Практическое задание к Уроку 6. Производная функции одной переменной. Часть 1

1. Найти производные указанных функций:

1) $y = x^3 \log_2 x$

$$y' = (x^3 \log_2 x)' = (x^3)' \cdot \log_2 x + x^3 \cdot (\log_2 x)' = 3x^2 \cdot \log_2 x + \frac{x^2}{ln\ 2} = x^2 \left(3 \log_2 x + \frac{1}{ln\ 2}\right)$$

2) $y = -10 arctg\ x + 7e^x$

$$y' = -10' \cdot arctg\ x + (-10) \cdot (arctg\ x)' + 7 \cdot e^{x'} =$$
$$= 0 \cdot arctg\ x - 10 \cdot \left(-\frac{1}{1+x^2}\right) + 7 \cdot e^x = \frac{10}{1+x^2} + 7e^x$$

3) $y = \frac{1}{\sqrt[3]{x^2}} - \frac{2}{x^3} + \sqrt{7} \cdot x = x^{-\frac{2}{3}} - 2 \cdot x^{-3} + 7^{\frac{1}{2}} \cdot x$

$$y' = \left(x^{-\frac{2}{3}} - 2 \cdot x^{-3} + 7^{\frac{1}{2}} \cdot x\right)' = \left(x^{-\frac{2}{3}}\right)' - 2 \cdot (x^{-3})' + 7^{\frac{1}{2}} \cdot (x)' =$$
$$= -\frac{2}{3} x^{-\frac{2}{3}-1} - 2 \cdot (-3) \cdot x^{-3-1} + 7^{\frac{1}{2}} \cdot 1 \cdot x^{1-1} = -\frac{2}{3} x^{-\frac{5}{3}} + 6 \cdot x^{-4} + 7^{\frac{1}{2}} = \frac{6}{x^4} - \frac{2}{3\sqrt[3]{x^5}} + \sqrt{7}$$
$$x' = 1 \cdot x^{1-1}$$

4) $y = \cos \frac{1-\sqrt{x}}{1+\sqrt{x}}$

$$y' = \left(\cos \frac{1-\sqrt{x}}{1+\sqrt{x}}\right)' = -sin \frac{1-\sqrt{x}}{1+\sqrt{x}} \cdot \left(\frac{1-\sqrt{x}}{1+\sqrt{x}}\right)' =$$
$$= -sin \frac{1-\sqrt{x}}{1+\sqrt{x}} \cdot \frac{(1-\sqrt{x})' \cdot (1+\sqrt{x}) - (1-\sqrt{x}) \cdot (1+\sqrt{x})'}{(1+\sqrt{x})^2} =$$
$$= -sin \frac{1-\sqrt{x}}{1+\sqrt{x}} \cdot \frac{\left(0 - \frac{1}{2\sqrt{x}}\right) \cdot (1+\sqrt{x}) - (1-\sqrt{x}) \cdot \left(0 + \frac{1}{2\sqrt{x}}\right)}{1 + 2\sqrt{x} + x} =$$
$$= -sin \frac{1-\sqrt{x}}{1+\sqrt{x}} \cdot \frac{-\frac{1}{2\sqrt{x}} - \frac{\sqrt{x}}{2\sqrt{x}} - \frac{1}{2\sqrt{x}} + \frac{\sqrt{x}}{2\sqrt{x}}}{1 + 2\sqrt{x} + x} = -sin \frac{1-\sqrt{x}}{1+\sqrt{x}} \cdot \frac{-\sqrt{x}^{-1}}{1 + 2\sqrt{x} + x}$$

5) $y = e^{sh^2 5x}$

$$y' = \left(e^{sh^2 5x}\right)' = e^{sh^2 5x} \cdot (sh^2 5x)' = e^{sh^2 5x} \cdot 2sh5x \cdot ch5x \cdot 5 = 5\ sh\ 10x\ e^{sh^2 5x}$$

6) $y = ln \frac{(x+1)(x+3)^3}{(x+2)^3(x+4)}$

$$y = ln \frac{(x+1)(x+3)^3}{(x+2)^3(x+4)} = ln(x+1) + ln(x+3)^3 - ln(x+2)^3 - ln(x+4) =$$
$$= ln(x+1) + 3ln(x+3) - 3ln(x+2) - ln(x+4)$$
$$y' = \left(ln(x+1) + 3ln(x+3) - 3ln(x+2) - ln(x+4)\right)' = \frac{1}{x+1} - \frac{3}{x+2} + \frac{3}{x+3} - \frac{1}{x+4} =$$
$$= \frac{(x+2)(x+3)(x+4) - 3(x+1)(x+3)(x+4) + 3(x+1)(x+2)(x+4) - (x+1)(x+2)(x+3)}{(x+1)(x+2)(x+3)(x+4)} =$$
$$= \frac{x^3 + 9x^2 + 26x + 24 - 3x^3 - 24x^2 - 57x - 36 + 3x^3 + 21x^2 + 42x + 24 - x^3 - 6x^2 - 11x - 6}{(x+1)(x+2)(x+3)(x+4)} =$$
$$= \frac{9x^2 - 24x^2 + 21x^2 - 6x^2 + 26x - 57x + 42x - 11x + 24 - 36 + 24 - 6}{(x+1)(x+2)(x+3)(x+4)} =$$
$$= \frac{6}{(x+1)(x+2)(x+3)(x+4)}$$

7) $y=\frac{sin^2x}{ctg\,x+1}+\frac{cos^2x}{tg\,x+1}$

$$y=\frac{sin^2x}{ctg\,x+1}+\frac{cos^2x}{tg\,x+1}=\frac{sin^2x}{\frac{\cos x}{\sin x}+1}+\frac{cos^2x}{\frac{\sin x}{\cos x}+1}=$$

$$y=\frac{sin^2x}{\frac{1}{1-\sin x}}+\frac{1-sin^2x}{\frac{1}{\sin x}}=sin^2x(1-\sin x)+\sin x\,(1-sin^2x)=sin^2x-sin^3x+\sin x-sin^3x=$$

$$=-2sin^3x+sin^2x+\sin x=-2\cdot\frac{3\sin x-\sin 3x}{4}+\frac{1-\cos 2x}{2}+\sin x=$$

$$=-\frac{3\sin x-\sin 3x+1-1+2sin^2x+2\sin x}{2}=-\frac{1}{2}(5\sin x-\sin 3x+2\sin^2 x)$$

$$y'=-\frac{1}{2}(5\sin x-\sin 3x+2\sin^2 x)'=-\frac{1}{2}\big(5\cos x+\cos 3x\cdot(3x)'+2(\sin x'\cdot\sin x+\sin x\cdot\sin x')\big)=$$

$$=-\frac{1}{2}\big(5\cos x+3\cos 3x+2(2\cos x\cdot\sin x)\big)=-\frac{1}{2}(5\cos x+3\cos 3x+2\sin x)$$

$$=-\frac{1}{2}(5\cos x+3\cos 3x+2\sin x)=$$

$=-\frac{1}{2}(5\cos x+3\cos 3x)-\sin x$ – первый вариант решения

$$y'=\left(\frac{sin^2x}{ctg\,x+1}+\frac{cos^2x}{tg\,x+1}\right)'=\frac{2\cdot\sin x\cdot\sin x\cdot ctg\,x+2\cdot\sin x\cdot\sin x+sin^2x\cdot\left(-\frac{1}{sin^2x}+0\right)}{2\cdot\left(-\frac{1}{sin^2x}+0\right)}+$$

$$+\frac{2\cdot\cos x\cdot\cos x\cdot tg\,x+2\cdot\cos x\cdot\cos x+cos^2x\cdot\frac{1}{cos^2x}+0}{2\cdot\left(\frac{1}{cos^2x}+0\right)}=$$

$$=\frac{2\cdot sin^2x\cdot ctg\,x+2\cdot sin^2x-1}{-\frac{2}{sin^2x}}+\frac{2\cdot cos^2x\cdot tg\,x+2\cdot cos^2x+1}{\frac{2}{cos^2x}}=$$

$=-sin^4x(ctg\,x+1)+cos^4x(tg\,x+1)=-\frac{sin^3x}{cos^3x}-\frac{sin^4x}{cos^4x}+\frac{\sin x}{\cos x}+1$– второй вариант решения

2. Найти производную данной функции в точке:

1) $y=\frac{ln\,x}{x},x_0=e$

$$y'=\left(\frac{ln\,x}{x}\right)'=\frac{ln\,x'\cdot x-ln\,x\cdot x'}{x^2}=\frac{1-\ln x}{x^2}$$

$\frac{1-\ln e}{e^2}=0$ – второй вариант решения

2) $y=\frac{\sqrt{x}}{\sqrt{x}+1},x_0=9$

$$y'=\left(\frac{\sqrt{x}}{\sqrt{x}+1}\right)'=\frac{\sqrt{x}'\cdot(\sqrt{x}+1)-\sqrt{x}\cdot(\sqrt{x}+1)'}{(\sqrt{x}+1)^2}=\frac{\frac{1}{2\sqrt{x}}\cdot(\sqrt{x}+1)-\sqrt{x}\cdot\frac{1}{2\sqrt{x}}}{(\sqrt{x}+1)^2}=\frac{\frac{1}{2\sqrt{x}}\cdot(\sqrt{x}+1-\sqrt{x})}{(\sqrt{x}+1)^2}=$$

$$=\frac{\frac{1}{2\sqrt{x}}}{(\sqrt{x}+1)^2}=\frac{(2\sqrt{x})^{-1}}{(\sqrt{x}+1)^2}=\frac{1}{2\sqrt{x}(\sqrt{x}+1)^2}$$

$$\frac{1}{2\sqrt{9}(\sqrt{9}+1)^2}=\frac{1}{6\cdot 16}=\frac{1}{96}$$

3. Используя логарифмическую производную, найти производные функций:

1) $y=x^{\ln x}$

$\ln y = \ln x^{\ln x} = \ln x \cdot \ln x$

$(\ln y)' = (\ln x \cdot \ln x)'$

$$\frac{y'}{y} = \ln x' \cdot \ln x + \ln x \cdot \ln x' = 2\frac{\ln x}{x}$$

$$y' = x^{\ln x} \cdot 2\frac{\ln x}{x} = 2\ln x \cdot x^{\ln x - 1}$$

2) $y = \frac{(x^3-2)\cdot\sqrt[3]{(x-1)}}{(x+5)^4}$

$$\ln y = \ln\frac{(x^3-2)\cdot\sqrt[3]{(x-1)}}{(x+5)^4} = ln(x^3-2) + \frac{1}{3}ln(x-1) - 4\,ln(x+5)$$

$$(\ln y)' = \left(ln(x^3-2) + \frac{1}{3}ln(x-1) - 4\,ln(x+5)\right)'$$

$$\frac{y'}{y} = \frac{3x^2}{x^3-2} + \frac{1}{3(x-1)} - \frac{4}{x+5}$$

$$y' = y\cdot\left(\frac{3x^2}{x^3-2} + \frac{1}{3(x-1)} - \frac{4}{x+5}\right) = \frac{(x^3-2)\cdot\sqrt[3]{(x-1)}}{(x+5)^4}\cdot\left(\frac{3x^2}{x^3-2} + \frac{1}{3(x-1)} - \frac{4}{x+5}\right)$$

3) $y = (tg\ x)^{\cos x}$

$\ln y = \ln(tg\ x)^{\cos x} = \cos x \cdot \ln tg\ x$

$(\ln y)' = (\cos x \cdot \ln tg\ x)'$

$$\frac{y'}{y} = \cos x' \cdot \ln tg\ x + \cos x \cdot \ln tg\ x' = -\sin x \cdot \ln tg\ x + \cos x \cdot \frac{1}{tg\ x}\cdot\frac{1}{\cos^2 x} = -\sin x \cdot \ln tg\ x + \frac{1}{sin\ x}$$

$$y' = y\cdot\left(\frac{1}{sin\ x} - \sin x \cdot \ln tg\ x\right) = (tg\ x)^{\cos x}\cdot\left(\frac{1}{sin\ x} - \sin x \cdot \ln tg\ x\right)$$

4. Найти производную неявно заданной функции:

1) $e^{xy} - \cos(x^2+y^2) = 0$

$e^{xy}\cdot(xy)' + \sin(x^2+y^2)\cdot(x^2+y^2)' = 0$

$e^{xy}\cdot(y+xy') + \sin(x^2+y^2)\cdot(2x+2yy') = 0$

$ye^{xy} + xe^{xy}y' + 2x\sin(x^2+y^2) + 2y\sin(x^2+y^2)\,y' = 0$

$y'(xe^{xy} + 2y\sin(x^2+y^2)) = -(ye^{xy} + 2x\sin(x^2+y^2))$

$$y' = -\frac{ye^{xy} + 2x\sin(x^2+y^2)}{xe^{xy} + 2y\sin(x^2+y^2)}$$

2) $x\sin y + y\sin x = 0$

$\sin y + x\cos y \cdot y' + y'\sin x + y\cos x = 0$

$y'(\sin x + x\cos y) = -(\sin y + y\cos x)$

$$y' = -\frac{(\sin y + y\cos x)}{(\sin x + x\cos y)}$$

5. Найти производную для заданных параметрически функций:

1) $x = t^3 + t,\ y = t^2 + t + 1$

$$y'(x) = \frac{y'(t)}{x'(t)} = \frac{(t^2 + t + 1)'}{(t^3 + t)'} = \frac{2 \cdot t^{2-1} + 1 \cdot t^{1-1} + 0}{3 \cdot t^{3-1} + 1 \cdot t^{1-1}} = \frac{2t + 1}{3t^2 + 1}$$

2) $x = e^t \sin t\,,\ y = e^t \cos t$

$$y'(x) = \frac{y'(t)}{x'(t)} = \frac{(e^t \cos t)'}{(e^t \sin t)'} = \frac{e^{t'} \cos t + e^t \cos t'}{e^{t'} \sin t + e^t \sin t'} = \frac{e^t \cos t + e^t(-\sin t)}{e^t \sin t + e^t \cos t} = \frac{\cos t - \sin t}{\cos t + \sin t}$$

6. Найти уравнения касательной и нормали к данной кривой в точке x_0:

$y = e^x, x_0 = 0$

$y' = e^x$

$y'(x_0) = y'(e^x) = 1$

$y - 1 = x\ \Rightarrow y = x + 1 -$ касательная

$y - 1 = -x\ \Rightarrow y = 1 - x -$ нормаль

7. Найти производные указанных порядков для следующих функций:

1) $y = -x \cdot \cos x\,, y'' = ?$

$y' = (-x \cdot \cos x)' = -x' \cdot \cos x - x \cdot \cos x' = -\cos x + x \cdot \sin x$

$y'' = (-\cos x + x \cdot \sin x)' = -\cos x' +\ x' \cdot \sin x + x \cdot \sin x' =$

$= \sin x + \sin x + x \cdot \cos x = 2 \sin x + x \cos x$

2) $y = e^{2x}, y^{(V)} = ?$

$y^I = e^{2x'} = e^{2x} \cdot 2x' = e^{2x} \cdot (2'x + 2x') = 2e^{2x}$

$y^{II} = (2e^{2x})' = 2e^{2x} \cdot 2x' = 2e^{2x} \cdot (2'x + 2x') = 2^2 e^{2x}$

$y^{III} = (2^2 e^{2x})' = 2^2 e^{2x} \cdot 2x' = 2^2 e^{2x} \cdot (2'x + 2x') = 2^3 e^{2x}$

$y^{IV} = (2^3 e^{2x})' = 2^3 e^{2x} \cdot 2x' = 2^3 e^{2x} \cdot (2'x + 2x') = 2^4 e^{2x}$

$y^{V} = (2^4 e^{2x})' = 2^4 e^{2x} \cdot 2x' = 2^4 e^{2x} \cdot (2'x + 2x') = 2^5 e^{2x} = 32e^{2x}$

3) $y = \ln(1 + x),\ y^{(n)} = ?$

$$y^I = \left(\ln(1 + x)\right)' = \frac{1}{1 + x} \cdot (1 + x)' = \frac{1 \cdot (0 + 1)}{1 + x} = \frac{1}{1 + x}$$

$$y^{II} = \left(\frac{1 \cdot 1}{1 + x}\right)' = \frac{1' \cdot (1 + x) - 1 \cdot (1 + x)'}{(1 + x)^2} = \frac{0 \cdot (1 + x) - 1 \cdot (0 + 1)}{(1 + x)^2} = -\frac{1}{(1 + x)^2}$$

$$y^{III} = \left(-\frac{1}{(1 + x)^2}\right)' = -\frac{1' \cdot (1 + x)^2 - 1 \cdot (1 + x)^{2'}}{(1 + x)^3} = -\frac{0 \cdot (1 + x)^2 - 1 \cdot 2(0 + 1)^{2-1}}{(1 + x)^3} = \frac{2}{(1 + x)^3}$$

$$y^n = (-1)^{n+1} \frac{(n - 1)!}{(1 + x)^n}$$