How to construct bizarre objects: from *Baire Category* to *Convex Integration*

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In this thesis we present two techniques that are useful in the construction of bizarre examples and counterexamples in the fields of analysis and geometry.

In the first chapter, we provide some classical examples that exploit the *Baire category theorem.* In particular, we prove the following facts.

- There exists a bounded function $f : \mathbb{R} \to \mathbb{R}$ that is 1/2-Hölder continuous in $\mathbb{R}$, but not Lipschitz continuous in any sub-interval $(a, b) \subseteq \mathbb{R}$.
- There exists a measurable set $E \subseteq [0, 1]$ such that, for every sub-interval $I \subseteq [0, 1]$, both $E \cap I$ and $E^c \cap I$ have strictly positive measure.
- In the space of bounded functions $f : \mathbb{R} \to \mathbb{R}$ that are Lipschitz continuous with constant ≤ 1, the set of functions with Lipschitz constant $= 1$ in any sub-interval is residual.
- In the space of functions $f \in L^2(a, b)$ with $|f(x)| \leq 1$ almost everywhere, the set of functions such that $f(x) \in \{-1, 1\}$ almost everywhere is residual with respect to the metric that induces the weak convergence in $L^2(a, b)$.

Furthermore, in the first two cases, we actually show that the examples are residual and explain how the same ingredients of a "Baire proof" can be used in order to construct an explicit solution.

In the next three chapters, we discuss three applications of the method which is now known as *Convex Integration.*

- In Chapter 2, we show that the set of curves $u : [0, 1] \to \mathbb{R}^d$ of class C^1 with $\|u'(t)\|_{\mathbb{R}^d} = 1$ for every $t \in [0, 1]$ is dense (with respect to C^0 norm) in the set of curves $u : [0, 1] \to \mathbb{R}^d$ of class C^1 with $\|u'(t)\|_{\mathbb{R}^d} \leq 1$ for every $t \in [0, 1]$. This set is also residual with respect to a suitable metric.
- Let $\Omega \subset \mathbb{R}^3$ be an open and bounded set. In Chapter 3, we show that there exist infinitely many $u \in L^\infty(\mathbb{R}^3, \mathbb{R}^3)$ such that *div* $u = 0$ in the sense of distributions and $\|u(x)\|_{\mathbb{R}^3} = \mathbb{1}_\Omega(x)$ for almost every $x \in \mathbb{R}^3$. We then consider a slightly more general case where $u(x) \in K$ for a certain compact set $K \subset \mathbb{R}^3$.
- In Chapter 4, we consider a special case of the seminal *Nash-Kuiper Theorem.* In particular, if we let D^2 be the 2-dimensional disk and $N \geq 3$, we apply the convex integration procedure in order to prove that the set of C^1 isometric immersions $\phi : D^2 \to \mathbb{R}^N$ is dense (with respect to the C^0 norm) in the set of C^∞ short immersions $\phi : D^2 \to \mathbb{R}^N$.

The convex integration method became fundamental recently, since it is possible to use the same machinery of Nash-Kuiper proof in order to construct many weak solutions to Incompressible Euler equations, which is related to Onsager's conjecture of turbulence in fluid dynamics. In our work, we try to convey the main ideas of the method without technicalities, and to highlight the common features of the approach to three different problems with increasing difficulty.