A Problem Set on

GOL 2 (Discrete Math.)

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(1) Use set builder notation to give a description of each of these sets.

(a) $\{0, 3, 6, 9, 12\}$.

(b) $\{-3, -2, -1, 0, 1, 2, 3\}$.

(2) Determine whether each of these pairs of sets are equal.

(a) $\emptyset$, $\{\emptyset\}$.

(b) $\{1, 3, 3, 3, 5, 5, 5, 5, 5\}$, $\{5, 1, 3\}$

(c) $\{\{1\}$, $\{1, \{1\}\}$, $\{1, \{1\}\}$.

(3) Consider the sets $S = \{a, b, \{c, d\}\}$ and $T = \{S, \{a\}, \{a, b\}, \emptyset\}$. Now answer the following:

(a) How many elements are in the set S?

(b) How many elements are in the set T?

(c) Is $\emptyset \subseteq S$?

(d) Is $\emptyset \subseteq T$?

(e) Is $\emptyset \in S$?

(f) Is $\emptyset \in T$?

(g) Is $S \in T$?

(h) Is $S \subseteq T$?

(i) Is $a \in S$?

(j) Is $a \in T$?

(k) Is $\{a\} \in S$?

(l) Is $\{a\} \in T$?

(m) Is $\{a\} \subseteq S$?

(n) Is $\{a\} \subseteq T$?

(o) List all subsets of S.

(4) The **symmetric difference** of A and B, denoted by $A \oplus B$, is the set containing those elements in either A or B, *but not in both* A and B. Show that

(a) $A \oplus B = (A \cup B) - (A \cap B)$.

(b) $A \oplus B = (A \cup B) - (A \cap B)$.

(5) Consider the following sets:

$$A = \{z \in \mathbb{Z} \,|\, z^7 + 7z^4 - 8z = 0\} \qquad B = \left\{ q \in \mathbb{Q} \,\middle|\, q = \frac{a_1}{a_2} \text{ for some } a_1, a_2 \in A \right\} \qquad C = \{1, 3, \{1, 3\}\}$$

Find each of the following. Answers should be given in *roster notation with no ellipsis*, and otherwise should be given in *set-builder notation with predicates* that are as simple as possible. Give a brief justification of each answer.[1]

[1]Recall that $\mathcal{P}(A) = \{S \,|\, S \subseteq A\}$ is the *power set* of A.

(a) $A \oplus C$
(b) $\mathcal{P}(B) \cap C$
(c) $(\{1, \{2\}\} \cup C) \cap B$
(d) $(A \cup C) \setminus \overline{B}$
(e) $\{\emptyset\} \setminus \mathcal{P}(A)$
(f) $B \cap \overline{(A \cup \overline{C})}$

(6) Let $A, B,$ and C be sets. Prove or disprove each of the following statements (no Venn diagrams or membership tables).

(a) $(A \setminus C) \setminus (B \setminus C) = A \setminus B$.
(b) $A \oplus (B \cap C) = (A \oplus B) \cap (A \oplus C)$.
(c) $\overline{A \cup \overline{B}} \cup \overline{A} = \overline{A}$.
(d) $A \oplus B = A \cup B$ if and only if $A \cap B = \emptyset$.

(7) Prove that $\mathcal{P}(A) \subseteq \mathcal{P}(B)$ if and only if $A \subseteq B$.

(8) (a) Let A be a set. Prove that $A \times \emptyset = \emptyset \times A = \emptyset$.
(b) Suppose that $A \times B = \emptyset$, where A and B are sets. What can you conclude?

(9) Show that $A \times B \neq B \times A$, when A and B are non-empty, unless $A = B$.

(10) Let A, B, and C be sets. Show that[2]

(a) $A - B = A \cap \overline{B}$.
(b) $(A \cap B) \cup (A \cap \overline{B}) = A$.
(c) $(A \cup B) \subseteq (A \cup B \cup C)$.
(d) $(A \cap B \cap C) \subseteq (A \cap B)$.
(e) $(A - B) - C \subseteq (A - C)$.
(f) $(A - B) \cap (B - C) = \emptyset$.
(g) $(B - A) \cup (C - A) = (B \cup C) - A$.

(11) Determine whether each of these functions (mappings!) from $\mathbb{Z}$ to $\mathbb{Z}$ is injective (= one-to-one).

(a) $f(n) = n^3$,
(b) $f(n) = n^2 + n + 41$,
(c) $f(n) = -n^5 - 3n^3 - 5n + 17$,
(d) $f(n) = \lceil \frac{n}{2} \rceil$.

(12) Determine whether $f : \mathbb{Z} \times \mathbb{Z} \leftrightarrow \mathbb{Z}$ is surjective (= onto) if

(a) $f(m, n) = 2m - n$,
(b) $f(m, n) = m^2 - n^2$,
(c) $f(m, n) = |m| - |n|$,
(d) $f(m, n) = \left\lceil \frac{m+n}{2} \right\rceil$,
(e) $f(m, n) = m^2 + n^2$.

(13) For each of the mappings f in the previous problem, determine $f^{-1}(S)$ (i.e., the inverse-image or the preimage) of S if (i) $S = \{0\}$, (ii) $S = \{1\}$, or (iii) $S = \{0, 1\}$.

(14) Let $g : A \leftrightarrow B$ and $f : B \leftrightarrow C$ be functions. Prove
(a) If f and g are injective, then $f \circ g$ is injective.
(b) If f and g are surjective, then $f \circ g$ is surjective.
(c) Assuming mappings, if f and g are bijections, then so is $f \circ g$.

[2]Remember: $A - B$ is the same as $A \setminus B = \{x \in A \mid x \notin B\}$.

(15) Let $g : A \leftrightarrow B$ and $f : B \leftrightarrow C$ be functions. Prove or disprove:
   (a) If $f \circ g$ and g are injective, then f is injective.
   (b) If $f \circ g$ and g are surjective, then f is surjective.
   (c) If $f \circ g$ is a bijection, then f is injective if and only if g is surjective.

(16) For each function, determine if it is injective and/or surjective. Prove your answers.
   (a) $g : \mathbb{Z} \times \mathbb{Z}^+ \to \mathbb{Q}$ given by $g((x, y)) = \frac{x}{y}$
   (b) $h : \mathbb{R}^+ \to \mathbb{R}^+$ given by $h(x) = x\lceil x \rceil$.

(17) How many strings of eight English letters are there

   (a) that contain no vowels, if letters can be repeated?
   (b) that contain no vowels, if letters cannot be repeated?
   (c) that start with a vowel, if letters can be repeated?
   (d) that start with a vowel, if letters cannot be repeated?
   (e) that contain at least one vowel, if letters can be repeated?
   (f) that contain exactly one vowel, if letters can be repeated?
   (g) that start with X and contain at least one vowel, if letters can be repeated?
   (h) that start and end with X and contain at least one vowel, if letters can be repeated?

(18) How many functions (= mappings) are there from the set $\{1, 2, ..., n\}$, where n is a positive integer, to the set $\{0, 1\}$

   (a) that are one-to-one (= injective)?
   (b) that assign 0 to both 1 and n?
   (c) that assign 1 to exactly one of the positive integers less than n?

(19) In how many ways can a photographer at a wedding arrange six people in a row, including the bride and groom, if

   (a) the bride must be next to the groom?
   (b) the bride is not next to the groom?
   (c) the bride is positioned somewhere to the left of the groom?

(20) How many 10-digit numbers:

   (a) are there?
   (b) how many of them are greater than 2,000,000,000?
   (c) how many of them are even?
   (d) how many of them are odd and have different digits?

(21) How many ways are there to arrange the letters a, b, c, and d such that a is not followed immediately by b?

(22) Consider the number 35964.

(a) How many 3-digit numbers can be formed using digits from 35964 if digits may be repeated?
(b) How many 3-digit numbers can be formed using digits from 35964 if no digits may be repeated?
(c) What is the sum of all of the 3-digit numbers in part (b)?
(d) Repeat (c) if the original number was 7238954 and if we wanted all the 5-digit numbers.

(23) Answer each of these enumeration problems:

(a) How many different ordering of the letters of the word MISSISSIPPI are there?
(b) How many different ordering of the letters of the word STATISTICS: **(i)** are there? **(ii)** begin and end with an S?
(c) How many different orderings of the letters of the word SUCCESSFULLY: **(i)** are there? **(ii)** are there if the 3 S's must be together in each ordering? **(iii)** are there, if the sub-word SUCCESS must appear in each ordering?

(24) How many permutations of the letters ABCDEFG contain

(a) the string BCD?
(b) the string CFGA?
(c) the strings BA and GF?
(d) the strings ABC and DE?
(e) the strings ABC and CDE?

(25) Find the value of each of these quantities.

(a) $P(6,3)$, $P(6,5)$, $P(8,1)$, $P(8,8)$, $P(n,1)$, $P(n,n)$ (n: a positive integer)
(b) $C(5,1)$, $C(5,4)$, $C(8,0)$, $C(8,8)$, $C(0,0)$, $C(n,0)$, $C(n,n-1)$ (n: a positive integer)

(26) Five people are due to speak at a conference. In how many different orders can they:

(a) speak?
(b) speak, if the keynote speaker must speak last?

(27) A scrabble player with 7 different letters decides to test all possible 5-letter orderings before playing. If he tests 1 ordering each second, how long will it be before he plays?

(28) If you have an alphabet of 26 letters, how many 3-letter words can you make? What if the three letters all have to be different? How many 5-letter words can you make, if you can repeat letters, but cannot have 2 in a row that are the same?

(29) To win the California lottery, you must choose 6 numbers correctly from a set of 51 numbers. How many ways are there to make your 6 choices?

(30) Answer each of these enumeration problems:

(a) How many ways can a group of 10 girls be divided into two basketball teams (A and B say) of 5 players each? What if we don't name the teams?

(b) How many ways can you split 14 people into 7 pairs?

(c) N boys and N girls are in a dance class. How many ways are there to pair them all up?

(31) How many bit strings of length 10 contain either five consecutive 0's or five consecutive 1's?

(32) How many bit strings of length 12 contain

(a) exactly three 1's?

(b) at least three 1's?

(c) at most three 1's?

(d) an equal number of 0's and 1's?

(e) more 0's than 1's?

(33) A person has 10 friends. Over days and months, he invites some of them to dinner parties in such a way that he never invites exactly the same group of friends. How many nights can he keep this up, assuming that one of the possibilities is to ask nobody to dinner?

(34) There are 7 steps in a flight of stairs. When going down, you can jump over some steps if you like, perhaps even all of them. In how many different ways can you go down the stairs?

(35) The following illustration is a map of a city, and you would like to travel from the lower left corner A to the upper right corner D along the roads in the *shortest possible distance* (one possible path is illustrated).

path.pdf

In how many ways can you do this if

(a) there are no (other) restrictions?

(b) you must pass through the point B?

(c) you must pass through the segment BC?

(d) you must not pass through the segment BC?

(36) A bowl contains 10 red balls and 10 blue balls. A woman selects balls at random without looking at them.

(a) How many balls must she select to be sure of having at least three balls of the same color?

(b) How many balls must she select to be sure of having at least three blue balls?

(37) Suppose that there are nine students in a discrete mathematics class at a small college. (a) Show that the class must have at least five male students or at least five female students. (b) Show that the class must have at least three male students or at least seven female students.

(38) In the 17th century, there were more than 800,000 inhabitants of Paris. At the time, it was believed that no one had more than 200,000 hairs on their head. Assuming these numbers are correct and that everyone has at least one hair on their head (that is, no one is completely bald), use the pigeonhole principle to show, as the French writer Pierre Nicole did, that there had to be two Parisians with the same number of hairs on their heads. Then use the generalized pigeonhole principle to show that there had to be at least five Parisians at that time with the same number of hairs on their heads.

(39) (a) Show that if seven integers are selected from the first 10 positive integers, there must be at least two pairs of these integers with the sum 11. (b) Is the conclusion in part (a) true if six integers are selected rather than seven?

(40) (a) Show that among any group of five (not necessarily consecutive) integers, there are two with the same remainder when divided by 4. (b) More generally, let d be a positive integer. Show that among any group of $d+1$ (not necessarily consecutive) integers there are two with exactly the same remainder when they are divided by d.

(41) Let n be a positive integer. Show that in any set of n consecutive integers there is exactly one divisible by n.

(42) Show that if f is a function (= mapping) from S to T, where S and T are finite sets with $|S| > |T|$, then there are elements s_1 and s_2 in S such that $f(s_1) = f(s_2)$, or in other words, f is not one-to-one (= injective).

(43) (a) Let (x_i, y_i), $i = 1, 2, 3, 4, 5$, be a set of five distinct points with integer coordinates in the xy-plane. Show that the midpoint of the line joining at least one pair of these points has integer coordinates. (b) Likewise, let (x_i, y_i, z_i), $i = 1, 2, 3, 4, 5, 6, 7, 8, 9$, be a set of nine distinct points with integer coordinates in xyz- space. Show that the midpoint of at least one pair of these points has integer coordinates.

(1) Use set builder notation to give a description of each of these sets.

(a) $\{0, 3, 6, 9, 12\}$.

(b) $\{-3, -2, -1, 0, 1, 2, 3\}$.

(2) Determine whether each of these pairs of sets are equal.

(a) $\emptyset$, $\{\emptyset\}$.

(b) $\{1, 3, 3, 3, 5, 5, 5, 5, 5\}$, $\{5, 1, 3\}$

(c) $\{\{1\}, \{1, \{1\}\}, \{1, \{1\}\}$.

(3) Consider the sets $S = \{a, b, \{c, d\}\}$ and $T = \{S, \{a\}, \{a, b\}, \emptyset\}$. Now answer the following:

(a) How many elements are in the set S?
(b) How many elements are in the set T?
(c) Is $\emptyset \subseteq S$?
(d) Is $\emptyset \subseteq T$?
(e) Is $\emptyset \in S$?
(f) Is $\emptyset \in T$?
(g) Is $S \in T$?
(h) Is $S \subseteq T$?
(i) Is $a \in S$?
(j) Is $a \in T$?
(k) Is $\{a\} \in S$?
(l) Is $\{a\} \in T$?
(m) Is $\{a\} \subseteq S$?
(n) Is $\{a\} \subseteq T$?
(o) List all subsets of S.

Solution.

(a) 3
(b) 4
(c) Yes, $\emptyset$ is a subset of every set.
(d) Yes, $\emptyset$ is a subset of every set.
(e) No, it is not one of the elements listed.
(f) Yes, it is one of the elements listed.
(g) Yes, it is one of the elements listed.
(h) No, there exists an element of S which is not an element of T (ex: $\{c, d\}$)
(i) Yes
(j) No
(k) No
(l) Yes
(m) Yes
(n) No
(o) $\emptyset, \big\{a\big\}, \big\{b\big\}, \big\{\{c, d\}\big\}, \big\{a, b\big\}, \big\{a, \{c, d\}\big\}, \big\{b, \{c, d\}\big\}, S$

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(4) The **symmetric difference** of A and B, denoted by $A \oplus B$, is the set containing those elements in either A or B, *but not in both* A and B. Show that

(a) $A \oplus B = (A \cup B) - (A \cap B)$. (b) $A \oplus B = (A \cup B) - (A \cap B)$.

Solution. Note that the *definition* of the symmetric difference of A and B essentially says:

$$\begin{aligned} A \oplus B &:= \{x \mid (x \in A \lor x \in B) \land x \notin (A \cap B)\} \\ &= \{x \mid (x \in A \land x \notin B) \lor (x \notin A \land x \in B)\} \end{aligned}$$

(the second line follows from the (logical) equivalence $(p \lor q) \land \neg(p \land q) \equiv (p \land \neg q) \lor (\neg p \land q)$). The first equality proves part (a), and the second equality proves part (b). $\square$

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(5) Consider the following sets:

$$A = \{z \in \mathbb{Z} \mid z^7 + 7z^4 - 8z = 0\} \qquad B = \left\{ q \in \mathbb{Q} \,\middle|\, q = \frac{a_1}{a_2} \text{ for some } a_1, a_2 \in A \right\} \qquad C = \{1, 3, \{1, 3\}\}$$

Find each of the following. Answers should be given in *roster notation with no ellipsis*, and otherwise should be given in *set-builder notation with predicates* that are <u>as simple as possible</u>. Give a brief justification of each answer.[3]

(a) $A \oplus C$
(b) $\mathcal{P}(B) \cap C$
(c) $(\{1, \{2\}\} \cup C) \cap B$
(d) $(A \cup C) \setminus \overline{B}$
(e) $\{\emptyset\} \setminus \mathcal{P}(A)$
(f) $B \cap \overline{(A \cup \overline{C})}$

Solution. Before beginning, it will be easier to rewrite each set in roster notation:

$$A = \{0, 1, -2\}, \qquad B = \left\{0, 1, -2, -\frac{1}{2}\right\}, \qquad C = \{1, 3, \{1, 3\}\}.$$

We now provide the answers:

(a) $A \oplus C = \{0, -2, 3, \{1, 3\}\}$
(b) $\mathcal{P}(B) \cap C = \emptyset$, since there is no subset of B which is also an element of C
(c) $(\{1, \{2\}\} \cup C) \cap B = \big\{1, \{2\}, 3, \{1, 3\}\big\} \cap B = \{1\}$
(d) $(A \cup C) \setminus \overline{B} = (A \cup C) \cap B = \{0, 1, -2\}$
(e) $\{\emptyset\} \setminus \mathcal{P}(A) = \emptyset$
(f) $B \cap \overline{(A \cup \overline{C})} = B \cap (\overline{A} \cap C) = (B \cap C) \cap \overline{A} = (B \cap C) \setminus A = \{1\} \setminus A = \emptyset$

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(6) Let A, B, and C be sets. Prove or disprove each of the following statements (no Venn diagrams or membership tables).

(a) $(A \setminus C) \setminus (B \setminus C) = A \setminus B$.
(b) $A \oplus (B \cap C) = (A \oplus B) \cap (A \oplus C)$.
(c) $\overline{A \cup \overline{B}} \cup \overline{A} = \overline{A}$.
(d) $A \oplus B = A \cup B$ if and only if $A \cap B = \emptyset$.

Solution.

[3] Recall that $\mathcal{P}(A) = \{S \mid S \subseteq A\}$ is the *power set* of A.

(a) This statement is false. For one counterexample, let $A = \{a, b\}, B = \emptyset, C = \{a\}$. Then $(A \setminus C) \setminus (B \setminus C) = \{b\} \setminus \emptyset = \{b\}$, but $A \setminus B = \{a, b\}$.

(b) This statement is false. Let $A = \{1, 2\}, B = \{2, 3\}, C = \{1, 3\}$. Then $A \oplus (B \cap C) = \{1, 2\} \oplus \{3\} = \{1, 2, 3\}$ but $(A \oplus B) \cap (A \oplus C) = \{1, 3\} \cap \{2, 3\} = \{3\}$.

(c) This statement is true; we prove it using identities.

$$\begin{aligned}
\overline{A \cup \overline{B}} &= (\overline{A} \cap \overline{\overline{B}}) \cup \overline{A} && \text{(by DeMorgans Law)} \\
&= (\overline{A} \cap B) \cup \overline{A} && \text{(by double negation)} \\
&= \overline{A}. \ \square && \text{(by absorption)}
\end{aligned}$$

(d) This statement is true. We first prove that $A \oplus B = A \cup B \ \Rightarrow \ A \cap B = \emptyset$ by proving the contrapositive. Suppose there exists some element $x \in A \cap B$ (in other words, that $A \cap B$ is nonempty). Then $x \in A \cup B$ and $x \notin A \oplus B$, so $A \oplus B \neq A \cup B$.
Now, we show that $A \oplus B \neq A \cup B \ \Rightarrow \ A \cap B \neq \emptyset$. Since $A \oplus B = (A \cup B) \setminus (A \cap B)$, it follows that $A \oplus B \subseteq (A \cup B)$. Thus, if the two sets are not equal, then there is some element $y \in A \cup B$ but $y \notin A \oplus B$. Since y is in at least one of A or B, but not exactly one, y is an element of both A and B. Thus $y \in A \cap B$ and so $A \cap B$ is nonempty. $\square$

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(7) Prove that $\mathcal{P}(A) \subseteq \mathcal{P}(B)$ if and only if $A \subseteq B$.

Solution. We give a direct prove by showing each inclusion relation implies the other one:

• Suppose that $\mathcal{P}(A) \subseteq \mathcal{B}$. Since $A \in \mathcal{P}(A)$, we also have $A \in \mathcal{P}(B)$. This, according to the very definition of power set, means $A \subseteq B$.

• Now, assume that $A \subseteq B$, and take an arbitrary element X of $\mathcal{P}(A)$. Since $X \subseteq A$, we get $X \subseteq B$, which means $X \in \mathcal{P}(B)$, done! $\square$

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(8) (a) Let A be a set. Prove that $A \times \emptyset = \emptyset \times A = \emptyset$.
(b) Suppose that $A \times B = \emptyset$, where A and B are sets. What can you conclude?

Solution. (a) We prove $A \times \emptyset = \emptyset$; the other equality is prove the same way. Note that we just need to show $A \times \emptyset \subseteq \emptyset$ (why?) Suppose otherwise (*this is proof by contradiction!*) If $A \times \emptyset \not\subseteq \emptyset$, there should exist some ordered pair (a, x) in $A \times \emptyset$ which is not in $\emptyset$. We then must have $x \in \emptyset$, which is a contradiction! $\square$

(b) The obvious conclusion is that at least one of A or B must be the empty set.

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(9) Show that $A \times B \neq B \times A$, when A and B are non-empty, unless $A = B$.

Solution. Working under the hypothesis $A, B \neq \emptyset$, let us equivalently prove that

$$A \times B = B \times A \Rightarrow A = B.$$

Take an arbitrary element x of A. Since $B \neq \emptyset$, there is some $b \in B$. This means that $(x, b) \in A \times B$. This in turn means $(x, b) \in B \times A$, which yields $x \in B$, hence $A \subseteq B$. The proof of $B \subseteq A$ is similar, thus left as an exercise to the hard working student! □

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(10) Let A, B, and C be sets. Show that[4]

(a) $A - B = A \cap \overline{B}$.
(b) $(A \cap B) \cup (A \cap \overline{B}) = A$.
(c) $(A \cup B) \subseteq (A \cup B \cup C)$.
(d) $(A \cap B \cap C) \subseteq (A \cap B)$.
(e) $(A - B) - C \subseteq (A - C)$.
(f) $(A - B) \cap (B - C) = \emptyset$.
(g) $(B - A) \cup (C - A) = (B \cup C) - A$.

Solution. In what follows, U is some universal set.

(a) We just need to apply the definitions:

$$A - B = \{x \,|\, x \in A \wedge x \notin B\} = \{x \,|\, x \in A \wedge x \in \overline{B}\} = A \cap \overline{B}.$$

(b) Using one of the distribution laws:

$$(A \cap B) \cup (A \cap \overline{B}) = A \cap (B \cup \overline{B}) = A \cap U = A.$$

(c) Using the tautology $\big[(p \vee q) \to (p \vee q \vee r)\big] \equiv \mathbf{T}$, we have

$$x \in (A \cup B) \Rightarrow x \in A \vee x \in B \Rightarrow x \in A \vee x \in B \vee x \in C \Rightarrow x \in (A \cup B \cup C),$$

which is equivalent to what we wished to show.

(d) Very similar to the previous case, except that one needs to apply another tautology, namely

$$\big[(p \wedge q \wedge r) \to (p \wedge q)\big] \equiv \mathbf{T}.$$

The details are left to the student.

(e) And this one is based on the tautology

$$\big[(p \wedge \neg q) \wedge \neg r) \to (p \wedge \neg r)\big] \equiv \mathbf{T}.$$

Finish the proof by yourself!

(f) And this one is based on the logical equivalence

$$\big[(p \wedge \neg q) \wedge (q \wedge \neg r)\big] \equiv \mathbf{F}.$$

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[4]Remember: $A - B$ is the same as $A \setminus B = \{x \in A \,|\, x \notin B\}$.

(g) Using part (a) *three times*, and using one of the distributions laws, we have

$$(B-A)\cup(C-A) = (B\cap\overline{A})\cup(C\cap\overline{A}) = (B\cup C)\cap\overline{A} = (B\cup C)-A,$$

as was to be shown.

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(11) Determine whether each of these functions (mappings!) from $\mathbb{Z}$ to $\mathbb{Z}$ is injective (= one-to-one).

(a) $f(n) = n^3$,
(b) $f(n) = n^2 + n + 41$,
(c) $f(n) = -n^5 - 3n^3 - 5n + 17$,
(d) $f(n) = \lceil \frac{n}{2} \rceil$.

Solution.
(a) The function f is indeed injective, as one can see more generally that the function $f : \mathbb{R} \leftrightarrow \mathbb{R}$, with the same rule $f(x) = x^3$ is so. (Note: the *real-valued* cousin f has an everywhere positive derivative!)

(b) This function is not injective: note for instance $g(0) = g(-1)$.

(c) The same story as part (a): note that the real-valued cousin has an everywhere negative derivative.

(d) The answer is a **no** (by inspection: $k(1) = k(2)$.)

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(12) Determine whether $f : \mathbb{Z}\times\mathbb{Z} \leftrightarrow \mathbb{Z}$ is surjective (= onto) if

(a) $f(m,n) = 2m - n$,
(b) $f(m,n) = m^2 - n^2$,
(c) $f(m,n) = |m| - |n|$,
(d) $f(m,n) = \left\lceil \dfrac{m+n}{2} \right\rceil$,
(e) $f(m,n) = m^2 + n^2$.

Solution. (a) The answer is Yes. To show this, let $k \in \mathbb{Z}$ be given, and note that $f(0,-k) = k$. [To show the surjectivity, it suffice to exhibit one (ordered) pair which is mapped to k, but it is worth noting that $(0,-k)$ is not the only pair that is mapped to k. More generally, the pair $(2m, 2m-k)$, for any $m \in \mathbb{Z}$, does the job!]

(b) The answer is No. To show the non-surjectivity, it would suffice to find one $k \in \mathbb{Z}$, for which there is no $(m,n) \mapsto k$, under f. Let us show this for $k = 2$. We must show that $m^2 - n^2 \neq 2$, for all pairs of integers m, n. Assume otherwise, namely suppose

$$m^2 - n^2 = 2.$$

Without loss of generality, we may assume that $m > n \geqslant 0$ (you must convince yourself that we can make such assumption(s)!) Now comparing the factorisation $m^2 - n^2 = (m+n)(m-n)$ to the *unique factorisation* $2 = (2)\cdot(1)$, we would infer that

$$m+n = 2 \quad \text{and} \quad m-n = 1,$$

(why?) But this leads to $m = 3/2$ and $n = 1/2$, which is absurd!

(c) We claim that this mapping is onto. To establish that, choose an arbitrary $k \in \mathbb{Z}$, and consider two cases:

• If $k \geqslant 0$, then we have $f(k, 0) = |k| - |0| = k$;

• And if $k < 0$, then we have $f(0, k) = |0| - |k| = -(-k) = k$. [Note that the pair $(0, -k)$ also does the job!]

(d) This map is also onto: $f(k, k) = \left\lceil \frac{k+k}{2} \right\rceil = \lceil k \rceil = k$, true for every $k \in \mathbb{Z}$.

(e) This time the assertion is that this map is not surjective: even though it is rather trivial that the expression $m^2 + n^2$ can never ever be equal to any negative integer (or even a negative real number for that matter), let us show that the *quadratic form* $m^2 + n^2$ cannot even represent the positive integer $k = 3$:

$$m^2 + n^2 \neq 3.$$

This can easily be done by examining all possible cases: we just need to try the possibilities for $m = 0, \pm 1$ and $n = 0, \pm 1$. None of the combinations of these integers work (check that!) We are done.

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(13) For each of the mappings f in the previous problem, determine $f^{-1}(S)$ (i.e., the inverse-image or the preimage) of S if (i) $S = \{0\}$, (ii) $S = \{1\}$, or (iii) $S = \{0, 1\}$.

Solution.

We shall only provide the answers, leaving the details to the interested student!

(i) (a) $f^{-1}(\{0\}) = \{(k, 2k) \mid k \in \mathbb{Z}\}$;
(b) $f^{-1}(\{0\}) = \{(\pm k, \pm k) \mid k \in \mathbb{Z}\}$, where the plus/minus signs are *independent*;
(c) The same answer is part (b);
(d) $f^{-1}(\{0\}) = \{(k, -k) \mid k \in \mathbb{Z}\} \cup \{(k, -k-1) \mid k \in \mathbb{Z}\}$;
(e) $f^{-1}(\{0\}) = \{(0, 0)\}$.

(ii) (a) $f^{-1}(\{1\}) = \{(k, 2k-1) \mid k \in \mathbb{Z}\}$;
(b) $f^{-1}(\{1\}) = \{(\pm 1, 0)\}$;
(c) $f^{-1}(\{1\}) = \{(k, \pm(|k| + 1)) \mid k \in \mathbb{Z}\}$;
(d) $f^{-1}(\{1\}) = \{(k, -k+1) \mid k \in \mathbb{Z}\} \cup \{(k, -k+2) \mid k \in \mathbb{Z}\}$;
(e) $f^{-1}(\{1\}) = \{(\pm 1, 0), (0, \pm 1)\}$.

(iii) Thanks to the general statement

$$f^{-1}(S \cup T) = f^{-1}(S) \cup f^{-1}(T),$$

proved in class, in each case, the answer is obtained by taking the union between $f^{-1}(\{0\})$ and $f^{-1}(\{1\})$.

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(14) Let $g : A \leftrightarrow B$ and $f : B \leftrightarrow C$ be functions. Prove
(a) If f and g are injective, then $f \circ g$ is injective.
(b) If f and g are surjective, then $f \circ g$ is surjective.
(c) Assuming mappings, if f and g are bijections, then so is $f \circ g$.

Solution.

(a) Assume that $(f \circ g)(x) = (f \circ g)(x')$, or that $f(g(x)) = f(g(x'))$. Using the injectivity of f, we infer that $g(x) = g(x')$, and using the injectivity of g, we deduce that $x = x'$, as was to be shown.

(b) Choose an arbitrary element z of C. Since f is onto, there exists some $y \in B$ such that $f(y) = z$. On the other hand, g is also surjective, which means that there exists some $x \in A$ with $g(x) = y$. Putting these things together, we see that $(f \circ g)(x) = f(g(x)) = f(y) = z$, which proves the assertion.

(c) After parts (a) and (b), this is obvious, by virtue of the fact that a mapping is bijective if and only if it is both injective and surjective.

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(15) Let $g : A \leftrightarrow B$ and $f : B \leftrightarrow C$ be functions. Prove or disprove:
- (a) If $f \circ g$ and g are injective, then f is injective.
- (b) If $f \circ g$ and g are surjective, then f is surjective.
- (c) If $f \circ g$ is a bijection, then f is injective if and only if g is surjective.

Solution.

(a) This statement is false. Let $A = \{a\}$, $B = \{r, s\}$, $C = \{x\}$, $g = \{(a, r)\}$ and $f = \{(r, x), (s, x)\}$. This means $f \circ g = \{(a, x)\}$. In this case, both $f \circ g$ and g are injective, but f is not.

(b) This is true; in fact, a “stronger” statement is true – we only need to assume that $f \circ g$ is surjective. Since $f \circ g$ is surjective, for every $c \in C$ there exists some $a \in A$ such that $(f \circ g)(a) = c$, or $f(g(a)) = c$. Since f is defined for $g(a)$, this means that $g(a) \in B$ ($g(a)$ is in the domain of f). In other words, we have exhibited a value $b \in B$ for which $f(b) = c$ for any $c \in C$, so f is surjective.

(c) Suppose that f is not injective, so there exists (at least) two elements $b_1, b_2 \in B$ and $c \in C$ such that $f(b_1) = f(b_2) = c$. If g is surjective, then there exist a_1, a_2 such that $g(a_1) = b_1$ and $g(a_2) = b_2$. However, this implies that $f(g(a_1)) = f(g(a_2)) = c$, contradicting the fact that $f \circ g$ is an injection by virtue of being a bijection. Thus g is not surjective.

Now, suppose that g is not surjective, so there exists some element $b \in B$ such that $g(a) \neq b$ for every $a \in A$. Since f is a function[5] from B to C, $f(b) = c$ for some $c \in C$, and since $f \circ g$ is surjective (by virtue of being a bijection), there is some element $a \in A$ such that $(f \circ g)(a) = c$. Since $g(a) \neq b$ and $f(g(a)) = f(b) = c$, this implies that f is not injective.

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(16) For each function, determine if it is injective and/or surjective. Prove your answers.
- (a) $g : \mathbb{Z} \times \mathbb{Z}^+ \to \mathbb{Q}$ given by $g((x, y)) = \frac{x}{y}$
- (b) $h : \mathbb{R}^+ \to \mathbb{R}^+$ given by $h(x) = x\lceil x \rceil$.

Solution.

(a) It is easy to see that g is not injective; for example, $g(4, 2) = g(2, 1) = \frac{1}{2}$. To see that g is surjective, let $q \in Q$ be arbitrary. We can write $q = \frac{m}{n}$ for integers m and n, and we can further assume that

[5] Here the term “function” means *mapping!*

$n > 0$ (if $q > 0$, m will be positive; if $q < 0$ then m will be negative). Thus $m \in \mathbb{Z}, n \in \mathbb{Z}^+$, and so there exists $(m, n) \in \mathbb{Z} \times \mathbb{Z}^+$ such that $g(m, n) = \frac{m}{n} = q$.

(b) To see that h is not surjective, we consider values of x by cases. If $0 < x \leqslant 1$, then $x\lceil x \rceil = x(1) \leqslant 1$. If $x > 1$, then $x\lceil x \rceil \geqslant x(2) > 2$. Thus, there exists no value of x for which $1 < h(x) \leqslant 2$ and so h is not surjective. We now show that h is injective. Let $x_1, x_2 \in \mathbb{R}^+$ be such that $x_1 \neq x_2$; we show that $h(x_1) \neq h(x_2)$. If $\lceil x_1 \rceil = \lceil x_2 \rceil$, then $x_1\lceil x_1 \rceil = x_1\lceil x_2 \rceil \neq x_2\lceil x_2 \rceil$. Now, if $\lceil x_1 \rceil \neq \lceil x_2 \rceil$, we may assume without loss of generality that $x_1 < x_2$ and so $\lceil x_1 \rceil < \lceil x_2 \rceil$ (since these values are not equal by assumption). This implies that $x_1\lceil x_1 \rceil < x_1\lceil x_2 \rceil < x_2\lceil x_2 \rceil$, and so $x_1\lceil x_1 \rceil \neq x_2\lceil x_2 \rceil$.

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(17) How many strings of eight English letters are there

(a) that contain no vowels, if letters can be repeated? (21^8);
(b) that contain no vowels, if letters cannot be repeated? ($21 \times 20 \times 19 \times \cdots \times 14$);
(c) that start with a vowel, if letters can be repeated? (5×26^7);
(d) that start with a vowel, if letters cannot be repeated? ($5 \times 25 \times 24 \times \cdots \times 19$);
(e) that contain at least one vowel, if letters can be repeated? ($26^8 - 21^8$);
(f) that contain exactly one vowel, if letters can be repeated? (5×22^7);
(g) that start with X and contain at least one vowel, if letters can be repeated? ($26^7 - 21^7$);
(h) that start and end with X and contain at least one vowel, if letters can be repeated? ($26^6 - 21^6$)

(18) How many functions (= mappings) are there from the set $\{1, 2, ..., n\}$, where n is a positive integer, to the set $\{0, 1\}$

(a) that are one-to-one (= injective)? (2, if $n \leqslant 2$; but **none**, otherwise!);
(b) that assign 0 to both 1 and n? (2^{n-2});
(c) that assign 1 to exactly one of the positive integers less than n? ($2(n-1)$)

(19) In how many ways can a photographer at a wedding arrange six people in a row, including the bride and groom, if

(a) the bride must be next to the groom? ($2 \times 5!$);
(b) the bride is not next to the groom? ($6! - 2 \times 5!$);
(c) the bride is positioned somewhere to the left of the groom? ($\frac{1}{2} \times 6!$)

(20) How many 10-digit numbers:

(a) are there? (9×10^9);
(b) how many of them are greater than 2,000,000,000? ($8 \times 10^8 - 1$);
(c) how many of them are even? ($9 \times 10^8 \times 5$);
(d) how many of them are odd and have different digits? ($5 \times 8 \times 8!$)

(21) How many ways are there to arrange the letters a, b, c, and d such that a is not followed immediately by b? $(4! - 3!)$

(22) Consider the number 35964.

(a) How many 3-digit numbers can be formed using digits from 35964 if digits may be repeated? (5^3);

(b) How many 3-digit numbers can be formed using digits from 35964 if no digits may be repeated? $(5 \times 4 \times 3)$;

(c) What is the sum of all of the 3-digit numbers in part (b)? $(12 \times (100 + 10 + 1) \times (3 + 5 + 9 + 6 + 4))$;

(d) Repeat (c) if the original number was 7238954 and if we wanted all the 5-digit numbers.

(23) Answer each of these enumeration problems:

(a) How many different ordering of the letters of the word MISSISSIPPI are there? $(\frac{11!}{4! \times 3 \times 2!})$;

(b) How many different ordering of the letters of the word STATISTICS: **(i)** are there? **(ii)** begin and end with an S? $(\frac{10!}{3! \cdot 3! \cdot 2!}$, and $\frac{8!}{3! \cdot 2!})$;

(c) How many different orderings of the letters of the word SUCCESSFULLY: **(i)** are there? **(ii)** are there if the 3 S's must be together in each ordering? **(iii)** are there, if the subword SUCCESS must appear in each ordering? $(\frac{12!}{3! \cdot 2! \cdot 2! \cdot 2!}, \quad \frac{10!}{(2!)^3}, \quad \frac{6!}{2!})$

(24) How many permutations of the letters ABCDEFG contain

(a) the string BCD? $(4!)$;

(b) the string CFGA? $(3!)$;

(c) the strings BA and GF? $(4!)$;

(d) the strings ABC and DE? $(3!)$;

(e) the strings ABC and CDE? $(2!)$.

(25) Find the value of each of these quantities.

(a) $P(6,3) = 120$, $P(6,5) = 720$, $P(8,1) = 8$, $P(8,8) = 8!$, $P(n,1) = n$, $P(n,n) = n!$ (n: a positive integer)

(b) $C(5,1) = 5$, $C(5,4) = 5$, $C(8,0) = 1$, $C(8,8) = 1$, $C(0,0) = 1$, $C(n,0) = 1$, $C(n, n-1) = n$ (n: a positive integer)

(26) Five people are due to speak at a conference. In how many different orders can they:

(a) speak? $(5!)$;

(b) speak, if the keynote speaker must speak last? $(4!)$.

(27) A scrabble player with 7 different letters decides to test all possible 5-letter orderings before playing. If he tests 1 ordering each second, how long will it be before he plays? ($P(7,5)$ seconds, or 42 minutes)

(28) If you have an alphabet of 26 letters, how many 3-letter words can you make? What if the three letters all have to be different? How many 5-letter words can you make, if you can repeat letters, but cannot have 2 in a row that are the same? (Only the last part: 26×25^4)

(29) To win the California lottery, you must choose 6 numbers correctly from a set of 51 numbers. How many ways are there to make your 6 choices? ($C(51, 6) = \binom{51}{6} = \frac{51!}{6! \cdot 45!}$)

(30) Answer each of these enumeration problems:

(a) How many ways can a group of 10 girls be divided into two basketball teams (A and B say) of 5 players each? What if we don't name the teams? ($\binom{10}{5}\binom{5}{5}$, and $\dfrac{1}{2}\dbinom{10}{5}$);

(b) How many ways can you split 14 people into 7 pairs? ($\binom{14}{2}\binom{12}{2}\binom{10}{2}\binom{8}{2}\binom{6}{2}\binom{4}{2}\binom{2}{2}$, or $\frac{1}{7!} \times \binom{14}{2}\binom{12}{2}\binom{10}{2}\binom{8}{2}\binom{6}{2}\binom{4}{2}\binom{2}{2} = 13 \times 11 \times \cdots \times 3 \times 1$, depending how we view the splitting process!);

(c) N boys and N girls are in a dance class. How many ways are there to pair them all up? ($(N!)^2$, or $N!$, depending how the pairings are done!)

(31) How many bit strings of length 10 contain either five consecutive 0's or five consecutive 1's? (This is rather hard, I will do it later!)

(32) How many bit strings of length 12 contain

(a) exactly three 1's? ($\binom{12}{3}$);

(b) at least three 1's? ($\sum_{k=3}^{12}\binom{12}{k}$, or equally $2^{12} - \sum_{k=0}^{2}\binom{12}{k}$);

(c) at most three 1's? ($\sum_{k=0}^{3}\binom{12}{k}$, or equally $2^{12} - \sum_{k=4}^{12}\binom{12}{k}$);

(d) an equal number of 0's and 1's? ($\binom{12}{6}$);

(e) more 0's than 1's? ($\frac{1}{2}\left(2^{12} - \binom{12}{6}\right)$)

(33) A person has 10 friends. Over days and months, he invites some of them to dinner parties in such a way that he never invites exactly the same group of friends. How many nights can he keep this up, assuming that one of the possibilities is to ask nobody to dinner? ($\sum_{k=0}^{10}\binom{10}{k}$, which is the same as 2^{10})

(34) There are 7 steps in a flight of stairs. When going down, you can jump over some steps if you like, perhaps even all of them. In how many different ways can you go down the stairs? ($\sum_{k=0}^{6}\binom{6}{k}$, which is the same as 2^6)

(35)

(36) A bowl contains 10 red balls and 10 blue balls. A woman selects balls at random without looking at them.

(a) How many balls must she select to be sure of having at least three balls of the same color? (7 balls);

(b) How many balls must she select to be sure of having at least three blue balls? (13 balls)

(37) Suppose that there are nine students in a discrete mathematics class at a small college. (a) Show that the class must have at least five male students or at least five female students. (b) Show that the class must have at least three male students or at least seven female students. (Rather simple; Write an argument for it!)

(38) In the 17th century, there were more than 800,000 inhabitants of Paris. At the time, it was believed that no one had more than 200,000 hairs on their head. Assuming these numbers are correct and that everyone has at least one hair on their head (that is, no one is completely bald), use the pigeonhole principle to show, as the French writer Pierre Nicole did, that there had to be two Parisians with the same number of hairs on their heads. Then use the generalized pigeonhole principle to show that there had to be at least five Parisians at that time with the same number of hairs on their heads.

(39) (a) Show that if seven integers are selected from the first 10 positive integers, there must be at least two pairs of these integers with the sum 11. (b) Is the conclusion in part (a) true if six integers are selected rather than seven?

(40)

(41)

(42)

(43)