Linear Algebra I

Solving Systems of Linear Equations

Gaussian Elimination

- things you can do:
  1. switch two equations
  2. multiply one equation by a nonzero number
  3. add a multiple of one equation to a second equation

ex - $x-2y=1$ ^ $4x+y=0$

③ $-4E_1+E_2 \Leftrightarrow$ $-4x+8y=-4$
$+\ 4x+y=0$
$9y=-4$
$\boxed{y=\frac{-4}{9}}$

substitute $x-2(\frac{-4}{9})=1$
$x+\frac{8}{9}=1$
$x=\frac{9}{9}-\frac{8}{9}$
$\boxed{x=\frac{1}{9}}$

ex - $x-y=1$ ^ $2x-2y=3$

③ $-2E_1+E_2 \Leftrightarrow$ $-2x+2y=-2$
$+\ 2x-2y=3$
$0=1$ contradiction $\boxed{\text{no solution}}$

ex - $7x+5y=2$ ^ $14x+10y=4$

③ $-2E_1+E_2 \Leftrightarrow$ $-14x-10y=-4$
$+\ 14x+10y=4$
$0=0$ second equation is redundant

when you can't eliminate, set rightmost variable $=t$ as a dummy and solve for all remaining variables

$7x+5y=2$ let $y=t$ ^ solve for x
$7x+5t=2 \Leftrightarrow 7x=2-5t \Leftrightarrow x=\frac{2-5t}{7}$
so $\boxed{(\frac{2-5t}{7},t) \text{ where } t\in\mathbb{R}}$

GE + Row Echelon Form

- for a system of 3 equations ^ 3 variables, solve in a similar fashion by getting rid of the variables one by one until we have a triangular shape

ex - $x+y+z=0$ ^ $-x+2y+3z=1$ ^ $3x-3y+z=-1$

$E_1+E_2 \Leftrightarrow$ $x+y+z=0$
$+\ -x+2y+3z=1$
$3y+4z=1 : E_4$

$-3E_1+E_3$ $-3x-3y-3z=0$
$+\ 3x-3y+z=-1$
$-6y-2z=-1 : E_5$

$2E_4+E_5 \Leftrightarrow$ $6y+8z=2$
$+\ -6y-2z=-1$
$6z=1$
$\boxed{z=1/6}$

3 equations w/ triangular shape w/ first coefficient each $=1$, which is row echelon form

row echelon form

$$\begin{aligned} x + y + z &= 0 \\ y + \tfrac{4}{3}z &= \tfrac{1}{3} \\ z &= \tfrac{1}{6} \end{aligned}$$

$\frac{1}{3}E_4 \Rightarrow$

ex - $x - 2y + 5z = 2 \wedge 3x + 2y - z = -2$

$-3E_1 + E_2 \Rightarrow$ $\cancel{-3x} + 6y - 15z = -6$
$+\ \cancel{3x} + 2y - z = -2$
$8y - 16z = -8$

let $z = t$ $\quad 8y - 16t = -8$
$8y = 16t - 8$
$y = 2t - 1$

$x - 2y + 5t = 2 \Rightarrow x - 2(2t-1) + 5t = 2 \Rightarrow x - 4t + 2 + 5t = 2 \Rightarrow x + t + 2 = 2 \Rightarrow x + t = 0 \Rightarrow$
$x = -t$ so $\boxed{(-t, 2t-1, t) \text{ when } t \in \mathbb{R}}$

ex - $x + y - z = 0 \wedge x - y + z = 1 \wedge 2x + y - z = 0$

$E_1 - E_2 \Rightarrow$ $\cancel{x} + y - z = 0$
$+\ \cancel{-x} + y - z = -1$
$2y - 2z = -1 : E_4$

$-2E_1 + E_3 \Rightarrow$ $\cancel{-2x} - 2y + 2z = 0$
$+\ \cancel{2x} + y - z = 0$
$-y + z = 0 : E_5$

$E_4 + 2E_5 \Rightarrow$ $2y - 2z = -1$
$+\ -2y + 2z = 0$
$0 = -1$ X contradiction
$\boxed{\text{no solution}}$

Problem Set

ex - $x + 2y = 0 \wedge -x + y = 10$

$E_1 + E_2 \Rightarrow$ $\cancel{x} + 2y = 0$
$+\ \cancel{-x} + y = 10$
$3y = 10$
$\boxed{y = \frac{10}{3}}$

E_1 $\quad x + 2\left(\frac{10}{3}\right) = 0$
$x + \frac{20}{3} = 0$
$\boxed{x = -\frac{20}{3}}$

ex - $3x - y = 3 \wedge -4x + 11y = 7$

$11E_1 + E_2$ $\quad 33x - \cancel{11y} = 33$
$+\ -4x + \cancel{11y} = 7$
$29x = 40$
$\boxed{x = \frac{40}{29}}$

E_1 $\quad 3\left(\frac{40}{29}\right) - y = 3$
$\frac{120}{29} - y = 3$
$-y = \frac{87}{29} - \frac{120}{29}$
$\boxed{y = \frac{33}{29}}$

ex - $8x-5y=20 \wedge -16x+10y=-40$

$2E_1+E_2 \Rightarrow$

$$\begin{aligned} 16x-10y&=40 \\ + \; -16x+10y&=-40 \\ \hline 0&=0 \end{aligned}$$

let $y=t$ $\quad 8x-5t=20 \Rightarrow 8x=5t+20 \Rightarrow x=\frac{5t+20}{8}$

$\boxed{\left(\frac{5t+20}{8}, t\right) \text{ where } t\in\mathbb{R}}$

ex - $2x+y=13 \wedge -4x-2y=4$

$2E_1+E_2$

$$\begin{aligned} 4x+2y&=26 \\ -4x-2y&=4 \\ \hline 0&=4 \end{aligned}$$

contradiction $\boxed{\text{no solution}}$

ex - $x-y-z=1 \wedge 2x+y+3z=0 \wedge 3x-y+z=-1$

$-2E_1+E_2 \Rightarrow$

$$\begin{aligned} -2x+2y+2z&=-2 \\ + \; 2x+y+3z&=0 \\ \hline 3y+5z&=-2 : E_4 \end{aligned}$$

$-3E_1+E_3 \Rightarrow$

$$\begin{aligned} -3x+3y+3z&=-3 \\ + \; 3x-y+z&=-1 \\ \hline 2y+4z&=-4 : E_5 \end{aligned}$$

$2E_4-3E_5 \Rightarrow$

$$\begin{aligned} 6y+10z&=-4 \\ + \; -6y-12z&=12 \\ \hline -2z&=8 \\ z&=-4 \end{aligned}$$

$E_4 \Rightarrow 3y+5(-4)=-2 \Rightarrow 3y-20=-2 \Rightarrow$

$3y=18 \Rightarrow \boxed{y=6}$

$E_1 \Rightarrow x-(6)-(-4)=1 \Rightarrow x-6+4=1 \Rightarrow$

$x-2=1 \Rightarrow \boxed{x=3}$

ex - $x+2y+2z=4 \wedge -y+z=-1 \wedge x+y=8$

$E_1-E_3 \Rightarrow$

$$\begin{aligned} x+2y+2z&=-4 \\ + \; -x-y&=-8 \\ \hline y+2z&=-4 : E_4 \end{aligned}$$

E_4+E_2

$$\begin{aligned} y+2z&=-4 \\ + \; -y+z&=-1 \\ \hline 3z&=-5 \end{aligned}$$

$\boxed{z=-\frac{5}{3}}$

$E_4 \Rightarrow y+2\left(-\frac{5}{3}\right)=-4 \Rightarrow y-\frac{10}{3}=-4 \Rightarrow y=-\frac{12}{3}+\frac{10}{3} \Rightarrow$

$\boxed{y=-\frac{2}{3}}$

$E_3 \Rightarrow x+\left(-\frac{2}{3}\right)=8 \Rightarrow x=\frac{27}{3}+\frac{2}{3} \Rightarrow \boxed{x=\frac{29}{3}}$

ex - $x-y-z=3 \wedge x-10y+10z=0$

$E_1-E_2 \Rightarrow$

$$\begin{aligned} x-y-z&=3 \\ + \; -x+10y-10z&=0 \\ \hline 9y-11z&=3 \end{aligned}$$

let $z=t$, solve for y $\quad 9y-11t=3 \Rightarrow$

$9y=3+11t \Rightarrow y=\frac{11t+3}{9}$

solve for x

$E_1 \Rightarrow x-\left(\frac{11t+3}{9}\right)-t=3 \Rightarrow x=\frac{11}{9}t+\frac{1}{3}+t+\frac{9}{3} \Rightarrow$

$x=\frac{20}{9}t+\frac{10}{3}$

$\boxed{\left(\frac{20}{9}t+\frac{10}{3}, \frac{11}{9}t+\frac{1}{3}, t\right) \text{ where } t\in\mathbb{R}}$

Elementary Row Operations

- we can rewrite a system of equations using a matrix
- ex: $x+y+z=0 \ \wedge\ -x+2y+3z=1 \ \wedge\ 3x-3y+z=-1$

$$\underbrace{\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ -1 & 2 & 3 & 1 \\ 3 & -3 & 1 & -1 \end{array}\right]}_{\text{augmented matrix}}$$

- elementary row operations (same steps):
  1. switch two rows
  2. multiply one row by a nonzero number
  3. add a multiple of one row to a second row

ex cont. – $R1+R2 \to R2$ $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 3 & 4 & 1 \\ 3 & -3 & 1 & -1 \end{bmatrix} \Rightarrow$ $-3R1+R3 \to R3$ $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 3 & 4 & 1 \\ 0 & -6 & -2 & -1 \end{bmatrix} \Rightarrow$

$2R2+R3 \to R3$ $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 3 & 4 & 1 \\ 0 & 0 & 6 & 1 \end{bmatrix} \Rightarrow$ $\frac{1}{6}R3 \to R3$ $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 3 & 4 & 1 \\ 0 & 0 & 1 & \frac{1}{6} \end{bmatrix} \Rightarrow$

$\frac{1}{3}R2 \to R2$ $\boxed{\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & \frac{4}{3} & \frac{1}{3} \\ 0 & 0 & 1 & \frac{1}{6} \end{bmatrix} \text{row-echelon form}}$ can solve for variables easily

ex – $y+z=0 \ \wedge\ x-y-z=1 \ \wedge\ 2x+2y-z=3$

$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & -1 & -1 & 1 \\ 2 & 2 & -1 & 3 \end{bmatrix}$ $R1 \leftrightarrow R2 \Rightarrow$ $\begin{bmatrix} 1 & -1 & -1 & 1 \\ 0 & 1 & 1 & 0 \\ 2 & 2 & -1 & 3 \end{bmatrix}$ $-2R1+R3 \to R3 \Rightarrow$ $\begin{bmatrix} 1 & -1 & -1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 4 & 1 & 1 \end{bmatrix}$ $-4R2+R3 \to R3$

$\begin{bmatrix} 1 & -1 & -1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -3 & 1 \end{bmatrix}$ $-\frac{1}{3}R3 \to R3 \Rightarrow$ $\boxed{\begin{bmatrix} 1 & -1 & -1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -\frac{1}{3} \end{bmatrix} \text{row-echelon form}}$

ex – $7x+7y-z=3 \ \wedge\ x+y+z=-1 \ \wedge\ -x-y+3z=0$

$\begin{bmatrix} 7 & 7 & -1 & 3 \\ 1 & 1 & 1 & -1 \\ -1 & -1 & 3 & 0 \end{bmatrix}$ $R1-7R2 \to R2 \Rightarrow$ $\begin{bmatrix} 7 & 7 & -1 & 3 \\ 0 & 0 & -8 & 10 \\ -1 & -1 & 3 & 0 \end{bmatrix}$ $R1+7R3 \to R3 \Rightarrow$ $\begin{bmatrix} 7 & 7 & -1 & 3 \\ 0 & 0 & -8 & 10 \\ 0 & 0 & 20 & 3 \end{bmatrix}$ $-\frac{1}{8}R_2 \to R_2 \Rightarrow$

$\begin{bmatrix} 7 & 7 & -1 & 3 \\ 0 & 0 & 1 & -\frac{5}{4} \\ 0 & 0 & 20 & 3 \end{bmatrix}$ $-20R_2+R_3 \to R_3 \Rightarrow$ $\begin{bmatrix} 7 & 7 & -1 & 3 \\ 0 & 0 & 1 & -\frac{5}{4} \\ 0 & 0 & 0 & 28 \end{bmatrix} \Rightarrow 0=28$ X contradiction $\boxed{\text{no solution}}$

ex – $3x-y+11z=7 \ \wedge\ 7x+y+7z=-3 \ \wedge\ 14x+2y+14z=-6$

-7 -3
77 -21
49 11

$\begin{bmatrix} 3 & -1 & 11 & 7 \\ 7 & 1 & 7 & -3 \\ 14 & 2 & 14 & -6 \end{bmatrix}$ $7R_1-3R_2 \to R_2 \Rightarrow$ $\begin{bmatrix} 3 & -1 & 11 & 7 \\ 0 & -10 & 56 & 58 \\ 14 & 2 & 14 & -6 \end{bmatrix} \leftarrow$ R_3 is double our old R_2 $\Rightarrow$ $\begin{bmatrix} 3 & -1 & 11 & 7 \\ 0 & -10 & 56 & 58 \\ 0 & -10 & 56 & 58 \end{bmatrix}$ same $\Rightarrow$

let $z=t$: $-10y+56t=58 \Rightarrow -10y=-56t+58 \Rightarrow y=\frac{56}{10}t-\frac{58}{10} \Rightarrow y=\frac{28}{5}t-\frac{29}{5}$

first row: $3x-y+11z=7$ plug $y \wedge z \Rightarrow 3x-(\frac{28}{5}t-\frac{29}{5})+11(t)=7 \Rightarrow 3x=\frac{28}{5}t-\frac{55}{5}t-\frac{29}{5}+\frac{35}{5} \Rightarrow$

$3x=-\frac{27}{5}t+\frac{6}{5} \Rightarrow x=-\frac{9}{5}t+\frac{2}{5}$

so $\boxed{(-\frac{9}{5}t+\frac{2}{5}, \frac{28}{5}t-\frac{29}{5}, t) \text{ when } t\in\mathbb{R}}$

ex - $x-11y-z=8 \wedge 8x+y-z=2 \wedge -7x-12y=-12$

$\begin{bmatrix} 1 & -11 & -1 & 8 \\ 8 & 1 & -1 & 2 \\ -7 & -12 & 0 & -12 \end{bmatrix} -8R_1+R_2 \to R_2 \Rightarrow \begin{bmatrix} 1 & -11 & -1 & 8 \\ 0 & 89 & 7 & -62 \\ -7 & -12 & 0 & -12 \end{bmatrix} 7R_1+R_3 \to R_3 \Rightarrow \begin{bmatrix} 1 & -11 & -1 & 8 \\ 0 & 89 & 7 & -62 \\ 0 & -89 & -7 & 44 \end{bmatrix} R_2+R_3 \to R_3 \Rightarrow$

$\begin{bmatrix} 1 & -11 & -1 & 8 \\ 0 & 89 & 7 & -62 \\ 0 & 0 & 0 & -18 \end{bmatrix}$ $0=-18$ X contradiction $\boxed{\text{no solution}}$

ex - $x-z=2 \wedge x+y+z=-3 \wedge x-y=0$

$\begin{bmatrix} 1 & 0 & -1 & 2 \\ 1 & 1 & 1 & -3 \\ 1 & -1 & 0 & 0 \end{bmatrix} R_1-R_2 \to R_2 \Rightarrow \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & -1 & -2 & 5 \\ 1 & -1 & 0 & 0 \end{bmatrix} -R_2 \to R_2 \Rightarrow \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 2 & -5 \\ 1 & -1 & 0 & 0 \end{bmatrix} R_1-R_3 \to R_3 \Rightarrow$

$\begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 2 & -5 \\ 0 & 1 & -1 & 2 \end{bmatrix} R_2-R_3 \to R_3 \Rightarrow \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 2 & -5 \\ 0 & 0 & 3 & -7 \end{bmatrix} \frac{1}{3}R_3 \to R_3 \Rightarrow \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 2 & -5 \\ 0 & 0 & 1 & -\frac{7}{3} \end{bmatrix}$

$1z = \boxed{-\frac{7}{3}}$

$1y+2(-\frac{7}{3})=-5 \Rightarrow y-\frac{14}{3}=-\frac{15}{3} \Rightarrow \boxed{y=-\frac{1}{3}}$

$1x+0(-\frac{1}{3})-1(-\frac{7}{3})=2 \Rightarrow x+\frac{7}{3}=\frac{6}{3} \Rightarrow \boxed{x=-\frac{1}{3}}$

ex - $x+z=8 \wedge y+z=-10$

$\begin{bmatrix} 1 & 0 & 1 & 8 \\ 0 & 1 & 1 & -10 \end{bmatrix}$

let $z=t$: $y+t=-10 \Rightarrow y=-t-10$

$x+t=8 \Rightarrow x=-t+8$

so $\boxed{(-t+8, -t-10, t) \text{ when } t \in \mathbb{R}}$

ex - $x-y+z=9 \wedge 9y-9z=3$

$\begin{bmatrix} 1 & -1 & 1 & 9 \\ 0 & 9 & -9 & 3 \end{bmatrix}$

let $z=t$: $9y-9t=3 \Rightarrow 9y=9t+3 \Rightarrow y=t+\frac{1}{3}$

$x-(\cancel{t}+\frac{1}{3})+\cancel{t}=9 \Rightarrow x-\frac{1}{3}=\frac{27}{3} \Rightarrow x=\frac{28}{3}$

so $\boxed{(\frac{28}{3}, t+\frac{1}{3}, t) \text{ when } t \in \mathbb{R}}$

Vectors

Vector Operations

Linear Combinations

- vector is a list of real numbers

$\bar{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ is a vector of $\mathbb{R}^n$

- $\bar{v} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$ is a vector in $\mathbb{R}^2$ = 2 entries

$\bar{u} = \begin{bmatrix} 0 \\ 3 \\ -12 \end{bmatrix}$ is a vector in $\mathbb{R}^3$

ex - $\bar{u}_1 = \begin{bmatrix} 0 \\ 3 \\ -22 \end{bmatrix}$ $\bar{u}_2 = \begin{bmatrix} 7 \\ 2 \\ 5 \end{bmatrix}$ $\bar{u}_1 + \bar{u}_2 = \boxed{\begin{bmatrix} 7 \\ 5 \\ -17 \end{bmatrix}}$

Vector addition: same dimensions, add same positions

ex - $c=8$ $\bar{v} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ $c\bar{v} = 8\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \boxed{\begin{bmatrix} 8 \\ 8 \\ 16 \end{bmatrix}}$

multiplying a vector by a scalar: multiply scalar by each value in vector/matrix

- if we have a set of vectors $\bar{v}_1, \bar{v}_2, \bar{v}_3 \ldots \bar{v}_k$ and scalars $c_1, c_2 \ldots c_k$, then $c_1\bar{v}_1 + c_2\bar{v}_2 \ldots + c_k\bar{v}_k$ is called a linear combination of $\bar{v}_1 \ldots \bar{v}_k$ ^ $c_1 \ldots c_k$ are called weights

ex - $\bar{v}_1 = \begin{bmatrix}1\\1\\2\end{bmatrix}$ $\bar{v}_2 = \begin{bmatrix}0\\-1\\0\end{bmatrix}$ $\bar{v}_3 = \begin{bmatrix}2\\2\\2\end{bmatrix}$ $c_1 = 1$ $c_2 = 3$ $c_3 = -5$ calculate the linear combination

$1\begin{bmatrix}1\\1\\2\end{bmatrix} + 3\begin{bmatrix}0\\-1\\0\end{bmatrix} + (-5)\begin{bmatrix}2\\2\\2\end{bmatrix} = \begin{bmatrix}1\\1\\2\end{bmatrix} + \begin{bmatrix}0\\-3\\0\end{bmatrix} + \begin{bmatrix}-10\\-10\\-10\end{bmatrix} = \begin{bmatrix}-9\\-12\\-8\end{bmatrix}$

ex - $A = \begin{bmatrix}1&0&2\\3&1&-1\\2&2&0\end{bmatrix}$ $\bar{x} = \begin{bmatrix}5\\2\\4\end{bmatrix}$ matrix-vector multiplication

$$\boxed{A\bar{x} = \begin{bmatrix}1&0&2\\3&1&-1\\2&2&0\end{bmatrix}\begin{bmatrix}5\\2\\4\end{bmatrix} = \begin{bmatrix}5+0+8\\15+2-4\\10+4+0\end{bmatrix} = \begin{bmatrix}13\\13\\14\end{bmatrix}}$$

ex - $\bar{u} = \begin{bmatrix}1\\0\end{bmatrix}$ $\bar{v} = \begin{bmatrix}-2\\1\end{bmatrix}$ $\bar{u}+\bar{v} \Rightarrow \begin{bmatrix}-1\\1\end{bmatrix}$

ex - $\bar{u} = \begin{bmatrix}1\\1\end{bmatrix}$ $\bar{v} = \begin{bmatrix}-1\\3\end{bmatrix}$ $\bar{u}+\bar{v} \Rightarrow \begin{bmatrix}0\\4\end{bmatrix}$

ex - $\bar{u} = \begin{bmatrix}1\\1\\1\end{bmatrix}$ $\bar{v} = \begin{bmatrix}2\\-3\\10\end{bmatrix}$ $\bar{u}+\bar{v} \Rightarrow \begin{bmatrix}3\\-2\\11\end{bmatrix}$

ex - $\bar{u} = \begin{bmatrix}-1\\1\end{bmatrix}$ $c = 8$ $\bar{u}c \Rightarrow \begin{bmatrix}-8\\8\end{bmatrix}$

ex - $\bar{u} = \begin{bmatrix}2\\2\end{bmatrix}$ $c = -3$ $\bar{u}c \Rightarrow \begin{bmatrix}-6\\-6\end{bmatrix}$

ex - $\bar{u} = \begin{bmatrix}-9\\9\\19\end{bmatrix}$ $c = -1$ $\bar{u}c \Rightarrow \begin{bmatrix}9\\-9\\-19\end{bmatrix}$

ex - simplify the linear combination of vectors $\bar{v}_1 = \begin{bmatrix}0\\1\end{bmatrix}$ $\bar{v}_2 = \begin{bmatrix}-3\\6\end{bmatrix}$ $\bar{v}_3 = \begin{bmatrix}11\\9\end{bmatrix}$ $c_1 = -1$ $c_2 = 3$ $c_3 = 4$

$-1\begin{bmatrix}0\\1\end{bmatrix} + 3\begin{bmatrix}-3\\6\end{bmatrix} + 4\begin{bmatrix}11\\9\end{bmatrix} = \begin{bmatrix}0\\-1\end{bmatrix} + \begin{bmatrix}-9\\18\end{bmatrix} + \begin{bmatrix}44\\36\end{bmatrix} = \begin{bmatrix}35\\53\end{bmatrix}$

ex - simplify the linear combination of vectors $\bar{v}_1 = \begin{bmatrix}2\\2\end{bmatrix}$ $\bar{v}_2 = \begin{bmatrix}-1\\2\end{bmatrix}$ $\bar{v}_3 = \begin{bmatrix}7\\7\end{bmatrix}$ $c_1 = 3$ $c_2 = -3$ $c_3 = 0$

$3\begin{bmatrix}2\\2\end{bmatrix} - 3\begin{bmatrix}-1\\2\end{bmatrix} + \cancel{0\begin{bmatrix}7\\7\end{bmatrix}} = \begin{bmatrix}6\\6\end{bmatrix} + \begin{bmatrix}3\\-6\end{bmatrix} = \begin{bmatrix}9\\0\end{bmatrix}$

ex - simplify the linear combination of vectors $\bar{v}_1 = \begin{bmatrix}0\\0\\1\end{bmatrix}$ $\bar{v}_2 = \begin{bmatrix}1\\1\\2\end{bmatrix}$ $\bar{v}_3 = \begin{bmatrix}9\\8\\4\end{bmatrix}$ $c_1 = 1$ $c_2 = 6$ $c_3 = -1$

$1\begin{bmatrix}0\\0\\1\end{bmatrix} + 6\begin{bmatrix}1\\1\\2\end{bmatrix} - 1\begin{bmatrix}9\\8\\4\end{bmatrix} = \begin{bmatrix}0\\0\\1\end{bmatrix} + \begin{bmatrix}6\\6\\12\end{bmatrix} + \begin{bmatrix}-9\\-8\\-4\end{bmatrix} = \begin{bmatrix}-3\\-2\\9\end{bmatrix}$

ex - simplify the linear combination of vectors $\bar{v}_1 = \begin{bmatrix}4\\-1\\4\end{bmatrix}$ $\bar{v}_2 = \begin{bmatrix}1\\1\\0\end{bmatrix}$ $\bar{v}_3 = \begin{bmatrix}0\\0\\0\end{bmatrix}$ $c_1 = 0$ $c_2 = 1$ $c_3 = 8$

$\cancel{0\begin{bmatrix}4\\-1\\4\end{bmatrix}} + 1\begin{bmatrix}1\\1\\0\end{bmatrix} + \cancel{8\begin{bmatrix}0\\0\\0\end{bmatrix}} = \begin{bmatrix}1\\1\\0\end{bmatrix}$

ex - $\bar{v} = \begin{bmatrix}9\\-9\end{bmatrix}$ $A = \begin{bmatrix}-2&3\\4&2\end{bmatrix}$ $A\bar{v}$

$\begin{bmatrix}-2&3\\4&2\end{bmatrix}\begin{bmatrix}9\\-9\end{bmatrix} = \begin{bmatrix}-18-27\\36-18\end{bmatrix} = \begin{bmatrix}-45\\18\end{bmatrix}$

ex - $\bar{v} = \begin{bmatrix}8\\-4\end{bmatrix}$ $A = \begin{bmatrix}1&0\\-1&0\end{bmatrix}$ $A\bar{v}$

$\begin{bmatrix}1&0\\-1&0\end{bmatrix}\begin{bmatrix}8\\-4\end{bmatrix} = \begin{bmatrix}8+0\\-8+0\end{bmatrix} = \begin{bmatrix}8\\-8\end{bmatrix}$

ex - $\bar{v} = \begin{bmatrix}-1\\0\\1\end{bmatrix}$ $A = \begin{bmatrix}2&3&4\\1&0&1\\-7&7&14\end{bmatrix}$ $A\bar{v}$

$\begin{bmatrix}2&3&4\\1&0&1\\-7&7&14\end{bmatrix}\begin{bmatrix}-1\\0\\1\end{bmatrix} = \begin{bmatrix}-2+0+4\\-1+0+1\\7+0+14\end{bmatrix} = \begin{bmatrix}2\\0\\21\end{bmatrix}$

ex - $\bar{v} = \begin{bmatrix}-1\\0\\3\end{bmatrix}$ $A = \begin{bmatrix}1&0&0\\0&-1&1\\1&1&2\end{bmatrix}$ $A\bar{v}$

$\begin{bmatrix}1&0&0\\0&-1&1\\1&1&2\end{bmatrix}\begin{bmatrix}-1\\0\\3\end{bmatrix} = \begin{bmatrix}-1+0+0\\0+0+3\\-1+0+6\end{bmatrix} = \begin{bmatrix}-1\\3\\5\end{bmatrix}$

Vector Equations ^ Matrix Equation $Ax = b$

ex - $x + y + z = 0$ ^ $-x + 2y + 3z = 1$ ^ $3x - 3y + z = -1$ ⇐ can be written as

$x\begin{bmatrix}1\\-1\\3\end{bmatrix} + y\begin{bmatrix}1\\2\\-3\end{bmatrix} + z\begin{bmatrix}1\\3\\1\end{bmatrix} = \begin{bmatrix}0\\1\\-1\end{bmatrix}$

- span of a set of vectors is set of all linear combinations of those vectors thus, we want to know if the vector $\begin{bmatrix}0\\1\\-1\end{bmatrix}$ lies in the span $\left\{\begin{bmatrix}1\\-1\\3\end{bmatrix}, \begin{bmatrix}1\\2\\-3\end{bmatrix}, \begin{bmatrix}1\\3\\1\end{bmatrix}\right\}$

ex - vector equation $x\begin{bmatrix}1\\-1\\3\end{bmatrix}+\dots$ can be written as a matrix equation:

$$\underbrace{\begin{bmatrix}1&1&1\\-1&2&3\\3&-3&1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}}=\begin{bmatrix}0\\1\\-1\end{bmatrix}$$

$\hookrightarrow$ LHS $=\begin{bmatrix}x+y+z\\-x+2y+3z\\3x-3y+z\end{bmatrix}=\begin{bmatrix}x\\-x\\3x\end{bmatrix}+\begin{bmatrix}y\\2y\\-3y\end{bmatrix}+\begin{bmatrix}z\\3z\\z\end{bmatrix}=x\begin{bmatrix}1\\-1\\3\end{bmatrix}+y\begin{bmatrix}1\\2\\-3\end{bmatrix}+z\begin{bmatrix}1\\3\\1\end{bmatrix}$

- our original system of equations can be rewritten as a matrix equation $A\bar{x}=\bar{b}$ where A is the coefficient matrix, $\bar{x}$ is the column vector of weights $\begin{bmatrix}x\\y\\z\end{bmatrix}$ and $\bar{b}$ is our column vector of constants $\begin{bmatrix}0\\1\\-1\end{bmatrix}$

Linear Independence
- let $\bar{v}_1 \dots \bar{v}_k \in \mathbb{R}^n$. then the set $\{\bar{v}_1 \dots \bar{v}_k\}$ is linearly independent just in case the vector equation $c_1\bar{v}_1+\dots c_k\bar{v}_k=\bar{0}$ has only the trivial solution $c_1=0$ $c_2=0 \dots c_k=0$. otherwise the set is said to be linearly dependent
- note the vector equation $c_1\bar{v}_1+\dots c_k\bar{v}_k=\bar{0}$ can be rewritten $A\bar{x}=\bar{0}$ where $A=[\bar{v}_1 \dots \bar{v}_k]$ $\bar{x}=\begin{bmatrix}c_1\\ \vdots \\c_k\end{bmatrix}$

ex - determine if the set $\{v_1, v_2, v_3\}$ is linearly independent $\bar{v}_1=\begin{bmatrix}1\\-1\\0\end{bmatrix}$ $\bar{v}_2=\begin{bmatrix}3\\4\\7\end{bmatrix}$ $\bar{v}_3=\begin{bmatrix}13\\-6\\7\end{bmatrix}$

$\begin{bmatrix}1&3&13&0\\-1&4&-6&0\\0&7&7&0\end{bmatrix}$ $R_1+R_2\to R_2 \Leftrightarrow \begin{bmatrix}1&3&13&0\\0&7&7&0\\0&7&7&0\end{bmatrix}$ $R_2-R_3\to R_3 \Leftrightarrow \begin{bmatrix}1&3&13&0\\0&7&7&0\\0&0&0&0\end{bmatrix}$ $\frac{1}{7}R_2\to R_2 \Leftrightarrow$

$\begin{bmatrix}1&3&13&0\\0&1&1&0\\0&0&0&0\end{bmatrix}$ let $c_3=t$

$c_2+t=0 \Leftrightarrow c_2=-t$

$c_1+3(-t)+13(t)=0 \Leftrightarrow c_1=-10t$

$\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}=\begin{bmatrix}-10t\\-t\\t\end{bmatrix}=t\begin{bmatrix}-10\\-1\\1\end{bmatrix}$ non-trivial solution, so the set $\{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$ $\boxed{\text{is linearly dependent}}$

let $t=1$, then $c_1=-10$ $c_2=-1$ $c_3=1$

$c_1\bar{v}_1+c_2\bar{v}_2+c_3\bar{v}_3=\bar{0} \Leftrightarrow \boxed{-10\bar{v}_1-\bar{v}_2+\bar{v}_3=\bar{0}}$

ex - determine if these vectors are linearly independent $\bar{v}_1=\begin{bmatrix}1\\0\\-1\end{bmatrix}$ $\bar{v}_2=\begin{bmatrix}3\\2\\0\end{bmatrix}$ $\bar{v}_3=\begin{bmatrix}4\\-4\\4\end{bmatrix}$

$\begin{bmatrix}1&3&4&0\\0&2&-4&0\\-1&0&4&0\end{bmatrix}$ $\frac{1}{2}R_2\to R_2 \Leftrightarrow \begin{bmatrix}1&3&4&0\\0&1&-2&0\\-1&0&4&0\end{bmatrix}$ $R_1+R_3\to R_3 \Leftrightarrow \begin{bmatrix}1&3&4&0\\0&1&-2&0\\0&3&8&0\end{bmatrix}$ $-3R_2+R_3\to R_3 \Leftrightarrow$

$\begin{bmatrix}1&3&4&0\\0&1&-2&0\\0&0&14&0\end{bmatrix}$

$14c_3=0 \Leftrightarrow c_3=0$

$c_2-2(0)=0 \Leftrightarrow c_2=0$

$c_1+3(0)+4(0)=0 \Leftrightarrow c_1=0$ $\boxed{c_1=c_2=c_3=0 \text{ linear independence}}$

ex - determine if the set $\{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$ is linearly independent $\bar{v}_1=\begin{bmatrix}6\\6\\-7\end{bmatrix}$ $\bar{v}_2=\begin{bmatrix}-1\\0\\8\end{bmatrix}$ $\bar{v}_3=\begin{bmatrix}2\\2\\-2\end{bmatrix}$

$\begin{bmatrix}6&-1&2&0\\6&0&2&0\\-7&8&-2&0\end{bmatrix}$ $R_1-R_2\to R_2 \Leftrightarrow \begin{bmatrix}6&-1&2&0\\0&-1&0&0\\-7&8&-2&0\end{bmatrix}$ $7R_1-6R_3\to R_3 \Leftrightarrow \begin{bmatrix}6&-1&2&0\\0&-1&0&0\\0&-55&26&0\end{bmatrix}$ $-55R_2+R_3\to R_3 \Leftrightarrow \begin{bmatrix}6&-1&2&0\\0&-1&0&0\\0&0&26&0\end{bmatrix}$

$26c_3=0 \Leftrightarrow c_3=0$

$-c_2+\cancel{0(0)}=0 \Leftrightarrow -c_2=0 \Leftrightarrow c_2=0$

$6c_1-\cancel{1(0)}+\cancel{2(0)}=0 \Leftrightarrow 6c_1=0 \Leftrightarrow c_1=0$ $\boxed{c_1=c_2=c_3=0 \text{ linear independence}}$

ex - determine if the set is linearly independent $\bar{v}_1=\begin{bmatrix}1\\1\\1\end{bmatrix}$ $\bar{v}_2=\begin{bmatrix}0\\0\\1\end{bmatrix}$ $\bar{v}_3=\begin{bmatrix}1\\1\\0\end{bmatrix}$

$\begin{bmatrix}1&0&1&0\\1&0&1&0\\1&1&0&0\end{bmatrix}$ $R_1-R_2\to R_2 \Leftrightarrow \begin{bmatrix}1&0&1&0\\0&0&0&0\\1&1&0&0\end{bmatrix}$ $R_1-R_3\to R_3 \Leftrightarrow \begin{bmatrix}1&0&1&0\\0&-1&1&0\end{bmatrix}$ let $c_3=t$

$-c_2+t=0 \Leftrightarrow -c_2=-t \Leftrightarrow c_2=t$

$c_1+\cancel{0(t)}+1(t)=0 \Leftrightarrow c_1=-t$ $\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}=\begin{bmatrix}-t\\t\\t\end{bmatrix}=t\begin{bmatrix}-1\\1\\1\end{bmatrix}$ where $t\in\mathbb{R}$ non-trivial solution $\boxed{\text{linearly dependent}}$

ex - determine if the set is linearly independent $\bar{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ $\bar{v}_2 = \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix}$ $\bar{v}_3 = \begin{bmatrix} 7 \\ 7 \\ 9 \end{bmatrix}$

$\begin{bmatrix} 1 & 2 & 7 & 0 \\ -1 & -2 & 7 & 0 \\ 0 & 0 & 9 & 0 \end{bmatrix}$ $R_1 + R_2 \to R_2 \Rightarrow \begin{bmatrix} 1 & 2 & 7 & 0 \\ 0 & 0 & 14 & 0 \\ 0 & 0 & 9 & 0 \end{bmatrix}$

$9c_3 = 0 \Rightarrow c_3 = 0$

let $c_2 = t \Rightarrow c_1 + 2(t) + 7(0) = 0 \Rightarrow c_1 = -2t$

$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} -2t \\ t \\ 0 \end{bmatrix} = \boxed{t \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}}$ non-trivial solutions, $\boxed{\text{linearly dependent}}$

Matrix Operations

Addition ^ - an $m \times n$ matrix has m rows ^ n columns

Scalar ex - $\begin{bmatrix} 2 & -1 & 0 \\ 1 & 1 & -1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 8 \\ 8 & 2 & 3 \end{bmatrix} = \boxed{\begin{bmatrix} 2 & -1 & 8 \\ 9 & 3 & 2 \end{bmatrix}}$

Multiplication ex - $A = \begin{bmatrix} -1 & 2 & 4 \\ 0 & 1 & 1 \end{bmatrix}$ $c = 3$ cA?

$3 \begin{bmatrix} -1 & 2 & 4 \\ 0 & 1 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} -3 & 6 & 12 \\ 0 & 3 & 3 \end{bmatrix}}$

ex - $A = \begin{bmatrix} 11 & 10 \\ 8 & 9 \\ 4 & 2 \end{bmatrix}$ $c = -2$ cA?

$-2 \begin{bmatrix} 11 & 10 \\ 8 & 9 \\ 4 & 2 \end{bmatrix} = \boxed{\begin{bmatrix} -22 & -20 \\ -16 & -18 \\ -8 & -4 \end{bmatrix}}$

ex - $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$ $A - B$?

$A - B$ as $A + (-1)B$

$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + (-1) \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} -3 & -2 \\ -1 & -4 \end{bmatrix} = \boxed{\begin{bmatrix} -3 & -1 \\ -2 & -4 \end{bmatrix}}$

Multiplication - we can multiply two matrices A ^ B as long as the number of columns of A is equal to the number of rows of B $m \times n = n \times n$

matrix multiplication results

$(m \times n)(m \times n) = m \times n$

ex - $A = \begin{bmatrix} 1 & 3 & -1 \\ 0 & 2 & 2 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ -1 & 1 \end{bmatrix}$ AB?

$\begin{bmatrix} 1+3-1 & 0+0-1 \\ 0+2-2 & 0+0+2 \end{bmatrix} = \boxed{\begin{bmatrix} 5 & -1 \\ 0 & 2 \end{bmatrix}}$

ex - $A = \begin{bmatrix} 2 & 0 \\ 1 & -2 \end{bmatrix}$ $B = \begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix}$ AB? BA?

$\begin{bmatrix} 6+0 & -2+0 \\ 3-4 & -1+0 \end{bmatrix} = \boxed{\begin{bmatrix} 6 & -2 \\ -1 & -1 \end{bmatrix}}$ $\begin{bmatrix} 6-1 & 0+2 \\ 4+0 & 0+0 \end{bmatrix} = \boxed{\begin{bmatrix} 5 & 2 \\ 4 & 0 \end{bmatrix}}$

- note that $AB \neq BA$ in general, matrix multiplication is not commutative

ex - $A = \begin{bmatrix} 3 & 4 \\ 5 & 1 \end{bmatrix}$ $B = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}$ $A+B$ ^ $A-B$?

$\boxed{\begin{bmatrix} 2 & 4 \\ 6 & 2 \end{bmatrix} \quad \begin{bmatrix} 4 & 4 \\ 4 & 0 \end{bmatrix}}$

ex - $A = \begin{bmatrix} 8 & 0 \\ -1 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 2 & 2 \\ 10 & -1 \end{bmatrix}$ $A+B$ ^ $A-B$?

$\boxed{\begin{bmatrix} 10 & 2 \\ 9 & -1 \end{bmatrix} \quad \begin{bmatrix} 6 & -2 \\ -11 & 1 \end{bmatrix}}$

ex - $c = 2$ $A = \begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix}$ cA?

$\boxed{\begin{bmatrix} -2 & 0 \\ -2 & 2 \end{bmatrix}}$

ex - $c = 8$ $A = \begin{bmatrix} 4 & 3 & 1 \\ 0 & 0 & 2 \\ -3 & -3 & 9 \end{bmatrix}$

$\boxed{\begin{bmatrix} 32 & 24 & 8 \\ 0 & 0 & 16 \\ -24 & -24 & 72 \end{bmatrix}}$

ex - $A=\begin{bmatrix}2&0\\-1&1\end{bmatrix}$ $B=\begin{bmatrix}3&3\\4&1\end{bmatrix}$ AB?

$\begin{bmatrix}6+0&6+0\\-3+4&-3+1\end{bmatrix}=\boxed{\begin{bmatrix}6&6\\1&-2\end{bmatrix}}$

ex - $A=\begin{bmatrix}3&0&1\\-1&1&1\\8&7&6\end{bmatrix}$ $B=\begin{bmatrix}2&-2&3\\6&6&2\\1&4&4\end{bmatrix}$ AB?

$\begin{bmatrix}6+0+1&-6+0+4&9+0+4\\-2+6+1&2+6+4&-3+2+4\\16+42+6&-16+42+24&24+14+24\end{bmatrix}=\boxed{\begin{bmatrix}7&-2&13\\5&12&3\\64&50&62\end{bmatrix}}$

ex - $A=\begin{bmatrix}0&1\\1&1\end{bmatrix}$ $B=\begin{bmatrix}3&1&-1\\1&1&3\\0&1&2\end{bmatrix}$ AB?

can't multiply b/c $\overset{A}{m\times n} \neq \overset{B}{m\times n}$

ex - $A=\begin{bmatrix}2&0&1\\1&-1&1\end{bmatrix}$ $B=\begin{bmatrix}3&-1&1\\1&1&4\\0&1&3\end{bmatrix}$ AB?

$\begin{bmatrix}6+0+0&-2+0+1&2+0+3\\3-1+0&-1-1+1&1-4+3\end{bmatrix}=\boxed{\begin{bmatrix}6&-1&5\\2&-1&0\end{bmatrix}}$

ex - $A=\begin{bmatrix}1&1\\2&-1\\1&0\end{bmatrix}$ $B=\begin{bmatrix}0&0&1\\7&6&5\end{bmatrix}$ AB?

$\begin{bmatrix}0+7&0+6&1+5\\0-7&0-6&2-5\\0+0&0+0&1+0\end{bmatrix}=\boxed{\begin{bmatrix}7&6&6\\-7&-6&-3\\0&0&1\end{bmatrix}}$

Properties of Matrix Addition ^ Scalar Multiplication

Commutativity, Associativity, ^

- matrix addition is commutative and associative; $A+B=B+A$ ^ $A+(B+C)=(A+B)+C$

ex - $A=\begin{bmatrix}1&0\\0&-1\end{bmatrix}$ $B=\begin{bmatrix}2&1\\3&4\end{bmatrix}$ $C=\begin{bmatrix}-1&7\\13&5\end{bmatrix}$ A+B? B+A? A+(B+C)? (A+B)+C?

Distributivity

$A+B=\boxed{\begin{bmatrix}3&1\\3&3\end{bmatrix}}$

$B+A=\boxed{\begin{bmatrix}3&1\\3&3\end{bmatrix}}$

$A+(B+C)=\begin{bmatrix}1&0\\0&-1\end{bmatrix}+\begin{bmatrix}1&8\\16&9\end{bmatrix}=\boxed{\begin{bmatrix}2&8\\16&8\end{bmatrix}}$

$(A+B)+C=\begin{bmatrix}3&1\\3&3\end{bmatrix}+\begin{bmatrix}-1&7\\13&5\end{bmatrix}=\boxed{\begin{bmatrix}2&8\\16&8\end{bmatrix}}$

- let c, d be scalars $(cd)A=c(dA)$

ex - $c=2$ $d=-1$ $A=\begin{bmatrix}1&0\\0&-1\end{bmatrix}$ $(cd)A$? $c(dA)$?

$(cd)A=-2\begin{bmatrix}1&0\\0&-1\end{bmatrix}=\boxed{\begin{bmatrix}-2&0\\0&2\end{bmatrix}}$

$c(dA)=2\begin{bmatrix}-1&0\\0&1\end{bmatrix}=\boxed{\begin{bmatrix}-2&0\\0&2\end{bmatrix}}$

- distributive property: $c(A+B)=cA+cB$ $(c+d)A=cA+dA$

ex - $c=2$ $A=\begin{bmatrix}1&0\\0&-1\end{bmatrix}$ $B=\begin{bmatrix}2&1\\3&4\end{bmatrix}$ $c(A+B)$? $cA+cB$?

$c(A+B)=2\begin{bmatrix}3&1\\3&3\end{bmatrix}=\boxed{\begin{bmatrix}6&2\\6&6\end{bmatrix}}$

$cA+cB=\begin{bmatrix}2&0\\0&-2\end{bmatrix}+\begin{bmatrix}4&2\\6&8\end{bmatrix}=\boxed{\begin{bmatrix}6&2\\6&6\end{bmatrix}}$

Identities,

- additive identity: $\begin{bmatrix}0&0\\0&0\end{bmatrix}+\begin{bmatrix}1&0\\0&-1\end{bmatrix}=\begin{bmatrix}1&0\\0&-1\end{bmatrix}$

Additive Inverses, Multiplicative

- additive inverses exist for matricies, the additive inverse of A is -A:

$\begin{bmatrix}1&0\\0&-1\end{bmatrix}+-\begin{bmatrix}1&0\\0&-1\end{bmatrix}=\begin{bmatrix}1&0\\0&-1\end{bmatrix}+\begin{bmatrix}-1&0\\0&1\end{bmatrix}=\begin{bmatrix}0&0\\0&0\end{bmatrix}$

Associativity, Distributivity

- associativity ^ distributivity holds for matrix multiplication: $A(BC)=(AB)C$ $A(B+C)=AB+AC$

$(A+B)C=AC+BC$ $c(AB)=(cA)B=A(cB)$

ex - $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ $B = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$ $C = \begin{bmatrix} -1 & 7 \\ 13 & 5 \end{bmatrix}$ $(AB)C$? $A(BC)$?

$(AB)C = \begin{bmatrix} 2+0 & 1+0 \\ 0-3 & 0-4 \end{bmatrix}\begin{bmatrix} -1 & 7 \\ 13 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -3 & -4 \end{bmatrix}\begin{bmatrix} -1 & 7 \\ 13 & 5 \end{bmatrix} = \begin{bmatrix} -2+13 & 14+5 \\ 3-52 & -21-20 \end{bmatrix} = \boxed{\begin{bmatrix} 11 & 19 \\ -49 & -41 \end{bmatrix}}$

$A(BC) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -2+13 & 14+5 \\ -3+52 & 21+20 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 11 & 19 \\ 49 & 41 \end{bmatrix} = \begin{bmatrix} 11+0 & 19+0 \\ 0-49 & 0-41 \end{bmatrix} = \boxed{\begin{bmatrix} 11 & 19 \\ -49 & -41 \end{bmatrix}}$

ex - $A = 11$ $B = 11$ $c = 3$ $c(AB)$? $(cA)B$? $A(cB)$?

$c(AB) = 3\begin{bmatrix} 2 & 1 \\ -3 & -4 \end{bmatrix} = \boxed{\begin{bmatrix} 6 & 3 \\ -9 & -12 \end{bmatrix}}$

$(cA)B = \begin{bmatrix} 3 & 0 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 6+0 & 3+0 \\ 0-9 & 0-12 \end{bmatrix} = \boxed{\begin{bmatrix} 6 & 3 \\ -9 & -12 \end{bmatrix}}$

$A(cB) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 6 & 3 \\ 9 & 12 \end{bmatrix} = \begin{bmatrix} 6+0 & 3-0 \\ 0-9 & 0-12 \end{bmatrix} = \boxed{\begin{bmatrix} 6 & 3 \\ -9 & -12 \end{bmatrix}}$

- identity matrix: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

ex - prove commutativity for addition $A = \begin{bmatrix} 1 & 7 & 8 \\ -7 & 6 & 10 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 1 & 2 \\ 2 & -2 & 1 \end{bmatrix}$

$A+B = \boxed{\begin{bmatrix} 1 & 8 & 10 \\ -5 & 4 & 11 \end{bmatrix}}$

$B+A = \boxed{\begin{bmatrix} 1 & 8 & 10 \\ -5 & 4 & 11 \end{bmatrix}}$

ex - prove multiplicative associativity holds $A = \begin{bmatrix} -1 & 0 \\ 1 & 2 \end{bmatrix}$ $B = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$ $C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$(AB)C = \begin{bmatrix} -2+0 & -2+0 \\ 2+4 & 2+4 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -2 & -2 \\ 6 & 6 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0-2 & -2+0 \\ 0+6 & 6+0 \end{bmatrix} = \boxed{\begin{bmatrix} -2 & -2 \\ 6 & 6 \end{bmatrix}}$

$A(BC) = \begin{bmatrix} -1 & 0 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 0+2 & 2+0 \\ 0+2 & 2+0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} -2+0 & -2+0 \\ 2+4 & 2+4 \end{bmatrix} = \boxed{\begin{bmatrix} -2 & -2 \\ 6 & 6 \end{bmatrix}}$

ex - show $AB \neq BA$ $A = \begin{bmatrix} 8 & -8 \\ 1 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$

$AB = \begin{bmatrix} 16-24 & 8-32 \\ 2+3 & 1+4 \end{bmatrix} = \boxed{\begin{bmatrix} -8 & -24 \\ 5 & 5 \end{bmatrix}}$

$BA = \begin{bmatrix} 16+1 & -16+1 \\ 24+4 & -24+4 \end{bmatrix} = \boxed{\begin{bmatrix} 17 & -15 \\ 28 & -20 \end{bmatrix}}$

ex - prove $(c+d)A = cA + dA$ $c = 4$ $d = 2$ $A = \begin{bmatrix} 1 & 0 & -3 \\ 3 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$

$(c+d)A = 6\begin{bmatrix} 1 & 0 & -3 \\ 3 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix} = \boxed{\begin{bmatrix} 6 & 0 & -18 \\ 18 & 12 & 6 \\ -6 & -6 & 0 \end{bmatrix}}$

$cA + dA = \begin{bmatrix} 4 & 0 & -12 \\ 12 & 8 & 4 \\ -4 & -4 & 0 \end{bmatrix} + \begin{bmatrix} 2 & 0 & -6 \\ 6 & 4 & 2 \\ -2 & -2 & 0 \end{bmatrix} = \boxed{\begin{bmatrix} 6 & 0 & -18 \\ 18 & 12 & 6 \\ -6 & -6 & 0 \end{bmatrix}}$

ex - prove $(A+B)C = AC + BC$ $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 0 & 1 \end{bmatrix}$ $B = \begin{bmatrix} -1 & 0 & 3 \\ 4 & 1 & 4 \end{bmatrix}$ $C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \end{bmatrix}$

$(A+B)C = \begin{bmatrix} 0 & 2 & 6 \\ 8 & 1 & 5 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0+0-6 & 0+2+0 \\ 8+0-5 & 0+1+0 \end{bmatrix} = \boxed{\begin{bmatrix} -6 & 2 \\ 3 & 1 \end{bmatrix}}$

$AC + BC = \begin{bmatrix} 1+0-3 & 0+2+0 \\ 4+0-1 & 0+0+0 \end{bmatrix} + \begin{bmatrix} -1+0-3 & 0+0+0 \\ 4+0-4 & 0+1+0 \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 3 & 0 \end{bmatrix} + \begin{bmatrix} -4 & 0 \\ 0 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} -6 & 2 \\ 3 & 1 \end{bmatrix}}$

ex - prove $AI = A$ $A = \begin{bmatrix} 1 & 8 & 8 \\ 13 & 24 & 6 \\ 6 & -6 & 8 \end{bmatrix}$

$AI = \begin{bmatrix} 1 & 8 & 8 \\ 13 & 24 & 6 \\ 6 & -6 & 8 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1+0+0 & 0+8+0 & 0+0+8 \\ 13+0+0 & 0+24+0 & 0+0+6 \\ 6+0+0 & 0-6+0 & 0+0+8 \end{bmatrix} = \boxed{\begin{bmatrix} 1 & 8 & 8 \\ 13 & 24 & 6 \\ 6 & -6 & 8 \end{bmatrix}}$

transpose of a matrix

ex - $A = \begin{bmatrix} 1 & 0 \\ -13 & 8 \end{bmatrix}$ A^T?   transpose flips columns ^ rows

$A^T = \boxed{\begin{bmatrix} 1 & -13 \\ 0 & 8 \end{bmatrix}}$

ex - $B = \begin{bmatrix} 7 & -1 & -3 \\ 4 & 1 & 1 \\ 8 & 9 & 3 \end{bmatrix}$ B^T?

$B^T = \boxed{\begin{bmatrix} 7 & 4 & 8 \\ -1 & 1 & 9 \\ -3 & 1 & 3 \end{bmatrix}}$

- transpose satisfies the following properties: $(A^T)^T = A$ $(A+B)^T = A^T + B^T$
$(cA)^T = c(A^T)$ $(AB)^T = B^T A^T$

ex - $A=\begin{bmatrix}1&0\\-13&8\end{bmatrix}$ $B=\begin{bmatrix}2&1\\1&2\end{bmatrix}$ $AB=\begin{bmatrix}2&1\\-18&3\end{bmatrix}$ $(AB)^T?$ $B^TA^T?$

$(AB)^T=\boxed{\begin{bmatrix}2&-18\\1&3\end{bmatrix}}$

$B^TA^T=\begin{bmatrix}2&1\\1&2\end{bmatrix}\begin{bmatrix}1&-13\\0&8\end{bmatrix}=\begin{bmatrix}2+0&-26+8\\1+0&-13+16\end{bmatrix}=\boxed{\begin{bmatrix}2&-18\\1&3\end{bmatrix}}$

ex - prove $(A+B)^T=A^T+B^T$ $A=\begin{bmatrix}1&0&-3\\3&6&8\end{bmatrix}$ $B=\begin{bmatrix}1&-1&1\\0&0&1\end{bmatrix}$

$(A+B)^T=\begin{bmatrix}2&-1&-2\\3&6&9\end{bmatrix}^T=\boxed{\begin{bmatrix}2&3\\-1&6\\-2&9\end{bmatrix}}$

$A^T+B^T=\begin{bmatrix}1&3\\0&6\\-3&8\end{bmatrix}+\begin{bmatrix}1&0\\-1&0\\1&1\end{bmatrix}=\boxed{\begin{bmatrix}2&3\\-1&6\\-2&9\end{bmatrix}}$

ex - prove $(cA)^T=c(A^T)$ $c=4$ $A=\begin{bmatrix}1&2\\0&2\\-1&2\end{bmatrix}$

$(cA)^T=\begin{bmatrix}4&8\\0&8\\-4&8\end{bmatrix}^T=\boxed{\begin{bmatrix}4&0&-4\\8&8&8\end{bmatrix}}$

$c(A^T)=4\begin{bmatrix}1&0&-1\\2&2&2\end{bmatrix}=\boxed{\begin{bmatrix}4&0&-4\\8&8&8\end{bmatrix}}$

ex - prove $(A^T)^T=A$ $A=\begin{bmatrix}0&1&6\\4&5&6\end{bmatrix}$

$(A^T)^T=\begin{bmatrix}0&4\\1&5\\6&6\end{bmatrix}^T=\boxed{\begin{bmatrix}0&1&6\\4&5&6\end{bmatrix}=A}$

Inverse of a Matrix

- $BA=AB=I$ ⇒ $AA^{-1}=A^{-1}A=I$

A^{-1} method 1

- inverse is denoted A^{-1}
- $A=\begin{bmatrix}a&b\\c&d\end{bmatrix}$ $A^{-1}=\frac{1}{ad-bc}\begin{bmatrix}d&-b\\-c&a\end{bmatrix}=\frac{1}{\det A}\begin{bmatrix}d&-b\\-c&a\end{bmatrix}$

ex - $A=\begin{bmatrix}4&0\\-1&2\end{bmatrix}$ $A^{-1}?$ $AA^{-1}?$ $A^{-1}A?$

$A^{-1}=\frac{1}{4(2)-0(-1)}\begin{bmatrix}2&-0\\1&4\end{bmatrix}=\frac{1}{8}\begin{bmatrix}2&0\\1&4\end{bmatrix}=\boxed{\begin{bmatrix}\frac{1}{4}&0\\\frac{1}{8}&\frac{1}{2}\end{bmatrix}}$

$AA^{-1}=\begin{bmatrix}4&0\\-1&2\end{bmatrix}\begin{bmatrix}1/4&0\\1/8&1/2\end{bmatrix}=\begin{bmatrix}1+0&0+0\\-1/4+1/4&0+1\end{bmatrix}=\boxed{\begin{bmatrix}1&0\\0&1\end{bmatrix}}$

$A^{-1}A=\begin{bmatrix}1/4&0\\1/8&1/2\end{bmatrix}\begin{bmatrix}4&0\\-1&2\end{bmatrix}=\begin{bmatrix}1+0&0+0\\1/2-1/2&0+1\end{bmatrix}=\boxed{\begin{bmatrix}1&0\\0&1\end{bmatrix}}$

A^{-1} method 2

Gauss-Jordan Elimination

- gauss-jordan elimination adjoins the identity matrix to A row operations until we transform A into I

ex - $A=\begin{bmatrix}4&0\\-1&2\end{bmatrix}$ $A^{-1}?$

$\left[\begin{array}{cc|cc}4&0&1&0\\-1&2&0&1\end{array}\right]$ $\frac{1}{4}R_1\to R_1$ ⇒ $\left[\begin{array}{cc|cc}1&0&1/4&0\\-1&2&0&1\end{array}\right]$ $R_1+R_2\to R_2$ ⇒ $\left[\begin{array}{cc|cc}1&0&1/4&0\\0&2&1/4&1\end{array}\right]$ $\frac{1}{2}R_2\to R_2$ ⇒

$\left[\begin{array}{cc|cc}1&0&1/4&0\\0&1&1/8&1/2\end{array}\right]$ $\boxed{A^{-1}=\begin{bmatrix}1/4&0\\1/8&1/2\end{bmatrix}}$

ex - $A=\begin{bmatrix}1&0&-1\\2&2&1\\0&0&3\end{bmatrix}$ $A^{-1}?$

$\left[\begin{array}{ccc|ccc}1&0&-1&1&0&0\\2&2&1&0&1&0\\0&0&3&0&0&1\end{array}\right]$ $-2R_1+R_2\to R_2$ ⇒ $\left[\begin{array}{ccc|ccc}1&0&-1&1&0&0\\0&2&3&-2&1&0\\0&0&3&0&0&1\end{array}\right]$ $\frac{1}{3}R_3\to R_3$ ⇒ $\left[\begin{array}{ccc|ccc}1&0&-1&1&0&0\\0&2&3&-2&1&0\\0&0&1&0&0&1/3\end{array}\right]$ $\frac{1}{2}R_2\to R_2$

$\left[\begin{array}{ccc|ccc}1&0&-1&1&0&0\\0&1&3/2&-1&1/2&0\\0&0&1&0&0&1/3\end{array}\right]$ $R_2-\frac{3}{2}R_3\to R_2$ ⇒ $\left[\begin{array}{ccc|ccc}1&0&-1&1&0&0\\0&1&0&-1&1/2&-1/2\\0&0&1&0&0&1/3\end{array}\right]$ $R_1+R_3\to R_1$ ⇒ $\left[\begin{array}{ccc|ccc}1&0&0&1&0&1/3\\0&1&0&-1&1/2&-1/2\\0&0&1&0&0&1/3\end{array}\right]$ $\boxed{A^{-1}=\begin{bmatrix}1&0&1/3\\-1&1/2&-1/2\\0&0&1/3\end{bmatrix}}$

ex - $A=\begin{bmatrix}2&1&4\\-1&1&1\\0&4&8\end{bmatrix}$ $A^{-1}?$

$\left[\begin{array}{ccc|ccc}2&1&4&1&0&0\\-1&1&1&0&1&0\\0&4&8&0&0&1\end{array}\right]$ $2R_2+R_1\to R_2$ ⇒ $\left[\begin{array}{ccc|ccc}2&1&4&1&0&0\\0&3&6&1&2&0\\0&4&8&0&0&1\end{array}\right]$ $\frac{1}{2}R_1\to R_1$ ⇒ $\left[\begin{array}{ccc|ccc}1&1/2&2&1/2&0&0\\0&3&6&1&2&0\\0&4&8&0&0&1\end{array}\right]$ $\frac{1}{3}R_2\to R_2$ ⇒

$\left[\begin{array}{ccc|ccc}1&1/2&2&1/2&0&0\\0&1&2&1/3&2/3&0\\0&4&8&0&0&1\end{array}\right]$ $\frac{1}{4}R_3\to R_3$ ⇒ $\left[\begin{array}{ccc|ccc}1&1/2&2&1/2&0&0\\0&1&2&1/3&2/3&0\\0&1&2&0&0&1/4\end{array}\right]$ $R_2-R_3\to R_3$ ⇒ $\left[\begin{array}{ccc|ccc}1&1/2&2&1/2&0&0\\0&1&2&1/3&2/3&0\\0&0&0&1/3&2/3&-1/4\end{array}\right]$ $\boxed{\text{can't transform}}$ A is not invertible ex - $A = \begin{bmatrix} 8 & -1 \\ 0 & 3 \end{bmatrix}$ A^{-1}? use gauss-jordan elimination

$\left[\begin{array}{cc|cc} 8 & -1 & 1 & 0 \\ 0 & 3 & 0 & 1 \end{array}\right]$ $\frac{1}{8}R_1 \to R_1 \Leftrightarrow \left[\begin{array}{cc|cc} 1 & -1/8 & 1/8 & 0 \\ 0 & 3 & 0 & 1 \end{array}\right]$ $\frac{1}{3}R_2 \to R_2 \Leftrightarrow \left[\begin{array}{cc|cc} 1 & -1/8 & 1/8 & 0 \\ 0 & 1 & 0 & 1/3 \end{array}\right]$ $R_1 + \frac{1}{8}R_2 \to R_1 \Leftrightarrow$

$\left[\begin{array}{cc|cc} 1 & 0 & 1/8 & 1/24 \\ 0 & 1 & 0 & 1/3 \end{array}\right]$ $A^{-1} = \begin{bmatrix} 1/8 & 1/24 \\ 0 & 1/3 \end{bmatrix}$

ex - $A = \begin{bmatrix} 1 & -1 \\ -1 & 0 \end{bmatrix}$ A^{-1}? use gauss-jordan elimination

$\left[\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{array}\right]$ $R_1 + R_2 \to R_2 \Leftrightarrow \left[\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ 0 & -1 & 1 & 1 \end{array}\right]$ $-R_2 \to R_2 \Leftrightarrow \left[\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & -1 \end{array}\right]$ $R_1 + R_2 \to R_1 \Leftrightarrow$

$\left[\begin{array}{cc|cc} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & -1 \end{array}\right]$ $A^{-1} = \begin{bmatrix} 0 & -1 \\ -1 & -1 \end{bmatrix}$

ex - $A = \begin{bmatrix} 2 & -1 & 1 \\ 0 & 0 & 5 \\ -6 & 6 & 2 \end{bmatrix}$ A^{-1}? use gauss-jordan elimination

[illegible] $R_2 \leftrightarrow R_3 \Leftrightarrow$ [illegible] $3R_1 + R_2 \to R_2 \Leftrightarrow$ [illegible] $\frac{1}{2}R_1 \to R_1 \Leftrightarrow$

[illegible] $\frac{1}{18}R_2 \to R_2 \Leftrightarrow$ [illegible] $\frac{1}{5}R_3 \to R_3 \Leftrightarrow$ [illegible] $R_1 - 2R_2 \to R_1 \Leftrightarrow$

[illegible] $R_2 - \frac{5}{8}R_3 \to R_2 \Leftrightarrow$ [illegible] $R_1 + \frac{1}{18}R_3 \to R_1 \Leftrightarrow$ [illegible] A^{-1} [illegible]

ex - $A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 2 & 3 \\ 5 & 2 & 1 \end{bmatrix}$ A^{-1}? gauss-jordan elimination

$\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 2 & 2 & 3 & 0 & 1 & 0 \\ 5 & 2 & 1 & 0 & 0 & 1 \end{array}\right]$ $-2R_1 + R_2 \to R_2 \Leftrightarrow$ [illegible] $5R_1 - R_3 \to R_3 \Leftrightarrow$ [illegible] $7R_2 + 4R_3 \to R_3 \Leftrightarrow$

[illegible] $\frac{1}{4}R_2 \to R_2$ ^ $\frac{1}{17}R_3 \to R_3 \Leftrightarrow$ [illegible] $R_1 + R_2 \to R_1 \Leftrightarrow$ [illegible] ...

ex - $A = \begin{bmatrix} 4 & -1 \\ 8 & 3 \end{bmatrix}$ A^{-1}?

$\frac{1}{4(3)-(-1)(8)} \begin{bmatrix} 3 & 1 \\ -8 & 4 \end{bmatrix} = \frac{1}{20} \begin{bmatrix} 3 & 1 \\ -8 & 4 \end{bmatrix} = \boxed{\begin{bmatrix} 3/20 & 1/20 \\ -2/5 & 1/5 \end{bmatrix}}$

Determinants

Determinant of 2×2 Matrix - $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $ad - bc$ denoted by $\det(A)$ ^ $|A|$

ex - $A = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}$ $\det(A)$?

$\det(A) = 2(4) - (-1)(3) = 8 + 3 = \boxed{11}$

ex - $B = \begin{bmatrix} -1 & 2 \\ 3 & -2 \end{bmatrix}$ $\det(B)$?

$\det(B) = -1(-2) - 2(3) = 2 - 6 = \boxed{-4}$

Cofactor Expansion ex - $A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 3 \\ -1 & 0 & 4 \end{bmatrix}$ $\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$ ~~det~~ cofactor of 1st row? = det A

cofactors of any row or column in the same matrix are all equal

$+1\begin{vmatrix} 1 & 3 \\ 0 & 4 \end{vmatrix} + 0(-1)\begin{vmatrix} 2 & 3 \\ -1 & 4 \end{vmatrix} + (-1)\begin{vmatrix} 2 & 1 \\ -1 & 0 \end{vmatrix}$

prefend 1st row column are deleted M_{11} | 1st row, 2nd column M_{12} | 1st row, 3rd column M_{13}

m_{ij} is minor of a_{ij} $(-1)^{i+j} m_{ij}$ is cofactor

we multiply each cofactor by entry in A

$= 1(1(4) - 3(0)) + 0 - (2(0) - 1(-1)) = 4 + -1 = \boxed{3}$

ex - same matricies, second row?

$2(-1)\begin{vmatrix} 0 & -1 \\ 0 & 4 \end{vmatrix} + \cancel{0}1(+1)\begin{vmatrix} 1 & -1 \\ -1 & 4 \end{vmatrix} + 3(-1)\begin{vmatrix} 1 & 0 \\ -1 & 0 \end{vmatrix} = -2\begin{vmatrix} 0 & -1 \\ 0 & 4 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ -1 & 4 \end{vmatrix} + -3\begin{vmatrix} 1 & 0 \\ -1 & 0 \end{vmatrix} = -2(0(4) - (-1)(0)) + (1(4) - (-1)(-1))$ $-3(1(0) - 0(-1)) = 0 + 3 + 0 = \boxed{3}$

ex - same matricies, third column?

$-1\begin{vmatrix} 2 & 1 \\ -1 & 0 \end{vmatrix} + 3(-1)\begin{vmatrix} 1 & 0 \\ -1 & 0 \end{vmatrix} + 4\begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} = -1(\cancel{2(0)}-1(-1)) - 3(\cancel{1(0)}-\cancel{0(-1)}) + 4(1(1)-\cancel{0(2)}) = -1+0+4 = \boxed{3}$

ex - $A = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}$ det(A)?

$\det(A) = -1(1) - \cancel{1(0)} = \boxed{-1}$

ex - $A = \begin{bmatrix} 8 & 6 \\ 7 & -10 \end{bmatrix}$ det(A)?

$\det(A) = 8(-10) - 6(7) = -80 - 42 = \boxed{-122}$

ex - $A = \begin{bmatrix} -1 & 0 & 1 \\ 2 & -1 & 0 \\ 3 & 4 & 6 \end{bmatrix}$ det(A) using cofactor expansion?

start w/ positive in top left column = $\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$

$-1\begin{vmatrix} -1 & 0 \\ 4 & 6 \end{vmatrix} + \cancel{0} + 1\begin{vmatrix} 2 & -1 \\ 3 & 4 \end{vmatrix} = -(-1(6) - \cancel{0(4)}) + (2(4) - (-1)(3)) = 6 + 11 = \boxed{17}$

ex - $A = \begin{bmatrix} -1 & 0 & 1 \\ 2 & -1 & 1 \\ 3 & 4 & 7 \end{bmatrix}$ det(A) using cofactor expansion? remember $\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$

$-1\begin{vmatrix} -1 & 1 \\ 4 & 7 \end{vmatrix} + \cancel{0} + 1\begin{vmatrix} 2 & -1 \\ 3 & 4 \end{vmatrix} = -(-1(7) - 1(4)) + (2(4) - (-1)(3)) = 11 + 11 = \boxed{22}$

ex - $A = \begin{bmatrix} 1 & 0 & 0 & 4 \\ -1 & 2 & 3 & 6 \\ 1 & 0 & 1 & -1 \\ 4 & 4 & 1 & 0 \end{bmatrix}$ det(A) using cofactor expansion? remember $\begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$

$1\begin{vmatrix} 2 & 3 & 6 \\ 0 & 1 & -1 \\ 4 & 1 & 0 \end{vmatrix} + \cancel{0} + \cancel{0} + 4(-1)\begin{vmatrix} -1 & 2 & 3 \\ 1 & 0 & 1 \\ 4 & 4 & 1 \end{vmatrix}$

$\begin{vmatrix} 2 & 3 & 6 \\ 0 & 1 & -1 \\ 4 & 1 & 0 \end{vmatrix} = \cancel{0} + 1\begin{vmatrix} 2 & 6 \\ 4 & 0 \end{vmatrix} - 1(-1)\begin{vmatrix} 2 & 3 \\ 4 & 1 \end{vmatrix} = (\cancel{2(0)} - 6(4)) + (2(1) - 3(4)) = -24 - 10 = -34$

$\begin{vmatrix} -1 & 2 & 3 \\ 1 & 0 & 1 \\ 4 & 4 & 1 \end{vmatrix} = 1(-1)\begin{vmatrix} 2 & 3 \\ 4 & 1 \end{vmatrix} + \cancel{0} + 1(-1)\begin{vmatrix} -1 & 2 \\ 4 & 4 \end{vmatrix} = -(2(1) - 3(4)) - (-1(4) - 2(4)) = 10 + 12 = 22$

$-34 - 4(22) = -34 - 88 = \boxed{-122}$

Properties of Determinants

- if A ^ B are $n \times n$ matricies, then $\det(AB) = \det A \det B$

ex - $A = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$ $B = \begin{bmatrix} 3 & 4 \\ 5 & 2 \end{bmatrix}$ det(AB)?

$\det A = (1(2) - \cancel{(0)(-1)}) = 2$ $\det B = (3(2) - (4)(5)) = 6 - 20 = -14$

$\det(AB) = 2(-14) = \boxed{-28}$

another way to solve is to perform matrix multiplication ^ find determinant of result

$AB = \begin{bmatrix} 3+0 & 4-0 \\ 3+10 & -4+4 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 7 & 0 \end{bmatrix}$

$\det\left(\begin{bmatrix} 3 & 4 \\ 7 & 0 \end{bmatrix}\right) = \cancel{3(0)} - 4(7) = \boxed{-28}$

- if c is a scalar ^ A is an $n \times n$ matrix, then $\det(cA) = c^n \det A$

ex - $c = 8$ $A = \begin{bmatrix} 1 & -1 \\ 3 & -4 \end{bmatrix}$

$cA = 8\begin{bmatrix} 1 & -1 \\ 3 & -4 \end{bmatrix} = \begin{bmatrix} 8 & -8 \\ 24 & -32 \end{bmatrix}$ $\det(cA) = 8(-32) - (-8)(24) = -256 + 192 = \boxed{-64}$ OR

$c^n \det A = 8^2(1(-4) - (-1)(3)) = 64(-4+3) = \boxed{-64}$

ex - $c = 3$ $A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 0 & 3 \\ 1 & 4 & 2 \end{bmatrix}$ det(cA)?

$cA = \begin{bmatrix} 3 & 0 & -3 \\ 6 & 0 & 9 \\ 3 & 12 & 6 \end{bmatrix}$

$\det(cA) = \cancel{0} + \cancel{0} + 12(-1)\begin{vmatrix} 3 & -3 \\ 6 & 9 \end{vmatrix}$ ~~$\begin{vmatrix} -36 & 36 \\ -72 & -108 \end{vmatrix}$ = -36(-108) - 36(-72) = 3888~~

$\hookrightarrow -12\begin{vmatrix} 3 & -3 \\ 6 & 9 \end{vmatrix} = -12(3(9) - (-3)(6)) = -12(27+18) = -12(45) = \boxed{-540}$ OR

$c^n \det A = 3^3\left(\cancel{0} + \cancel{0} + 4(-1)\begin{vmatrix}1&-1\\2&3\end{vmatrix}\right) = 27(-4(1(3)-(-1)(2))) = 27(-4(3+2)) = 27(-20) = \boxed{-540}$

Determinants & Invertibility

- A is invertible if $\det(A) \neq 0$

ex - determine if the matrix is invertible $A = \begin{bmatrix}-2&3&1\\4&2&1\\2&5&2\end{bmatrix}$

$\det A = -2\begin{vmatrix}2&1\\5&2\end{vmatrix} + 3(-1)\begin{vmatrix}4&1\\2&2\end{vmatrix} + 1\begin{vmatrix}4&2\\2&5\end{vmatrix} = -2(2(2)-1(5)) - 3(4(2)-1(2)) + (4(5)-2(2)) = -2(-1)-3(6)+16 =$

$2-18+16 = \boxed{0}$ $\boxed{\text{not invertible}}$

ex - " $A = \begin{bmatrix}1&-1&9\\0&2&2\\1&3&2\end{bmatrix}$

$\det A = 1\begin{vmatrix}2&2\\3&2\end{vmatrix} + \cancel{0} + 1\begin{vmatrix}-1&9\\2&2\end{vmatrix} = (2(2)-2(3)) + (-1(2)-9(2)) = -2-20 = \boxed{-22}$ $\boxed{\text{is invertible}}$

- $\det(A^{-1}) = \frac{1}{\det A}$

- for last problem, $\det(A^{-1}) = \boxed{\frac{1}{-22}}$

Determinants & Transpose

- $\det(A^T) = \det A$

ex - $A = \begin{bmatrix}1&-1&9\\0&2&2\\1&3&2\end{bmatrix}$ $\det A^T$?

$A^T = \begin{bmatrix}1&0&1\\-1&2&3\\9&2&2\end{bmatrix}$ $\det A^T = 1\begin{vmatrix}2&3\\2&2\end{vmatrix} + \cancel{0} + 1\begin{vmatrix}-1&2\\9&2\end{vmatrix} = (2(2)-3(2)) + (-1(2)-2(9)) = -2-20 = \boxed{-22}$

last problem, $\det A = \boxed{-22}$

ex - find $\det(AB)$ using $\det(AB) = \det(A)\det(B)$

a. $A = \begin{bmatrix}6&-6\\7&8\end{bmatrix}$ $B = \begin{bmatrix}100&-1\\0&10\end{bmatrix}$

$\det(AB) = (6(8)-(-6)(7))(100(10)-(-1)(0)) = 90(1000) = \boxed{90000}$

b. $A = \begin{bmatrix}1&6&-7\\0&1&2\\0&3&4\end{bmatrix}$ $B = \begin{bmatrix}-2&-2&3\\0&1&2\\1&-1&3\end{bmatrix}$

$\det(AB) = \left(1\begin{vmatrix}1&2\\3&4\end{vmatrix} + \cancel{0} + \cancel{0}\right)\left(-2\begin{vmatrix}1&2\\-1&3\end{vmatrix} + \cancel{0} + 1\begin{vmatrix}-2&3\\1&2\end{vmatrix}\right) = (1(4)-2(3))(-2(1(3)-2(-1)) + (-2(2)-2(1))) =$

$-2(-2(5)-6) = -2(-16) = \boxed{32}$

ex - find $\det(cA)$ using $\det(cA) = c^n \det(A)$

a. $c = 10$ $A = \begin{bmatrix}1&-1\\8&6\end{bmatrix}$

$\det(cA) = 10^2(1(6)-(-1)(8)) = 100(6+8) = 100(14) = \boxed{1400}$

b. $c = 6$ $A = \begin{bmatrix}1&1&1\\2&2&2\\3&3&3\end{bmatrix}$

$\begin{vmatrix}2&2\\3&3\end{vmatrix} = (2(3)-2(3)) = 6-6 = 0$

$\det(cA) = 6^3\left(1\begin{vmatrix}2&2\\3&3\end{vmatrix} + 1(-1)\begin{vmatrix}2&2\\3&3\end{vmatrix} + 1\begin{vmatrix}2&2\\3&3\end{vmatrix}\right) = 216(0+0+0) = \boxed{0}$

ex - determine if matrix is invertible using determinants

a. $A = \begin{bmatrix}0&100\\10&0\end{bmatrix}$

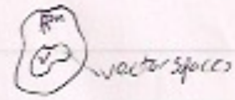
$\det A = 0(0)-100(10) = \boxed{-1000}$ $\boxed{\text{yes}}$

b. $A = \begin{bmatrix}6&1&3\\-6&0&-1\\2&0&7\end{bmatrix}$

$\det A = 1(-1)\begin{vmatrix}-6&-1\\2&7\end{vmatrix} + \cancel{0} + \cancel{0} = -1(-6(7)-(-1)(2)) = -1(-42+2) = \boxed{40}$ $\boxed{\text{yes}}$

c. $A = \begin{bmatrix}6&1&9\\-6&0&-6\\2&0&2\end{bmatrix}$

$\det A = 1(-1)\begin{vmatrix}-6&-6\\2&2\end{vmatrix} + \cancel{0} + \cancel{0} = -1(-6(2)-(-6)(2)) = -1(-12+12) = -1(0) = \boxed{0}$ $\boxed{\text{no}}$



Vector Spaces

$+ \cdot$

- a vector space is a set V, together with addition ^ scalar multiplication, such that the following 10 properties hold:
- let $u, v, w \in V$ ^ $c, d \in \mathbb{R}$

addition rules:

① $\bar{u} + \bar{v} \in V$ closure under addition
② $\bar{u} + \bar{v} = \bar{v} + \bar{u}$ commutativity under addition
③ $\bar{u} + (\bar{v} + \bar{w}) = (\bar{u} + \bar{v}) + \bar{w}$ associativity under addition
④ there is a zero vector $\bar{0}$ such that $\bar{u} + \bar{0} = \bar{u}$ additive identity
⑤ For each $\bar{u} \in V$ there is an additive inverse $-\bar{u}$ such that $\bar{u} + (-\bar{u}) = \bar{0}$ additive inverse

multiplication rules:

⑥ $c\bar{u} \in V$ closure under scalar multiplication
⑦ $c(\bar{u} + \bar{v}) = c\bar{u} + c\bar{v}$ distributivity
⑧ $(c + d)\bar{u} = c\bar{u} + d\bar{u}$ "
⑨ $c(d\bar{u}) = (cd)\bar{u}$ associativity
⑩ $1u = u$ scalar identity

- $\mathbb{R}^2$ is a vector space, let $u = (u_1, u_2)$, $v = (v_1, v_2)$, $w = (w_1, w_2)$ lie in $\mathbb{R}^2$ note $u_1, u_2, v_1, v_2, w_1, w_2 \in \mathbb{R}$

① $u + v$ (1+2) $\to (u_1 + v_1, u_2 + v_2)$ ^ $u_1 + v_1 \in \mathbb{R}$ ^ $u_2 + v_2 \in \mathbb{R}$ by closure of real numbers (adding two $\mathbb{R} \in \mathbb{R}$)

$(u_1, u_2) + (v_1, v_2) \to (u_1 + v_1, u_2 + v_2) \in \mathbb{R}^2$ $\quad u_1 + v_1 \in \mathbb{R}$, $u_2 + v_2 \in \mathbb{R}$ by closure of the real numbers ① ✓

② $= (v_1 + u_1, v_2 + u_2) = (v_1, v_2) + (u_1, u_2) = v + u$

by commutativity under addition of real numbers $u_1 + v_1 = v_1 + u_1$ ^ $u_2 + v_2 = v_2 + u_2$

③ $u + (v + w) = (u + v) + w$

$u + (v + w) = (u_1, u_2) + ((v_1, v_2) + (w_1, w_2)) = (u_1, u_2) + (v_1 + w_1, v_2 + w_2) = (u_1 + (v_1 + w_1), u_2 + (v_2 + w_2))$

$((u_1 + v_1) + w_1, (u_2 + v_2) + w_2) = (u_1 + v_1, u_2 + v_2) + (w_1, w_2)$ by associativity under addition of real numbers

④ $(0, 0)$ is the additive identity

$u + \bar{0} = (u_1, u_2) + (0, 0) = (u_1 + 0, u_2 + 0) = (u_1, u_2) = u$ because 0 is the additive identity in $\mathbb{R}$

⑤ $-u$ is the additive inverse of u

$u + (-u) = (u_1, u_2) + (-u_1, -u_2) = (u_1 - u_1, u_2 - u_2) = (0, 0)$ because $-u_1$ is the additive inverse of u_1 in $\mathbb{R}$; similarly, $-u_2$ is the additive inverse of u_2 in $\mathbb{R}$

⑥ cu

$c(u_1, u_2) = (cu_1, cu_2) \in \mathbb{R}^2$ $\quad cu_1, cu_2 \in \mathbb{R}$ by closure under multiplication of $\mathbb{R}$

⑦ $c(u + v)$

$c((u_1, u_2) + (v_1, v_2)) = c(u_1 + v_1, u_2 + v_2) = (c(u_1 + v_1), c(u_2 + v_2)) = (cu_1 + cv_1, cu_2 + cv_2) =$

$(cu_1, cu_2) + (cv_1, cv_2) = c(u_1, u_2) + c(v_1, v_2) = cu + cv$ by distributivity in $\mathbb{R}$

⑬ $(c+d)u$

$(c+d)(u_1,u_2) = ((c+d)u_1, (c+d)u_2) = (cu_1+du_1, cu_2+du_2) = (cu_1, cu_2)+(du_1, du_2) =$
$c(u_1,u_2) + d(u_1,u_2) = cu+du$ by distributivity in $\mathbb{R}$

⑭ $c(du)$

$c(d(u_1,u_2)) = c(du_1, du_2) = (c(du_1), c(du_2)) = ((cd)u_1, (cd)u_2) = (cd)(u_1,u_2) = (cd)u$
by associativity in $\mathbb{R}$

⑮ $1u$

$1\cdot(u_1,u_2) = (1\cdot u_1, 1\cdot u_2) = (u_1,u_2) = u$ because 1 is the multiplicative identity in $\mathbb{R}$

- $\mathbb{R}^2$ is a vector space; $\mathbb{R}^n$ is a vector space for any $n \geq 2$

ex - P_2, the set of all polynomials of degree less than or equal to 2, is a vector space

let $f,g,h \in P_2$ and $c,d \in \mathbb{R}$

$f(x) = a_2x^2+a_1x+a_0 \quad a_0,a_1,a_2 \in \mathbb{R}$

$g(x) = b_2x^2+b_1x+b_0 \quad b_0,b_1,b_2 \in \mathbb{R}$

$h(x) = c_2x^2+c_1x+c_0 \quad c_0,c_1,c_2 \in \mathbb{R}$

- the set $M_{m,n}$ of all m by n matricies with matrix ~~multiplicati~~ addition and scalar multiplication, forms a vector space

- the set of all polynomials of degree less than or equal to n also forms a vector space

①→ex $-(f+g)(x) = ?$ (1^2) $\quad \in f+g \in P_2$

① $f(x)+g(x) = (a_2x^2+a_1x+a_0)+(b_2x^2+b_1x_0+b_0) = (a_2+b_2)x^2+(a_1+b_1)x+(a_0+b_0) =$

② $(b_2+a_2)x^2+(b_1+a_1)x+(b_0+a_0) = (b_2x^2+b_1x+b_0)+(a_2x^2+a_1x+a_0) =$

ex $g(x)+f(x) = (g+f)(x) \Rightarrow f+g = g+f$

ex ③ $f+(g+h) = (f+g)+h$

LHS $= (f+(g+h))x$

RHS $= f(x)+(g+h)x = f(x)+(g(x)+h(x)) = (a_2x^2+a_1x+a_0)+((b_2x^2+b_1x+b_0)+(c_2x^2+c_1x+c_0)) =$
$(a_2x^2+a_1x+a_0)+(b_2+c_2)x^2+(b_1+c_1)x+(b_0+c_0) = (a_2+(b_2+c_2))x^2+(a_1+(b_1+c_1))x+$
$(a_0+(b_0+c_0)) = ((a_2+b_2)+c_2)x^2+((a_1+b_1)+c_1)x+((a_0+b_0)+c_0) = (a_2+b_2)x^2+(a_1+b_1)x+$
$(a_0+b_0)+(c_2x^2+c_1x+c_0) = ((a_2x^2+a_1x+a_0)+(b_2x^2+b_1x+b_0))+(c_2x^2+c_1x+c_0) =$
$(f(x)+g(x))+h(x) = (f+g)(x)+h(x) = ((f+g)+h)(x)$

LHS = RHS $\Rightarrow (f+(g+h))(x) = ((f+g)+h)(x)$

ex ④ let $\bar{0}(x)=0$ then $(\bar{0}+f)(x) = ?$

$\bar{0}(x)+f(x) = 0+a_2x^2+a_1x+a_0 = a_2x^2+a_1x+(a_0+0) = a_2x^2+a_1x+a_0 = f(x)$

$\Rightarrow \bar{0}+f = f$

ex ⑤ - the additive inverse of f is $-f$ defined by $(-f)(x) = -f(x)$

$(f+(-f))(x) = f(x) + (-f)(x) = f(x) - f(x) = 0 = \bar{0}(x) \Rightarrow f + (-f) = \bar{0}$

ex - prove the vector space properties 6-10 for P_2, the set of all polynomials of degree less than or equal to 2

let: $f, g, h \in P_2$ ^ $c, d \in \mathbb{R}$, $f(x) = a_2x^2 + a_1x + a_0$, $g(x) = b_2x^2 + b_1x + b_0$;

$h(x) = c_2x^2 + c_1x + c_0$ where all a's, b's, ^ c's $\in \mathbb{R}$

⑥ $cf \in P_2$

$(cf)(x) = cf(x) = c(a_2x^2 + a_1x + a_0) = (ca_2)x^2 + (ca_1)x + (ca_0) \Rightarrow cf \in P_2$

⑦ $c(f+g) = cf + cg$

$(c(f+g))(x) = c((f+g)(x)) = c(f(x) + g(x)) = cf(x) + cg(x) = (cf)(x) + (cg)(x) = (cf+cg)(x)$

$\Rightarrow c(f+g) = cf + cg$

⑧ $(c+d)f = cf + df$

$((c+d)f)(x) = LHS$

$RHS = (c+d)f(x) = cf(x) + df(x) = (cf)(x) + (df)(x) = (cf + df)(x)$

$\Rightarrow (c+d)f = cf + df$

⑨ $c(df) = (cd)f$

$LHS = (c(df))(x)$

$RHS = c(df)(x) = c(df(x)) = (cd)f(x) = ((cd)f)(x) \Rightarrow c(df) = (cd)f$

⑩ $1 \cdot f = f$

$(1 \cdot f)(x) = 1 \cdot f(x) = f(x) \Rightarrow 1 \cdot f = f$

Sets that aren't Vector Spaces

ex - the set of polynomials that have degree exactly 2

let $f(x) = 3x^2 - x + 2$ $g(x) = -3x^2 + 4x + 5$

$(f+g)(x) = \cancel{3x^2} - x + 2 \cancel{-3x^2} + 4x + 5 = 3x + 7$

$\Rightarrow f+g$ has degree 1 $\Rightarrow f+g$ is not in the set of polynomials that have degree 2

$\Rightarrow$ closure under addition does not hold

ex - $\mathbb{Z}^2$ is not a vector space $\mathbb{Z}^2 = \{(m,n) \mid m, n \in \mathbb{Z}\}$

let $u = (2,3)$ ^ $c = \frac{1}{3}$

then $cu = \frac{1}{3}(2,3) = (\frac{2}{3}, 1) \notin \mathbb{Z}^2$

$\Rightarrow cu \notin \mathbb{Z}^2$ since $\frac{2}{3} \notin \mathbb{Z}$ $\Rightarrow$ closure under scalar multiplication does not hold

ex - prove that the set of all invertible 2 by 2 matricies, with matrix multiplication ^ scalar multiplication is not a vector space let $A = \begin{bmatrix} 1 & 8 \\ 7 & 2 \end{bmatrix}$ ^ $B = \begin{bmatrix} 1 & -8 \\ -7 & -2 \end{bmatrix}$

$\det A = 1(2) - 8(7) = 2 - 56 = -54 \neq 0$

$\det B = 1(-2) - (-8)(7) = -2 + 56 = 54 \neq 0$ both invertible

$A + B = \begin{bmatrix} 2 & 0 \\ 14 & 0 \end{bmatrix}$

$\det(A+B) = 2(0) - 0(14) = 0 - 0 = 0$ - not invertible

ex - determine if the set $\{(x,-x) \mid x \in \mathbb{R}\}$ with the standard addition ^ scalar multiplication in $\mathbb{R}^2$, is a vector space. if so, prove all vector space properties. if not, identify which vector space properties fail.

yes it is a vector space

let $(x,-x)$ ^ $(y,-y)$ ^ $(z,-z)$ lie in V ^ $c, d \in \mathbb{R}$

① $(x,-x)+(y,-y) = (x+y, -x-y) = (x+y, -(x+y)) \in V$

② $(x,-x)+(y,-y) = (x+y, -x-y) = (y+x, -y-x) = (y,-y)+(x,-x)$

③ $(x,-x)+((y,-y)+(z,-z)) = (x,-x)+(y+z,-y-z) = (x+(y+z), -x+(-y-z)) = ((x+y)+z, (-x-y)-z) =$
$(x+y,-x-y)+(z,-z) = ((x,-x)+(y,-y))+(z,-z)$

④ $0 = (0,0) \in V$ ^ $(x,-x)+(0,0) = (x+0, -x+0) = (x,-x)$

⑤ the inverse of $(x,-x)$ is $(-x,x)$ ⑤ $(x,-x)+(-x,x) = (x-x, -x+x) = (0,0)$

⑥ $c(x,-x) = (cx, c(-x)) = (cx,-cx) \in V$

⑦ $c((x,-x)+(y,-y)) = c(x+y, -(x+y)) = (c(x+y), -c(x+y)) = (cx+cy, -cx-cy) = (cx,-cx)+(cy,-cy) =$
$c(x,-x)+c(y,-y)$

⑧ $(c+d)(x,-x) = ((c+d)x, -(c+d)x) = (cx+dx, -cx-dx) = (cx,-cx)+(dx,-dx) = c(x,-x)+d(x,-x)$

⑨ $(cd)(x,-x) = ((cd)x, -(cd)x) = (c(dx), -c(dx)) = (c(dx), c(-dx)) = c(dx,-dx) = c(dx, d(-x)) =$
$c(d(x,-x))$

⑩ $1 \cdot (x,-x) = (1 \cdot x, 1 \cdot (-x)) = (x,-x)$

ex - consider $\mathbb{R}^2$ with the following addition ^ scalar multiplication:

$(x_1,y_1)+(x_2,y_2) = (x_1y_1, x_2y_2)$ ^ $c(x_1,y_1) = (cx_1, cy_1)$

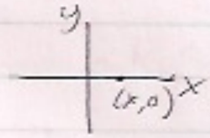
determine if $\mathbb{R}^2$, with the above addition ^ scalar multiplication, is a vector space. if so, prove all 10 vector space properties. if not, identify which ones fail.

its not a vector space b/c the commutativity ^ distributivity properties fail

Subspaces

- a subset W of vector space V is a subspace of V if W is nonempty and a vector space itself with the same operations as V

- for example, the set $W = \{(x,0) \mid x \in \mathbb{R}\}$ is a subspace of $V = \mathbb{R}^2$



- subspace properties:

① $\bar{0} \in W$

② $\bar{u} + \bar{v} \in W$ for all $\bar{u}, \bar{v} \in W$ closure under addition

③ $c\bar{u} \in W$ for all $c \in \mathbb{R}$, $\bar{u} \in W$ closure under scalar multiplication

ex - $W = \{(x,0) \mid x \in \mathbb{R}\}$ $V = \mathbb{R}^2$

① let $x = 0 \Leftrightarrow$ then $(x,0) = (0,0)$ and $0 \in \mathbb{R}$

so $\bar{0} = (0,0) \in W$

② let $\bar{u} = (x,0) \wedge \bar{v} = (y,0)$ where $x, y \in \mathbb{R}$

then $\bar{u} + \bar{v} = (x,0) + (y,0) = (x+y, 0) \in W$

③ let $c \in \mathbb{R} \wedge \bar{u} = (x,0) \in W$

then $c\bar{u} = c(x,0) = (cx, c0) = (cx, 0) \in W$ $cx \in \mathbb{R}$

so $\boxed{W \text{ is a subspace of } \mathbb{R}^2}$

Trivial^ Nontrivial Subspace

- trivial subspaces of V $\begin{cases} \{\bar{0}\} & \text{zero subspace} \\ V \text{ itself} \end{cases}$

ex - $W = \{(x,0) \mid x \in \mathbb{R}\}$ $V = \mathbb{R}^2$

$(1,0) \neq (0,0)$ W is not the zero subspace $\boxed{\overset{\text{non}}{\text{not}} \text{ trivial subspace}}$

ex - $(2,3) \in \mathbb{R}^2$ but $(2,3) \in W$ W is a $\boxed{\text{nontrivial subspace}}$ of $\mathbb{R}^2$

ex - the set W of all matricies of the form $\begin{bmatrix} a & 0 \\ b & c \end{bmatrix}$ is a subspace of $M_{2,2}$.

① let $a = b = c = 0$, then the zero matrix $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ lies in W

② let $\bar{u} = \begin{bmatrix} a & 0 \\ b & c \end{bmatrix}$ $\bar{v} = \begin{bmatrix} d & 0 \\ e & f \end{bmatrix}$ be in W, show $\bar{u} + \bar{v} \in W$

$\bar{u} + \bar{v} = \begin{bmatrix} a & 0 \\ b & c \end{bmatrix} + \begin{bmatrix} d & 0 \\ e & f \end{bmatrix} = \begin{bmatrix} a+d & 0 \\ b+e & c+f \end{bmatrix} \in W$

③ let $\bar{u} = \begin{bmatrix} a & 0 \\ b & c \end{bmatrix} \in W \wedge k = \mathbb{R}$

then $k\bar{u} = k\begin{bmatrix} a & 0 \\ b & c \end{bmatrix} = \begin{bmatrix} ka & 0 \\ kb & kc \end{bmatrix} \in W$ W is a $\boxed{\text{nontrivial subspace}}$ of $M_{2,2}$

ex - show that the set $W = \{(x, 2x) \mid x \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^2$ using subspace properties. Show that W is a nontrivial subspace.

① let $x = 0 \Leftrightarrow (0, 2(0)) = (0,0)$ and $0 \in \mathbb{R}$

so $\bar{0} = (0,0) \in W$

② let $\bar{u} = (x, 2x)$ $\bar{v} = (y, 2y)$ where $x, y \in \mathbb{R}$

then $\bar{u} + \bar{v} = (x, 2x) + (y, 2y) = (x+y, 2x+2y) = (x+y, 2(x+y)) \in W$

③ let $c \in \mathbb{R}$ $\bar{u} = (x, 2x) \in W$

then $c\bar{u} = c(x, 2x) = (cx, 2(cx)) \in W$ W is a $\boxed{\text{nontrivial subspace}}$ of $\mathbb{R}^2$

show that ① W contains more than just zero vector, ^ ② there is some element in $\mathbb{R}^2$ that is not in W

① $x=1 \iff (1, 2(1)) = (1,2)$ so $W \neq \{(0,0)\}$

② $(5,5) \in \mathbb{R}^2$ but $x=5 \iff (5, 2(5)) = (5,10) \notin W$

the set W is a subspace of $\mathbb{R}^2$ ^ nontrivial

ex - show that the set $W = \{k \cdot \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} \mid k \in \mathbb{R}\}$ is a subspace of $M_{2,2}$. Show that W is a nontrivial subspace

subspace properties:

① let $k=0$ ^ $0 \in \mathbb{R}$

$0\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in W$

② let $\bar{u} = k_1\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$ ^ $\bar{v} = k_2\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} \in W$

then $\bar{u}+\bar{v} = k_1\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} + k_2\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} = (k_1+k_2)\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} \in W$

③ let $c \in \mathbb{R}$ ^ $\bar{u} = k\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} \in W$

then $c\bar{u} = c(k\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}) = (ck)\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} \in W$

nontrivial properties:

① let $k=1 \iff 1\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$ so $W \neq \{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\}$

② $\begin{bmatrix} 5 & 5 \\ 5 & 15 \end{bmatrix} \in \mathbb{R}^2$ but $k=5 \Rightarrow 5\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 5 & -5 \\ 10 & 15 \end{bmatrix} \notin W$ so $W \neq M_{2,2}$

the set W is a subspace of $M_{2,2}$ ^ nontrivial

bsets that ar(e) not Subspaces - let $W = \{(x, x^2) \mid x \in \mathbb{R}\}$ then $W \subseteq \mathbb{R}^2$ ← is a subset of

① let $x=0$ ^ $0 \in \mathbb{R}$

then $(x, x^2) = (0, 0^2) = (0,0)$ so $\vec{0} \in W$

② let $\bar{u} = (x, x^2)$ $\bar{v} = (y, y^2)$

then $\bar{u}+\bar{v} = (x,x^2)+(y,y^2) = (x+y, x^2+y^2) \in W$?

usually $x^2+y^2 \neq (x+y)^2$ so $\bar{u}+\bar{v} \notin W$, closure under addition fails

③ let $\bar{u} = (x, x^2) \in W$ ^ $c \in \mathbb{R}$

then $c\bar{u} = c(x,x^2) = (cx, cx^2)$, but $(cx)^2 \overset{\text{usually}}{\neq} c^2x^2$ closure under scalar multiplication fails

ex - let W be the set of all 2x2 matricies that are not invertible. show that W is not a subspace of $M_{2,2}$

① $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in W$

② let $A = \begin{bmatrix} 2 & 6 \\ 1 & 3 \end{bmatrix}$ $B = \begin{bmatrix} -1 & -6 \\ -1 & -6 \end{bmatrix}$

$\det A = 2(3) - 6(1) = 6-6 = 0$

$\det B = -1(-6) - (-6)(-1) = 6-6 = 0$ both A ^ B are not invertible

$A+B = \begin{bmatrix} 2 & 6 \\ 1 & 3 \end{bmatrix} + \begin{bmatrix} -1 & -6 \\ -1 & -6 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix}$ $\det(A+B) = \begin{vmatrix} 1 & 0 \\ 0 & -3 \end{vmatrix} = 1(-3) - 0(0) = -3 \neq 0$ A+B ~~not~~ is invertible, A+B $\notin W$ fails ex - show that the set $W=\{(x,y) \mid x,y \in \mathbb{R}, y \geq 0\}$, the upper half of a plane, is not a subspace of $\mathbb{R}^2$

① let $x=y=0 \quad 0 \in \mathbb{R}$

then $(x,y)=(0,0) \in W$

② let $\bar{u}=(x,y) \quad \bar{v}=(x_2,y_2)$

then $\bar{u}+\bar{v}=(x,y)+(x_2,y_2)=(x+x_2, y+y_2) \in W$

③ let $x=y=1$ ^ $c=-1$

then $c\bar{u}=-1(1,1)=(-1,-1) \notin W$ b/c upper half of plane

ex - show that the set $W=\{(x,y) \mid x,y \in \mathbb{R} \wedge x^2+y^2=1\}$, the circle of radius 1 in the plane, is not a subspace of $\mathbb{R}^2$

① $x \wedge y \neq 0$ b/c $x^2+y^2=1$ fail

ex - show that the set W of all n by n matricies with determinant zero is not a subspace of $M_{n,n}$

① $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in W$

② $A=\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \quad B=\begin{bmatrix} -1 & 0 \\ 6 & 0 \end{bmatrix}$

$\det A = 1(2)-1(2)=0 \qquad \det B = -1(0)-0(6)=0$

$A+B=\begin{bmatrix} 0 & 1 \\ 8 & 2 \end{bmatrix} \quad \det(A+B)=\begin{vmatrix} 0 & 1 \\ 8 & 2 \end{vmatrix}=0(2)-1(8)=-8 \neq 0 \quad A+B \notin W$ fails

Span ^ Linear Independence

Span - say $S=\{\bar{v}_1, \bar{v}_2, \dots \bar{v}_k\}$ is a subset of V ^ S spans V → to solve for, multiply each value in S by $c_1, c_2, c_3 \dots c_n$ where $\in V$ ^ sum them to see if result $\in V$ ^ no contradictions

ex - show that $S=\{(1,0),(0,1)\}$ spans $\mathbb{R}^2$

let $(u_1,u_2) \in \mathbb{R}^2$ ^ $u_1,u_2 \in \mathbb{R}$

$(u_1,u_2)=u_1(1,0)+u_2(0,1)=(u_1,0)+(0,u_2)=(u_1,u_2)$ so S spans $\mathbb{R}^2$

ex - $S=\left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$ spans $M_{2,2}$

let $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{2,2}$ ^ $a,b,c,d \in \mathbb{R}$

$a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}+b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}+c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}+d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}=\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}+\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}+\begin{bmatrix} 0 & 0 \\ c & 0 \end{bmatrix}+\begin{bmatrix} 0 & 0 \\ 0 & d \end{bmatrix}=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ so S spans $M_{2,2}$

ex - $S=\{(1,0),(3,0)\}$ does not span $\mathbb{R}^2$ $\qquad c_1 \wedge c_2 =$ scalars

consider $(2,2) \Rightarrow c_1(1,0)+c_2(3,0)=(2,2) \Rightarrow (c_1,0)+(3c_2,0)=(2,2) \Rightarrow (c_1+3c_2,0)=(2,2) \Rightarrow$

$c_1+3c_2=2$ ^ $0=2$ contradiction S does not span $\mathbb{R}^2$

ex - $S=\left\{\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 2 & -3 \\ 0 & -5 \end{bmatrix}\right\}$ does not span $M_{2,2}$

consider $\begin{bmatrix} 1 & 7 \\ 4 & 1 \end{bmatrix} \Rightarrow c_1\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}+c_2\begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}+c_3\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}+c_4\begin{bmatrix} 2 & -3 \\ 0 & -5 \end{bmatrix}=\begin{bmatrix} 1 & 7 \\ 4 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} c_1 & c_1 \\ 0 & 0 \end{bmatrix}+\begin{bmatrix} c_2 & -c_2 \\ 2c_2 & 0 \end{bmatrix}+\begin{bmatrix} 0 & 0 \\ c_3 & c_3 \end{bmatrix}+\begin{bmatrix} 2c_4 & -3c_4 \\ 0 & -5c_4 \end{bmatrix}$

$\begin{bmatrix} 1 & 7 \\ 4 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} c_1+c_2+2c_4 & c_1-c_2-3c_4 \\ 2c_2+c_3 & c_3-5c_4 \end{bmatrix}=\begin{bmatrix} 1 & 7 \\ 4 & 1 \end{bmatrix} \Rightarrow$

$\Rightarrow$ system of equations $\Rightarrow$ augmented matrix $\Rightarrow$ row echelon form $\Rightarrow$ look for contradictions

$c_1 + c_2 + 2c_4 = 1 \wedge c_1 - c_2 - 3c_4 = 2 \wedge 2c_2 + c_3 = 4 \wedge c_3 - 5c_4 = 1$

$\begin{bmatrix} 1 & 1 & 0 & 2 & 1 \\ 1 & -1 & 0 & -3 & 2 \\ 0 & 2 & 1 & 0 & 4 \\ 0 & 0 & 1 & -5 & 1 \end{bmatrix} R_1 - R_2 \to R_2 \Leftrightarrow \begin{bmatrix} 1 & 1 & 0 & 2 & 1 \\ 0 & 2 & 0 & 5 & -1 \\ 0 & 2 & 1 & 0 & 4 \\ 0 & 0 & 1 & -5 & 1 \end{bmatrix} R_2 - R_3 \to R_3 \Leftrightarrow \begin{bmatrix} 1 & 1 & 0 & 2 & 1 \\ 0 & 2 & 0 & 5 & -1 \\ 0 & 0 & -1 & 5 & -5 \\ 0 & 0 & 1 & -5 & 1 \end{bmatrix} R_3 + R_4 \to R_4 \Leftrightarrow$

$\begin{bmatrix} 1 & 1 & 0 & 2 & 1 \\ 0 & 2 & 0 & 5 & -1 \\ 0 & 0 & -1 & 5 & -5 \\ 0 & 0 & 0 & 0 & -4 \end{bmatrix} \Leftarrow$ contradiction b/c $0 \neq -4$ $\begin{bmatrix} 1 & 2 \\ 4 & 1 \end{bmatrix}$ does not span $M_{2,2}$

Span of a Subset of a Vector Space

- $S = \{\bar{v}_1, \bar{v}_2 \ldots \bar{v}_k\}$ span of S == span(S)

ex - $S = \{(1,0), (3,0)\}$ $V = \mathbb{R}^2$ span(S)?

$\text{span}(S) = \{c_1(1,0) + c_2(3,0) \mid c_1, c_2 \in \mathbb{R}\} = \{k(1,0) \mid k \in \mathbb{R}\}$?

if $c_1(1,0) + c_2(3,0) \in \text{span}(S)$, then $= (c_1, 0) + (3c_2, 0) = (c_1 + 3c_2, 0) = (c_1 + 3c_2)(1,0)$

let $k = c_1 + 3c_2$?

$\text{span}(S) \subseteq \{k(1,0) \mid k \in \mathbb{R}\} \Leftrightarrow k(1,0) = k(1,0) + 0(3,0)$ let $c_1 = k \wedge c_2 = 0$

$k(1,0) = c_1(1,0) + c_2(3,0) \in \text{span}(S) \Leftrightarrow \{k(1,0) \mid k \in \mathbb{R}\} \subseteq \text{span}(S) \Leftrightarrow \text{span}(S) = \{k(1,0) \mid k \in \mathbb{R}\}$

Linear Independence

- when $S = \{\bar{v}_1, \ldots \bar{v}_k\}$ is a subset of V $\wedge$ $c_1\bar{v}_1 + \ldots + c_k\bar{v}_k = \bar{0}$ has only the trivial solution $c_1 = c_2 = \ldots c_k =$

so S linearly independent, otherwise is linearly dependent → to solve, multiply each value by $c_1, c_2, \ldots c_n \in V$ and add them, & st if $= \bar{0}$ = linearly independent or $\neq \bar{0}$ = nontrivial solution = linearly dependent

ex - suppose $c_1(1,0) + c_2(0,1) = \bar{0}$, show that $c_1 = c_2 = 0$

$c_1(1,0) + c_2(0,1) = (0,0) \Leftrightarrow (c_1, 0) + (0, c_2) = (0,0) \Leftrightarrow (c_1, c_2) = (0,0) \Leftrightarrow c_1 = c_2 = 0$

so S is $\boxed{\text{linearly independent}}$

ex - $S = \{(1,0), (3,0)\}$ subset of $\mathbb{R}^2$, show that S is linearly dependent let $c_2 = t$

$c_1(1,0) + c_2(3,0) = (0,0) \Leftrightarrow (c_1, 0) + (3c_2, 0) = (0,0) \Leftrightarrow (c_1 + 3c_2, 0) = (0,0) \Leftrightarrow 0 = 0 \wedge c_1 + 3c_2 = 0$

$c_1 + 3t = 0 \Leftrightarrow c_1 = -3t \wedge c_2 = t$ let $t = 1 \Leftrightarrow c_1 = -3 \wedge c_2 = 1$ non-trivial solution so $\boxed{\text{linearly dependent}}$

ex - $S = \{(1,-1,0), (2,1,1), (3,3,2)\}$ subset of $\mathbb{R}^3$, is S linearly in/dependent?

$c_1(1,-1,0) + c_2(2,1,1) + c_3(3,3,2) = \bar{0} \Leftrightarrow c_1 + 2c_2 + 3c_3 = 0 \wedge -c_1 + c_2 + 3c_3 = 0 \wedge c_2 + 2c_3 = 0$

$\begin{bmatrix} 1 & 2 & 3 & 0 \\ -1 & 1 & 3 & 0 \\ 0 & 1 & 2 & 0 \end{bmatrix} R_1 + R_2 \to R_2 \Leftrightarrow \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 3 & 6 & 0 \\ 0 & 1 & 2 & 0 \end{bmatrix} \frac{1}{3}R_2 \to R_2 \Leftrightarrow \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \end{bmatrix} R_2 - R_3 \to R_3 \Leftrightarrow \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ let $c_3 = t$

$c_2 + 2t = 0 \Leftrightarrow c_2 = -2t$ system of equations ⇔ augmented matrix ⇔ row echelon form ⇔ look for contradictions/nontrivial answers

$c_1 + 2c_2 + 3c_3 = 0 \Leftrightarrow c_1 + 2(-2t) + 3t = 0 \Leftrightarrow c_1 - t = 0 \Leftrightarrow c_1 = t$ let $t = 1$

$c_1 = 1, c_2 = -2, c_3 = 1$ non trivial solution, S is $\boxed{\text{linearly dependent}}$

ex - $S = \{1 + 2x + 3x^2, x + 2x^2, -2 + x^2\}$ subset of P_2, is S linearly in/dependent?

$c_1(1 + 2x + 3x^2) + c_2(x + 2x^2) + c_3(-2 + x^2) = \bar{0} \Leftrightarrow (c_1 + 2c_1x + 3c_1x^2) + (c_2x + 2c_2x^2) + (-2c_3 + c_3x^2) = 0 + 0x + 0x^2$

polynomials factor out x & augmented matrix etc.

$\Leftrightarrow (c_1 - 2c_3) + (2c_1 + c_2)x + (3c_1 + 2c_2 + c_3)x^2 = 0 + 0x + 0x^2 \Leftrightarrow c_1 - 2c_3 = 0 \wedge 2c_1 + c_2 = 0 \wedge 3c_1 + 2c_2 + c_3 = 0$

$\begin{bmatrix} 1 & 0 & -2 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix} -2R_1 + R_2 \to R_2 \Leftrightarrow \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 4 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix} -3R_1 + R_3 \to R_3 \Leftrightarrow \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 2 & 7 & 0 \end{bmatrix} -2R_2 + R_3 \to R_3 \Leftrightarrow$

$\begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix} -R_3 \to R_3 \Leftrightarrow \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ $c_3 = 0$

$c_2 + 4(0) = 0 \Leftrightarrow c_2 = 0$

$c_1 + 0(0) - 2(0) = 0 \Leftrightarrow c_1 = 0$ $c_1 = c_2 = c_3 = 0$ S is $\boxed{\text{linearly independent}}$

ex - show that $S = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$ spans $M_{2,3}$ where $M_{2,3}$ is the set of all 2×3 matrices

let $\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \in M_{2,3}$ where $a, b, c, d, e, f \in \mathbb{R}$

$a \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \ldots + f \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$ $\boxed{S \text{ spans } M_{2,3}}$

ex - show that $S = \{1, x, 1+x\}$ does not span P_2, where P_2 is the set of all polynomials of degree ≤ 2

consider $p(x) = x^2 \Rightarrow c_1(1) + c_2(x) + c_3(1+x) = x^2 \Leftrightarrow c_1 + c_2 x + c_3 + c_3 x = x^2 \Leftrightarrow (c_1 + c_3) + (c_2 + c_3)x + (0)x^2$

$c_1 + c_3 = 0 \wedge c_2 + c_3 = 0 \wedge 1 = 0$ X contradiction $\boxed{S \text{ does not span } P_2}$

ex - let $S = \{1, x - x^2, x + x^2\}$ a subset of P_2, show that S is linearly independent

$c_1(1) + c_2(x - x^2) + c_3(x + x^2) = 0 \Leftrightarrow c_1 + c_2 x - c_2 x^2 + c_3 x + c_3 x^2 = 0 \Leftrightarrow (c_1) + (c_2 + c_3)x + (-c_2 + c_3)x^2 = 0$

$c_1 = 0 \wedge c_2 + c_3 = 0 \wedge -c_2 + c_3 = 0$

$c_3 = c_2 \Leftrightarrow c_2 + (c_2) = 0 \Leftrightarrow 2c_2 = 0 \Leftrightarrow c_2 = 0$ $\quad c_1 = c_2 = c_3 = 0$ $\boxed{\text{linearly independent}}$

ex - let $S = \left\{ \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -5 & 0 \\ 2 & -1 \end{bmatrix} \right\}$ a subset of $M_{2,2}$, show that S is linearly dependent

$c_1 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \ldots = \vec{0} \Leftrightarrow \begin{bmatrix} 0 & 0 \\ c_1 & c_1 \end{bmatrix} + \begin{bmatrix} c_2 & 0 \\ 0 & c_2 \end{bmatrix} + \begin{bmatrix} -c_3 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} -5c_4 & 0 \\ 2c_4 & -c_4 \end{bmatrix} = \vec{0} \Leftrightarrow \begin{bmatrix} c_2 - c_3 - 5c_4 & 0 \\ c_1 + 2c_4 & c_1 + c_2 - c_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Leftrightarrow$

$c_2 - c_3 - 5c_4 = 0 \wedge c_1 + 2c_4 = 0 \wedge c_1 + c_2 - c_4 = 0 \wedge 0 = 0$

$\begin{bmatrix} 0 & 1 & -1 & -5 \\ 1 & 0 & 0 & 2 \\ 1 & 1 & 0 & -1 \end{bmatrix} R_1 \leftrightarrow R_2 \Leftrightarrow \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & -1 & -5 \\ 1 & 1 & 0 & -1 \end{bmatrix} R_1 - R_3 \to R_3 \Leftrightarrow \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & -1 & -5 \\ 0 & -1 & 0 & 3 \end{bmatrix} R_2 + R_3 \to R_3 \Leftrightarrow \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & -1 & -5 \\ 0 & 0 & -1 & -2 \end{bmatrix} -R_3 \to R_3 \Leftrightarrow$
$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & -1 & -5 \\ 0 & 0 & 1 & 2 \end{bmatrix}$

$c_3 + 2c_4 = 0$ $\quad$ let $c_4 = t \Leftrightarrow c_3 + 2t = 0 \Leftrightarrow c_3 = -2t$

$c_2 - (-2t) - 5(t) = 0 \Leftrightarrow c_2 - 3t = 0 \Leftrightarrow c_2 = 3t$

$c_1 + 2(t) = 0 \Leftrightarrow c_1 = -2t$

let $t = 1 \Leftrightarrow$ $c_1 = -2, c_2 = 3, c_3 = -2, c_4 = 1$ non-trivial solution so $\boxed{\text{linearly dependent}}$

ex - let $S = \left\{ \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \right\}$ a subset of $M_{2,2}$, is S linearly in/dependent?

$c_1 \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \ldots = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Leftrightarrow \begin{bmatrix} c_1 & 0 \\ c_1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & c_2 \\ 0 & c_2 \end{bmatrix} + \begin{bmatrix} c_3 & c_3 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ c_4 & c_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Leftrightarrow \begin{bmatrix} c_1 + c_3 & c_2 + c_3 \\ c_1 + c_4 & c_2 + c_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Leftrightarrow$

$c_1 + c_3 = 0 \wedge c_2 + c_3 = 0 \wedge c_1 + c_4 = 0 \wedge c_2 + c_4 = 0$

$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} R_1 - R_3 \to R_3 \Leftrightarrow \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 \end{bmatrix} R_2 - R_4 \to R_4 \Leftrightarrow \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix} R_3 - R_4 \to R_4 \Leftrightarrow \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

let $c_4 = t \Leftrightarrow c_3 - (t) = 0 \Leftrightarrow c_3 = t$

$c_2 + (t) = 0 \Leftrightarrow c_2 = -t$

$c_1 + (t) = 0 \Leftrightarrow c_1 = -t$

let $t = 1$ $\quad c_1 = -1, c_2 = -1, c_3 = 1, c_4 = 1$ non trivial solution, $\boxed{\text{linearly dependent}}$

Basis ^ Dimension

↑ to solve, test for linear independence; then test for span by instead of setting $=\vec{0}$ set $= c_1, c_2 \ldots c_n$ matter ^ solve to find nontrivial solution w/ no contradictions

Basis

ex - show that $S = \{(1,2), (-2,3)\}$ is a non-standard basis for $\mathbb{R}^2$

$c_1(1,2) + c_2(-2,3) = (0,0) \Leftrightarrow (c_1, 2c_1) + (-2c_2, 3c_2) = (0,0) \Leftrightarrow (c_1 - 2c_2, 2c_1 + 3c_2) = (0,0) \Leftrightarrow$

$c_1 - 2c_2 = 0$ ^ $2c_1 + 3c_2 = 0 \Leftrightarrow \begin{bmatrix} 1 & -2 & 0 \\ 2 & 3 & 0 \end{bmatrix}$ $-2R_1 + R_2 \to R_2 \Leftrightarrow \begin{bmatrix} 1 & -2 & 0 \\ 0 & 7 & 0 \end{bmatrix}$ $\frac{1}{7}R_2 \to R_2 \Leftrightarrow \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

$\underline{c_2 = 0} \Leftrightarrow c_1 - 2(0) = 0 \Leftrightarrow \underline{c_1 = 0}$

instead of setting $= (0,0)$, set $= (u_1, u_2) \Leftrightarrow \begin{bmatrix} 1 & -2 & u_1 \\ 2 & 3 & u_2 \end{bmatrix}$ $-2R_1 + R_2 \to R_2 \Leftrightarrow \begin{bmatrix} 1 & -2 & u_1 \\ 0 & 7 & -2u_1 + u_2 \end{bmatrix}$ $\frac{1}{7}R_2 \to R_2$

$\begin{bmatrix} 1 & -2 & u_1 \\ 0 & 1 & \frac{-2u_1 + u_2}{7} \end{bmatrix}$ $R_1 + 2R_2 \to R_1 \Leftrightarrow \begin{bmatrix} 1 & 0 & u_1 + \frac{2(-2u_1 + u_2)}{7} \\ 0 & 1 & \frac{-2u_1 + u_2}{7} \end{bmatrix} = \begin{bmatrix} 1 & 0 & \frac{3u_1 + 2u_2}{7} \\ 0 & 1 & \frac{-2u_1 + u_2}{7} \end{bmatrix}$

so $c_1 = \frac{3u_1 + 2u_2}{7}$ $c_2 = \frac{-2u_1 + u_2}{7}$ S spans $\mathbb{R}^2$

ex - show that $S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ is a basis for $M_{2,2}$

$c_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \ldots = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Leftrightarrow \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ $c_1 = c_2 = c_3 = c_4 = 0$ S is linearly independent ^ basis for $M_{2,2}$

- P_n $\{1, x, x^2 \ldots, x^n\}$ standard basis; $\{\vec{v}_1, \ldots \vec{v}_k\}$ basis for V

Dimension

- $\{1, x, x^2\}$ is basis for P_2; $\{1 + 2x + 3x^2, x + 2x^2, -2 + x^2\}$ also a basis for P_2

- if V has a basis consisting of n vectors, then the dimension of V is n:

i.e. $\{(1,0), (0,1)\}$ $\mathbb{R} \to \dim(\mathbb{R}^2) = 2$

ex - $W = \{k(4,6) \mid k \in \mathbb{R}\}$

$\{(4,6)\}$ spans W ^ linearly independent

$c(4,6) = (0,0) \Leftrightarrow (4c, 6c) = (0,0) \Leftrightarrow 4c = 0$ ^ $6c = 0 \Leftrightarrow c = 0$

$\{(4,6)\}$ basis for $W \to \boxed{\dim W = 1}$

ex - $W = \left\{ \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 0 \end{bmatrix} \middle| a, b \in \mathbb{R} \right\}$

W is subspace of $M_{3,3}$, find the dimension of W

$\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 0 \end{bmatrix} = a \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Leftrightarrow \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\}$ spans W, forms a basis for W, $\boxed{\dim W = 2}$

ex - show that $S = \left\{ \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \right\}$ is a basis for $M_{2,2}$

show S is linearly independent:

$c_1 \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + \ldots = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Leftrightarrow \begin{bmatrix} c_1 & 0 \\ c_1 & 0 \end{bmatrix} + \begin{bmatrix} c_2 & 0 \\ -c_2 & 0 \end{bmatrix} + \begin{bmatrix} 0 & c_3 \\ 0 & c_3 \end{bmatrix} + \begin{bmatrix} 0 & c_4 \\ 0 & -c_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Leftrightarrow \begin{bmatrix} c_1 + c_2 & c_3 + c_4 \\ c_1 - c_2 & c_3 - c_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Leftrightarrow$

$c_1 + c_2 = 0$ ^ $c_3 + c_4 = 0$ ^ $c_1 - c_2 = 0$ ^ $c_3 - c_4 = 0 \Leftrightarrow$

$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$ $R_2 \leftrightarrow R_3 \Leftrightarrow \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$ $R_1 - R_2 \to R_2 \Leftrightarrow \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$ $R_3 - R_4 \to R_4 \Leftrightarrow \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix} \Leftrightarrow$

$2c_4 = 0 \Leftrightarrow \underline{c_4 = 0}$

$c_3 + (0) = 0 \Leftrightarrow \underline{c_3 = 0}$ $c_1 = c_2 = c_3 = c_4 = 0$ linearly independent

$2c_2 = 0 \Leftrightarrow \underline{c_2 = 0}$

$c_1 + (0) = 0 \Leftrightarrow \underline{c_1 = 0}$

$\begin{bmatrix} 3/7 & 2/7 \\ -2/7 & 1/7 \end{bmatrix}\begin{bmatrix} 4 \\ 15 \end{bmatrix} = \left[\frac{12}{7} + \frac{30}{7} - \frac{4?}{7}\right.$

$a - \frac{1}{3}(a-c) = a - \frac{1}{3}a + \frac{1}{3}c$

show that S spans $M_{2,2}$ $\Leftrightarrow$ instead $= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \to = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$c_1 + c_2 = a \wedge c_1 - c_2 = c \wedge c_3 + c_4 = b \wedge c_3 - c_4 = d \Leftrightarrow$

$\begin{bmatrix} 1 & 1 & 0 & 0 & a \\ 1 & -1 & 0 & 0 & c \\ 0 & 0 & 1 & 1 & b \\ 0 & 0 & 1 & -1 & d \end{bmatrix} R_1 - R_2 \to R_2 \Leftrightarrow \begin{bmatrix} 1 & 1 & 0 & 0 & a \\ 0 & 2 & 0 & 0 & a-c \\ 0 & 0 & 1 & 1 & b \\ 0 & 0 & 1 & -1 & d \end{bmatrix} R_3 - R_4 \to R_4 \Leftrightarrow \begin{bmatrix} 1 & 1 & 0 & 0 & a \\ 0 & 2 & 0 & 0 & a-c \\ 0 & 0 & 1 & 1 & b \\ 0 & 0 & 0 & 2 & b-d \end{bmatrix} R_1 - \frac{1}{2}R_2 \to R_1 \Leftrightarrow$

$\begin{bmatrix} 1 & 0 & 0 & 0 & \frac{a+c}{2} \\ 0 & 2 & 0 & 0 & a-c \\ 0 & 0 & 1 & 1 & b \\ 0 & 0 & 0 & 2 & b-d \end{bmatrix} \frac{1}{2}R_2 \to R_2 \Leftrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & \frac{a+c}{2} \\ 0 & 1 & 0 & 0 & \frac{a-c}{2} \\ 0 & 0 & 1 & 1 & b \\ 0 & 0 & 0 & 2 & b-d \end{bmatrix} \frac{1}{2}R_4 \to R_4 \Leftrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & \frac{a+c}{2} \\ 0 & 1 & 0 & 0 & \frac{a-c}{2} \\ 0 & 0 & 1 & 1 & b \\ 0 & 0 & 0 & 1 & \frac{b-d}{2} \end{bmatrix} R_3 - R_4 \to R_3 \Leftrightarrow$

$\begin{bmatrix} 1 & 0 & 0 & 0 & \frac{a+c}{2} \\ 0 & 1 & 0 & 0 & \frac{a-c}{2} \\ 0 & 0 & 1 & 0 & \frac{b+d}{2} \\ 0 & 0 & 0 & 1 & \frac{b-d}{2} \end{bmatrix}$ there are non-trivial solutions so $\boxed{S \text{ spans } M_{2,2} \wedge \text{ is a basis for } M_{2,2}}$

ex - let $W = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \middle| a, b, c \in \mathbb{R} \right\}$, W is a subspace of $M_{2,2}$, find the dimension of W

$\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

every matrix in W can be written as a linear combination of $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ thus $\{\,\}$ spans W

show the set is linearly independent:

$\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ $a = b = c = 0$ is linearly independent so $\{\ldots\}$ forms a basis for W

$\boxed{\dim W = 3}$

Coordinates - if B is a basis for V, $B = \{\bar{v}_1 \ldots \bar{v}_n\}$, $\bar{x} = c_1\bar{v}_1 + \ldots + c_n\bar{v}_n$, scalars $c_1 \ldots c_n$

coordinates of $\bar{x}$ relative to the basis B $\quad \bar{y} \in V$

$\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$ coordinate matrix of $\bar{x}$ relative to B $\quad [\bar{x}]_B \to$ to solve, [illegible] use scalars $c_1 \ldots c_n \wedge$ multiply by each vector in B, add, [illegible] and set $= \bar{x}$, then, use system of equations to solve, then use result like scalars $\wedge$ multiply by B', add, and set $= \bar{y}$

Change of Basis - $B = \{(1,0), (0,1)\}$ basis for $\mathbb{R}^2$

$B' = \{(1,2), (-2,3)\}$ $\quad \bar{x} = (4, 15)$

$\bar{x} = 4(1,0) + 15(0,1) \Leftrightarrow [\bar{x}]_B = \begin{bmatrix} 4 \\ 15 \end{bmatrix}$

$c_1(1,2) + c_2(-2,3) = (4,15) \Leftrightarrow (c_1, 2c_1) + (-2c_2, 3c_2) = (4,15) \Leftrightarrow (c_1 - 2c_2, 2c_1 + 3c_2) = (4,15) \Leftrightarrow$

$c_1 - 2c_2 = 4 \wedge 2c_1 + 3c_2 = 15$

$c_1 = 2c_2 + 4 \Leftrightarrow 2(2c_2 + 4) + 3c_2 = 15 \Leftrightarrow 4c_2 + 8 + 3c_2 = 15 \Leftrightarrow 7c_2 = 7 \Leftrightarrow \underline{c_2 = 1}$

$c_1 - 2(1) = 4 \Leftrightarrow \underline{c_1 = 6}$

$(4,15) = 6(1,2) + 1(-2,3)$ the coordinates for $\bar{x}$ relative to B' are $6 \wedge 1 \Leftrightarrow \boxed{[\bar{x}]_{B'} = \begin{bmatrix} 6 \\ 1 \end{bmatrix}}$

- V n-dim vector-space, $B \wedge B'$ bases for V, P transition matrix from B to B' $\Leftrightarrow P[\bar{x}]_B = [\bar{x}]_{B'}$

$[B' \vdots B]$ perform Gauss-Jordan elimination $[I_n \vdots P]$

Finding Transition Matrices ex - find the transition matrix from B to B' where $B = \{(1,0), (0,1)\} \wedge B' = \{(1,2), (-2,3)\}$

form $[B' \vdots B] \Leftrightarrow \left[\begin{array}{cc|cc} 1 & -2 & 1 & 0 \\ 2 & 3 & 0 & 1 \end{array}\right] -2R_1 + R_2 \to R_2 \Leftrightarrow \left[\begin{array}{cc|cc} 1 & -2 & 1 & 0 \\ 0 & 7 & -2 & 1 \end{array}\right] \frac{1}{7}R_2 \to R_2 \Leftrightarrow \left[\begin{array}{cc|cc} 1 & -2 & 1 & 0 \\ 0 & 1 & -\frac{2}{7} & \frac{1}{7} \end{array}\right] 2R_2 + R_1 \to R_1 \Leftrightarrow$

$\left[\begin{array}{cc|cc} 1 & 0 & 3/7 & 2/7 \\ 0 & 1 & -2/7 & 1/7 \end{array}\right] \quad P = \begin{bmatrix} 3/7 & 2/7 \\ -2/7 & 1/7 \end{bmatrix} \quad \begin{bmatrix} 3/7 & 2/7 \\ -2/7 & 1/7 \end{bmatrix}\begin{bmatrix} 4 \\ 15 \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \end{bmatrix}$

$\hookrightarrow = \begin{bmatrix} \frac{12}{7} + \frac{30}{7} \\ -\frac{8}{7} + \frac{15}{7} \end{bmatrix} = \begin{bmatrix} \frac{42}{7} \\ \frac{7}{7} \end{bmatrix}$

form $[B' \vdots B]$, after elimination, becomes $[I \vdots P]$

ex – find the transition matrix from B to B' $\quad B = \{(1,3), (-2,-2)\}$ $\quad B' = \{(-12,0), (-4,4)\}$

form $[B' \vdots B] \Leftrightarrow \left[\begin{array}{cc|cc} -12 & -4 & 1 & -2 \\ 0 & 4 & 3 & -2 \end{array}\right]$ $-\frac{1}{12}R_1 \to R_1 \Leftrightarrow \left[\begin{array}{cc|cc} 1 & 1/3 & -1/12 & 1/6 \\ 0 & 4 & 3 & -2 \end{array}\right]$ $\frac{1}{4}R_2 \to R_2 \Leftrightarrow \left[\begin{array}{cc|cc} 1 & 1/3 & -1/12 & 1/6 \\ 0 & 1 & 3/4 & -1/2 \end{array}\right]$ $-\frac{1}{3}R_2 + R_1 \to R_1 \Leftrightarrow$

$\left[\begin{array}{cc|cc} 1 & 0 & -1/3 & 1/3 \\ 0 & 1 & 3/4 & -1/2 \end{array}\right]$ $\quad P = \begin{bmatrix} -1/3 & 1/3 \\ 3/4 & -1/2 \end{bmatrix}$ $\quad [\bar{x}]_B = \begin{bmatrix} -1 \\ 5 \end{bmatrix}$ find $[\bar{x}]_{B'}$

$P[\bar{x}]_B = [\bar{x}]_{B'} = \begin{bmatrix} -1/3 & 1/3 \\ 3/4 & -1/2 \end{bmatrix}\begin{bmatrix} -1 \\ 5 \end{bmatrix} = \begin{bmatrix} 2 \\ -13/4 \end{bmatrix}$

$\bar{x} = -1(1,3) + 5(-2,-2) = (-1,-3) + (-10,-10) = (-11,-13)$ ✓

$2(-12,0) + \left(-\frac{13}{4}\right)(-4,4) = (-24,0) + (13,-13) = (-11,-13)$ ✓

– P = transition matrix from B to B'; P^{-1} = transition matrix from B' to B

$\det P = \begin{vmatrix} -1/3 & 1/3 \\ 3/4 & -1/2 \end{vmatrix} = -\frac{1}{3}\left(-\frac{1}{2}\right) - \frac{1}{3}\left(\frac{3}{4}\right) = \frac{1}{6} - \frac{3}{12} = \frac{2}{12} - \frac{3}{12} = -\frac{1}{12}$

$P^{-1} = \frac{1}{\det P}(P) = \frac{1}{(-1/12)}\begin{bmatrix} -1/3 & 1/3 \\ 3/4 & -1/2 \end{bmatrix} = \begin{bmatrix} 6 & 4 \\ 9 & 4 \end{bmatrix}$

ex – find the coordinate matrix of x relative to the basis B' $\quad \bar{x} = (5,3)$ $\quad B' = \{(2,4), (1,3)\}$

$c_1(2,4) + c_2(1,3) = (5,3) \Leftrightarrow (2c_1, 4c_1) + (c_2, 3c_2) = (5,3) \Leftrightarrow (2c_1 + c_2, 4c_1 + 3c_2) = (5,3) \Leftrightarrow$

$2c_1 + c_2 = 5 \wedge 4c_1 + 3c_2 = 3 \Leftrightarrow$

$c_2 = -2c_1 + 5 \Leftrightarrow 4c_1 + 3(-2c_1 + 5) = 3 \Leftrightarrow 4c_1 - 6c_1 + 15 = 3 \Leftrightarrow -2c_1 = -12 \Leftrightarrow c_1 = 6$

$c_2 = -2(6) + 5 = -12 + 5 = -7$ $\quad$ so $[\bar{x}]_{B'} = \begin{bmatrix} 6 \\ -7 \end{bmatrix}$

ex – find the transition matrix from B to B' where $B = \{(1,0), (0,1)\} \wedge B' = \{(2,4), (1,3)\}$, then verify that $[\bar{x}]_{B'} = P[\bar{x}]_B$ for $x = (5,3)$, using the answer from problem 1

$\left[\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 4 & 3 & 0 & 1 \end{array}\right]$ $-2R_1 + R_2 \to R_2 \Leftrightarrow \left[\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{array}\right]$ $\frac{1}{2}R_1 \to R_1 \Leftrightarrow \left[\begin{array}{cc|cc} 1 & 1/2 & 1/2 & 0 \\ 0 & 1 & -2 & 1 \end{array}\right]$ $R_1 - \frac{1}{2}R_2 \to R_1 \Leftrightarrow$

$\left[\begin{array}{cc|cc} 1 & 0 & 3/2 & -1/2 \\ 0 & 1 & -2 & 1 \end{array}\right]$ $\quad P = \begin{bmatrix} 3/2 & -1/2 \\ -2 & 1 \end{bmatrix}$

verify $[\bar{x}]_{B'} = P[\bar{x}]_B$ for $\bar{x} = (5,3)$

$P[\bar{x}]_B = \begin{bmatrix} 3/2 & -1/2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} \frac{15}{2} - \frac{3}{2} \\ -10 + 3 \end{bmatrix} = \begin{bmatrix} 6 \\ -7 \end{bmatrix} = [\bar{x}]_{B'}$ ✓

ex – find the transition matrix from B to B' where $B = \{(2,-2), (6,3)\} \wedge B' = \{(1,1), (32,31)\}$

verify that $[\bar{x}]_{B'} = P[\bar{x}]_B$ for the vector whose coordinate matrix relative to B is given by $[\bar{x}]_B = [$

$\left[\begin{array}{cc|cc} 1 & 32 & 2 & 6 \\ 1 & 31 & -2 & 3 \end{array}\right]$ $R_1 - R_2 \to R_2 \Leftrightarrow \left[\begin{array}{cc|cc} 1 & 32 & 2 & 6 \\ 0 & 1 & 4 & 3 \end{array}\right]$ $R_1 - 32R_2 \to R_1 \Leftrightarrow \left[\begin{array}{cc|cc} 1 & 0 & -126 & -90 \\ 0 & 1 & 4 & 3 \end{array}\right]$ $\quad P = \begin{bmatrix} -126 & -90 \\ 4 & 3 \end{bmatrix}$

$P[\bar{x}]_B = \begin{bmatrix} -126 & -90 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -162 \\ 5 \end{bmatrix}$

$2(2,-2) + (-1)(6,3) = (4,-4) + (-6,-3) = (-2,-7)$ ✓

$-162(1,1) + 5(32,31) = (-162,-162) + (160,155) = (-2,-7)$ ✓