$$

   Hence, $v_1(C) > 0$ and we have the thesis.

□

Consequently, we have proved the following Theorem.

Theorem 3.3. *There exists a unique stationary probability measure $\pi(\cdot)$ for $\{P^k\}_{k=0}^{\infty}$. Moreover, for any $p^0 \in \mathbb{R}_+^N$*

$$\lim_{n \to \infty} \left\| P_{p^0}^n(\cdot) - \pi(\cdot) \right\|_{TV} = 0,$$

where $P_{p^0}^n(\cdot)$ denotes the probability measure of the state at time n, starting at p^0.

4 Conclusion

In this seminar we presented a game-theoretic formulation of power control in wireless communications in the deterministic and stochastic case. In the first one, we provided a new version of Tarski's fixed point theorem that helped us to achieve the existence and uniqueness of the Nash equilibrium in "large-power" regime. Moreover, we presented the synchronous and asynchronous best response dynamics in order to obtain convergence and we studied the Price of Anarchy bounds for the power control game to measure the Nash equilibrium efficiency. In the second case, we focused on a worst-case stability characterization of the best response update under such random environment and we investigated the stochastic stability, gaining the existence of a unique stationary probability distribution.

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