ESTIMATING AREAS AND VOLUMES BY MONTE CARLO TECHNIQUES

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WHO IS MONTE CARLO?

- Invented in the late 1940 by Stanislaw Ulam & John von Neumann (nuclear weapon projects)

- Monte Carlo - City in Monaco famed for it casinos and games of chance (like roulette, dice, slot machines)

- Main idea is to use **random samples** of parameters or inputs to explore the behavior of a complex system or process
- The greater the number of samles the more accuate is the evaluation of this area
- When are models are too complex for an analytical solution - so they had to be evaluated numerically

(method is only competitive for omplicated and/ or multi-dimensional functions)

RANDOM?

Randomness is difficult to define.

1. the values are uniformly distributed over a defined interval or set
2. it is impossible to predict future values based on past or present ones

Pseudo-random Numbers

- Computer algorithms for generating random # are deterministic; a sequence generated may appear random
- They are quite predictable and reproducible
- Any sequence must eventually repeat

THE MAIN STEPS OF MONTE CARLO SIMULATION

1. Model the inputs and process.
2. Draw a vector of random varieties
3. Evaluate the function of interest
4. Repeat the last two steps many times, aggregating the results

MONTE CARLO INTEGRATION

$$\int f\,dV = \bar{f}V$$

- The integral of a function, *f*, over a volume, V, is equal to the mean of *f* times V

$$\int f\,dV \approx V(f) \pm V\sqrt{\frac{(f^2) - (f)^2}{N}}$$

- *Where*

$$(f) = \frac{1}{N}\sum_{i=1}^{N} f(x_i)$$

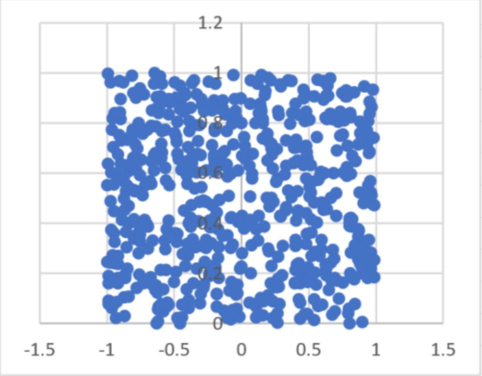
$$(f^2) = \frac{1}{N}\sum_{i=1}^{N} f^2(x_i)$$

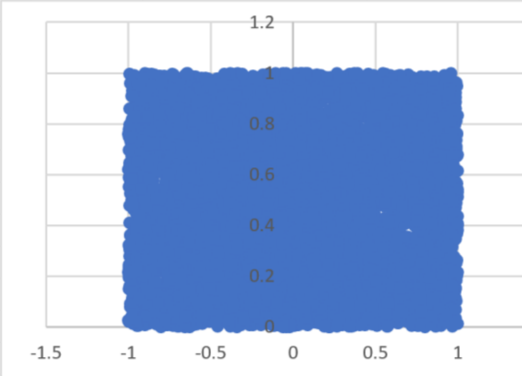
Y=X²

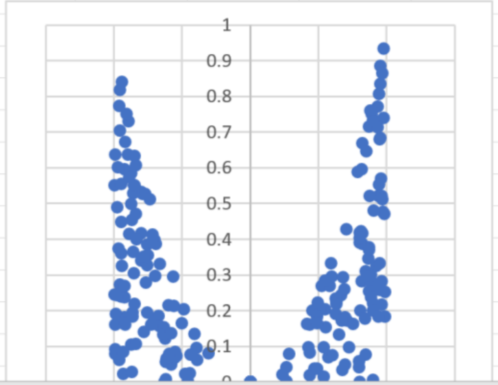
- $\int_{-1}^{1} x^2 \mathbf{dx} = \frac{2}{3}$
- $\int_{-1}^{1} x^2 \mathbf{dx} \approx 2(x^2) \pm 2\sqrt{\frac{((x^2)^2) - (x^2)^2}{N}}$

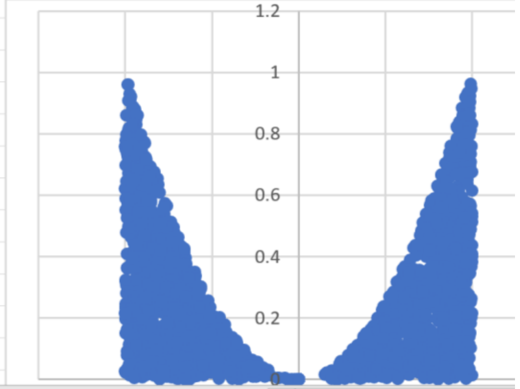
					Total Points =	6161	600
range for x	-1	1		y=x^2	Points under =	2068	209
range for y	0	1		x^3/3	Total area =	2	2
				0.666667	Area under =	0.67132	0.696667
	X	Y					
1	-0.7335	0.756213	0	0	0		
1	-0.19903	0.766698	0	0	0		
1	0.284896	0.078322	0.284896	0.078322	1		
1	-0.7223	0.913894	0	0	0		
1	0.984764	0.471263	0.984764	0.471263	1		
1	0.074248	0.313422	0	0	0		
1	-0.96231	0.062792	-0.96231	0.062792	1		
1	-0.33042	0.21882	0	0	0		
1	-0.3189	0.407289	0	0	0		
1	0.666239	0.242965	0.666239	0.242965	1		
1	0.820753	0.387597	0.820753	0.387597	1		
1	0.635759	0.816171	0	0	0		
1	-0.92046	0.181748	-0.92046	0.181748	1		
1	-0.94877	0.360971	-0.94877	0.360971	1		
1	0.00542	0.428287	0	0	0		
1	-0.77297	0.69967	0	0	0		
1	-0.75531	0.327914	-0.75531	0.327914	1		
1	0.285287	0.953975	0	0	0		
1	0.73199	0.748992	0	0	0		
1	-0.98639	0.190513	-0.98639	0.190513	1		
1	0.658538	0.977965	0	0	0		
1	0.868613	0.715151	0.868613	0.715151	1		

Sheet2 Sheet1





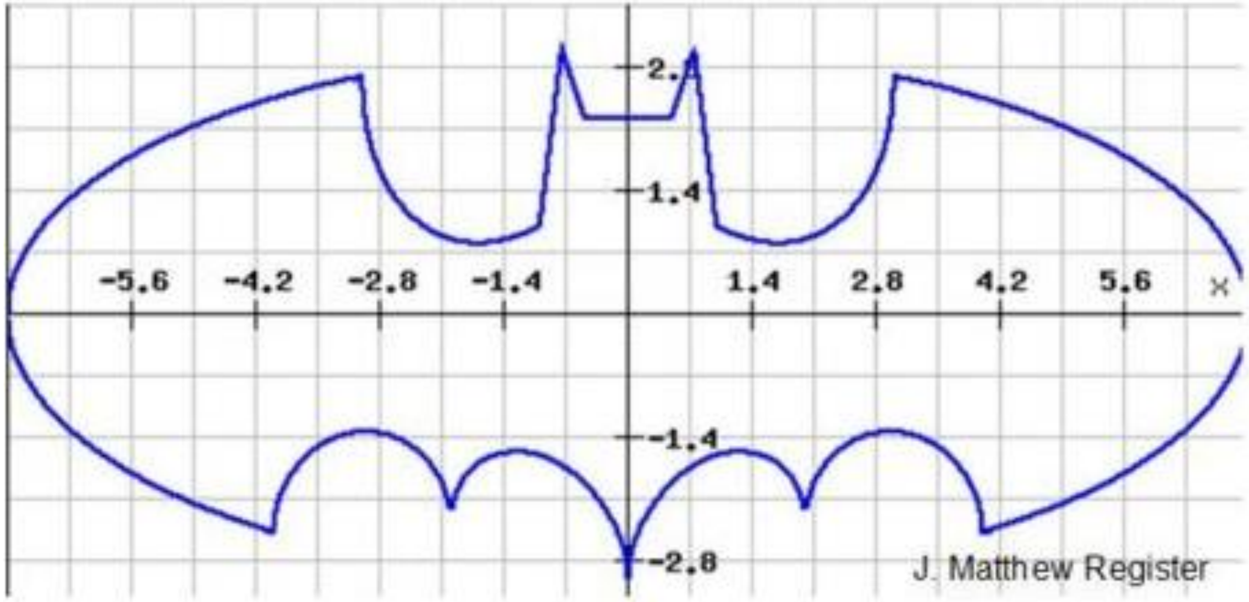




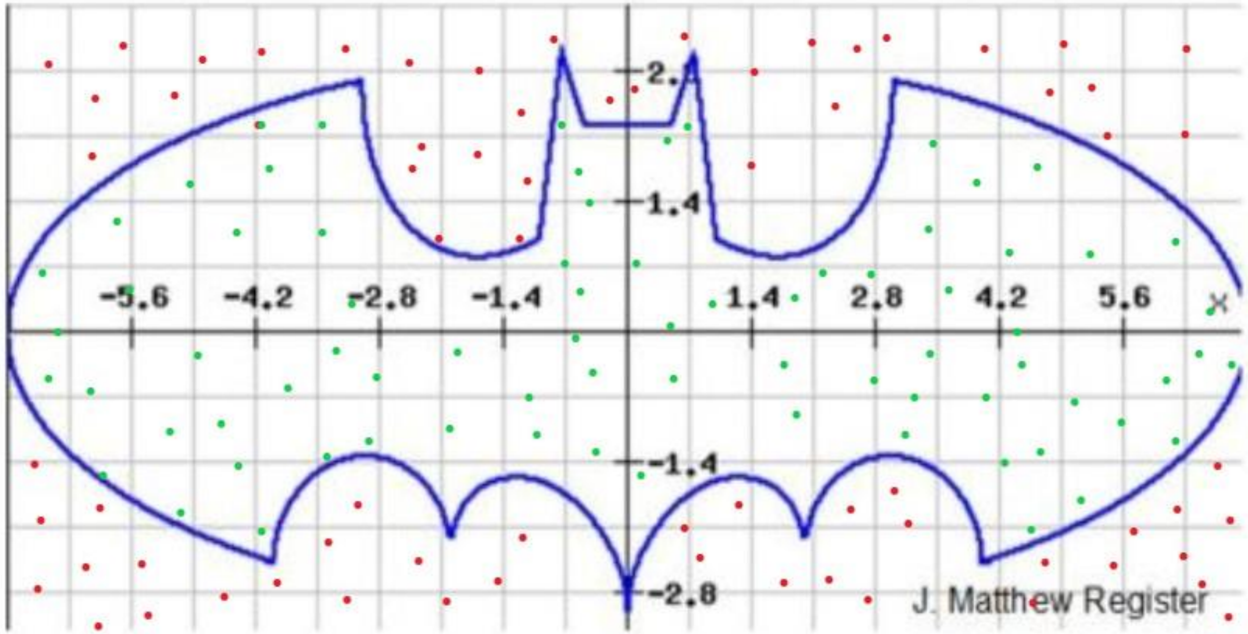
AREA OF PI

- Randomly selected points x_i, y_i i=1,2…..N in unique square
- R=$\frac{M}{N}$ - ratio
- Where M – number of points that satisfy $x_i^2 + y_i^2 \leq 1$

$$\left(\left(\frac{x}{7}\right)^2\sqrt{\frac{\left||x|-3\right|}{|x|-3}}+\left(\frac{y}{3}\right)^2\sqrt{\frac{\left|y+\frac{3\sqrt{33}}{7}\right|}{y+\frac{3\sqrt{33}}{7}}}-1\right)\cdot\left(\left|\frac{x}{2}\right|-\left(\frac{3\sqrt{33}-7}{112}\right)x^2-3+\sqrt{1-\left(\left||x|-2\right|-1\right)^2}-y\right)$$
$$\cdot\left(9\sqrt{\frac{\left|(|x|-1)(|x|-.75)\right|}{(1-|x|)(|x|-.75)}}-8|x|-y\right)\cdot\left(3|x|+.75\sqrt{\frac{\left|(|x|-.75)(|x|-.5)\right|}{(.75-|x|)(|x|-.5)}}-y\right)$$
$$\cdot\left(2.25\sqrt{\frac{\left|(x-.5)(x+.5)\right|}{(.5-x)(.5+x)}}-y\right)\cdot\left(\frac{6\sqrt{10}}{7}+(1.5-.5|x|)\sqrt{\frac{\left||x|-1\right|}{|x|-1}}-\frac{6\sqrt{10}}{14}\sqrt{4-(|x|-1)^2}-y\right)=0$$

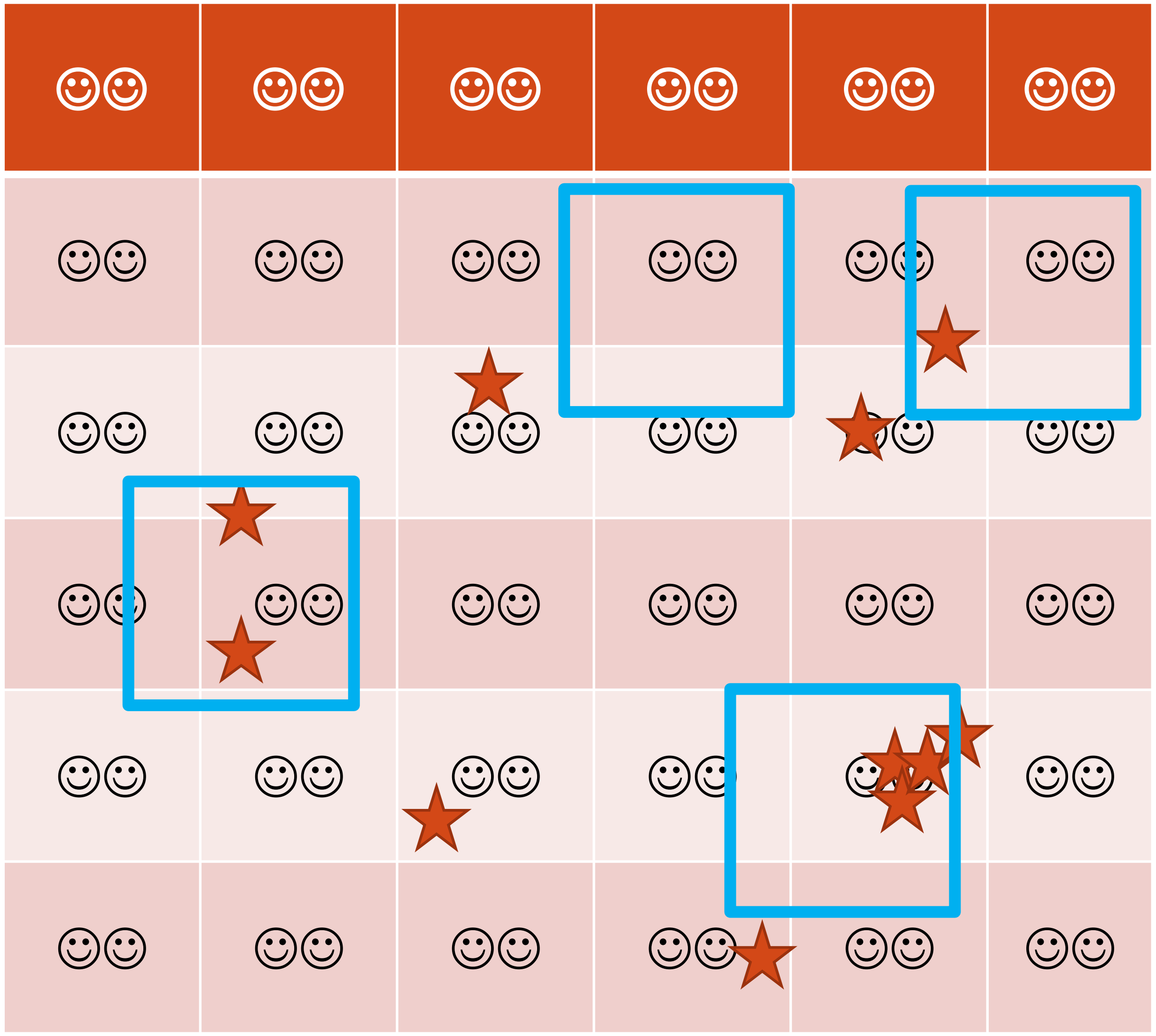


$$\frac{area\ B}{area\ A}\approx\frac{\#\ green}{\#\ all}\approx\frac{n}{N}$$



$$Area\ B\approx\frac{area\ A*\#\ green}{\#\ all}$$

$$Area\ B\approx\frac{(12x6)*75}{(70+75)}\approx 37.2413$$



\# of counted noise = (sq1.noise+sq2.noise+…+sqN.noise

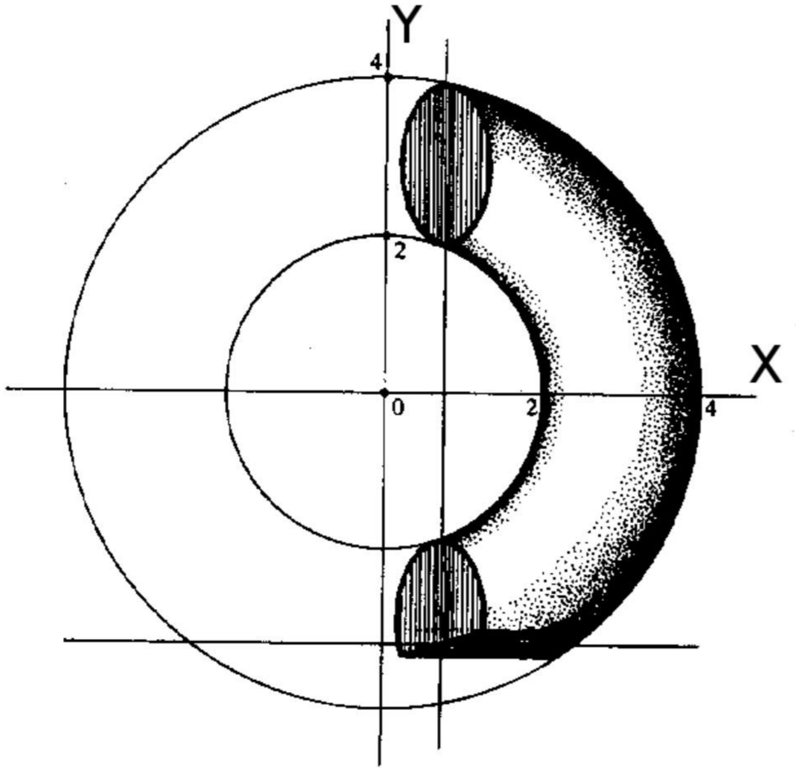
\# of counted objects = (sq1.ob+sq2.ob+…+sqN.ob)

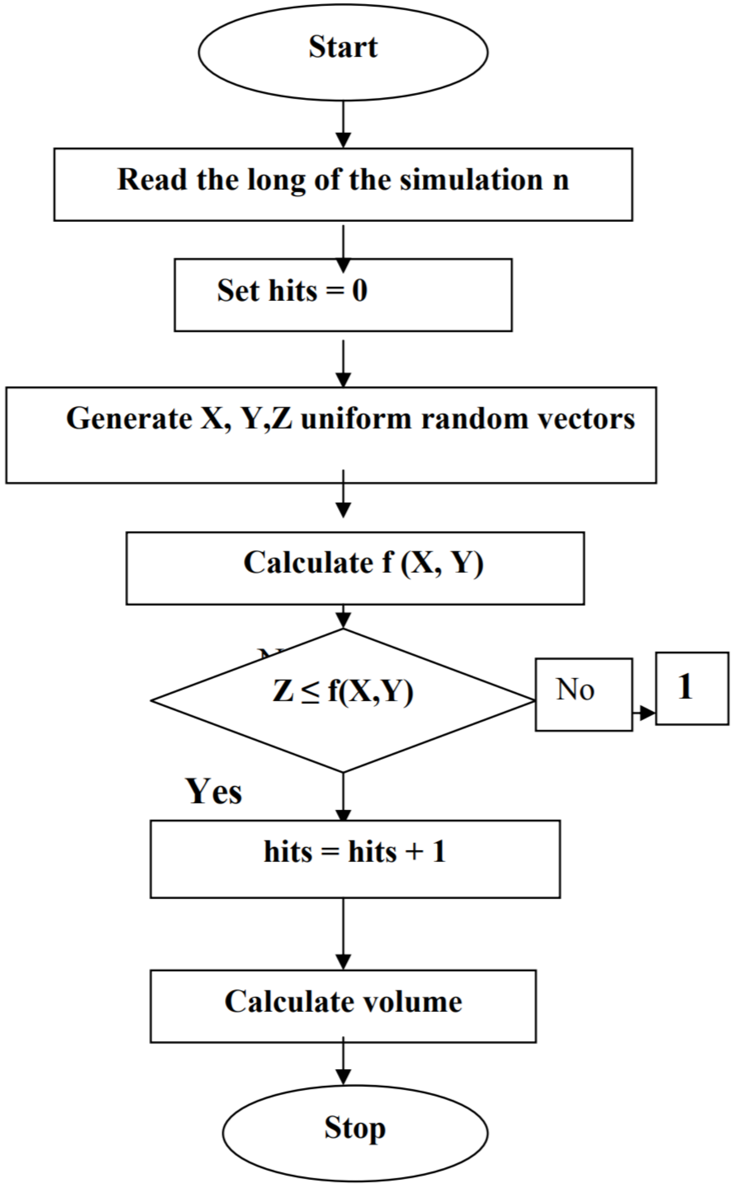
$$\frac{area\ B*N}{area\ A}=\frac{\#\ of\ counted\ noise}{noise}$$

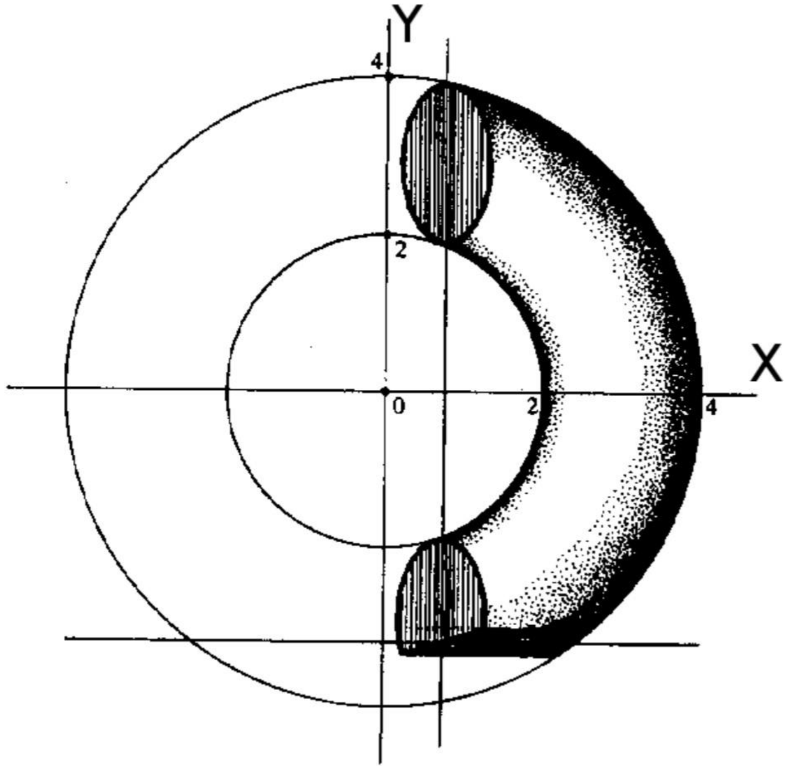
$$noise=\frac{area\ A*\#\ of\ counted\ noise}{area\ B*N}$$

MORE COMPLICATED INTEGRA'

- Integrate $f\ (x\ , y\ , z) = e^{x+y+z}$ inside the volume V, a torus







- Pick a volume, W, that encloses the integration volume, V.

1. e.g. 1 < x < 4, -3 < y < 4, and -1 < z < 1
2. The actual volume isn't very important, as long as it is larger than the integration region.

- Redefine the integration function to "include limits" and integrate over W
- $z^2 + (\sqrt{x^2+y^2} - 3)^2 \leq 1, x > 1 \; and \; y > -3$

$$f^1(x,y,z) = \begin{cases} e^{x+y+z} \; if \; z^2 + (\sqrt{x^2+y^2} - 3)^2 \leq 1, x > 1 \; and \; y > -3 \\ 0, otherwise \end{cases}$$

$$\int f dV = \int f^1 dW$$

ADVANTAGE

1. **Simple** to implement (we don't need to know the domain, we only need to know where we are)
2. Do not rely on **smoothness** for convergence (only think as 1 and 0)
3. The convergence rate **does not degrade** in higher dimensions (volume of a region in a three-dimensional space)
4. MC methods provide a **result**, but takes a lot of CPU time to get an accurate answer, but CPU is **cheap**

REFERENCES:

- Engels, H. (1980),"Numerical Quadrature and Cubature", Academic Press, New York.
- Stokes, V. P. (2004), "Monte Carlo simulation: basics", Survey Report, Uppsala Univ. Press.
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- Smaldone, Tony, 2001, "Modeling and Simulation", McGraw Hill, New York.