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Linear hyperbolic equations with time-dependent propagation speed

Candidato
Alessandro La Farciola

Relatore
Prof. Massimo Gobbino

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A nonna

Lo duca e io per quel cammino ascoso
intrammo a ritornar nel chiaro mondo;
e sanza cura aver d'alcun riposo,

salimmo sù, el primo e io secondo,
tanto ch'i' vidi de le cose belle
che porta 'l ciel, per un pertugio tondo.

E quindi uscimmo a riveder le stelle.

Dante

Contents

Introduction **i**

1 Functional Spaces **1**
1.1 Preliminaries 1
1.2 Functional spaces 5
1.3 Compatibility with classic definitions 6

2 Approximated energy estimates **15**
2.1 Preliminary energy estimates 15
2.2 Kovaleskian estimates depending on $c(t)$ regularity 18

3 Well-posedness Theorems **25**

4 Swing effect **35**
4.1 Heuristic search for counterexamples 35
4.2 General counterexample 37

Bibliography **44**

Introduction

We consider a second order linear equation with a time-dependent coefficient $c(t)$. Specifically, we take a separable Hilbert space H and a maximal multiplication operator A defined in H and we consider the second order linear evolution equation

$$\ddot{u}(t) + c(t)Au = 0$$

with initial data

$$u(0) = u_0, \qquad \dot{u}(0) = u_1.$$

We study the regularity of solutions $u(t)$ depending on the space-regularity of initial data and on the time-regularity of the propagation speed $c(t)$. We show that higher space-regularity of initial data compensates a lower time-regularity of $c(t)$.

The main results that we exhibit throughout this thesis were presented for the first time in the paper *"Sur les équations hyperboliques avec des coefficients qui ne dépendent que du temps"* published by Ennio De Giorgi, Ferruccio Colombini and Sergio Spagnolo in *"Annali della Scuola Normale Superiore di Pisa"* in 1979.

In the first chapter we clarify the funcional setting. We introduce some classes of functional spaces depending on the operator A, such as Sobolev spaces, distributions, Gevrey spaces, Gevrey distributions, analytic functions and ultradistributions. We show the compatibility of these abstract notions with classic definitions in some significant cases.

In the second chapter, we consider the following family of ordinary differential equations

$$\ddot{u}_\lambda(t) + \delta(t)\dot{u}_\lambda(t) + \lambda^2 c(t)u_\lambda(t) = 0$$

and we estimate the growth of the solution $u_\lambda(t)$ as $\lambda \to +\infty$. The argument relies on estimates on the so-called Kovaleskian energy and hyperbolic energy. The main trick is that different energies are used depending on λ and on the time-regularity of $c(t)$. Using these estimates we prove the well-posedness of the original problem in suitable functional spaces, depending on the time-regularity of the propagation speed.

In the fourth chapter we present some counterexamples that show that the previous results are optimal.

Chapter 1

Functional Spaces

We start from some preliminaries about Hilbert spaces and we define multiplication operators. Then, we give some definitions of different functional spaces and we discuss about their compatibility with classic definitions.

1.1 Preliminaries

Definition 1.1.1 (Hilbert Space)**.** A real vectoral space with a scalar product is called *Hilbert space* if it is a complete metric space with respect to the distance induced by the scalar product.

Definition 1.1.2. A Hilbert space is called *separable* if contains a countable dense subset.

Definition 1.1.3 (Countable Hilbert basis)**.** Let H be a Hilbert space with infinite dimension. A *Hilbert basis* (sometimes also called *orthonormal basis* or *complete orthonormal system*) is a sequence $\{e_n\}_{n\in\mathbb{N}} \subseteq H$ such that

- the element of the sequence are orthonormal vectors, namely

$$\langle e_i, e_j \rangle = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases},$$

- $Span\left(\{e_n\}\right)$ is dense in H.

Theorem 1.1.4 (existence of a countable orthonormal basis)**.** *Let H be a Hilbert space such that*

- *the (algebraic) dimension of H is infinite,*
- *H is separable.*

Than there exists an orthonormal basis in H.

Remark 1.1.5. The previous statement is actually an "if and only if" result. Indeed, if a Hilbert space H admits a (countable) orthonormal basis $\{e_n\}$, then necessarily H is a separable and has infinite dimension.

From the theory of Hilbert spaces we know that given any $u \in H$, where H is a separable Hilbert space with $\{e_n\}_{n\in\mathbb{N}}$ an its Hilbert basis, set $u_n := \langle u, e_n\rangle$, it is true that

$$u = \sum_{n\in\mathbb{N}} u_n e_n \quad \textit{and} \quad \sum_{n\in\mathbb{N}} u_n^2 = \|u\|^2 < \infty.$$

The elements $\{u_n\}$ are called Fourier components of $u \in H$ and the previous result means that we can identify each $v \in H$ with a sequence $\{v_n\}$ such that $\{v_n\} \in \ell^2$, where ℓ^2 represent the Hilbert space of square-summable sequences.

We give an important example of Hilbert space that we will use in later results.

Proposition 1.1.6. *Let $\{w_n\}$ be a sequence contained in $\mathbb{R}^+$. Then*

$$H := \left\{ \{u_n\} \subseteq \mathbb{R} \mid \sum_{n=1}^{\infty} w_n u_n^2 < +\infty \right\}$$

is a Hilbert space with respect to the following scalar product

$$\langle \{u_n\}, \{v_n\}\rangle := \sum_{k=1}^{\infty} w_k u_k v_k.$$

Proof. Firstly, we can think about H as

$$H := \left\{ f : \mathbb{N} \longrightarrow \mathbb{R} \mid \sum_{k=1}^{\infty} w_k f(k)^2 < \infty \right\}.$$

We have to show that:

1. The scalar product is well defined,

2. H with the induced distance is a complete metric space.

1. Firstly, we show that $\forall\ f, g \in H$ then $\langle f, g\rangle \in \mathbb{R}$. In fact,

$$\begin{aligned} |\langle f, g\rangle| &\le \sum_{k\in\mathbb{N}} w_k |f(k)g(k)| \le \sum_{k\in\mathbb{N}} w_k \left(\frac{f(k)^2 + g(k)^2}{2}\right) \\ &\le \frac{1}{2}\sum_{k\in\mathbb{N}} w_k f(k)^2 + \frac{1}{2}\sum_{k\in\mathbb{N}} w_k g(k)^2 < \infty. \end{aligned}$$

Now, we show that it is well defined, i.e. it respects the three properties of the scalar product:

- $\langle f, g\rangle = \sum_{k\in\mathbb{N}} w_k f(k)g(k) = \sum_{k\in\mathbb{N}} w_k g(k)f(k) = \langle g, f\rangle \ \forall f, g \in H$.
- $\langle af+bg, h\rangle = \sum_{k\in\mathbb{N}} w_k(af(k)+bg(k))h(k) = a\sum_{k\in\mathbb{N}} w_k f(k)h(k)+b\sum_{k\in\mathbb{N}} w_k g(k)h(k) = a\langle f, h\rangle + b\langle g, h\rangle \quad \forall f, g, h \in H$.
- $\langle f, f\rangle = \sum_{k\in\mathbb{N}} w_k f(k)^2 \geq 0 \quad \forall f \in H$ and it is $= 0 \Leftrightarrow f(k) = 0 \ \forall k \in \mathbb{N}$

2. H is complete.
   Let $\{f_n\} \subseteq H$ be a Cauchy sequence. The proof is divided in different steps.
   - There exists $f_\infty : \mathbb{N} \longrightarrow \mathbb{R}$ such that $f_n(k) \longrightarrow f_\infty(k) \ \forall k \in \mathbb{N}$. Indeed,
   $$|f_n(k) - f_m(k)|^2 \leq \frac{1}{w_k} \|f_n - f_m\|^2 .$$
   Now, the last term is arbitrary small because $\{f_n\}$ is a Cauchy sequence and so also $\{f_n(k)\} \subseteq \mathbb{R}$ is a Cauchy sequence $\Rightarrow f_n(k)$ converges. Then we can define $f_\infty(k)$ as the punctual limit for all $k \in \mathbb{N}$.
   - We show that $f_\infty \in H$. Indeed, $\forall m \in \mathbb{N}$
   $$\sum_{k=1}^{m} w_k f_\infty(k)^2 = \sum_{k=1}^{m} w_k \left(\lim_{n\to+\infty} f_n(k)\right)^2 = \lim_{n\to+\infty} \sum_{k=1}^{m} w_k f_n(k)^2 \leq \limsup_{n\to+\infty} \|f_n\|$$
   and the last term is bounded because $\{f_n\}$ is a Cauchy sequence and so also $\|f_n\|$ is a Cauchy sequence. We conclude because it is true for all $m \in \mathbb{N}$.
   - We want to show that
   $$\forall \epsilon > 0 \ \exists M \in \mathbb{N} \ : \ \forall n \ \sum_{k=M}^{\infty} w_k f_n(k)^2 \leq \epsilon.$$
   Take $\epsilon > 0$. Firstly, let $n_0 \in \mathbb{N}$ be such that
   $$\|f_n - f_m\| \leq \frac{\epsilon}{4} \quad \forall n \geq n_0 \ \forall m \geq n_0 \tag{1.1}$$
   which is possible because it is a Cauchy sequence.
   Secondly, let $M \in \mathbb{N}$ be such that
   $$\sum_{k=M}^{\infty} w_k f_n(k)^2 \leq \frac{\epsilon}{4} \quad \forall n \leq n_0 \tag{1.2}$$
   which is is possible because they are in a finite number so we can take the maximum. We are now ready to show that the same M is a good choice also for $n \geq n_0$. Indeed,
   $$\begin{aligned}\sum_{k=M}^{\infty} w_k f_n(k)^2 &= \sum_{k=M}^{\infty} w_k \left(f_n(k) - f_{n_0}(k) + f_{n_0}(k)\right)^2 \\ &\leq 2\sum_{k=M}^{\infty} w_k|f_n(k) - f_{n_0}(k)|^2 + 2\sum_{k=M}^{\infty} w_k f_{n_0}(k)^2 \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon\end{aligned}$$

where the last inequalities follows since the second term is less than or equal to $\epsilon/2$ for (1.2), while for the first term

$$2\sum_{k=M}^{\infty} w_k|f_n(k)-f_{n_0}(k)|^2 \leq 2\underbrace{\sum_{k=1}^{\infty} w_k|f_n(k)-f_{n_0}(k)|^2}_{\|f_n-f_{n_0}\|} \leq \frac{\epsilon}{2} \quad for\ (1.1).$$

- We show that M is a good choice also for f_∞. In fact, for all $m > M$

$$\begin{aligned}\sum_{k=M}^{m} w_k f_\infty(k)^2 &= \lim_{n\to+\infty}\sum_{k=M}^{m} w_k f_n(k)^2 \\ &\leq \lim_{n\to+\infty}\sum_{k=M}^{\infty} w_k f_n(k)^2 \leq \epsilon\end{aligned}$$

  because each term of the last sum is $\leq \epsilon$ and we conclude because it is true for all $m > M$.

- Now, we show that $f_n \longrightarrow f_\infty \in H$. Take $\epsilon > 0$. Let $M \in \mathbb{N}$ be as in the previous steps such that

$$\sum_{k=M}^{\infty} w_k f_n(k)^2 \leq \frac{\epsilon}{8} \quad and \quad \sum_{k=M}^{\infty} w_k f_\infty(k)^2 \leq \frac{\epsilon}{8}.$$

  Then,

$$\begin{aligned}d_H(f_n,f_\infty)^2 &= \sum_{k=1}^{\infty} w_k|f_n(k)-f_\infty(k)|^2 \\ &= \underbrace{\sum_{k=1}^{M-1} w_k|f_n(k)-f_\infty(k)|^2}_{(i)} + \underbrace{\sum_{k=M}^{\infty} w_k|f_n(k)-f_\infty(k)|^2}_{(ii)}\end{aligned}$$

  and both addends are less than or equal to $\epsilon/2$. Indeed, (i) has a finite number of terms and $f_n(k) \longrightarrow f_\infty(k)$ punctually, and so we can find $n_0 \in \mathbb{N}$ such that

$$|f_n(k)-f_\infty(k)| \leq \frac{\epsilon}{2} \quad \forall n \geq n_0 \quad \forall k \leq M-1.$$

  On the other hand,

$$(ii) \leq 2\sum_{k=M}^{\infty} w_k f_n(k)^2 + 2\sum_{k=M}^{\infty} w_k f_\infty(k)^2 \leq \frac{\epsilon}{2}.$$

  This concludes the proof.

□

We can now define multiplication operators in a Hilbert space.

Definition 1.1.7 (Multiplication operator)**.** Let H be a Hilbert space. An operator $A : D(A) \longrightarrow H$, where $D(A) \subseteq H$ is its domain, is called a *multiplication operator* if there exists a Hilbert basis $\{e_k\}$ and a sequence $\{\lambda_k\} \subseteq \mathbb{R}$ such that for all $u = \sum_{k\in\mathbb{N}} u_k e_k \ \in D(A)$

$$Au = \sum_{k\in\mathbb{N}} \lambda_k u_k e_k.$$

Moreover, we say that A is a *maximal multiplication operator* when $D(A) = \{u \in H \ : \sum_{k\in\mathbb{N}} \lambda_k^2 u_k^2 < +\infty\}$.

Definition 1.1.8 (Coercive operator)**.** Let $A : D(A) \longrightarrow H$ be a self-adjoint operator, i.e. $\langle Au, v\rangle = \langle u, Av\rangle$ for all u and $v \in D(A)$, we say that A is a *coercive operator* if there exists a constant $\gamma > 0$ such that

$$\langle Au, u\rangle \geq \gamma \|u\|^2 \qquad \forall u \in D(A).$$

1.2 Functional spaces

Let H be a separable Hilbert space with $\{e_k\}$ an its Hilbert basis and A a self-adjoint maximal multiplication operator. We assume that A is a coercive operator with $\gamma \geq 1$. Let $\{\lambda_k^2\}$ be the sequence of eigenvalues of A such that

$$\forall u \in H \quad u = \sum_{k\in\mathbb{N}} u_k e_k \quad \textit{and} \quad Au = \sum_{k\in\mathbb{N}} \lambda_k^2 u_k e_k.$$

Moreover, due to the coercivity of the operator A, we have that for each e_k

$$\lambda_k^2 \|e_k\|^2 = \langle \lambda_k^2 e_k, e_k\rangle = \langle Ae_k, e_k\rangle \geq \gamma \|e_k\|^2$$

and so

$$\lambda_k^2 \geq \gamma \geq 1.$$

Definition 1.2.1 (Functional spaces relative to A)**.** We define the following spaces:

- *Sobolev spaces.* Let α be a real number. We say that $u \in D(A^\alpha)$ if

$$\|u\|_{D(A^\alpha)}^2 := \sum_{k=1}^{\infty} \lambda_k^{4\alpha} u_k^2 \ < +\infty, \tag{1.3}$$

- *Distributions.* Let α be a real number. We say that $u \in D(A^{-\alpha})$ if

$$\|u\|_{D(A^{-\alpha})}^2 := \sum_{k=1}^{\infty} \lambda_k^{-4\alpha} u_k^2 \ < +\infty, \tag{1.4}$$

- *Gevrey spaces.* Let α be a real number, let r be positive real number and let $\varphi : [0,+\infty) \longrightarrow (0,+\infty)$. We say that $u \in \mathcal{G}_{\varphi,r,\alpha}(A)$ if

$$\|u\|^2_{\mathcal{G}_{\varphi,r,\alpha}(A)} := \sum_{k=1}^{\infty} \lambda_k^{4\alpha} \exp\left(2r\varphi(\lambda_k)\right) u_k^2 \; < +\infty, \tag{1.5}$$

- *Gevrey ultradistributions.* Let α be a real number, let R be positive real number and let $\psi : [0,+\infty) \longrightarrow (0,+\infty)$. We say that $u \in \mathcal{G}_{-\psi,R,\alpha}(A)$ if

$$\|u\|^2_{\mathcal{G}_{-\psi,R,\alpha}(A)} := \sum_{k=1}^{\infty} \lambda_k^{4\alpha} \exp\left(-2R\psi(\lambda_k)\right) u_k^2 \; < +\infty. \tag{1.6}$$

The quantities defined in (1.3) through (1.6) are actually norms which induce a Hilbert space structure on $D(A^{\alpha})$, $D(A^{-\alpha})$, $\mathcal{G}_{\varphi,r,\alpha}(A)$, $\mathcal{G}_{-\psi,R,\alpha}(A)$, respectively, and this follows by Proposition 1.1.6.
Moreover, for every $\alpha \geq 0$ and for every admissible choice of φ, ψ, r, R, it turns out that

$$\mathcal{G}_{\varphi,r,\alpha}(A) \subseteq D(A^{\alpha}) \subseteq H \subseteq D(A^{-\alpha}) \subseteq \mathcal{G}_{-\psi,R,\alpha}(A).$$

The general philosophy is that the sequence w_k of the Proposition 1.1.6 can be read as a weight and larger the weight, harder is for the series to converge and smaller is the functional space. On the other hand, fewer the weight, easier is for the series to converge and bigger is the functional space. All inclusions are strict if $\{\lambda_k\}$ is unbounded.
In addition, for the same reason, also the following containments are true

$$\mathcal{G}_{\varphi,r_2,\alpha}(A) \subseteq \mathcal{G}_{\varphi,r_1,\alpha}(A) \quad if \quad r_1 < r_2$$

$$\mathcal{G}_{-\psi,R_2,\alpha}(A) \supseteq \mathcal{G}_{-\psi,R_1,\alpha}(A) \quad if \quad R_1 < R_2.$$

1.3 Compatibility with classic definitions

Let us consider a concrete case to show the compatibility of the previous definitions with the classic ones.
To achieve this aim we should, firstly, exhibit the usual definition of Sobolev spaces and we start from the concept of weak derivative.

Definition 1.3.1. Let $(a,b) \subseteq \mathbb{R}$ be a real interval, let u and v two functions in $L^1_{loc}\left((a,b)\right)$, i.e. u and v are in $L^1\left(\Omega'\right)$ for all Ω' such that $clos(\Omega')$ *is a compact* $\subseteq \Omega$. We say that v is the *W-weak derivative* of u if

$$\int_a^b u(x)\varphi'(x)dx = -\int_a^b v(x)\varphi(x)dx \qquad \forall \varphi \in C_c^{\infty}((a,b))$$

where $C_c^{\infty}((a,b))$ represents the space of C^{∞} functions of compact support.

Definition 1.3.2. Let $(a, b) \subseteq \mathbb{R}$ be a real interval, let $p \in [1, +\infty]$ be an exponent. We put

$$W^{1,p}((a, b))$$

the set of the functions $u \in L^p((a, b))$ for whom there exists a W-weak derivative $v \in L^p((a, b))$.

Definition 1.3.3. Let $(a, b) \subseteq \mathbb{R}$ be a real interval, let u and v two functions in $L^1_{loc}((a, b))$. We say that v is the *H-weak derivative* of u if there exists a sequence $\{u_n\} \subseteq C^1((a, b))$ such that

$$u_n \longrightarrow u \quad \textit{and} \quad \dot{u}_n \longrightarrow v \quad \textit{in} \quad L^1_{loc}((a, b)).$$

Definition 1.3.4. Let $(a, b) \subseteq \mathbb{R}$ be a real interval, let $p \in [1, +\infty]$ be an exponent. We put

$$H^{1,p}((a, b))$$

the set of the functions $u \in L^p((a, b))$ such that admit an H-weak derivative $v \in L^p((a, b))$. I.e. $u \in H^{1,p}((a, b))$ if there exists $\{u_n\} \subseteq C^{1,p}((a, b))$, which means that for all $n \in \mathbb{N}$ we have that $u_n \in C^1((a, b))$ and more both u_n and $\dot{u}_n$ are in $L^p((a, b))$, such that

- $u_n \longrightarrow u \;\; \textit{in} \;\; L^p$,
- $\dot{u}_n \longrightarrow v \;\; \textit{in} \;\; L^p$.

Thanks to the previous definitions we have an important statement, namely the spaces $W^{1,p}((a, b))$ and $H^{1,p}((a, b))$ are the same and this turns out from the the following result.

Theorem 1.3.5. *The two definitions of weak derivative are equivalent. I.e. given* $u \in L^1_{loc}((a, b))$

v is its H-weak derivative if and only if v is its W-weak derivative.

Remark 1.3.6. The *if* side of the theorem is obvious, just passing to the limit from the W-weak derivative definition for u_n that hold because of the H-weak derivative definition.

We are now ready to consider the space $L^2((0, \pi))$; we know that it is a separable Hilbert space. Let $\{e_k\} = \{a \sin(kx)\}$ with $a = \sqrt{2/\pi}$ the normalizing constant; then $\{e_k\}$ is a Hilbert basis for $L^2((0, \pi))$. So for all $u \in L^2((0, \pi))$, set $u_k := a \langle u, e_k \rangle$, we can write

$$u(x) = \sum_{k=1}^{\infty} u_k \sin(kx) \quad \textit{with} \quad \sum_{k=1}^{\infty} u_k^2 < \infty.$$

Let $A : D(A) \longrightarrow H$ be the operator such that $Au = -\ddot{u}$, then the component of the previous basis e_k are eigenvectors with k^2 as eigenvalues, so A is a multiplication operator. Therefore we have that

$$D(A) = \left\{ \sum_{k=1}^{\infty} u_k e_k \ : \ \sum_{k=1}^{\infty} k^4 u_k^2 < \infty \right\}$$

because we want that for all $u \in D(A)$ it is well defined Au and $\sum_{k=1}^{\infty} k^2 u_k e_k \in H$. In addition, we can note that $D(A)$ coincides with definition (1.3) with $\alpha = 1$.

We can now prove the following result.

Theorem 1.3.7. *Let A be the previous operator and let $H^p((0,\pi))$ the usual Sobolev space. Therefore*

$$u \in D(A) \Leftrightarrow u \in H^2 \textit{ and } u \textit{ is null on its edges.}$$

Proof. We will prove both sides.
$\Rightarrow$) Let be $u(x) = \sum_{k=1}^{\infty} u_k \sin(kx)$. We put $S_n = \sum_{k=1}^{n} u_k \sin(kx)$. We know that $S_n \longrightarrow u$ in L^2. So,

$$S_n' = \sum_{k=1}^{n} k u_k \cos(kx) \quad \textit{and} \quad S_n'' = -\sum_{k=1}^{n} k^2 u_k \sin(kx).$$

Therefore,

$$S_n'' \textit{ converges in } L^2$$

if and only if

$$\sum_{k=1}^{\infty} k^4 u_k^4 < +\infty,$$

namely

$$u \in D(A).$$

This means that if $u \in D(A)$, there exists $v \in L^2((0,\pi))$ such that $S_n'' \longrightarrow v$ in L^2. Hence, for all $\varphi \in C_c^{\infty}$, from integration by part, it turns out that

$$\int_0^{\pi} S_n''(x)\varphi(x)dx = \int_0^{\pi} S_n(x)\varphi''(x)dx$$

and thanks to the previous convergences,

$$\int_0^{\pi} v(x)\varphi(x)dx = \int_0^{\pi} u(x)\varphi''(x)dx$$

which means that v is the weak derivative of u. Moreover, we have to show that u is null on its edges.
Indeed, since $S_n(x)$ is a finite linear combination of $\sin(kx)$, it is null in 0 and π (each

$\sin(kx)$ is null in 0 and π). If $S_n(x)$ converges uniformly we conclude. But it is true because

$$\begin{aligned}\sum_{k=1}^{\infty} |u_k||\sin(kx)| \leq \sum_{k=1}^{\infty} |u_k| &= \sum_{k=1}^{\infty} k^2 |u_k| \frac{1}{k^2} \\ &\leq \left(\sum_{k=1}^{\infty} k^4 u_k^2\right)^{1/2} \left(\sum_{k=1}^{\infty} \frac{1}{k^4}\right)^{1/2} < \infty.\end{aligned}$$

So, we showed that the series is totally convergent.
$\Leftarrow$) Suppose that $u \in H^2$ and it is null on its edges and we want to show that $\sum_{k=1}^{\infty} k^4 u_k^2 < \infty$. By definition of Fourier coefficient and integration by part,

$$u_k = a \int_0^{\pi} u(x) sin(kx)dx = \frac{a}{k} \int_0^{\pi} u'(x)\cos(kx)dx = -\frac{1}{k^2} \underbrace{a \int_0^{\pi} u''(x) sin(kx)dx}_{=:v_k}.$$

So, v_k is the k-th coefficient of the Fourier series of $u''(x)$ which is in $L^2((0,\pi))$. Hence,

$$\sum_{k=1}^{\infty} k^4 u_k^2 = \sum_{k=1}^{\infty} v_k^2 < \infty.$$

□

Remark 1.3.8. In the same way we could consider the general case and by induction we could prove that $u \in D(A^{\alpha/2}) \Leftrightarrow u \in H^{\alpha}$ and all the even derivative of order smaller than α are null on their edges.

Let A be the previous operator and let $\mathcal{G}_{\varphi,r,\alpha}(A)$ be the Gevrey space with $\alpha = 0$, r a positive real number and $\varphi : [0,+\infty) \longrightarrow (0,+\infty)$ such that $\varphi(\lambda) = \lambda$. We want to show that $\mathcal{G}_{\varphi,r,\alpha}(A)$ coincides with the space of analytic functions. To do that we need the following lemma.

Lemma 1.3.9. *Let be $x_0 \in \mathbb{R}$, $r > 0$, $f \in C^{\infty}((x_0 - r, x_0 + r))$. Suppose that there exists a constant M such that*

$$|f^{(k)}(x)| \leq M \frac{k!}{r^k} \quad \forall k \in \mathbb{N}, \forall x \in (x_0 - r, x_0 + r).$$

Then,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n \quad \forall x \in (x_0 - r, x_0 + r),$$

i.e. $f(x)$ is analytic in x_0 with radius r.

Proof. Let $S_n(x)$ be the partial sums of the last series. Due to the Taylor-Lagrange theorem,

$$f(x) = S_n(x) + \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}$$

where $c \in (x_0 - r, x_0 + r)$. And so,

$$|f(x) - S_n(x)| \leq \frac{|f^{(n+1)}(c)|}{(n+1)!}|x - x_0|^{n+1}.$$

Using the hypothesis, it turns out that

$$|f(x) - S_n(x)| \leq \frac{M}{r^{n+1}}|x - x_0|^{n+1} = M\left|\frac{x - x_0}{r}\right|^{n+1}$$

and for $n \longrightarrow +\infty$ we have that $RHS \longrightarrow 0$ because $|x - x_0| < r$. Hence, it follows that $S_n(x) \longrightarrow f(x)$ uniformly on compacts of the type $[x_0 - a, x_0 + a]$ with $0 < a < r$. $\square$

Theorem 1.3.10. *Let $u(x) = \sum_{k=1}^{\infty} u_k \sin(kx)$ such that $\sum_{k=1}^{\infty} u_k^2 e^{2rk^2} < \infty$, therefore $u(x)$ is analytic with radius r.*

Proof. The claim of the proof is to estimate the k-th derivative of u and use the previous lemma. Firstly, we have that

$$u^{(n)}(x) = \begin{cases} \sum_{k=1}^{\infty} k^n u_k \sin(kx) & if\ n \equiv 0 \mod 4 \\ \sum_{k=1}^{\infty} k^n u_k \cos(kx) & if\ n \equiv 1 \mod 4 \\ -\sum_{k=1}^{\infty} k^n u_k \sin(kx) & if\ n \equiv 2 \mod 4 \\ -\sum_{k=1}^{\infty} k^n u_k \cos(kx) & if\ n \equiv 3 \mod 4 \end{cases}$$

In any case,

$$\begin{aligned} |u^{(n)}(x)| \leq \sum_{k=1}^{\infty} k^n |u_k| &= \sum_{k=1}^{\infty} k^n e^{-rk^2} e^{rk^2} |u_k| \\ &\leq \left(\sum_{k=1}^{\infty} k^{2n} e^{-2rk^2}\right)^{1/2} \left(\sum_{k=1}^{\infty} e^{2rk^2} u_k^2\right)^{1/2}. \end{aligned}$$

The last term of the previous product is convergent by hypothesis and so

$$|u^{(n)}(x)| \leq c\left(\sum_{k=1}^{\infty} k^{2n} e^{-2rk^2}\right)^{1/2} \leq c\left(\int_0^{\infty} x^n e^{-2rx} dx\right)^{1/2},$$

where the last step follows from the integral test for convergence. Now, we can compute the last integral. Indeed, by induction on n, thanks to a change of variable

$$\begin{aligned} \int_0^{\infty} x^n e^{-2rx} dx = \frac{1}{(2r)^{n+1}} \int_0^{\infty} t^n e^{-t} dt &= \frac{1}{(2r)^{n+1}} \left\{ \left[-t^n e^{-t}\right]_0^{\infty} + n \int_0^{\infty} t^{n-1} e^{-t} dt \right\} \\ &= \frac{n!}{(2r)^{n+1}} \end{aligned}$$

Therefore, we obtained that for any $n \in \mathbb{N}$ and suitable constants $N, M > 0$

$$|u^{(n)}(x)| \leq \; N \left(\frac{n!}{r^n}\right)^{1/2} \; \leq \; M \frac{n!}{r^n} \; .$$

The thesis follows by the previous lemma.

$\square$

On the other side, the compatibility is not perfect but, as we'll see, there is only a small difference between the radius of the two spaces.

Theorem 1.3.11. *Let $u(x) = \sum_{k=1}^{\infty} u_k \sin(kx)$ be an analytic function with radius r. Hence, $u(x) \in \mathcal{G}_{\varphi,\rho,0}(A)$ for any $0 < \rho < r$, i.e. $\sum_{k=1}^{\infty} u_k^2 e^{2\rho k} < \infty$.*

Proof. As we did to show the compatibility of Sobolev spaces, we can consider the Fourier component u_k. Indeed, for any $i \in \mathbb{N}$,

$$u_k = \int_0^{\pi} u(x) sin(kx) dx = \pm \frac{1}{k^i} \int_0^{\pi} u^{(i)}(x) \sin^{(i)}(kx) dx$$

where with $\sin^{(i)}(kx)$ we mean $\sin(kx)$ or $\cos(kx)$. Now, since $u(x)$ is analytic

$$\exists M > 0 \; \exists r > 0 \; \forall x \in [0, \pi] \; \forall i \in \mathbb{N} \quad |u^{(i)}(x)| \leq \frac{i!}{r^i}$$

and more,

$$u^{(2i)}(x) = 0 \quad \forall x \in \{0, \pi\} \;\; \forall i \in \mathbb{N}.$$

Therefore,

$$|u_k(x)| \leq \frac{1}{k^i} \int_0^{\pi} M \frac{i!}{r^i} \; dx \leq N \frac{i!}{r^i k^i} \; ,$$

where $N = M\pi$. Our aim is to show that $\sum_{k=1}^{\infty} u_k^2 e^{2\rho k} < \infty$, with $\rho < r$. Using the result of the following Lemma 1.3.12, for any $i \in \mathbb{N}$ it turns out that

$$|u_k(x)|^2 \leq N^2 \frac{(i!)^2}{r^{2i} k^{2i}} \leq N^2 \left(\frac{i}{e}\right)^{2i} i^2 e^2 \frac{1}{r^{2i} k^{2i}}.$$

And now, using the previous result with $i = \lceil rk \rceil$,

$$\begin{aligned} u_k^2 \leq N^2 \left(\frac{i}{e}\right)^{2i} i^2 e^2 \frac{1}{r^{2i} k^{2i}} &= N^2 \left(\frac{\lceil rk \rceil}{e}\right)^{2\lceil rk \rceil} \lceil rk \rceil^2 e^2 \frac{1}{r^{2\lceil rk \rceil} k^{2\lceil rk \rceil}} \\ &\leq N^2 \left(\frac{rk+1}{e}\right)^{2(rk+1)} (rk+1)^2 e^2 \frac{1}{r^{2(rk+1)} k^{2(rk+1)}} =: a_k^2 \; . \end{aligned}$$

This means that $\sum_{k=1}^{\infty} u_k^2 e^{2\rho k}$ is convergent if $\sum_{k=1}^{\infty} (a_k/N)^2 e^{2\rho k}$ is convergent. The last series is convergent because $\lim_{k\to\infty} \sqrt[k]{(a_k/N)^2 e^{2\rho k}} < 1$. Indeed,

$$\sqrt[k]{(a_k/N)^2 e^{2\rho k}} = \left(\frac{rk+1}{e}\right)^{\frac{2rk+2}{k}} (rk+1)^{\frac{2}{k}} e^{\frac{2}{k}} \frac{e^{2\rho}}{r^{\frac{2rk+2}{k}} k^{\frac{2rk+2}{k}}}$$

Now,

- $\lim_{k\to\infty}(rk+1)^{\frac{2}{k}} = \lim_{k\to\infty} e^{\frac{2}{k}} = 1$;
- $\lim_{k\to\infty} e^{\frac{2rk+2}{k}} = e^{2r}$;
- $r^{\frac{2rk+2}{k}} k^{\frac{2rk+2}{k}} = r^{2r}k^{2r}r^{\frac{2}{k}}k^{\frac{2}{k}}$ and $\lim_{k\to\infty} r^{\frac{2}{k}}k^{\frac{2}{k}} = 1$.

And so, we have that

$$\lim_{k\to\infty} \sqrt[k]{(a_k/N)^2 e^{2\rho k}} = \lim_{k\to\infty} (rk+1)^{\frac{2rk+2}{k}} \frac{1}{r^{2r}k^{2r}} \frac{e^{2\rho}}{e^{2r}} < \lim_{k\to\infty} (rk+1)^{\frac{2}{k}} \left(\frac{rk+1}{rk}\right)^{2r} = 1$$

where the inequality occurs since by hypothesis $\rho < r$.

□

Lemma 1.3.12. *For any $i \in \mathbb{N}$*

$$i! \;\leq\; \left(\frac{i}{e}\right)^i ie \;.$$

Proof. We will prove this lemma by induction on $i \geq 1$.
If $i = 1$ it is obvious.
Suppose that the thesis occurs for i, then

$$(i+1)! = i!(i+1) \leq \left(\frac{i}{e}\right)^i ie(i+1).$$

Now,

$$\left(\frac{i}{e}\right)^i ie(i+1) \leq \left(\frac{i+1}{e}\right)^{i+1}(i+1)e$$

if and only if

$$i^{i+1} \leq \frac{(i+1)^{i+1}}{e} \;,$$

namely

$$e \leq \left(1+\frac{1}{i}\right)^{i+1}$$

which is trivial considering that the last term is greater than or equal to 4 for any $i \geq 1$.

□

Remark 1.3.13. Because of this compatibility with the analytic functions we refer to the parameter r as the "radius" of the Gevrey space.

We conclude our first chapter with definition of the continuity modulus that we use throughout this work.

Definition 1.3.14 (Continuity modulus)**.** Let (X, d_x) and (Y, d_y) be metric spaces. Let $\omega : [0, +\infty) \longrightarrow [0, +\infty)$ and $f : X \longrightarrow Y$ be functions. We say that f is ω *continuous* in X if

$$d_y(f(b), f(a)) \leq \omega(d_x(b, a)) \quad \forall (a, b) \in X^2.$$

Example 1.3.15. We show some examples of continuity moduli:

- (Lipschitz continuity) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ and $\omega(x) = L|x|$, so

  $$|f(b) - f(a)| \leq L|b - a| \quad \forall (a, b) \in \mathbb{R}^2$$

  and L is called *Lipschitz* constant.

- (Hölder continuity) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ and $\omega(x) = H|x|^\alpha$ with $\alpha \in (0, 1)$, so

  $$|f(b) - f(a)| \leq H|b - a|^\alpha \quad \forall (a, b) \in \mathbb{R}^2$$

  and H is called *Hölder* constant.

- (log-Lipschitz continuity) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ and $\omega(x) = L|x||\log x|$, so

  $$|f(b) - f(a)| \leq L|b - a| \, |\log |b - a|| \quad \forall (a, b) \in \mathbb{R}^2$$

  and L is called *log-Lipschitz* constant.

Throughout this paper we assume that the *continuity modulus* respects the following conditions:

1. $\omega(0) = 0$;
2. $\omega(x)$ is a nondecreasing function;
3. $\frac{\omega(x)}{x}$ is a nonincreasing function.

Chapter 2

Approximated energy estimates

In this chapter we consider the following family of ordinary differential equations

$$\ddot{u}_\lambda(t) + \delta(t)\dot{u}_\lambda(t) + \lambda^2 c(t) u_\lambda(t) = 0 \tag{2.1}$$

with initial data

$$u_\lambda(0) = u_0 \qquad \dot{u}_\lambda(0) = u_1 \tag{2.2}$$

where $\delta(t) \geq 0$, λ is a real parameter and $c : [0, +\infty) \longrightarrow [0, +\infty)$ is a given function that we call *propagation speed* because of its meaning in this abstract setting of a wave equation. We always assume that $c(t)$ satisfies the strict hyperbolicity condition, i.e. there exist two constants μ_1 and μ_2 such that

$$0 < \mu_1 \leq c(t) \leq \mu_2. \tag{2.3}$$

Our aim is to estimate the growth of the solution $u_\lambda(t)$ as $\lambda \to +\infty$.
To do that we start to consider different kind of energy estimates.

2.1 Preliminary energy estimates

Definition 2.1.1. In view of the previous setting

- we call *Simple energy* the quantity

$$S_\lambda(t) := \dot{u}_\lambda^2(t) + u_\lambda^2(t), \tag{2.4}$$

- we call *Kovaleskian energy* the quantity

$$E_\lambda(t) := \dot{u}_\lambda^2(t) + \lambda^2 u_\lambda^2(t), \tag{2.5}$$

- let $\gamma(t)$ be a nonnegative function of class C^1, we call *Hyperbolic energy* the quantity

$$F_\lambda(t) := \dot{u}_\lambda^2(t) + \lambda^2 \gamma(t) u_\lambda^2(t). \tag{2.6}$$

In almost all of results we obtain in this chapter we use the following lemma that show the equivalence between Kovaleskian energy and Hyperbolic energy.

Lemma 2.1.2. *Let $\gamma(t)$ be a nonnegative function of class C^1. Assume the strict hyperbolicity condition (2.3) for $\gamma(t)$. Then, there exist two constants m and M such that both are > 0 and*

$$\frac{F_\lambda(t)}{M} \leq E_\lambda(t) \leq \frac{F_\lambda(t)}{m}.$$

Proof. For any t we have that

$$\begin{aligned} F_\lambda(t) &= \dot{u}_\lambda^2(t) + \lambda^2\gamma(t)u_\lambda^2(t) \\ &\geq \dot{u}_\lambda^2(t) + \lambda^2\mu_1 u_\lambda^2(t) \\ &\geq m\left(\dot{u}_\lambda^2(t) + \lambda^2 u_\lambda^2(t)\right) = mE_\lambda(t) \end{aligned}$$

where $m := \min\{1, \mu_1\}$. On the other hand,

$$\begin{aligned} F_\lambda(t) &= \dot{u}_\lambda^2(t) + \lambda^2\gamma(t)u_\lambda^2(t) \\ &\leq \dot{u_\lambda}^2(t) + \lambda^2\mu_2 u_\lambda^2(t) \\ &\leq M\left(\dot{u}_\lambda^2(t) + \lambda^2 u_\lambda^2(t)\right) = ME_\lambda(t) \end{aligned}$$

where $M := \max\{1, \mu_2\}$

□

Proposition 2.1.3. *In the same setting, the previous energies satisfy the following inequalities:*

1.
$$S_\lambda(t) \leq S_\lambda(0)\exp\left(\int_0^t \left(1 + \lambda^2|c(s)|\right) ds\right), \tag{2.7}$$

2.
$$E_\lambda(t) \leq E_\lambda(0)\exp\left(\lambda\int_0^t (1 + |c(s)|)\, ds\right), \tag{2.8}$$

3.
$$F_\lambda(t) \leq F_\lambda(0)\exp\left(\lambda\int_0^t \frac{|\gamma(s) - c(s)|}{\sqrt{\gamma(s)}} ds + \int_0^t \frac{|\dot{\gamma}(s)|}{\gamma(s)} ds\right). \tag{2.9}$$

Proof. 1. From the equation (2.1), it follows that

$$\begin{aligned} S_\lambda'(t) &= 2\dot{u}_\lambda(t)\ddot{u}_\lambda(t) + 2u_\lambda(t)\dot{u}_\lambda(t) \\ &= 2\dot{u}_\lambda(t)\left(-\delta(t)\dot{u}_\lambda(t) - \lambda^2 c(t)u_\lambda(t)\right) + 2u_\lambda(t)\dot{u}_\lambda(t) \\ &= -2\delta(t)\dot{u}_\lambda^2(t) + \left(-\lambda^2 c(t) + 1\right) 2u_\lambda(t)\dot{u}_\lambda(t). \end{aligned}$$

Now, due to the first component $-2\delta(t)\dot{u}_\lambda^2(t) \leq 0$, the second term satisfies $(-\lambda^2 c(t) + 1) \leq 1 + \lambda^2|c(t)|$ and for the other term we have that $2u_\lambda(t)\dot{u}_\lambda(t) \leq \dot{u}_\lambda^2(t) + u_\lambda^2(t) = S(t)$, consequently it is true that

$$S'_\lambda(t) \leq \Big(1 + \lambda^2|c(t)|\Big)\, S_\lambda(t).$$

Thanks to the last inequality we obtain thesis.

2. As before, we have that

$$\begin{aligned} E'_\lambda(t) &= 2\dot{u}_\lambda(t)\ddot{u}_\lambda(t) + 2\lambda^2 u_\lambda(t)\dot{u}_\lambda(t) \\ &= 2\dot{u}_\lambda(t)\Big(-\delta(t)\dot{u}_\lambda(t) - \lambda^2 c(t)u_\lambda(t)\Big) + 2\lambda^2 u_\lambda(t)\dot{u}_\lambda(t) \\ &= -2\delta(t)\dot{u}_\lambda^2(t) + \lambda\,(1 - c(t))\, 2\lambda u_\lambda(t)\dot{u}_\lambda(t). \end{aligned}$$

Now, thanks to $-2\delta(t)\dot{u}_\lambda^2(t) \leq 0$ and $(1 - c(t)) \leq (1 + |c(t)|)$ and $2\lambda u_\lambda(t)\dot{u}_\lambda(t) \leq \dot{u}_\lambda^2(t) + \lambda^2 u_\lambda^2(t) = E_\lambda(t)$, we obtain that

$$E'_\lambda(t) \leq \lambda\,(1 + |c(t)|)\, E_\lambda(t)$$

and so the thesis occurs.

3. As before,

$$\begin{aligned} F'_\lambda(t) &= 2\dot{u}_\lambda(t)\ddot{u}_\lambda(t) + 2\lambda^2\gamma(t)u_\lambda(t)\dot{u}_\lambda(t) + \lambda^2\dot{\gamma}(t)u_\lambda^2(t) \\ &= 2\dot{u}_\lambda(t)\Big(-\delta(t)\dot{u}_\lambda(t) - \lambda^2 c(t)u_\lambda(t)\Big) + 2\lambda^2\gamma(t)u_\lambda(t)\dot{u}_\lambda(t) + \lambda^2\dot{\gamma}(t)u_\lambda^2(t) \\ &= -2\delta(t)\dot{u}_\lambda^2(t) + \lambda\frac{(\gamma(t) - c(t))}{\sqrt{\gamma(t)}}2\sqrt{\gamma(t)}\lambda u_\lambda(t)\dot{u}_\lambda(t) + \frac{\dot{\gamma}(t)}{\gamma(t)}\gamma(t)\lambda^2 u_\lambda^2(t). \end{aligned}$$

Now, we use the following inequalities.

- $-2\delta(t)\dot{u}_\lambda^2(t) \leq 0$,
- $\dfrac{(\gamma(t) - c(t))}{\sqrt{\gamma(t)}} \leq \dfrac{|\gamma(t) - c(t)|}{\sqrt{\gamma(t)}}$,
- $2\sqrt{\gamma(t)}\lambda u_\lambda(t)\dot{u}_\lambda(t) \leq \dot{u}_\lambda^2(t) + \lambda^2\gamma(t)u_\lambda^2(t) = F_\lambda(t)$,
- $\dfrac{\dot{\gamma}(t)}{\gamma(t)} \leq \dfrac{|\dot{\gamma}(t)|}{\gamma(t)}$ and $\gamma(t)\lambda^2 u_\lambda^2(t) \leq F_\lambda(t)$.

Then we obtain that

$$F'_\lambda(t) \leq \left(\frac{|\gamma(t) - c(t)|}{\sqrt{\gamma(t)}}\lambda + \frac{|\dot{\gamma}(t)|}{\gamma(t)}\right) F_\lambda(t)$$

and so we concluded.

□

Remark 2.1.4. Due to equivalence showed in Lemma 2.1.2 we can gain the following inequality for kovaleskian energy too. In fact, using (2.9) it turns out that

$$E_\lambda(t) \leq \frac{F_\lambda(t)}{m} \leq \frac{F_\lambda(0)}{m} \exp\left(g(\lambda,t)\right) \leq \frac{M}{m} E_\lambda(0) \exp\left(g(\lambda,t)\right) \tag{2.10}$$

where

$$g(\lambda,t) := \lambda \int_0^t \frac{|\gamma(s)-c(s)|}{\sqrt{\gamma(s)}} ds + \int_0^t \frac{|\dot{\gamma}(s)|}{\gamma(s)} ds.$$

2.2 Kovaleskian estimates depending on $c(t)$ regularity

Due to the previous facts, we are now ready to prove important results.

Theorem 2.2.1. *Let T be a positive constant. Let the propagation speed $c(t) \in L^1((0,T))$. Hence, the solution $u_\lambda(t)$ of the problem (2.1)–(2.2) satisfies the following inequality*

$$\dot{u}_\lambda^2(t) + \lambda^2 u_\lambda^2(t) \ \leq \ K \exp\left(\lambda t + \lambda \int_0^t |c(s)| ds\right) \tag{2.11}$$

where

$$K := \left(u_1^2 + \lambda^2 u_0^2\right).$$

Proof. The thesis turns out just replacing the expression (2.5) in the inequality (2.8). Moreover, the RHS $\sim \exp(\lambda t)$ since $\int_0^t |c(s)| ds$ is finite by hypothesis.

□

Theorem 2.2.2. *Let T be a positive constant. Let $c(t)$ be a propagation speed satisfying the strict hyperbolicity condition (2.3). Let us assume that $c(t) \in L^1((0,T))$. Hence, for all $\epsilon > 0$ there exists a constant N_ϵ such that the solution $u_\lambda(t)$ of the problem (2.1)–(2.2) satisfies the following inequality*

$$\dot{u}_\lambda^2(t) + \lambda^2 u_\lambda^2(t) \leq K \exp\left\{\frac{\lambda\epsilon}{\mu_1} + N_\epsilon t\right\} \tag{2.12}$$

where, taking M and m as in (2.10),

$$K := \frac{M}{m}\left(u_1^2 + \lambda^2 u_0^2\right).$$

Proof. Take $\epsilon > 0$ and let $\gamma(t)$ be a C^1 function such that $\|\gamma(t) - c(t)\|_{L^1} \leq \epsilon$ and we can assume that $\gamma(t) \geq \mu_1^2 > 0$. Now, using the definition in (2.6) and the inequality obtained in (2.9), we have that

$$F_\lambda(t) \leq F_\lambda(0) \exp\left(\lambda \int_0^t \frac{|\gamma(s) - c(s)|}{\sqrt{\gamma(s)}} ds + \int_0^t \frac{|\dot{\gamma}(s)|}{\gamma(s)} ds\right).$$

But $\gamma(s) \geq \mu_1^2$ and $\int_0^t |\gamma(s) - c(s)| ds \leq \epsilon$, then we call

$$N_\epsilon := \frac{sup\left\{|\dot{\gamma}(s)| \ : \ s \in (0,t)\right\}}{\mu_1^2}.$$

Hence, N_ϵ is well defined because $\dot{\gamma}$ is a continuous function in an interval and we have that

$$F_\lambda(t) \leq F_\lambda(0) \exp\left\{\frac{\lambda\epsilon}{\mu_1} + N_\epsilon t\right\}.$$

Now, using the result obtained in (2.10)

$$\dot{u}_\lambda^2(t) + \lambda^2 u_\lambda^2(t) = E_\lambda(t) \leq \frac{M}{m} E_\lambda(0) \exp\left\{\frac{\lambda\epsilon}{\mu_1} + N_\epsilon t\right\}$$

and taking $K := \frac{M}{m} E(0)$ we have the thesis.

$\square$

Remark 2.2.3. In the previous proof we used

$$\int_0^t \frac{|\dot{\gamma}(s)|}{\gamma(s)} ds \leq N_\epsilon t$$

using that $|\dot{\gamma}(t)|$ is less than or equal to its sup. But it is important to remark that what we want to estimate from above is the integral and it could happen that the sup of the function $|\dot{\gamma}(t)|$ is very high, but the integral from 0 to t is the same bounded. Moreover, we must note that the constant N_ϵ is independent of λ.

Now, we consider the more general case in which we assume higher regularity on $c(t)$. But first we need the following lemma.

Lemma 2.2.4. *Let $c(t)$ be an ω-continuous function. Let us assume the strict hyperbolicity for $c(t)$. For every $\epsilon > 0$ take*

$$c_\epsilon(t) = \frac{1}{\epsilon} \int_t^{t+\epsilon} c(s) \ ds$$

Consequently,

1. $|c_\epsilon'(t)| \leq \dfrac{\omega(\epsilon)}{\epsilon}$,

2. $|c(t) - c_\epsilon(t)| \le \omega(\epsilon)$.

Proof. We'll prove both points.

1. Consider the derivative of $c_\epsilon(t)$ and we obtain

$$c'_\epsilon(t) = \frac{c(t+\epsilon) - c(t)}{\epsilon}.$$

And so,

$$|c'_\epsilon(t)| = \frac{|c(t+\epsilon) - c(t)|}{\epsilon} \le \frac{\omega(\epsilon)}{\epsilon}.$$

2. We have that

$$\begin{aligned} |c_\epsilon(t) - c(t)| &= \left| \left(\frac{1}{\epsilon} \int_t^{t+\epsilon} c(s)\ ds \right) - c(t) \right| \\ &= \left| \frac{1}{\epsilon} \int_t^{t+\epsilon} (c(s) - c(t))\, ds \right| \\ &\le \frac{1}{\epsilon} \int_t^{t+\epsilon} |c(s) - c(t)| ds \\ &\le \omega(\epsilon). \end{aligned}$$

$\square$

Theorem 2.2.5. *Let $c(t)$ be a propagation speed satisfying the strict hyperbolicity condition* (2.3)*. Let us assume that $c(t)$ is ω-continuous. Thus, there exists a constant $N > 0$ such that the solution $u_\lambda(t)$ of the problem* (2.1)–(2.2) *satisfies the following inequality*

$$\dot{u}_\lambda^2(t) + \lambda^2 u_\lambda^2(t) \le K \exp\left\{ N\lambda\omega\left(\frac{1}{\lambda}\right) t \right\} \tag{2.13}$$

where

$$K := \frac{M}{m}\left(u_1^2 + \lambda^2 u_0^2\right).$$

Proof. We use the estimate about the hyperbolic energy. Indeed, for all $\epsilon > 0$ we can take $\gamma(t) := c_\epsilon(t)$ where $c_\epsilon(t)$ is the function defined in the previous lemma. Thanks to the strict hyperbolicity of $c(t)$ and the definition of $c_\epsilon(t)$ it turns out that also $\gamma(t)$ satisfies $\mu_1 \le \gamma(t) \le \mu_2$. Now, using the results obtained in (2.9) and using the inequalities get in the previous lemma, it turns out that

$$F'_\lambda(t) \le \left(\frac{|\gamma(t) - c(t)|}{\sqrt{\gamma(t)}}\lambda + \frac{|\dot{\gamma}(t)|}{\gamma(t)} \right) F_\lambda(t) \le \left(\frac{\omega(\epsilon)}{\sqrt{\mu_1}}\lambda + \frac{\omega(\epsilon)}{\epsilon}\frac{1}{\mu_1} \right) F(t).$$

Moreover, there exists $H > 0$ (it is enough taking $H \ge \max\left\{ \frac{1}{\sqrt{\mu_1}}, \frac{1}{\mu_1} \right\}$) such that

$$\frac{\omega(\epsilon)}{\sqrt{\mu_1}}\lambda + \frac{\omega(\epsilon)}{\epsilon}\frac{1}{\mu_1} \le H\left(\lambda\omega(\epsilon) + \frac{\omega(\epsilon)}{\epsilon} \right).$$

Now, if we take $\epsilon = 1/\lambda$, we obtain that

$$F'_\lambda(t) \leq H\left(\lambda\omega\left(\frac{1}{\lambda}\right) + \lambda\omega\left(\frac{1}{\lambda}\right)\right) F_\lambda(t) = N\lambda\omega\left(\frac{1}{\lambda}\right) F_\lambda(t),$$

where $N = 2H$. And so,

$$F_\lambda(t) \leq F_\lambda(0) \exp\left\{N\lambda\omega\left(\frac{1}{\lambda}\right) t\right\}.$$

In conclusion, using the result obtained in (2.10)

$$\dot{u}_\lambda^2(t) + \lambda^2 u_\lambda^2(t) = E_\lambda(t) \leq \frac{M}{m} E_\lambda(0) \exp\left\{N\lambda\omega\left(\frac{1}{\lambda}\right) t\right\}$$

and taking $K := \frac{M}{m} E_\lambda(0)$ we have the thesis.

$\square$

Remark 2.2.6. The key idea of the previous proof is to put $\epsilon = 1/\lambda$ and this firstly appeared in the paper [1] published in 1979 by De Giorgi, Colombini and Spagnolo.

Corollary 2.2.7. *Let $c(t)$ be a propagation speed satisfying the strict hyperbolicity condition* (2.3)*. Assume that $c(t)$ is an α-Hölder function, so $\omega(x) = H|x|^\alpha$. Thus, there exists a constant $\hat{N} > 0$ such that the solution $u_\lambda(t)$ of the problem* (2.1)–(2.2) *satisfies the following inequality*

$$\dot{u}_\lambda^2(t) + \lambda^2 u_\lambda^2(t) \leq K \exp\left(\hat{N}\lambda^{1-\alpha} t\right). \tag{2.14}$$

Proof. Due to the previous Theorem there exists N such that

$$E_\lambda(t) \leq K \exp\left(N\lambda\omega\left(\frac{1}{\lambda}\right) t\right).$$

And so in this case,

$$E_\lambda(t) \leq K \exp\left(\hat{N}\lambda^{1-\alpha} t\right)$$

where $\hat{N}$ is an appropriate constant, in which we have included the *Hölder* constant too.

$\square$

Corollary 2.2.8. *Let $c(t)$ be a propagation speed satisfying the strict hyperbolicity condition* (2.3)*. Assume that $c(t)$ is a log-Lipschitz continuous function, so $\omega(x) = L|x||\log x|$. Thus, there exists a constant $\tilde{N} > 0$ such that the solution $u_\lambda(t)$ of the problem* (2.1)–(2.2) *satisfies the following inequality*

$$\dot{u}_\lambda^2(t) + \lambda^2 u_\lambda^2(t) \leq K\lambda^{\tilde{N}t}. \tag{2.15}$$

Proof. Due to the previous Theorem there exists N such that

$$E_\lambda(t) \leq K \exp\left(N\lambda\omega\left(\frac{1}{\lambda}\right)t\right).$$

And so in this case,

$$E_\lambda(t) \leq K \exp\left(\tilde{N}t\left|\log\left(\frac{1}{\lambda}\right)\right|\right) = K \exp\left\{\log\left(\lambda^{\tilde{N}t}\right)\right\} = K\lambda^{\tilde{N}t}$$

where $\tilde{N}$ is an appropriate constant, in which we have included the log-Lipschitz constant too.

$\square$

Corollary 2.2.9. *Let $c(t)$ be a propagation speed satisfying the strict hyperbolicity condition* (2.3). *Assume that $c(t)$ is a Lipschitz continuous function, so $\omega(x) = L|x|$. Thus, there exists a constant $\tilde{N} > 0$ such that the solution $u_\lambda(t)$ of the problem* (2.1)–(2.2) *satisfies the following inequality*

$$\dot{u}_\lambda^2(t) + \lambda^2 u_\lambda^2(t) \leq K \exp\left(\tilde{N}t\right). \tag{2.16}$$

Proof. Thanks to the previous Theorem there exists N such that

$$E_\lambda(t) \leq K \exp\left(N\lambda\omega\left(\frac{1}{\lambda}\right)t\right).$$

And so,

$$E_\lambda(t) \leq K \exp\left(\tilde{N}t\right)$$

where $\tilde{N}$ is an appropriate constant, in which we have included the Lipschitz constant too.

$\square$

Theorem 2.2.10. *Let $c(t)$ be a propagation speed satisfying the strict hyperbolicity condition* (2.3). *Let $c(t)$ be Lipschitz continuous and C^1. Therefore, there exists a constant $\bar{N} > 0$ such that the solution $u_\lambda(t)$ of the problem* (2.1)–(2.2) *satisfies the following inequality*

$$\dot{u}_\lambda^2(t) + \lambda^2 u_\lambda^2(t) \;\leq\; K \exp(\bar{N}t) \tag{2.17}$$

where

$$K := \frac{M}{m}\left(u_1^2 + \lambda^2 u_0^2\right).$$

Proof. We can use the estimate with the hyperbolic energy done before at (2.9) with $\gamma(t) = c(t)$. Indeed, put $F_\lambda(t) = \dot{u}_\lambda^2(t) + \lambda^2 c(t) u_\lambda^2(t)$ as in (2.6), we have that

$$F_\lambda'(t) \leq \frac{|\dot{c}(t)|}{c(t)} F(t) \leq \frac{L}{\mu_1} F(t) = \bar{N} F(t).$$

where L is the Lipschitz constant and $\bar{N} := \dfrac{L}{\mu_1}$. And so, using as before the (2.10),

$$\dot{u}_\lambda^2(t) + \lambda^2 u_\lambda^2(t) \le \dot{u}_\lambda^2(t) + \lambda^2 c(t) u_\lambda^2(t) \le \frac{M}{m} E(0) \exp(\bar{N} t).$$

Taking $K := \frac{M}{m} E(0)$, as before, we conclude.

□

Remark 2.2.11. In the last proof, thanks to $c(t)$ was C^1 by hypothesis, we used $\gamma(t) = c(t)$ and we obtained the inequality

$$E_\lambda(t) = \dot{u}_\lambda^2(t) + \lambda^2 u_\lambda^2(t) \ \le\ K \exp(Nt).$$

We can note that $E_\lambda(t)$ is bounded by a function that does not depend on the parameter λ. This result is consistent with the Corollary 2.2.9. The difference between the Corollary 2.2.9 and the Theorem 2.2.10 is that in the Theorem we assumed that $c(t)$ was C^1 as well as the Lipschitz continuity, as in Corollary. But, consistently, the estimate obtained is the same in both cases, but the constants results different. Indeed, in Corollary the constant $\tilde{N}$ is $\tilde{N} = 2HL$, where H is defined in the Theorem 2.2.5 as $H = \max\left\{\frac{1}{\sqrt{\mu_1}}, \frac{1}{\mu_1}\right\}$. While, the constant $\tilde{N}$ used in Theorem 2.2.10 is equal to L/μ_1.

And so, if $\mu_1 \le 1$ it turns out that $\tilde{N} = 2\bar{N}$; if $\mu_1 > 1$ it turns out that $\tilde{N} = 2\sqrt{\mu_1}\bar{N}$.

Chapter 3

Well-posedness Theorems

Firstly, in this chapter, we are going to define the meaning of a ”well-posed” problem and then we are going to discuss the well-posedness of our problem.

Definition 3.0.1. Given a mathematical problem, we say that it is *well-posed* if it meets the following proprieties:

1. There exists a solution,
2. The solution is unique,
3. The solution's behaviour changes continuously with the initial conditions.

Resuming the definitions given in the first chapter, let H be a separable Hilbert space with $\{e_k\}$ an its Hilbert basis and let A be a maximal multiplication operator. We assume that A is a coercive operator with $\gamma \geq 1$. Let $\{\lambda_k^2\}$ be the sequence of eigenvalues of A such that

$$\forall u \in H \quad u = \sum_{k \in \mathbb{N}} u_k e_k \quad and \quad Au = \sum_{k \in \mathbb{N}} \lambda_k^2 u_k e_k.$$

We consider the second order linear evolution equation

$$\ddot{u}(t) + c(t)Au = 0 \tag{3.1}$$

with initial data

$$u(0) = u_0, \qquad \dot{u}(0) = u_1. \tag{3.2}$$

Due to the assumption on the operator A, the original equation (3.1) is equivalent to the system of ODEs like

$$\ddot{u}_k(t) + \lambda_k^2 c(t) u_k(t) = 0 \tag{3.3}$$

which is the family of ordinary differential equations that we discussed in Chapter 2 with $\delta = 0$. Because of this reduction, in the next findings the uniqueness of the solution of problem (3.1)–(3.2) follows directly from the uniqueness of the solution of problem (3.3) for every k, that is a classic result about ODEs.

Moreover, to show the regularity of the solution we use the following statements.

Theorem 3.0.2 (Interchange of limit and derivative)**.** *Let X be a metric space. Let $f_n : [0,T] \longrightarrow X$ a sequence of functions. Assume that there exists $f : [0,T] \longrightarrow X$ and $g : [0,T] \longrightarrow X$ such that*

- $f_n \in C^1([0,T])$ *for each* $n \in \mathbb{N}$,
- $f_n \longrightarrow f$ *punctually,*
- $f'_n \longrightarrow g$ *uniformly.*

Therefore,

- f *is of class* C^1 *and* $f'(t) = g(t)$,
- $f_n \longrightarrow f$ *uniformly.*

Proposition 3.0.3. *Let H be a separable Hilbert space with $\{e_k\}$ an its Hilbert basis. Let $u : [0,T] \longrightarrow H$ be a function and for any $t \in [0,T]$ take $u(t) = \sum_{k\in\mathbb{N}} u_k(t)e_k$. Let $\{w_k\}$ be a sequence contained in $\mathbb{R}^+$ such that $w_k \geq 1$ $\forall k \in \mathbb{N}$ and let be $W := \{u(t) \mid \sum_{k=1}^{\infty} w_k u_k(t)^2 < +\infty \ \forall t \in [0,T]\}$. Therefore, each $u(t) \in W$ is a continuous function.*

Moreover, let $\{h_k\}$ be another sequence contained in $\mathbb{R}^+$ such that $h_k \geq 1$ $\forall k \in \mathbb{N}$ and let $K := \{u(t) \in W \mid \sum_{k=1}^{\infty} h_k \dot{u}_k(t)^2 < +\infty \ \forall t \in [0,T]\}$. Hence, $u(t) \in C^1([0,T])$ and $u'(t) = \sum_{k\in\mathbb{N}} \dot{u}_k(t)e_k$.

Proof. To prove the continuity of $u(t)$ we need that the series $\sum_{k\in\mathbb{N}} u_k(t)e_k$ converges uniformly. This follows because the series is totally convergent. Indeed,

$$\sum_{k\in\mathbb{N}} \left(\sup_{t\in[0,T]} u_k(t) \right) e_k < \infty$$

if and only if

$$\sum_{k\in\mathbb{N}} \left(\sup_{t\in[0,T]} u_k(t) \right)^2 < \infty$$

and the last series is convergent since by hypothesis $\sum_{k=1}^{\infty} w_k u_k(t)^2$ is convergent for all $t \in [0,T]$ and $w_k \geq 1$ $\forall k \in \mathbb{N}$.

For the second part of the proof, similar to what we did above,

$$\sum_{k\in\mathbb{N}} \left(\sup_{t\in[0,T]} \dot{u}_k(t) \right) e_k < \infty,$$

and so the series $\sum_{k\in\mathbb{N}} \dot{u}_k(t)e_k$ is totally convergent. This means that thanks to the theorem of interchange of limit and derivative it turns out that

$$u(t) \in C^1([0,T]) \quad \textit{and} \quad u'(t) = \sum_{k\in\mathbb{N}} \dot{u}_k(t)e_k.$$

□

We want to exhibit the important findings about the well-posedness of the problem introduced above which are included in [1].

The first result concerns the well-posedness in huge spaces such as analytic ultra-distributions, with minimal assumptions on $c(t)$.

Theorem 3.0.4. *Let us consider problem* (3.1)–(3.2) *under the following assumptions:*

- $c(t) \in L^1((0,T))$ *for every* $T > 0$ *(without sign condition),*
- *initial conditions satisfy*

$$(u_0, u_1) \in \mathcal{G}_{-\psi,R_0,1/2}(A) \times \mathcal{G}_{-\psi,R_0,0}(A)$$

for some $R_0 > 0$ *and* $\psi : (0,+\infty) \longrightarrow (0,+\infty)$ *such that* $\psi(x) = x$.

Then, there exists a nondecreasing function $R : [0,+\infty) \longrightarrow [0,+\infty)$ *with* $R(0) = R_0$, *such that problem* (3.1)–(3.2) *admits a unique solution*

$$u \in C^0\left([0,+\infty), \mathcal{G}_{-\psi,R(t),1/2}(A)\right) \cap C^1\left([0,+\infty), \mathcal{G}_{-\psi,R(t),0}(A)\right). \tag{3.4}$$

Condition (3.4) *simply means that*

$$u \in C^0\left([0,\tau], \mathcal{G}_{-\psi,R(\tau),1/2}(A)\right) \cap C^1\left([0,\tau], \mathcal{G}_{-\psi,R(\tau),0}(A)\right) \quad \forall \tau \geq 0.$$

Proof. Due to result obtained in Theorem 2.2.1, we have that for every $k \in \mathbb{N}$

$$\dot{u}_k^2(t) + \lambda_k^2 u_k^2(t) \ \leq \ \left(u_{1k}^2 + \lambda_k^2 u_{0k}^2\right) \exp\left(\lambda_k t + \lambda_k \int_0^t |c(s)|ds\right).$$

But, the RHS of the previous inequality, multiplying and dividing by $\exp(2\psi(\lambda_k)R_0)$, is equal to

$$\left(u_{1k}^2 + \lambda_k^2 u_{0k}^2\right) \exp(-2\psi(\lambda_k)R_0) \exp\left(2\psi(\lambda_k)R_0 + \lambda_k t + \lambda_k \int_0^t |c(s)|ds\right).$$

Now,

- The series of the first part of the previous term, i.e. $(u_{1k}^2 + \lambda_k^2 u_{0k}^2)\exp(-2\psi(\lambda_k)R_0)$, converges by assumption on (u_0, u_1).
- About the second part of the previous term, we can say that

$$\begin{aligned}\exp\left(2\psi(\lambda_k)R_0 + \lambda_k t + \lambda_k \int_0^t |c(s)|ds\right) &= \exp\left(2\lambda_k\left(R_0 + \frac{t}{2} + \frac{C(t)}{2}\right)\right)\\ &= \exp(2\lambda_k R(t))\end{aligned}$$

where we used that $\psi(x) = x$ and we set

$$C(t) := \int_0^t |c(s)|ds, \qquad R(t) := \left(R_0 + \frac{t}{2} + \frac{C(t)}{2}\right).$$

And so,

$$\sum_{k\in\mathbb{N}}(\dot{u}_k^2(t)+\lambda_k^2u_k^2(t))\exp(-2\lambda_kR(t))\leq\sum_{k\in\mathbb{N}}\left(u_{1k}^2+\lambda_k^2u_{0k}^2\right)\exp\left(-2\lambda_kR_0\right)$$

that, as noted above, converges by assumption. Then, we got that $u(t)$ belongs to the spaces of ultradistributions as we wanted and the time-regularity of the solution follows by Proposition 3.0.3.

□

In the second result we assume strict hyperbolicity and ω-continuity of the coefficient, and we obtain well-posedness in a suitable class of Gevrey ultradistributions.

Theorem 3.0.5. *Let us consider problem* (3.1)–(3.2) *under the following assumptions:*

- $c:[0,+\infty)\longrightarrow\mathbb{R}$ *satisfies the strict hyperbolicity, i.e.* $0<\mu_1\leq c(t)\leq\mu_2$ *for some constants* $0<\mu_1<\mu_2$*, and the* ω*-continuity assumption for some continuity modulus,*
- *initial conditions satisfy*

$$(u_0,u_1)\in\mathcal{G}_{-\psi,R_0,1/2}(A)\times\mathcal{G}_{-\psi,R_0,0}(A)$$

for some $R_0>0$ *and* $\psi:[0,+\infty)\longrightarrow(0,+\infty)$ *such that*

$$\limsup_{x\to+\infty}\frac{x}{\psi(x)}\omega\left(\frac{1}{x}\right)<+\infty. \tag{3.5}$$

Let u *be the unique solution provided by Theorem 3.0.4.*
Then, there exists $R>0$ *such that*

$$u\in C^0\left([0,+\infty),\mathcal{G}_{-\psi,R_0+Rt,1/2}(A)\right)\cap C^1\left([0,+\infty),\mathcal{G}_{-\psi,R_0+Rt,0}(A)\right). \tag{3.6}$$

Condition (3.6)*, as before, means that*

$$u\in C^0\left([0,\tau],\mathcal{G}_{-\psi,R_0+R\tau,1/2}(A)\right)\cap C^1\left([0,\tau],\mathcal{G}_{-\psi,R_0+R\tau,0}(A)\right)\quad\forall\tau\geq 0.$$

Proof. Due to result obtained in Theorem 2.2.5, we have that for any $k\in\mathbb{N}$

$$\dot{u}_k^2(t)+\lambda_k^2u_k^2(t)\ \leq\ K\left(u_{1k}^2+\lambda_k^2u_{0k}^2\right)\exp\left\{N\lambda_k\omega\left(\frac{1}{\lambda_k}\right)t\right\}.$$

Now, by assumption we know that

$$\frac{\lambda_k}{\psi(\lambda_k)}\omega\left(\frac{1}{\lambda_k}\right)\quad \textit{is bounded,}$$

so it is smaller than a certain constant $M > 0$. Thus, taking $R > 0$ such that $R > HM$ where $H = N/2$, it turns out that

$$\sum_{k\in\mathbb{N}} \left(\dot{u}_k^2(t) + \lambda_k^2 u_k^2(t)\right) \exp\left\{-2\psi(\lambda_k)(R_0 + Rt)\right\} \tag{3.7}$$

$$\leq K \sum_{k\in\mathbb{N}} \left(u_{1k}^2 + \lambda_k^2 u_{0k}^2\right) \exp\left\{-2\psi(\lambda_k)R_0\right\} \exp\left\{\left(-2\psi(\lambda_k)R + 2H\lambda_k\omega\left(\frac{1}{\lambda_k}\right)\right)t\right\}. \tag{3.8}$$

And, the last part of the previous term satisfies that

$$\begin{aligned}
-2\psi(\lambda_k)R + 2H\lambda_k\omega\left(\frac{1}{\lambda_k}\right) &= \psi(\lambda_k)\left(-2R + 2H\frac{\lambda_k}{\psi(\lambda_k)}\omega\left(\frac{1}{\lambda_k}\right)\right) \\
&\leq \psi(\lambda_k)\left(-2R + 2HM\right) = 2\psi(\lambda_k)(HM - R) \\
&\leq 0
\end{aligned}$$

where the last inequality follows by the choice of R.
And so,

$$(3.8) \leq K \sum_{k\in\mathbb{N}} \left(u_{1k}^2 + \lambda_k^2 u_{0k}^2\right) \exp\left\{-2\psi(\lambda_k)R_0\right\}$$

that is a convergent series by assumption. Consequently, also the series (3.7) is convergent and we concluded since the time-regularity of the solution follows by Proposition 3.0.3.

□

In the third result, the assumption on $c(t)$ are the same of the Theorem 3.0.5, but initial data are significantly more regular (Gevrey spaces instead of Gevrey ultradistributions).

Theorem 3.0.6. *Let us consider problem* (3.1)–(3.2) *under the following assumptions:*

- $c : [0, +\infty) \longrightarrow \mathbb{R}$ *satisfies the strict hyperbolicity, i.e.* $0 < \mu_1 \leq c(t) \leq \mu_2$ *for some constants* $0 < \mu_1 < \mu_2$*, and the* ω*-continuity assumption for some continuity modulus,*
- *initial conditions satisfy*

$$(u_0, u_1) \in \mathcal{G}_{\varphi, r_0, 1/2}(A) \times \mathcal{G}_{\varphi, r_0, 0}(A)$$

for some $r_0 > 0$ *and* $\varphi : [0, +\infty) \longrightarrow (0, +\infty)$ *such that*

$$\limsup_{x\to+\infty} \frac{x}{\varphi(x)}\omega\left(\frac{1}{x}\right) < +\infty. \tag{3.9}$$

Let u be the unique solution provided by Theorem 3.0.4.
Then, there exist $T > 0$ and $r > 0$ such that $rT < r_0$ and

$$u \in C^0\left([0,T], \mathcal{G}_{\varphi, r_0 - rt, 1/2}(A)\right) \cap C^1\left([0,T], \mathcal{G}_{\varphi, r_0 - rt, 0}(A)\right). \tag{3.10}$$

Proof. As done in the previous proof, thanks to result obtained in Theorem 2.2.5, we have that for every $k \in \mathbb{N}$

$$\dot{u}_k^2(t) + \lambda_k^2 u_k^2(t) \ \leq \ K\left(u_{1k}^2 + \lambda_k^2 u_{0k}^2\right) \exp\left\{2H\lambda_k \omega\left(\frac{1}{\lambda_k}\right) t\right\}$$

Now, by assumption we know that

$$\frac{\lambda_k}{\varphi(\lambda_k)}\omega\left(\frac{1}{\lambda_k}\right) \quad \textit{is bounded,}$$

so it is smaller than a certain constant $M > 0$. Thus, taking $r > 0$ such that $r > HM$ and $T > 0$ such that $rT < r_0$, until $t \leq T$ it turns out that

$$\sum_{k \in \mathbb{N}} \left(\dot{u}_k^2(t) + \lambda_k^2 u_k^2(t)\right) \exp\left\{2\varphi(\lambda_k)(r_0 - rt)\right\} \tag{3.11}$$

$$\leq K \sum_{k \in \mathbb{N}} \left(u_{1k}^2 + \lambda_k^2 u_{0k}^2\right) \exp\left\{2\varphi(\lambda_k) r_0\right\} \exp\left\{\left(-2\varphi(\lambda_k) r + 2H\lambda_k \omega\left(\frac{1}{\lambda_k}\right)\right) t\right\}. \tag{3.12}$$

And, the last part of the previous term satisfies that

$$\begin{aligned} -2\varphi(\lambda_k) r + 2H\lambda_k \omega\left(\frac{1}{\lambda_k}\right) &= \varphi(\lambda_k)\left(-2r + 2H\frac{\lambda_k}{\varphi(\lambda_k)}\omega\left(\frac{1}{\lambda_k}\right)\right) \\ &\leq \varphi(\lambda_k)\left(-2r + 2HM\right) = 2\varphi(\lambda_k)(HM - r) \\ &\leq 0 \end{aligned}$$

where the last inequality follows by the choice of r.

And so,

$$(3.12) \leq K \sum_{k \in \mathbb{N}} \left(u_{1k}^2 + \lambda_k^2 u_{0k}^2\right) \exp\left\{2\varphi(\lambda_k) r_0\right\}.$$

that is a convergent series by assumption. Consequently, also the series (3.11) is convergent and we concluded. As before, the time-regularity of the solution follows by Proposition 3.0.3.

□

Remark 3.0.7. We can note that in the last proof it is possible to choose r like we wanted, i.e. $HM < r < r_0/T$, because also T is a free variable. But, it happens that if HM is bigger we have to choose a smaller T and so the solution is defined for a shorter time interval.

Remark 3.0.8. The key assumptions of Theorems 3.0.5 and 3.0.6 are conditions (3.5) and (3.9) respectively, that represent the exact compensation between space-regularity of initial data and time-regularity of the propagation speed $c(t)$ required in order to obtain well-posedness. In the next chapter we prove that this conditions are optimal.

Corollary 3.0.9. *Let us consider problem* (3.1)–(3.2) *under the same assumptions of the two previous Theorems. Let* $\omega(x) = Hx^{\alpha}$*, with* $\alpha \in (0, 1)$*, be the Hölder continuity modulus and let initial data belong, respectively, to standard Gevrey ultradistribution and standard Gevrey space, i.e. with* $\psi(x) = \varphi(x) = x^{1-\alpha}$*, which satisfy the assumptions* (3.5) *and* (3.9)*. Therefore, the problem is well-posed in standard Gevrey ultradistributions and standard Gevrey space of order* $(1-\alpha)^{-1}$.

Now, we are going to discuss some particular cases. In the first one, we assume the log-Lipschitz continuity which produces well-posedness in Sobolev spaces with a derivative loss by time. In the second one, we assume $\omega(x) = Lx|\log x|^{\beta}$ as continuity modulus and we obtain well-posedness in Sobolev spaces with an arbitrarily small derivative loss. In the last one, we assume Lipschitz continuity which produces well-posedness in the same spaces of initial conditions, i.e. without derivative loss.

Theorem 3.0.10 (Derivative loss)**.** *Let us consider problem* (3.1)–(3.2) *under the following assumptions:*

- $c : [0, T] \longrightarrow \mathbb{R}$ *satisfies the strict hyperbolicity, i.e.* $0 < \mu_1 \leq c(t) \leq \mu_2$ *for some constants* $0 < \mu_1 < \mu_2$*, and the* ω*-continuity assumption for* $\omega(x) = Lx|\log x|$*,*
- *initial conditions satisfy*
$$(u_0, u_1) \in D(A^{\alpha+1/2}) \times D(A^{\alpha}).$$

Then, there exists $r > 0$ *such that the problem admits a unique solution*

$$u \in C^0\left([0, T], D(A^{\alpha+1/2-rt})\right) \cap C^1\left([0, T], D(A^{\alpha-rt})\right). \tag{3.13}$$

Proof. Thanks to result obtained in Theorem 2.2.5, we have that for every $k \in \mathbb{N}$

$$\dot{u}_k^2(t) + \lambda_k^2 u_k^2(t) \;\leq\; K\left(u_{1k}^2 + \lambda_k^2 u_{0k}^2\right) \exp\left\{2H\lambda_k\omega\left(\frac{1}{\lambda_k}\right)t\right\}.$$

And so,

$$\dot{u}_k^2(t) + \lambda_k^2 u_k^2(t) \;\leq\; K\left(u_{1k}^2 + \lambda_k^2 u_{0k}^2\right) \exp\left\{2HL\left|\log\left(\lambda_k\right)\right| t\right\}.$$

Then, taking $r = HL/2$ and due to $\exp\left\{2HL\log\left(\lambda_k\right)t\right\} = \lambda_k^{4rt}$, it turns out that

$$\lambda_k^{4\alpha}\left(\dot{u}_k^2(t) + \lambda_k^2 u_k^2(t)\right) \leq K\lambda_k^{4\alpha}\left(u_{1k}^2 + \lambda_k^2 u_{0k}^2\right)\lambda_k^{4rt}.$$

And so,

$$\sum_{k\in\mathbb{N}} \lambda_k^{4(\alpha-rt)}\left(\dot{u}_k^2(t) + \lambda_k^2 u_k^2(t)\right) \leq K\sum_{k\in\mathbb{N}}\lambda_k^{4\alpha}\left(u_{1k}^2 + \lambda_k^2 u_{0k}^2\right)$$

that is a convergent series by assumption. As above, the time-regularity of the solution follows by the Proposition 3.0.3.

□

Theorem 3.0.11 (Arbitrarily small derivative loss)**.** *Let us consider problem* (3.1)–(3.2) *under the following assumptions:*

- $c : [0, T] \longrightarrow \mathbb{R}$ *satisfies the strict hyperbolicity, i.e.* $0 < \mu_1 \leq c(t) \leq \mu_2$ *for some constants* $0 < \mu_1 < \mu_2$*, and the* ω*-continuity assumption for* $\omega(x) = Lx|\log x|^\beta$*, with* $\beta \in (0, 1)$*,*
- *initial conditions satisfy*

$$(u_0, u_1) \in D(A^{\alpha+1/2}) \times D(A^\alpha).$$

Then, for every $\epsilon > 0$ *the problem admits a unique solution*

$$u \in C^0\left([0, T], D(A^{\alpha+1/2-\epsilon})\right) \cap C^1\left([0, T], D(A^{\alpha-\epsilon})\right). \tag{3.14}$$

Proof. Due to result obtained in Theorem 2.2.5, we have that for every $k \in \mathbb{N}$

$$\dot{u}_k^2(t) + \lambda_k^2 u_k^2(t) \;\leq\; K\left(u_{1k}^2 + \lambda_k^2 u_{0k}^2\right) \exp\left\{2H\lambda_k \omega\left(\frac{1}{\lambda_k}\right) t\right\}.$$

And so,

$$\dot{u}_k^2(t) + \lambda_k^2 u_k^2(t) \;\leq\; K\left(u_{1k}^2 + \lambda_k^2 u_{0k}^2\right) \exp\left\{2HL\left|\log\left(\lambda_k\right)\right|^\beta t\right\}.$$

Then, for any $\epsilon > 0$, due to $\exp\{-4\epsilon \log \lambda_k\} = \lambda_k^{-4\epsilon}$,

$$\lambda_k^{4\alpha}\left(\dot{u}_k^2(t) + \lambda_k^2 u_k^2(t)\right) \leq K\lambda_k^{4\alpha+4\epsilon}\left(u_{1k}^2 + \lambda_k^2 u_{0k}^2\right) \exp\{\underbrace{2HL\left|\log\left(\lambda_k\right)\right|^\beta t - 4\epsilon \log \lambda_k}_{(i)}\}.$$

But, thanks to $t \leq T$, for any $\epsilon > 0$ there exists a constant M_ϵ such that (i) is smaller than M_ϵ. Thus,

$$\sum_{k\in\mathbb{N}} \lambda_k^{4(\alpha-\epsilon)}\left(\dot{u}_k^2(t) + \lambda_k^2 u_k^2(t)\right) \leq K \exp(M_\epsilon) \sum_{k\in\mathbb{N}} \lambda_k^{4\alpha}\left(u_{1k}^2 + \lambda_k^2 u_{0k}^2\right)$$

that is a convergent series by assumption. As always, the time-regularity of the solution follows by Proposition 3.0.3.

□

Theorem 3.0.12. *Let us consider problem* (3.1)–(3.2) *under the following assumptions:*

- $c : [0, T] \longrightarrow \mathbb{R}$ *satisfies the strict hyperbolicity, i.e.* $0 < \mu_1 \leq c(t) \leq \mu_2$ *for some constants* $0 < \mu_1 < \mu_2$*, and the* ω*-continuity assumption for* $\omega(x) = Lx$*,*

- *initial conditions satisfy*

$$(u_0, u_1) \in D(A^{\alpha+1/2}) \times D(A^{\alpha}).$$

Then, the problem admits a unique solution

$$u \in C^0 \left([0, T], D(A^{\alpha+1/2})\right) \cap C^1 \left([0, T], D(A^{\alpha})\right). \tag{3.15}$$

Proof. Thanks to result obtained in Corollary 2.2.9, we have that

$$\dot{u}_k^2(t) + \lambda_k^2 u_k^2(t) \leq K \left(u_{1k}^2 + \lambda_k^2 u_{0k}^2\right) \exp\left(\tilde{N}t\right).$$

And so, taking $M := \exp(\tilde{N}T)$, it turns out that

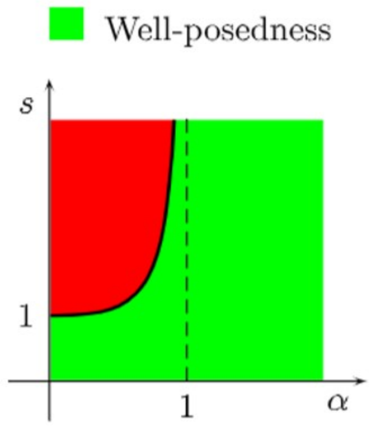
$$\sum_{k\in\mathbb{N}} \left(\dot{u}_k^2(t) + \lambda_k^2 u_k^2(t)\right) \lambda_k^{4\alpha} \leq KM \sum_{k\in\mathbb{N}} \lambda_k^{4\alpha} \left(u_{1k}^2 + \lambda_k^2 u_{0k}^2\right)$$

that is a convergent series by assumption. As always, the time-regularity of the solution follows by Proposition 3.0.3.

□

To conclude this chapter, we could combine some of the previous results in a rough graph which could explain intuitively what we obtained.
Consider the standard Gevrey space with $\varphi(x) = x^{1/s}$. In Corollary 3.0.9 we obtained that the problem (3.1)–(3.2) is *well-posed* in standard Gevrey spaces with $s = (1-\alpha)^{-1}$.

Then, take on the x-axis the value of α that represents the time-regularity of the function $c(t)$, i.e. if $\alpha \in (0, 1)$ it means that $c(t)$ is α-*Hölder*: $\alpha = 1$ means Lipschitz and $\alpha = 7$ means C^7, and so on. In particular, $\alpha = 3.14$ means that $c(t) \in C^3$ and $\dot{c}(t)$ is a α-*Hölder* function, with $\alpha = 0.14$.
On the other hand, we take on the y-axis the space-regularity of initial data, where the value s stands for the Gevrey space of order s (so that higher values of s mean lower regularity). Thus, the curve that describes where we have well-posedness or not is $s = (1 - \alpha)^{-1}$.


Well-posedness
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Chapter 4

Swing effect

4.1 Heuristic search for counterexamples

In the previous chapters, we discussed the well-posedness of the original problem. Starting from the basic ordinary differential equations

$$\ddot{u}_\lambda + \lambda^2 c(t) u_\lambda = 0 \tag{4.1}$$

with initial data

$$u_\lambda(0) = 0 \qquad \dot{u}_\lambda(0) = b\lambda, \quad b > 0 \tag{4.2}$$

our aim was to estimate the growth of the solution $u_\lambda(t)$ as $\lambda \to +\infty$. To get it we performed fundamental estimates on Kovaleskian energy of the solution and the basic one that we obtained was

$$E(t) \leq E(0) \exp\left(M\lambda t\right). \tag{4.3}$$

In this chapter, we want to prove that the inequalities found in the second chapter are optimal. Indeed, we are going to construct suitable solutions that realize the equality in (4.3) .

Our idea is to find a solution that grows exponentially with time. If we think of problem (4.1) as a model of a harmonic oscillator we need to find an opportune spring's strength $c(t)$ which allows the function to grow exponentially. The basic idea is to use a stronger spring when we are moving toward the origin and a weaker spring when we are going to the opposite side, exactly how children do on a swing.

Mathematically, to achieve this goal, it seems reasonable to look for a function of type

$$u(t) = \sin(\lambda t) e^{a(t)} \tag{4.4}$$

with a function $a(t)$ possibly increasing with time. Starting from the definition of $u(t)$ we find that

$$\dot{u}(t) = \lambda \cos(\lambda t) e^{a(t)} + \sin(\lambda t) \dot{a}(t) e^{a(t)}$$

$$\ddot{u}(t) = e^{a(t)} \left\{ -\lambda^2 \sin(\lambda t) + 2\lambda \cos(\lambda t) \dot{a}(t) + \sin(\lambda t) \ddot{a}(t) + \sin(\lambda t) \dot{a}(t)^2 \right\}.$$

And so, plugging the last expressions into (4.1) we obtain that

$$\ddot{u} + \lambda^2 c(t) u = 0$$

if and only if

$$-\lambda^2 \sin(\lambda t) + 2\lambda \cos(\lambda t)\dot{a}(t) + \sin(\lambda t)\ddot{a}(t) + \sin(\lambda t)\dot{a}(t)^2 + c(t)\lambda^2 \sin(\lambda t) = 0,$$

namely

$$\sin(\lambda t)\left\{-\lambda^2 + \ddot{a}(t) + \dot{a}(t)^2 + c(t)\lambda^2\right\} + 2\lambda\cos(\lambda t)\dot{a}(t) = 0.$$

Due to the last equality it can be a reasonable choice taking $a(t)$ such that

$$\dot{a}(t) = \alpha \sin^2(\lambda t)$$

for a suitable constant $\alpha \in \mathbb{R}$. Hence, we have that

$$\ddot{a}(t) = 2\alpha\lambda \sin(\lambda t)\cos(\lambda t),$$

and so

$$-\lambda^2 + 4\alpha\lambda \sin(\lambda t)\cos(\lambda t) + \alpha^2 \sin^4(\lambda t) + c(t)\lambda^2 = 0.$$

At this point, we set $\alpha = \epsilon\lambda$ for a reasonable choice of $\epsilon > 0$ and we obtain that

$$\lambda^2\left(-1 + 4\epsilon \sin(\lambda t)\cos(\lambda t) + \epsilon^2 \sin^4(\lambda t) + c(t)\right) = 0.$$

Now, we can take

$$c(t) = 1 - 4\epsilon \sin(\lambda t)\cos(\lambda t) - \epsilon^2 \sin^4(\lambda t). \tag{4.5}$$

We can choose the value of ϵ, so if we take ϵ small enough, in particular $0 < \epsilon < 1/5$, it turns out that $1/2 \leq c(t) \leq 3/2$, which is the strict hyperbolicity condition. Therefore, due to $\dot{a}(t) = \epsilon\lambda \sin^2(\lambda t)$, we achieve that

$$a(t) = \frac{1}{2}\epsilon\lambda t - \frac{1}{4}\epsilon \sin(2\lambda t) + c\ ,$$

where c is a real constant which depends on initial conditions.

Going back to the first assumption on $u(t)$ we can now write the right expression for this function

$$u(t) = b \sin(\lambda t) \exp\left\{\frac{1}{2}\epsilon\lambda t - \frac{1}{4}\epsilon \sin(2\lambda t)\right\} \tag{4.6}$$

where $b = e^c$.

Remark 4.1.1. In the assumption (4.4) we could replace $\sin(\lambda t)$ with $\cos(\lambda t)$ and we would achieve the same outcome, i.e. finding a solution that grows exponentially with time.

The solution that we have just found depends on λ, in particular the formula that describes the spring's strength $c(t)$ written in (4.5) heavily depends on λ. Because of this reason we refer to it as $c_\lambda(t)$.

4.2 General counterexample

Let us remember our general problem. We have a separable Hilbert space H with $\{e_k\}$ an its Hilbert basis and A a maximal multiplication operator. We assume that A is a coercive operator with $\gamma \geq 1$. Let $\{\lambda_k^2\}$ be the sequence of eigenvalues of A such that

$$\forall u \in H \quad u = \sum_{k \in \mathbb{N}} u_k e_k \quad \textit{and} \quad Au = \sum_{k \in \mathbb{N}} \lambda_k^2 u_k e_k.$$

We considered the equation

$$\ddot{u}(t) + c(t)Au = 0 \tag{4.7}$$

with initial data

$$u(0) = u_0, \qquad \dot{u}(0) = u_1. \tag{4.8}$$

Due to the assumption on the operator A, we noted that the original equation (4.7) is equivalent to the system of ODEs like

$$\ddot{u}_k(t) + \lambda_k^2 c(t) u_k(t) = 0. \tag{4.9}$$

Then we have discussed the well-posedness of the problem (4.7)–(4.8) with respect to the regularity of the propagation speed.

Now, we will show that there exists a propagation speed $c(t)$ that is ω-continuous with "good" initial condition, i.e. such that at time $t = 0$ the solution belongs to a suitable *Gevrey space*, which starts growing out of control by time and it does not belong to any *Gevrey ultradistributions* for any radius and for any $t > 0$. More precisely, we will prove the following result.

Theorem 4.2.1. *In our usual general setting, let* $\varphi : (0, +\infty) \longrightarrow (0, +\infty)$ *and* $\psi : (0, +\infty) \longrightarrow (0, +\infty)$ *be two functions such that*

$$\lim_{x \to \infty} \frac{x}{\varphi(x)} \omega\left(\frac{\pi}{x}\right) = \lim_{x \to \infty} \frac{x}{\psi(x)} \omega\left(\frac{\pi}{x}\right) = +\infty. \tag{4.10}$$

Then there exists an ω*-continuous function* $c : \mathbb{R} \longrightarrow \mathbb{R}$ *such that*

$$\frac{1}{2} \leq c(t) \leq \frac{3}{2} \quad \forall t \in \mathbb{R} \tag{4.11}$$

and a solution $u(t)$ *to equation* (4.7) *such that*

$$(u(0), \dot{u}(0)) \in \mathcal{G}_{\varphi,r,1/2}(A) \times \mathcal{G}_{\varphi,r,0}(A) \quad \forall r > 0 \tag{4.12}$$

$$(u(t), \dot{u}(t)) \notin \mathcal{G}_{-\psi,R,1/2}(A) \times \mathcal{G}_{-\psi,R,0}(A) \quad \forall R > 0, \ \forall t > 0. \tag{4.13}$$

Proof. Before proceeding with technical details let us describe the general idea and the strategy of the proof.

Roughly speaking, we are looking for a solution $u(t)$ whose components $u_k(t)$ are small at time $t = 0$ and huge when $t > 0$. Moreover, we want that the solution grows exactly as the right-hand side of (2.13). For any $k \in \mathbb{N}$ we can find a function like in (4.6), but, unfortunately, this is not enough because we need to realize a similar growth for all the countable components with the same coefficient $c(t)$.

Then, we introduce a suitable decreasing sequence $t_k \to 0^+$ and in each interval $[t_k, t_{k-1}]$ we design the coefficient $c(t)$ so that $u_k(t_k)$ is small and $u_k(t_{k-1})$ is huge. Then, we check that the function $c(t)$ as defined above, has the required time-regularity, and that each component $u_k(t)$ remains small for $t \in [0, t_k]$ and huge for $t \geq t_{k-1}$.

Mathematically speaking, fixed the suitable sequence t_k, for any $k \in \mathbb{N}$ we consider the problem (4.1) depending on λ_k, i.e. the equation (4.9). Now, we define $c(t)$ for $t \in [t_k, t_{k-1}]$ as in (4.5) and this assumption is necessary in order to increase the component $u_k(t)$, as we have seen at the beginning of this chapter. So, at this point, we have four different sequences that we are free to choose in order to achieve our aim:

- $\{\lambda_k\}$, that is a subsequence of eigenvalues of the operator A;
- $\{t_k\}$, that is the sequence of times that we have just discussed;
- $\{\epsilon_k\}$, that is the sequence of amplitude of the oscillation of the suitable propagation speed, as we introduced in (4.5).
- $\{b_k\}$, that is the sequence of constants that appear in (4.6), which depends on initial conditions.

Let us now define the sequences and then check that they are good.

Definitions of sequences. Let us consider the sequence $\{\lambda_k\}$, which we assumed to be unbounded. Due to (4.10) we can find a suitable subsequence (that we still call λ_k) such that the following inequalities hold true. We set $\{\lambda_k\}$ such that

$$\lambda_k \geq 2\lambda_{k-1} \quad \forall k \geq 0\ ; \tag{4.14}$$

$$\omega\left(\frac{\pi}{\lambda_k}\right)\ \lambda_k\ \geq\ \frac{48}{5}\log k\ \lambda_{k-1}\ ; \tag{4.15}$$

$$\omega\left(\frac{\pi}{\lambda_k}\right)\ \frac{\lambda_k}{\varphi(\lambda_k)}\ \geq\ \frac{48}{5}k\lambda_{k-1}\ ; \tag{4.16}$$

$$\omega\left(\frac{\pi}{\lambda_k}\right)\ \lambda_k\ \geq\ \frac{24\pi}{5}\lambda_{k-1}\ ; \tag{4.17}$$

$$\omega\left(\frac{\pi}{\lambda_k}\right)\ \lambda_k\ \geq\ \frac{24}{5\pi}\ \omega\left(\frac{\pi}{\lambda_{k-1}}\right)\lambda_{k-1}t\ ; \tag{4.18}$$

$$\omega\left(\frac{\pi}{\lambda_k}\right)\ \frac{\lambda_k}{\psi(\lambda_k)}\ \geq\ \frac{48}{5}k\lambda_{k-1}. \tag{4.19}$$

In fact, we define the subsequence recursively and it is possible because conditions (4.16) and (4.19) follows immediately from (4.10) and the others occurs since $\omega(\pi/x)\cdot x$ is also unbounded because of (4.10).

Now let us set

$$t_k = \frac{2\pi}{\lambda_k}, \qquad \epsilon_k = \frac{\omega(\pi/\lambda_k)}{24\pi} \tag{4.20}$$

Firstly, we notice that the function $c(t)$ defined above respects the strict hyperbolicity condition (4.11): indeed, as we have seen before, $c(t)$ fulfils those inequalities in each interval $[t_k, t_{k-1}]$.

Modulus of continuity of $c(t)$. Set $x, y \in [0, t]$ and we have three different cases:

- there exists $k \in \mathbb{N}$ such that $t_k < x < y < t_{k-1}$;
- there exists $k \in \mathbb{N}$ such that $t_{k+1} < x < t_k < y < t_{k-1}$;
- there exist more than one $k \in \mathbb{N}$ such that $x < t_k < y$.

For the first case, let us show that

$$|c(x) - c(y)| \leq \frac{1}{2}\,\omega(|x-y|).$$

Since $c(t)$ is periodic, we can take $a, b \in [t_k, t_{k-1}]$ such that $|a-b|$ is smaller than the period of $c(t)$ and $c(x) = c(a)$ and $c(y) = c(b)$.
We remind the formula that describes $c(t)$ in $[t_k, t_{k-1}]$, i.e.

$$c(t) = 1 - 4\epsilon_k \sin(\lambda_k t)\cos(\lambda_k t) - \epsilon_k^2 \sin^4(\lambda_k t). \tag{4.21}$$

Hence, it is obviously Lipschitz continuous and

$$\begin{align}
|\dot{c}(t)| &= \left|-4\epsilon_k\lambda_k \cos^2(\lambda_k t) + 4\epsilon_k\lambda_k \sin^2(\lambda_k t) - 4\epsilon_k^2\lambda_k \sin^3(\lambda_k t)\cos(\lambda_k t)\right| \tag{4.22}\\
&\leq 4\epsilon_k\lambda_k\,(2+\epsilon_k) \leq 12\epsilon_k\lambda_k \tag{4.23}
\end{align}$$

because $\epsilon_k \leq 1$ for any k. Therefore,

$$|c(x) - c(y)| = |c(a) - c(b)| \leq 12\epsilon_k\lambda_k|a-b|.$$

Now, $c(t)$ is ω-continuous if the last term

$$12\epsilon_k\lambda_k|a-b| \leq \frac{1}{2}\,\omega(|a-b|),$$

namely

$$\epsilon_k\lambda_k \frac{|a-b|}{\omega(|a-b|)} \leq \frac{1}{24}.$$

Since the assumptions we did at the end of the first chapter on the ω-continuity function we know that $\omega(z)$ is nondecreasing, $\omega(z)/z$ is nonincreasing, which means that $z/\omega(z)$ is nondecreasing, and noting that the period of the function in (4.21) is equal to π/λ_k, it turns out that

$$\frac{|a-b|}{\omega(|a-b|)} \leq \frac{\pi/\lambda_k}{\omega(\pi/\lambda_k)}.$$

Thus, we obtained that

$$\epsilon_k\lambda_k\frac{|a-b|}{\omega(|a-b|)} \leq \frac{\epsilon_k\pi}{\omega(\pi/\lambda_k)} = \frac{1}{24}$$

where the last equality holds true due to definition of ϵ_k in (4.20). And so, it turns out that

$$|c(x)-c(y)| = |c(a)-c(b)| \leq \frac{1}{2}\,\omega(|a-b|) \leq \frac{1}{2}\,\omega(|x-y|)$$

where the last inequality follows by the fact that $\omega(z)$ is nondecreasing.

In the second case, we have that there exists $k \in \mathbb{N}$ such that $t_{k+1} < x < t_k < y < t_{k-1}$, and so

$$\begin{aligned}|c(x)-c(y)| \leq |c(x)-c(t_k)| + |c(t_k)-c(y)| &\leq \frac{1}{2}\omega(|x-t_k|) + \frac{1}{2}\omega(|y-t_k|)\\ &\leq \frac{1}{2}\omega(|x-y|) + \frac{1}{2}\omega(|x-y|)\\ &= \omega(|x-y|)\end{aligned}$$

thanks to the first case.

In the third case, let $i, j \in \mathbb{N}$ such that $x < t_i < t_j < y$ and $|x-t_i| \leq \pi/\lambda_k$ (length of period of $c(t)$) and similarly $|y-t_j| \leq \pi/\lambda_k$. So,

$$|c(x)-c(y)| \leq |c(x)-c(t_i)| + |c(t_i)-c(t_j)| + |c(t_j)-c(y)|.$$

The middle term is equal to 0 since $c(t_k) = 0$ fon any $k \in \mathbb{N}$, while for the other two we are covered by the first case. Then,

$$\begin{aligned}|c(x)-c(y)| \leq |c(x)-c(t_i)| + |c(t_j)-c(y)| &\leq \frac{1}{2}\omega(|x-t_i|) + \frac{1}{2}\omega(|y-t_j|)\\ &\leq \frac{1}{2}\omega(|x-y|) + \frac{1}{2}\omega(|x-y|)\\ &= \omega(|x-y|).\end{aligned}$$

Energy estimates Here we want to prove the "regularity" of the solution at $t = 0$ and its "irregularity" when $t > 0$. To do that, let us set $k \in \mathbb{N}$. As we said before our definition of $c(t)$ has the growth of the component $u_k(t)$ as its goal. Now, we are ready to show that before t_k the growth of u_k becomes more regular, while for times greater than or equal of t_{k-1} it is still irregular.

Indeed, as we have seen in the second chapter about the Kovaleskian estimate in (2.8) with $\delta = 0$, we have that

$$E_k'(t) = \lambda_k \left(1 - c(t)\right) 2\lambda_k u_k(t)\dot{u}_k(t) \geq \lambda_k \left(c(t) - 1\right) E_k(t) \geq -\frac{1}{2}\lambda_k E_k(t),$$

thanks to hyperbolicity condition on $c(t)$. Then, it follows that

$$E_k(t) \geq E_k(0) \exp\left(-\frac{1}{2}\lambda_k t\right) \quad \forall t \geq 0$$

which for $t = t_k$ becomes

$$E_k(t_k) \geq E_k(0) \exp\left(-\frac{1}{2}\lambda_k t_k\right),$$

and so

$$E_k(0) \leq E_k(t_k) \exp\left(\frac{1}{2}\lambda_k t_k\right). \tag{4.24}$$

On the other side, we remind that in the case in which $c(t)$ is of class C^1 the hyperbolic estimate was

$$E(t) \leq KE(0) \exp\left(\frac{|\dot{c}(t)|}{c(t)} t\right)$$

and the previous inequality follows by the estimate on hyperbolic energy $F_\lambda(t) = u_\lambda(t)^2 + \lambda^2 c(t) u_\lambda(t)^2$. We can replicate that proof considering the inequalities on the other side. Indeed,

$$F_\lambda'(t) \geq -\frac{|\dot{c}(t)|}{c(t)} F(t)$$

and, as we did in Chapter 2 thanks to Lemma 2.1.2

$$E_\lambda(t) \geq \frac{F_\lambda(t)}{M} \geq \frac{F_\lambda(0)}{M} \exp\left(-\int \frac{|\dot{c}(s)|}{c(s)} ds\right) \geq \frac{m}{M} E_\lambda(0) \exp\left(-\int \frac{|\dot{c}(s)|}{c(s)} ds\right).$$

So, in our case, as we did above in (4.22), after t_{k-1} we can estimate $\dot{c}(t)$ as follows

$$|\dot{c}(t)| \leq 12\epsilon_{k-1} t_{k-1}.$$

Moreover we know that $c(t) \geq 1/2$, then

$$E_k(t) \geq KE_k(t_{k-1}) \exp\left\{-24\, \epsilon_{k-1}\lambda_{k-1}(t - t_{k-1})\right\} \geq KE_k(t_{k-1}) \exp\left\{-24\epsilon_{k-1}\lambda_{k-1} t\right\} \tag{4.25}$$

due to $0 < t_{k-1} \leq t$.

In addition, using definition (4.6) we can estimate the energy $E_k(t_{k-1}) = \dot{u_k}^2(t_{k-1}) + \lambda_k^2 u_k^2(t_{k-1})$ from below. Indeed, with some simple computation and obvious estimate of $\sin(t)$ and $\cos(t)$ we obtain that

$$E_k(t_{k-1}) \geq \; NE_k(t_k) \exp\left\{\epsilon_k \lambda_k (t_{k-1} - t_k)\right\}$$

for a suitable constant $N > 0$. Now, using that $t_k = 2\pi/\lambda_k$ it turns out that

$$t_{k-1} - t_k = 2\pi \left(\frac{1}{\lambda_{k-1}} - \frac{1}{\lambda_k} \right) \geq \frac{1}{2} \frac{2\pi}{\lambda_{k-1}} = \frac{1}{2} t_{k-1}$$

where the last inequality holds true because

$$\left(\frac{1}{\lambda_{k-1}} - \frac{1}{\lambda_k} \right) \geq \frac{1/2}{\lambda_{k-1}}$$

if and only if

$$\left(\frac{\lambda_k - \lambda_{k-1}}{\lambda_k} \right) \geq \frac{1}{2} ,$$

namely

$$\left(1 - \frac{\lambda_{k-1}}{\lambda_k} \right) \geq \frac{1}{2}$$

that holds true thanks to condition (4.14) on the subsequence $\{\lambda_k\}$.
Thus,

$$E_k(t_{k-1}) \geq N E_k(t_k) \exp\left\{ \frac{1}{2} \epsilon_k \lambda_k t_{k-1} \right\}. \tag{4.26}$$

Now, in order to verify conditions (4.12) and (4.13) we need that

$$\sum_{k \geq 1} E_k(0) \exp\left(2r\varphi(\lambda_k)\right) < \infty \quad \forall r > 0$$

that follows from

$$\sum_{k \geq 1} E_k(0) \exp\left(2k\varphi(\lambda_k)\right) < \infty; \tag{4.27}$$

on the other hand,

$$\sum_{k \geq 1} E_k(t) \exp\left(-2R\psi(\lambda_k)\right) = \infty \quad \forall R > 0 \quad \forall t > 0$$

that follows from

$$\sum_{k \geq 1} E_k(t) \exp\left(-2k\psi(\lambda_k)\right) = \infty. \tag{4.28}$$

Putting together condition (4.24)-(4.27) and (4.25)-(4.26)-(4.28) it is enough to show that

$$\sum_{k \geq 0} E_k(t_k) \exp\left\{ \frac{1}{2} \lambda_k t_k + 2k\varphi(\lambda_k) \right\} < \infty, \tag{4.29}$$

$$\sum_{k \geq 1} E_k(t_k) \exp\left\{ \frac{1}{2} \epsilon_k \lambda_k t_{k-1} - 24\epsilon_{k-1}\lambda_{k-1} t - 2k\psi(\lambda_k) \right\} = \infty. \tag{4.30}$$

Indeed, replacing the expressions defined in (4.20) into (4.29), the series becomes

$$\sum_{k\geq 1} E_k(t_k)\exp\left(\pi + 2k\varphi(\lambda_k)\right)$$

which converges if we take

$$E_k(t_k) = \exp\left(-2\log k - 2k\varphi(\lambda_k) - \pi\right) \tag{4.31}$$

because, since this choice, the previous series becomes the series of $\{k^{-2}\}$ which converges. We have free choice on $E_k(t_k)$ because it depends on initial conditions (4.2) for the ordinary differential equation that describes $c(t)$ in $[t_k, t_{k-1}]$. More specifically, we set the sequence b_k such that $E_k(t_k)$ is equal to expression in (4.31).
Regarding the condition (4.30), due to definitions in (4.20) and (4.31), it becomes

$$\sum_{k\geq 1}\exp\left\{-2\log k - 2k\varphi(\lambda_k) - \pi + \frac{1}{2}\epsilon_k\lambda_k t_{k-1} - 24\epsilon_{k-1}\lambda_{k-1}t - 2k\psi(\lambda_k)\right\}. \tag{4.32}$$

By definition, we have that

$$\epsilon_k\lambda_k t_{k-1} = \frac{\omega(\pi/\lambda_k)}{24\pi}\ \lambda_k\ \frac{2\pi}{\lambda_{k-1}} = \frac{1}{12}\ \omega\left(\frac{\pi}{\lambda_k}\right)\ \frac{\lambda_k}{\lambda_{k-1}}.$$

Now, thanks to inequalities (4.15) through (4.19) term by term, it turns out that

$$\frac{1}{2}\epsilon_k\lambda_k t_{k-1} \geq 2\log k + 2k\varphi(\lambda_k) + \pi + 24\epsilon_{k-1}\lambda_{k-1}t + 2k\psi(\lambda_k)$$

and so the series (4.32) diverges.

$\square$

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