Linear hyperbolic equations with time-dependent propagation speed

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We consider a second order linear equation with a time-dependent coefficient $c(t)$. Specifically, we take a separable Hilbert space H and a maximal multiplication operator A defined in H and we consider the second order linear evolution equation

$$\ddot{u}(t) + c(t)Au = 0$$

with initial data

$$u(0) = u_0, \qquad \dot{u}(0) = u_1.$$

We study the regularity of solutions $u(t)$ depending on the space-regularity of initial data and on the time-regularity of the propagation speed $c(t)$. We show that higher space-regularity of initial data compensates a lower time-regularity of $c(t)$.

The main results that we exhibit throughout this thesis were presented for the first time in the paper *"Sur les équations hyperboliques avec des coefficients qui ne dépendent que du temps"* published by Ennio De Giorgi, Ferruccio Colombini and Sergio Spagnolo in *"Annali della Scuola Normale Superiore di Pisa"* in 1979.

In the first chapter we clarify the funcional setting. We introduce some classes of functional spaces depending on the operator A, such as Sobolev spaces, distributions, Gevrey spaces, Gevrey distributions, analytic functions and ultradistributions. We show the compatibility of these abstract notions with classic definitions in some significant cases.

In the second chapter, we consider the following family of ordinary differential equations

$$\ddot{u}_\lambda(t) + \delta(t)\dot{u}_\lambda(t) + \lambda^2 c(t)u_\lambda(t) = 0$$

and we estimate the growth of the solution $u_\lambda(t)$ as $\lambda \to +\infty$. The argument relies on estimates on the so-called Kovaleskian energy and hyperbolic energy. The main trick is that different energies are used depending on λ and on the time-regularity of $c(t)$. Using these estimates we prove the well-posedness of the original problem in suitable functional spaces, depending on the time-regularity of the propagation speed.

In the fourth chapter we present some counterexamples that show that the previous results are optimal.