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How to construct bizarre objects: from *Baire Category* to *Convex Integration*

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Ai miei genitori

S'io avessi, lettor, più lungo spazio
da scrivere, i' pur cantere' in parte
lo dolce ber che mai non m'avria sazio;

ma perché piene son tutte le carte
ordite a questa cantica seconda,
non mi lascia più ir lo fren de l'arte.

Io ritornai da la santissima onda
rifatto sì come piante novelle
rinovellate di novella fronda,

puro e disposto a salire a le stelle.

Dante

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Introduction

> *"Nash, like Columbus, unwillingly discovered a new land. [...] It may be hard to decide what this land is but it is easy to say what it is not:*
> *what Nash discovered is not any part of Riemannian geometry, neither has it much (if anything at all) to do with classical PDE".*
>
> – Michail Leonidovič Gromov[1]

The method of *convex integration* is recently being discussed in several areas of mathematics. The most relevant results are related to differential geometry and fluid dynamics.

The first example of convex integration method is due to John Nash and Nicolaas Kuiper in 1954 and 1955. Nash considered the classic problem of isometric embedding according to which it was well-known that C^2 isometric immersions of positively curved spheres into $\mathbb{R}^3$ are uniquely determined up to rigid motions. Such result was a consequence of *rigidity* studies of Cohn-Vossen and Hergolotz (see [2] - [12]) and it was natural to expect that a similar result did not depend on the regularity of the immersion. However, Nash and Kuiper proved a surprising result that can be state in its general form in the following Theorem.

Theorem 0.0.1 (Nash-Kuiper)**.** *Let (M^d, g) be a smooth closed d-dimensional Riemannian manifold and $\phi : M^d \to \mathbb{R}^n$ a C^∞ strictly short immersion (or embedding) with $n \geq d+1$. Then, for any $\epsilon > 0$ there exists a C^1 isometric immersion (or embedding) $\psi : M^d \to \mathbb{R}^n$ such that $\|\phi - \psi\|_{C^0} \leq \epsilon$.*

Namely, we are saying that the C^1 isometric immersions are C^0-dense in the set of C^∞ short immersions. We recall that given a smooth d-dimensional Riemannian manifold, a map $\phi : M^d \to \mathbb{R}^n$ is *short* if it "shrinks" the length of curves, i.e.

$$\ell_e(\phi \circ \gamma) \leq \ell_g(\gamma) \qquad \textit{for any } C^1 \textit{ curve } \gamma \subset M^d,$$

where ℓ_g denotes the length of curves according to the metric g. Of course, the map ϕ is said to be *isometric* if it preserves the length of curves and we have equality in

[1] M. Gromov; Geometric, algebraic, and analytic descendants of Nash isometric embedding theorems, *Bull. Amer. Math. Soc.(N.S.)* **54** (2017), no. 2, 173–245.

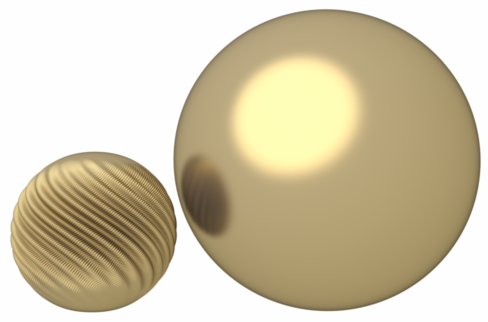

Figure 1: The reduced sphere

previous condition. One way to "visualize" this theorem is to consider the existence of C^1 isometric embeddings of S^2 into a ball B_ϵ with arbitrarily small radius $\epsilon > 0$. In particular, one can take, as the short embedding, the map that shrinks the standard sphere S^2 into ϵS^2. Therefore, the theorem implies that there exist C^1 isometric embeddings arbitrarily C^0-close to the previous one; or even better, there exist C^1 isometric images of S^2 in an arbitrarily small neighbourhood of ϵS^2. Figure 1 show such effects due to Nash-Kuiper theorem[2].

Actually, Nash proved the Theorem for $n \geq d+2$ in [15] and the next year Kuiper gave a proof for the case $n = d+1$ in [14]. A result such as Nash-Kuiper theorem is known as *flexibility* result since it is opposed to rigidity previous discussed. Moreover, in local coordinates we can consider the metric $g = g_{ij}dx_i \otimes dx_j$ and for a C^1 map ϕ we have that isometric condition is equivalent to the system of partial differential equations

$$\partial_i \phi \cdot \partial_j \phi = g_{ij}. \tag{1}$$

In other words, condition (1) means that g is the pullback of the Euclidean metric through the map ϕ. In addition, it is possible to say that condition (1) is a system of $\frac{1}{2}d(d+1)$ equations for n unknown functions. Therefore, if $n \leq \frac{1}{2}d(d+1)$, such system is overdetermined and so the existence of solutions given by Nash-Kuiper theorem can be considered quite astonishing. This is the reason why sometimes it is also called *Nash-Kuiper Paradox*. The main idea in order to prove this counterintuitive result is to start from the strictly short immersion and then add successively highly oscillations in order to increase the distances back to the Euclidean ones and reach isometry. Such iterative procedure is considered the heart of convex integration method.

Nevertheless, the term *convex integration* related to Nash and Kuiper is used *ante litteram*, since it was suggested by Gromov later. Indeed, Gromov formalised a more general theory in which Nash-Kuiper Theorem is a primary example (see [11]). He realized that this type of flexibility appears in a variety of different geometric context

[2]Images showing Nash-Kuiper effects, such as Figure 1, can be found on the website `https://hevea-project.fr/ENIndexHevea.html`, thanks to the work of V. Borrelli et al.

and he used the idea of convex integration in order to solve partial differential relations. This theory is known as *h-principle*, but it is not discussed in this thesis[3]. Indeed, the problem of curves presented in Chapter 3, where we discuss some differential inclusions, can be considered also as introductory examples for Gromov theory, but it is not the main purpose of our work.

One of the main discussion come from Nash-Kuiper result was to investigate the relationship *rigidity vs flexibility*. On one hand we know the rigidity of C^2 isometries, on the other hand we have flexibility of C^1 isometries thanks to Nash-Kuiper Theorem. Therefore, it was natural to ask whether there is a threshold regularity that could separate the two different behaviours. Of course, in order to measure regularity between C^1 and C^2 it is possible to consider the Hölder space. In particular, we say that a map ϕ belongs to $C^{1,\alpha}$ for $0 < \alpha \leq 1$ if

$$|D\phi(x) - D\phi(y)| \leq H|x-y|^\alpha$$

for some constant $H > 0$ independent on x, y. In this direction it was studied which threshold $0 < \alpha_0 < 1$ could distinguish flexibility phenomena à la Nash-Kuiper for $\alpha < \alpha_0$ and rigidity phenomena for $\alpha > \alpha_0$. Relevant contributions to such problem are due to Camillo De Lellis and László Székelyhidi that proved some general statements in this direction in [4]. The latter result represents a key role in this story since the same authors in [6] pointed out a strong analogy between isometric immersions problem and Onsager's conjecture in the theory of mathematical fluid dynamics.

The previous conjecture was proposed in 1949 by Lars Onsager and it refers to the behaviour of solutions of incompressible Euler equations. Such equations describe the motion of a perfect incompressible fluid and can be written as

$$\begin{cases} \partial_t v + (v \cdot \nabla)v + \nabla p = 0 \\ div\ v = 0, \end{cases} \tag{2}$$

where $v(x,t)$ is the velocity and $p(x,t)$ is the pressure. The classical setting is the 3-dimensional case with periodic boundary conditions. It is known that for classical solutions of (2), i.e. $(v,p) \in C^1(\mathbb{T}^3 \times [0,T]; \mathbb{R}^3 \times \mathbb{R})$ where $\mathbb{T}^3$ is the flat 3-dimensional torus that represents the periodic boundary conditions, the total energy

$$e(t) := \frac{1}{2}\int_{\mathbb{T}^3} |v(x,t)|^2\ dx$$

is conserved, so that $e(t) = e(0)$. Indeed, thanks to C^1 hypothesis of v we have that

$$\frac{d}{dt}\int_{\mathbb{T}^3} \frac{|v(x,t)|^2}{2}\ dx = \int_{\mathbb{T}^3} \frac{d}{dt}\frac{|v(x,t)|^2}{2}\ dx = \int_{\mathbb{T}^3} v \cdot \partial_t v\ dx.$$

[3] It is possible to find a detailed discussion of Convex Integration Theory related to *h-principle* in geometry and topology in Spring's book [19].

Now, since v is divergence free it turns out that

$$\begin{aligned} div\left(v\left(\frac{|v|^2}{2}+p\right)\right) &= div\ v\cdot\left(\frac{|v|^2}{2}+p\right)+v\cdot\nabla\left(\frac{|v|^2}{2}+p\right) \\ &= v\cdot\nabla\left(\frac{|v|^2}{2}+p\right). \end{aligned}$$

Now, due to divergence theorem

$$\int_{\mathbb{T}^3} div\left(v\left(\frac{|v|^2}{2}+p\right)\right)\ dx = 0,$$

and so,

$$\begin{aligned} \frac{d}{dt}\int_{\mathbb{T}^3}\frac{|v(x,t)|^2}{2}\ dx &= \int_{\mathbb{T}^3} v\cdot\partial_t v + div\left(v\left(\frac{|v|^2}{2}+p\right)\right)\ dx \\ &= \int_{\mathbb{T}^3} v\cdot\partial_t v + v\cdot\nabla\left(\frac{|v|^2}{2}+p\right)\ dx \\ &= \int_{\mathbb{T}^3} v\cdot(\partial_t v + (v\cdot\nabla)v + \nabla p)\ dx\ = 0, \end{aligned}$$

thanks to Euler equation.

The note conjecture due to Onsager is referred to the existence of weak solutions to (2) that could dissipate the energy. More formally,

Conjecture 0.0.1. *Consider periodic 3-dimensional weak solutions of (2), where the the velocity v satisfies the Hölder condition*

$$|v(x,t)-v(y,t)| \le H|x-y|^\theta,$$

for certain constants H,θ independent of x,y,t. Then,

(a) if $\theta > \frac{1}{3}$, then the total energy of v is constant;

(b) for any $\theta < \frac{1}{3}$ there exist solutions v for which the total energy is not constant.

It is not our purpose to discuss in details the results related to previous Conjecture, but we can say that the part (a) of the Conjecture was proved in [3] in 1994. More interesting for us is what happened in part (b) of the Conjecture. Indeed, even if it was possible to construct an L^2 solution that dissipates energy (see Scheffer in [17] and Shnirelman in [18]), the fundamental work of De Lellis and Székelyhidi (see [6] and [9]) gave a machinery in order to construct bounded weak solutions that violate the conservation energy. Such machinery is based on convex integration method and the procedure used is very similar to Nash-Kuiper approach to isometric immersions problem.

More technical details of what discussed so far can be found in the paper [5] of De Lellis where he deeply describes the previous results. Finally, in order to conclude the story, it is possible to say that the research started from De Lellis and Székelyhidi works brought recently to solve the Onsager's conjecture, thanks also to P. Isett (see [13]), even if the case $\theta = \frac{1}{3}$ is still open. Of course the same approach could be used to attach other similar problems in fluid dynamics, as for instance Navier-Stokes equations.

All previous arguments provide the main motivation behind this thesis and it is clear now why the method of convex integration represents a fundamental tool. In our work, we want to isolate the key ideas of the method and try to convey them without technicalities. In particular, our purpose is to present two techniques that are useful in the construction of bizarre examples and counterexamples in the fields of analysis and geometry, based on **Baire category** theorem and clearly **convex integration** method.

In the first chapter, we provide some classical examples that exploit the *Baire category theorem*. In particular, we prove the following facts.

- There exists a bounded function $f : \mathbb{R} \to \mathbb{R}$ that is 1/2-Hölder continuous in $\mathbb{R}$, but not Lipschitz continuous in any sub-interval $(a, b) \subseteq \mathbb{R}$.
- There exists a measurable set $E \subseteq [0, 1]$ such that, for every sub-interval $I \subseteq [0, 1]$, both $E \cap I$ and $E^c \cap I$ have strictly positive measure.
- In the space of bounded functions $f : \mathbb{R} \to \mathbb{R}$ that are Lipschitz continuous with constant ≤ 1, the set of functions with Lipschitz constant $= 1$ in any sub-interval is residual.

Furthermore, in the first two cases, we actually show that the examples are residual and explain how the same ingredients of a "Baire proof" can be used in order to construct an explicit solution. Then, we discuss the metrizability of weak topology in a separable Hilbert space and we show that

- in the space of functions $f \in L^2(a, b)$ with $|f(x)| \leq 1$ almost everywhere, the set of functions such that $f(x) \in \{-1, 1\}$ almost everywhere is residual with respect to the metric that induces the weak convergence in $L^2(a, b)$.

In the next three chapters, we discuss three applications of the method which is now known as *Convex Integration*.

In Chapter 2, we show that the set of curves $u : [0, 1] \to \mathbb{R}^d$ of class C^1 with $\|u'(t)\|_{\mathbb{R}^d} = 1$ for every $t \in [0, 1]$ is dense (with respect to C^0 norm) in the set of curves

$u : [0, 1] \to \mathbb{R}^d$ of class C^1 with $\|u'(t)\|_{\mathbb{R}^d} \leq 1$ for every $t \in [0, 1]$. Then, we show that this set is also residual with respect to a suitable metric.

In Chapter 3, we consider an introductory example in the context of Euler equations of fluid dynamics. Let $\Omega \subset \mathbb{R}^3$ be an open and bounded set. we show that there exist infinitely many $u \in L^\infty(\mathbb{R}^3, \mathbb{R}^3)$ such that $div\ u = 0$ in the sense of distributions and $\|u(x)\|_{\mathbb{R}^3} = \mathbb{1}_\Omega(x)$ for almost every $x \in \mathbb{R}^3$. We then consider a slightly more general case where $u(x) \in K$ for a certain compact set $K \subset \mathbb{R}^3$.

In Chapter 4, we consider a special case of the seminal *Nash-Kuiper Theorem*. In particular, we prove the following result.

Theorem 0.0.2 (Nash-Kuiper). *Let D^2 represents the 2-dimensional disk and let $N \geq 3$. Let $\phi : D^2 \to \mathbb{R}^N$ be a C^∞ strictly short immersion.*
Then, for any $\epsilon > 0$ there exists a C^1 isometric immersion $\phi_\epsilon : D^2 \to \mathbb{R}^N$ such that $\|\phi - \phi_\epsilon\|_{C^0} \leq \epsilon$.

In the proof, we present previously the construction of Nash "spirals" in the case of codimension 2. Then, we discuss how to extend the previous result in codimension 1, where we can replace them with Kuiper "corrugations".

Chapter 1

Baire category theorem and metrizability of weak topology

1.1 Baire category theorem

In the first section, we remind the fundamental notions about *Baire spaces*. In particular, we use Baire category theorem in order to construct some specific objects.

Let recall before some initial important definitions that will be useful throughout this work.

Definition 1.1.1. Let X be a topological space. Then, $Y \subseteq X$ is called

- **Nowhere dense** if $Int(Clos(Y)) = \emptyset$;
- **Meager** or **First Category** if it is a countable union of nowhere dense subsets;
- **Residual** if $X \setminus Y$ is meager.

Now, we can provide the general definition of a Baire space.

Definition 1.1.2 (Baire space)**.** A **Baire space** is a topological space that satisfies one of the following equivalent properties:

1. If $\{A_i\}$ is a sequence of open dense subsets, then $\bigcap_{i=1}^{\infty} A_i$ is dense.
2. If $\{C_i\}$ is a sequence of closed subsets with $Int(C_i) = \emptyset$, then $\bigcup_{i=1}^{\infty} C_i$ has empty interior.
3. If $\{C_i\}$ is a sequence of closed subsets such that $Int(\bigcup_{i=1}^{\infty} C_i) \neq \emptyset$, then there exists $i_0 \geq 1$ such that $Int(C_{i_0}) \neq \emptyset$.

Remark 1.1.1. Let us make some important remarks.

1. It is possible to note that in a Baire space a meager set has empty interior.

2. A fundamental property of Baire spaces which we will use throughout this work is that we cannot write the whole space as a countable union of closed sets with empty interior, since it would contradict the second equivalent definition of a Baire space.

3. Intuitively, we can think of nowhere dense sets as *very very small* sets from the topological point of view. While, a meager set can be seen as a *small* set from the topological point of view. On the other side, a residual set fill almost all the space, topologically speaking, and in particular, in a Baire space a residual set is dense.

Now, we are ready to state one of the most useful result that allow us to use Baire spaces properties.

Theorem 1.1.1 (Baire Category)**.** *If X is a topological space in one of the three following categories, then it is a Baire space:*

(i) Complete metric spaces;

(ii) Locally compact topological spaces;

(iii) Open subsets of Baire Spaces.

The *Baire Category Theorem* (and in particular category (i)) is a powerful result that establishes the prevalence of certain properties in Baire spaces. In particular, it represents a huge opportunity to interpret some specific space, such as complete metric spaces, as Baire spaces and use the characterization of a Baire space in order to prove some important statements. It has numerous applications, including proving the existence of solutions to mathematical problems, establishing fixed-point theorems, and providing insights into the topological properties of various spaces. It has several important applications in various areas of mathematics: for instance the Banach-Steinhaus Theorem and the Open Mapping Theorem can be proved thanks to Baire spaces setting. On the other hand, the following statement is an example of classical constructions using Baire category theorem.

Proposition 1.1.1. *There exists a continuous function $f : \mathbb{R} \longrightarrow \mathbb{R}$ that is not differentiable at any $x \in \mathbb{R}$.*

Remark 1.1.2. We can make a couple of remarks:

1. The main idea in order to prove the previous results with Baire category theorem is to make qualitative properties, such as differentiability, into quantitative properties, such as quantitative bounds on the derivative.

2. It is important to note that such existence results can be proved also by hand, exhibiting an explicit example satisfying required properties.

Now, we can show how to use Baire category in the following standard result in functional analysis that will be also useful later in this work, i.e. the residual continuity of pointwise limit of a sequence of continuous function.

Proposition 1.1.2. *Let X be a complete metric space and let Y be a metric space. Let $f_n : X \longrightarrow Y$ be a sequence of continuous functions and let $f_\infty : X \longrightarrow Y$ be the pointwise limit, i.e. $f_n(x) \longrightarrow f_\infty(x)$ for any $x \in X$.*
Then, the set of continuity points of f_∞ is residual in X.

Proof. Let us note that by Baire category theorem, X is a Baire space. Now, let us define

$$D_\epsilon := \{x \in X \ : \ \forall r > 0 \ \ \exists \, y, z \in B(x,r) \ \ s.t. \ \ d_Y(f_\infty(y), f_\infty(z)) \geq \epsilon\}.$$

It is straightforward to see that if we call D the set of discontinuity points of f_∞, then

$$D = \bigcup_{\epsilon > 0} D_\epsilon = \bigcup_{k=1}^{\infty} D_{\frac{1}{k}} \ .$$

The thesis follows proving that D_ϵ is closed and with empty interior for any $\epsilon > 0$.

It is closed since if we take $x_n \to x \in X$ and consider $B(x,r)$ for any $r > 0$, then for n large enough $B(x_n, \frac{r}{2}) \subseteq B(x,r)$. Therefore, since in $B(x_n, \frac{r}{2})$ there are two points y, z such that $d_Y(f_\infty(x), f_\infty(z)) \geq \epsilon$ by definition of D_ϵ, the same points are also in $B(x,r)$ and the closure holds.

Now, let us suppose by contradiction that there exists $\epsilon > 0$ such that $Int(D_\epsilon) \neq \emptyset$. Hence, there exists a certain non empty open set $A_0 \subseteq D_\epsilon$. We can observe that A_0 is again a Baire space. For any $k \geq 1$, we consider

$$\begin{aligned} C_k &:= \{x \in A_0 \ : \ d_Y(f_n(x), f_m(x)) \leq \tfrac{\epsilon}{3} \ \forall n \geq k, \ \forall m \geq k \ \} \\ &= \bigcap_{n \geq k, \ m \geq k} \{x \in A_0 \ : \ d_Y(f_n(x), f_m(x)) \leq \tfrac{\epsilon}{3}\}. \end{aligned}$$

Due to the fact that each set of the last intersection is closed, then it easily follows that also C_k is closed. Moreover, $\bigcup_{k=1}^{\infty} C_k = A_0$ and

$$d_Y(f_n(x), f_\infty(x)) \leq \frac{\epsilon}{3} \qquad \forall n \geq k, \ \forall x \in C_k \ . \tag{1.1}$$

Then, again thanks to Baire category theorem, we have that there exists an open set $A_1 \subseteq C_{k_1}$, for a certain $k_1 \geq 1$. Now, let us consider the function f_{k_1}. It is a continuous function and so

$$\exists \ A_2 \subseteq A_1 \ \ s.t. \ \ d_Y(f_{k_1}(y), f_{k_1}(z)) \leq \frac{\epsilon}{3} \quad \forall y \in A_2, \ \forall z \in A_2 \ .$$

Therefore, for any $y, z \in A_2$, we have that

$$d_Y(f_\infty(y), f_\infty(z)) \le d_Y(f_\infty(y), f_{k_1}(y)) + d_Y(f_{k_1}(y), f_{k_1}(z)) + d_Y(f_{k_1}(z), f_\infty(z)) \le \epsilon \ .$$

Hence, we have concluded. □

We can now begin the detailed presentation of the first technique discussed in this thesis. Specifically, we utilize Baire category theorem in order to construct some specific objects and we draw a comparison between the "Baire proof" and the explicit construction of these objects.

Proposition 1.1.3. *There exists a bounded function $f : \mathbb{R} \to \mathbb{R}$ that is $1/2$-Hölder continuous in $\mathbb{R}$, but not Lipschitz continuous in any sub-interval $(a, b) \subseteq \mathbb{R}$.*

Before proving the Proposition, we need a lemma. In particular, we want to prove that the set of bounded function from a generic set to a complete metric space, provided by the sup distance, is complete.

Lemma 1.1.1. *Let us consider S a set and X a complete metric space. Let us take*

$$B(S, X) := \{f : S \to X \ \ \textit{bounded}\}, \tag{1.2}$$

where f is bounded if its image is contained in a ball of X. Let us equipe $B(S, X)$ with the sup *distance, i.e.*

$$d_{\sup}(f, g) := \sup_{s \in S} \ d_X(f(s), g(s)).$$

Then, $(B(S, X), d_{sup})$ is a complete metric space.

Proof. First of all, we claim that if $\{f_n\}$ is a Cauchy sequence with $d_{\sup}$ distance, then $\{f_n(s)\}$ is a Cauchy sequence in X for any s. This follows easily, since $\forall \epsilon > 0$ there exists $N \in \mathbb{N}$ such that for any $n, m \ge N$ we have that $d_{sup}(f_n, f_m) \le \epsilon$. Hence, for any $s \in S$ and $n, m \ge N$

$$d_X(f_n(s), f_m(s)) \le d_{sup}(f_n, f_m) \le \epsilon.$$

Thus, due to completeness of X, we have that for any $s \in S$ the sequence $f_n(s)$ converges and we call $f(s) \in X$ such limit. In this way, we proved that the pointwise limit function $f : S \to X$ is well defined. Now, we must prove that f is bounded and f_n converges to f with *sup* distance. In order to prove boundedness, we can take N large enough such that $d_{sup}(f_n, f_N) \le 1$ for any $n \ge N$. Moreover, for any $s \in S$

$$\begin{aligned} d_X(f(s), 0) &\le d_X(f(s), f_N(s)) + d_X(f_N(s), 0) \\ &\le d_{sup}(f, f_N) + d_X(f_N(s), 0). \end{aligned}$$

Now, by continuity of a distance, it turns out that for any $s \in S$

$$d_X(f(s), f_N(s)) = \lim_{n\to\infty} d_X(f_n(s), f_N(s)) \leq 1,$$

and so

$$d_{sup}(f, f_N) = \sup_{s\in S} \; d_X(f(s), f_N(s)) \leq 1.$$

Hence, taking $M > 0$ such that $d_X(f_N(s), 0) \leq M$ for any $s \in S$ due to boundedness of f_N, we have that

$$\begin{aligned} d_X(f(s), 0) &\leq d_{sup}(f, f_N) + d_X(f_N(s), 0) \\ &\leq 1 + M, \end{aligned}$$

which means that f is bounded. Finally, let $\epsilon > 0$ and let us take the usual N large enough such that $d_{sup}(f_n, f_m) \leq \epsilon$ for any $n, m \geq N$. Then, for all $n \geq N$ and for all $s \in S$

$$d_X(f_n(s), f(s)) = \lim_{m\to\infty} d_X(f_n(s), f_m(s)) \leq \epsilon.$$

Therefore, also $d_{\mathrm{sup}}(f_n, f) \leq \epsilon$ and we have concluded. □

We are now ready for the proof of previous Proposition 1.1.3 based on Baire category theorem..

Proof 1 of Proposition 1.1.3 . Let us V be the space of bounded 1/2-Hölder functions with L^∞-norm. Namely, consider

$$V := \{f : \mathbb{R} \to \mathbb{R} \;\; bounded \; : \; |f(x) - f(y)| \leq |x - y|^{1/2} \;\; \forall x, y \in \mathbb{R}\}$$

provided by the sup norm. It is possible to note that it is a complete metric space. Indeed, thanks to previous lemma, the set of bounded functions from $\mathbb{R}$ to $\mathbb{R}$ is complete, and, in addition, $V \subset B(\mathbb{R}, \mathbb{R})$ is closed since we have fixed the Hölder constant equal to 1. Hence, it is also a Baire space thanks to Baire category theorem. Due to the latter characterization, the claim of the proof is to show that the set of functions that are Lipschitz at least one sub-interval is a countable union of closed sets with empty interior. This means that, as we noticed before, this set can't be the whole space and so the complementary is not empty. That gives us our thesis.
In order to do this, let us previously state that being Lipschitz in at least in one sub-interval means that there exist $a, b \in \mathbb{R}$ and a positive constant L such that $|f(x) - f(y)| \leq L|x - y| \;\; \forall x, y \in (a, b)$. Let us define

$$C_k := \left\{ f \in V \; : \; \exists x_f \in [-k, k] \; s.t. \; |f(x) - f(y)| \leq k|x - y| \; \forall x, y \in [x_f, x_f + \tfrac{1}{k}] \right\}. \tag{1.3}$$

It is clear that if f is Lipschitz continuous in one sub-interval, then it is in a C_k for a certain $k \in \mathbb{N}$. So, it is just necessary to show that C_k is closed and with empty interior for any k.

- In order to show that C_k is closed, let $\{f_n\} \subseteq C_k$ be a sequence that converges uniformly to a certain function f_∞. For each n, there exists x_n according to (1.3). Since $\{x_n\} \subseteq [-k, k]$, up to subsequences we can suppose that x_n converges to $x_\infty \in [-k, k]$. Now, any $x, y \in [x_\infty, x_\infty + \frac{1}{k}]$ can be seen as limit of sequences $\{\alpha_n\}$ and $\{\beta_n\}$, respectively, such that $\alpha_n, \beta_n \in [x_n, x_n + \frac{1}{k}]$ $\forall n$ (for example, if $x = x_\infty + \alpha$ with $\alpha \leq \frac{1}{k}$, we can take $\alpha_n = x_n + \alpha$; the same for y). Therefore,

$$|f_n(\alpha_n) - f_n(\beta_n)| \leq k|\alpha_n - \beta_n| \qquad \forall n \in \mathbb{N},$$

and so passing to the (double) limit, thanks to uniform convergence, we obtain that
$$|f_\infty(x) - f_\infty(y)| \leq k|x - y|,$$
which means that $f_\infty \in C_k$.

- Let us suppose by contradiction that there exists an open ball $B_V(f_0, r_0) \subseteq C_{k_0}$, for certain $f_0 \in V,\ r_0 > 0,\ k_0 \geq 1$. Without loss of generality, up to reducing the radius or multiplying f_0 by a constant, we can assume that f_0 is of class C^1 (and so L-Lipschitz in at least in $[-k_0 - 1, k_0 + 1]$) and it is 1/2-Hölder with constant $H_0 \leq (1 - \epsilon_0)$, with ϵ_0 small enough. Let us, now, define the *sawtooth* function $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ as in the following figure:

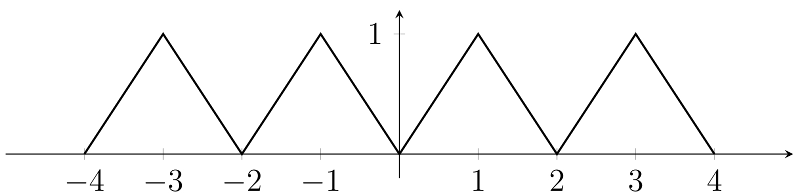

Figure 1.1: The *sawtooth* function.

This $\varphi(x)$ is a piecewise affine function with derivative equal to ± 1. Let's take
$$\varphi_n(x) := \frac{1}{n}\varphi(n^2 x),$$
that is a sequence of piecewise affine functions with derivative equal to $\pm n$. In particular, it is 1/2-Hölder and let H be its 1/2-Hölder constant. Now, considering $f_n(x) := f_0(x) + \dfrac{\epsilon_0}{H}\varphi_n(x)$, we have that the 1/2-Hölder constant of f_n is less than or equal to $H_0 + \epsilon_0 \leq 1$, which means that $f_n(x) \in V$ for any $n \in \mathbb{N}$. Furthermore, if n is large enough, we obtain that $f_n(x)$ is in $B_V(f_0, r_0) \subseteq C_{k_0}$, since $\|\varphi_n\|_\infty \to 0$ as $n \to \infty$. On the other hand, f_n should be in C_{k_0} for n large, i.e. there exists $x_{f_n} \in [-k_0, k_0]$ s.t.

$$|f_n(x) - f_n(y)| \leq k_0|x - y| \qquad \forall x, y \in [x_{f_n}, x_{f_n} + \tfrac{1}{k_0}]. \tag{1.4}$$

But, for every $x \in [-k_0, k_0]$ and for any n, we can find x_n and y_n fairly close in $[x, x + \frac{1}{k_0}]$ such that

$$|\varphi_n(x_n) - \varphi_n(y_n)| \geq n|x_n - y_n|.$$

Hence,

$$\begin{aligned} |f_n(x_n) - f_n(y_n)| &\geq \frac{\epsilon_0}{H} |\varphi_n(x_n) - \varphi_n(y_n)| - |f_0(x_n) - f_0(y_n)| \\ &\geq \frac{\epsilon_0}{H} n|x_n - y_n| - L|x_n - y_n| \\ &= \left(\frac{\epsilon_0}{H} n - L\right) |x_n - y_n|. \end{aligned}$$

It is clear that for n large it contradicts (1.4).

□

Remark 1.1.3. In the previous proof we showed that the set of functions that are Lipschitz at least in one sub-interval is a countable union of closed sets with empty interior, saying that it is a meager set in V. Therefore, not only there exists a bounded 1/2-Hölder function that is not Lipschitz in any sub-interval, but we are also saying that the set of such functions is residual in V.

Now, we can present a second proof of the same result, providing an explicit solution.

Proof 2 of Proposition 1.1.3. In the second proof we want to construct an explicit counterexample, i.e. exhibiting a 1/2-Hölder function that it is not a Lipschitz function in any sub-interval. Consider the function φ as in Figure 1.1. Let us define

$$f(x) := \sum_{k=1}^{\infty} \epsilon_k \varphi(i_k x). \tag{1.5}$$

Our goal is now to choose appropriate sequences $\{\epsilon_k\}$ and $\{i_k\}$ in such a way that f has the required properties. Firstly, in order to obtain a well defined function, it is sufficient to take $\epsilon_k > 0$ such that

$$\sum_{k=1}^{\infty} \epsilon_k < +\infty. \tag{1.6}$$

Moreover, we want that f is 1/2-Hölder continuous, that is implied by taking

$$\sum_{k=1}^{\infty} \epsilon_k i_k^{1/2} < +\infty. \tag{1.7}$$

Indeed, if the previous condition holds, for any $x, y \in \mathbb{R}$

$$|f(x) - f(y)| \le \sum_{k=1}^{\infty} \epsilon_k |\varphi(i_k x) - \varphi(i_k y)| \le \sum_{k=1}^{\infty} \epsilon_k i_k^{1/2} H \ |x-y|^{1/2},$$

where we used that if H is the 1/2-Hölder constant of φ, then $i_k^{1/2} H$ is the 1/2-Hölder constant of $\varphi(i_k x)$. Hence, if (1.7) holds, then f is 1/2-Hölder continuous.

Now, in order to exclude Lipschitz continuity, following the same idea used in the Baire proof, fixed any $x_0 \in \mathbb{R}$, then for any $n \in \mathbb{N}$ there exist $x_n, y_n \in [x_0, x_0 + 1/i_n]$ such that

$$|f(x_n) - f(y_n)| \ge n|x_n - y_n|.$$

If the latter is true, then $f(x)$ cannot be Lipschitz in any sub-interval.

Fixed $n \ge 1$, we can write

$$f(x) = S_{n-1}(x) + \epsilon_n \varphi(i_n x) + R_n(x),$$

where S_{n-1} is the partial sum until $k \le n-1$ and R_n is the remainder of the series, i.e. the sum for $k \ge n+1$. Now, we can remark that $\varphi(i_n x)$ is affine along sections of length $1/i_n$, with slope equal to i_n. Now, we choose x_n and y_n in $[x_0, x_0 + 1/i_n]$ with $|x_n - y_n| = \dfrac{1}{2i_n}$ such that they are on the same interval in which $\varphi(i_n x)$ is affine. Hence,

$$|\epsilon_n \varphi(i_n x_n) - \epsilon_n \varphi(i_n y_n)| \ge \epsilon_n i_n |x_n - y_n|.$$

Furthermore, we have that

$$|f(x_n) - f(y_n)| \ge |\epsilon_n \varphi(i_n x_n) - \epsilon_n \varphi(i_n y_n)| - |S_{n-1}(x_n) - S_{n-1}(y_n)| - |R_n(x_n) - R_n(y_n)|.$$

Now, since S_{n-1} is a finite sum of Lipschitz functions, it is Lipschitz and so

$$|S_{n-1}(x_n) - S_{n-1}(y_n)| \le L_{n-1}|x_n - y_n|,$$

where L_{n-1} is the Lipschitz constant on S_{n-1} and satisfies

$$L_{n-1} \le \sum_{k=1}^{n-1} L_k = \sum_{k=1}^{n-1} \epsilon_k i_k.$$

Moreover,

$$|R_n(x_n) - R_n(y_n)| \le 2 \sup_{x \in \mathbb{R}} |R_n(x)| = 2 \sum_{k=n+1}^{\infty} \epsilon_k = 4 i_n \sum_{k=n+1}^{\infty} \epsilon_k \ \ |x_n - y_n|,$$

where in the last equality we have multiplied and divided by $|x_n - y_n| = \frac{1}{2i_n}$. Putting all together we obtain

$$|f(x_n) - f(y_n)| \ge \left(\epsilon_n i_n - \sum_{k=1}^{n-1} \epsilon_k i_k - 4 i_n \sum_{k=n+1}^{\infty} \epsilon_k \right) |x_n - y_n|. \tag{1.8}$$

In order to obtain our goal, it is sufficient that the quantity in the brackets diverges. Namely, we need that

$$\epsilon_n i_n \to \infty, \qquad \sum_{k=1}^{n-1} \epsilon_k i_k \ll \epsilon_n i_n, \qquad i_n \sum_{k=n+1}^{\infty} \epsilon_k \ll \epsilon_n i_n,$$

together with conditions (1.6)-(1.7). Now, if we take $\epsilon_k = 1/10^k$ and $i_k = 81^k$, we can see that they satisfy all the conditions required. The existence condition (1.6) and the divergence of $\epsilon_n i_n$ are trivial. For the other terms,

$$\begin{aligned}
\sum_{k=1}^{\infty} \epsilon_k i_k^{1/2} &= \sum_{k=1}^{\infty} \left(\frac{9}{10}\right)^k < \infty; \\
i_n \sum_{k=n+1}^{\infty} \epsilon_k &= i_n \sum_{k=n+1}^{\infty} \frac{1}{10^k} = i_n \frac{1}{9}\frac{1}{10^n} = \frac{1}{9}\epsilon_n i_n; \\
\sum_{k=1}^{n-1} \epsilon_k i_k &= \sum_{k=1}^{n-1} \left(\frac{81}{10}\right)^k \leq \frac{\left(\frac{81}{10}\right)^n - 1}{\frac{81}{10} - 1} \leq \frac{\left(\frac{81}{10}\right)^n}{7} = \frac{1}{7}\epsilon_n i_n.
\end{aligned}$$

Replacing in (1.8), it turns out that

$$\begin{aligned}
|f(x_n) - f(y_n)| &\geq \left(\left(\frac{81}{10}\right)^n - \frac{1}{7}\left(\frac{81}{10}\right)^n - \frac{4}{9}\left(\frac{81}{10}\right)^n\right)|x_n - y_n| \\
&= \frac{26}{63}\left(\frac{81}{10}\right)^n |x_n - y_n|,
\end{aligned}$$

and this concludes the proof. □

Remark 1.1.4. In the explicit proof, we have followed an heuristic way in order to point out the similarity to the Baire proof, but at the end we showed that the function

$$f(x) := \sum_{k=1}^{\infty} \frac{1}{10^k}\varphi(81^k x)$$

is 1/2-Hölder continuous, but it is not Lipschitz in any sub-interval.

Following the same proving structure used above, it is possible to show some other relevant results. Let us make an useful definition before.

Definition 1.1.3. Let us $A \subseteq [0,1]$ be a set. Let us consider the uniform partition of $[0,1]$ with size equal to $\frac{1}{n}$, for a certain $n \in \mathbb{N}$, i.e. $0 < \frac{1}{n} < \cdots < \frac{n-1}{n} < 1$.
We define the ***n*-modification of** A (and we refer to such set as A_n) the set obtained after the following procedure: for any interval of the partition $[\frac{i}{n}, \frac{i+1}{n}]$, if $x \in [\frac{i}{n}, \frac{i}{n} + \frac{1}{n^2}]$ then $x \in A_n$; if $x \in [\frac{i+1}{n} - \frac{1}{n^2}, \frac{i+1}{n}]$ then $x \notin A_n$ (see Figure 1.2); otherwise, if $x \in A$ then $x \in A_n$, if not $x \notin A_n$.

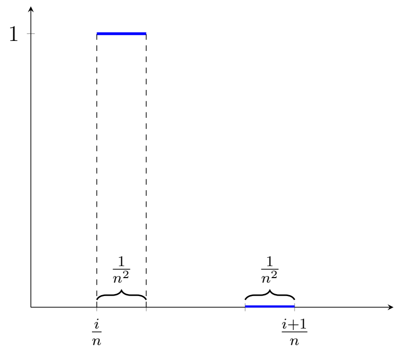

Figure 1.2: How to construct the indicator function of A_n restricted to interval $[\frac{i}{n}, \frac{i+1}{n}]$.

In previous definition we are saying that for any interval of the partition we always add the first small part of length $\frac{1}{n^2}$ to the set and we always remove the last small part of the same length, independently from E. In the next proposition, every time we refer to a measure it will be the Lesbegue measure.

Proposition 1.1.4. *There exists a measurable set $E \subseteq [0,1]$ such that for every subinterval $I \subseteq [0,1]$, both $E \cap I$ and $E^c \cap I$ have strictly positive measure.*

Proof 1. As above, the idea of the first proof is to use Baire category theorem in order to show the existence of such measurable set.

Let $V := \{E \subseteq [0,1] : E \text{ measurable}\}$ be the space of measurable set in $[0,1]$ with the metric induced by the L^1 norm for the respectevely indicator functions. Namely,

$$d(E,F) = \|\mathbb{1}_E - \mathbb{1}_F\|_{L^1} \quad \forall E, F \in V.$$

This is a complete metric space and so a Baire space. Let us now consider the following sets for any $k \geq 1$,

$$C_k := \{E \in V \ : \ \exists x_E \in [0,1] \ s.t. |E \cap [x_E, x_E + 1/k]| = 0 \ or \ |E^c \cap [x_E, x_E + 1/k]| = 0\}.$$

It is clear that if a measurable set does not satisfy the required property, then it is in C_k for a certain k. If we show that C_k's are closed and with empty interior, as usual, we obtain thesis.

- Let $\{E_n\} \subseteq \{C_k\}$ such that $E_n \to E$, i.e. $\|\mathbb{1}_{E_n} - \mathbb{1}_E\|_{L^1} \to 0$. For each $n \in \mathbb{N}$, there exists x_n such that or $|E_n \cap [x_n, x_n + \frac{1}{k}]| = 0$ or $|E_n^c \cap [x_n, x_n + \frac{1}{k}]| = 0$. Since, $\{x_n\} \subseteq [0,1]$, up to subsequences, $x_n \to x_\infty \in [0,1]$. Now, we have three cases:

  1. $|E_n \cap [x_n, x_n + \frac{1}{k}]| = 0$ eventually, i.e. $\int \mathbb{1}_{E_n} \mathbb{1}_{[x_n, x_n+1/k]} dx = 0$ eventually. In this case, passing to the limit, thanks to dominated convergence, we have that also $|E \cap [x_\infty, x_\infty + \frac{1}{k}]| = 0$, which means that $E \in C_k$.

2. $|E_n^c \cap [x_n, x_n + \frac{1}{k}]| = 0$ eventually, that is analogous to the previous point. In this case we obtain that $|E^c \cap [x_\infty, x_\infty + 1/k]| = 0$, which also means that $E \in C_k$.
3. $|E_n \cap [x_n, x_n + \frac{1}{k}]| = 0$ frequently and $|E_n^c \cap [x_n, x_n + \frac{1}{k}]| = 0$ frequently. Actually, this is not a real case, since we can find two different subsequences where we can apply case 1 and case 2 respectively, and so we can obtain that $|E \cap [x_\infty, x_\infty + \frac{1}{k}]| = 0$ and $|E^c \cap [x_\infty, x_\infty + \frac{1}{k}]| = 0$, which imply that $|[0,1] \cap [x_\infty, x_\infty + \frac{1}{k}]| = 0$, which is a contradiction.

- Let us suppose by contradiction that there exists an open ball $B_V(E_0, r_0) \subseteq C_{k_0}$, for certain $E_0 \in V,\ r_0 > 0,\ k_0 \geq 1$. Since $E_0 \in C_{k_0}$ we know that there exists $x_0 \in (0,1)$ such that $|E_0 \cap [x_0, x_0 + \frac{1}{k}]| = 0$ or $|E_0^c \cap [x_0, x_0 + \frac{1}{k}]| = 0$. It is possible to make some adjustments to the set E_0 in order to obtain a set F that is sufficiently close to E_0, but it is not in C_{k_0}. Firstly, for any $n \in \mathbb{N}$ let us take a partition of $[0,1]$ with size equal to $\frac{1}{n}$. Now, we can define a sequence of measurable sets E_n according to Definition 1.1.3, i.e. for any $n \in \mathbb{N}$ let us take E_n as the n-modification of E_0. Thanks to such procedure, it is possible to prove that for n large enough E_n is close to E_0 and it is not in C_{k_0}. Indeed, for any $n \in \mathbb{N}$, we have that

$$\|\mathbb{1}_{E_n} - \mathbb{1}_{E_0}\|_{L^1} = \ |E_n - E_0| \leq \ \frac{2}{n} \ .$$

Thus, it is possible to take n sufficiently large in order to obtain $\|\mathbb{1}_{E_n} - \mathbb{1}_{E_0}\|_{L^1} \leq r_0$, which means that $E_n \in B(E_0, r_0)$. On the other hand, if n is such that $\frac{1}{n} \leq \frac{1}{2k_0}$, any interval $I_{k_0} \subset [0,1]$ of length $\frac{1}{k_0}$ contains at least one sub-interval of type $[\frac{i}{n}, \frac{i+1}{n}]$. This implies that

$$|E_n \cap I_{k_0}| \geq |\ [\tfrac{i}{n}, \tfrac{i}{n} + \tfrac{i}{n^2}]\ | = \frac{1}{n^2} > 0,$$

and

$$|E_n^c \cap I_{k_0}| \geq |\ [\tfrac{i+1}{n} - \tfrac{1}{n^2}, \tfrac{i+1}{n}]\ | = \frac{1}{n^2} > 0.$$

Hence, it could not be in C_{k_0} and it is a contradiction.

□

Remark 1.1.5. Similarly to the previous result, thanks to the Baire proof, we have showed not only the existence of a measurable set such that both itself and its complementary have positive measure in each sub-interval, but also that it is residual in V.

As before, we can present a second explicit proof of the same result presented in Proposition 1.1.4.

Proof 2 of Proposition 1.1.4. In the second proof, our goal is to construct an explicit solution, i.e. a measurable set that satisfies the required properties. The main idea can be caught from the construction of the contradiction at the end of *proof 1*.

Let us start from the empty set, i.e. at first step, *Step 0*, we take $E_0 = \emptyset$. We are searching for an increasing sequence of indices $\{n_k\}$ in order to define the target set. At *Step 1*, let us consider the uniform partition of $[0, 1]$ of size n_1 and define E_1 as the n_1-modification of E_0. At *Step 2*, we define E_2 as the n_2-modification of E_1. Similarly, at *Step k*, we define E_k as the n_k-modification of E_{k-1}. We continue in the same way for infinite steps and we define E_∞ as the limit set obtained. Figure 1.3 illustrates the first steps of the construction.

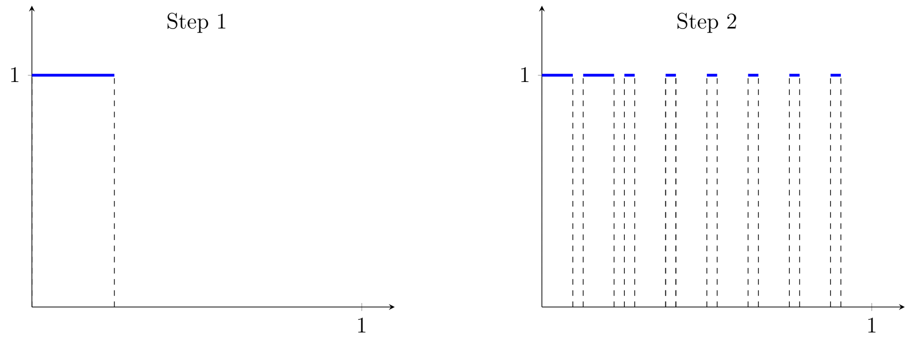

Figure 1.3: How to construct the indicator function of E_∞: the first two steps.

The set E_∞ obtained is the target set. Indeed, to conclude the proof, we need to take suitable $\{n_k\}$ in order to show, firstly, that E_∞ is measurable and then that for every sub-interval $I \subseteq [0, 1]$, both $E_\infty \cap I$ and $E_\infty^c \cap I$ have strictly positive measure.

- Measurability of E_∞ follows by taking $\{\mathbb{1}_{E_k}(x)\}$ as a Cauchy sequence in L^1. In particular, it is sufficient that

$$\sum_{j=0}^{\infty} \frac{1}{n_j} < \infty . \tag{1.9}$$

  Indeed, for any $\epsilon > 0$, for $n < m$ large enough,

$$\|\mathbb{1}_{E_m}(x) - \mathbb{1}_{E_n}(x)\|_{L^1} \leq \sum_{j=n+1}^{m} n_j \frac{2}{n_j^2} \leq 2 \sum_{j=n+1}^{m} \frac{1}{n_j} \leq \epsilon ,$$

  where in the last inequality we used that $\sum_{j=n+1}^{m} \frac{1}{n_j} \leq \frac{\epsilon}{2}$ for $n < m$ large enough, since it is a Cauchy sequence thanks to (1.9).

- Let us consider I any interval of $[0, 1]$. If we call ℓ the length of the interval I, then for k large enough, $\frac{1}{n_k} \leq \frac{\ell}{2}$ and we are sure that at least one section of type

$[\frac{i}{n_k}, \frac{i+1}{n_k}]$ is contained in I. Hence, since E_{n_k} is a n_k-modification of $E_{n_{k-1}}$, we know that an entire sub-interval S_1 of length $\frac{1}{n_k^2}$ is contained in E_{n_k} and another sub-interval S_0 of the same length is contained in $E^c_{n_k}$. Now, if $\frac{1}{n_k}$ converges to 0 sufficiently fast we can prove that the series of intervals contained in S_1 that will be removed after *Step* k in the construction of E_∞ is small and S_1 still remains of positive measure. The same argument holds for S_0 and the complementary set E^c_∞, too. To this aim, it is sufficient to consider the condition

$$\sum_{j=k+1}^{\infty} \frac{1}{n_j} \leq \frac{1}{2} \quad \forall k \geq 1. \tag{1.10}$$

Indeed, it turns out that

$$|E_\infty \cap I| \geq \frac{1}{n_k^2} - \sum_{j=k+1}^{\infty} \frac{n_j}{n_k^2} \cdot \frac{1}{n_j^2} \geq \frac{1}{2n_k^2} > 0.$$

The same estimate holds for $|E^c_\infty \cap I|$.

In conclusion, it is sufficient to take the sequence of indices $\{n_k\}$ such that conditions (1.9) and (1.10) hold true. For example, taking $n_k = 10^k$ we have that (1.9) follows easily and for any $k \geq 1$,

$$\sum_{k+1}^{\infty} \frac{1}{n_j} = \sum_{k+1}^{\infty} \frac{1}{10^j} = \frac{1}{9 \cdot 10^k} \leq \frac{1}{2}.$$

This concludes the proof. □

Finally, we can present a last example of the Baire proof procedure. In this case, our goal is to show that the set of Lipschitz functions with constant equal to M in any sub-interval, for a certain $M > 0$, is residual in the space of Lipschitz function with constant less than or equal to M.

Proposition 1.1.5. *Let M be a positive constant. Let V be the space of bounded functions $f : \mathbb{R} \to \mathbb{R}$ that are Lipschitz continuous with constant $\leq M$.*
Then, the set of functions with Lipschitz constant $= M$ in any sub-interval is residual.

Proof. Our goal is to prove that the set of functions for which there exists an interval where it is Lipschitz with constant less than M is meager.

First of all we notice that the space V is a complete metric space with the sup norm, and so it is a Baire space. Now, we can define

$$C_k := \left\{ f \in V \ : \ \exists x_f \in [-k,k] \ s.t. \ |f(x) - f(y)| \leq \left(M - \frac{1}{k}\right) |x-y| \ \forall x, y \in [x_f, x_f + \tfrac{1}{k}] \right\}.$$

It is clear that if f is a function for which there exists an interval where it is Lipschitz with constant less than M, then it is in C_k for a certain k. It remains to show that each C_k is closed and with empty interior for any k. The conclusion follows as previous results.

- Similarly to what done in Proposition 1.1.3, let $\{f_n\} \subseteq C_k$ be a sequence that converges uniformly to a certain function f_∞. For each n, there exists x_n according to the above definition of C_k. Since $\{x_n\} \subseteq [-k, k]$, up to subsequences we can suppose that $x_n \to x_\infty \in [-k, k]$. Now, any $x, y \in [x_\infty, x_\infty + \frac{1}{k}]$ can be seen as limit of sequences $\{\alpha_n\}$ and $\{\beta_n\}$, respectively, such that $\alpha_n, \beta_n \in [x_n, x_n + \frac{1}{k}]$ $\forall n$. Hence,
$$|f_n(\alpha_n) - f_n(\beta_n)| \leq \left(M - \frac{1}{k}\right) |\alpha_n - \beta_n| \quad \forall n \in \mathbb{N},$$
and so passing to the (double) limit, thanks to uniform convergence, we obtain that
$$|f_\infty(x) - f_\infty(y)| \leq \left(M - \frac{1}{k}\right) |x - y|,$$
which means that $f_\infty \in C_k$.
- As before, let us suppose by contradiction that there exists an open ball $B_V(f_0, r_0) \subseteq C_{k_0}$, for certain $f_0 \in V$, $r_0 > 0$, $k_0 \geq 1$. In particular, f_0 is a Lipschitz function with a constant $L \leq M$ on $\mathbb{R}$ and a suitable constant $L_k \leq M - \frac{1}{k_0} < M$ on $[x_0, x_0 + \frac{1}{k_0}]$ for a certain $x_0 \in [-k_0, k_0]$. Now, it is possible to make some adjustments to the function f_0 in order to obtain a function g that is sufficiently close to f_0, but without sub-interval in $[-k_0, k_0]$ of length $\frac{1}{k_0}$ in which g can be Lipschitz with constant $< M$. In order to do that, let us consider a partition of $[-k_0, k_0]$ with size equal to $\frac{1}{n}$. Namely let us take
$$-k_0 = y_0 < y_1 < \cdots < y_n = k_0, \qquad y_{i+1} - y_i = \frac{1}{n} \quad \forall i = 0, \ldots, n-1.$$
Now, for n large enough such that $\frac{1}{n} \leq \frac{1}{2k_0}$, let us construct $g_n(x)$ as follows:
  1. take $g_n(x) = f_0(x)$ for all $x \in \mathbb{R}$;
  2. for any y_i of the partition if $f_0(y_{i+1}) \geq f_0(y_i)$ (Case 1) let us modify $g_n(x)$ in the sub-interval $[y_i, y_{i+1}]$ as in Figure 1.4, otherwise (Case 2) do as Figure 1.5.

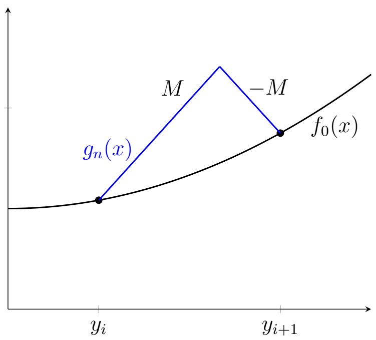

Figure 1.4: Case 1.

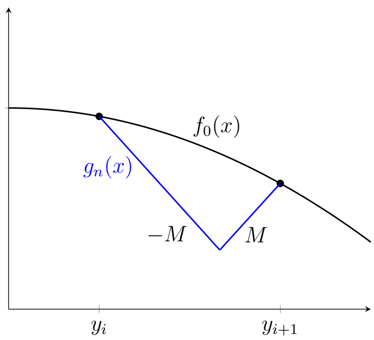

Figure 1.5: Case 2.

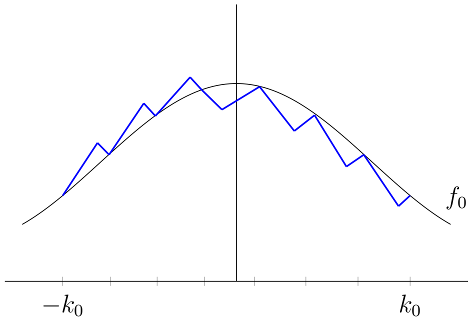

Figure 1.6: Construction of g_n in the interval $[-k_0, k_0]$.

It is possible to do that because $f_0(x)$ is Lipschitz with constant $\leq M$ and so we are sure that the function cannot go above (below, respectively) the affine function with derivative equal to M.

With this construction we are sure that for any interval of length $\frac{1}{k_0}$, there is a sub-interval of type $[y_i, y_{i+1}]$ in which the function $g_n(x)$ is Lipschitz with constant equal to M and so it cannot be in C_{k_0}. On the other hand, we have that $\|g_n - f_0\|_{L^\infty}$ is getting closer to 0 for $n \to \infty$. Indeed, let us consider for a moment one interval of the partition, i.e. $[y_i, y_{i+1}]$. We have that

$$\sup_{x\in[y_i,y_{i+1}]} |g_n(x) - f_0(x)| \leq \frac{2M}{n}.$$

Since, the previous equation holds for any interval, we have also that

$$\begin{aligned}\|g_n - f_0\|_{L^\infty} = \sup_{x\in\mathbb{R}} |g_n(x) - f_0(x)| &= \sup_{x\in[-k_0,k_0]} |g_n(x) - f_0(x)| \\ &= \sup_{x\in[y_i,y_{i+1}]} |g_n(x) - f_0(x)| \leq \frac{2M}{n},\end{aligned}$$

for any $i \leq n$. Hence, for n large enough, it turns out that $\|g_n - f_0\|_{L^\infty} \leq r_0$, and so g_n should be in $B_V(f_0, r_0) \subset C_{k_0}$, but it is a contradiction.

□

Remark 1.1.6. Using the same idea of Figures 1.4 and 1.5, it is possible to prove directly that the subset of functions that have Lipschitz constant equal to M in any sub-interval is dense in the space of Lipschitz functions with constant $\leq M$. Indeed, to make it simpler, let us suppose that $f : [a, b] \to \mathbb{R}$. Then, it is possible to consider for any $n \in \mathbb{N}$ a partition $\{x_i\}$ of $[a, b]$ with size equal to $\frac{1}{n}$. Now, we can define $f_n(x)$ in each interval $[x_i, x_{i+1}]$ making the right adjustments on $f(x)$ as in Figure 1.4 or in Figure 1.5. By construction, $f_n(x)$ is a Lipschitz function with constant equal to M for any $n \in \mathbb{N}$ and, as we proved above, $\|f_n - f\|_\infty \to 0$. We conclude saying that in this remark we have noticed how to prove density, but of course Proposition 1.1.5 is a stronger result, since we gained residuality.

1.2 Metrizability of weak topology

In this section, we want to remind some important results about weak topology in a separable Hilbert space that will be useful throughout this work.

First of all, we want to to present under which conditions there exists a metric that induces the weak convergence.

Theorem 1.2.1. *Let H be a separable Hilbert space. Let $B \subset H$ be a closed ball. Then, there exists a metric that induces the weak convergence. Moreover, B equipped with such metric is a complete metric space.*

Proof. Let us suppose that $\|v\| \leq M$ for any $v \in B$ for a certain $M > 0$. Now, let $\{e_n\}_{n\in\mathbb{N}}$ be a Hilbert basis, i.e. an orthornormal system with $Span\{e_i\}$ dense in H. We know that every $v \in H$ can be seen as $v = \sum_{n=1}^{\infty} v_n e_n$, where $v_n = \langle v, e_n \rangle$ for each $n \in \mathbb{N}$. Now, taking any $v = \sum_{n=1}^{\infty} v_n e_n$ and $w = \sum_{n=1}^{\infty} w_n e_n$ in H, we can define

$$d(v,w) := \sum_{n=1}^{\infty} 2^{-n} |v_n - w_n|. \tag{1.11}$$

It is possible to show that $d(v,w)$ as in (1.11) is actually a distance. Firstly, it is well defined for any choice of v and w, since

$$\sum_{n=1}^{\infty} 2^{-n} |v_n - w_n| \leq 2M \sum_{n=1}^{\infty} 2^{-n},$$

which is a convergent series. Furthermore, it is trivial to see that $d(v,v) = 0$, $d(v,w) > 0$ if $v \neq w$. Symmetry is also clear. About triangle inequality, it holds since taken $v, w, z \in H$, for any $n \in \mathbb{N}$

$$|v_n - z_n| \leq |v_n - w_n| + |w_n - z_n|,$$

due to triangle inequality of absolute value. And so the inequality holds also in the sum. Now, we should prove that the notion of convergence induced by the previous metric is the weakly convergence. This follows since for bounded sequences in a Hilbert space the weakly convergence is fully characterize by the strong convergence of components. Namely, let $v^{(k)} \subset B$ be a sequence that converges weakly to a certain $v^{(\infty)} \in B$, therefore

$$v^{(k)} \rightharpoonup v^{(\infty)} \;\Leftrightarrow\; v_n^{(k)} \longrightarrow v_n^{(\infty)} \;\; \forall n \in \mathbb{N} \;\;\Leftrightarrow\; d(v^{(k)}, v^{(\infty)}) \longrightarrow 0,$$

where $v_n^{(k)}$ and $v_n^{(\infty)}$ denote the n-th component of $v^{(k)}$ and $v^{(\infty)}$ respectively.

In conclusion, we can show that B with the previous metric is a complete metric space. Let $\{v^{(k)}\} \subset H$ be a Cauchy sequence with respect to the distance in (1.11), then $\{v_n^{(k)}\}$ is a Cauchy sequence in $\mathbb{R}$ for each $n \in \mathbb{N}$. Hence, the sequences of components converge and the limits of components will be the components of the limit. Therefore, $v^{(k)}$ admits a weak limit which is in B by closedness. $\square$

Remark 1.2.1. More in general, it is possible to say that if X is a separable Banach space and $K \subset X$ is *weakly** compact, then K is metrizable in the *weak** topology. In this case, let $\{x_n\}_{n\in\mathbb{N}}$ be a dense subset of X, a distance can be defined as

$$d(\ell,\ell') = \sum_{n=1}^{\infty} 2^{-n} \min\{1, |\ell(x_n), \ell'(x_n)|\} \quad \forall \ell,\ell' \in X^*,$$

where X^* is the dual of X.

The key point in order to obtain the metrizability is taking compact subset of H and we used this information in order to characterize weakly convergence with components convergence. On the other hand, if we consider the whole Hilbert space (with infinite dimension), it is possible to show that it is not metrizable and this is the result of the following theorem.

Theorem 1.2.2. *Let H be a separable Hilbert space with infinite dimension. Then, it doesn't exist any metric which induces the weak convergence.*

Proof. Suppose by contradiction that there exists a metric that induces the weak convergence. Let us call d such a distance and let us consider $\{e_n\}_{n\in\mathbb{N}}$ a Hilbert basis. It is known that

$$\begin{aligned} e_n &\rightharpoonup 0 \\ 2e_n &\rightharpoonup 0 \\ &\vdots \\ ke_n &\rightharpoonup 0, \end{aligned}$$

and so on for every $k \in \mathbb{N}$. This means, of course, that for every k it turns out that $d(ke_n, 0) \longrightarrow 0$. Therefore, it is possible to take an increasing subsequence of indices n_k such that

$$\begin{aligned} d(e_{n_1}, 0) &\leq 1 \\ d(2e_{n_2}, 0) &\leq \frac{1}{2} \\ &\vdots \\ d(ke_{n_k}, 0) &\leq \frac{1}{k}, \end{aligned}$$

and so on. In this way we have constructed a subsequence $\{ke_{n_k}\}_{k\in\mathbb{N}}$ such that $d(ke_{n_k}, 0) \leq \frac{1}{k}$, which means that ke_{n_k} converges to 0 according to the metric induced by d. However $\{ke_{n_k}\}$ is an unbounded sequence that doesn't converge weakly to 0 and so we have the contradiction. $\square$

Now, we can discuss one interesting property of the weak topology. In particular, we can characterize the weak closure of the unite sphere.

Proposition 1.2.1. *Let us consider the unit sphere $S = \{v \in H \ : \ \|v\| = 1\}$. Then, the weak closure of S is actually the unit ball $B = \{v \in H \ : \ \|v\| \leq 1\}$. Equivalently, each v in the unit ball is the weak limit of a sequence contained in the unit sphere.*

Proof. Let us take $v \in B$. If $v = 0$, it is known that taking any orthonormal basis $\{e_n\}$, then $e_n \rightharpoonup 0$. Now, if $v \neq 0$, let us consider $\alpha = \|v\|$. So, $\frac{v}{\alpha}$ is of unit norm and taking $e_1 = \frac{v}{\alpha}$, it is possible to extend it into a Hilbert basis $\{e_n\}_{n\geq 1}$. Consider

$$v_n = v + \sqrt{1-\alpha^2}\, e_n, \quad n \geq 2.$$

We can prove that $\{v_n\} \subset S$ and $v_n \rightharpoonup v$. Indeed, the second property is trivial since $e_n \rightharpoonup 0$; while, for the first one, it is sufficient saying that, recalling that $\alpha = \|v\|$, for any $n \geq 2$, it turns out that

$$\|v_n\|^2 = \|v\|^2 + (1-\alpha^2) + 2\sqrt{1-\alpha^2}\langle v, e_n\rangle = 1,$$

where we used that v and e_n are orthogonal. □

Remark 1.2.2. It is clear that the same result obtained in the previous Proposition holds also for any ball $B \subset H$ of radius $r > 0$ and its boundary. The proof is substantially the same.

After investigating the metrizability of weak topology in a Hilbert space, we can link this information to previous results on Baire spaces. One example is the following proposition where we can show some properties that hold with the weak metric, but not with the strong one. Before proving this fact, we need some preliminary lemmas. In the first one, we can state a classical result about approximation of L^2 function.

Lemma 1.2.1. *Let us consider $L^2((a,b))$ provided by the usual metric induced by the L^2-norm. Then, the set of piecewise constant functions is dense.*

Remark 1.2.3. It is possible to note that if the set of piecewise constant functions is dense with respect to the strong metric, it is dense also with respect to the weak metric.

In the next lemma, we recall the property of the lower semicontinuity of the norm under weak convergence in a Hilbert space that will be often useful in this thesis.

Lemma 1.2.2. *Let us H be a Hilbert space and let us consider a sequence $\{v_n\}$ such that $v_n \rightharpoonup v_\infty$ weakly in H. Then,*

$$\liminf_{n\to\infty} \|v_n\| \geq \|v_\infty\|.$$

Proof. The proof is straightforward and follows from the following steps.

$$\begin{aligned}\|v_n\|^2 &= \|(v_n - v_\infty) + v_\infty\|^2 \\ &= \|(v_n - v_\infty)\|^2 + \|v_\infty\|^2 + 2\langle v_n - v_\infty, v_\infty\rangle \\ &\geq \|v_\infty\|^2 + 2\langle v_n - v_\infty, v_\infty\rangle.\end{aligned}$$

Since the last inner product converges to 0 thanks to weak convergence, it turns out that

$$\liminf_{n\to\infty} \|v_n\|^2 \geq \|v_\infty\|^2 .$$

The thesis follows. □

Remark 1.2.4. We will use the previous Lemma with Hilbert space of functions $H = L^2$.

Finally, we can present another preliminary result before considering the main proposition.

Lemma 1.2.3. *Let us consider the set of functions $f : (a, b) \longrightarrow [-1, 1]$. Then, it is a closed set in $L^2((a, b))$ with both strong and weak topology.*

Proof. First of all we remind that if $\{f_n\}$ is a sequence of functions such that f_n converges to f_∞ in L^2, then it converges pointwise almost everywhere, up to subsequences. This important fact suggests us that if $f_n \longrightarrow f_\infty$ strongly in L^2, it converges pointwise, up to subsequences, and so, since f_n is valued in $[-1, 1]$, the same holds also for f_∞. The previous argument implies that it is closed with respect to strong topology.

About weak topology, we can say that if $f_n \rightharpoonup f_\infty$ weakly in L^2, then

$$\int_a^b f_n(x)g(x)dx \longrightarrow \int_a^b f_\infty(x)g(x)dx, \quad \forall g \in L^2((a, b)).$$

Suppose by contradiction that there exists $A \subset (a, b)$ with $|A| > 0$ and $|f_\infty(x)| \geq 1+\alpha$ with $\alpha > 0$ for any $x \in A$. Therefore, taking $g = \frac{1}{|A|}\mathbb{1}_A sgn(f_\infty) \in L^2((a, b))$,

$$\int_a^b f_\infty(x)g(x)dx = \frac{1}{|A|}\int_A |f_\infty(x)|dx \geq 1 + \alpha > 1.$$

Hence, for n large enough, thanks to convergence of previous integrals, it turns out that

$$\frac{1}{|A|}\int_A f_n(x)sgn(f_\infty(x))dx = \int_a^b f_n(x)g(x)dx > 1.$$

But,

$$\frac{1}{|A|}\int_A f_n(x)sgn(f_\infty(x))dx \leq \frac{1}{|A|}\int_A |f_n(x)|dx \leq 1,$$

since $f_n(x) \leq 1$ for any $x \in (a, b)$. This brings to a contradiction. □

We are now ready for the main following result about residuality.

Proposition 1.2.2. *In the space of functions $f \in L^2(a,b)$ with $|f(x)| \leq 1$ a.e., the set of functions such that $f(x) \in \{-1,1\}$ a.e. is residual with respect to the weak metric, but it's not with respect to the strong metric.*

Proof. Let us call

$$V := \{f \in L^2((a,b)) \ : \ |f(x)| \leq 1 \ \ a.e.\},$$

and

$$W := \{f \in L^2((a,b)) \ : \ |f(x)| = 1 \ \ a.e.\}.$$

Firstly, in order to see that it cannot be a residual set with respect to the strong topology, we can notice that the sets of functions with image in $\{-1,1\}$ can't be dense. In fact, if it was, it would have been possible to find a sequence of functions with image equal to 1 or -1 that converges to any function in V strongly in L^2. But it is a contradiction since, up to subsequences, it should converge also pointwise, almost everywhere.

Now, we can prove that actually with the weak metric discussed above it is a residual set. The idea of the proof is the same presented in the first part of this work. Due to the fact that $|f(x)| \leq 1$ for any $x \in (a,b)$, we can notice that

$$\|f\|_{L^2} = \int_a^b |f(x)|^2 dx \ \leq (b-a) \quad \forall f \in V,$$

which means that $V \subseteq B_{L^2}(0, b-a)$, i.e. the ball of radius equal to $b-a$ in L^2. Hence, thanks to previous Lemma 1.2.3 and initial Theorem 1.2.1, if we consider V provided by the weak metric, it is complete since it is a closed set in a complete metric space, and then it is a Baire space. Furthermore, for any $f \in V$,

$$\|f\|_{L^2} = b-a \ \Leftrightarrow \ Im(f) \in \{-1,1\} \ \ a.e. \tag{1.12}$$

Therefore, let us take

$$C_k = \left\{ f \in V \ : \ \|f\|_{L^2} \leq (b-a)\left(1 - \frac{1}{k}\right) \right\}. \tag{1.13}$$

It is clear that, calling $B_V(0,r) := B_{L^2}(0,r) \cap V$ for any radius $r > 0$, the set $\bigcup_{k\geq 1} C_k$ is equal to the open ball $B_V(0, b-a)$. Namely, if f is in $V \setminus W$, then it is in a C_k for a certain k. As usual, we want to to prove that C_k's are closed and with empty interior (with respect to weak topology).

- Closedness follows thanks to Lemma 1.2.2. Indeed, let us take $\{f_n\} \subseteq C_k$ such that $f_n \rightharpoonup f_\infty$. Due to lower semicontinuity of the norm, we have that

$$\begin{aligned}(b-a)\left(1-\frac{1}{k}\right) &\geq \liminf_{n\to\infty} \left(\int_a^b |f_n(x)|^2 \, dx\right)^{\frac{1}{2}} \\ &\geq \left(\int_a^b |f_\infty(x)|^2 \, dx\right)^{\frac{1}{2}} = \|f_\infty\|_{L^2} \ .\end{aligned}$$

Consequently, $f_\infty \in C_k$ and C_k is closed.

- Let us suppose by contradiction that there exists an open ball $B(f_0, r_0) \subseteq C_{k_0}$ (with respect to weak topology), for certain $f_0 \in V$, $r_0 > 0$, $k_0 \geq 1$. Due to Lemma 1.2.1 and Remark 1.2.3 there exists a piecewise constant function $\tilde{f}(x)$ such that
$$d(f_0, \tilde{f}) \leq \frac{r_0}{2},$$
where $d(\cdot,\cdot)$ is the distance inducing the weak convergence.

  First of all, since $|f_0(x)| \leq 1$, we can assume that also $|\tilde{f}(x)| \leq 1$, namely there exist k intervals I_j contained in (a,b) such that
$$\tilde{f}(x) = \alpha_j \quad x \in I_j, \quad j = 1, \dots, k,$$
with $|\alpha_j| \leq 1$ for any $j \leq k$. Now, for each j we can find a sequence of functions defined in I_j valued in $\{-1, 1\}$ that converges weakly to α_j. More specifically, let us consider for a moment one small interval I_j for a certain j. We know that $\tilde{f}(x) = \alpha_j$ for a.e. $x \in I_j$. Moreover, we can write α_j as a convex combination of 1 and -1, i.e.
$$\alpha_j = \lambda_j - (1-\lambda_j) = 2\lambda_j - 1,$$
for a certain $0 \leq \lambda_j \leq 1$. Let us take a uniform partition of the interval I_j of size equal to $\frac{1}{n}$. Namely, if $I_j = [y_j, y_{j+1}]$, we can take $\{x_r\}$ such that
$$y_j = x_0 < \; x_2 < \ldots < \; x_n = y_{j+1}, \quad with \;\; |x_{r+1} - x_r| = \frac{1}{n}.$$
For any $r = 0, \dots, n-1$, we can define $x_{\lambda_j}^{(r)} := \lambda_j x_{r+1} + (1-\lambda_j) x_r$ and
$$g_n^{(j)}(x) := \begin{cases} 1 & if \; x \in [x_r, x_{\lambda_j}^{(r)}] \\ -1 & if \; x \in (x_{\lambda_j}^{(r)}, x_{r+1}] \end{cases} . \tag{1.14}$$
The following Figure 1.7 illustrates the construction in (1.14).

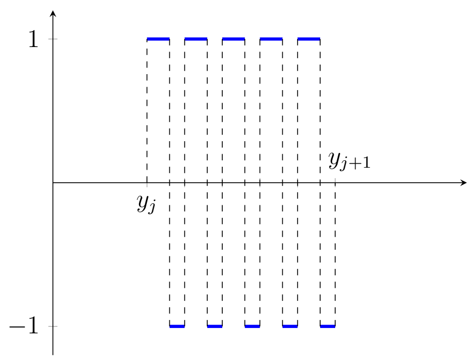

Figure 1.7: Definition of $g_n^{(j)}(x)$ with $\lambda_j = \frac{3}{5}$ and $n = 5$.

It is known that such $g_n^{(j)}(x)$ converges weakly to the function constantly equal to the convex combination of extremes with weight equal to λ_j. Therefore, by construction, we are saying that

$$g_n^{(j)} \rightharpoonup \tilde{f}|_{I_j} \qquad \text{as} \quad n \to \infty. \tag{1.15}$$

Thanks to what discussed so far,it is possible to construct a sequence of function $\{g_n\} \subset W$ such that $g_n \rightharpoonup \tilde{f}$. In particular, in order to do that, it is sufficient to define $g_n(x)$ according to (1.14) for each interval I_j in which $\tilde{f}(x)$ is constantly equal to α_j. Hence, we can find a certain n large enough such that

$$d(g_n, \tilde{f}) \leq \frac{r_0}{2},$$

and so

$$d(g_n, f_0) \leq d(g_n, \tilde{f}) + d(\tilde{f}, f_0) \leq \frac{r_0}{2} + \frac{r_0}{2} = r_0.$$

Therefore, g_n should be in $B(f_0, r_0)$ but it is not in C_{k_0} since it is valued in $\{-1, 1\}$ and (1.12) holds. This brings to a contradiction.

$\square$

Chapter 2

Curves with unit speed

In this chapter, we want to start presenting the **convex integration** method. More specifically, we consider one first problem in which we discuss the density of curves with unit norm derivative into the set of curves with derivative of norm less than 1[1]. Such density can be proved through a one dimensional version of the convex integration procedure and its deep meaning is enclosed within Fundamental Lemma 2.1.1. Indeed, the idea presented in the Fundamental Lemma, and even more in the Figure 2.1, can be considered a fundamental point of convex integration and it will appear also later in the last chapter. All the other results follow the same idea, but with more technical details.

It is possible to present the same result in different levels of generality. In order to be more clear, we prefer to highlight the main idea before in easier cases and later in more general ones. In particular, we will discuss the problem of curves taking into account the ball as *differential relation* in two dimensions and then in more dimensions. After that, we consider for a moment one more general case with a convex set as *differential relation*.

Finally, we can present one first connection between convex integration and Baire category theorem. In particular, we use the density result in a Baire category argument to show residuality with a suitable metric.

2.1 Density

2.1.1 2-dimensional ball as differential relation

Now, we can start stating the first theorem that can show the meaning and the power of convex integration.

[1]We refer to the set where the curves are valued in as *differential relation*. For example, in Section 2.1.1 it is the 2-dimensional ball, in Section 2.1.2 it is the d-dimensional ball, and so on.

Theorem 2.1.1. *Let us consider V the set of curves*

$$V := \{u : [0,1] \longrightarrow \mathbb{R}^2 \ : \ u \in C^1, \ \|u'(t)\|_{\mathbb{R}^2} \leq 1 \ \forall t \in [0,1]\}.$$

Then, the set of curves with unit speed, i.e.

$$W := \{u : [0,1] \to \mathbb{R}^2 \ : \ u \in C^1, \ \|u'(t)\|_{\mathbb{R}^2} = 1 \ \forall t \in [0,1]\},$$

is C^0-dense in V. Namely, for any $u \in V$ and for any $\epsilon > 0$, there exists a curve $u_\epsilon : [0,1] \to \mathbb{R}^2$ with

(i) $\|u'_\epsilon(t)\|_{\mathbb{R}^2} = 1, \quad \forall t \in [0,1]$;

(ii) $\|u_\epsilon(t) - u(t)\|_{\mathbb{R}^2} \leq \epsilon, \quad \forall t \in [0,1]$.

Remark 2.1.1. It is possible to note that the last condition (ii) is equivalent to say that

$$(ii') \quad \|u_\epsilon - u\|_{C^0} \leq \epsilon.$$

In order to prove the previous result, as we said above, we want to highlight the main idea that there is behind the proof. In particular, we show the following Fundamental Lemma, in which we prove that it is possible to find a loop that lies on the boundary of the unit ball and in average it is equal to a certain vector that lies in the interior part of unit ball, for any choice of the vector inside the ball. After that, it will be possible to construct the approximated curve according to Theorem 2.1.1 using the Fundamental Lemma for the derivative, moving along the loop enough times in order to remain sufficiently close to the target curve.

Lemma 2.1.1 (Fundamental Lemma)**.** *Let us consider a vector $v \in B(0,1) \subseteq \mathbb{R}^2$. Then, there exists a loop $h : [0,1] \to \mathcal{S}^1$ such that*

$$v = \int_0^1 h(s)ds. \tag{2.1}$$

Proof. Let us prove this lemma considering different cases.

Step 1: Let us, firstly, consider the case in which the vector v lies on the positive part of x-axis, i.e. $v = (\alpha, 0)$ for a certain $0 < \alpha < 1$. Our goal is to find a loop

$$h(s) := (cos(\theta(s)), sin(\theta(s))) \quad \forall s \in [0,1], \tag{2.2}$$

such that it lies on $\mathcal{S}^1$ and its average is equal to v component-wise. The main idea is shown in Figure 2.1.

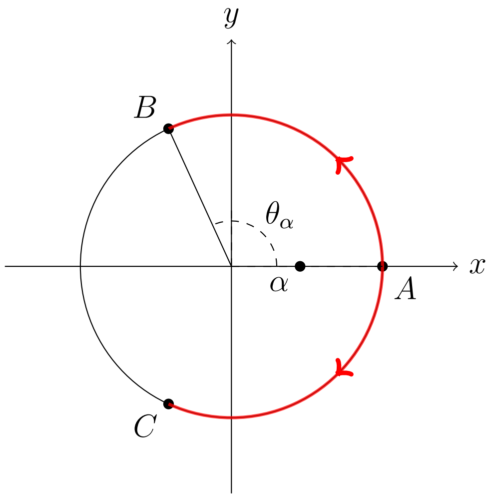

Figure 2.1: Convex integration: construction of the loop.

Fixed $\alpha \in (0,1)$ according to the previous definition, it is possible to find a certain $\theta_\alpha \in (0,\pi)$ such that $\dfrac{\sin(\theta_\alpha)}{\theta_\alpha} = \alpha$. Such θ_α exists since the function $\dfrac{\sin(x)}{x}$ from $[0,\pi]$ to $[0,1]$ is decreasing. Now, according to Figure 2.1, we can construct the loop starting to point A at time $t=0$, moving to point B at time $t=1/4$, then passing again through A at time $t=1/2$, moving to point C at time $t=3/4$ and finally coming back to A at time $t=1$. It is possible to define such curve according to the following expression

$$h(s) := \begin{cases} (\cos(4\theta_\alpha s), \sin(4\theta_\alpha s)) & s \in [0, \frac{1}{4}] \\ (\cos(2\theta_\alpha - 4\theta_\alpha s), \sin(2\theta_\alpha - 4\theta_\alpha s)) & s \in [\frac{1}{4}, \frac{3}{4}] \\ (\cos(-4\theta_\alpha + 4\theta_\alpha s), \sin(-4\theta_\alpha + 4\theta_\alpha s)) & s \in [\frac{3}{4}, 1] \end{cases}.$$

We must prove that $v = \int_0^1 h(s)ds$, i.e. if we call $h_x(s)$ and $h_y(s)$ its components, we want that $\int_0^1 h_x(s)ds = \alpha$ and $\int_0^1 h_y(s)ds = 0$.

- Regarding the second component, it follows easily by symmetry that $\int_0^1 h_y(s)ds = 0$.
- Regarding the first component, it is possible to note that $h_x(s)$ is the same

in each of the four intervals of $[0,1]$. Consequently, it turns out that

$$\begin{aligned}\int_0^1 h_x(s)ds &= 4\int_0^{\frac{1}{4}} h_x(s)ds \\ &= 4\int_0^{\frac{1}{4}} \cos(4\theta_\alpha s)ds \\ &= \frac{1}{\theta_\alpha}\sin(4\theta_\alpha s)\Big|_0^{\frac{1}{4}} \\ &= \frac{\sin(\theta_\alpha)}{\theta_\alpha} = \alpha.\end{aligned}$$

Step 2: Consider the simple case in which $v=(0,0)$. In this case we can take for example $h(s) := (cos(2\pi s), sin(2\pi s))$ with $s\in[0,1]$, as the loop around $\mathcal{S}^1$. It is clear by symmetry that $\int_0^1 h(s)ds = 0$.

Step 3: Now, let us take a generic $v=(\alpha,\beta)\in B(0,1)$. Thanks to a rotation it is possible to consider v lying on the new first axis and we can proceed as in Step 1. The following Figure 2.2 illustrates effectively such construction.

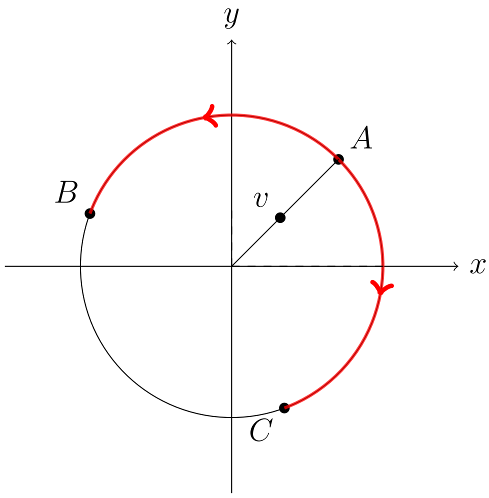

Figure 2.2: Convex integration: construction of the loop for generic v.

□

Thanks to previous construction, the main idea in order to obtain an approximated curve with unit norm derivative is to take as derivative the loop provided by the lemma and moving along that loop many times. The approximated curve will be the primitive function of such derivative. Before proceeding with the proof, we can state a classical result about approximation of continuous function that will be useful (similar to Lemma 1.2.1 of the first chapter).

Lemma 2.1.2. *Let us consider $C([0,1],\mathbb{R}^d)$ the set of continuous function from $[0,1]$ to $\mathbb{R}^d$. Let us provide $C([0,1],\mathbb{R}^d)$ with C^0-norm, i.e. for any $f,g \in C([0,1],\mathbb{R}^d)$*

$$\|f-g\|_{C^0} := \sup_{t\in[0,1]} \|f(t)-g(t)\|_{\mathbb{R}^d}.$$

Then, the set of piecewise linear functions is dense.

We are ready to prove Theorem 2.1.1.

Proof of Theorem 2.1.1. We want to prove the Theorem taking into account different steps. First of all, we consider the case of curves with constant derivative. Then, we look at piecewise constant derivative functions and finally we achieve the general case.

Step 1: Let us suppose that $u'(t)$ is constant. Following what was done in the previous Fundamental Lemma, we can consider without loss of generality, up to rotations, that $u'(t) \equiv (\alpha, 0) \in B(0,1)$, as in Step 1 of the proof of Lemma 2.1.1. Thanks to the lemma, there exists $h(t)$ such that $\int_0^1 h(s)ds = (\alpha, 0)$. Now, for each $n \in \mathbb{N}$, we can take a loop $h_n(t)$ that repeat the same loop $h(s)$ for n times by taking $h_n(s) := h(\{ns\})$, where $\{\cdot\}$ represents the fractional part. More precisely, following the same construction done in the lemma, for each $k = 0, 1, \dots, n-1$, we have

$$h_n(s) = \begin{cases} (\cos(4\theta_\alpha(ns-k), \sin(4\theta_\alpha(ns-k))) & s \in [\frac{k}{n}, \frac{4k+1}{4n}] \\ (\cos(2\theta_\alpha - 4\theta_\alpha(ns-k)), \sin(2\theta_\alpha - 4\theta_\alpha(ns-k))) & s \in [\frac{4k+1}{4n}, \frac{4k+3}{4n}] \\ (\cos(-4\theta_\alpha + 4\theta_\alpha(ns-k)), \sin(-4\theta_\alpha + 4\theta_\alpha(ns-k))) & s \in [\frac{4k+3}{4n}, \frac{k+1}{n}] \end{cases}.$$

Hence, we can define

$$u_n(t) := u(0) + \int_0^t h_n(s)ds. \tag{2.3}$$

We say that such $u_n(t)$ is obtained from $u(t)$ by a **convex integration** process. First of all, we can notice that for each $n \in \mathbb{N}$

$$\|u_n'(t)\|_{\mathbb{R}^2} = \|h_n(t)\|_{\mathbb{R}^2} = 1,$$

by construction. Therefore, it is sufficient to prove that for n large enough $u_n(t)$ is sufficiently close to $u(t)$ in C^0-norm. Let us fix $n \in \mathbb{N}$ and $t \in [0,1]$, there exists $k_n \in \{0, 1, \dots, n-1\}$ such that $t \in [\frac{k_n}{n}, \frac{k_n+1}{n}]$. In particular, we can write $t = \frac{k_n}{n} + \tau$ for a certain $\tau < \frac{1}{n}$.

Now, we can note that for any $i = 0, \dots, n-1$

$$\begin{aligned}\int_{\frac{i}{n}}^{\frac{i+1}{n}} h_n(s)ds &= \left(\int_{\frac{i}{n}}^{\frac{i+1}{n}} h_{nx}(s)\ ds\ ,\ 0 \right) \\ &= \left(4 \int_{\frac{i}{n}}^{\frac{4i+1}{4n}} \cos(4\theta_\alpha(ns-i))\ ds\ ,\ 0 \right) \\ &= \left(\frac{1}{n\theta_\alpha} \sin(4\theta_\alpha(ns-i) \Big|_{\frac{i}{n}}^{\frac{4i+1}{4n}}\ ,\ 0 \right) \\ &= \left(\frac{\sin(\theta_\alpha)}{n\theta_\alpha}\ ,\ 0 \right) = \left(\frac{\alpha}{n}\ ,\ 0 \right).\end{aligned}$$

Hence, thanks to latter calculation, it turns out that

$$\begin{aligned}\|u_n(t) - u(t)\|_{\mathbb{R}^2} &= \left\| \int_0^t h_n(s)ds - \int_0^t u'(s)ds \right\|_{\mathbb{R}^2} \\ &= \left\| \sum_{i=0}^{k_n-1} \int_{\frac{i}{n}}^{\frac{i+1}{n}} h_n(s)ds + \int_{\frac{k_n}{n}}^t h_n(s)ds - \left(\alpha \frac{k_n}{n}, 0 \right) - (\alpha\tau, 0) \right\|_{\mathbb{R}^2} \\ &= \left\| k_n \left(\frac{\alpha}{n}, 0 \right) + \int_{\frac{k_n}{n}}^t h_n(s)ds - \left(\alpha \frac{k_n}{n}, 0 \right) - (\alpha\tau, 0) \right\|_{\mathbb{R}^2} \\ &\leq \int_{\frac{k_n}{n}}^t \|h_n(s)\|_{\mathbb{R}^2}\, ds + \|(\alpha\tau, 0)\|_{\mathbb{R}^2} \leq \frac{2}{n}\ .\end{aligned}$$

Thus, there exists n large enough such that

$$\|u_n(t) - u(t)\|_{\mathbb{R}^2} \leq \epsilon,$$

and we can take such $u_n(t)$ as $u_\epsilon(t)$ required for the theorem.

Step 2: Let us take $u(t)$ continuous with piecewise constant derivative function. Let us suppose that u' is constant in k different intervals. Without loss of generality we can consider that the endpoints of such intervals are rationals. Namely, there exist intervals $I_j = [a_j, a_{j+1}]$, with $a_j \in \mathbb{Q}$, for $j = 1, \dots, k$ such that

$$u'(t) = v_j \quad if \quad t \in I_j, \quad with \ \ \|v_j\|_{\mathbb{R}^2} \leq 1.$$

Let us call $u_j(t) := u(t)\Big|_{I_j}$ for $t \in I_j$, that is the curve with constant derivative equal to v_j. Thus, for each j, thanks to Step 1, we know that it is possible to find a sequence of curves with unit norm derivative and such that they are sufficiently close to u_j. From such sequences, we can construct the suitable approximated function for $u(t)$. More specifically, let us fix $\epsilon > 0$ and let us consider $u_1(t)$; thanks to Step 1 there exists $u_{1,n}(t)$ such that

$$\left\| u'_{1,n}(t) \right\|_{\mathbb{R}^2} = 1 \ \ and \ \ \|u_{1,n}(t) - u_1(t)\|_{\mathbb{R}^2} \leq \epsilon \ \ \forall t \in I_1,\ \forall n \geq n_1,$$

for a suitable n_1 large enough. Similarly, considering $u_2(t)$, we can find $u_{2,n}(t)$ such that

$$\left\|u'_{2,n}(t)\right\|_{\mathbb{R}^2} = 1 \quad and \quad \|u_{2,n}(t) - u_2(t)\|_{\mathbb{R}^2} \leq \epsilon \quad \forall t \in I_2,\ \forall n \geq n_2,$$

for a suitable $n_2 \geq n_1$ large enough. It is possible to proceed until $j = k$. Now, we can take $n \geq n_k$ such that all the previous conditions hold and the uniform partition of the interval $[0,1]$ with interval length equal to $\frac{1}{n}$ is such that each a_j is a node of the partition. Now, we can define

$$u_\epsilon(t) := u_{j,n}(t) \qquad \forall t \in I_j \quad and \quad j = 1, \dots, k.$$

Hence, for ant $t \in [0,1]$, we have that $t \in I_j$ and

$$\|u'_\epsilon(t)\|_{\mathbb{R}^2} = \left\|u'_{j,n}(t)\right\|_{\mathbb{R}^2} = 1.$$

Moreover, for $t \in I_j$

$$\|u_\epsilon(t) - u(t)\|_{\mathbb{R}^2} = \|u_{j,n}(t) - u(t)\|_{\mathbb{R}^2} \leq \epsilon, \tag{2.4}$$

from which the thesis follows. Actually, the approximated curve u_ϵ obtained so far is not C^1. Indeed, in each interval I_j, u'_ϵ starts and ends in the same base point of the loop, but in general from one interval to the following the base point change and we have a jump in the derivative function. However, it is possible to fix it easily by considering for any interval I_j with $j = 2, \dots, k$, the first sub-interval of the partition of length $\frac{1}{n}$ and just defining u_ϵ as the arc of the unitary circle that connect the two base points. In this way, $u'_\epsilon(t)$ becomes continuous and we lose a negligible C^0-distance between u and u_ϵ. Therefore, (2.4) still holds and we have concluded.

Step 3: Finally, let us consider the general case. Taking $u(t) \in V$, for any $\epsilon > 0$, thanks to Lemma 2.1.2, there exists $\tilde{u}_\epsilon(t)$ piecewise linear such that

$$\|u - \tilde{u}_\epsilon\|_{C^0} \leq \frac{\epsilon}{2}.$$

Hence, for any $t \in [0,1]$, we have that $\|u(t) - \tilde{u}_\epsilon(t)\|_{\mathbb{R}^2} \leq \frac{\epsilon}{2}$. Moreover, since $\tilde{u}'_\epsilon(t)$ is piecewise constant, if we take any section $(\frac{i}{n}, \frac{i+1}{n})$ where it is constant, it turns out that

$$\begin{aligned} \|\tilde{u}'_\epsilon(t)\|_{\mathbb{R}^2} = \left\|\frac{u(\frac{i+1}{n}) - u(\frac{i}{n})}{\frac{1}{n}}\right\|_{\mathbb{R}^2} &= n\left\|\int_{\frac{i}{n}}^{\frac{i+1}{n}} u'(s)ds\right\|_{\mathbb{R}^2} \\ &\leq n\int_{\frac{i}{n}}^{\frac{i+1}{n}} \|u'(s)\|_{\mathbb{R}^2}\, ds \leq 1. \end{aligned} \tag{2.5}$$

Thus, thanks to Step 2 there exists $u_\epsilon(t) \in W$ such that $\|u_\epsilon(t) - \tilde{u}_\epsilon(t)\|_{\mathbb{R}^2} \leq \frac{\epsilon}{2}$ for all $t \in [0,1]$. Therefore,

$$\begin{aligned} \forall t \in [0,1] \qquad \|u_\epsilon(t) - u(t)\|_{\mathbb{R}^2} &\leq \|u_\epsilon(t) - \tilde{u}_\epsilon(t)\|_{\mathbb{R}^2} + \|\tilde{u}_\epsilon(t) - u(t)\|_{\mathbb{R}^2} \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

□

Remark 2.1.2. Looking more carefully to previous proof, we can notice that it is possible to find a C^1 approximated curve with unit norm derivative not only for any $u \in V$, but also for some other functions, say for instance the piecewise linear functions, that are not in V (see Step 2). More specifically, with the previous proof we showed that W is C^0-dense not only in V, but in the C^0-closure of piecewise linear functions for which (2.5) still holds. Indeed, such set coincides with $W^{1,\infty}([0,1];\mathbb{R}^2)$ and due to the bound on derivative it is also the set of 1-Lipschitz functions.

2.1.2 d-dimensional ball as differential relation

In this section, we want to bring the same idea into a more general case. First of all, we present the same result for curves in dimension d and we will see that the procedure is very similar to what was done above.

Theorem 2.1.2. *Let us consider V the set of curves*

$$V := \{u : [0,1] \longrightarrow \mathbb{R}^d \;:\; u \in C^1,\; \|u'(t)\|_{\mathbb{R}^d} \leq 1 \;\forall t \in [0,1]\}.$$

Then, the set of curves with unit speed, i.e.

$$W := \{u : [0,1] \to \mathbb{R}^d \;:\; u \in C^1,\; \|u'(t)\|_{\mathbb{R}^d} = 1 \;\forall t \in [0,1]\},$$

is C^0-dense in V. Namely, for any $u \in V$ and for any $\epsilon > 0$, there exists a curve $u_\epsilon : [0,1] \to \mathbb{R}^d$ with

(i) $\|u'_\epsilon(t)\|_{\mathbb{R}^d} = 1, \quad \forall t \in [0,1]$;

(ii) $\|u_\epsilon(t) - u(t)\|_{\mathbb{R}^d} \leq \epsilon, \quad \forall t \in [0,1]$.

Also in this case, we need the result of the Fundamental Lemma.

Lemma 2.1.3 (Fundamental Lemma)**.** *Let us consider a vector $v \in B(0,1) \subseteq \mathbb{R}^d$. Therefore, there exists a loop $h : [0,1] \to \mathcal{S}^{d-1}$ such that*

$$v = \int_0^1 h(s)ds.$$

Proof. Following the same structure of the proof of the Fundamental Lemma in the previous section, we can consider different cases.

Step 1: Let us consider the case in which $v = (\alpha, 0, \ldots, 0)$ for a certain $0 < \alpha < 1$. This is analogous to Step 1 of Lemma 2.1.1. Indeed, it is possible to reduce the construction to the first two components. In particular, we have again a vector of the type $(\alpha, 0)$ that lies on the plane generated by the first two axis. Hence, the procedure explained in Figure 2.1 still holds and it is possible to find a loop $h(s)$ on the boundary of the of the two dimensional ball obtained in this restriction. Of course, such loop lies also on $\mathcal{S}^{d-1}$ and $v = \int_0^1 h(s)ds$ component-wise, since for the first two components the same argument of Lemma 2.1.1 holds, and the other ones are always equal to 0.

Step 2: If $v = 0$, we can restrict again the discussion for example to the first two components and the Step 2 of the usual Lemma implies thesis.

Step 3: Let us consider a generic $v \in B(0, 1)$. Similarly to what was done in the previous section, we can reduce this case to Step 1. Indeed, let us define $e_1 := \frac{v}{\|v\|}$ and let us complete e_1 into a orthonormal basis $\{e_i\}$ of $\mathbb{R}^d$, with $i = 2, \ldots, d$. Therefore, it turns out that in the new basis v lies on the first axis and so it is possible to apply Step 1 in order to construct the loop $h(s) \subset \mathcal{S}^{d-1}$ such that $v = \int_0^1 h(s)ds$.

$\square$

Now, we can say that it is possible to prove Theorem 2.1.2 using the latter version of Fundamental Lemma. Actually, due to the Lemma, the proof is basically the same of Theorem 2.1.1 with the difference of $\|\cdot\|_{\mathbb{R}^d}$ instead of $\|\cdot\|_{\mathbb{R}^2}$.

2.1.3 Convex set as differential relation

Another more general setting in which we can discuss the same result is considering curves valued in a convex set of $\mathbb{R}^d$, that is relevant since the term *convex* in the name *convex integration* comes from differential relations of this type. In particular, for simplicity, we prove an easier case that follows directly and similarly to previous results. Later we can state a more general setting and it is possible to go further and find more details in the notes made by Vincent Borrelli about one dimensional convex integration (see [1]).

Theorem 2.1.3. *Let us consider $C \subseteq \mathbb{R}^d$ a compact convex set and let us take V as the set of curves*

$$V := \{u : [0,1] \longrightarrow C \ : \ u \in C^1, \ u'(t) \in Int(C) \ \ \forall t \in [0,1]\}.$$

Then, for any $u \in V$ and for any $\epsilon > 0$, there exists $u_\epsilon : [0,1] \to \mathbb{R}^d$ with piecewise constant derivative function such that

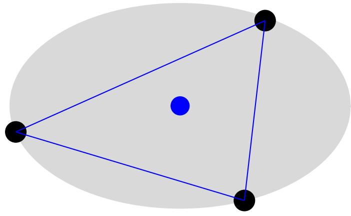

Figure 2.3: Convex combination.

(i) $u'_\epsilon(t) \in \partial C, \quad \forall t \in [0,1]$;

(ii) $\|u_\epsilon(t) - u(t)\|_{\mathbb{R}^d} \leq \epsilon, \quad \forall t \in [0,1]$.

As usual, we firstly present the Fundamental Lemma in this setting.

Lemma 2.1.4 (Fundamental Lemma). *Let us consider $C \subseteq \mathbb{R}^d$ a compact convex set and let us consider a vector $v \in C \subseteq \mathbb{R}^d$.*
Then, there exists a pointwise constant function $h : [0,1] \to \partial C$ such that

$$v = \int_0^1 h(s)\ ds.$$

Proof. Since C is convex, there exists a finite number of points that lie on the boundary such that v is equal to the convex combination of such points. In particular, thanks to *Carathéodory's theorem*, we know that there exist at most $d+1$ points $z_1, z_2, \ldots, z_{d+1}$ on ∂C, such that $v = \lambda_1 z_1 + \cdots + \lambda_{d+1} z_{d+1}$ for certain $0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{d+1} \leq 1$ and such that $\sum_{i=1}^{i+1} \lambda_i = 1$ (see Figure 2.3).
Now, calling $\tilde{\lambda}_0 = 0,\ \tilde{\lambda}_{d+1} = 1,\ \tilde{\lambda}_i = \sum_{j=1}^{i} \lambda_j$ for any $i = 1, \ldots, d$, we can easily define

$$h(s) := z_{i+1} \qquad if\ \ s \in [\tilde{\lambda}_i, \tilde{\lambda}_{i+1}], \quad i = 0, \ldots, d\ .$$

Therefore, it turns out that

$$\int_0^1 h(s)\ ds = \sum_{i=0}^{d} \int_{\tilde{\lambda}_i}^{\tilde{\lambda}_{i+1}} h(s)\ ds = \sum_{i=0}^{d} z_{i+1}(\tilde{\lambda}_{i+1} - \tilde{\lambda}_i) = \sum_{i=1}^{d+1} z_i \lambda_i = v \quad .$$

$\square$

The proof of Theorem 2.1.3 follows in the same way of what done in previous sections and in order to obtain closeness it is possible to integrate the function gived by Fundamental Lemma 2.1.4. However, in the proof equation (2.5) needed the ball setting as differential relation. In the new convex setting the same result holds thanks to the following lemma.

Lemma 2.1.5. *Let us take $C \subseteq \mathbb{R}^d$ a compact convex set and a function $u : [a,b] \to C$. Then, also the mean is valued on that set, i.e.*

$$\frac{1}{b-a}\int_a^b u(s)\ ds\ \in C.$$

Proof. Let us suppose by contradiction that $x_0 := \frac{1}{b-a}\int_a^b u(s)ds$ is a point that lies outside C. Then thanks to *Hyperplane separation theorem* there exist $\delta_1 < \delta_2$ and a linear function $f : \mathbb{R}^d \to \mathbb{R}$ such that

$$f(x) \leq \delta_1 < \delta_2 = f(x_0)\ \ \forall x \in C,$$

as Figure 2.4 illustrates. On the other hand, by linearity of f, it turns out that

$$\delta_2 = f\left(\frac{1}{b-a}\int_a^b u(s)\ ds\right) = \frac{1}{b-a}\int_a^b f(u(s))\ ds \leq \delta_1,$$

from which we obtain that $\delta_2 \leq \delta_1$ that is a contradiction. $\square$

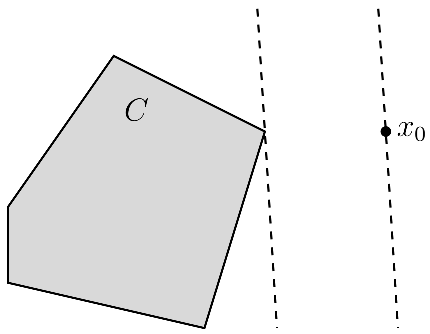

Figure 2.4: Hyperplane separation theorem between a compact convex set and a point.

In conclusion, we can state the general convex setting discussed in the notes by Borrelli (see [1]).

Theorem 2.1.4. *Let $\mathcal{R} \subset \mathbb{R}^d$ be a path-connected subset (= our differential relation) and $u : [0,1] \to \mathbb{R}^d$ of class C^1 be such that*

$$\forall t \in [0,1]\ \ u'(t) \in IntConv(\mathcal{R}),$$

where $IntConv(\mathcal{R})$ denotes the interior of the convex hull of $\mathcal{R}$.
Then, for any $\epsilon > 0$, there exists $u_\epsilon : [0,1] \to \mathbb{R}^d$ of class C^1 such that

(i) $u_\epsilon'(t) \in \mathcal{R}, \quad \forall t \in [0,1]$;

(ii) $\|u_\epsilon - u\|_{C^0} \leq \epsilon$.

2.2 Residuality

In this section, we would like to connect the problem of curves discussed so far with the procedure about Baire category theorem presented in the first chapter. In particular, until now we have considered curves of class C^1 and thanks to convex integration we proved that the set of curves with derivative valued in the boundary of the unit ball of $\mathbb{R}^d$ is C^0-dense in the set of curves with derivative valued in the interior part of the ball. Now, we would like to obtain something more and, in particular, to reach residuality according to definitions in a Baire space.

To achieve this goal, we need to consider the weak definition of derivative and we must move from C^1 space to Sobolev space H^2, i.e. the set of L^2 function with a L^2 weak derivative. More specifically, we equip H^2 with the strong-weak topology according to which taking $\{u_n\} \subset H^2$ it follows that $u_n \xrightarrow{SW} u$ if and only if $u_n \xrightarrow{C^0} u$ and $u_n' \overset{L^2}{\rightharpoonup} u'$. Moreover, it is possible to prove that H^2 with a limitation of weak derivative norm, provided by the strong-weak topology, is a complete metric space, and so a Baire space. We need to take a restriction for the weak derivative in order to obtain metrizability, as we discussed in the previous chapter (see Theorem 1.2.1). In particular, the strong-weak topology is induced by the sum of the C^0 metric on functions and L^2 weak metric on weak derivatives, which exists again thanks to Theorem 1.2.1. This is the argument of the following preliminary theorem.

Theorem 2.2.1. *Let us consider the set of functions*

$$H^2_r([0,1],\mathbb{R}^d) := \{u : [0,1] \longrightarrow \mathbb{R}^d \ : \ u \in H^2, \quad \|u'\|_{L^2} \le r\},$$

for a certain $r > 0$. Let us provide $H^2_r([0,1],\mathbb{R}^d)$ by the strong-weak metric.
Then, it is a complete metric space.

Proof. Thanks to Theorem 1.2.1 discussed in the previous chapter, there exists a distance d' on $B_{L^2}(0,r)$ that induces the weak convergence (for derivative functions). Then, the strong-weak metric on $H^2_r([0,1],\mathbb{R}^d)$ is defined as follows:

$$d(u,v) := \|u-v\|_{C^0} + d'(u',v'), \qquad \forall u,v \in H^2_r([0,1],\mathbb{R}^d).$$

Now, let us take $\{u_n\} \subset H^2_r([0,1],\mathbb{R}^d)$ a Cauchy sequence. By definition of distance d, it turns out that also $\{u_n\}$ is a Cauchy sequence with respect to the uniform metric and $\{u_n'\}$ is a Cauchy sequence with respect to the weak L^2 metric. Since both latter metrics are complete, there exist $u, v \in L^2([0,1],\mathbb{R}^d)$ such that

$$u_n \xrightarrow{C^0} u \quad \textit{and} \quad u_n' \overset{L^2}{\rightharpoonup} v.$$

Now it remains to show that such v is actually the weak derivative of u and its L^2 norm is $\le r$. Indeed, about the first condition, for any $n \in \mathbb{N}$, we have that

$$\int_0^1 u_n(x)\cdot\varphi'(x)\,dx = -\int_0^1 u_n'(x)\cdot\varphi(x)\,dx, \qquad \forall\varphi(x) \in C_0^\infty([0,1]),$$

where the product in the previous equation represents the usual scalar product on $\mathbb{R}^d$. Therefore, we can pass to the limit in the previous equation and we obtain

$$\int_0^1 u(x) \cdot \varphi'(x)\ dx = -\int_0^1 v(x) \cdot \varphi(x)\ dx, \qquad \forall \varphi(x) \in C_0^\infty([0,1]),$$

where the left hand side convergence follows by uniform convergence of u_n, while the right hand side follows by definition of weak convergence of u_n'. Therefore $v = u'$.

About the last condition, thanks to lower semicontinuity of the norm (see Lemma 1.2.2), since $u_n' \rightharpoonup u'$, therefore

$$\|u'\|_{L^2} \leq \liminf_{n\to\infty} \|u_n'\|_{L^2} \leq r,$$

and we have concluded. □

From the previous Theorem, an useful corollary follows.

Corollary 2.2.1. *Let us consider the set*

$$V := \{u : [0,1] \longrightarrow \mathbb{R}^d \ : \ u \in H^2, \quad \|u'(t)\|_{\mathbb{R}^d} \leq 1 \ \ a.e.\} \tag{2.6}$$

equipped with the strong-weak metric.
Then, it is a complete metric space.

Proof. First of all, we can notice that the condition $\|u'(t)\|_{\mathbb{R}^d} \leq 1$ *a.e.* implies that

$$\|u'\|_{L^2} = \int_0^1 \|u'(t)\|_{\mathbb{R}^d}\ dt \leq 1.$$

Therefore $u'(t) \in B_{L^2}(0,1)$ which implies that $V \subset H_1^2([0,1],\mathbb{R}^d)$ and, thanks to previous lemma, it is sufficient to show that V is closed. In particular, let $\{u_n\} \subseteq V$ be a sequence such that $u_n \xrightarrow{SW} u$ for a certain $u \in H^2$, then we have to prove that $u \in V$, namely $\|u'(t)\|_{\mathbb{R}^d} \leq 1$ a.e. But, by hypothesis $u_n' \rightharpoonup u'$ and $\|u_n'(t)\|_{\mathbb{R}^d} \leq 1$ a.e. for each $n \in \mathbb{N}$. Moreover, thanks to weak convergence, we know that

$$\int_0^1 u_n'(s) \cdot g(s)\ ds \longrightarrow \int_0^1 u'(s) \cdot g(s)\ ds, \tag{2.7}$$

for any $g \in L^2([0,1])$.

Let us suppose by contradiction that there exists $A \subseteq [0,1]$ of positive measure such that $\|u'(t)\|_{\mathbb{R}^d} \geq 1 + \alpha$ with $t \in A$ and a certain $\alpha > 0$. Let us take

$$g(s) = \frac{1}{|A|} \frac{u'(s)}{\|u'(s)\|_{\mathbb{R}^d}} \mathbb{1}_A \ \in L^2\ .$$

Therefore,

$$\int_0^1 u'(s) \cdot g(s)\ ds = \frac{1}{|A|} \int_A \|u'(s)\|_{\mathbb{R}^2}\ \ ds \geq 1 + \alpha > 1. \tag{2.8}$$

Hence due to (2.7), for n large enough, we have that also

$$\frac{1}{|A|} \int_A u_n'(s) \cdot \frac{u'(s)}{\|u'(s)\|_{\mathbb{R}^d}}\ ds = \int_0^1 u_n'(s) \cdot g(s)\ ds > 1.$$

But, for Cauchy-Schwarz inequality (inside the integral), it turns out that

$$\frac{1}{|A|} \int_A u_n'(s) \cdot \frac{u'(s)}{\|u'(s)\|_{\mathbb{R}^d}}\ ds\ \leq\ \frac{1}{|A|} \int_A \|u_n'(s)\|_{\mathbb{R}^2}\, ds\ \leq 1. \tag{2.9}$$

Therefore, since for n large enough (2.8) and (2.9) are in contradiction, we have thesis. □

Before proving the main theorem about residuality, we need an approximation result, that is the argument of the following lemma.

Lemma 2.2.1. *Let us consider V according to definition in* (2.6), *equipped with the distance d inducing the strong-weak topology (that exists thanks to Theorem 2.2.1). Let us take any $u \in V$. Then, for any $\epsilon > 0$, there exists $u_\epsilon \in V$ continuous with piecewise constant derivative function such that*

$$d(u_\epsilon, u) \leq \epsilon.$$

Proof. Let us fix a certain $\epsilon > 0$. Since u is a Sobolev function, then it is a continuous function. Moreover, it is also uniformly continuous. Therefore there exists $\delta > 0$ such that for any $t, s \in [0, 1]$ with $|t - s| \leq \delta$ it turns out that $\|u(t) - u(s)\|_{\mathbb{R}^d} \leq \frac{\epsilon}{2}$. Let us consider a uniform partition of the interval $[0, 1]$ of size equal to $\frac{1}{n}$ with n large enough in order to satisfy $\frac{1}{n} \leq \delta$. Now, it is possible to define a sequence of piecewise linear continuous functions $u_n(t)$ such that

$$u_n \left(\frac{i}{n}\right) := u \left(\frac{i}{n}\right) \qquad \forall i = 0, 1, \ldots, n\ .$$

More specifically, we can explicitly construct such u_n by connecting $u\left(\frac{i}{n}\right)$ to $u\left(\frac{i+1}{n}\right)$ with a straight line segment for each i. In order to conclude the proof we need to show that, for n large enough, the function u_n is sufficiently close to u according to C^0 norm and the weak derivatives are sufficiently close according to weak metric.

Firstly, we can say that such sequence u_n converges uniformly to u. Indeed, for any $t \in [0, 1]$, let us consider $k \leq n$ such that $t \in [\frac{k}{n}, \frac{k+1}{n}]$. Hence, calling $t_j := \frac{j}{n}$ for any $j = 0, \ldots, n$, we have that

$$\|u_n(t) - u_n(t_k)\|_{\mathbb{R}^d} \leq \|u(t_{k+1}) - u(t_k)\|_{\mathbb{R}^d} \leq \frac{\epsilon}{2}\ ,$$

since $|t_{k+1} - t_k| \leq \delta$.

On the other hand, we can notice that, if we refer to the weak derivative of u as u', it turns out that for any $n \in \mathbb{N}$

$$\|u_n'(t)\|_{\mathbb{R}^d} = \left\| \frac{u(\frac{i+1}{n}) - u(\frac{i}{n})}{\frac{1}{n}} \right\|_{\mathbb{R}^d} = n \left\| \int_{\frac{i}{n}}^{\frac{i+1}{n}} u'(s)\ ds \right\|_{\mathbb{R}^d}$$
$$\leq n \int_{\frac{i}{n}}^{\frac{i+1}{n}} \|u'(s)\|_{\mathbb{R}^d}\ ds \leq 1,$$

which means that $u_n \in V \subset H_1^2([0,1], \mathbb{R}^d)$. Therefore, we have that $\|u_n'\|_{L^2} \leq 1$ and so the sequence $\{u_n'\} \subseteq B_{L^2}(0,1)$, that is the unit ball in L^2 space with L^2 strong metric. Therefore, due to Banach-Alaoglu theorem[2], the unit ball is compact with respect to the weak metric, and so there exists a subsequence $\{u_{n_k}\}$ such that u_{n_k} converges weakly to a certain $v : [0,1] \to \mathbb{R}^d \ \in B_{L^2}(0,1)$. Now, we should prove that $v(t)$ coincides with $u'(t)$ almost everywhere. Indeed, similarly to proof of Theorem 2.2.1, we have that

$$\int_0^1 u_{n_k}(t) \cdot \varphi'(t)\ dt = -\int_0^1 u_{n_k}'(t) \cdot \varphi(t)\ dt, \quad \forall \varphi(t) \in C_0^\infty([0,1]), \tag{2.10}$$

by definition of weak derivative of u_{n_k}, for any $k \in \mathbb{N}$. Hence, we can pass to the limit in the previous equation and we obtain

$$\int_0^1 u(t) \cdot \varphi'(t)\ dt = -\int_0^1 v(t) \cdot \varphi(t)\ dt, \quad \forall \varphi(t) \in C_0^\infty([0,1]), \tag{2.11}$$

where the left hand side convergence follows by uniform convergence of u_{n_k}, while the right hand side follows by definition of weak convergence of u_{n_k}'. Therefore $v = u'$ almost everywhere. Actually, it is possible to note that the entire sequence $\{u_n'\}$ converges weakly to u' and we do not need to extract a convergent subsequence. In fact, for any subsequence $\{u_{n_k}'\}$, thanks to weak compactness of unit ball, admits a (sub)subsequence that converges weakly to u' (see (2.10) - (2.11)), and so it is known that the whole sequence u_n' converges weakly to the same limit u'.

In conclusion, it is sufficient to take $u_\epsilon := u_n$, with n large enough in order to obtain that also $d'(u_n', u') \leq \frac{\epsilon}{2}$. Therefore, according to the strong-weak metric discussed above, it turns out that $d(u_\epsilon, u) \leq \epsilon$ and we have concluded. □

We are, now, ready to state and prove the main theorem of this section about residuality.

[2]It is possible to find statement and proof for example in Section V.5 of Conway, *A Course in Functional Analysis* (1990).

Theorem 2.2.2. *Let us consider the set*

$$V := \{u : [0,1] \longrightarrow \mathbb{R}^d \ : \ u \in H^2, \ \|u'(t)\|_{\mathbb{R}^d} \leq 1 \ \ a.e.\}.$$

Then, the set of curves with unit norm weak derivative, i.e.

$$W := \{u : [0,1] \to \mathbb{R}^d \ : \ u \in H^2, \ \|u'(t)\|_{\mathbb{R}^d} = 1 \ \ a.e.\},$$

is residual with respect to the strong-weak metric.

Proof. The proof follows the same procedure presented in the first chapter. Firstly, we remark that thanks to Corollary 2.2.1, V is a Baire space. Then, we define a family of sets $\{C_k\}$ closed and with empty interior such that $\bigcup_k C_k = V \setminus W$. Therefore, $V \setminus W$ turns out to be meager and so, thanks to Baire spaces characterization, W is a residual set. Let us define

$$C_k := \left\{ u \in V \ : \ \int_0^1 \|u'(t)\|^2_{\mathbb{R}^d} \ dt \ \leq \left(1 - \frac{1}{k}\right)\right\}.$$

It is clear that $\bigcup_k C_k = V \setminus W$ since, for any $u \in V$, we have that $\int_0^1 \|u'(t)\|^2_{\mathbb{R}^d} \, dt = 1$ if and only if $\|u'(t)\|_{\mathbb{R}^d} = 1$ almost everywhere.

- We can prove that for each k, the set C_k is closed. Indeed, fix $k \in \mathbb{N}$ and let us take $\{u_n\} \subseteq C_k$ such that $u_n \xrightarrow{SW} u$. Due to lower semicontinuity of the norm (see Lemma 1.2.2), we have that

$$\left(1 - \frac{1}{k}\right) \geq \liminf_{n\to\infty} \int_0^1 \|u_n'(t)\|^2_{\mathbb{R}^d} \, dt \geq \int_0^1 \|u'(t)\|^2_{\mathbb{R}^d} \, dt.$$

  Consequently, $u \in C_k$ and C_k is closed.

- As usual, let us suppose by contradiction that there exists an open ball $B_V(u_0, r_0) \subseteq C_{k_0}$, for certain $u_0 \in V$, $r_0 > 0$, $k_0 \geq 1$. Without loss of generality, we can suppose that u_0 is regular according to previous Lemma 2.2.1, i.e. u_0 with piecewise constant derivative function. It is possible to do this assumption up to reduce the radius r_0 of the open ball $B_V(u_0, r_0) \subseteq C_{k_0}$.

  Now, we want to use the result about convex integration discussed above in order to prove that it is possible to find a curve with unit norm derivative sufficiently close that brings to contradiction. Indeed, due to Theorem 2.1.1, it is possible to find a sequence $\{u_n\}$, with $\|u_n'(t)\|_{\mathbb{R}^d} = 1$ such that u_n converges uniformly to u_0. In particular, thanks to construction of *Step 2*, we can find a sequence of curves $\{u_n(t)\} \subset W$ such that $\|u_n - u\|_{C^0}$ tends to 0, as $n \to \infty$.

  Our goal is to show also that derivatives $\{u_n'\}$ converges weakly in L^2 to weak derivative u'. In this way, for n large enough, $d(u_n, u) \leq r_0$, but, on the other hand, u_n could not be in C_{k_0} because $\|u_n'(t)\|_{\mathbb{R}^d} = 1$ implies that also $\|u_n'(t)\|_{L^2} = 1$. This is a contradiction. To this aim, let us consider the sequence u_n'. Since $\|u_n'(t)\|_{\mathbb{R}^d} = 1$, we have that $\{u_n'\} \subseteq B_{L^2}(0,1)$, that is the unit ball in L^2 space with L^2 strong metric. Therefore, similarly to what done in the proof of previous lemma, due to Banach-Alaoglu theorem, there exists a subsequence $\{u_{n_k}\}$ such that u_{n_k} converges weakly to a certain $v \in L^2$. Now, thanks to (2.10) - (2.11) we can prove that $v(t)$ coincides with $u'(t)$ almost everywhere. Actually, as we noticed above, due to the (sub)subsequences result, the entire sequence $\{u_n'\}$ converges weakly to u' and we do not need to extract a convergent subsequence.

□

Chapter 3

Divergence-free vector fields

In this chapter, we continue our discussion about the method of *convex integration*. In particular, we present a theorem regarding divergence-free vector fields, which is interesting since it demonstrates the main idea of the method while being elementary. Previously, we introduced the convex integration procedure in order to solve the problem of approximation of curves. Here, the setting is a little bit different and we consider the weak topology, as we discussed in the last result of previous chapter. However, unlike before, we consider the case of vector fields and instead of considering weak derivatives, we refer to weak definition of divergence. Moreover, it is possible to say that the problem of divergence-free vector fields that we are going to discuss can be seen as a preliminary result compared to the context of three-dimensional incompressible Euler equations in fluid dynamics. In particular, the method of *convex integration* represent a key role in the construction of infinitely many compactly supported weak solutions to the Euler equations. We recall that such constructions are discussed in different works of De Lellis e Székelyhidi and it is related to the famous Onsager's conjecture presented in the introduction of the thesis.

3.1 3-dimensional vector fields: isotropic case

From now on let us consider $\Omega \subset \mathbb{R}^3$ an open and bounded set. One of the main results of this section is the following theorem.

Theorem 3.1.1. *There exist infinitely many $u \in L^\infty(\mathbb{R}^3, \mathbb{R}^3)$ such that*

$$\begin{cases} div\ u = 0 & in\ \mathcal{D}'(\mathbb{R}^3), \\ \|u(x)\|_{\mathbb{R}^3} = \mathbb{1}_\Omega(x) & for\ almost\ every\ x \in \mathbb{R}^3. \end{cases} \tag{3.1}$$

Remark 3.1.1. It is possible to make some remarks about the previous statement. First of all, $\|u(x)\|_{\mathbb{R}^3} = \mathbb{1}_\Omega(x)$ denotes that $\|u(x)\|_{\mathbb{R}^3} = 1$ for a.e. $x \in \Omega$ and it is null outside. Furthermore, condition *div* $u = 0$ in $\mathcal{D}'(\mathbb{R}^3)$ must be intended as the weak definition of divergence. In particular, $\mathcal{D}'(\mathbb{R}^3)$ represents the space of distributions on $\mathbb{R}^3$ and so the first condition is equivalent to ask

$$\int_{\mathbb{R}^3} u \cdot \nabla\varphi = 0 \qquad \forall \varphi \in C_0^\infty(\mathbb{R}^3).$$

Before proving the previous theorem we need a lemma.

Lemma 3.1.1. *Let $\{x_n\}$ be a sequence in a Hilbert space H.*
If $x_n \rightharpoonup x$, then

$$x_n \to x \quad \textit{if and only if} \quad \|x_n\|_H \to \|x\|_H .$$

Proof. We have that

$$\|x_n - x\|_H^2 = \|x_n\|_H^2 - 2\langle x_n, x\rangle_{L^2} + \|x\|_H^2 .$$

Therefore, since $\langle x_n, x\rangle_{L^2} \to \|x\|_H^2$ as $n \to \infty$ by hypothesis, it turns out that $\|x_n - x\|_H^2 \to \|x_n\|_H^2 - \|x\|_H^2$ as $n \to \infty$, and so

$$x_n \to x \quad \Leftrightarrow \quad \|x_n\|_H \to \|x\|_H .$$

□

We can now prove the main theorem.

Proof of Theorem 3.1.1. The proof is very instructive and we can divide it into different steps.

Firstly, let us set the functional setup. Let us define

$$X_0 := \{u \in C_0^\infty(\Omega) \ : \ \textit{div } u = 0 \textit{ and } \|u(x)\|_{\mathbb{R}^3} < 1 \ \forall x \in \Omega\}. \tag{3.2}$$

Then, let us consider

$$X := \textit{weak closure of } X_0 \textit{ in } L^2(\Omega). \tag{3.3}$$

We can note that X is bounded in $L^2(\Omega)$; in particular $X \subset B\left(0, \sqrt{|\Omega|}\right)$ since by lower semicontinuity of the norm with respect to weak convergence (see Lemma 1.2.2), if $u_n \rightharpoonup u$ in L^2, then

$$\|u\|_{L^2} \le \liminf_{n\to\infty} \|u_n\|_{L^2} = \liminf_{n\to\infty} \left(\int_\Omega \|u_n(x)\|_{\mathbb{R}^3}^2 \ dx\right)^{\frac{1}{2}} \le \sqrt{|\Omega|} .$$

Hence, due to Theorem 1.2.1, $B\left(0, \sqrt{|\Omega|}\right)$ equipped with the weak L^2 topology is metrizable and X is its closed subset. Let d denote the corresponding metric and we know that X with distance d is a complete metric space (and so a Baire space).

Now, let $I : X \to \mathbb{R}$ be the functional

$$I(u) := \int_{\Omega} (1 - \|u(x)\|_{\mathbb{R}^3}^2)\ dx\ . \tag{3.4}$$

It is possible to notice that if for a certain $u \in X$ we have that $I(u) = 0$, then it is a solution to problem (3.1). Our goal from now on is to show that

$$\{I(u) = 0\}\ \textit{is dense in } X.$$

From the latter result, not only follows directly the existence of many solutions to problem (3.1) (our thesis), but also that the set of such solutions is dense in X.

Step 1: Given $u \in X_0$ and $\tilde{\Omega} \Subset \Omega$, it is possible to construct a sequence $\{u_n\}$ such that

(i) $u_n \in X_0$ for n large enough;

(ii) $u_n \rightharpoonup u$ in $L^2(\Omega)$;

(iii) there exists a constant $M_u > 0$ such that $\liminf_{n\to\infty} \|u_n\|_{L^2}^2 \geq \|u\|_{L^2}^2 + M_u$.

It is possible to note that, due to Lemma 3.1.1, (iii) gives in particular that u_n cannot converge strongly in L^2 to u.

In order to construct u_n, let us take $\xi, \eta \in \mathbb{R}^3$ such that $\|\xi\|_{\mathbb{R}^3} = \|\eta\|_{\mathbb{R}^3} = 1$ and $\xi \perp \eta$. For example, we can consider $\xi := (1, 0, 0)$ and $\eta := (0, 1, 0)$. Then, let us take $\phi \in C_0^\infty(\Omega, [0, 1])$ such that $\phi = 1$ on $\tilde{\Omega}$. Now, calling $x = (x_1, x_2, x_3)$ a generic vector in $\mathbb{R}^3$, let us define

$$\begin{aligned} v_n(x) &:= \frac{\eta}{2n}(1 - \|u(x)\|_{\mathbb{R}^3}^2)\phi(x)\sin(nx \cdot \xi) \\ &= \left(0, \tfrac{1}{2n}(1 - \|u(x)\|_{\mathbb{R}^3}^2)\phi(x)\sin(nx_1), 0\right)\ . \end{aligned} \tag{3.5}$$

We will show that

$$u_n(x) := u(x) + \textit{curl}\ v_n(x) \tag{3.6}$$

satisfies previous conditions. The main idea is to add some high oscillations to u on $\tilde{\Omega}$ through definitions of *curl*, since we recall the identity $\textit{div curl} = 0$. From the latter remark, we have that $\textit{div}\ u_n = 0$ for any $n \in \mathbb{N}$. In order to obtain the first condition (i), we need to show that $\|u_n(x)\|_{\mathbb{R}^3} < 1$ for n sufficiently large. Indeed, firstly we can notice that trivially $\|u_n(x)\|_{\mathbb{R}^3} = \|u(x)\|_{\mathbb{R}^3} < 1$ outside *supp* ϕ. Moreover, let $\delta > 0$ be such that $\|u_n(x)\|_{\mathbb{R}^3} \leq 1 - \delta$ on *supp* ϕ, and note that

$$\textit{curl}\ v_n(x) = \left(0, 0, \tfrac{1}{2}(1 - \|u(x)\|_{\mathbb{R}^3}^2)\phi(x)\cos(nx_1)\right) + O\left(\tfrac{1}{n}\right). \tag{3.7}$$

The latter expression follows directly by some algebra, after noticing that the only term without the factor $\frac{1}{n}$ appears deriving $\sin(nx_1)$. Thus, taking a certain constant $C > 0$, on *supp* ϕ we have that

$$\begin{aligned}\|u_n(x)\|_{\mathbb{R}^3} &\leq \|u(x)\|_{\mathbb{R}^3} + \tfrac{1}{2}(1 - \|u(x)\|_{\mathbb{R}^3})(1 + \|u(x)\|_{\mathbb{R}^3}) + \frac{C}{n} \\ &\leq \|u(x)\|_{\mathbb{R}^3} + (1 - \|u(x)\|_{\mathbb{R}^3})(1 - \tfrac{\delta}{2}) + \frac{C}{n} \\ &= 1 + \tfrac{\delta}{2}(\|u(x)\|_{\mathbb{R}^3} - 1) + \frac{C}{n} \\ &\leq 1 - \frac{\delta^2}{2} + \frac{C}{n},\end{aligned}$$

and so for n large enough it follows that $\|u_n(x)\|_{\mathbb{R}^3} < 1$.

The second condition $u_n \rightharpoonup u$ in L^2 follows since it is known that $cos(nx_1) \rightharpoonup 0$ in L^2 as $n \to \infty$ and so due to equation (3.7), we have that $curl\ v_n \rightharpoonup 0$ (the first part thanks to previous remark and the second part tends strongly to 0).

The third condition follows by a direct calculation. Similarly as the previous point, we have that $\langle curl\ v_n, u\rangle_{L^2} \to 0$ as $n \to \infty$ and so

$$\begin{aligned}\|u_n\|^2_{L^2} &= \|u\|^2_{L^2} + 2\langle curl\ v_n, u\rangle_{L^2} + \|curl\ v_n\|^2_{L^2} \\ &= \|u\|^2_{L^2} + \frac{1}{4}\int_\Omega (1 - \|u(x)\|^2_{\mathbb{R}^3})^2 \phi(x)^2 \cos(nx_1)^2\ dx + h_n,\end{aligned}$$

where h_n denotes the other terms that converge to 0 as $n \to \infty$. Now, due to the identity $\cos\alpha^2 = \frac{1+cos(2\alpha)}{2}$, it turns out that

$$\int_\Omega (1-\|u(x)\|^2_{\mathbb{R}^3})^2\phi(x)^2\cos(nx_1)^2 dx = \frac{1}{2}\left(\int_{\tilde\Omega}(1 - \|u(x)\|^2_{\mathbb{R}^3})^2 dx + \int_{\tilde\Omega}(1 - \|u(x)\|^2_{\mathbb{R}^3})^2\cos(2nx_1)dx\right),$$

where the last integral term tends to 0 thanks to weakly L^2 convergence of $\cos(2nx_1)$. Therefore, including the latter term in h_n, we have that

$$\begin{aligned}\|u_n\|^2_{L^2} &= \|u\|^2_{L^2} + \frac{1}{8}\int_{\tilde\Omega}(1 - \|u(x)\|^2_{\mathbb{R}^3})^2\ dx + h_n \\ &\geq \|u\|^2_{L^2} + \frac{1}{8|\tilde\Omega|}\left(\int_{\tilde\Omega}(1 - \|u(x)\|^2_{\mathbb{R}^3})\ dx\right)^2 + h_n\end{aligned}$$

thanks to Cauchy-Schwarz inequality. Now, taking $\liminf_{n\to\infty}$ and just defining $M_u := I_{\tilde\Omega}(u)^2/8|\tilde\Omega|$, where

$$I_{\tilde\Omega}(u) := \int_{\tilde\Omega}(1 - \|u(x)\|^2_{\mathbb{R}^3})\ dx,$$

we obtain condition (iii).

Step 2: Let us consider the set

$$S := \{u \in X \ : \ \text{any sequence weakly convergent to } u \text{ in } L^2(\Omega) \text{ converges strongly}\}.$$

The set S is called the set of *stable* elements of X, which means that such elements cannot be weakly perturbed. We claim that S is dense in X with respect to the weak metric d.

Letting $J : X \to \mathbb{R}$ such that $J(u) := \|u\|_{L^2}$, it is possible to show that J is a pointwise limit of the sequence

$$J_k(u) := \|\rho_{\epsilon_k} * u\|_{L^2},$$

where we extended u by zero outside Ω, ρ is a standard mollyfing kernel such that

$$\rho_\epsilon * u(x) = \frac{1}{\epsilon^3}\int_{\mathbb{R}^3} u(z)\rho\left(\frac{z-x}{\epsilon}\right)\, dz,$$

and $\epsilon_k \to 0$ as $k \to \infty$. Indeed, it is known that if $u \in L^2$ then $\rho_\epsilon * u \to u$ in L^2 as $\epsilon \to 0$. Therefore, fix any $u \in L^2$, since the L^2 norm is continuous we have that as $k \to \infty$, $J_k(u) \to J(u)$.

Now, we can say that each J_k is a continuous function according to the weak metric d. Indeed, if $w_n \rightharpoonup w$, then by definition of weak convergence, since $\rho \in C_0^\infty$, it turns out that $\rho_{\epsilon_k} * w_n(x) \longrightarrow \rho_{\epsilon_k} * w(x)$ pointwise as $n \to \infty$. Since they are bounded, due to dominated convergence we have that the L^2 norm is also convergent. Thus, we obtain continuity of J_k for any k.

In conclusion, we can note that $S \supset \{u \ : \ J(u) \textit{ is continuous}\}$, where continuity is with respect to the weak metric. Indeed, if $u_n \rightharpoonup u$ in L^2 and J is continuous in u with respect to the weak metric, then $\|u_n\|_{L^2} \to \|u\|_{L^2}$. But, due to Lemma 3.1.1, $u_n \to u$ strongly in L^2 and so $u \in S$. Hence, the thesis follows since due to Proposition 1.1.2 the set of points where the pointwise limit function is continuous is dense.

Step 3: We can conclude showing that $S \subset \{I(u) = 0\}$. Let us suppose by contradiction that there exists $u \in S$ such that $I(u) > 0$. Then, there exists a certain $\tilde{\Omega} \Subset \Omega$ such that $I_{\tilde{\Omega}}(u) > 0$. Since $u \in X$, u is a weak limit of a certain sequence $\{u_n\} \subset X_0$. Therefore for each $n \in \mathbb{N}$ it is possible to apply Step 1 for u_n. In particular, we find another sequence u_{n_k} that satisfies conditions $(i) - (iii)$. Therefore, thanks to a diagonal process we can define a sequence u'_n such that for each $n \in \mathbb{N}$

$$d(u'_n, u_n) \leq \tfrac{1}{n} \quad \textit{and} \quad \liminf_{n\to\infty} \|u'_n\|^2_{L^2} \geq \|u_n\|^2_{L^2} + M_{u_n} - \tfrac{1}{n}. \tag{3.8}$$

We can note that also u'_n converges weakly L^2 to u and since $u \in S$, it converges also strongly. Hence, taking the limit for $n \to \infty$ in the last inequality (3.8) and since $M_{u_n} \to M_u$ as $n \to \infty$, we obtain that

$$\|u\|_{L^2}^2 \geq \|u\|_{L^2}^2 + M_u > \|u\|_{L^2}^2 ,$$

that is a contradiction.

□

Remark 3.1.2. In the previous Theorem, we proved that $\{I(u) = 0\}$ is dense in X, which means that the set of possible solution to problem (3.1) is dense in X. Actually, in the proof we obtained more than density, we obtained residuality, according to definition discussed in the first chapter since X is a Baire space. This is a simple consequence of Step 2 where we used Proposition 1.1.2. In such Proposition we proved that the set of continuity points for the pointwise limit of continuous function is indeed residual.

The method presented in previous proof rely on the same main principle: solutions are constructed from suitable functions that are usually called “subsolutions” by adding an highly oscillatory perturbation. In a sort of way, it is possible to see subsolutions as relaxation of the original problem. This idea is the same of what we will see in order to prove the Nash-Kuiper theorem. In that case the “subsolutions” will be the strictly short immersions and we are able to construct the C^1 isometric immersions by adding highly oscillatory term.

Furthermore, the previous procedure discussed to prove the existence of many solutions to problem of divergence-free vector fields in dimension 3 can be considered as a general machinery in order to generate weak solutions to suitable conditions. Actually, it represents the same idea that De Lellis et al. used for the Euler equations, with of course several technical details and different settings.

3.2 2-dimensional vector fields

In this section, we can show how the same method can be applied to the problem in dimension 2. The idea of the proof is exactly the same of 3-dimensional vector fields; however, we need to modify the previous Step 1 in order to construct the approximating sequence. Previously, we used the known identity *div curl* $= 0$, while in 2-dimensional case we can use the fact that the gradient of a function rotated of 90 degrees has null divergence. In particular, similarly to Theorem 3.1.1, it holds the following result.

Theorem 3.2.1. *Let us consider $\Omega \subset \mathbb{R}^2$ an open and bounded set. There exist infinitely many $u \in L^\infty(\mathbb{R}^2, \mathbb{R}^2)$ such that*

$$\begin{cases} div\ u = 0 & in\ \mathcal{D}'(\mathbb{R}^2), \\ \|u(x)\|_{\mathbb{R}^2} = \mathbb{1}_\Omega(x) & for\ almost\ every\ x \in \mathbb{R}^2. \end{cases} \tag{3.9}$$

Proof. The proof follows the same construction of Theorem 3.1. Firstly, we define

$$X_0 := \{u \in C_0^\infty(\Omega) \ : \ div\ u = 0 \ and \ \|u(x)\|_{\mathbb{R}^2} < 1 \ \forall x \in \Omega\}. \tag{3.10}$$

Then, as before, we consider

$$X := weak\ closure\ of\ X_0\ in\ L^2(\Omega). \tag{3.11}$$

Due to L^2 boundedness of X and Theorem 1.2.1, X equipped with the weak L^2 topology is metrizable and let d denote the corresponding metric.

Now, let $I : X \to \mathbb{R}$ be the functional

$$I(u) := \int_\Omega (1 - \|u(x)\|^2_{\mathbb{R}^2})\ dx\ . \tag{3.12}$$

The thesis follows showing that

$$\{I(u) = 0\}\ is\ dense\ in\ X.$$

The key point is again to prove that given $u \in X_0$ and $\tilde{\Omega} \Subset \Omega$, it is possible to construct a sequence $\{u_n\}$ such that

(i) $u_n \in X_0$ for n large enough;

(ii) $u_n \rightharpoonup u$ in $L^2(\Omega)$;

(iii) there exists a constant $M_u > 0$ such that $\liminf_{n\to\infty} \|u_n\|^2_{L^2} \geq \|u\|^2_{L^2} + M_u$.

In order to construct such sequence, we can replace the notion of *curl* with the gradient of function rotated of 90 degrees. More specifically, if we consider any C^1 function $F : \mathbb{R}^2 \to \mathbb{R}$ and we define $\nabla_R F := (\partial_y F, -\partial_x F)$, it turns out that

$$div(\nabla_R F) = 0.$$

Therefore, taking $\phi \in C_0^\infty(\Omega, [0,1])$ such that $\phi = 1$ on $\tilde{\Omega}$ and calling $x = (x_1, x_2)$ a generic vector in $\mathbb{R}^2$, it is possible to define for any $n \in \mathbb{N}$ the function

$$V_n := \frac{1}{2n}(1 - \|u(x)\|^2_{\mathbb{R}^2})\phi(x)\sin(nx_2).$$

Now, we can show that the sequence

$$u_n := u + \nabla_R V_n$$

satisfies conditions $(i) - (iii)$. Indeed, firstly we can note that

$$\nabla_R V_n = \left(\tfrac{1}{2}(1 - \|u(x)\|^2_{\mathbb{R}^2})\phi(x)\cos(nx_2), 0\right) + O\left(\tfrac{1}{n}\right).$$

Therefore it is clear that $u_n \rightharpoonup u$ in L^2 since it is known that $cos(nx_2) \rightharpoonup 0$ in L^2 as $n \to \infty$.

In order to see that for n large enough $u_n \in X_0$, as we did the 3-dimensional case, we can say that outside *supp* ϕ it is trivial. On the other hand, on *supp* ϕ, there exists $\delta > 0$ such that $\|u_n(x)\|_{\mathbb{R}^2} \leq 1 - \delta$ and, taking a certain constant $C > 0$, as we did above, we have that

$$\begin{aligned}
\|u_n(x)\|_{\mathbb{R}^2} &\leq \|u(x)\|_{\mathbb{R}^2} + \tfrac{1}{2}(1 - \|u(x)\|_{\mathbb{R}^2})(1 + \|u(x)\|_{\mathbb{R}^2}) + \frac{C}{n} \\
&\leq \|u(x)\|_{\mathbb{R}^2} + (1 - \|u(x)\|_{\mathbb{R}^2})(1 - \tfrac{\delta}{2}) + \frac{C}{n} \\
&= 1 + \tfrac{\delta}{2}(\|u(x)\|_{\mathbb{R}^2} - 1) + \frac{C}{n} \\
&\leq 1 - \frac{\delta^2}{2} + \frac{C}{n},
\end{aligned}$$

and so for n large enough it follows that $\|u_n(x)\|_{\mathbb{R}^2} < 1$ and $u_n \in X_0$.

Finally, also condition (iii) follows directly from the same computations already done in previous case, taking $M_u := I_{\tilde{\Omega}}(u)^2/8|\tilde{\Omega}|$ and noting that $\langle \nabla_R V_n, u \rangle_{L^2} \to 0$ thanks to weakly L^2 convergence.

From now on the proof is exactly the same of 3-dimensional case. □

Remark 3.2.1. It is possible to note that a similar result of residuality as discussed in Remark 3.1.2 can be obtained also in the latter case of 2-dimensional vector fields. It follows directly from the proof.

3.3 3-dimensional vector fields: anisotropic case

After previous discussion about 2-dimensional vector fields, we can come back to the 3-dimensional case and consider a slight more general result. Let us consider again $\Omega \subset \mathbb{R}^3$ an open and bounded set and also $K \subset \mathbb{R}^3$ a compact set such that $0 \in IntConv(K)$. Then, we can prove the following result.

Theorem 3.3.1. *There exist infinitely many $u \in L^\infty(\mathbb{R}^3, \mathbb{R}^3)$ such that*

$$\begin{cases} div\ u = 0 & in\ \mathcal{D}'(\Omega), \\ u(x) \in K & for\ almost\ every\ x \in \Omega. \end{cases} \tag{3.13}$$

Remark 3.3.1. The previous case discussed in Theorem 3.1.1 corresponds to take $K = \mathcal{S}^2$. However, one particular case that is included in this new setting could be to take K as a finite set of points. Therefore, Theorem 3.3.1 tell us that there exists a certain vector field valued only in this finite set with free divergence everywhere. We can make an example in 2 dimensions.

Example 3.3.1. Let us consider $K = \{(1,0),(-1,0),(0,1),(0,-1)\}$ a set of four vertices. Then, $IntConv(K)$ is the closed unit ball according to the L^1 norm for vectors in $\mathbb{R}^2$. While, let us denote by Ω the unit square (open unit ball according to infinity norm). If we define the field $u : \Omega \to K$ according to the following Figure 3.1, we can show that it satisfies conditions (3.13).

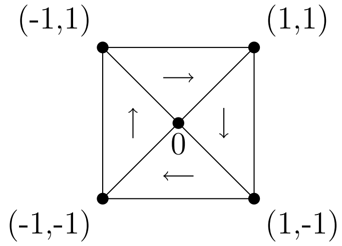

Figure 3.1: Definition of the vector field u.

Indeed, of course $u(x) \in K$ a.e. and we just need to prove that $div\ u = 0$ in the sense of distributions. Then, let $\varphi : \Omega \to \mathbb{R}$ any $C_0^\infty(\Omega)$ and we need to show that

$$\iint_\Omega u(x,y) \cdot \nabla\varphi(x,y)\ dxdy \ = \ 0\ . \tag{3.14}$$

Let us call

$$u(x,y) = (u_1(x,y), u_2(x,y)) \quad and \quad \nabla\varphi(x,y) = (\varphi_x(x,y), \varphi_y(x,y)),$$

then, calling Ω_1 the triangle in the top of the square and enumerate the others clockwise, it turns out that

$$\iint_\Omega u(x,y) \cdot \nabla\varphi(x,y)\ dxdy \ = \iint_\Omega u_1(x,y)\varphi_x(x,y)\ dxdy \ + \iint_\Omega u_2(x,y)\varphi_y(x,y)\ dxdy.$$

Now, we have that

$$\begin{aligned}\iint_\Omega u_1(x,y)\varphi_x(x,y)\ dxdy &= \iint_{\Omega_1} \varphi_x(x,y)\ dxdy - \iint_{\Omega_3} \varphi_x(x,y)\ dxdy \\ &= \int_0^1 dy \int_{-y}^{y} \varphi_x(x,y)dy - \int_{-1}^{0} dx \int_{y}^{-y} \varphi_x(x,y)dx \\ &= \int_0^1 \varphi(y,y) - \varphi(-y,y)dy - \int_{-1}^{0} \varphi(-y,y) - \varphi(y,y)dy.\end{aligned}$$

Similarly, considering the second term of the previous sum, it follows that

$$\begin{aligned}\iint_\Omega u_2(x,y)\varphi_y(x,y)\ dxdy &= \iint_{\Omega_4} \varphi_y(x,y)\ dxdy - \iint_{\Omega_2} \varphi_y(x,y)\ dxdy \\ &= \int_{-1}^{0} \varphi(x,-x) - \varphi(x,x)dx - \int_0^1 \varphi(x,x) - \varphi(x,-x)dx.\end{aligned}$$

Summing up the two parts obtained, thanks to a change of variable, each term vanishes and we end up with the condition (3.14), as we wanted.

Before proceeding with the proof of Theorem 3.3.1, we state and show a geometric lemma that will be fundamental in the proof of the Theorem.

Lemma 3.3.1 (Geometric lemma). *Let $K \subset \mathbb{R}^d$ be compact. For any $z \in IntConv(K)$ there exists $\hat{z} \in \mathbb{R}^d$ and a constant $c > 0$ such that*

(i) $z + t\hat{z} \ \in \ IntConv(K) \quad for\ all\ t \in [-1, 1]$;

(ii) $\|\hat{z}\|_{\mathbb{R}^d} \ \geq \ \frac{1}{cd} dist(z, K)$.

Proof. Due to Caratheodory's theorem, since $z \in IntConv(K)$, there exist $z_1, \dots, z_{d+1} \in K$ such that

$$z = \sum_{i=1}^{d+1} \lambda_i z_i,$$

with $0 < \lambda_i < 1$ for any i and $\sum_i \lambda_i = 1$. We can also assume that $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{d+1}$. Then, we have that

$$z - z_1 = \sum_{i=2}^{d+1} \lambda_i (z_i - z_1) \ .$$

Hence,

$$dist(z, K) \leq \|z - z_1\|_{\mathbb{R}^d} \leq d \max_{i=2,\dots,d+1} \lambda_i \|z_i - z_1\|_{\mathbb{R}^d} \, .$$

Now, if we choose $\hat{z} := \frac{1}{c}\lambda_j(z_1 - z_j)$ for j that gives the maximum and for a certain constant $c > 0$, we obtain that

$$dist(z, K) \leq cd \|\hat{z}\|_{\mathbb{R}^d} \, ,$$

that gives condition (ii). Now, we should see that for any $t \in [-1, 1]$ and for a suitable small constant c, it turns out that $z + t\hat{z} \ \in \ IntConv(K)$. Indeed, for any $j = 2, \dots, d+1$ and for $t \in [-1, 1]$,

$$z + \frac{t}{c}\lambda_j(z_1 - z_j) = \sum_{i \neq 1, j} \lambda_i z_i + (\lambda_1 + \frac{t}{c}\lambda_j)z_1 + (\lambda_j - \frac{t}{c}\lambda_j)z_j. \tag{3.15}$$

that is also a strictly convex combination of $z_1, \dots, z_{d+1}$, since in the sum of coefficients we are summing and subtracting the same quantity $\frac{t}{c}\lambda_j$. Everything is fine if $\lambda_1 + \frac{t}{c}\lambda_j$ and $\lambda_j - \frac{t}{c}\lambda_j$ are both < 1 for any $t \in [-1, 1]$. But, in order to satisfy such condition, it is sufficient to take a constant $c > 0$ small enough. □

We are now ready to prove the main Theorem.

Proof of Theorem 3.3.1. The proof of Theorem 3.3.1 is again an application of convex integration method as in Theorem 3.1.1. More specifically, we define

$$X_0 := \{u \in C_0^\infty(\Omega) \ : \ div\ u = 0 \ and\ u(x) \in IntConv(K)\ a.e.\ in\ \Omega\}.$$

Firstly, we can note that it is not empty since $0 \in IntConv(K)$ by hypothesis. Then, as before, we consider

$$X := weak\ closure\ of\ X_0\ in\ L^2(\Omega).$$

The proof follows the same steps of the isotropic case with some modifications.

First of all, we define the functional $I : X \to \mathbb{R}$ according to

$$I(u) := \int_\Omega dist(u(x), K)^2\ dx.$$

The thesis follows by proving the residuality of $\{I(u) = 0\}$ in X. Now, fixed $u \in X_0$, the only step to change from previous proof is Step 1, in order to construct the suitable sequence $\{u_n\}$ such that

(i) $u_n \in X_0$ for n large enough;

(ii) $u_n \rightharpoonup u$ in $L^2(\Omega)$;

(iii) there exists a constant $M_u > 0$ such that $\liminf_{n\to\infty} \|u_n\|^2_{L^2} \geq \|u\|^2_{L^2} + M_u$.

Our aim is to consider again highly perturbations of the type (3.5) -(3.6) in order to define $\{u_n\}$. However, this time we want to add oscillations to suitable directions that might depend on x. Therefore, the idea is to define

$$v_n(x) := \frac{\eta(x)}{n}\phi(x)\sin(nx \cdot \xi(x)),$$

where we let $\eta(x)$ and $\xi(x)$ vary with respect to $x \in \Omega$. In order to choose them appropriately we can use previous lemma.

Let us fix for a moment $x_0 \in \Omega$ and $r > 0$ such that $B_r(x_0) \subset \Omega$. Let us take $\phi \in C_0^\infty(B_r(x_0), [0,1])$ a cut-off function such that $\phi \equiv 1$ on $B_{r/2}(x_0)$. Thanks to geometric lemma above, there exist $\eta, \xi \in \mathbb{R}^3$ such that if

$$v_n(x) := \frac{\eta}{n}\phi(x)\sin(nx \cdot \xi),$$

then

$$\begin{aligned} u(x_0) + curl\ v_n(x) \in IntConv(K) \quad &\text{for } n \text{ large enough and } \forall x \in \Omega \\ \|\xi \times \eta\|_{\mathbb{R}^3} \geq C\ dist(u(x_0), K), \end{aligned}$$

for a certain constant $C > 0$. This is possible, since we applied previous lemma with $z = u(x_0)$, finding $\hat{z} \in \mathbb{R}^3$, rewriting $\hat{z} = \xi \times \eta$ and noticing that by simple computations

$$curl\ v_n(x) = (\xi \times \eta)\phi(x)\cos(nx \cdot \xi) + O\left(\frac{1}{n}\right),$$

and $|\phi(x)\cos(nx\cdot\xi)| \leq 1$. Now, since u is continuous, then we can choose an eventually smaller $r > 0$ such that

$$u(x) + curl\ v_n(x) \in IntConv(K) \quad for\ n\ large\ enough\ and\ \forall x \in \Omega$$

$$\frac{1}{|B_r(x_0)|}\int_{B_r(x_0)} \|curl\ v_n(x)\|^2_{\mathbb{R}^3}\ dx \geq C\ dist(u(x_0), K)^2,$$

for a suitable constant $C > 0$, possibly different.

Now, due to uniform continuity of u in Ω, it is possible to find a finite family of pairwise disjoints balls $B_{r_j}(x_j) \subset \Omega$, suitable vectors $\xi_j, \eta_j \in \mathbb{R}^3$ and cut-off functions $\phi_j \in C_0^\infty(B_{r_j}(x_j); [0,1])$ such that if

$$\begin{aligned} v_n(x) &= \sum_j \tfrac{\eta_j}{n}\phi_j(x)\sin(nx \cdot \xi_j) \\ u_n(x) &= u(x) + curl\ v_n(x), \end{aligned}$$

then $u_n \in X_0$ if n is large enough (it is satisfied locally thanks to what discussed so far), and

$$\begin{aligned} \int_\Omega dist(u(x), K)^2\ dx &= \sum_j \int_{B_{r_j}(x_j)} dist(u(x), K)^2\ dx \\ &\leq \sum_j \int_{B_{r_j}(x_j)} 2dist(u(x_j), K)^2\ dx \\ &\leq 2\sum_j |B_{r_j}(x_j)| dist(u(x_j), K)^2 \\ &\leq C\sum_j \int_{B_{r_j}(x_j)} \|curl\ v_n(x)\|^2_{\mathbb{R}^3}\ dx = C\int_\Omega \|curl\ v_n(x)\|^2_{\mathbb{R}^3}\ dx, \end{aligned}$$

for a certain constant $C > 0$. Therefore, since $curl\ v_n \rightharpoonup 0$ in L^2, we have that

$$\begin{aligned} \liminf_{n\to\infty} \int_\Omega \|u_n(x)\|^2_{\mathbb{R}^3}\ dx &= \int_\Omega \|u(x)\|^2_{\mathbb{R}^3}\ dx + \liminf_{n\to\infty}\int_\Omega \|curl\ v_n(x)\|^2_{\mathbb{R}^3}\ dx \\ &\geq \int_\Omega \|u(x)\|^2_{\mathbb{R}^3}\ dx + C\int_\Omega dist(u(x), K)^2\ dx, \end{aligned}$$

for another suitable constant $C > 0$. Hence, defining

$$M_u := C\int_\Omega dist(u(x), K)^2\ dx\ > 0,$$

condition (iii) is satisfied.

Finally, thanks to what done so far it is clear that $u_n \rightharpoonup u$ in $L^2(\Omega)$ and $u_n \in X_0$ for n large enough. Therefore, Step 1 is now completed and it is possible to proceed with next steps of above proof in order to conclude. $\square$

Chapter 4

The Nash-Kuiper Theorem

In this chapter we present a special case of *Nash-Kuiper Theorem* and in particular we discuss the application of the convex integration technique in the proof.

As we argued in the introduction, the *Nash-Kuiper Theorem* dates back to 1954–1955 and in a sort of way was considered an astonishing result in differential geometry, so much that it is also known as *Nash-Kuiper Paradox*. Indeed, in its general form, it states that it is possible to find many C^1 isometric immersions (or embeddings) of a smooth closed d-dimensional Riemannian manifold into $\mathbb{R}^N$ with $N \geq d+1$. Such theorem was considered quite surprising, as already discussed, since it is opposed to the previous classical rigidity result about C^2 isometric immersions.

Here, we consider a local and simplier case of *Nash-Kuiper Theorem* in order to expose the convex integration procedure in the proof in a clearer setting. Our purpose, indeed, is to isolate the key ideas and try to remark the similarities with problems discussed in previous chapters. As already mentioned, such method became fundamental in recent literature since, thanks to De Lellis and Székelyhidi, it was applied in order to construct many weak solutions to Incompressible Euler equations and to solve the note Onsager's conjecture of turbulence in fluid dynamics.

4.1 Preliminaries

Before discussing the Nash-Kuiper Theorem in detail, we need some definitions. From now on, let us call $\Omega \subseteq \mathbb{R}^d$ a bounded open set with boundary of class C^1.

Definition 4.1.1. Let us consider $\phi : \bar{\Omega} \to \mathbb{R}^N$ of class C^1. Then, we define the map $M_\phi : \bar{\Omega} \to \mathbb{R}^{d\times d}$ as

$$M_\phi(x) := J\phi(x)^T J\phi(x), \tag{4.1}$$

where, $J\phi$ represents the Jacobian matrix. We say that ϕ is an **immersion** if $M_\phi(x)$ is non singular for any $x \in \bar{\Omega}$.

Remark 4.1.1. It is possible to note that if ϕ is an immersion, then M_ϕ is symmetric and positive definite. Indeed, for any $\xi \in \mathbb{R}^d$ with $\xi \neq 0$,

$$\xi^T M_\phi \, \xi = \xi^T J\phi^T J\phi \, \xi = \|J\phi \, \xi\|^2_{\mathbb{R}^d} > 0.$$

Definition 4.1.2. Let us $\phi : \bar{\Omega} \to \mathbb{R}^N$ be an immersion. Then, we say that ϕ is **short** if it reduces the length of curves, i.e. for any rectifiable curve $\gamma : [0,1] \to \bar{\Omega}$

$$\ell(\phi \circ \gamma) \leq \ell(\gamma),$$

where ℓ represents the length of the curve with respect to the usual metric. Moreover, we say that ϕ is **isometric** if it preserves such quantity, i.e. $\ell(\phi \circ \gamma) = \ell(\gamma)$.

Remark 4.1.2. Let us consider the case in which $d = 2$ and $n = 3$. We can denote with $(u, v) \in \bar{\Omega} \subseteq \mathbb{R}^2$ the input variable and let us denote an immersion between $\bar{\Omega}$ and $\mathbb{R}^3$ with

$$\phi(u, v) := (X(u, v), Y(u, v), Z(u, v)).$$

Therefore, M_ϕ becomes

$$M_\phi = \begin{pmatrix} X_u^2 + Y_u^2 + Z_u^2 & X_u X_v + Y_u Y_v + Z_u Z_v \\ X_u X_v + Y_u Y_v + Z_u Z_v & X_v^2 + Y_v^2 + Z_v^2 \end{pmatrix}.$$

Lemma 4.1.1. *Let us take $\Omega \subset \mathbb{R}^2$ and consider $\phi : \bar{\Omega} \to \mathbb{R}^3$ an immersion. Then, ϕ is strictly short if and only if there exists a map $R : \bar{\Omega} \to \mathbb{R}^{2\times 2}$ symmetric and positive definite (i.e. $R(u, v)$ is symmetric and positive definite for any $(u, v) \in \bar{\Omega}$) such that*

$$M_\phi + R = Id \; . \tag{4.2}$$

On the other hand, ϕ is isometric if and only if $M_\phi = Id$.

Proof. We recall that being short means reducing length of curves. Then, let us take $\gamma : [0,1] \to \bar{\Omega}$ a certain rectifiable curve of class C^1 and let us denote

$$\gamma(t) := (u(t), v(t)) \quad for \; t \in [0,1].$$

Then, it follows that

$$\ell(\gamma) = \int_0^1 \|\gamma'(t)\|^2_{\mathbb{R}^2} \; dt.$$

On the other hand, we have that

$$[\phi \circ \gamma](t) = (X(u(t), v(t)), Y(u(t), v(t)), Z(u(t), v(t))) \, .$$

Hence,

$$[\phi \circ \gamma]' = (X_u u' + X_v v', Y_u u' + Y_v v', Z_u u' + Z_v v') = J\phi \cdot \gamma',$$

and
$$\ell(\phi \circ \gamma) = \int_0^1 \|[\phi \circ \gamma]'(t)\|^2_{\mathbb{R}^3} \; dt.$$
Thus, supposing that ϕ is strictly short we know that there exists a positive definite matrix R such that $M_\phi = Id - R$. Therefore,

$$\begin{aligned}\|J\phi \cdot \gamma'\|^2_{\mathbb{R}^3} &= (X_u u' + X_v v')^2 + (Y_u u' + Y_v v')^2 + (Z_u u' + Z_v v')^2 \\ &= \gamma'\,^T M_\phi \gamma' \\ &= \gamma'\,^T (Id - R)\gamma' \\ &= \|\gamma'\|^2_{\mathbb{R}^2} - \gamma'\,^T R\gamma' < \|\gamma'\|^2_{\mathbb{R}^2}\,,\end{aligned}$$

which implies that it reduces the length of γ.

On the other hand, we can always write $M_\phi = Id - R$, for a certain R. But, from the same computations of before in order to reduce the length of curves we need that R is positive definite.

Moreover, it is clear that we obtain an isometry with $=$ instead of $<$ in previous passages if and only if $R = 0 \Leftrightarrow M_\phi = Id$.

□

Remark 4.1.3.

1. From previous lemma it is possible to deduce that if $\bar{\Omega} \subseteq \mathbb{R}^2$, then $\phi : \bar{\Omega} \to \mathbb{R}^3$ is an isometric immersion if and only if the following conditions hold
$$\begin{cases} X_u^2 + Y_u^2 + Z_u^2 = 1 \\ X_v^2 + Y_v^2 + Z_v^2 = 1 \\ X_u X_v + Y_u Y_v + Z_u Z_v = 0 \end{cases}.$$

2. We will refer to a general strictly short immersion as a couple (ϕ, R), where ϕ is the immersion and R is the positive definite map associated according to previous Lemma.

4.2 Theorem and proof

Now, we are ready to state and prove the main theorem.

Theorem 4.2.1 (Nash-Kuiper)**.** *Let D^2 represents the 2-dimensional disk and let $N \geq 3$. Let $\phi : D^2 \to \mathbb{R}^N$ be a C^∞ strictly short immersion.*
Then, for any $\epsilon > 0$ there exists a C^1 isometric immersion $\phi_\epsilon : D^2 \to \mathbb{R}^N$ such that $\|\phi - \phi_\epsilon\|_{C^0} \leq \epsilon$.

Equivalently, the previous theorem tells us that C^1 isometric immersions are C^0-dense in the set of C^∞ short immersions.

In order to prove Nash-Kuiper Theorem we proceed by steps and we exploit the method of convex integration in detail. The first key result is the following iterative step.

Proposition 4.2.1 (Iterative Step). *Let $N \geq 3$. Let us consider (ϕ, R) a strictly short immersion, with $\phi : D^2 \to \mathbb{R}^N$.*
Then, for any $\epsilon > 0$, there exists another strictly short immersion (ϕ', R') and a certain constant $c > 0$ that depends only on dimension N, such that

(i) $\|\phi - \phi'\|_{C^0} \leq \epsilon$;

(ii) $\|R'\|_{C^0} \leq \epsilon$;

(iii) $\|J\phi - J\phi'\|_{C^0} \leq \epsilon + c\,\|R\|_{C^0}^{\frac{1}{2}}$.

From the Iterative Step we can deduce thesis.

Proof Proposition 4.2.1 $\Rightarrow$ Nash-Kuiper Theorem. Let us fix a certain $\epsilon > 0$. Now, let us define $(\phi^{(0)}, R^{(0)}) := (\phi, R)$ the first strictly short immersion. We can apply the above proposition iteratively in order to obtain a sequence of strictly short immersions $(\phi^{(n)}, R^{(n)})$ such that

(i) $\left\|\phi^{(n)} - \phi^{(n-1)}\right\|_{C^0} \leq \dfrac{\epsilon}{2^n}$;

(ii) $\left\|R^{(n)}\right\|_{C^0} \leq \dfrac{\epsilon^2}{c^2 4^n}$;

(iii) $\left\|J\phi^{(n)} - J\phi^{(n-1)}\right\|_{C^0} \leq \dfrac{\epsilon}{2^n} + c\left\|R^{(n-1)}\right\|_{C^0}^{\frac{1}{2}}$.

We can note that the constant $c > 0$ is good for any n since from previous proposition we know that it just only depend on dimension N.

Now, due to condition (ii), the last inequality becomes

$$\left\|J\phi^{(n)} - J\phi^{(n-1)}\right\|_{C^0} \leq \frac{\epsilon}{2^n} + \frac{\epsilon}{2^{n-1}}.$$

Therefore, thanks to (i) the sequence $\{\phi^{(n)}\}$ is a Cauchy sequence in $C(D^2, \mathbb{R}^N)$ with sup norm, and by completness we know that it converges to a certain continuous function ϕ_ϵ. Moreover, due to (iii) also $\{J\phi^{(n)}\}$ is a Cauchy sequence in $C(D^2, \mathbb{R}^{N\times 2})$ and so it converges to a certain continuous map $K : D^2 \to \mathbb{R}^{N\times 2}$. It is known that, due to uniform convergence, K corresponds to the $J\phi_\epsilon$ and so ϕ_ϵ is of class C^1. Furthermore, due to (ii) it is clear that $R^{(n)} \xrightarrow{C^0} 0$ as $n \to \infty$, which means that $M_{\phi^{(n)}} \xrightarrow{C^0} Id$. Hence, M_{ϕ_ϵ} is the identity matrix and so ϕ_ϵ is an isometric immersion.

Finally, in order to conclude we can note that

$$\|\phi - \phi_\epsilon\|_{C^0} \leq \sum_{i=1}^{\infty} \left\|\phi^{(i)} - \phi^{(i-1)}\right\|_{C^0} \leq \epsilon \sum_{i=1}^{\infty} \frac{1}{2^i} = \epsilon.$$

□

The previous Iterative Step can be seen as a consequence of the actual iterative procedure used by Nash, that in the terminology of the same Nash is known as *stage*.

Proposition 4.2.2 (Stage: reducing error)**.** *Let $N \geq 3$. Let us consider $\phi : D^2 \to \mathbb{R}^N$ a C^∞ immersion and $R : D^2 \to \mathbb{R}^{2\times 2}$ a C^∞ symmetric and positive definite map. Then, there exists a sequence of C^∞ immersions $\phi_n : D^2 \to \mathbb{R}^N$ such that*

1. $\phi_n \xrightarrow{C^0} \phi$;
2. $M_{\phi_n} = M_\phi + R + o(1)$;
3. $J\phi_n = J\phi + O\left(\|R\|_{C^0}^{\frac{1}{2}}\right) + o(1)$;

where $o(1)$ represents a quantity that goes to 0 uniformly on D^2 as $n \to \infty$ and the implied constant in the last condition depends only on dimension N.

Thanks to latter Proposition 4.2.2, it is possible to prove Proposition 4.2.1.

Proof Proposition 4.2.2 ⇒ Proposition 4.2.1. Let us fix $\epsilon > 0$. It is possible to apply Proposition 4.2.2 with $R = R - \tilde{\epsilon}Id$, which is symmetric and positive definite for $\tilde{\epsilon}$ small enough. Then, we can find a sequence ϕ_n such that

1. $\|\phi - \phi_n\|_{C^0} \leq \epsilon$, for n large enough;
2. $\|M_{\phi_n} - M_\phi - R + \tilde{\epsilon}Id\|_{C^0} \leq \frac{\epsilon}{2}$, for n large enough and $\tilde{\epsilon}$ small enough;
3. $\|J\phi_n - J\phi\|_{C^0} \leq c\,\|R - \tilde{\epsilon}Id\|_{C^0}^{\frac{1}{2}} + (\epsilon - c\sqrt{\tilde{\epsilon}})$, for n large enough and $\tilde{\epsilon}$ small enough.

First of all it is clear that condition 1. implies (i). Moreover, we can say that the constant c is good for any n since it depends only on dimension N. Hence, from condition 3. it follows that

$$\|J\phi_n - J\phi\|_{C^0} \leq c\,\|R - \tilde{\epsilon}Id\|_{C^0}^{\frac{1}{2}} + (\epsilon - c\sqrt{\tilde{\epsilon}}) \leq c\,\|R\|^{\frac{1}{2}} + \epsilon.$$

Furthermore, if we call $R_n := M_\phi + R - M_{\phi_n}$, it is symmetric and positive definite for n large enough by continuity, thanks to condition 2. In addition, each ϕ_n is strictly short since we have that

$$M_{\phi_n} + R_n = M_\phi + R = Id,$$

by hypothesis. Moreover, due to condition 2. and reverse triangle inequality, it follows that

$$\|R_n\|_{C^0} = \|M_\phi + R - M_{\phi_n}\|_{C^0} \leq \epsilon,$$

for n large enough and we have concluded. $\square$

Now, for Proposition 4.2.2 it is necessary to decompose the positive definite map R into smaller components, each represented by rank-one matrices. To achieve this, we first present the following lemma.

Lemma 4.2.1 (Rank one decomposition)**.** *Let $\mathcal{S}$ be the set of symmetric positive definite $d \times d$ matrices.*
Then, there exist a sequence $\{v_k\}_{k\in\mathbb{N}} \subseteq \mathbb{R}^d$ of unit vectors and a sequence $\{a_k\}_{k\in\mathbb{N}}$ of C^∞ functions $a_k : \mathcal{S} \to \mathbb{R}$ such that for any $A \in \mathcal{S}$

$$A = \sum_{k=1}^{\infty} a_k(A)^2 v_k \otimes v_k \ .$$

Moreover, there exists a certain constant $C_d \in \mathbb{N}$ (that depends only on dimension d) such that at most C_d of the $a_k(A)$ are non-zero for any $A \in \mathcal{S}$ (hence the previous sum is always a finite sum).

Proof. First of all, we can observe that $\mathcal{S}_1 := \mathcal{S} \cap \{S \in \mathcal{S} : \mathrm{tr}(S) = 1\}$ is an open convex subset of $L := \left\{A \in \mathbb{R}^{d\times d}_{sym} \ : \ \mathrm{tr}(A) = 1\right\}$, with $\dim L = \frac{d(d+1)}{2} - 1$. Therefore, by Caratheodory's theorem, each element of $\mathcal{S}_1$ can be written as a convex combination of matrices in L and in particular it is contained in $IntConv\left\{A_1, \dots, A_{\frac{d(d+1)}{2}}\right\} \subset L$, of course with different A_i's for different elements of $\mathcal{S}_1$.

Now, let $\left\{S^{(i)}\right\}_{i\in\mathbb{N}}$ be a locally finite covering of $\mathcal{S}_1$ such that each $S^{(i)}$ has the form

$$S^{(i)} = IntConv\left\{A_1^{(i)}, \dots, A^{(i)}_{\frac{d(d+1)}{2}}\right\},$$

for certain $A_j^{(i)} \in \mathcal{S}_1$. Moroever, there exists $N_d \in \mathbb{N}$ (depending only on d) such that each $A \in \mathcal{S}_1$ is contained in at most N_d of the $S^{(i)}$. In each $S^{(i)}$ then there exist C^∞ functions $\mu_{i,j} : S^{(i)} \longrightarrow (0,1)$ such that

$$A = \sum_{j=1}^{\frac{d(d+1)}{2}} \mu_{i,j}^2(A) A_j^{(i)} \ \text{ for any } A \in S^{(i)}.$$

with $\sum_j \mu_{i,j}^2(A) = 1$. Moreover, each $A_j^{(i)}$ is diagonalizable and it can be written as

$$A_j^{(i)} = \left(\lambda_{j,1}^{(i)}\right)^2 v_{j,1}^{(i)} \otimes v_{j,1}^{(i)} + \cdots + \left(\lambda_{j,d}^{(i)}\right)^2 v_{j,d}^{(i)} \otimes v_{j,d}^{(i)}$$

where $\lambda_{j,k}^{(i)} \in \mathbb{R}$ and $v_{j,k}^{(i)}$ are unit vectors of $\mathbb{R}^d$ that form an orthonormal basis of eigenvectors of $A_j^{(i)}$.

Finally, let ψ_i be a partition of unity such that

(i) $\operatorname{supp} \psi_i \Subset S^{(i)}$;

(ii) $\sum_i \psi_i^2(A) = 1$ for any $A \in \mathcal{S}_1$.

Then given $A \in \mathcal{S}_1$, it is possible to make the following decomposition:

$$A = \sum_{i=1}^{\infty} \sum_{j=1}^{\frac{d(d+1)}{2}} \sum_{k=1}^{d} \left(\psi_i(A)\mu_{i,j}(A)\lambda_{j,k}^{(i)}\right)^2 v_{j,k}^{(i)} \otimes v_{j,k}^{(i)} .$$

For general $A \in \mathcal{S}$, since $\frac{1}{\operatorname{tr}(A)}A$ is of unit trace, we can obtain the required decomposition according to

$$A = \sum_{i=1}^{\infty} \sum_{j=1}^{\frac{d(d+1)}{2}} \sum_{k=1}^{d} \operatorname{tr}(A) \left(\psi_i\left(\frac{1}{\operatorname{tr}(A)}A\right)\mu_{i,j}\left(\frac{1}{\operatorname{tr}(A)}A\right)\lambda_{j,k}^{(i)}\right)^2 v_{j,k}^{(i)} \otimes v_{j,k}^{(i)},$$

with $C_d = \frac{1}{2}N_d\, d^2(d+1)$. $\square$

Remark 4.2.1. If we consider a compact set of matrices contained in $\mathcal{S}$, it is possible to note that there exists a finite number K such that it is sufficient to take unit vectors v_k and C^∞ functions a_k for $k = 1, \dots, K$ in order to obtain the previous decomposition for any matrix in the compact set.

The idea of the proof of the main result is now to show the *Stage* step for a rank one matrix. Then, we can pass from "local" version to "global" case thanks to Rank one decomposition Lemma.

Proposition 4.2.3 (Stage: reducing error, rank one version)**.** *Let $N \geq 3$ and $\phi : D^2 \to \mathbb{R}^N$ a C^∞ immersion. Let v be a unit vector in $\mathbb{R}^2$ and $a : D^2 \to \mathbb{R}$ a C^∞ function. Then, there exists a sequence of C^∞ immersions $\phi_n : D^2 \to \mathbb{R}^N$ such that*

1. $\phi_n \xrightarrow{C^0} \phi$;
2. $M_{\phi_n} = M_\phi + a^2 v \otimes v + o(1)$;
3. $J\phi_n = J\phi + O\left(\|a\|_{C^0}\right) + o(1)$.

Furthermore, the support of $\phi_n - \phi$ is contained in the support of a.

Before proving Proposition 4.2.3, that represents the key point of convex integration method, we can use it in order to show the validity of Proposition 4.2.2.

Proof Proposition 4.2.3 ⇒ Proposition 4.2.2. Starting from rank one decomposition due to Lemma 4.2.1 and Remark 4.2.1, it is possible to write

$$R(x) = \sum_{k=1}^{K} a_k(x)^2 v_k \otimes v_k \qquad \forall x \in D^2, \tag{4.3}$$

for certain unit vectors $v_1, \dots, v_K$ in $\mathbb{R}^2$, C^∞ functions $a_1, \dots, a_K : D^2 \to \mathbb{R}$ and $K > 0$. Now, by taking the trace from (4.3), thanks to Cauchy-Schwarz inequality, we obtain that

$$\sum_{k=1}^{K} a_k^2 = \operatorname{tr}(R) \leq \sqrt{2}(R_{11}^2 + R_{22}^2)^{\frac{1}{2}} \leq \sqrt{2}\, \|R\|_{C^0},$$

from which it follows that

$$\|a_k(x)\|_{C^0} \leq c\, \|R\|_{C^0}^{\frac{1}{2}} \qquad for\ any\ k = 1, \dots, K\ ,$$

for a suitable constant $c > 0$. Now, we can apply K times Proposition 4.2.3, each time with the respectively matrix $a_k(x)^2 v_k \otimes v_k$. Diagonalising and thanks to latter inequality, we obtain sequences of immersions $\phi_n^{(k)} : D^2 \to \mathbb{R}^N$ for $k = 1, \dots, K$ such that $\phi_n^{(0)} = \phi$ and

(A) $\left\|\phi_n^{(k)} - \phi_n^{(k-1)}\right\|_{C^0} = o(1)$;

(B) $\left\|M_{\phi_n^{(k)}} - M_{\phi_n^{(k-1)}} - a_k(x)^2 v_k \otimes v_k\right\|_{C^0} = o(1)$;

(C) $\left\|J\phi_n^{(k)} - J\phi_n^{(k-1)}\right\|_{C^0} \leq c\, \|R\|_{C^0}^{\frac{1}{2}} + o(1)$.

Therefore, we can show that the final sequence $\phi_n^{(K)}$ satisfies conditions of Proposition 4.2.2. Indeed,

1. Fixed any $\epsilon > 0$, for n large enough and for any $k = 1, \dots, K$, we have that

$$\left\|\phi_n^{(k)} - \phi_n^{(k-1)}\right\|_{C^0} \leq \frac{\epsilon}{K}\ .$$

   Hence,

$$\left\|\phi_n^{(K)} - \phi\right\|_{C^0} \leq \sum_{k=1}^{K} \left\|\phi_n^{(k)} - \phi_n^{(k-1)}\right\|_{C^0} \leq \epsilon\ .$$

2. Similarly, fixed any $\epsilon > 0$, for n large enough and for any $k = 1, \dots, K$, we have that

$$\left\|M_{\phi_n^{(k)}} - M_{\phi_n^{(k-1)}} - a_k(x)^2 v_k \otimes v_k\right\|_{C^0} \leq \frac{\epsilon}{K}.$$

Hence,

$$\begin{aligned}\left\|M_{\phi_n^{(K)}} - M_\phi - R\right\|_{C^0} &= \left\|M_{\phi_n^{(K)}} - M_{\phi^{(0)}} - \sum_{k=1}^K a_k(x)^2 v_k \otimes v_k\right\|_{C^0} \\ &= \left\|\sum_{k=1}^K \left(M_{\phi_n^{(k)}} - M_{\phi_n^{(k-1)}}\right) - \sum_{k=1}^K a_k(x)^2 v_k \otimes v_k\right\|_{C^0} \\ &\leq \sum_{k=1}^K \left\|M_{\phi_n^{(k)}} - M_{\phi_n^{(k-1)}} - a_k(x)^2 v_k \otimes v_k\right\|_{C^0} \leq \epsilon\ .\end{aligned}$$

3. Again, fixed any $\epsilon > 0$, for n large enough and for any $k = 1, \ldots, K$, we have that,

$$\left\|J\phi_n^{(k)} - J\phi_n^{(k-1)}\right\|_{C^0} \leq c\,\|R\|_{C^0}^{\frac{1}{2}} + \frac{\epsilon}{K}$$

Hence, it turns out that

$$\left\|J\phi_n^{(K)} - J\phi\right\|_{C^0} \leq \sum_{k=1}^K \left\|J\phi_n^{(k)} - J\phi_n^{(k-1)}\right\|_{C^0} \leq c\,\|R\|_{C^0}^{\frac{1}{2}} + \epsilon\ ,$$

calling again c the new constant.

Thus, we have concluded. □

4.2.1 Stage in codimension 2

What we discussed so far does not depend on the dimension N of the codomain, provided that $N \geq 3$. However, we can now separate the proof in two cases. Before we consider the Nash proof supposing that $\phi : D^2 \to \mathbb{R}^4$ (of course we can consider a generic $N \geq 4$ and the proof is the same); then, we need to slightly modify the proof in order to reach immersions to space of codimension 1 instead of 2 (as Kuiper did).

In order to show the *stage* proposition in codimension 2, we need the following lemma.

Lemma 4.2.2 (Orthogonal frame)**.** *Let $D^2 \subset \mathbb{R}^2$ be the unitary disk. Let $f_i : D^2 \to \mathbb{R}^4$ with $i = 1, 2$ of class C^r such that $f_1(x)$ and $f_2(x)$ are linearly independent for any $x \in D^2$.*
Then there exist vector fields $\xi : D^2 \to \mathbb{R}^4$ and $\eta : D^2 \to \mathbb{R}^4$ of class C^r such that for any $x \in D^2$

1. $\|\xi(x)\|_{\mathbb{R}^4} = \|\eta(x)\|_{\mathbb{R}^4} = 1$;
2. $\langle \xi(x), \eta(x) \rangle = 0$;
3. $\langle \xi(x), f_i(x) \rangle = \langle \eta(x), f_i(x) \rangle = 0$ *for* $i = 1, 2$.

Proof. It is possible to apply Gram-Schmidt process to $\{f_1(x), f_2(x)\}$ and obtain maps $\{w_1(x), w_2(x)\}$ of class C^r in x, such that for any $x \in D^2$ it is an orthonormal system whose span is the same as the one generated by f_i's. Now, it is enough to find $\xi, \eta : D^2 \to \mathbb{R}^4$ such that 1. and 2. hold as before, and 3. becomes

$$\langle \xi(x), w_i(x)\rangle = \langle \eta(x), w_i(x)\rangle = 0 \quad for\ i = 1, 2. \tag{4.4}$$

Let us proceed by steps.

Step 1: Let us fix $x = 0$. Since we are in codimension 2, it is clear, thanks to Gram-Schmidt, that it is possible to find two unit vectors ξ_0 and η_0 that are orthogonal between them and between $w_i(0)$ for $i = 1, 2$.

Step 2: Now, we want to extend it in the e_1 direction, in particular we evolve the fields ξ and η along the segment $\{te_1 : |t| \leq 1\}$ according to the following ODE equations:

$$\begin{cases} \partial_{x_1}\xi = -\langle \xi, \partial_{x_1} w_1\rangle w_1 - \langle \xi, \partial_{x_1} w_2\rangle w_2 \\ \xi(0) = \xi_0 \end{cases} \qquad \begin{cases} \partial_{x_1}\eta = -\langle \eta, \partial_{x_1} w_1\rangle w_1 - \langle \eta, \partial_{x_1} w_2\rangle w_2 \\ \eta(0) = \eta_0 \end{cases}$$

The previous Cauchy problems admit a unique C^r solutions ξ, η thanks to the classical existence and uniqueness theorem of ODE. Now, since conditions 1. to 3. are satisfied in $x = 0$ and they are all regular functions, we can show that the derivatives of such quantities are equal to 0. More specifically,

- We first consider the third condition. Since w_1 and w_2 are orthogonal, for any x we have that

$$\begin{aligned} \partial_{x_1}\langle \xi, w_1\rangle &= \langle \partial_{x_1}\xi, w_1\rangle + \langle \xi, \partial_{x_1} w_1\rangle \\ &= \langle (-\langle \xi, \partial_{x_1} w_1\rangle w_1 - \langle \xi, \partial_{x_1} w_2\rangle w_2), w_1\rangle + \langle \xi, \partial_{x_1} w_1\rangle \\ &= -\langle \xi, \partial_{x_1} w_1\rangle + \langle \xi, \partial_{x_1} w_1\rangle = 0. \end{aligned}$$

  The same holds also for w_2 and in both cases for η too.

- For the first condition, we can say that

$$\begin{aligned} \partial_{x_1}\langle \xi, \xi\rangle = 2\langle \partial_{x_1}\xi, \xi\rangle &= 2\langle (-\langle \xi, \partial_{x_1} w_1\rangle w_1 - \langle \xi, \partial_{x_1} w_2\rangle w_2), \xi\rangle \\ &= -2\langle \xi, \partial_{x_1} w_1\rangle \langle w_1, \xi\rangle - 2\langle \xi, \partial_{x_1} w_2\rangle \langle w_2, \xi\rangle = 0, \end{aligned}$$

  thanks to previous point. The same holds also for η.

- Finally, about second condition, it turns out that

$$\begin{aligned} \partial_{x_1}\langle \xi, \eta\rangle &= \langle \partial_{x_1}\xi, \eta\rangle + \langle \xi, \partial_{x_1}\eta\rangle \\ &= \langle (-\langle \xi, \partial_{x_1} w_1\rangle w_1 - \langle \xi, \partial_{x_1} w_2\rangle w_2), \eta\rangle + \langle \xi, (-\langle \eta, \partial_{x_1} w_1\rangle w_1 - \langle \eta, \partial_{x_1} w_2\rangle w_2)\rangle \\ &= -\langle \xi, \partial_{x_1} w_1\rangle \langle w_1, \eta\rangle - \langle \xi, \partial_{x_1} w_2\rangle \langle w_2, \eta\rangle - \langle \eta, \partial_{x_1} w_1\rangle \langle \xi, w_1\rangle - \langle \eta, \partial_{x_1} w_2\rangle \langle \xi, w_2\rangle \\ &= 0, \end{aligned}$$

  thanks to first point.

Step 3: Now, we can iterate the same procedure in order to extend the vector fields ξ, η to all $\{te_1 + se_2 \ : \ t^2 + s^2 \leq 1\}$, solving the following equations

$$\begin{cases} \partial_{x_2}\xi = -\langle \xi, \partial_{x_2} w_1 \rangle w_1 - \langle \xi, \partial_{x_2} w_2 \rangle w_2 \\ \partial_{x_2}\eta = -\langle \eta, \partial_{x_2} w_1 \rangle w_1 - \langle \eta, \partial_{x_2} w_2 \rangle w_2 \end{cases},$$

using as initial conditions the previous extension on the line segment. In particular, we are extending the field in the same way of previous step but on the vertical axis inside the disk. We should verify that such ξ, η satisfy again conditions 1. to 3. But, this follows with the same computations of before. Moreover, such extension preserve regularity according to the regularity dependence from initial conditions result about ODE.

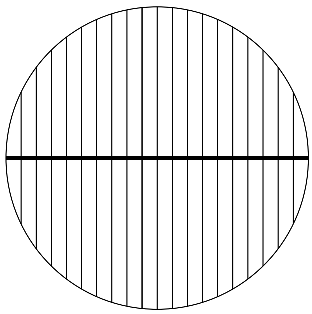

Figure 4.1: Propagation of vector fields ξ, η to the entire disk.

□

Finally, we can proceed with the proof of the *stage* for the rank one case in Nash's version.

Proof of Proposition 4.2.3 Case $N = 4$. The idea of the proof is to define the sequence of immersions as $\phi_n := \phi + \psi_n$ for suitable maps $\psi_n : D^2 \to \mathbb{R}^4$ with $supp(\psi_n) \subseteq supp(a)$ and such that

(a) $\|\psi_n\|_{C^0} \to 0$;

(b) $\left\| J\psi_n^T J\phi + J\phi^T J\psi_n + J\psi_n^T J\psi_n - a^2 v \otimes v \right\|_{C^0} \to 0.$

(c) $\|J\psi_n - c\,\|a\|_{C^0}\|_{C^0} \to 0$;

Indeed, supposing that there exists a certain sequence ψ_n which satisfies $(a) - (c)$, we obtain that

- $(a) \Rightarrow 1.$, by definition of ϕ_n;

- $(b) \Rightarrow 2.$, since by computations it is straightforward to see that

$$M_{\phi_n} = M_\phi + J\psi_n^T J\phi + J\phi^T J\psi_n + J\psi_n^T J\psi_n.$$

  Therefore,

$$\left\| M_{\phi_n} - M_\phi - a^2 v \otimes v \right\|_{C^0} = \left\| J\psi_n^T J\phi + J\phi^T J\psi_n + J\psi_n^T J\psi_n - a^2 v \otimes v \right\|_{C^0} \to 0.$$

- $(c) \Rightarrow \left\| J\phi_n - J\phi - c\,\|a\|_{C^0} \right\|_{C^0} = \left\| J\psi_n - c\,\|a\|_{C^0} \right\|_{C^0} \to 0;$

It is possible to note that such maps ϕ_n are actually immersions for n large enough due to condition 2. Indeed, M_ϕ is positive definite, $o(1)$ does not affect it for n large by continuity and the rank one matrix $a^2 v \otimes v$ is positive semi-definite.

Now, we need to define a suitable sequence ψ_n such that $(a) - (c)$ hold. Let us denote every $x \in D^2$ as the couple $x = (x_1, x_2)$. We consider C^∞ maps ξ, $\eta : \mathbb{R}^2 \to \mathbb{R}^4$ with $\|\xi(x)\|_{\mathbb{R}^4} = \|\eta(x)\|_{\mathbb{R}^4} = 1$ for any $x \in D^2$ and such that

$$\xi(x) \perp \eta(x) \perp \partial_{x_1}\phi(x) \perp \partial_{x_2}\phi(x) \quad for\ any\ x \in D^2. \tag{4.5}$$

Such maps are given by previous Lemma 4.2.2 with dimension $d = 2$ and $f_1 = \partial_{x_1}\phi$ and $f_2 = \partial_{x_2}\phi$. Now, let us define

$$\psi_n(x) := \frac{a(x)}{n}\Big(\xi(x)\cos(nx \cdot v) + \eta(x)\sin(nx \cdot v)\Big). \tag{4.6}$$

Up to rotations, we can suppose that $v = (1, 0)$. Therefore, ψ_n becomes

$$\psi_n(x_1, x_2) := \frac{a(x_1, x_2)}{n}\Big(\xi(x_1, x_2)\cos(nx_1) + \eta(x_1, x_2)\sin(nx_1)\Big). \tag{4.7}$$

Hence, we have that

(a) $\|\psi_n\|_{C^0} \to 0$ clearly thanks to the factor $\frac{1}{n}$;

(b) Let us consider the partial derivatives of ψ_n. It turns out that

$$\partial_{x_1}\psi_n = a(x_1, x_2)\left(-\xi(x_1, x_2)\sin(nx_1) + \eta(x_1, x_2)\cos(nx_1)\right) + O\left(\frac{1}{n}\right);$$
$$\partial_{x_2}\psi_n = O\left(\frac{1}{n}\right).$$

Now, thanks to conditions (4.5) we have that

- $J\phi^T J\psi_n = \begin{pmatrix} \partial_{x_1}\phi \cdot \partial_{x_1}\psi_n & \partial_{x_1}\phi \cdot \partial_{x_2}\psi_n \\ \partial_{x_2}\phi \cdot \partial_{x_1}\psi_n & \partial_{x_2}\phi \cdot \partial_{x_2}\psi_n \end{pmatrix} = O\left(\frac{1}{n}\right).$

- $J\psi_n^T J\phi = (J\phi^T J\psi_n)^T = O\left(\frac{1}{n}\right)$;
- By simple computations it is possible to see that for any $(x_1, x_2) \in D^2$

$$\|a(x_1,x_2)\left(-\xi(x_1,x_2)\sin(nx_1) + \eta(x_1,x_2)\cos(nx_1)\right)\|^2_{\mathbb{R}^4} = a^2(x_1,x_2).$$

  Hence,

$$J\psi_n^T J\psi_n = \begin{pmatrix} \|\partial_{x_1}\psi_n\|^2_{\mathbb{R}^4} & \partial_{x_1}\psi_n \cdot \partial_{x_2}\psi_n \\ \partial_{x_2}\psi_n \cdot \partial_{x_1}\psi_n & \|\partial_{x_2}\psi_n\|^2_{\mathbb{R}^4} \end{pmatrix} = \underbrace{\begin{pmatrix} a^2 & 0 \\ 0 & 0 \end{pmatrix}}_{a^2 v\otimes v} + O\left(\frac{1}{n}\right).$$

Therefore, putting together the three previous points, it turns out that

$$J\psi_n^T J\phi + J\phi^T J\psi_n + J\psi_n^T J\psi_n - a^2 v\otimes v = O\left(\frac{1}{n}\right),$$

from which (b) follows.

(c) Considering the partial derivatives, we have that

$$\begin{aligned} J\psi_n &\le \|a\|_{C^0}\,\|-\xi(x_1,x_2)\sin(nx_1) + \eta(x_1,x_2)\cos(nx_1)\|_{C^0} + O\left(\frac{1}{n}\right) \\ &\le c\,\|a\|_{C^0} + O\left(\frac{1}{n}\right), \end{aligned}$$

for a certain constant $c > 0$. Therefore, condition (c) follows.

Thus, we have concluded. □

4.2.2 Stage in codimension 1

Now, our goal is to discuss how we can change previous proof in order to show Kuiper's version of the Theorem, considering $\phi : D^2 \to \mathbb{R}^3$. In particular, we can note that the construction of orthogonal frame of Lemma 4.2.2 does not work anymore in the new setting since the codimension of the immersion is 1 and no more 2. Indeed, in the new case we cannot find two orthogonal vector fields as before; however, by repeating the same arguments mentioned previously, we might still be able to obtain one such vector field. Let us call $\eta(x_1, x_2)$ such normal vector field, i.e. such that it is of unit norm, of class C^∞ and orthogonal to $\partial_{x_1}\phi$ and $\partial_{x_2}\phi$. Our goal in the proof is to modify the high oscillations term converting "Nash spirals" into "Kuiper corrugations". Before proceeding with the proof, we need a simple lemma in order to supplant the lack of the second codimension.

Lemma 4.2.3. *Let $f_i : D^2 \to \mathbb{R}^3$ with $i = 1, 2$ of class C^r such that $f_1(x)$ and $f_2(x)$ are linearly independent for any $x \in D^2$. Let $\eta : D^2 \to \mathbb{R}^3$ of class C^r according to Lemma 4.2.2. Then, there exists a C^∞ function $\zeta : D^2 \to \mathbb{R}^3$ of class C^r such that for any $x \in D^2$*

1. $\|\zeta(x)\|_{\mathbb{R}^3} = 1$;
2. $\langle \zeta(x), f_2(x) \rangle = 0$;
3. $\langle \zeta(x), \eta(x) \rangle = 0$.

Moreover, $\langle \zeta(x), f_1(x) \rangle$ is always positive or always negative.

Proof. The thesis is a simple consequence of Gram-Schmidt. Indeed for any $x \in D^2$ applying the procedure to $f_2(x)$ and $\eta(x)$ we can find the respectively orthogonal vector with unit norm and we can call it $\zeta(x)$.

Moreover, it is known that such vector depend directly by the others due to an explicit regualr formula and so $\zeta(x)$ as a function of x preserves the same regularity of f_2 and η.

In conclusion, $\langle \zeta(x), f_1(x) \rangle$ can't be 0 by hypothesis since we are in codimension 1 and so by continuity it is always positive, or always negative. □

We are ready to prove the main result in codimension 1.

Proof of Proposition 4.2.3 Case $N = 3$. We need to define a certain sequence ψ_n similarly to what done in (4.7). In particular, we want to construct two suitable families of curves

$$\gamma_i : D^2 \times [0, 1] \to \mathbb{R} \qquad for \quad i = 1, 2,$$

such that $\gamma_i((x_1, x_2), 0) = \gamma_i((x_1, x_2), 1)$ for any $(x_1, x_2) \in D^2$ and $i = 1, 2$, that satisfy certain conditions. Similarly to codimension 2 case let us define

$$\psi_n(x_1, x_2) := \frac{1}{n}\Big(\zeta(x_1, x_2)\gamma_1((x_1, x_2), \{nx_1\}) + \eta(x_1, x_2)\gamma_2((x_1, x_2), \{nx_1\})\Big), \tag{4.8}$$

where $\{\cdot\}$ indicates the fractional part, $\zeta : D^2 \to \mathbb{R}^3$ is given by previous Lemma 4.2.3 and $\eta : D^2 \to \mathbb{R}^3$, as already noticed, is given by Lemma 4.2.2. As before, we can construct the target sequence as $\phi_n := \phi + \psi_n$. Now, we need to find the right conditions for γ_i such that the sequence ψ_n satisfies previous conditions $(a) - (c)$.

Thanks to the usual factor $\frac{1}{n}$, condition (a) is trivial. Now, if we compute the partial derivatives of ψ_n, denoting with $\dot{\gamma_i}$ its derivative with respect to t, we obtain that

$$\partial_{x_1}\psi_n = \zeta\dot{\gamma_1} + \eta\dot{\gamma_2} + O\left(\frac{1}{n}\right);$$
$$\partial_{x_2}\psi_n = O\left(\frac{1}{n}\right).$$

Hence, in order to obtain condition (b), it is sufficient to require that

$$\|\zeta\dot{\gamma}_1 + \eta\dot{\gamma}_2\|_{C^0} \leq c\,\|a\|_{C^0}\,, \tag{4.9}$$

for a certain constant $c > 0$.

Moreover, for condition (c) let us do some computations.

- Firstly, we have that

$$J\phi^T J\psi_n = \begin{pmatrix} \partial_{x_1}\phi \cdot [\zeta\dot{\gamma}_1 + \eta\dot{\gamma}_2] & 0 \\ \partial_{x_2}\phi \cdot [\zeta\dot{\gamma}_1 + \eta\dot{\gamma}_2] & 0 \end{pmatrix} + O\left(\frac{1}{n}\right).$$

  But, due to orthogonality assumptions on ζ and η, it turns out that

$$J\phi^T J\psi_n = \begin{pmatrix} (\partial_{x_1}\phi \cdot \zeta)\dot{\gamma}_1 & 0 \\ 0 & 0 \end{pmatrix} + O\left(\frac{1}{n}\right).$$

- Of course, $J\psi_n^T J\phi = (J\phi^T J\psi_n)^T$ that is the same of previous point.
- Finally, again thanks to assumptions on ζ and η, we have that

$$J\psi_n^T J\psi_n = \begin{pmatrix} \dot{\gamma}_1{}^2 + \dot{\gamma}_2{}^2 & 0 \\ 0 & 0 \end{pmatrix} + O\left(\frac{1}{n}\right).$$

Therefore, putting them together we obtain that

$$J\phi^T J\psi_n + J\psi_n^T J\phi + J\psi_n^T J\psi_n = \begin{pmatrix} 2(\partial_{x_1}\phi \cdot \zeta)\dot{\gamma}_1 + \dot{\gamma}_1{}^2 + \dot{\gamma}_2{}^2 & 0 \\ 0 & 0 \end{pmatrix} + O\left(\frac{1}{n}\right).$$

Let us define the function $\alpha := (\partial_{x_1}\phi \cdot \zeta)$. Now, in order to conclude, we can require that

$$2\alpha\dot{\gamma}_1 + \dot{\gamma}_1{}^2 + \dot{\gamma}_2{}^2 = a^2\,. \tag{4.10}$$

In conclusion, it is sufficient to prove that there exist a positive constant c and two C^∞ functions $\gamma_i : D^2 \times [0,1] \to \mathbb{R}$, $i = 1,2$ such that

(i) $\gamma_i((x_1,x_2),0) = \gamma_i((x_1,x_2),1)$ for any $(x_1,x_2) \in D^2$, $i = 1,2$;

(ii) $\|\dot{\gamma}_i\|_{C^0} \leq c\,\|a\|_{C^0}$ for $i = 1,2$;

(iii) $2\alpha\dot{\gamma}_1 + \dot{\gamma}_1{}^2 + \dot{\gamma}_2{}^2 = a^2$.

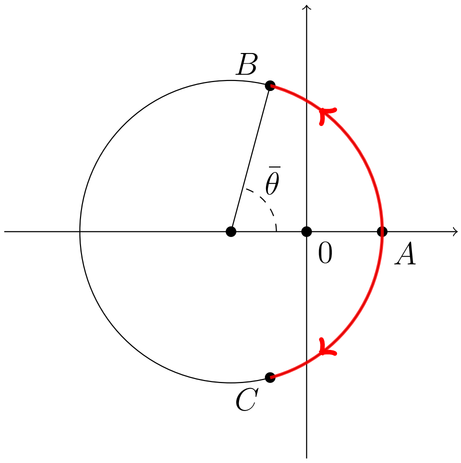

Figure 4.2: Convex integration: construction of the loop.

Firstly, it is possible to note that the condition of periodicity (i) is a consequence of supposing that $\dot{\gamma}_i((x_1, x_2), 0) = \dot{\gamma}_i((x_1, x_2), 1)$ and $\dot{\gamma}_i((x_1, x_2), t)$ with null average. Indeed, for $i = 1, 2$ it turns out that

$$\gamma_i((x_1, x_2), 1) = \gamma_i((x_1, x_2), 0) + \underbrace{\int_0^1 \dot{\gamma}_i((x_1, x_2), t)\ dt}_{=\ 0} = \gamma_i((x_1, x_2), 0)\ .$$

The idea of constructing the two curves is enclosed in Figure 4.2, exactly as we did in Chapter 2. More specifically, once one fixed $(x_1, x_2) \in D^2$, he have fixed the radius of the circle that depends on $a(x_1, x_2)$ and $\alpha(x_1, x_2)$. Then, it is possible to construct the red loop of the Figure, which shows the loop when $\alpha > 0$. Of course, when $\alpha < 0$ it is possible to do the same in the reverse way in order to obtain null average.

Hence, it is clear that for any $(x_1, x_2) \in D^2$ the respectively loop satisfies the differential relation (iii). Therefore, it remains to note regularity and condition (ii). For this reason, we can use the explicit formula describing such loop.

First of all, fixed $(x_1, x_2) \in D^2$, it is possible to find a certain angle $\bar{\theta}$ that depends on $\alpha(x_1, x_2)$ and $a(x_1, x_2)$, such that

$$\frac{\sin(\bar{\theta})}{\bar{\theta}} = \frac{\alpha}{\sqrt{\alpha^2 + a^2}}\ . \tag{4.11}$$

Such $\bar{\theta}$ exists since the function $\dfrac{\sin(x)}{x}$ from $[0, \pi]$ to $[0, 1]$ is decreasing. Now, according to Figure 4.2, we can construct the loop starting to point A at time $t = 0$, moving to point B at time $t = 1/4$, then passing again through A at time $t = 1/2$, moving to point C at time $t = 3/4$ and finally coming back to A at time $t = 1$. It is possible to define such curve with component $\dot{\gamma_1}((x_1,x_2),\cdot)$ and $\dot{\gamma_2}((x_1,x_2),\cdot)$ in the following way:

$$\begin{cases} \dot{\gamma_1}((x_1,x_2),t) := \sqrt{\alpha^2+a^2}\cos(\theta(t)) - \alpha \\ \dot{\gamma_2}((x_1,x_2),t) := \sqrt{\alpha^2+a^2}\sin(\theta(t)) \end{cases} . \tag{4.12}$$

In the previous expression, the function $\theta(t)$ can be given by

$$\theta(t) := \begin{cases} 4\bar{\theta}t & t \in [0,\frac{1}{4}] \\ 2\bar{\theta} - 4\bar{\theta}t & t \in [\frac{1}{4},\frac{3}{4}] \\ -4\bar{\theta} + 4\bar{\theta}t & t \in [\frac{3}{4},1] \end{cases} . \tag{4.13}$$

Now, we can prove that

$$\int_0^1 \dot{\gamma_1}((x_1,x_2),t)\; dt \;=\; \int_0^1 \dot{\gamma_2}((x_1,x_2),t)\; dt \;=\; 0 \;.$$

Regarding $\dot{\gamma_2}$ it follows easily by symmetry. While, regarding $\dot{\gamma_1}$, we can say that

$$\begin{aligned} \int_0^1 \dot{\gamma_1}((x_1,x_2),t)\; dt &= 4\int_0^{\frac{1}{4}} \dot{\gamma_1}((x_1,x_2),t)\; dt \\ &= 4\int_0^{\frac{1}{4}} \sqrt{\alpha^2+a^2}\cos(4\bar{\theta}t)\; dt - \alpha \\ &= \frac{\sqrt{\alpha^2+a^2}}{\bar{\theta}}\;\sin(4\bar{\theta}t)\Big|_0^{\frac{1}{4}} - \alpha \\ &= \sqrt{\alpha^2+a^2}\;\frac{\sin(\bar{\theta})}{\bar{\theta}} - \alpha = 0, \end{aligned}$$

where the last equality follows by (4.11). Now, it is clear that it is possible to take γ_i integrating the curves constructed so far. Actually, the curves γ_i obtained are not C^∞ but only C^1. However, it is possible to fix such problem just considering the C^∞ function made from $\theta(t)$ smoothing out the angles.

Now, thanks to definitions (4.12) - (4.13), we would like to have that for $i = 1,2$

$$\|\dot{\gamma_i}\|_{C^0} \le c\,\|a\|_{C^0}\,, \tag{4.14}$$

for a certain $c > 0$. This is a simple consequence if we write

$$\begin{aligned} \dot{\gamma_1}((x_1,x_2),t) &= a\left(\sqrt{\left(\tfrac{\alpha}{a}\right)^2+1}\;\cos(\theta(t)) - \tfrac{\alpha}{a}\right), \\ \dot{\gamma_1}((x_1,x_2),t) &= a\left(\sqrt{\left(\tfrac{\alpha}{a}\right)^2+1}\;\sin(\theta(t))\right). \end{aligned}$$

We can note that if $a = 0$, then circle of Figure 4.2 touches the origin and so γ_i must be identically 0 for any t in order to obtain null average. Indeed, also looking at condition

(4.11), it turns out that $\theta(t)$ tends to 0 as $a \to 0$. Then it is follows that the entire previous quantity in brackets is bounded and it is sufficient to take c as its C^0 norm in order to satisfy condition (4.14). Indeed, from (4.11), when $a \to 0$ and so $\bar{\theta} \to 0$, it turns out that

$$\frac{\alpha^2}{\alpha^2 + a^2} = \frac{\sin\bar{\theta}}{\bar{\theta}} \approx 1 - \frac{1}{3}\bar{\theta}^2 \quad \Rightarrow \quad \bar{\theta}^2 \approx \frac{3a^2}{\alpha^2 + a^2}.$$

Hence, for a suitable constant $C > 0$, when $a \to 0$ (and $\bar{\theta} \to 0$) we have that

$$\left|\sqrt{\frac{\alpha^2}{a^2} + 1}\ \cos(4\bar{\theta}t) - \frac{\alpha}{a}\right| = \left|\frac{\cos^2(4\bar{\theta}t) - \frac{\alpha^2}{a^2}\sin^2(4\bar{\theta}t)}{\sqrt{\frac{\alpha^2}{a^2} + 1}\ \cos(4\bar{\theta}t) + \frac{\alpha}{a}}\right| \lesssim \frac{\alpha^2\left(\frac{4\sqrt{3}t}{\sqrt{\alpha^2+a^2}}\right)^2}{\frac{\alpha}{a}} \le C\ a\ .$$

It is clear, now, that the last term tends to 0 as $a \to 0$ and so we have that the required quantity is actually bounded. The same follows also for $\dot{\gamma}_2$.
This concludes the proof of the Nash-Kuiper Theorem. $\square$

Remark 4.2.2. We can notice that the last construction of curves γ_1 and γ_2 represents the key point of the proof in codimension 1 and it is strictly related to the problem of curves discussed in Chapter 2 of the thesis. In particular, condition (iii) required can be viewed as a differential inclusion and we ask to find a loop valued in the boundary of a circle with null average. This is the same type of problem discussed in Theorem 2.1.1 and even more in Fundamental Lemma 2.1.1.

Remark 4.2.3. It is possible to note that the special case of Nash-Kuiper theorem considered in our discussion it is not so restrictive. In particular, in the proof we considered the 2-dimensional disk as domain of the immersion and dimensions 3 or 4 as codomain. From the proof it is quite clear that the same procedure of Nash spirals in dimension 4 works also for dimension $N \geq 4$, since the only requirement is to allow at least two degrees of freedom.

Furthermore, in a similar way, instead of D^2 it is possible to consider a generic ball $B \subseteq \mathbb{R}^d$ as domain of the immersion and the proof is almost the same. One should put more attention in some technical details; for example in the construction of the orthogonal frame (see Lemma 4.2.2) one should iterate the identical procedure until reaching dimension desired. This version of the theorem is quite relevant since it is a sort of local version of the general Nash-Kuiper Theorem (see Theorem 0.0.1 in the Introduction) if we think of B as a coordinate patch on a smooth closed Riemannian manifold. Moreover, if we consider a general smooth metric g on B, i.e. $(g_i(x))_{1\leq i,j\leq d}$ symmetric and positive definite for any $x \in B$, the isometric condition considered above becomes $M_\phi = g$, and the identity (4.2) that describes the strictly short immersions becomes $M_\phi + R = g$.

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