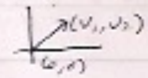
Linear Algebra II

Inner Product Spaces

Norms for R_n

- $\mathbb{R}^2$ is a vector space of two $\overset{\text{real}}{\mathbb{R}}$ numbers (v_1, v_2) where $v_1, v_2 \in \mathbb{R}$ 
- distance formula for 2 points: (u_1, u_2) ^ (v_1, v_2) $d = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2}$
- length of vector: $d = \sqrt{(0 - v_1)^2 + (0 - v_2)^2} = \sqrt{v_1^2 + v_2^2}$
- length of n-vector: $d = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$ $\bar{v} = (v_1, v_2, \ldots v_n)$
- $\|\bar{v}\|$ = norm of $\bar{v}$

ex – $\bar{v} = (3,4)$ $\|\bar{v}\|$?
$\|\bar{v}\| = \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = \boxed{5}$

- when multiply $\bar{v}$ w/ scalar c: if c is positive, then $c\bar{v}$ will point in the same direction as $\bar{v}$; if c is negative, $c\bar{v}$ points in the opposite direction

ex – $c = 2$ ^ $\bar{v} = (3,4)$
$c\bar{v} = 2(3,4) = (6,8)$ ⟹ twice as long

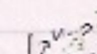
- scalar c ~~to~~ makes $\bar{v}$ $|c|$ times as long as $\bar{v}$

ex – $c = -2$ ^ $\bar{v} = (3,4)$
$c\bar{v} = -2(3,4) = (-6,-8)$ ⟹ opposite direction, twice as long

- if $\bar{v} \in \mathbb{R}^n$ ^ $c \in \mathbb{R}$, then $\|c\bar{v}\| = |c| \cdot \|\bar{v}\|$
- if $\bar{v}$ is nonzero vector in $\mathbb{R}^n$, then $\bar{u} = \underbrace{\frac{1}{\|\bar{v}\|}}_{\text{scaled}} \cdot \bar{v}$ unit vector in the direction of $\bar{v} = \bar{u}$
  ↳ vector w/ length 1

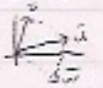
ex – find the unit vector in the direction of $(3,4)$
$\bar{u} = \frac{1}{5} \cdot (3,4) = \left(\frac{3}{5}, \frac{4}{5}\right)$

Distance btwn Vectors in R_n

- $\bar{u}$ ^ $\bar{v}$ two vectors in $\mathbb{R}^2$, then $\bar{u} + \bar{v}$ 

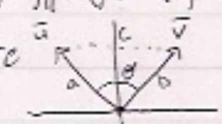
parallelogram

ex – $\bar{u} = (6,2)$ ^ $\bar{v} = (1,4)$ $\bar{u} + \bar{v}$?
$\bar{u} + \bar{v} = (7,6)$

-  $\bar{w}$ such that $\bar{v} + \bar{w} = \bar{u}$ so $\bar{w} = \bar{u} - \bar{v}$
$d(\bar{u}, \bar{v}) = \|\bar{w}\| = \|\bar{u} - \bar{v}\|$
- generalizing to vectors in $\mathbb{R}^n$: $\bar{u}, \bar{v} \in \mathbb{R}^n$ $d(\bar{u}, \bar{v}) = \|\bar{u} - \bar{v}\|$

ex – $\bar{u} = (6,2)$ ^ $\bar{v} = (1,4)$
$d(\bar{u}, \bar{v}) = \|\bar{u} - \bar{v}\| = \|(5,-2)\| = \sqrt{5^2 + (-2)^2} = \sqrt{25 + 4} = \boxed{\sqrt{29}}$

Angle btwn Vectors in R_n

- say $\bar{u}$ ^ $\bar{v}$ in $\mathbb{R}^2$ where 

$c = \|\bar{u} - \bar{v}\|$

$c^2 = a^2 + b^2 - 2ab\cos\theta$
$\|\bar{u} - \bar{v}\|^2 = \|\bar{u}\|^2 + \|\bar{v}\|^2 - 2\|\bar{u}\| \cdot \|\bar{v}\| \cos\theta$
$\|(u_1 - v_1, u_2 - v_2)\|^2 = u_1^2 + u_2^2 + v_1^2 + v_2^2 - 2\sqrt{u_1^2 + u_2^2}\sqrt{v_1^2 + v_2^2}\cos\theta$
definition of norm: $(u_1 - v_1)^2 + (u_2 - v_2)^2 =$

$\cancel{u_1^2} - 2u_1v_1 + \cancel{v_1^2} + \cancel{u_2^2} - 2u_2v_2 + \cancel{v_2^2} = \cancel{u_1^2} + \cancel{u_2^2} + \cancel{v_1^2} + \cancel{v_2^2} - 2\sqrt{u_1^2+u_2^2}\sqrt{v_1^2+v_2^2}\cos\theta$

$-2u_1v_1 - 2u_2v_2 = -2\sqrt{u_1^2+u_2^2}\sqrt{v_1^2+v_2^2}\cos\theta$

$u_1v_1 + u_2v_2 = \sqrt{u_1^2+u_2^2}\sqrt{v_1^2+v_2^2}\cos\theta$

$\cos\theta = \frac{u_1v_1 + u_2+v_2}{\sqrt{u_1^2+u_2^2}\sqrt{v_1^2+v_2^2}}$ OR $= \frac{u_1v_1+u_2v_2}{\|\bar{u}\|\cdot\|\bar{v}\|}$ OR $= \frac{\bar{u}\cdot\bar{v}}{\|\bar{u}\|\cdot\|\bar{v}\|}$ ← dot product of $\bar{u}$ ^ $\bar{v}$ denoted $\bar{u}\cdot\bar{v}$ is $u_1v_1+u_2v_2$ (in $\mathbb{R}^n$)

↑ defines the angle θ between $\bar{u}$ ^ $\bar{v}$ in $\mathbb{R}^n$

↖ generalizes to $\mathbb{R}^n$

ex - $\bar{u} = (6,2)$ $\bar{v} = (1,4)$

$\cos\theta = \frac{\bar{u}\cdot\bar{v}}{\|\bar{u}\|\|\bar{v}\|} = \frac{6(1)+2(4)}{\sqrt{6^2+2^2}\sqrt{1^2+4^2}} = \frac{6+8}{\sqrt{36+4}\sqrt{1+16}} = \frac{14}{\sqrt{40}\sqrt{17}} = \frac{\cancel{14}}{\cancel{2}\sqrt{10}\sqrt{17}} \Leftrightarrow \theta = \cos^{-1}\left(\frac{7}{\sqrt{10}\sqrt{17}}\right) \approx \boxed{57.53°}$

ex - $\bar{u} = (-3,3)$ ^ $\bar{v} = (5,5)$

$\cos\theta = \frac{-3(5)+3(5)}{\|\bar{u}\|\cdot\|\bar{v}\|} = 0 \Leftrightarrow \theta = \cos^{-1}(0) = \boxed{90°}$

ex - find the norm of v

a) $\bar{v} = (3,5)$ $\|\bar{v}\|$?

$\|\bar{v}\| = \sqrt{3^2+5^2} = \sqrt{9+25} = \boxed{\sqrt{34}}$

b) $\bar{v} = (-2,4)$ $\|\bar{v}\|$?

$\|\bar{v}\| = \sqrt{(-2)^2+4^2} = \sqrt{4+16} = \sqrt{20} = \boxed{2\sqrt{5}}$

c) $\bar{v} = (-1,0,1)$ $\|\bar{v}\|$?

$\|\bar{v}\| = \sqrt{(-1)^2+0^2+1^2} = \sqrt{1+1} = \boxed{\sqrt{2}}$

ex - find the unit vector in the direction of v

a) $\bar{v} = (5,-6)$

$\|\bar{v}\| = \sqrt{5^2+(-6)^2} = \sqrt{25+36} = \sqrt{61}$

$u = \frac{1}{\sqrt{61}}(5,-6) = \boxed{\left(\frac{5}{\sqrt{61}}, -\frac{6}{\sqrt{61}}\right)}$

b) $\bar{v} = (-3,-4)$

$\|\bar{v}\| = \sqrt{(-3)^2+(-4)^2} = \sqrt{9+16} = \sqrt{25} = 5$

$u = \frac{1}{5}(-3,-4) = \boxed{\left(-\frac{3}{5}, -\frac{4}{5}\right)}$

ex - find the norm of $c\bar{v}$

a) $c = 8$ $\bar{v} = (-1,1)$ $\|c\bar{v}\|$

$\|\bar{v}\| = \sqrt{(-1)^2+1^2} = \sqrt{1+1} = \sqrt{2}$

$|c|\cdot\|\bar{v}\| = \boxed{8\sqrt{2}}$

b) $c = -100$ $\bar{v} = (3,-2)$

$\|\bar{v}\| = \sqrt{3^2+(-2)^2} = \sqrt{9+4} = \sqrt{13}$

$|c|\cdot\|\bar{v}\| = \boxed{100\sqrt{13}}$

ex - find the distance between u ^ v

a) $\bar{u}=(1,0)$ $\bar{v}=(0,1)$

$$d(u,v)=\|u-v\|=\sqrt{(1-0)^2+(0-1)^2}=\sqrt{1+1}=\boxed{\sqrt{2}}$$

b) $\bar{u}=(2,0)$ $\bar{v}=(-3,4)$

$$d(u,v)=\sqrt{(2--3)^2+(0-4)^2}=\sqrt{25+16}=\boxed{\sqrt{41}}$$

c) $\bar{u}=(1,1,2)$ $\bar{v}=(0,-3,5)$

$$d(u,v)=\sqrt{(1-0)^2+(1--3)^2+(2-5)^2}=\sqrt{1+16+9}=\boxed{\sqrt{26}}$$

ex - find the angle between u ^ v

a) $\bar{u}=(1,0)$ $\bar{v}=(0,1)$

$$\cos\theta=\frac{1(0)+0(1)}{\sqrt{1^2+0^2}\sqrt{0^2+1^2}}=\frac{0}{\sqrt{1}\sqrt{1}}=0 \Rightarrow \theta=\cos^{-1}(0)=\boxed{90^\circ}$$

b) $\bar{u}=(\frac{\sqrt{3}}{2},\frac{1}{2})$ $\bar{v}=(1,0)$

$$\cos\theta=\frac{\frac{\sqrt{3}}{2}(1)+\frac{1}{2}(0)}{\sqrt{(\frac{\sqrt{3}}{2})^2+(\frac{1}{2})^2}\sqrt{1^2+0^2}}=\frac{\frac{\sqrt{3}}{2}}{\sqrt{\frac{3}{4}+\frac{1}{4}}\sqrt{1}}=\frac{\frac{\sqrt{3}}{2}}{\sqrt{1}\sqrt{1}}=\frac{\sqrt{3}}{2} \Rightarrow \theta=\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)=\boxed{30^\circ}$$

c) $\bar{u}=(5,-3,7)$ $\bar{v}=(-1,-2,8)$

$$\cos\theta=\frac{5(-1)+(-3)(-2)+7(8)}{\sqrt{5^2+(-3)^2+7^2}\sqrt{(-1)^2+(-2)^2+8^2}}=\frac{-5+6+56}{\sqrt{25+9+49}\sqrt{1+4+64}}=\frac{57}{\sqrt{83}\sqrt{69}} \Rightarrow \theta=\cos^{-1}\left(\frac{57}{\sqrt{83}\sqrt{69}}\right)=\boxed{41.13^\circ}$$

Inner Product Spaces

- properties:
  1. $\bar{u}\cdot\bar{v}=\bar{v}\cdot\bar{u}$ commutativity
  2. $\bar{u}\cdot(\bar{v}+\bar{w})=\bar{u}\bar{v}+\bar{u}\bar{w}$ distributivity
  3. $c(\bar{u}\cdot\bar{v})=(c\bar{u})\cdot\bar{v}$
  4. $\bar{v}\cdot\bar{v}\geq 0$ & $\bar{v}\cdot\bar{v}=0$ if $\bar{v}=\bar{0}$
- inner product is $V\cdot V\to\mathbb{R}$ $(\bar{u},\bar{v})\mapsto\langle\bar{u},\bar{v}\rangle$; for any $\bar{u},\bar{v},\bar{w}\in V$ and $c\in\mathbb{R}$, the following four properties hold:
  1. $\langle\bar{u},\bar{v}\rangle=\langle\bar{v},\bar{u}\rangle$
  2. $\langle\bar{u},\bar{v}+\bar{w}\rangle=\langle\bar{u},\bar{v}\rangle+\langle\bar{u},\bar{w}\rangle$
  3. $c\langle\bar{u},\bar{v}\rangle=\langle c\bar{u},\bar{v}\rangle$
  4. $\langle\bar{v},\bar{v}\rangle\geq 0$ & $\langle\bar{v},\bar{v}\rangle=0$ if $\bar{v}=\bar{0}$
- a vector space equipped with an inner product is called an inner product space
- $\mathbb{R}^n$ equipped with dot product is euclidean n-space

ex - P_2 defines the function $H: P_2\times P_2\to\mathbb{R}$ by $(a_0+a_1x+a_2x^2,\ b_0+b_1x+b_2x^2)\mapsto a_0b_0+a_1b_1+a_2b_2$

denote this by $\langle p,q\rangle$ where $p=a_0+a_1x+a_2x^2$ $q=b_0+b_1x+b_2x^2$

ex - let $p(x)=-3+4x+2x^2$ ^ $q(x)=1+7x-3x^2$ find $\langle p,q\rangle$

$$\langle p,q\rangle=-3(1)+4(7)+2(-3)=-3+28-6=\boxed{19}$$

- $M_{2,2}$ define an inner product between two matricies $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ ^ $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$
  by $\langle A, B \rangle = a_{11}b_{11} + a_{12}b_{12} + a_{21}b_{21} + a_{22}b_{22}$

ex - let $A = \begin{bmatrix} -10 & 13 \\ 2 & 6 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 7 \\ 18 & 0 \end{bmatrix}$ find $\langle A, B \rangle$
  $\langle A, B \rangle = -10(1) + 13(7) + 2(18) + 6(0) = -10 + 91 + 36 = \boxed{117}$

- $c[0,1]$ set of all real-valued continuous functions on $[0,1]$; define an inner product between two functions f ^ g in $c[0,1]$ by $\langle f, g \rangle = \int_0^1 f(x)\,g(x)\,dx$

ex - let $f(x) = x+1$ $g(x) = 2+x^2$ find $\langle f, g \rangle$
  $\langle f, g \rangle = \int_0^1 (x+1)(2+x^2)\,dx = \int_0^1 2x + x^3 + 2 + x^2\,dx = \frac{x^4}{4} + \frac{x^3}{3} + \frac{2x^2}{2} + 2x \Big|_0^1 = \frac{1}{4} + \frac{1}{3} + 1 + 2 =$
  $\frac{3}{12} + \frac{4}{12} + \frac{36}{12} = \boxed{\frac{43}{12}}$

Norm, Distance, ^ Angle
- V inner product space, $\bar{u}, \bar{v} \in V$
- norm of $\bar{u} = \sqrt{\langle \bar{u}, \bar{u} \rangle} \Leftrightarrow \|\bar{u}\| = \sqrt{\bar{u} \cdot \bar{u}} = \sqrt{u_1^2 + u_2^2 \dots u_n^2}$
- distance between $\bar{u}$ ^ $\bar{v}$ $d(\bar{u}, \bar{v}) = \|\bar{u} - \bar{v}\|$
- angle between $\bar{u}$ ^ $\bar{v}$ $(\bar{u}, \bar{v} \neq 0)$ is given by $\cos\theta = \frac{\langle \bar{u}, \bar{v} \rangle}{\|\bar{u}\| \cdot \|\bar{v}\|}$
  $\bar{u}$ ^ $\bar{v}$ orthogonal just in case $\langle \bar{u}, \bar{v} \rangle = 0$

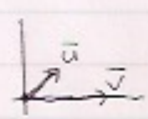
ex - let $p(x) = -3 + 4x + 2x^2$ $q(x) = 1 + 7x - 3x^2$ find $\|p\|$, $\|q\|$, $\|p-q\|$ ^ angle btwn p ^ q
  $\|p\| = \langle p, p \rangle = \sqrt{-3(-3) + 4(4) + 2(2)} = \sqrt{9+16+4} = \boxed{\sqrt{29}}$
  $\|q\| = \sqrt{1(1) + 7(7) - 3(-3)} = \sqrt{1+49+9} = \boxed{\sqrt{59}}$

  $p - q = -3 + 4x + 2x^2 - 1 - 7x + 3x^2 = -4 - 3x + 5x^2$
  $\|p-q\| = \sqrt{(-4)^2 + (-3)^2 + 5^2} = \sqrt{16+9+25} = \sqrt{50} = \boxed{5\sqrt{2}}$
  $\langle p, q \rangle = -3(1) + 4(7) + 2(-3) = -3 + 28 - 6 = 19$
  $\cos\theta = \frac{19}{\sqrt{29}\sqrt{59}} \Rightarrow \theta = \boxed{\cos^{-1}\left(\frac{19}{\sqrt{29}\sqrt{59}}\right)}$

ex - show that $r(x) = x$ ^ $s(x) = x^2$ are orthogonal
  $\langle r, s \rangle = 0(0) + 1(0) + 0(1) = 0$ $\{1, x, x^2\}$ for P_2

ex - let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ find $\|A\|$, $\|B\|$, $\|A-B\|$, ^ angle between A ^ B
  $\|A\| = \sqrt{1(1) + 0(0) + 0^2 + 1^2} = \sqrt{1+1} = \boxed{\sqrt{2}}$
  $\|B\| = \sqrt{0^2 + 1^2 + 0^2 + 1^2} = \sqrt{1+1} = \boxed{\sqrt{2}}$
  $A - B = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ $\|A-B\|$ $\sqrt{1^2 + (-1)^2 + (-1)^2 + 1^2} = \sqrt{1+1+1+1} = \sqrt{4} = \boxed{2}$
  $\theta = \cos^{-1}\left(\frac{0}{\sqrt{2}\sqrt{2}}\right) = \cos^{-1}(0) = \boxed{90^\circ}$
  $\langle A, B \rangle = 1(0) + 0(1) + 0(1) + 1(0) = 0$

Orthogonal Projection
- orthogonal projection of $\bar{u}$ onto $\bar{v}$, denoted by $\text{proj}_{\bar{v}}\bar{u}$
  $\text{proj}_{\bar{v}}\bar{u} = \frac{\bar{u} \cdot \bar{v}}{\bar{v} \cdot \bar{v}} \cdot \bar{v} = \frac{\langle \bar{u}, \bar{v} \rangle}{\langle \bar{v}, \bar{v} \rangle} \cdot \bar{v}$



ex - let $p(x) = x + 3x^2$ $\quad q(x) = x^2$ $\quad r(x) = -5x$ $\quad$ find $proj_q\, p$ ^ $proj_r\, p$

$\langle p,q \rangle = \cancel{1(0)} + 3(1) = 3$ $\qquad \langle p,r \rangle = 1(-5) + \cancel{3(0)} = -5$

$\langle q,q \rangle = 1(1) = 1$ $\qquad \langle r,r \rangle = -5(-5) = 25$

$\frac{3}{1}(x^2) = \boxed{3x^2} = proj_q\, p$ $\qquad proj_r\, p = \frac{-5}{25}(-5x) = \boxed{x}$

ex - find $\langle p,q \rangle$, $\|p\|$, $\|q\|$, $\|p-q\|$, and the angle between p ^ q

a) $p(x) = 2 - 3x + 5x^2$ $\quad q(x) = 1 - 2x - 3x^2$

$\langle p,q \rangle = 2(1) + (-3)(-2) + 5(-3) = 2 + 6 - 15 = \boxed{-7}$

$\|p\| = \sqrt{2^2 + (-3)^2 + 5^2} = \sqrt{4+9+25} = \boxed{\sqrt{38}}$

$\|q\| = \sqrt{1^2 + (-2)^2 + (-3)^2} = \sqrt{1+4+9} = \boxed{\sqrt{14}}$

$p - q = 2 - 3x + 5x^2 - 1 + 2x + 3x^2 = 1 - x + 8x^2$

$\|p-q\| = \sqrt{1^2 + (-1)^2 + 8^2} = \sqrt{1+1+64} = \boxed{\sqrt{66}}$

$\theta = \boxed{\cos^{-1}\left(\frac{-7}{\sqrt{38}\sqrt{14}}\right)}$

b) $p(x) = 1 - x$ $\quad q(x) = 3x^2$

$\langle p,q \rangle = \cancel{1(0)} - \cancel{1(0)} + \cancel{0(3)} = 0$

$\|p\| = \sqrt{1^2 + (-1)^2 + \cancel{0^2}} = \sqrt{1+1} = \boxed{\sqrt{2}}$

$\|q\| = \sqrt{\cancel{0^2} + \cancel{0^2} + 3^2} = \sqrt{9} = \boxed{3}$

$p - q = 1 - x - 3x^2$

$\|p-q\| = \sqrt{1^2 + (-1)^2 + (-3)^2} = \sqrt{1+1+9} = \boxed{\sqrt{11}}$

$\theta = \cos^{-1}\left(\frac{0}{3\sqrt{2}}\right) = \cos^{-1}(0) = \boxed{90^\circ}$

ex - find $\langle A, B \rangle$, $\|A\|$, $\|B\|$, $\|A-B\|$, and the angle between A and B

a) $A = \begin{bmatrix} -1 & 0 \\ 2 & 6 \end{bmatrix}$ $\quad B = \begin{bmatrix} 1 & 3 \\ 9 & 0 \end{bmatrix}$

$\langle A, B \rangle = -1(1) + \cancel{0(3)} + 2(9) + \cancel{6(0)} = -1 + 18 = \boxed{17}$

$\|A\| = \sqrt{(-1)^2 + \cancel{0^2} + 2^2 + 6^2} = \sqrt{1+4+36} = \boxed{\sqrt{41}}$

$\|B\| = \sqrt{1^2 + 3^2 + 9^2 + \cancel{0^2}} = \sqrt{1+9+81} = \boxed{\sqrt{91}}$

$A - B = \begin{bmatrix} -2 & -3 \\ -7 & 6 \end{bmatrix}$

$\|A-B\| = \sqrt{(-2)^2 + (-3)^2 + (-7)^2 + 6^2} = \sqrt{4+9+49+36} = \sqrt{98} = \boxed{7\sqrt{2}}$

$\theta = \boxed{\cos^{-1}\left(\frac{17}{\sqrt{41}\sqrt{91}}\right)}$

b) $A = \begin{bmatrix} -4 & 4 \\ 6 & 2 \end{bmatrix}$ $\quad B = \begin{bmatrix} 1 & 3 \\ -2 & 2 \end{bmatrix}$

$\langle A, B \rangle = -4(1) + 4(3) + 6(-2) + 2(2) = -4 + 12 - 12 + 4 = \boxed{0}$

$\|A\| = \sqrt{(-4)^2 + 4^2 + 6^2 + 2^2} = \sqrt{16+16+36+4} = \sqrt{72} = \boxed{6\sqrt{2}}$

$\|B\| = \sqrt{1^2 + 3^2 + (-2)^2 + 2^2} = \sqrt{1+9+4+4} = \boxed{3\sqrt{2}}$

$\|A-B\| = \left\| \begin{bmatrix} -5 & 1 \\ 8 & 0 \end{bmatrix} \right\| = \sqrt{(-5)^2 + 1^2 + 8^2 + \cancel{0^2}} = \sqrt{25+1+64} = \sqrt{90} = \boxed{3\sqrt{10}}$

$\theta = \cos^{-1}\left(\frac{0}{6\sqrt{2}\, 3\sqrt{2}}\right) = \cos^{-1}(0) = \boxed{90^\circ}$

ex - find $\text{proj}_v u$

a) $u = (4,5) \quad v = (10,0)$

$\text{proj}_v u = \frac{4(10)+5(0)}{10(10)+0(0)}(10,0) = \frac{40}{100}(10,0) = \frac{2}{5}(10,0) = \boxed{(4,0)}$

b) $u = (1,3) \quad v = (5,5)$

$\text{proj}_v u = \frac{1(5)+3(5)}{5^2+5^2}(5,5) = \frac{20}{50}(5,5) = \boxed{(2,2)}$

c) $u = (1,1,1) \quad v = (0,2,0)$

$\text{proj}_v u = \frac{1(0)+1(2)+1(0)}{0^2+2^2+0^2}(0,2,0) = \frac{2}{4}(0,2,0) = \boxed{(0,1,0)}$

ex - find $\text{proj}_q p$

a) $p(x) = 3-2x+4x^2 \quad q(x) = x-2x^2$

$\text{proj}_q p = \frac{3(0)+(-2)(1)+4(-2)}{0^2+1^2+(-2)^2}(x-2x^2) = \frac{-2-8}{1+4}(x-2x^2) = \boxed{-2(x-2x^2)}$

b) $p(x) = 8+5x^2 \quad q(x) = -33x$

$\text{proj}_q p = \frac{8(0)+0(-33)+5(0)}{(-33)^2}(-33x) = \boxed{0}$

ex - find $\text{proj}_B A$

a) $A = \begin{bmatrix} 3 & -4 \\ 8 & 9 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

$\text{proj}_B A = \frac{3(0)+(-4)(0)+8(1)+9(0)}{0^2+0^2+1^2+0^2}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = 8\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \boxed{\begin{bmatrix} 0 & 0 \\ 8 & 0 \end{bmatrix}}$

b) $A = \begin{bmatrix} -3 & 0 \\ 2 & 6 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$\text{proj}_B A = \frac{-3(0)+0(1)+2(1)+6(0)}{0^2+1^2+1^2+0^2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \boxed{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}$

c) $A = \begin{bmatrix} 3 & 0 \\ 8 & 9 \end{bmatrix} \quad B = \begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix}$

$\text{proj}_B A = \frac{3(0)+0(-2)+8(0)+9(0)}{0^2+(-2)^2+0^2+0^2}\begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix} = \boxed{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}$

Orthonormal Bases

- $\mathbb{R}^2$ standard basis $\{(1,0),(0,1)\}$ non-standard basis $\{(1,1),(3,2)\}$

$\|(1,1)\| = \sqrt{1^2+1^2} = \sqrt{2}$

$\|(3,2)\| = \sqrt{3^2+2^2} = \sqrt{9+4} = \sqrt{13}$

- $\mathbb{R}^3$ $\{(1,0,0),(0,1,0),(0,0,1)\}$

$\|(1,0,0)\| = \|(0,1,0)\| = \|(0,0,1)\| = \sqrt{1} = 1$ orthogonal basis

- orthonormal basis $\{(1,-1,2),(0,0,1),(1,1,0)\}$

$(0,0,1)\cdot(1,1,0) = 0+\cancel{0}+\cancel{0} = 0$

$(1,-1,2)\cdot(0,0,1) = 0+\cancel{0}+2 = 2 \neq 0$

$\|(0,0,1)\| = \sqrt{\cancel{0^2}+\cancel{0^2}+1^2} = 1$

$\|(1,-1,2)\| = \sqrt{1^2+(-1)^2+2^2} = \sqrt{6} \neq 1$

Coordinates Relative to an Orthonormal Basis

- V inner product space, $B = \{\vec{v}_1, \ldots \vec{v}_n\}$ orthonormal basis for V
- $\vec{w}$ in V can be written as $\vec{w} = \langle \vec{w}, \vec{v}_1 \rangle \vec{v}_1 + \langle \vec{w}, \vec{v}_2 \rangle \vec{v}_2 + \ldots + \langle \vec{w}, \vec{v}_n \rangle \vec{v}_n$

ex - $B=\{(\frac{3}{5},\frac{4}{5},0),(-\frac{4}{5},\frac{3}{5},0),(0,0,1)\}$ orthonormal basis for $\mathbb{R}^3$

find coordinates of $\bar{w}=(10,15,-1)$ relative to the basis B

$\bar{w}=\langle\bar{w},\bar{v}_1\rangle\bar{v}_1+\ldots+\langle\bar{w},\bar{v}_n\rangle\bar{v}_n$

$\langle\bar{w},\bar{v}_1\rangle=10(\frac{3}{5})+15(\frac{4}{5})\cancel{-1(0)}=6+12=18$

$\langle\bar{w},\bar{v}_2\rangle=10(-\frac{4}{5})+15(\frac{3}{5})\cancel{-1(0)}=-8+9=1$

$\langle\bar{w},\bar{v}_3\rangle=\cancel{10(0)}+\cancel{15(0)}-1(1)=-1$

$\bar{w}=18\bar{v}_1+1\bar{v}_2-1v_3$ $\boxed{[\bar{w}]_B=\begin{bmatrix}18\\1\\-1\end{bmatrix}}$

ex - $B=\{\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}x,\ x^2,\ \frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}x\}$ orthonormal basis for P_2

find coordinates of $\bar{f}=-4+7x+11x^2$ relative to the basis B

$\langle\bar{f},\bar{v}_1\rangle=$ ~~[scribbled out]~~ $-4(\frac{1}{\sqrt{2}})+7(-\frac{1}{\sqrt{2}})+\cancel{11(0)}=-\frac{4}{\sqrt{2}}-\frac{7}{\sqrt{2}}=-\frac{11}{\sqrt{2}}$ multiply corresponding coefficients

$\langle\bar{f},\bar{v}_2\rangle=\cancel{-4(0)}+\cancel{7(0)}+11(1)=11$

$\langle\bar{f},\bar{v}_3\rangle=-4(\frac{1}{\sqrt{2}})+7(\frac{1}{\sqrt{2}})+\cancel{11(0)}=-\frac{4}{\sqrt{2}}+\frac{7}{\sqrt{2}}=\frac{3}{\sqrt{2}}$

$\bar{f}=-\frac{11}{\sqrt{2}}\bar{v}_1+11\bar{v}_2+\frac{3}{\sqrt{2}}\bar{v}_3$ $\boxed{[\bar{f}]_B=\begin{bmatrix}-11/\sqrt{2}\\11\\3/\sqrt{2}\end{bmatrix}}$

Gram-Schmidt Process

- gram-schmidt process:

① $B=\{\bar{v}_1,\ldots\bar{v}_n\}$ is a basis for an inner product space V

② Construct the following vectors:

$\bar{w}_1=\bar{v}_1$

$\bar{w}_2=\bar{v}_2-\frac{\langle\bar{v}_2,\bar{w}_1\rangle}{\langle\bar{w}_1,\bar{w}_1\rangle}\bar{w}_1$

$\bar{w}_3=\bar{v}_3-\frac{\langle\bar{v}_3,\bar{w}_1\rangle}{\langle\bar{w}_1,\bar{w}_1\rangle}\bar{w}_1-\frac{\langle\bar{v}_3,\bar{w}_2\rangle}{\langle\bar{w}_2,\bar{w}_2\rangle}\bar{w}_2$

$\vdots$

$\bar{w}_n=\bar{v}_n-$ " " $\ldots-\frac{\langle\bar{v}_n,\bar{w}_{n-1}\rangle}{\langle\bar{w}_{n-1},\bar{w}_{n-1}\rangle}\bar{w}_{n-1}$

$B'=\{\bar{w}_1,\ldots,\bar{w}_n\}$ is an orthogonal basis for V

③ Normalize each vector in B' to get an orthonormal basis for V

let $\bar{u}_i=\frac{\bar{w}_i}{\|\bar{w}_i\|}$ then $\{\bar{u}_1,\ldots\bar{u}_n\}$ is an orthonormal basis for V

ex - $B=\{(0,1,1),(2,2,0),(-1,0,-1)\}$ basis for $\mathbb{R}^3$; apply gram-schmidt process to B

$\bar{w}_1=(0,1,1)$

$\bar{w}_2=(2,2,0)-\frac{2(0)+2(1)+0(1)}{0^2+1^2+1^2}(0,1,1)=(2,2,0)-(0,1,1)=(2,1,-1)$

$\bar{w}_3=(-1,0,-1)-\frac{-1(0)+0(1)-1(1)}{0^2+1^2+1^2}(0,1,1)-\frac{-1(2)+0(1)-1(-1)}{2^2+1^2+(-1)^2}(2,1,-1)=(-1,0,-1)+(0,\frac{1}{2},\frac{1}{2})+(\frac{1}{3},\frac{1}{6},-\frac{1}{6})$

$(-\frac{2}{3},\frac{8}{12},-\frac{8}{12})=\boxed{(-\frac{2}{3},\frac{2}{3},-\frac{2}{3})}$

given $B'=\{(0,1,1),(2,1,-1),(-\frac{2}{3},\frac{2}{3},-\frac{2}{3})\}$, normalize each vector

$\bar{u}_1=\frac{1}{\sqrt{1^2+1^2}}(0,1,1)=\frac{1}{\sqrt{2}}(0,1,1)=\boxed{(0,\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})}$

$\bar{u}_2=\frac{1}{\sqrt{2^2+1^2+(-1)^2}}(2,1,-1)=\frac{1}{\sqrt{6}}(2,1,-1)=\boxed{(\frac{2}{\sqrt{6}},\frac{1}{\sqrt{6}},-\frac{1}{\sqrt{6}})}$

$\bar{u}_3=\frac{1}{\sqrt{(-\frac{2}{3})^2+\frac{2}{3}^2+(-\frac{2}{3})^2}}(-\frac{2}{3},\frac{2}{3},-\frac{2}{3})=\frac{1}{\sqrt{\frac{4}{9}+\frac{4}{9}+\frac{4}{9}}}(-\frac{2}{3},\frac{2}{3},-\frac{2}{3})=\frac{1}{\sqrt{4/3}}(-\frac{2}{3},\frac{2}{3},-\frac{2}{3})=\boxed{(\frac{-2}{3\sqrt{4/3}},\frac{2}{3\sqrt{4/3}},\frac{-2}{3\sqrt{4/3}})}$ $\frac{\sqrt{3}}{3}$

$\{(0,\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}),(\frac{2}{\sqrt{6}},\frac{1}{\sqrt{6}},-\frac{1}{\sqrt{6}}),(-\frac{\sqrt{3}}{3},\frac{\sqrt{3}}{3},-\frac{\sqrt{3}}{3})\}$ is an orthonormal basis for $\mathbb{R}^3$

ex – $B=\{2+4x-x^2, 3x, 5x^2\}$ for P_2; apply the gram-schmidt process to B

$\bar{w}_1 = 2+4x-x^2$

$\bar{w}_2 = 3x - \frac{2(0)+3(4)+0(-1)}{2^2+4^2+(-1)^2}(2+4x-x^2) = 3x - \frac{4}{7}(2+4x-x^2) = 3x - \frac{8}{7} - \frac{16}{7}x + \frac{4}{7}x^2 = -\frac{8}{7}+\frac{5}{7}x+\frac{4}{7}x^2$

$\bar{w}_3 = 5x^2 - \frac{2(0)+4(0)+5(-1)}{2^2+4^2+(-1)^2}(2+4x-x^2) - \frac{0(-\frac{8}{7})+0(\frac{5}{7})+5(\frac{4}{7})}{(\frac{8}{7})^2+(\frac{5}{7})^2+(\frac{4}{7})^2}(-\frac{8}{7}+\frac{5}{7}x+\frac{4}{7}x^2) = 5x^2+\frac{5}{21}(2+4x-x^2) - \frac{\frac{20}{7}}{\frac{64}{49}+\frac{25}{49}+\frac{16}{49}}(-\frac{8}{7}+\frac{5}{7}x+\frac{4}{7}x^2) =$

$5x^2+\frac{10}{21}+\frac{20}{21}x-\frac{5}{21}x^2+\frac{32}{21}-\frac{20}{21}x-\frac{16}{21}x^2 = \boxed{\frac{42}{21}+4x^2} = 2+4x^2$

$\frac{\frac{20}{7}}{\frac{105}{49}} = \frac{4}{3}$

given $B' = \{2+4x-x^2, -\frac{8}{7}+\frac{5}{7}x+\frac{4}{7}x^2, 2+4x^2\}$ normalize each vector

$\bar{u}_1 = \frac{1}{\sqrt{2^2+4^2+(-1)^2}}(2+4x-x^2) = \frac{1}{\sqrt{21}}(2+4x-x^2) = \boxed{\frac{2}{\sqrt{21}}+\frac{4}{\sqrt{21}}x-\frac{1}{\sqrt{21}}x^2}$

$\bar{u}_2 = \frac{1}{\sqrt{(\frac{8}{7})^2+(\frac{5}{7})^2+(\frac{4}{7})^2}}(-\frac{8}{7}+\frac{5}{7}x+\frac{4}{7}x^2) = \frac{7}{\sqrt{105}}(-\frac{8}{7}+\frac{5}{7}x+\frac{4}{7}x^2) = \boxed{-\frac{8}{\sqrt{105}}+\frac{5}{\sqrt{105}}x+\frac{4}{\sqrt{105}}x^2}$

$\bar{u}_3 = \frac{1}{\sqrt{(2)^2+4^2}}(2+4x^2) = \frac{1}{2\sqrt{5}}(2+4x^2) = \boxed{\frac{1}{\sqrt{5}}+\frac{2}{\sqrt{5}}x^2}$

$\{\frac{2}{\sqrt{21}}+\frac{4}{\sqrt{21}}x-\frac{1}{\sqrt{21}}x^2, -\frac{8}{\sqrt{105}}+\frac{5}{\sqrt{105}}x+\frac{4}{\sqrt{105}}x^2, \frac{1}{\sqrt{5}}+\frac{2}{\sqrt{5}}x^2\}$ orthonormal basis for P_2

ex – find the coordinates of $\bar{w}$ relative to the orthonormal basis B for $\mathbb{R}^3$

a) $\bar{w}=(2,-2,1)$ $B=\{(\frac{\sqrt{10}}{10},0,\frac{3\sqrt{10}}{10}),(0,1,0),(-\frac{3\sqrt{10}}{10},0,\frac{\sqrt{10}}{10})\}$

$\langle\bar{w},\bar{v}_1\rangle = 2(\frac{\sqrt{10}}{10})+\cancel{(-2)(0)}+1(\frac{3\sqrt{10}}{10}) = \frac{2\sqrt{10}}{10}+\frac{3\sqrt{10}}{10} = \frac{5\sqrt{10}}{10} = \frac{\sqrt{10}}{2}$

$\langle\bar{w},\bar{v}_2\rangle = \cancel{2(0)}+(-2)(1)+\cancel{1(0)} = -2$

$\langle\bar{w},\bar{v}_3\rangle = 2(\frac{-3\sqrt{10}}{10})+\cancel{(-2)(0)}+1(\frac{\sqrt{10}}{10}) = \frac{-6\sqrt{10}}{10}+\frac{\sqrt{10}}{10} = \frac{-5\sqrt{10}}{10} = -\frac{\sqrt{10}}{2}$

$\bar{w} = \frac{\sqrt{10}}{2}\bar{v}_1 - 2\bar{v}_2 - \frac{\sqrt{10}}{2}$ $\quad \boxed{[\bar{w}]_B = \begin{bmatrix}\sqrt{10}/2\\-2\\-\sqrt{10}/2\end{bmatrix}}$

b) $\bar{w}=(3,-5,8)$ $B=\{(\frac{5}{13},0,\frac{12}{13}),(-\frac{12}{13},0,\frac{5}{13}),(0,1,0)\}$

$\langle\bar{w},\bar{v}_1\rangle = 3(\frac{5}{13})\cancel{-5(0)}+8(\frac{12}{13}) = \frac{15}{13}+\frac{96}{13} = \frac{111}{13}$

$\langle\bar{w},\bar{v}_2\rangle = 3(-\frac{12}{13})\cancel{-5(0)}+8(\frac{5}{13}) = -\frac{36}{13}+\frac{40}{13} = \frac{4}{13}$

$\langle\bar{w},\bar{v}_3\rangle = \cancel{3(0)}-5(1)+\cancel{8(0)} = -5$

$\bar{w} = \frac{111}{13}\bar{v}_1+\frac{4}{13}\bar{v}_2-5\bar{v}_3$ $\quad \boxed{[\bar{w}]_B = \begin{bmatrix}\frac{111}{13}\\\frac{4}{13}\\-5\end{bmatrix}}$

ex – find the coordinates of $\bar{f}$ relative to the orthonormal basis B for P_2

a) $\bar{f}=8-3x+5x^2$ $B=\{\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}x^2, x, \frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}x^2\}$

$\langle\bar{f},\bar{v}_1\rangle = 8(\frac{1}{\sqrt{2}})\cancel{-3(0)}+5(-\frac{1}{\sqrt{2}}) = \frac{8}{\sqrt{2}}-\frac{5}{\sqrt{2}} = \frac{3}{\sqrt{2}}$

$\langle\bar{f},\bar{v}_2\rangle = \cancel{8(0)}-3(1)+\cancel{5(0)} = -3$

$\langle\bar{f},\bar{v}_3\rangle = 8(\frac{1}{\sqrt{2}})\cancel{-3(0)}+5(\frac{1}{\sqrt{2}}) = \frac{8}{\sqrt{2}}+\frac{5}{\sqrt{2}} = \frac{13}{\sqrt{2}}$

$\bar{f} = \frac{3}{\sqrt{2}}\bar{v}_1-3\bar{v}_2+\frac{13}{\sqrt{2}}\bar{v}_3$ $\quad [\bar{f}]_B = \begin{bmatrix}3/\sqrt{2}\\-3\\13/\sqrt{2}\end{bmatrix}$

b) $\bar{f}=8-3x+5x^2$ $B=\{\frac{3}{5}+\frac{4}{5}x, -\frac{4}{5}+\frac{3}{5}x, x^2\}$

$\langle\bar{f},\bar{v}_1\rangle = 8(\frac{3}{5})-3(\frac{4}{5})+\cancel{5(0)} = \frac{24}{5}-\frac{12}{5} = \frac{12}{5}$

$\langle\bar{f},\bar{v}_2\rangle = 8(-\frac{4}{5})-3(\frac{3}{5})+\cancel{5(0)} = -\frac{32}{5}-\frac{9}{5} = \frac{-41}{5}$

$\langle\bar{f},\bar{v}_3\rangle = \cancel{8(0)}-\cancel{3(0)}+5(1) = 5$

$\bar{f} = \frac{12}{5}-\frac{41}{5}x+5x^2$ $\quad \boxed{[\bar{f}]_B = \begin{bmatrix}\frac{12}{5}\\-\frac{41}{5}\\5\end{bmatrix}}$

ex - apply the gram-schmidt process to the basis B for $\mathbb{R}^3$

a) $B = \{(1,-2,2),(2,2,1),(2,0,1)\}$

$\bar{w}_1 = (1,-2,2)$

$\bar{w}_2 = (2,2,1) - \frac{2(1)+2(-2)+1(2)}{1^2+(-2)^2+2^2}(1,-2,2) = (2,2,1) - \cancel{\frac{0}{9}(1,-2,2)} = (2,2,1)$

$\bar{w}_3 = (2,0,1) - \frac{2(1)+\cancel{0(-2)}+1(2)}{1^2+(-2)^2+2^2}(1,-2,2) - \frac{2(2)+\cancel{0(2)}+1(1)}{2^2+2^2+1^2}(2,2,1) = (2,0,1) - \frac{4}{9}(1,-2,2) - \frac{5}{9}(2,2,1) =$
$(2,0,1) + (-\frac{4}{9}, \frac{8}{9}, -\frac{8}{9}) + (-\frac{10}{9}, -\frac{10}{9}, -\frac{5}{9}) = (\frac{4}{9}, -\frac{2}{9}, -\frac{4}{9})$

$u_1 = \frac{1}{\sqrt{1^2+(-2)^2+2^2}}(1,-2,2) = \frac{1}{3}(1,-2,2) = \boxed{(\frac{1}{3}, -\frac{2}{3}, \frac{2}{3})}$

$u_2 = \frac{1}{\sqrt{2^2+2^2+1^2}}(2,2,1) = \frac{1}{3}(2,2,1) = \boxed{(\frac{2}{3}, \frac{2}{3}, \frac{1}{3})}$

$u_3 = \frac{1}{\sqrt{(\frac{4}{9})^2+(-\frac{2}{9})^2+(-\frac{4}{9})^2}}(\frac{4}{9}, -\frac{2}{9}, -\frac{4}{9}) = \frac{1}{\sqrt{\frac{16}{81}+\frac{4}{81}+\frac{16}{81}}}(\frac{4}{9}, -\frac{2}{9}, -\frac{4}{9}) = \frac{1}{\sqrt{\frac{36}{81}}}(\frac{4}{9}, -\frac{2}{9}, -\frac{4}{9}) = \frac{3}{2}(\frac{4}{9}, -\frac{2}{9}, -\frac{4}{9}) =$
$\boxed{(\frac{2}{3}, -\frac{1}{3}, -\frac{2}{3})}$

b) $B = \{(4,-3,0),(1,2,0),(1,1,4)\}$

$\bar{w}_1 = (4,-3,0)$

$\bar{w}_2 = (1,2,0) - \frac{1(4)+2(-3)+\cancel{0(0)}}{4^2+(-3)^2+\cancel{0^2}}(4,-3,0) = (1,2,0) - \frac{-2}{25}(4,-3,0) = (1,2,0) + (\frac{8}{25}, -\frac{6}{25}, 0) = (\frac{33}{25}, \frac{44}{25}, 0)$

$\bar{w}_3 = (1,1,4) - \frac{1(4)+1(-3)+4(0)}{4^2+(-3)^2+0^2}(4,-3,0) - \frac{1(\frac{33}{25})+1(\frac{44}{25})+4(0)}{(\frac{33}{25})^2+(\frac{44}{25})^2+0^2}(\frac{33}{25}, \frac{44}{25}, 0) = (1,1,4) - \frac{1}{25}(4,-3,0) - \frac{77/25}{3025/625}(\frac{33}{25}, \frac{44}{25}, 0) =$
$(1,1,4) + (-\frac{4}{25}, \frac{3}{25}, 0) + (-\frac{21}{25}, -\frac{28}{25}, 0) = (0,0,4)$

$u_1 = \frac{1}{\sqrt{4^2+(-3)^2+\cancel{0^2}}}(4,-3,0) = \frac{1}{5}(4,-3,0) = \boxed{(\frac{4}{5}, -\frac{3}{5}, 0)}$

$u_2 = \frac{1}{\sqrt{(\frac{33}{25})^2+(\frac{44}{25})^2+\cancel{0^2}}}(\frac{33}{25}, \frac{44}{25}, 0) = \frac{1}{11/5}(\frac{33}{25}, \frac{44}{25}, 0) = \boxed{(\frac{3}{5}, \frac{4}{5}, 0)}$

$u_3 = \frac{1}{\sqrt{\cancel{0^2}+\cancel{0^2}+4^2}}(0,0,4) = \frac{1}{4}(0,0,4) = \boxed{(0,0,1)}$

ex - apply the gram-schmidt process to the basis B for P_2

a) $B = \{x^2, x^2+2x, x^2+2x+1\}$

$\bar{w}_1 = x^2$

$\bar{w}_2 = x^2+2x - \frac{0(0)+2(0)+1(1)}{\cancel{0^2}+\cancel{0^2}+1^2}(x^2) = x^2+2x-x^2 = 2x$

$\bar{w}_3 = x^2+2x+1 - \frac{1(0)+2(0)+1(1)}{0^2+0^2+1^2}(x^2) - \frac{1(0)+2(2)+1(0)}{0^2+2^2+0^2}(2x) = \cancel{x^2}+\cancel{2x}+1-\cancel{x^2}-\cancel{2x} = 1$

$u_1 = \frac{1}{\sqrt{0^2+0^2+1^2}}(x^2) = \boxed{x^2}$

$u_2 = \frac{1}{\sqrt{0^2+2^2+0^2}}(2x) = \frac{1}{2}(2x) = \boxed{x}$

$u_3 = \frac{1}{\sqrt{1^2+0^2+0^2}}(1) = \boxed{1}$

b) $B = \{x^2-1, x^2+2, x+3\}$

$\bar{w}_1 = x^2-1$

$\bar{w}_2 = x^2+2 - \frac{2(-1)+\cancel{0(0)}+1(1)}{(-1)^2+\cancel{0^2}+1^2}(x^2-1) = x^2+2+\frac{1}{2}(x^2-1) = x^2+2+\frac{1}{2}x^2-\frac{1}{2} = \frac{3}{2}x^2+\frac{3}{2}$

$\bar{w}_3 = x+3 - \frac{3(-1)+1(0)+0(1)}{(-1)^2+\cancel{0^2}+1^2}(x^2-1) - \frac{3(\frac{3}{2})+1(0)+0(\frac{3}{2})}{(\frac{3}{2})^2+0^2+(\frac{3}{2})^2}(\frac{3}{2}x^2+\frac{3}{2}) = x+3+\frac{3}{2}(x^2-1) - (\frac{3}{2}x^2+\frac{3}{2}) = x+3+\cancel{\frac{3}{2}x^2}-\frac{3}{2}$
$-\cancel{\frac{3}{2}x^2}-\frac{3}{2} = x+\cancel{\frac{6}{2}}-\cancel{\frac{6}{2}} = x$

$u_1 = \frac{1}{\sqrt{1^2+(-1)^2}}(x^2-1) = \frac{1}{\sqrt{2}}(x^2-1) = \boxed{\frac{1}{\sqrt{2}}x^2 - \frac{1}{\sqrt{2}}}$

$u_2 = \frac{1}{\sqrt{(\frac{3}{2})^2+(\frac{3}{2})^2}}\left(\frac{3}{2}x^2+\frac{3}{2}\right) = \frac{1}{\sqrt{\frac{18}{4}}}\left(\frac{3}{2}x^2+\frac{3}{2}\right) = \frac{2}{3\sqrt{2}}\left(\frac{3}{2}x^2+\frac{3}{2}\right) = \boxed{\frac{1}{\sqrt{2}}x^2 + \frac{1}{\sqrt{2}}}$

$u_3 = \frac{1}{\sqrt{1^2}}(x) = \boxed{x}$

Least Squares Problem

- $(2,1)$ $(3,4)$ $(5,9)$ $y = c_0 + c_1 x$

$c_0 + 2c_1 = 1$ $c_0 + 3c_1 = 4$ $c_0 + 5c_1 = 9$

$A\bar{x} = \bar{b}$ where $A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 1 & 5 \end{bmatrix}$ $\bar{x} = \begin{bmatrix} c_0 \\ c_1 \end{bmatrix}$ $\bar{b} = \begin{bmatrix} 1 \\ 4 \\ 9 \end{bmatrix}$

~~eq2 - eq1 = 4 - 1 = 3~~ $eq2 - eq1 = \cancel{c_0} + 3c_1 - 4 - (\cancel{c_0} + 2c_1 - 1) = 0 \Rightarrow c_1 - 3 = 0 \Rightarrow c_1 = 3$

~~eq3 - eq2 =~~ $eq3 - eq2 = \cancel{c_0} + 5c_1 - 9 - (\cancel{c_0} + 3c_1 - 4) = 0 \Rightarrow 2c_1 - 5 = 0 \Rightarrow 2c_1 = 5 \Rightarrow c_1 = \frac{5}{2}$

X contradiction c_1

no $\bar{x}$ such that $A\bar{x} = \bar{b}$

- least squares problem - find $\bar{x}$ such that $A\bar{x}$ is close to $\bar{b}$, $\|A\bar{x} - \bar{b}\|$ minimized

$A\bar{x} = \bar{b}$ $\quad$ $\bar{x}$ minimizes $\|A\bar{x} - \bar{b}\|$ least squares solution of $A\bar{x} = \bar{b}$

$A^T A\bar{x} = A^T\bar{b}$ $\quad$ normal equations for $A\bar{x} = \bar{b}$

ex - $A\bar{x} = \bar{b}$ $\begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 1 & 5 \end{bmatrix}\begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 9 \end{bmatrix}$ $\quad (2,1), (3,4), (5,9)$ solve $A^T A\bar{x} = A^T\bar{b}$

$A^T A\bar{x} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 5 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 1 & 5 \end{bmatrix}\bar{x} = \begin{bmatrix} 1+1+1 & 2+3+5 \\ 2+3+5 & 4+9+25 \end{bmatrix}\bar{x} = \begin{bmatrix} 3 & 10 \\ 10 & 38 \end{bmatrix}\bar{x}$

$A^T\bar{b} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 4 \\ 9 \end{bmatrix} = \begin{bmatrix} 1+4+9 \\ 2+12+45 \end{bmatrix} = \begin{bmatrix} 14 \\ 59 \end{bmatrix}$

$\begin{bmatrix} 3 & 10 \\ 10 & 38 \end{bmatrix}\bar{x} = \begin{bmatrix} 14 \\ 59 \end{bmatrix} \Rightarrow \begin{bmatrix} 3 & 10 & 14 \\ 10 & 38 & 59 \end{bmatrix}$ $-10R_1 + 3R_2 \to R_2 \Rightarrow \begin{bmatrix} 3 & 10 & 14 \\ 0 & 14 & 37 \end{bmatrix}$ $\frac{1}{14}R_2 \to R_2$ ^ $\frac{1}{3}R_1 \to R_1 \Rightarrow$

$\begin{bmatrix} 1 & 10/3 & 14/3 \\ 0 & 1 & 37/14 \end{bmatrix}$ $-\frac{10}{3}R_2 + R_1 \to R_1 \Rightarrow \begin{bmatrix} 1 & 0 & -29/7 \\ 0 & 1 & 37/14 \end{bmatrix}$ $\quad c_0 = -29/7$ $\quad c_1 = 37/14$ $\quad \boxed{y = \frac{-29}{7} + \frac{37}{14}x}$ best fit line

ex - find the solution of the least squares problem $A\bar{x} = \bar{b}$

a) $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 2 \end{bmatrix}$ $\quad \bar{x} = \begin{bmatrix} c_0 \\ c_1 \end{bmatrix}$ $\quad \bar{b} = \begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix}$

$A^T A\bar{x} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 2 \end{bmatrix}\bar{x} = \begin{bmatrix} 0+1+1 & 0+0+2 \\ 0+0+2 & 1+0+4 \end{bmatrix}\bar{x} = \begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix}\bar{x}$

$A^T\bar{b} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix}\begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0-1+3 \\ -1+0+6 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$

$\begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix}\bar{x} = \begin{bmatrix} 2 \\ 5 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 2 & 2 \\ 2 & 5 & 5 \end{bmatrix}$ $R_1 - R_2 \to R_2 \Rightarrow \begin{bmatrix} 2 & 2 & 2 \\ 0 & -3 & -3 \end{bmatrix}$ $\frac{1}{2}R_1 \to R_1$ ^ $-\frac{1}{3}R_2 \to R_2 \Rightarrow$

$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ $R_1 - R_2 \to R_1 \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ $\quad c_0 = 0$ $\quad c_1 = 1$ $\quad \boxed{y = x}$ best fit line

b) $A = \begin{bmatrix} 1 & 2 \\ 1 & -1 \\ -2 & 1 \\ 1 & 0 \end{bmatrix}$ $\quad \bar{x} = \begin{bmatrix} c_0 \\ c_1 \end{bmatrix}$ $\quad \bar{b} = \begin{bmatrix} 1 \\ 4 \\ 0 \\ -1 \end{bmatrix}$

$A^T A\bar{x} = \begin{bmatrix} 1 & 1 & -2 & 1 \\ 2 & -1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 1 & -1 \\ -2 & 1 \\ 1 & 0 \end{bmatrix}\bar{x} = \begin{bmatrix} 1+1+4+1 & 2-1-2+0 \\ 2-1-2+0 & 4+1+1+0 \end{bmatrix}\bar{x} = \begin{bmatrix} 7 & -1 \\ -1 & 6 \end{bmatrix}\bar{x}$

$A^T\bar{b} = \begin{bmatrix} 1 & 1 & -2 & 1 \\ 2 & -1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 4 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1+4+0-1 \\ 2-4+0+0 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$

$\begin{bmatrix} 7 & -1 \\ -1 & 6 \end{bmatrix}\bar{x} = \begin{bmatrix} 4 \\ -2 \end{bmatrix} \Rightarrow \begin{bmatrix} 7 & -1 & 4 \\ -1 & 6 & -2 \end{bmatrix}$ $\frac{1}{7}R_1 \to R_1 \Rightarrow \begin{bmatrix} 1 & -1/7 & 4/7 \\ -1 & 6 & -2 \end{bmatrix}$ $R_1 + R_2 \to R_2 \Rightarrow \begin{bmatrix} 1 & -1/7 & 4/7 \\ 0 & 41/7 & -10/7 \end{bmatrix}$ $\frac{7}{41}R_2 \to R_2 \Rightarrow$

$\begin{bmatrix} 1 & -1/7 & 4/7 \\ 0 & 1 & -10/41 \end{bmatrix}$ $R_1 + \frac{1}{7}R_2 \to R_1 \Rightarrow \begin{bmatrix} 1 & 0 & 22/41 \\ 0 & 1 & -10/41 \end{bmatrix}$ $\quad c_0 = \frac{22}{41}$ $\quad c_1 = -\frac{10}{41}$ $\quad \boxed{y = \frac{22}{41} - \frac{10}{41}x}$ best fit line

$\frac{4}{7} + \left(\frac{-10}{41}\right)\left(\frac{1}{7}\right) = \frac{4}{7} - \frac{10}{287}$

$\frac{164}{287} - \frac{10}{287} = \frac{154}{287}$

ex - find the least squares regression line for the set of points

a) $(-1,1), (1,0), (3,-4)$

$c_0 - 1c_1 = 1 \quad c_0 + c_1 = 0 \quad c_0 + 3c_1 = -4$

$\begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -4 \end{bmatrix}$

$A^TA\bar{x} = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 3 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 3 \end{bmatrix}\bar{x} = \begin{bmatrix} 1+1+1 & -1+1+3 \\ -1+1+3 & 1+1+9 \end{bmatrix}\bar{x} = \begin{bmatrix} 3 & 3 \\ 3 & 11 \end{bmatrix}\bar{x}$

$A^T\bar{b} = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ -4 \end{bmatrix} = \begin{bmatrix} 1+0-4 \\ -1+0-12 \end{bmatrix} = \begin{bmatrix} -3 \\ -13 \end{bmatrix}$

$\begin{bmatrix} 3 & 3 \\ 3 & 11 \end{bmatrix}\bar{x} = \begin{bmatrix} -3 \\ -13 \end{bmatrix} \Rightarrow \begin{bmatrix} 3 & 3 & -3 \\ 3 & 11 & -13 \end{bmatrix}$ $R_1 - R_2 \to R_2 \Rightarrow \begin{bmatrix} 3 & 3 & -3 \\ 0 & -8 & 10 \end{bmatrix}$ $\frac{1}{3}R_1 \to R_1 \wedge -\frac{1}{8}R_2 \to R_2 \Rightarrow$

$\begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & -5/4 \end{bmatrix}$ $R_1 - R_2 \to R_1 \Rightarrow \begin{bmatrix} 1 & 0 & 1/4 \\ 0 & 1 & -5/4 \end{bmatrix}$ $c_0 = \frac{1}{4} \quad c_1 = -\frac{5}{4}$ $\boxed{y = \frac{1}{4} - \frac{5}{4}x}$ best fit line

b) $(-1,1), (1,0), (3,-4), (2,-1)$

$c_0 - c_1 = 1 \quad c_0 + c_1 = 0 \quad c_0 + 3c_1 = -4 \quad c_0 + 2c_1 = -1$ $\begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -4 \\ -1 \end{bmatrix}$

$A^TA\bar{x} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 3 & 2 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 3 \\ 1 & 2 \end{bmatrix}\bar{x} = \begin{bmatrix} 1+1+1+1 & -1+1+3+2 \\ -1+1+3+2 & 1+1+9+4 \end{bmatrix}\bar{x} = \begin{bmatrix} 4 & 5 \\ 5 & 15 \end{bmatrix}\bar{x}$

$A^Tb = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 3 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ -4 \\ -1 \end{bmatrix} = \begin{bmatrix} 1+0-4-1 \\ -1+0-12-2 \end{bmatrix} = \begin{bmatrix} -4 \\ -15 \end{bmatrix}$

$\begin{bmatrix} 4 & 5 \\ 5 & 15 \end{bmatrix}\bar{x} = \begin{bmatrix} -4 \\ -15 \end{bmatrix} \Rightarrow \begin{bmatrix} 4 & 5 & -4 \\ 5 & 15 & -15 \end{bmatrix}$ $R_1 \leftrightarrow R_2 \wedge R_1 - R_2 \to R_1 \Rightarrow \begin{bmatrix} 1 & 10 & -11 \\ 4 & 5 & -4 \end{bmatrix}$ $-4R_1 + R_2 \to R_2 \Rightarrow$

$\begin{bmatrix} 1 & 10 & -11 \\ 0 & -35 & 40 \end{bmatrix}$ $-\frac{1}{35}R_2 \to R_2 \Rightarrow \begin{bmatrix} 1 & 10 & -11 \\ 0 & 1 & -8/7 \end{bmatrix}$ $R_1 - 10R_2 \to R_1 \Rightarrow \begin{bmatrix} 1 & 0 & 3/7 \\ 0 & 1 & -8/7 \end{bmatrix}$ $-11 - 10(-\frac{8}{7}) = -\frac{77}{7} + \frac{80}{7} = \frac{3}{7}$

$c_0 = \frac{3}{7} \quad c_1 = -\frac{8}{7}$ $\boxed{y = \frac{3}{7} - \frac{8}{7}x}$ best fit line

ex - find the least squares quadratic polynomial for the set of points

$(-1,1), (0,0), (1,-1), (2,2)$

$c_0 - c_1 + c_2 = 1 \quad c_0 = 0 \quad c_0 + c_1 + c_2 = -1 \quad c_0 + 2c_1 + 4c_2 = 2$ $\begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix}\begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \end{bmatrix}$

$A^TA\bar{x} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 \\ 1 & 0 & 1 & 4 \end{bmatrix}\begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix}\bar{x} = \begin{bmatrix} 1+1+1+1 & -1+0+1+2 & 1+0+1+4 \\ -1+0+1+2 & 1+0+1+4 & -1+0+1+8 \\ 1+0+1+4 & -1+0+1+8 & 1+0+1+16 \end{bmatrix}\bar{x} = \begin{bmatrix} 4 & 2 & 6 \\ 2 & 6 & 8 \\ 6 & 8 & 18 \end{bmatrix}\bar{x}$

$A^Tb = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 \\ 1 & 0 & 1 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1+0-1+2 \\ -1+0-1+4 \\ 1+0-1+8 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 8 \end{bmatrix}$

$\begin{bmatrix} 4 & 2 & 6 \\ 2 & 6 & 8 \\ 6 & 8 & 18 \end{bmatrix}\bar{x} = \begin{bmatrix} 2 \\ 2 \\ 8 \end{bmatrix} \Rightarrow \begin{bmatrix} 4 & 2 & 6 & 2 \\ 2 & 6 & 8 & 2 \\ 6 & 8 & 18 & 8 \end{bmatrix}$ $R_1 - 2R_2 \to R_2 \Rightarrow \begin{bmatrix} 4 & 2 & 6 & 2 \\ 0 & -10 & -10 & -2 \\ 6 & 8 & 18 & 8 \end{bmatrix}$ $3R_1 - 2R_3 \to R_3 \Rightarrow \begin{bmatrix} 4 & 2 & 6 & 2 \\ 0 & -10 & -10 & -2 \\ 0 & -10 & -18 & -10 \end{bmatrix}$ $R_2 - R_3 \to R_3 \Rightarrow$

$\begin{bmatrix} 4 & 2 & 6 & 2 \\ 0 & -10 & -10 & -2 \\ 0 & 0 & 8 & 8 \end{bmatrix}$ $\frac{1}{4}R_1 \to R_1 \wedge -\frac{1}{10}R_2 \to R_2 \wedge \frac{1}{8}R_3 \to R_3 \Rightarrow \begin{bmatrix} 1 & 1/2 & 3/2 & 1/2 \\ 0 & 1 & 1 & 1/5 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ $R_1 - \frac{1}{2}R_2 \to R_1 \Rightarrow \begin{bmatrix} 1 & 0 & 1 & 2/5 \\ 0 & 1 & 1 & 1/5 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ $R_2 - R_3 \to R_2 \Rightarrow$

$\begin{bmatrix} 1 & 0 & 1 & 3/5 \\ 0 & 1 & 0 & -4/5 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ $R_1 - R_3 \to R_1 \Rightarrow \begin{bmatrix} 1 & 0 & 0 & -2/5 \\ 0 & 1 & 0 & -6/5 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

$c_0 = -\frac{2}{5} \quad c_1 = -\frac{6}{5} \quad c_2 = 1$ $\boxed{y = -\frac{2}{5} - \frac{6}{5}x_1 + x_2}$ best fit line

Linear Transformations

- linear transformation $T: V \to W$
  1. $T(\bar{u} + \bar{v}) = T(\bar{u}) + T(\bar{v}) \quad \forall\, \bar{u}, \bar{v} \in V$
  2. $T(c\bar{u}) = cT(\bar{u}) \quad \forall\, \bar{u} \in V \;\; \forall\, c \in \mathbb{R}$

ex - $T: \mathbb{R}^2 \to \mathbb{R}^2$ $T(v_1, v_2) = (-v_1 + 3v_2, 4v_1 - 2v_2)$ show that T is a linear transformation

① $T(\bar{u}+\bar{v}) = T(u_1+v_1, u_2+v_2)$ b/c $\bar{u} = (u_1, u_2) \wedge \bar{v} = (v_1, v_2)$

$= (-(u_1+v_1)+3(u_2+v_2), 4(u_1+v_1)-2(u_2+v_2)) = (-u_1+3u_2-v_1+3v_2, 4u_1-2u_2+4v_1-2v_2) =$

$(-u_1+3u_2, 4u_1-2u_2) + (-v_1+3v_2, 4v_1-2v_2) = T(\bar{u}) + T(\bar{v})$ ✓

② show that $T(c\bar{u}) = cT(\bar{u})$

$T(cu_1, cu_2) = (-(cu_1)+3(cu_2), 4(cu_1)-2(cu_2)) = (c(-u_1+3u_2), c(4u_1-2u_2)) =$

$c(-u_1+3u_2, 4u_1-2u_2) = cT(\bar{u})$ ✓

ex - $T: \mathbb{R}^2 \to \mathbb{R}^2$ $T(v_1, v_2) = (v_1+v_2, 3)$ show that T is not a linear transformation

① $T(\bar{u}+\bar{v}) = T(u_1+v_1, u_2+v_2) = (u_1+v_1+u_2+v_2, 3)$

$T(\bar{u})+T(\bar{v}) = (u_1+u_2, 3) + (v_1+v_2, 3) = (u_1+u_2+v_1+v_2, 6)$

$T(\bar{u}+\bar{v}) \neq T(\bar{u}) + T(\bar{v})$ ✗

② $T(c\bar{u}) = T(cu_1, cu_2) = (cu_1+cu_2, 3)$

$cT(\bar{u}) = c(u_1+u_2, 3) = (cu_1+cu_2, 3c)$

$T(c\bar{u}) \neq cT(\bar{u})$ ✗

Linear transformations + Matrices

ex - $T: \mathbb{R}^2 \to \mathbb{R}^2$ $T(v_1, v_2) = (-v_1+3v_2, 4v_1-2v_2)$ ⇒ augmented matrix (?)

$T(\bar{v}) = A\bar{v} = \begin{bmatrix} -1 & 3 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

$T: \mathbb{R}^n \to \mathbb{R}^m$ $T(\bar{v}) = A\bar{v}$ is a linear transformation

① $T(\bar{u}+\bar{v}) = A(\bar{u}+\bar{v}) = A\bar{u} + A\bar{v} = T(\bar{u}) + T(\bar{v})$ ✓

② $T(c\bar{u}) = A(c\bar{u}) = cA\bar{u} = cT(\bar{u})$ ✓

ex - $T: \mathbb{R}^2 \to \mathbb{R}^2$ $T(\bar{v}) = A\bar{v}$ where $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ counter clockwise rotation by θ degrees

find $T(4,1)$ if $\theta = 60°$

$A = \begin{bmatrix} \cos 60° & -\sin 60° \\ \sin 60° & \cos 60° \end{bmatrix} = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}$

$T(4,1) = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \end{bmatrix} = \boxed{\begin{bmatrix} 2 & -\sqrt{3}/2 \\ 2\sqrt{3} & 1/2 \end{bmatrix}}$

ex - determine if T is a linear transformation

a) $T: \mathbb{R}^3 \to \mathbb{R}^3$ is defined by $T(x,y,z) = (x-y, x+z, -y)$

① $T(\bar{u}+\bar{v}) = T(u_1+v_1, u_2+v_2, u_3+v_3)$

$= (u_1+v_1-(u_2+v_2), u_1+v_1+u_3+v_3, -(u_2+v_2)) = (u_1-u_2+v_1-v_2, u_1+u_3+v_1+v_3, -u_2-v_2) =$

$(u_1-u_2, u_1+u_3, -u_2) + (v_1-v_2, v_1+v_3, -v_2) = T(\bar{u}) + T(\bar{v})$ ✓

② $T(cu_1, cu_2, cu_3) = (cu_1-cu_2, cu_1+cu_3, -cu_2) = c(u_1-u_2, u_1+u_3, -u_2) = cT(\bar{u})$ ✓

b) $T: \mathbb{R}^2 \to \mathbb{R}^3$ is defined by $T(x,y,z) = (x+y, 3, xy)$

① $T(\bar{u}+\bar{v}) = T(u_1+v_1, u_2+v_2, u_3+v_3)$

$= (u_1+v_1+u_2+v_2, 3, (u_1+v_1)(u_2+v_2))$

$T(\bar{u})+T(\bar{v}) = (u_1+u_2, 3, u_1u_2) + (v_1+v_2, 3, v_1v_2) = (u_1+v_1+u_2+v_2, 6, u_1u_2+v_1v_2)$ ✗

c) $T: M_{2,2} \to M_{2,2}$ is defined by $T(A) = 3A$

① $T(A+B) = 3(A+B) = 3A + 3B = T(A) + T(B)$ ✓

② $T(cA) = 3(cA) = c(3A) = cT(A)$ ✓

d) $T: M_{2,2} \to M_{2,2}$ is defined by $T(A) = A^2$

① $T(A+B) = (A+B)^2 = A^2 + 2AB + B^2$

$T(A) + T(B) = A^2 + B^2$ X

Kernel - $T: V \to W$ linear transformation kernel of T $\vec{v} \in V$ such that $T(\vec{v}) = \vec{0}$

$\vec{0} \in V$ ker(T) not empty

ex - $T: \mathbb{R}^3 \to \mathbb{R}^3$ $T(\vec{x}) = A\vec{x}$ where $A = \begin{bmatrix} 1 & 3 & 5 \\ -1 & 3 & 1 \\ 0 & 2 & 2 \end{bmatrix}$ find the kernel of T ($\vec{x} \in \mathbb{R}^3$ such that $A\vec{x} = \vec{0}$)

$\begin{bmatrix} 1 & 3 & 5 \\ -1 & 3 & 1 \\ 0 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $R_1 + R_2 \to R_2 \Rightarrow \begin{bmatrix} 1 & 3 & 5 & 0 \\ 0 & 6 & 6 & 0 \\ 0 & 2 & 2 & 0 \end{bmatrix}$ $R_2 - 3R_3 \to R_3 \Rightarrow \begin{bmatrix} 1 & 3 & 5 & 0 \\ 0 & 6 & 6 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $\frac{1}{6}R_2 \to R_2 \Rightarrow$

$\begin{bmatrix} 1 & 3 & 5 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ let $x_3 = t \Rightarrow$

$x_2 + t = 0 \Rightarrow x_2 = -t$

$x_1 + 3(-t) + 5(t) = 0 \Rightarrow x_1 + 2t = 0 \Rightarrow x_1 = -2t$

so $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2t \\ -t \\ t \end{bmatrix} = t\begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$ $\boxed{\ker(T) = \left\{ t\begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} \middle| t \in \mathbb{R} \right\}}$

ex - $T: \mathbb{R}^2 \to \mathbb{R}^3$ $T(\vec{x}) = A\vec{x}$ where $A = \begin{bmatrix} 1 & -1 \\ -5 & 6 \\ 2 & 0 \end{bmatrix}$ find ker(T) ($\vec{x} \in \mathbb{R}^2$ such that $A\vec{x} = \vec{0}$)

$\begin{bmatrix} 1 & -1 \\ -5 & 6 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 & 0 \\ -5 & 6 & 0 \\ 2 & 0 & 0 \end{bmatrix}$ $5R_1 + R_2 \to R_2 \Rightarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 0 \end{bmatrix}$ $-2R_1 + R_3 \to R_3 \Rightarrow$

$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 0 \end{bmatrix}$ $-2R_2 + R_3 \to R_3 \Rightarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $x_2 = 0$

$x_1 - (0) = 0 \Rightarrow x_1 = 0$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \ker(T) = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$

Kernel of T as Subspace of V ex - $T: \mathbb{R}^4 \to \mathbb{R}^3$ $T(\vec{x}) = A\vec{x}$ $A = \begin{bmatrix} 1 & -3 & -2 & 4 \\ 2 & 1 & -4 & 1 \end{bmatrix}$ find a basis for ker(T)

$\begin{bmatrix} 1 & -3 & -2 & 4 & 0 \\ 2 & 1 & -4 & 1 & 0 \end{bmatrix}$ $-2R_1 + R_2 \to R_2 \Rightarrow \begin{bmatrix} 1 & -3 & -2 & 4 & 0 \\ 0 & 7 & 0 & -7 & 0 \end{bmatrix}$ $\frac{1}{7}R_2 \to R_2 \Rightarrow \begin{bmatrix} 1 & -3 & -2 & 4 & 0 \\ 0 & 1 & 0 & -1 & 0 \end{bmatrix}$ let $x_4 = t$

$x_2 - (t) = 0 \Rightarrow x_2 = t$

$x_1 - 3(t) - 2x_3 + 4(t) = 0$ let $x_3 = s \Rightarrow x_1 + t - 2s = 0 \Rightarrow x_1 = 2s - t$

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2s-t \\ t \\ s \\ t \end{bmatrix} = \begin{bmatrix} 2s-t \\ 0s+t \\ s+0t \\ 0s+t \end{bmatrix} = s\begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$ $\ker(T) = \boxed{\text{span}\left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}}$

Range - ker(T) subspace of V, range(T) subspace of W $T: V \to W$

$w \in W$ $w = T(\vec{v})$ for some $\vec{v} \in V$. $T(\vec{x}) = A\vec{x}$, $\vec{b}$ such that $A\vec{x} = \vec{b}$ for some $\vec{x} \in V$

$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$ range of T = column space of A $x_1\begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix} + \ldots + x_n\begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix}$

Basis for Range(T) ex - $T: \mathbb{R}^3 \to \mathbb{R}^3$ $T(\vec{x}) = A\vec{x}$ $A = \begin{bmatrix} 1 & 3 & 5 \\ -1 & 3 & 1 \\ 0 & 2 & 2 \end{bmatrix}$ find a basis for range(T)

$A \to \text{reduce} \to \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ pivot position, pivot column

$\dim(\text{range}(T)) = \boxed{2}$

Rank ^ Nullity

- rank of T rank(T)

rank(T) = 2 $\ker(T) = \{t\begin{bmatrix}-2\\-1\\1\end{bmatrix} \mid t \in \mathbb{R}\}$

basis for ker(T)

- nullity of T nullity(T)

nullity(T) = 1 $\Rightarrow$ rank(T) + nullity(T) = 2 + 1 = 3 = dim(V)

- if $T: V \to W$ is a linear transformation and dim(V) = n, then rank(T) + nullity(T) = n ^

dim(range(T)) + dim(ker(T)) = dim(dom(T))

show

ex - $T: \mathbb{R}^4 \to \mathbb{R}^3$ $T(\bar{x}) = A\bar{x}$ $A = \begin{bmatrix} 1 & -3 & -2 & 4 \\ 2 & 1 & -4 & 1 \end{bmatrix}$

$A \to$ reduce $\to \begin{bmatrix} 1 & -3 & -2 & 4 \\ 0 & 1 & 0 & -1 \end{bmatrix}$

first 2 columns of A $\Rightarrow \{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}-3\\1\end{bmatrix}\}$ basis for range(T) $\Rightarrow$ dim(range(T)) = 2

from other problem, $\ker(T) = \{\begin{bmatrix}2\\0\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\1\\0\\1\end{bmatrix}\}$ $\Rightarrow$ dom(T) = $\mathbb{R}^4$ $\Rightarrow$ dim(dom(T)) = 4

dim(range(T)) + dim(ker(T)) = dim(dom(T)) $\Rightarrow$ 2 + 2 = 4 ✓

One-to-One ^ Onto Properties

- $T: V \to W$ properties:

① one-to-one: if $T(\bar{u}) = T(\bar{v})$, then $\bar{u} = \bar{v}$

② onto: for each $\bar{w} \in W$, there is a $\bar{v} \in V$ such that $T(\bar{v}) = \bar{w}$

ex - $T: \mathbb{R}^2 \to P_2$ $T(v_1, v_2) = p$ where $p(x) = v_1 + v_2x$ show that T is one-to-one but not onto

① $T(\bar{u}) = T(\bar{v}) \Rightarrow u_1 + u_2x = v_1 + v_2x \Rightarrow u_1 = v_1$ ^ $u_2 = v_2 \Rightarrow (u_1, u_2) = (v_1, v_2) \Rightarrow \bar{u} = \bar{v}$

② let $w(x) = 1 + x + x^2$, then there is no $\bar{v} \in \mathbb{R}^2$ such that $T(\bar{v}) = w$; suppose there is though

then $T(\bar{v}) = v_1 + v_2x = 1 + x + x^2$ X so T is not onto

ex - $T: \mathbb{R}^2 \to P_0$ $T(v_1, v_2) = p$ where $p(x) = v_1$ show that T is onto but not one-to-one

② let $w \in P_0$ then $w(x) = a_0$ for some $a_0 \in \mathbb{R}$

let $\bar{v} = (a_0, 0)$ ^ $T(\bar{v}) = w$ $\bar{v} \in \mathbb{R}^2$ $T(\bar{v}) = p$ $p(x) = a_0$ $w(x) = a_0$ T is onto

① $T(1,0) = T(1,1)$ $T(1,0) = p$ where $p(x) = 1$ but $(1,0) \neq (1,1)$ ^ $T(1,1) = q$ $q(x) = 1$

ex - $T: \mathbb{R}^2 \to P_1$ $T(v_1, v_2) = p$ where $p(x) = v_1 + v_2x$ show that T is one-to-one and onto

① $T(\bar{u}) = T(\bar{v}) \Rightarrow u_1 + u_2x = v_1 + v_2x \Rightarrow u_1 = v_1$ ^ $u_2 = v_2 \Rightarrow (u_1, u_2) = (v_1, v_2) \Rightarrow \bar{u} = \bar{v}$ ✓

② let $w \in P_1$ $w(x) = a_0 + a_1x$ $a_0, a_1 \in \mathbb{R}$

let $\bar{v} = (a_0, a_1)$ $\bar{v} \in \mathbb{R}^2$ ^ $T(\bar{v}) = a_0 + a_1x = w(x) \Rightarrow T(\bar{v}) = w \Rightarrow$ T is onto

Isomorphisms

- $T: V \to W$ V, W are isomorphic

$\mathbb{R}^2$ isomorphic to P_1 $\{(1,0), (0,1)\}$ $\{1, x\}$ $\dim(\mathbb{R}^2) = \dim(P_1) = 2$

- two finite-dimensional vector spaces are isomorphic if they are of the same dimension

P_0 $\dim(P_0) = 1$ P_2 $\dim(P_2) = 3$

ex - find the kernel of the linear transformation T then find a basis for ker(T)

a) $T: \mathbb{R}^3 \to \mathbb{R}^3$ is defined by $T(\bar{x}) = A\bar{x}$ where $A = \begin{bmatrix} 2 & 0 & -2 \\ 1 & 5 & 4 \\ 1 & -6 & -7 \end{bmatrix}$

augmented matrix $\Rightarrow \begin{bmatrix} 2 & 0 & -2 & 0 \\ 1 & 5 & 4 & 0 \\ 1 & -6 & -7 & 0 \end{bmatrix}$ $\frac{1}{2}R_1 \to R_1 \wedge R_2 - R_3 \to R_3 \Rightarrow \begin{bmatrix} 1 & 0 & -1 & 0 \\ 1 & 5 & 4 & 0 \\ 0 & 11 & 11 & 0 \end{bmatrix}$ $R_1 - R_2 \to R_2 \Rightarrow$

$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & -5 & -5 & 0 \\ 0 & 11 & 11 & 0 \end{bmatrix}$ $-\frac{1}{5}R_2 \to R_2 \Rightarrow \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 11 & 11 & 0 \end{bmatrix}$ $11R_2 - R_3 \to R_3 \Rightarrow \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ let $x_3 = t$

$x_2 + (t) = 0 \Rightarrow x_2 = -t$

$x_1 - (t) = 0 \Rightarrow x_1 = t$

basis for ker(T)

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ -t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ so $\ker(T) = \boxed{\left\{ t\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \,\middle|\, t \in \mathbb{R} \right\}}$

b) $T: \mathbb{R}^2 \to \mathbb{R}^3$ is defined by $T(\bar{x}) = A\bar{x}$ where $A = \begin{bmatrix} 1 & 2 \\ -1 & -2 \\ 1 & 1 \end{bmatrix}$

augmented matrix $\Rightarrow \begin{bmatrix} 1 & 2 & 0 \\ -1 & -2 & 0 \\ 1 & 1 & 0 \end{bmatrix}$ $R_1 + R_2 \to R_2 \Rightarrow \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$ $R_1 - R_3 \to R_3 \Rightarrow \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

$x_2 = 0$

$x_1 + 2(0) = 0 \Rightarrow x_1 = 0$

basis for ker(T)

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\ker(T) = \boxed{\left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}}$

ex - find the range of linear transformation T. Then find a basis for range(T)

a) $T: \mathbb{R}^3 \to \mathbb{R}^2$ is defined by $T(\bar{x}) = A\bar{x}$ where $A = \begin{bmatrix} 0 & -2 & 3 \\ 4 & 0 & 11 \end{bmatrix}$

$\begin{bmatrix} 0 & -2 & 3 \\ 4 & 0 & 11 \end{bmatrix}$ $R_1 \leftrightarrow R_2 \wedge \frac{1}{4}R_1 \to R_1 \wedge -\frac{1}{2}R_2 \to R_2 \Rightarrow \begin{bmatrix} 1 & 0 & 11/4 \\ 0 & 1 & -3/2 \end{bmatrix}$ first two columns are pivot columns

$\text{range}(T) = \text{span}\boxed{\left\{ \begin{bmatrix} 0 \\ 4 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \end{bmatrix} \right\}}$ $\leftarrow$ basis for range(T)

b) $T: \mathbb{R}^3 \to \mathbb{R}^3$ is defined by $T(\bar{x}) = A\bar{x}$ where $A = \begin{bmatrix} -1 & -2 & 1 \\ 0 & 0 & 0 \\ 1 & 2 & -1 \end{bmatrix}$

$\begin{bmatrix} -1 & -2 & 1 \\ 0 & 0 & 0 \\ 1 & 2 & -1 \end{bmatrix}$ $R_2 \leftrightarrow R_3 \wedge R_1 + R_2 \to R_2$ $\begin{bmatrix} -1 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $-R_1 \to R_1 \Rightarrow \begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ first column is pivot column

$\text{range}(T) = \text{span}\boxed{\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}}$ $\leftarrow$ basis for range(T)

ex - find ker(T), nullity(T), range(T), and rank(T) for the linear transformation T. then show that rank(T) + nullity(T) = dim(dom(T)).

a) $T: \mathbb{R}^3 \to \mathbb{R}^2$ is defined by $T(\bar{x}) = A\bar{x}$ where $A = \begin{bmatrix} 0 & -2 & 3 \\ 4 & 0 & 11 \end{bmatrix}$

From previous problem, $\text{range}(T) = \boxed{\text{span}\left\{ \begin{bmatrix} 0 \\ 4 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \end{bmatrix} \right\}} \Rightarrow \text{rank}(T) = \boxed{2}$

$\begin{bmatrix} 0 & -2 & 3 & 0 \\ 4 & 0 & 11 & 0 \end{bmatrix}$ $R_1 \leftrightarrow R_2 \wedge \frac{1}{4}R_1 \to R_1 \wedge -\frac{1}{2}R_2 \to R_2 \Rightarrow \begin{bmatrix} 1 & 0 & 11/4 & 0 \\ 0 & 1 & -3/2 & 0 \end{bmatrix}$ let $x_3 = t$

$x_2 - \frac{3}{2}(t) = 0 \Rightarrow x_2 = \frac{3}{2}t$

$x_1 + \frac{11}{4}(t) = 0 \Rightarrow x_1 = -\frac{11}{4}t$

basis for ker(T)

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\frac{11}{4}t \\ \frac{3}{2}t \\ t \end{bmatrix} = t\begin{bmatrix} -11/4 \\ 3/2 \\ 1 \end{bmatrix}$ so $\ker(T) = \boxed{\left\{ t\begin{bmatrix} -11/4 \\ 3/2 \\ 1 \end{bmatrix} \,\middle|\, t \in \mathbb{R} \right\}}$ $\Rightarrow \text{nullity}(T) = \boxed{1}$

$\text{rank}(T) + \text{nullity}(T) = 2 + 1 = 3 = \dim(\text{dom}(T))$

b) $T: \mathbb{R}^3 \to \mathbb{R}^3$ is defined by $T(\bar{x}) = A\bar{x}$ where $A = \begin{bmatrix} -1 & -2 & 1 \\ 0 & 0 & 0 \\ 1 & 2 & -1 \end{bmatrix}$

from previous problem, $\text{range}(T) = \boxed{\text{span}\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}} \Rightarrow \text{rank}(T) = \boxed{1}$

$\begin{bmatrix} -1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 2 & -1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ let $x_3 = t \wedge x_2 = s$

$x_1 + 2s - t = 0 \Rightarrow x_1 = t - 2s$

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2s+t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ $\ker(T) = \left\{ \overbrace{s\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}^{\text{basis}} \,\middle|\, s,t \in \mathbb{R} \right\} \Rightarrow \text{nullity}(T) = \boxed{2}$

$\text{rank}(T) + \text{nullity}(T) = 1 + 2 = 3 = \dim(\text{dom}(T))$

ex – $T: \mathbb{R}^3 \to M_{2,2}$ is defined by $T(v_1, v_2, v_3) = \begin{bmatrix} v_1 & v_2 \\ v_3 & 0 \end{bmatrix}$ show that T is one-to-one but not onto

① suppose $T(\bar{u}) = T(\bar{v})$ where (u_1, u_2, u_3) ^ (v_1, v_2, v_3)

$\begin{bmatrix} u_1 & u_2 \\ u_3 & 0 \end{bmatrix} = \begin{bmatrix} v_1 & v_2 \\ v_3 & 0 \end{bmatrix} \Rightarrow u_1 = v_1$ ^ $u_2 = v_2$ ^ $u_3 = v_3$ so $\bar{u} = \bar{v}$ ^ T is one-to one ✓

② let $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \in M_{2,2}$ there is no $\bar{v} \in \mathbb{R}^3$ such that $T(\bar{v}) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ b/c $0 = 1$ X contradiction

ex – let $T: M_{2,2} \to P_2$ be defined by $T\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right) = p$ where $p(x) = a_{11} + a_{12}x + a_{21}x^2$

show that T is not one-to-one but is onto

① let $p \in P_2$ then $p(x) = a_0 + a_1 x + a_2 x^2$ for some $a_0, a_1, a_2 \in \mathbb{R}$

then $T\left(\begin{bmatrix} a_0 & a_1 \\ a_2 & 0 \end{bmatrix}\right) = q$ where $q(x) = a_0 + a_1 x + a_2 x^2 \Rightarrow q = p \Rightarrow T$ is onto ✓

② note that $T\left(\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\right) = p$ ^ $T\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right) = q$ where $p(x) = q(x) = 1 + x + x^2$

$p = q \Rightarrow T\left(\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\right) = T\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right) \Rightarrow$ X contradiction $0 = 1$

ex – show that $M_{2,2}$ is isomorphic to $\mathbb{R}^4$

$T\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right) = (a_{11}, a_{12}, a_{21}, a_{22})$

① $T\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right) = T\left(\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}\right) \Rightarrow a_{11} = b_{11}$ ^ $a_{12} = b_{12}$ ^ $a_{21} = b_{21}$ ^ $a_{22} = b_{22}$ ✓

② " ^ $(a_1, a_2, a_3, a_4) \in \mathbb{R}^4$ ✓

because T is one-to-one ^ onto, T is also an isomorphism

ex – show that $M_{2,2}$ is isomorphic to P_3

$T\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right) = p$ where $p(x) = a_{11} + a_{12}x + a_{21}x^2 + a_{22}x^3$

① $T\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right) = T\left(\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}\right) \Rightarrow a_{11} + a_{12}x + a_{21}x^2 + a_{22}x^3 = b_{11} + b_{12}x + b_{21}x^2 + b_{22}x^3 \Rightarrow a_{11} = b_{11}$ ^

$a_{12} = b_{12}$ ^ $a_{21} = b_{21}$ ^ $a_{22} = b_{22}$ ✓

② let $p \in P_3$ then $p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$ where $a_0, a_1, a_2, a_3 \in \mathbb{R}$

then $T\left(\begin{bmatrix} a_0 & a_1 \\ a_2 & a_3 \end{bmatrix}\right) = p$ ✓

because T is one-to-one ^ onto, T is also isomorphic

Matrix Representations

– $T: \mathbb{R}^n \to \mathbb{R}^n$ $T(\bar{v}) = A\bar{v}$ linear transformation

– If $T: \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then there is a matrix A such that $T(\bar{v}) = A\bar{v}$

– $\bar{e}_1, \dots, \bar{e}_n$ basis vectors for $\mathbb{R}^n \Rightarrow [T(\bar{e}_1) \vdots T(\bar{e}_2) \vdots \cdots \vdots T(\bar{e}_n)]$ standard matrix for T

ex – $T: \mathbb{R}^2 \to \mathbb{R}^3$ $T(x,y) = (x+y, x-y, 3x+4y)$ find the standard matrix for T

$\bar{e}_1 = (1,0)$ $\bar{e}_2 = (0,1)$ $\Rightarrow T(\bar{e}_1) = (1,1,3)$ ^ $T(\bar{e}_2) = (1,-1,4)$

form $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 3 & 4 \end{bmatrix}$ $T(5,-2) = A\begin{bmatrix} 5 \\ -2 \end{bmatrix}$ $T(5,-2) = (3,7,7)$ ← plug in

or multiply

$A\begin{bmatrix} 5 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 5 \\ -2 \end{bmatrix} = \begin{bmatrix} 5-2 \\ 5+2 \\ 15-8 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \\ 7 \end{bmatrix}$ ✓

- V & W finite-dimensional vector spaces
- if $T: V \to W$ is a linear transformation, then there is a matrix A such that A tells us what the image of $\bar{v} \in V$ under T looks like
- let $B = \{\bar{v}_1, \dots \bar{v}_n\}$ basis for V. B' a basis for W.
- let $\left[[T(\bar{v}_1)]_{B'} \vdots \dots [T(\bar{v}_n)]_{B'} \right]$ be the columns of A so it is the matrix of T relative to the bases B and B'

ex - $T: \mathbb{R}^2 \to \mathbb{R}^3$ $T(x,y) = (x-3y, x+y, x-y)$ $B = \{(1,2), (1,1)\}$ $B' = \{(2,0,1), (0,2,1), (1,2,1)\}$

find the matrix of T relative to the bases B and B'

$T(1,2) = (1-6, 3, 1-2) = (-5, 3, -1)$ $T(1,1) = (1-3, 1+1, 1-1) = (-2, 2, 0)$

$B_0 = \{(1,0,0), (0,1,0), (0,0,1)\}$ $[T(1,2)]_{B'}$ & $[T(1,1)]_{B'}$

P from B_0 to B' $[B' \vdots B_0] \to [I \vdots P]$

$\left[\begin{array}{ccc|ccc} 2 & 0 & 1 & 1 & 0 & 0 \\ 0 & 2 & 2 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$ $\frac{1}{2}R_1 \to R_1$ & $\frac{1}{2}R_2 \to R_2 \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 1 & 0 & \frac{1}{2} & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$ $-R_3 + R_1 \to R_3 \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 1 & 0 & \frac{1}{2} & 0 \\ 0 & -1 & -\frac{1}{2} & \frac{1}{2} & 0 & -1 \end{array}\right]$ $R_2 + R_3 \to R_3 \Rightarrow$

$\left[\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 1 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -1 \end{array}\right]$ $2R_3 \to R_3 \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 1 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 1 & 1 & -2 \end{array}\right]$ $R_2 - R_3 \to R_2 \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & -1 & -\frac{1}{2} & 2 \\ 0 & 0 & 1 & 1 & 1 & -2 \end{array}\right]$ $R_1 - \frac{1}{2}R_3 \to R_1 \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -\frac{1}{2} & 1 \\ 0 & 1 & 0 & -1 & -\frac{1}{2} & 2 \\ 0 & 0 & 1 & 1 & 1 & -2 \end{array}\right]$

$= P$

$[T(1,2)]_{B'} = P[T(1,2)]_{B_0} = \begin{bmatrix} 0 & -\frac{1}{2} & 1 \\ -1 & -\frac{1}{2} & 2 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} -5 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} -\frac{5}{2} \\ \frac{3}{2} \\ 0 \end{bmatrix}$

$[T(1,1)]_{B'} = P[T(1,1)]_{B_0} =$ " $\begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

$\Rightarrow A = \begin{bmatrix} -\frac{5}{2} & -1 \\ \frac{3}{2} & 1 \\ 0 & 0 \end{bmatrix}$

$[\bar{v}]_B = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$ $[T(\bar{v})]_{B'} = A[\bar{v}]_B$ so $A[\bar{v}]_B = \begin{bmatrix} -\frac{5}{2} & -1 \\ \frac{3}{2} & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ \frac{3}{2} \\ 0 \end{bmatrix}$

$T(\bar{v}) = -\frac{1}{2}(2,0,1) + \frac{3}{2}(0,2,1) + \cancel{0(1,2,1)} = (-1, 0, -\frac{1}{2}) + (0, 3, \frac{3}{2}) = \boxed{(-1, 3, 1)}$

$\bar{v} = -1(1,2) + 3(1,1) = (-1,-2) + (3,3) = (2,1)$

check work: $T(\bar{v}) = T(2,1) = (2-3, 2+1, 2-1) = (-1, 3, 1)$ ✓

ex - $T: \mathbb{R}^2 \to P_1$ $T(v_1, v_2) = p$ $p(x) = v_1 + v_2 x$ find the matrix of T relative to the bases

$B = \{(1,0), (0,1)\}$ $B' = \{1, x\}$ $\Rightarrow$ $T(1,0) = 1$ $T(0,1) = x$

$[T(1,2)]_{B'} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $[T(0,1)]_{B'} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ so $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ✓

$[T(\bar{v})]_{B'} = A[\bar{v}]_B$ test for $\bar{v} = (3,-8) \Rightarrow [\bar{v}]_B = \begin{bmatrix} 3 \\ -8 \end{bmatrix} \Rightarrow A[\bar{v}]_B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ -8 \end{bmatrix} = \begin{bmatrix} 3 \\ -8 \end{bmatrix}$

$T(\bar{v}) = T(3,-8) = 3 - 8x = 3(1) + (-8)(x)$ $[T(\bar{v})]_{B'} = \begin{bmatrix} 3 \\ -8 \end{bmatrix}$ ✓

ex - $T: P_2 \to P_1$ $T(p) = p'$ find the matrix of T relative to the bases $B = \{1, x, x^2\}$ & $B' = \{1, x\}$

$T(1) = 0$ $T(x) = 1$ $T(x^2) = 2x$

$[T(1)]_{B'} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $[T(x)]_{B'} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $[T(x^2)]_{B'} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$ $\Rightarrow A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ let $p(x) = a_0 + a_1 x + a_2 x^2$

$[p]_B = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$ $A[p]_B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 + a_1 + 0 \\ 0 + 0 + 2a_2 \end{bmatrix} = \begin{bmatrix} a_1 \\ 2a_2 \end{bmatrix}$ ✓

$T(p) = p' = a_1 + 2a_2 x \Rightarrow [T(p)]_{B'} = \boxed{\begin{bmatrix} a_1 \\ 2a_2 \end{bmatrix}}$ ✓

ex - find the standard matrix for the linear transformation T

a) $T: \mathbb{R}^2 \to \mathbb{R}^3$ is defined by $T(x,y) = (2x-3y, x-y, y-4x)$

$T(e_1) = T(1,0) = (2(1) - \cancel{3(0)}, 1 - \cancel{0}, \cancel{0} - 4(1)) = (2, 1, -4)$

$T(e_2) = T(0,1) = (\cancel{2(0)} - 3(1), \cancel{0} - 1, 1 - \cancel{4(0)}) = (-3, -1, 1)$

$\Rightarrow \boxed{A = \begin{bmatrix} 2 & -3 \\ 1 & -1 \\ -4 & 1 \end{bmatrix}}$

b) $T:\mathbb{R}^3 \to \mathbb{R}^3$ is defined by $T(x,y,z) = (5x-3y+z,\ 2z+4y,\ 5x+3y)$

$T(e_1) = T(1,0,0) = (5(1)-\cancel{3(0)}+\cancel{0},\ \cancel{2(0)}+\cancel{4(0)},\ 5(1)+\cancel{3(0)}) = (5,0,5)$

$T(e_2) = T(0,1,0) = (\cancel{5(0)}-3(1)+\cancel{0},\ \cancel{2(0)}+4(1),\ \cancel{5(0)}+3(1)) = (-3,4,3)$ $\Rightarrow \boxed{A = \begin{bmatrix} 5 & -3 & 1 \\ 0 & 4 & 2 \\ 5 & 3 & 0 \end{bmatrix}}$

$T(e_3) = T(0,0,1) = (\cancel{5(0)}-\cancel{3(0)}+1,\ 2(1)+\cancel{4(0)},\ \cancel{5(0)}+\cancel{3(0)}) = (1,2,0)$

ex - find the matrix of T relative to the bases B and B'

a) $T:\mathbb{R}^2 \to \mathbb{R}^3$ is defined by $T(x,y) = (x-y,\ 0,\ x+y)$ $B = \{(1,2),(1,1)\}$ $B' = \{(1,1,1),(1,1,0),(0,1,1)\}$

$T(1,2) = (1-2,0,1+2) = (-1,0,3)$ $T(1,1) = (1-1,0,1+1) = (0,0,2)$

gauss-jordan elimination on $[B' : B_0/I]$

$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$ $R_1 - R_2 \to R_2$ ^ $R_1 - R_3 \to R_3 \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & -1 & 0 \\ 0 & 1 & -1 & 1 & 0 & -1 \end{array}\right]$ $R_2 \leftrightarrow R_3$ ^ $-R_3 \to R_3 \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 & 1 & 0 \end{array}\right]$

$R_1 - R_2 \to R_1 \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 & 1 & 0 \end{array}\right]$ $R_2 + R_3 \to R_2 \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 & 1 & 0 \end{array}\right]$ $R_1 - R_3 \to R_1 \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 & 1 & 0 \end{array}\right] = P$

$[T(1,2)]_{B'} = P[T(1,2)] = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ -1 & 1 & 0 \end{bmatrix}\begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} -1+0+3 \\ 0+0-3 \\ 1+0+0 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$ $\Rightarrow \boxed{A = \begin{bmatrix} 2 & 2 \\ -3 & -2 \\ 1 & 0 \end{bmatrix}}$

$[T(1,1)]_{B'} = P[T(1,1)] = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ -1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0+0+2 \\ 0+0-2 \\ 0+0+0 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix}$

b) $T:\mathbb{R}^3 \to \mathbb{R}^2$ is defined by $T(x,y,z) = (2x-z,\ y-2x)$

$B = \{(2,0,1),(0,2,1),(1,2,1)\}$ $B' = \{(1,1),(2,0)\}$

$T(2,0,1) = (2(2)-(1),\ \cancel{0}-2(2)) = (3,-4)$ $T(0,2,1) = (\cancel{2(0)}-(1),\ (2)-\cancel{2(0)}) = (-1,2)$

$T(1,2,1) = (2(1)-(1),\ (2)-2(1)) = (1,0)$

$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{array}\right]$ $R_1 - R_2 \to R_2 \Rightarrow \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 2 & 1 & -1 \end{array}\right]$ $\frac{1}{2}R_2 \to R_2 \Rightarrow \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & \frac{1}{2} & -\frac{1}{2} \end{array}\right]$ $R_1 - 2R_2 \to R_1 \Rightarrow \left[\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 1 & \frac{1}{2} & -\frac{1}{2} \end{array}\right] = P$

$[T(2,0,1)]_{B'} = \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} 3 \\ -4 \end{bmatrix} = \begin{bmatrix} 0-4 \\ \frac{3}{2}+2 \end{bmatrix} = \begin{bmatrix} -4 \\ \frac{7}{2} \end{bmatrix}$

$[T(0,2,1)]_{B'} = \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0+2 \\ -\frac{1}{2}-1 \end{bmatrix} = \begin{bmatrix} 2 \\ -\frac{3}{2} \end{bmatrix}$ $\Rightarrow \boxed{A = \begin{bmatrix} -4 & 2 & 0 \\ \frac{7}{2} & -\frac{3}{2} & \frac{1}{2} \end{bmatrix}}$

$[T(1,2,1)]_{B'} = \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0+0 \\ \frac{1}{2}+0 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix}$

c) $T:\mathbb{R}^2 \to P_1$ is defined by $T(v_1,v_2) = p$ where $p(x) = v_2 + 4v_1x$ $B = \{(1,0),(0,1)\}$ $B' = \{1,x\}$

$T(1,0) = \cancel{0} + 4(1)x = 4x$ $T(0,1) = 1 + \cancel{4(0)x} = 1$

$[T(1,0)]_{B'} = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$ $[T(0,1)]_{B'} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\Rightarrow \boxed{A = \begin{bmatrix} 0 & 1 \\ 4 & 0 \end{bmatrix}}$

d) $T:P_3 \to P_2$ is defined by $T(p) = p'$ where p' is the derivative of p $B = \{1,x,x^2,x^3\}$ $B' = \{1,x,x^2\}$

$T(1) = 0$ $T(x) = 1$ $T(x^2) = 2x$ $T(x^3) = 3x^2$

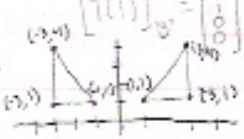
$[T(1)]_{B'} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $[T(x)]_{B'} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $[T(x^2)]_{B'} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$ $[T(x^3)]_{B'} = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$ $\Rightarrow \boxed{A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}}$

e) $T:P_2 \to P_3$ is defined by $T(p) = xp$ $B = \{1,x,x^2\}$ $B' = \{1,x,x^2,x^3\}$

$T(1) = x$ $T(x) = x^2$ $T(x^2) = x^3$

$[T(1)]_{B'} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ $[T(x)]_{B'} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ $[T(x^2)]_{B'} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ $\boxed{A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$

Applications ex -

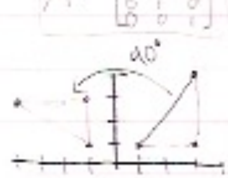


$T:\mathbb{R}^2 \to \mathbb{R}^2$ $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ (x,y)

$T(1,1) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1+0 \\ 0+1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$T(3,1) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -3+0 \\ 0+1 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$

$T(3,4) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} -3+0 \\ 0+4 \end{bmatrix} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$



- reflection across x-axis $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

  reflection across line $y=x$ $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

  rotation counter-clockwise by θ $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ ie. $90° \Leftrightarrow \begin{bmatrix} \cos 90 & -\sin 90 \\ \sin 90 & \cos 90 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

Composition ex - $T_1 : \mathbb{R}^2 \to \mathbb{R}^2$ $T_2 : \mathbb{R}^2 - \mathbb{R}^2$ $T_1 \circ T_2$ (1) reflection about x-axis followed by (2) rotation by 180°

$\begin{bmatrix} \cos 180 & -\sin 180 \\ \sin 180 & \cos 180 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ => multiply by coordinates to get coordinates reflected/rotated

Application for Computer Graphics - rotation about x-axis $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$

rotation about y-axis $\begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$

rotation about z-axis $\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

ex - (0,0,0), (1,0,0), (0,3,0), (1,3,0), (0,0,6), (1,0,6), (0,3,6), (1,3,6) ⇐ coordinates for cube

find matrix that rotates the box by (1) 90° about the x-axis, and then (2) rotates the box about z-axis by 90°

$\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ = rotation matrix = r

$r\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $r\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $r\begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$ $r\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$ $r\begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix}$ $r\begin{bmatrix} 1 \\ 0 \\ 6 \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \\ 0 \end{bmatrix}$

$r\begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 3 \end{bmatrix}$ $r\begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \\ 3 \end{bmatrix}$ ⇒ these are new coordinates

ex - find the matrix that corresponds to a reflection about the line $y=x$ followed by a rotation by 90°. then apply the matrix to the triangle with vertices (0,0), (1,0), (1,1)

Eigenvalues ^ Eigenvectors

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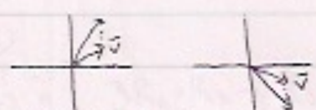
- $\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$ where $k > 1$ ⇒



scales vector

- $k=2$ ⇒ $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ ⇒ $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

  $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ ⇒ same as $\underset{\lambda}{2} \underset{\vec{v}}{\begin{bmatrix} x \\ y \end{bmatrix}}$

  $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \end{bmatrix}$

- $A\vec{v} = \lambda\vec{v}$ where $\vec{v}$ = nonzero vector, λ = scalar <u>OR</u>

  $\vec{v}$ = eigenvector of A corresponding to λ, λ = eigenvalue of A

  i.e. $\begin{bmatrix} 0 \\ 3 \end{bmatrix}$ eigenvector of $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, corresponding to eigenvalue $\lambda = 2$

- $\begin{bmatrix} 0 \\ t \end{bmatrix}$ $t \neq 0$ eigenvector of A corresponding to $\lambda = 2$
- $\begin{bmatrix} x \\ 0 \end{bmatrix}$ $x \neq 0$ eigenvector of A corresponding to $\lambda = 1$

  $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix} = 1 \cdot \begin{bmatrix} x \\ 0 \end{bmatrix}$

- eigenspace of $\lambda = 2$ is the y-axis
- eigen space of $\lambda = 1$ is the x-axis

- $A\vec{v} = \lambda\vec{v} \Leftrightarrow \lambda\vec{v} - A\vec{v} = \vec{0} \Leftrightarrow \lambda(I\vec{v}) - A\vec{v} = \vec{0} \Leftrightarrow (\lambda I)\vec{v} - A\vec{v} = \vec{0} \Leftrightarrow (\lambda I - A)\vec{v} = \vec{0}$
- $\vec{v}$ is a nonzero solution to $A\vec{v} = \lambda\vec{v}$ if and only if $\vec{v}$ is a nonzero solution to $(\lambda I - A)\vec{v} = \vec{0}$
- $M\vec{x} = \vec{0}$ has only the trivial solution iff M is invertible
- $(\lambda I - A)\vec{v} = \vec{0}$ has a nontrivial solution iff $(\lambda I - A)$ is not invertible
- M is invertible iff $\det(M) \neq 0$
- $(\lambda I - A)\vec{v} = \vec{0}$ has a nontrivial solution iff $\det(\lambda I - A) = 0$
- conclusion: $A\vec{v} = \lambda\vec{v}$ has a nontrivial solution iff $\det(\lambda I - A) = 0$ &

  λ is an eigenvalue of A iff $\det(\lambda I - A) = 0$.

  to solve for lambda, solve $\longrightarrow$

ex - find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$

$\lambda I - A = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix} = \begin{bmatrix} \lambda-2 & -3 \\ -3 & \lambda+6 \end{bmatrix}$

$\det(\lambda I - A) = (\lambda-2)(\lambda+6) - (-3)(-3) = \lambda^2 + 6\lambda - 2\lambda - 12 - 9 = \lambda^2 + 4\lambda - 21$ characteristic polynomial of A

$= (\lambda+7)(\lambda-3) \Leftrightarrow \quad \lambda + 7 = 0 \quad \lambda - 3 = 0$

$\lambda = -7 \quad \lambda = 3$

plug λ's into $\lambda I - A$:

$\lambda = -7 \Leftrightarrow \begin{bmatrix} (-7)-2 & -3 \\ -3 & (-7)+6 \end{bmatrix} = \begin{bmatrix} -9 & -3 \\ -3 & -1 \end{bmatrix} \Leftrightarrow \begin{bmatrix} -9 & -3 \\ -3 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Leftrightarrow$

$\begin{bmatrix} -9 & -3 \\ -3 & -1 \end{bmatrix} \ \frac{1}{3}R_1 \to R_1 \Leftrightarrow \begin{bmatrix} -3 & -1 \\ -3 & -1 \end{bmatrix} \ R_2 - R_1 \to R_2 \Leftrightarrow \begin{bmatrix} -3 & -1 \\ 0 & 0 \end{bmatrix} \ -\frac{1}{3}R_1 \to R_1 \Leftrightarrow \begin{bmatrix} 1 & 1/3 \\ 0 & 0 \end{bmatrix}$ let $v_2 = t$

$v_1 + \frac{1}{3}t = 0 \Leftrightarrow v_1 = -\frac{1}{3}t$

$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{3}t \\ t \end{bmatrix} = \boxed{t\begin{bmatrix} -\frac{1}{3} \\ 1 \end{bmatrix} \quad t \neq 0}$

$\lambda = 3 \Leftrightarrow \begin{bmatrix} (3)-2 & -3 \\ -3 & (3)+6 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ -3 & 9 \end{bmatrix} \ 3R_1 + R_2 \to R_2 \Leftrightarrow \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix}$ let $v_2 = t$

$v_1 - 3t = 0 \Leftrightarrow v_1 = 3t$

$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 3t \\ t \end{bmatrix} = \boxed{t\begin{bmatrix} 3 \\ 1 \end{bmatrix} \quad t \neq 0}$

ex - find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 2 & 5 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

$\lambda I - A = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 2 & 5 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} \lambda-2 & -5 & -1 \\ 0 & \lambda-3 & 0 \\ 0 & 0 & \lambda-3 \end{bmatrix}$

$\det(\lambda I - A) \overset{\text{cofactor expansion along 1st column}}{=} = (\lambda-2)\cdot\begin{vmatrix} \lambda-3 & 0 \\ 0 & \lambda-3 \end{vmatrix} = (\lambda-2)(\lambda-3)^2 - \cancel{0} \Leftrightarrow \lambda - 2 = 0 \quad \lambda - 3 = 0$

$\lambda = 3$ has multiplicity 2, $\lambda = 2$ has multiplicity 1 $\Rightarrow \lambda = 2 \quad \lambda = 3$

$\lambda = 2 \Rightarrow \begin{bmatrix} 2-2 & -5 & -1 \\ 0 & 2-3 & 0 \\ 0 & 0 & 2-3 \end{bmatrix} = \begin{bmatrix} 0 & -5 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \ -\frac{1}{5}R_1 \to R_1 \Leftrightarrow \begin{bmatrix} 0 & 1 & \frac{1}{5} \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \ -R_2 \to R_2 \ \wedge \ -R_3 \to R_3 \Leftrightarrow \begin{bmatrix} 0 & 1 & \frac{1}{5} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$R_1 - R_2 \to R_2 \Leftrightarrow \begin{bmatrix} 0 & 1 & 1/5 \\ 0 & 0 & 1/5 \\ 0 & 0 & 1 \end{bmatrix} \ 5R_2 \to R_2 \Rightarrow \begin{bmatrix} 0 & 1 & 1/5 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \ R_2 - R_3 \to R_3 \Leftrightarrow \begin{bmatrix} 0 & 1 & 1/5 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$v_3 = 0$

$v_2 + \frac{1}{5}v_3 = 0 \Rightarrow v_2 = 0 \quad$ let $v_1 = t$

$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = \boxed{t\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad t \neq 0} \ \wedge \ \boxed{\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right\} \text{ basis for eigenspace of } \lambda = 2}$

$\lambda = 3 \Rightarrow \lambda I - A = \begin{bmatrix} 3-2 & -5 & -1 \\ 0 & 3-3 & 0 \\ 0 & 0 & 3-3 \end{bmatrix} = \begin{bmatrix} 1 & -5 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ let $v_2 = s \wedge v_3 = t$

$v_1 - 5s - t = 0 \Rightarrow v_1 = 5s + t$

$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 5s+t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ where $s \wedge t$ are not both 0

$\boxed{\left\{\begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right\} \text{ basis for eigenspace of } \lambda = 3}$

note the λ's were also in $A = \begin{bmatrix} 2 & 5 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, generally true when all zeros after diagonal

ex - find the eigenvalues and eigenvectors of A. then find a basis for the eigenspace of each eigenvalue:

a) $A = \begin{bmatrix} 1 & -4 \\ -2 & 8 \end{bmatrix}$

$\lambda I - A = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & -4 \\ -2 & 8 \end{bmatrix} = \begin{bmatrix} \lambda-1 & 4 \\ 2 & \lambda-8 \end{bmatrix}$

$\det(\lambda I - A) = (\lambda-1)(\lambda-8) - (4)(2) = \lambda^2 - 8\lambda - \lambda \cancel{+8} \cancel{-8} = \lambda^2 - 9\lambda \Rightarrow \lambda(\lambda - 9) = 0 \Rightarrow$

$\lambda - 9 = 0 \Rightarrow \lambda = 9 \wedge \lambda = 0$

$\underline{\lambda = 0} \Rightarrow \begin{bmatrix} 0-1 & 4 \\ 2 & 0-8 \end{bmatrix} = \begin{bmatrix} -1 & 4 \\ 2 & -8 \end{bmatrix}$ $2R_1 + R_2 \to R_2 \Rightarrow \begin{bmatrix} -1 & 4 \\ 0 & 0 \end{bmatrix}$ $-R_1 \to R_1 \Rightarrow \begin{bmatrix} 1 & -4 \\ 0 & 0 \end{bmatrix}$ let $v_2 = t$

$v_1 - 4t = 0 \Rightarrow v_1 = 4t$

$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 4t \\ t \end{bmatrix} = t\begin{bmatrix} 4 \\ 1 \end{bmatrix}, t \neq 0$ $\boxed{\left\{\begin{bmatrix} 4 \\ 1 \end{bmatrix}\right\} \text{ is a basis for the eigenspace of } \lambda = 0}$

$\underline{\lambda = 9} \Rightarrow \begin{bmatrix} 9-1 & 4 \\ 2 & 9-8 \end{bmatrix} = \begin{bmatrix} 8 & 4 \\ 2 & 1 \end{bmatrix}$ $R_1 - 4R_2 \to R_2 \Rightarrow \begin{bmatrix} 8 & 4 \\ 0 & 0 \end{bmatrix}$ $\frac{1}{8}R_1 \to R_1 \Rightarrow \begin{bmatrix} 1 & 1/2 \\ 0 & 0 \end{bmatrix}$ let $v_2 = t$

$v_1 + \frac{1}{2}t = 0 \Rightarrow v_1 = -\frac{1}{2}t$

$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -1/2\,t \\ t \end{bmatrix} = t\begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$ $t \neq 0$

$\boxed{\left\{\begin{bmatrix} 1/2 \\ 1 \end{bmatrix}\right\} \text{ is a basis for the eigenspace } \lambda = 9}$

b) $A = \begin{bmatrix} 7 & 2 \\ -2 & 3 \end{bmatrix}$

$\lambda I - A = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 7 & 2 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} \lambda-7 & -2 \\ 2 & \lambda-3 \end{bmatrix}$

$\det(\lambda I - A) = (\lambda-7)(\lambda-3) - (-2)(2) = \lambda^2 - 3\lambda - 7\lambda + 21 + 4 = \lambda^2 - 10\lambda + 25 = (\lambda-5)^2$

$\lambda - 5 = 0 \Rightarrow \lambda = 5$ with multiplicity 2

$\underline{\lambda = 5} \Rightarrow \begin{bmatrix} 5-7 & -2 \\ 2 & 5-3 \end{bmatrix} = \begin{bmatrix} -2 & -2 \\ 2 & 2 \end{bmatrix}$ $R_1 + R_2 \to R_2 \Rightarrow \begin{bmatrix} -2 & -2 \\ 0 & 0 \end{bmatrix}$ $-\frac{1}{2}R_1 \to R_1 \Rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ let $v_2 = t$

$v_1 + t = 0 \Rightarrow v_1 = -t$

$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix} = t\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ $t \neq 0$ $\boxed{\left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\} \text{ is a basis for the eigenspace of } \lambda = 5}$

c) $A = \begin{bmatrix} 1 & 2 & -2 \\ -2 & 5 & -2 \\ -6 & 6 & -3 \end{bmatrix}$

$\lambda I - A = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & 2 & -2 \\ -2 & 5 & -2 \\ -6 & 6 & -3 \end{bmatrix} = \begin{bmatrix} \lambda-1 & -2 & 2 \\ 2 & \lambda-5 & 2 \\ 6 & -6 & \lambda+3 \end{bmatrix}$

$\det(\lambda I - A) =$ cofactor expansion across 1st row $= (\lambda-1)\begin{vmatrix} \lambda-5 & 2 \\ -6 & \lambda+3 \end{vmatrix} + (-1)(-2)\begin{vmatrix} 2 & 2 \\ 6 & \lambda+3 \end{vmatrix} + (2)\begin{vmatrix} 2 & \lambda-5 \\ 6 & -6 \end{vmatrix} = (\lambda-1)((\lambda-5)(\lambda+3) -$

$(2)(-6)) + 2(2(\lambda+3) - (2)(6)) + 2((2)(-6) - (\lambda-5)(6)) = (\lambda-1)(\lambda^2 + 3\lambda - 5\lambda - 15 + 12) + 2(2\lambda + 6 - 12) +$

$2(-12 - 6\lambda + 30) = (\lambda-1)(\lambda^2 - 2\lambda - 3) + 2(2\lambda - 6) + 2(-6\lambda + 18) = \lambda^3 - 2\lambda^2 - 3\lambda - \lambda^2 + 2\lambda + 3 + 4\lambda - 12$

$-12\lambda + 36 = \lambda^3 - 3\lambda^2 - 9\lambda + 27 = (\lambda+3)(\lambda-3)^2 \Rightarrow \lambda = -3$ (multiplicity 1) $\wedge$ $\lambda = 3$ (multiplicity 2)

$\frac{1}{2}+\frac{1}{2}$

$-\frac{1}{3}(-\frac{1}{3})+(-\frac{1}{6}) = \frac{2}{6}-\frac{1}{2}$

$\frac{1}{6}-\frac{3}{6}=\frac{-2}{6}=-\frac{1}{3}$

for $\lambda=3 \Rightarrow \lambda I - A = \begin{bmatrix}2&-2&2\\2&-2&2\\6&-6&6\end{bmatrix}$ $R_1-R_2\to R_2 \Rightarrow \begin{bmatrix}2&-2&2\\0&0&0\\6&-6&6\end{bmatrix}$ $-3R_1+R_3\to R_3 \Rightarrow$

$\begin{bmatrix}2&-2&2\\0&0&0\\0&0&0\end{bmatrix}$ $\frac{1}{2}R_1\to R_1 \Rightarrow \begin{bmatrix}1&-1&1\\0&0&0\\0&0&0\end{bmatrix}$ let $v_2=s$ ^ $v_3=t$

$v_1-s+t=0 \Rightarrow v_1=s-t$

$\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}=\begin{bmatrix}s-t\\s\\t\end{bmatrix}=s\begin{bmatrix}1\\1\\0\end{bmatrix}+t\begin{bmatrix}-1\\0\\1\end{bmatrix}$ where s ^ t are not both 0

$\boxed{\left\{\begin{bmatrix}1\\1\\0\end{bmatrix},\begin{bmatrix}-1\\0\\1\end{bmatrix}\right\} \text{ is the basis for the eigenspace of } \lambda=3}$

for $\lambda=-3 \Rightarrow \lambda I - A = \begin{bmatrix}-4&-2&2\\2&-8&2\\6&-6&0\end{bmatrix}$ $R_1+2R_2\to R_2 \Rightarrow \begin{bmatrix}-4&-2&2\\0&-18&6\\6&-6&0\end{bmatrix}$ $-\frac{1}{4}R_1\to R_1$ ^ $-\frac{1}{18}R_2\to R_2$ ^ $\frac{1}{6}R_3\to R_3 \Rightarrow$

$\begin{bmatrix}1&1/2&-1/2\\0&1&-1/3\\1&-1&0\end{bmatrix}$ $R_1-R_3\to R_3 \Rightarrow \begin{bmatrix}1&1/2&-1/2\\0&1&-1/3\\0&3/2&-1/2\end{bmatrix}$ $-\frac{3}{2}R_2+R_3\to R_3 \Rightarrow \begin{bmatrix}1&1/2&-1/2\\0&1&-1/3\\0&0&0\end{bmatrix}$ let $v_3=t$

$v_2-\frac{1}{3}t=0 \Rightarrow v_2=\frac{1}{3}t$

$v_1+\frac{1}{2}(\frac{1}{3}t)-\frac{1}{2}(t)=0 \Rightarrow v_1+\frac{1}{6}t-\frac{1}{2}t=0 \Rightarrow v_1-\frac{1}{3}t=0 \Rightarrow v_1=\frac{1}{3}t$

$\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}=\begin{bmatrix}\frac{1}{3}t\\\frac{1}{3}t\\t\end{bmatrix}=t\begin{bmatrix}1/3\\1/3\\1\end{bmatrix}$ $t\neq 0$ $\quad \left\{\begin{bmatrix}1/3\\1/3\\1\end{bmatrix}\right\}$ is a basis for the eigenspace of $\lambda=-3$

Diagonalization

- A, B are similar if there exists an invertible matrix P such that $B=P^{-1}AP$, and $P^{-1}AP=D$ where D is diagonal, we say A is diagonizable

ex - $A=\begin{bmatrix}1&2&-2\\-2&5&-2\\-6&6&-3\end{bmatrix}$ ^ $P=\begin{bmatrix}1&1&1\\1&0&1\\3&-1&0\end{bmatrix}$ show that $P^{-1}AP$ is a diagonal matrix

$P^{-1} \Rightarrow [P \,\vdots\, I] \to [I \,\vdots\, P^{-1}] \quad \Rightarrow P^{-1}=\begin{bmatrix}1/3&-1/3&1/3\\1&-1&0\\-1/3&4/3&-1/3\end{bmatrix}$

$P^{-1}AP=\begin{bmatrix}1/3&-1/3&1/3\\1&-1&0\\-1/3&4/3&-1/3\end{bmatrix}\begin{bmatrix}1&2&-2\\-2&5&-2\\-6&6&-3\end{bmatrix}\begin{bmatrix}1&1&1\\1&0&1\\3&-1&0\end{bmatrix}=\begin{bmatrix}1/3&-1/3&1/3\\1&-1&0\\-1/3&4/3&-1/3\end{bmatrix}\begin{bmatrix}-3&2&3\\-3&2&3\\-9&-3&0\end{bmatrix}=\begin{bmatrix}-3&0&0\\0&3&0\\0&0&3\end{bmatrix}$ $\boxed{\text{A is diagonizable}}$ ✓

- similarity is an equivalence relation:
  1. A is similar to itself (reflexivity)
  2. If A is similar to B, then B is similar to A (symmetry)
  3. If A is similar to B, and B is similar to C, then A is similar to C. (transitivity)

- let $Q=\begin{bmatrix}0&1&0\\1&0&1\\0&0&2\end{bmatrix}$ $\quad Q^{-1}=\begin{bmatrix}0&1&-1/2\\1&0&0\\0&0&1/2\end{bmatrix}$ let $B=Q^{-1}AQ$ where $A=\begin{bmatrix}1&2&-2\\-2&5&-2\\-6&6&-3\end{bmatrix}$

$B=Q^{-1}AQ=[\ldots][\ldots][\ldots]=\begin{bmatrix}0&1&-1/2\\1&0&0\\0&0&1/2\end{bmatrix}\begin{bmatrix}2&1&-2\\5&-2&1\\6&-6&0\end{bmatrix}=\begin{bmatrix}2&1&1\\2&1&-2\\3&-3&0\end{bmatrix} \Rightarrow$ A is similar to B, B is similar to A

A ^ B are similar to $D=\begin{bmatrix}-3&0&0\\0&3&0\\0&0&3\end{bmatrix}$ B has the same eigenvalues

Condition for Diagonalization

- an $n\times n$ matrix is diagonizable iff it has ~~been~~ n linearly independent eigenvectors
- $A=\begin{bmatrix}1&2&-2\\-2&5&-2\\-6&6&-3\end{bmatrix}$ is diagonizable $\Rightarrow \lambda I - A = \begin{bmatrix}\lambda-1&-2&2\\2&\lambda-5&2\\6&-6&\lambda+3\end{bmatrix}$

$\det(\lambda I - A)=(\lambda-1)\begin{vmatrix}\lambda-5&2\\-6&\lambda+3\end{vmatrix}+(-2)(-1)\begin{vmatrix}2&2\\6&\lambda+3\end{vmatrix}+2\begin{vmatrix}2&\lambda-5\\6&-6\end{vmatrix}=(\lambda-1)((\lambda-5)(\lambda+3)+12)+2(2(\lambda+3)-12)+$

$2(-12-(\lambda-5)(6))=\ldots=(\lambda-3)^2(\lambda+3)=0 \Rightarrow \lambda=3$ (multiplicity 2), $\lambda=-3$ (multiplicity 1)

for $\underline{\lambda=3} \Rightarrow \lambda I - A=\begin{bmatrix}2&-2&2\\2&-2&2\\6&-6&6\end{bmatrix}$ all equivalent $\Rightarrow \begin{bmatrix}1&-1&1\\0&0&0\\0&0&0\end{bmatrix}$ let $v_2=s$, $v_3=t$

$v_1-(s)+(t)=0 \Rightarrow v_1=s-t$

$\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}=\begin{bmatrix}s-t\\s\\t\end{bmatrix}=\boxed{s\begin{bmatrix}1\\1\\0\end{bmatrix}+t\begin{bmatrix}-1\\0\\1\end{bmatrix}}$ where s ^ t are not both 0

for $\lambda=-3 \Rightarrow \lambda I - A=\begin{bmatrix}-4&-2&2\\2&-8&2\\6&-6&0\end{bmatrix}$ $-\frac{1}{4}R_1\to R_1$ ^ $\frac{1}{2}R_2\to R_2$ ^ $\frac{1}{6}R_3\to R_3 \Rightarrow \begin{bmatrix}1&1/2&-1/2\\1&-4&1\\1&-1&0\end{bmatrix}$ $R_1-R_2\to R_2$ ^ $R_1-R_3\to R_3$

$\begin{bmatrix}1&1/2&-1/2\\0&9/2&-3/2\\0&3/2&-1/2\end{bmatrix}$ $\frac{2}{9}R_2\to R_2$ ^ $\frac{2}{3}R_3\to R_3 \Rightarrow \begin{bmatrix}1&1/2&-1/2\\0&1&-1/3\\0&1&-1/3\end{bmatrix}$ equivalent $\Rightarrow \begin{bmatrix}1&1/2&-1/2\\0&1&-1/3\\0&0&0\end{bmatrix}$ let $v_3=t$

$v_2-\frac{1}{3}(t)=0 \Rightarrow v_2=\frac{1}{3}t$

$v_1+\frac{1}{2}(\frac{1}{3}t)-\frac{1}{2}(t)=0 \Rightarrow v_1+\frac{1}{6}t-\frac{1}{2}t=0 \Rightarrow v_1-\frac{2}{6}t=0 \Rightarrow v_1=\frac{1}{3}t$

$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{3}t \\ \frac{1}{3}t \\ t \end{bmatrix} = \boxed{t \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ 1 \end{bmatrix} \quad t \neq 0}$

$\lambda = 3 \Rightarrow \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$ $\quad \lambda = -3$ let $t = 3 \Rightarrow t \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ 1 \end{bmatrix} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} \right\} \Rightarrow \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} \right\}$

$\Rightarrow$ form $P = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 3 \end{bmatrix} \to I \Rightarrow R_1 - R_2 \to R_2 \Rightarrow \begin{bmatrix} 1 & -1 & 1 \\ 0 & -1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$ $R_2 + R_3 \to R_3 \Rightarrow \begin{bmatrix} 1 & -1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ $\frac{1}{3}R_3 \to R_3 \wedge -R_2 \to R_2 \Rightarrow$

$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} R_1 + R_2 \to R_1 \Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} R_1 - R_3 \to R_1 \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Diagonalizing a Matrix

$P^{-1}AP = \begin{bmatrix} -\frac{1}{3} & \frac{4}{3} & -\frac{1}{3} \\ -1 & 1 & 0 \\ \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 & 2 & -2 \\ -2 & 5 & -2 \\ -6 & 6 & -3 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} & \frac{4}{3} & -\frac{1}{3} \\ -1 & 1 & 0 \\ \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 3 & -3 & -3 \\ 3 & 0 & -3 \\ 0 & 3 & -9 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -3 \end{bmatrix}$ ✓

ex – find an invertible matrix P that diagonalizes A. then show that $P^{-1}AP$ is a diagonal matrix with the eigenvalues of A along the diagonal $A = \begin{bmatrix} 2 & 0 & -2 \\ 0 & 2 & -2 \\ 3 & 0 & -3 \end{bmatrix}$

$\lambda I - A = \begin{bmatrix} \lambda-2 & 0 & 2 \\ 0 & \lambda-2 & 2 \\ -3 & 0 & \lambda+3 \end{bmatrix}$ = cofactor expansion along 2nd column $= \det(\lambda I - A) = (\lambda-2)\begin{vmatrix} \lambda-2 & 2 \\ -3 & \lambda+3 \end{vmatrix} = (\lambda-2)\big((\lambda-2)(\lambda+3) - (2)(-3)\big) =$

$(\lambda-2)(\lambda^2 + 3\lambda - 2\lambda \cancel{-6} \cancel{+6}) = (\lambda-2)(\lambda^2+\lambda) = (\lambda-2)(\lambda)(\lambda+1) \Rightarrow \lambda = 0, -1, 2$

$\underline{\lambda = 0} \Rightarrow A = \begin{bmatrix} -2 & 0 & 2 \\ 0 & -2 & 2 \\ -3 & 0 & 3 \end{bmatrix}$ $-\frac{1}{2}R_1 \to R_1 \wedge -\frac{1}{2}R_2 \to R_2 \wedge -\frac{1}{3}R_3 \to R_3 \Rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix}$ $R_1 - R_3 \to R_3 \Rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ let $v_3 = t$

$v_2 - (t) = 0 \Rightarrow v_2 = t$

$v_1 - (t) = 0 \Rightarrow v_1 = t$

$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} t \\ t \\ t \end{bmatrix} = \boxed{t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad t \neq 0}$

$\underline{\lambda = 2} \Rightarrow A = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 2 \\ -3 & 0 & 5 \end{bmatrix}$ $R_1 \leftrightarrow R_3 \wedge R_2 - R_3 \to R_3 \Rightarrow \begin{bmatrix} -3 & 0 & 5 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ $\frac{1}{2}R_2 \to R_2 \Rightarrow \begin{bmatrix} -3 & 0 & 5 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$v_3 = 0$

$-3v_1 + 5(0) = 0 \Rightarrow -3v_1 = 0 \Rightarrow v_1 = 0$ $\qquad$ let $v_2 = t$

$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ t \\ 0 \end{bmatrix} = \boxed{t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad t \neq 0}$

$\underline{\lambda = -1} \Rightarrow A = \begin{bmatrix} -3 & 0 & 2 \\ 0 & -3 & 2 \\ -3 & 0 & 2 \end{bmatrix}$ $R_1 - R_3 \to R_3 \Rightarrow \begin{bmatrix} -3 & 0 & 2 \\ 0 & -3 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ let $v_3 = t$

$-3v_2 + 2(t) = 0 \Rightarrow -3v_2 = -2t \Rightarrow v_2 = \frac{2}{3}t$

$-3v_1 + 2(t) = 0 \Rightarrow -3v_1 = -2t \Rightarrow v_1 = \frac{2}{3}t$

$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} \frac{2}{3}t \\ \frac{2}{3}t \\ t \end{bmatrix} = \boxed{t \begin{bmatrix} \frac{2}{3} \\ \frac{2}{3} \\ 1 \end{bmatrix} \quad t \neq 0}$

$\lambda = 0 \Rightarrow$ let $t = 1 \Rightarrow \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$\lambda = 2 \Rightarrow$ let $t = 1 \Rightarrow \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $\Big\}$ eigenvector of A corresponding to $\lambda = 0$, $\lambda = 2$, $\lambda = -1$

$\lambda = -1 \Rightarrow$ let $t = 3 \Rightarrow \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$

form $P = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 2 \\ 1 & 0 & 3 \end{bmatrix}$ $\quad$ check that $P^{-1}AP = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

$P^{-1}AP = \begin{bmatrix} 3 & 0 & -2 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & -2 \\ 0 & 2 & -2 \\ 3 & 0 & -3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 2 \\ 1 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 0 & -2 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & -2 \\ 0 & 2 & -2 \\ 0 & 0 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ ✓

ex – show that A is not diagonalizable. $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 1 & 0 & 2 \end{bmatrix}$

$\lambda I - A = \begin{bmatrix} \lambda-1 & 0 & 0 \\ -1 & \lambda-2 & -1 \\ -1 & 0 & \lambda-2 \end{bmatrix} \Rightarrow$ cofactor expansion across 1st row $\Rightarrow \det(\lambda I - A) = (\lambda-1)\begin{vmatrix} \lambda-2 & -1 \\ 0 & \lambda-2 \end{vmatrix} = (\lambda-1)\big((\lambda-2)^2 - \cancel{(-1)(0)}\big) =$

$(\lambda-2)^2(\lambda-1) \quad \lambda = 2, 1$

$\underline{\lambda = 1} \Rightarrow A = \begin{bmatrix} 0 & 0 & 0 \\ -1 & -1 & -1 \\ -1 & 0 & -1 \end{bmatrix}$ $R_1 \leftrightarrow R_3 \wedge R_1 - R_2 \to R_2 \Rightarrow \begin{bmatrix} -1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $-R_1 \to R_1 \Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$v_2 = 0 \qquad$ let $v_3 = t$

$v_1 + (t) = 0 \Rightarrow v_1 = -t$

$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -t \\ 0 \\ t \end{bmatrix} = \boxed{t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \quad t \neq 0}$

$\underline{\lambda=2} \Rightarrow A=\begin{bmatrix}1&0&0\\-1&0&-1\\-1&0&0\end{bmatrix}$ $R_1+R_3\to R_3 \Rightarrow \begin{bmatrix}1&0&0\\-1&0&-1\\0&0&0\end{bmatrix}$ $R_1+R_2\to R_2 \Rightarrow \begin{bmatrix}1&0&0\\0&0&-1\\0&0&0\end{bmatrix}$ $-R_2\to R_2 \Rightarrow \begin{bmatrix}1&0&0\\0&0&1\\0&0&0\end{bmatrix}$

$v_3=0 \wedge v_1=0$ let $v_2=t$

$\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}=\begin{bmatrix}0\\t\\0\end{bmatrix}=\boxed{t\begin{bmatrix}0\\1\\0\end{bmatrix} \quad t\neq 0}$

there are not 3 linearly independent eigenvectors of A [X not diagonizable]

Application to Differential Equations

- $y_1' = a_{11}y_1 + a_{12}y_2 + \dots + a_{1n}y_n$

$y_2' = a_{21}y_1 + a_{22}y_2 + \dots + a_{2n}y_n$

$\vdots$

$y_n' = a_{n1}y_1 + a_{n2}y_2 + \dots + a_{nn}y_n$ where $y_1, y_2, \dots, y_n$ are functions of t ^ $y_1' = \frac{dy_1}{dt}, y_2' = \frac{dy_2}{dt}, \dots, y_n' = \frac{dy_n}{dt}$

$\Downarrow$

$\begin{bmatrix}y_1'\\y_2'\\\vdots\\y_n'\end{bmatrix} = \begin{bmatrix}a_{11}&a_{12}&\cdots&a_{1n}\\a_{21}&a_{22}&\cdots&a_{2n}\\&&\vdots&\\a_{n1}&a_{n2}&&a_{nn}\end{bmatrix}\begin{bmatrix}y_1\\y_2\\\vdots\\y_n\end{bmatrix}$ $\bar{y}' = A\bar{y}$ where $\bar{y}=\begin{bmatrix}y_1\\y_2\\\vdots\\y_n\end{bmatrix}$ ^ $\bar{y}'=\begin{bmatrix}y_1'\\y_2'\\\vdots\\y_n'\end{bmatrix}$

ex - $y_1' = -3y_1$ $y_2' = 2y_2$ $y_3' = 5y_3$

$y_1 = c_1 e^{3t}$ $y_2 = c_2 e^{2t}$ $y_3 = c_3 e^{5t}$ c's are constants

$\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}=\begin{bmatrix}c_1e^{-3t}\\c_2e^{2t}\\c_3e^{5t}\end{bmatrix}$ $\begin{bmatrix}y_1'\\y_2'\\y_3'\end{bmatrix}=\begin{bmatrix}3&0&0\\0&2&0\\0&0&5\end{bmatrix}\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}$ $\bar{y}'=A\bar{y}$ assume A is diagonizable, invertible matrix P such that $P^{-1}AP$ is a diagonal matrix. change of variables by letting $\bar{y}=P\bar{w}$ $\bar{y}'=P\bar{w}'$

$\bar{y}'=A\bar{y} \Rightarrow \bar{y}'=A(P\bar{w}) \Rightarrow P\bar{w}'=A(P\bar{w}) \Rightarrow P\bar{w}'=(AP)\bar{w} \Rightarrow \bar{w}'=P^{-1}((AP)\bar{w}) \Rightarrow \bar{w}'=\underbrace{(P^{-1}(AP))}_{}\bar{w}$

$P^{-1}AP$ is a diagonizable matrix

ex - $y_1' = y_1 - y_2$ $y_2' = 2y_1 + 4y_2$

$\begin{bmatrix}y_1'\\y_2'\end{bmatrix}=\begin{bmatrix}1&-1\\2&4\end{bmatrix}\begin{bmatrix}y_1\\y_2\end{bmatrix}$ $\bar{y}'=A\bar{y}$ P diagonalizes A

$\lambda I - A = \begin{bmatrix}\lambda-1&1\\-2&\lambda-4\end{bmatrix} \Rightarrow \det(\lambda I - A) = (\lambda-1)(\lambda-4)-(1)(-2) = \lambda^2-4\lambda-\lambda+4+2 = \lambda^2-5\lambda+6 =$

$(\lambda-3)(\lambda-2) \Rightarrow \lambda = 3, 2$

$\underline{\lambda=2} \Rightarrow \lambda I - A = \begin{bmatrix}1&1\\2&2\end{bmatrix}$ $2R_1+R_2\to R_2 \Rightarrow \begin{bmatrix}1&1\\0&0\end{bmatrix}$ let $v_2=t$

$v_1+(t)=0 \Rightarrow v_1=-t$

$\begin{bmatrix}v_1\\v_2\end{bmatrix}=\begin{bmatrix}-t\\t\end{bmatrix}=\boxed{t\begin{bmatrix}-1\\1\end{bmatrix} \quad t\neq 0}$ let $t=1 \Rightarrow \begin{bmatrix}-1\\1\end{bmatrix}$ eigenvector corresponding to $\lambda=2$

$\underline{\lambda=3} \Rightarrow \lambda I - A = \begin{bmatrix}2&1\\-2&1\end{bmatrix}$ $R_1+R_2\to R_2 \Rightarrow \begin{bmatrix}2&1\\0&0\end{bmatrix}$ let $v_2=t$

$2v_1+t=0 \Rightarrow 2v_1=-t \Rightarrow v_1=-\frac{t}{2}$

$\begin{bmatrix}v_1\\v_2\end{bmatrix}=\begin{bmatrix}-1/2\,t\\t\end{bmatrix}=\boxed{t\begin{bmatrix}-1/2\\1\end{bmatrix} \quad t\neq 0}$ let $t=2 \Rightarrow \begin{bmatrix}-1\\2\end{bmatrix}$ eigenvector corresponding to $\lambda=3$

form $P=\begin{bmatrix}-1&-1\\1&2\end{bmatrix}$ P diagonalizes A and $P^{-1}AP=\begin{bmatrix}2&0\\0&3\end{bmatrix}$

$\bar{w}'=(P^{-1}AP)\bar{w} \Rightarrow \begin{bmatrix}w_1'\\w_2'\end{bmatrix}=\begin{bmatrix}2&0\\0&3\end{bmatrix}\begin{bmatrix}w_1\\w_2\end{bmatrix} \Rightarrow w_1=c_1e^{2t}$ $w_2=c_2e^{3t}$

$\bar{y}=P\bar{w} \Rightarrow \begin{bmatrix}y_1\\y_2\end{bmatrix}=\begin{bmatrix}-1&-1\\1&2\end{bmatrix}\begin{bmatrix}w_1\\w_2\end{bmatrix} \Rightarrow y_1=-w_1-w_2 \Rightarrow y_1=-c_1e^{2t}-c_2e^{3t}$

$y_2=w_1+2w_2 \qquad y_2=c_1e^{2t}+2c_2e^{3t}$

$y_1'=y_1-y_2=\boxed{-2c_1e^{2t}-3c_2e^{3t}}$

$y_2'=2y_1+4y_2=-2c_1e^{2t}-2c_2e^{3t}+4c_1e^{2t}+8c_2e^{3t}=\boxed{2c_1e^{2t}+6c_2e^{3t}}$

solve the system of linear differential equations

a) $y_1' = 5y_1 \quad y_2' = -3y_2$

$y_1 = c_1 e^{5t} \quad y_2 = c_2 e^{-3t} \Rightarrow \boxed{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} c_1 e^{5t} \\ c_2 e^{-3t} \end{bmatrix}}$

b) $y_1' = y_1 + 2y_2 \quad y_2' = 2y_1 + y_2$

matrix form $\Rightarrow \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ so $y' = Ay$

find a matrix P that diagonalizes A $\wedge$ make a change of variables $y = Pw$

$\lambda I - A = \begin{bmatrix} \lambda-1 & -2 \\ -2 & \lambda-1 \end{bmatrix} \quad \det(\lambda I - A) = (\lambda-1)^2 - (2)^2 = \lambda^2 - 2\lambda + 1 - 4 = (\lambda-3)(\lambda+1) \quad \lambda = 3, -1$

$\underline{\lambda = 3} \Rightarrow \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} R_1 + R_2 \to R_2 \Rightarrow \begin{bmatrix} 2 & -2 \\ 0 & 0 \end{bmatrix} \frac{1}{2}R_1 \to R_1 \Rightarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ let $v_2 = t$

$v_1 - (t) = 0 \Rightarrow v_1 = t$

$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix} = \boxed{t \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad t \neq 0}$

$\lambda = -1 \Rightarrow \begin{bmatrix} -2 & -2 \\ -2 & -2 \end{bmatrix} R_1 - R_2 \to R_2 \Rightarrow \begin{bmatrix} -2 & -2 \\ 0 & 0 \end{bmatrix} -\frac{1}{2}R_1 \to R_1 \Rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ let $v_2 = t$

$v_1 + t = 0 \Rightarrow v_1 = -t$

$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix} = \boxed{t \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad t \neq 0}$

letting $t = 1$ for both $\Rightarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

form $P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ P diagonalizes A and $P^{-1}AP = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$

$w' = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} w \Rightarrow w = \begin{bmatrix} c_1 e^{3t} \\ c_2 e^{-t} \end{bmatrix}$

$y = Pw = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 e^{3t} \\ c_2 e^{-t} \end{bmatrix} = \boxed{\begin{bmatrix} c_1 e^{3t} - c_2 e^{-t} \\ c_1 e^{3t} + c_2 e^{-t} \end{bmatrix}}$

c) $y_1' = y_1 - 4y_2 \quad y_2' = -2y_1 + 8y_2$

matrix form $\Rightarrow \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 1 & -4 \\ -2 & 8 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ so $y' = Ay$

find a matrix P that diagonalizes A $\wedge$ makes a change of variables $y = Pw$

$\lambda I - A = \begin{bmatrix} \lambda-1 & 4 \\ 2 & \lambda-8 \end{bmatrix} \quad \det(\lambda I - A) = (\lambda-1)(\lambda-8) - (4)(2) = \lambda^2 - 9\lambda = \lambda(\lambda-9) \quad \lambda = 0, 9$

$\underline{\lambda = 0} \Rightarrow A = \begin{bmatrix} -1 & 4 \\ 2 & -8 \end{bmatrix} 2R_1 + R_2 \to R_2 \Rightarrow \begin{bmatrix} -1 & 4 \\ 0 & 0 \end{bmatrix} -R_1 \to R_1 \Rightarrow \begin{bmatrix} 1 & -4 \\ 0 & 0 \end{bmatrix}$ let $v_2 = t$

$v_1 - 4(t) = 0 \Rightarrow v_1 = 4t$

$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 4t \\ t \end{bmatrix} = t \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

$\underline{\lambda = 9} \Rightarrow \begin{bmatrix} 8 & 4 \\ 2 & 1 \end{bmatrix} R_1 - 4R_2 \to R_2 \Rightarrow \begin{bmatrix} 8 & 4 \\ 0 & 0 \end{bmatrix} \frac{1}{8}R_1 \to R_1 \Rightarrow \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 0 \end{bmatrix}$ let $v_2 = t$

$v_1 + \frac{1}{2}(t) = 0 \Rightarrow v_1 = -\frac{1}{2}t$

$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}t \\ t \end{bmatrix} = \boxed{t \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} \quad t \neq 0}$

letting $t = 1 \wedge t = 2 \Rightarrow \begin{bmatrix} 4 \\ 1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

from $P = \begin{bmatrix} 4 & -1 \\ 1 & 2 \end{bmatrix}$ P diagonalizes A and $P^{-1}AP = \begin{bmatrix} 0 & 0 \\ 0 & 9 \end{bmatrix}$

$w' = \begin{bmatrix} 0 & 0 \\ 0 & 9 \end{bmatrix} w \Rightarrow w = \begin{bmatrix} c_1 e^{0t} \\ c_2 e^{9t} \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 e^{9t} \end{bmatrix}$

$y = Pw = \begin{bmatrix} 4 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 e^{9t} \end{bmatrix} = \boxed{\begin{bmatrix} 4c_1 - c_2 e^{9t} \\ c_1 + 2c_2 e^{9t} \end{bmatrix}}$

Symmetric Matricies and Orthogonal Diagonalization

Symmetric Matricies

- a matrix is symmetric if $A = A^T$ ie. $A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & -1 & 6 \\ 4 & 6 & 3 \end{bmatrix}$
- suppose A is a symmetric matrix:
  1. A is diagonizable
  2. all eigenvalues of A are real
  3. if λ is an eigenvalue of A, the dimension of the eigenspace of λ is equal to the multiplicity of λ

Verify Properties

ex - find the eigenvalues of the symmetric matrix A and find the dimensions of the corresponding eigenspaces $A = \begin{bmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{bmatrix}$

$\lambda I - A = \begin{bmatrix} \lambda & -2 & -2 \\ -2 & \lambda & -2 \\ -2 & -2 & \lambda \end{bmatrix}$ $\det(\lambda I - A) = (\lambda)\begin{vmatrix} \lambda & -2 \\ -2 & \lambda \end{vmatrix} + (-2)(-1)\begin{vmatrix} -2 & -2 \\ -2 & \lambda \end{vmatrix} + (-2)\begin{vmatrix} -2 & \lambda \\ -2 & -2 \end{vmatrix} = \lambda(\lambda^2 - 4) + 2(-2\lambda - 4)$

$-2(4 + 2\lambda) = \lambda^3 - 4\lambda - 4\lambda - 8 - 8 - 4\lambda = \lambda^3 - 12\lambda - 16 = \ldots = (\lambda + 2)^2(\lambda - 4)$ $\lambda = -2, 4$ $m=1$

for $\lambda = 4 \Leftrightarrow A = \begin{bmatrix} 4 & -2 & -2 \\ -2 & 4 & -2 \\ -2 & -2 & 4 \end{bmatrix}$ $2R_2 + R_1 \to R_2 \Leftrightarrow \begin{bmatrix} 4 & -2 & -2 \\ 0 & 6 & -6 \\ -2 & -2 & 4 \end{bmatrix}$ $2R_3 + R_1 \to R_3 \Leftrightarrow \begin{bmatrix} 4 & -2 & -2 \\ 0 & 6 & -6 \\ 0 & -6 & 6 \end{bmatrix}$ $R_2 + R_3 \to R_3 \Leftrightarrow$

$\begin{bmatrix} 4 & -2 & -2 \\ 0 & 6 & -6 \\ 0 & 0 & 0 \end{bmatrix}$ $\frac{1}{4}R_1 \to R_1$ ^ $\frac{1}{6}R_2 \to R_2 \Leftrightarrow \begin{bmatrix} 1 & -1/2 & -1/2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ let $v_3 = t$

$v_2 - (t) = 0 \Leftrightarrow v_2 = t$

$v_1 - \frac{1}{2}(t) - \frac{1}{2}(t) = 0 \Leftrightarrow v_1 = t$

$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} t \\ t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $t \neq 0$ dimension of the eigenspace of $\lambda = 4$ is 1

for $\lambda = -2 \Leftrightarrow A = \begin{bmatrix} -2 & -2 & -2 \\ -2 & -2 & -2 \\ -2 & -2 & -2 \end{bmatrix}$ $R_1 - R_2 \to R_2$ ^ $R_1 - R_3 \to R_3 \Leftrightarrow \begin{bmatrix} -2 & -2 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $-\frac{1}{2}R_1 \to R_1 \Leftrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ let $v_3 = t$ ^ $v_2 = s$

$v_1 + (s) + (t) = 0 \Leftrightarrow v_1 = -s - t$

$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -s-t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ dimension of the eigenspace of $\lambda = -2$ is 2

form $P \Leftrightarrow \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ and $P^{-1}AP = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$ A is diagonizable

Orthogonal Diagonalization

- orthogonally diagonizable - orthogonal matrix P such that $P^{-1}AP$ is a diagonal matrix
- orthogonal is a square matrix ^ $P^{-1} = P^T$
- the colums of an orthogonal matrix form an orthonormal set. a square matrix whose columns form an orthonormal set is orthogonal; ie. $\begin{bmatrix} \sqrt{3}/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & \sqrt{3}/2 \end{bmatrix}$
- if a square matrix is orthogonally diagonizable, then it is symmetric $A = \begin{bmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{bmatrix}$

^ $\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right\}$ basis for eigenspace of $\lambda = 4$ ^ $\left\{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right\}$ basis for eigenspace of $\lambda = -2$

$\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right\}$ linearly independent set

for a symmetric matrix, two eigenvectors corresponding to distinct eigenvalues are orthogonal to each other.

so $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ is not orthogonal to $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ ← apply gram-schmidt $\bar{v}_1, \bar{v}_2, \bar{v}_3$

let $\bar{w}_2 = \bar{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

$\bar{w}_3 = \bar{v}_3 - \left(\frac{\bar{v}_3 \cdot \bar{w}_2}{\bar{w}_2 \cdot \bar{w}_2}\right)\bar{w}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1/2 \\ -1/2 \\ 0 \end{bmatrix} = \begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix}$

normalize $\bar{v}_1, \bar{w}_2, \bar{w}_3$

$\bar{u}_1 = \frac{\bar{v}_1}{\|\bar{v}_1\|} = \frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}1/\sqrt{3}\\1/\sqrt{3}\\1/\sqrt{3}\end{bmatrix}$

$\bar{u}_2 = \frac{\bar{w}_2}{\|\bar{w}_2\|} = \frac{1}{\sqrt{2}}\begin{bmatrix}-1\\1\\0\end{bmatrix} = \begin{bmatrix}-1/\sqrt{2}\\1/\sqrt{2}\\0\end{bmatrix}$ $\gg$ $P = \begin{bmatrix}1/\sqrt{3} & -1/\sqrt{2} & -\sqrt{6}/6\\1/\sqrt{3} & 1/\sqrt{2} & -\sqrt{6}/6\\1/\sqrt{3} & 0 & \sqrt{6}/3\end{bmatrix}$ P is an orthogonal matrix

$\bar{u}_3 = \frac{\bar{w}_3}{\|\bar{w}_3\|} = \frac{1}{\sqrt{3/2}}\begin{bmatrix}-1/2\\-1/2\\1\end{bmatrix} = \frac{\sqrt{6}}{3}\begin{bmatrix}-1/2\\-1/2\\1\end{bmatrix} = \begin{bmatrix}-\sqrt{6}/6\\-\sqrt{6}/6\\\sqrt{6}/3\end{bmatrix}$

$P^{-1}AP = \begin{bmatrix}4&0&0\\0&-2&0\\0&0&-2\end{bmatrix}$ $P^{-1} = P^T = \begin{bmatrix}1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3}\\-1/\sqrt{2} & 1/\sqrt{2} & 0\\-\sqrt{6}/6 & -\sqrt{6}/6 & \sqrt{6}/3\end{bmatrix}$

$P^{-1}AP = [\ ][A][\ ] = \begin{bmatrix}4&0&0\\0&-2&0\\0&0&-2\end{bmatrix}$

Summary
- ① find the eigenvalues and corresponding eigenvectors
- ② if an eigenvalue λ has multiplicity 1, then a basis for the eigenspace of λ consists of a single eigenvector. this basis is orthogonal. Normalize the single eigenvector.
- ③ if an eigenvalue λ has multiplicity $k \geq 2$, then a basis for the eigenspace of λ consists of k eigenvectors. if this basis is not already orthonormal, apply the gram-schmidt orthonormalization process to this basis.
- ④ After performing the above steps for each eigenvalue, we'll end up with n eigenvectors that form an orthonormal set. Use these n eigenvectors to form the columns of P. P will be an orthogonal matrix that diagonalizes A.

Spectral Theorem for Symmetric Matrices
- ① A is orthogonally diagonizable
- ② all eigenvalues of A are real.
- ③ If λ is an eigenvalue of A, the dimension of the eigenspace of λ is equal to the multiplicity of λ
- ④ eigenvectors corresponding to different eigenvalues are orthogonal.

- spectral theorem for symmetric matrices - spectrum of A = set of eigenvalues of A

ex - find an orthogonal matrix P that diagonalizes A. then show that $P^{-1}AP$ is a diagonal matrix with the eigenvalues of A along the diagonal.

a) $A = \begin{bmatrix}4&2\\2&4\end{bmatrix}$

$\lambda I - A = \begin{bmatrix}\lambda-4 & -2\\-2 & \lambda-4\end{bmatrix}$ $\det(\lambda I - A) = (\lambda-4)^2 - (-2)^2 = \lambda^2 - 8\lambda + 16 - 4 = (\lambda-6)(\lambda-2)$ $\lambda = 2, 6$

$\underline{\lambda = 2} \Leftrightarrow A = \begin{bmatrix}-2&-2\\-2&-2\end{bmatrix}$ $R_1 + R_2 \to R_2$ ^ $-\frac{1}{2}R_1 \to R_1 \Leftrightarrow \begin{bmatrix}1&1\\0&0\end{bmatrix}$ let $v_2 = t$

$v_1 + (t) = 0 \Leftrightarrow v_1 = -t$

$\begin{bmatrix}v_1\\v_2\end{bmatrix} = \begin{bmatrix}t\\-t\end{bmatrix} = t\begin{bmatrix}1\\-1\end{bmatrix}$ $t \neq 0 \Leftrightarrow \left\{\begin{bmatrix}1\\-1\end{bmatrix}\right\}$ basis for eigenspace $\lambda = 2$

$\lambda = 6 \Leftrightarrow \begin{bmatrix}2&-2\\-2&2\end{bmatrix}$ $R_1 + R_2 \to R_2$ ^ $\frac{1}{2}R_1 \to R_1 \Leftrightarrow \begin{bmatrix}1&-1\\0&0\end{bmatrix}$ let $v_2 = t$

$v_1 - (t) = 0 \Leftrightarrow v_1 = t$

$\begin{bmatrix}v_1\\v_2\end{bmatrix} = \begin{bmatrix}t\\t\end{bmatrix} = t\begin{bmatrix}1\\1\end{bmatrix}$ $t \neq 0 \Leftrightarrow \left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}$ basis for eigenspace $\lambda = 6$

$\begin{bmatrix}-1\\1\end{bmatrix}$ ^ $\begin{bmatrix}1\\1\end{bmatrix}$ correspond to distinct eigenvalues so are orthogonal to eachother ^ just need to normalize

$u_1 = \frac{1}{\sqrt{2}}\begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}-1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix}$ $u_2 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix}$ $\Leftrightarrow \left\{\begin{bmatrix}-1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix}, \begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix}\right\}$ is an orthonormal set of 2 eigenvectors

form $P = \begin{bmatrix}-1/\sqrt{2} & 1/\sqrt{2}\\1/\sqrt{2} & 1/\sqrt{2}\end{bmatrix}$ P is an orthogonal matrix that diagonalizes A

$P^{-1}AP = P^TAP = \overset{\text{in this case}}{P}AP = \begin{bmatrix}-4/\sqrt5 & 2/\sqrt5\\ 6/\sqrt5 & 6/\sqrt5\end{bmatrix}\begin{bmatrix}-2/\sqrt5 & 1/\sqrt5\\ 1/\sqrt5 & 2/\sqrt5\end{bmatrix} = \begin{bmatrix}2 & 0\\ 0 & 6\end{bmatrix}$

b) $A = \begin{bmatrix}3 & -2 & 4\\ -2 & 6 & 2\\ 4 & 2 & 3\end{bmatrix}$

$\lambda I - A = \begin{bmatrix}\lambda-3 & 2 & -4\\ 2 & \lambda-6 & -2\\ -4 & -2 & \lambda-3\end{bmatrix}$ $\det(\lambda I - A) = \underset{\text{along 1st row}}{\text{cofactor expansion}} = (\lambda-3)\begin{vmatrix}\lambda-6 & -2\\ -2 & \lambda-3\end{vmatrix} + (-1)(2)\begin{vmatrix}2 & -2\\ -4 & \lambda-3\end{vmatrix} + (-4)\begin{vmatrix}2 & \lambda-6\\ -4 & -2\end{vmatrix} =$

$\ldots = (\lambda-7)^2(\lambda+2) \Leftrightarrow \lambda = \underset{\text{am}=2}{7}, -2 \ \text{am}=1$

$\lambda = 7 \Rightarrow A = \begin{bmatrix}4 & 2 & -4\\ 2 & 1 & -2\\ -4 & -2 & 4\end{bmatrix}$ $2R_2 + R_3 \to R_3 \Leftrightarrow \begin{bmatrix}4 & 2 & -4\\ 2 & 1 & -2\\ 0 & 0 & 0\end{bmatrix}$ $R_1 - 2R_2 \to R_2$ ^ $\frac{1}{4}R_1 \to R_1 \Leftrightarrow \begin{bmatrix}1 & 1/2 & -1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$ let $v_3 = t$, $v_2 = s$

$v_1 + \frac{1}{2}(s) - (t) = 0 \Rightarrow v_1 = -\frac{1}{2}s + t$

$\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix} = \begin{bmatrix}-1/2 s + t\\ s\\ t\end{bmatrix} = s\begin{bmatrix}-1/2\\ 1\\ 0\end{bmatrix} + t\begin{bmatrix}1\\ 0\\ 1\end{bmatrix}$ where s ^ t are not both $0 \Leftrightarrow \left\{\begin{bmatrix}-1/2\\ 1\\ 0\end{bmatrix}, \begin{bmatrix}1\\ 0\\ 1\end{bmatrix}\right\}$ basis for eigenspace $\lambda = 7$

$\lambda = -2 \Rightarrow A = \begin{bmatrix}-5 & 2 & -4\\ 2 & -8 & -2\\ -4 & -2 & -5\end{bmatrix}$ $2R_2 + R_3 \to R_3 \Leftrightarrow \begin{bmatrix}-5 & 2 & -4\\ 2 & -8 & -2\\ 0 & -18 & -9\end{bmatrix}$ $\frac{1}{2}R_2 \to R_2$ ^ $R_1 + 5R_2 \to R_2 \Leftrightarrow \begin{bmatrix}-5 & 2 & -4\\ 0 & -18 & -9\\ 0 & -18 & -9\end{bmatrix}$ $R_2 - R_3 \to R_3$ ^

$-\frac{1}{18}R_2 \to R_2 \Leftrightarrow \begin{bmatrix}-5 & 2 & -4\\ 0 & 1 & 1/2\\ 0 & 0 & 0\end{bmatrix}$ let $v_3 = t$

$v_2 + \frac{1}{2}(t) = 0 \Leftrightarrow v_2 = -\frac{1}{2}t$

$-5v_1 - 2(\frac{1}{2}t) - 4(t) = 0 \Leftrightarrow -5v_1 = t + 4t \Leftrightarrow v_1 = -t$

$\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix} = \begin{bmatrix}-t\\ -1/2 t\\ t\end{bmatrix} = t\begin{bmatrix}-1\\ -1/2\\ 1\end{bmatrix}$ $t \neq 0 \Leftrightarrow \left\{\begin{bmatrix}-1\\ -1/2\\ 1\end{bmatrix}\right\}$ basis for eigenspace $\lambda = -2$ $\Leftrightarrow$ the basis are orthogonal | normalize

this one $u_1 = \frac{1}{\sqrt{9/4}}\begin{bmatrix}-1\\ -1/2\\ 1\end{bmatrix} = \begin{bmatrix}-2/3\\ -1/3\\ 2/3\end{bmatrix}$

for $\lambda = 7$ not orthonormal so gram-schmidt

$v_2 = \begin{bmatrix}-1/2\\ 1\\ 0\end{bmatrix}$ ^ $v_3 = \begin{bmatrix}1\\ 0\\ 1\end{bmatrix} \Leftrightarrow w_2 = v_2$

$w_3 = v_3 - \frac{v_3 \cdot v_2}{v_2 \cdot v_2} v_2 = \begin{bmatrix}1\\ 0\\ 1\end{bmatrix} - \frac{(-1/2)}{(5/4)}\begin{bmatrix}-1/2\\ 1\\ 0\end{bmatrix} = \begin{bmatrix}4/5\\ 2/5\\ 1\end{bmatrix}$

normalize w_2 ^ $w_3 \Rightarrow u_2 = \frac{1}{\|w_2\|}w_2 = \frac{1}{\sqrt{5/4}}\begin{bmatrix}-1/2\\ 1\\ 0\end{bmatrix} = \begin{bmatrix}-1/\sqrt5\\ 2/\sqrt5\\ 0\end{bmatrix}$ $u_3 = \frac{1}{\sqrt{9/5}}\begin{bmatrix}4/5\\ 2/5\\ 1\end{bmatrix} = \begin{bmatrix}4/3\sqrt5\\ 2/3\sqrt5\\ 5/3\sqrt5\end{bmatrix}$

now $\{u_1, u_2, u_3\}$ = orthonormal set of 3 eigenvectors

form $P = \begin{bmatrix}-2/3 & -1/\sqrt5 & 4/3\sqrt5\\ -1/3 & 2/\sqrt5 & 2/3\sqrt5\\ 2/3 & 0 & 5/3\sqrt5\end{bmatrix}$ P is an orthogonal matrix that diagonalizes A

check that $P^{-1}AP = \begin{bmatrix}-2 & 0 & 0\\ 0 & 7 & 0\\ 0 & 0 & 7\end{bmatrix}$

$P^{-1}AP = P^TAP = \begin{bmatrix}4/3 & 2/3 & -4/3\\ -7/\sqrt5 & 14/\sqrt5 & 0\\ 28/3\sqrt5 & 14/3\sqrt5 & 35/3\sqrt5\end{bmatrix}\begin{bmatrix}-2/3 & -1/\sqrt5 & 4/3\sqrt5\\ -1/3 & 2/\sqrt5 & 2/3\sqrt5\\ 2/3 & 0 & 5/3\sqrt5\end{bmatrix} = \begin{bmatrix}-2 & 0 & 0\\ 0 & 7 & 0\\ 0 & 0 & 7\end{bmatrix}$ ✓

Quadratic Forms

- a quadric form on $\mathbb{R}^n$ $Q: \mathbb{R}^n \to \mathbb{R}$ given by $Q(\vec{x}) = \vec{x}^T A \vec{x}$ where A is a symmetric matrix

ex - $A = \begin{bmatrix}2 & 0\\ 0 & 6\end{bmatrix}$ find $Q(x_1, x_2)$

$Q(x_1, x_2) = [x_1\ x_2]\begin{bmatrix}2 & 0\\ 0 & 6\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} = [x_1\ x_2]\begin{bmatrix}2x_1\\ 6x_2\end{bmatrix} = \boxed{2x_1^2 + 6x_2^2}$

ex - $A = \begin{bmatrix}-4 & 7\\ 7 & 5\end{bmatrix}$ find $Q(x_1, x_2)$

$Q(x_1, x_2) = [x_1\ x_2]\begin{bmatrix}-4 & 7\\ 7 & 5\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} = [x_1\ x_2]\begin{bmatrix}-4x_1 + 7x_2\\ 7x_1 + 5x_2\end{bmatrix} = x_1(-4x_1 + 7x_2) + x_2(7x_1 + 5x_2) = -4x_1^2 + 7x_1x_2$
$+ 7x_1x_2 + 5x^2 = -4x_1^2 + 14x_1x_2 + 5x^2$

ex - $A = \begin{bmatrix}-1 & 2 & 5\\ 2 & 4 & -3\\ 5 & -3 & 3\end{bmatrix}$ find $Q(x_1, x_2, x_3)$

$Q(x_1, x_2, x_3) = [x_1\ x_2\ x_3]\begin{bmatrix}-1 & 2 & 5\\ 2 & 4 & -3\\ 5 & -3 & 3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} = [x_1\ x_2\ x_3]\begin{bmatrix}-x_1 + 2x_2 + 5x_3\\ 2x_1 + 4x_2 - 3x_3\\ 5x_1 - 3x_2 + 3x_3\end{bmatrix} = -x_1^2 + 2x_1x_2 + 5x_1x_3 + 2x_2x_1 + 4x_2^2 - 3x_2x_3$
$+ 5x_3x_1 - 3x_3x_2 + 3x_3^2 = -x_1^2 + 4x_1x_2 + 4x_2^2 + 10x_1x_3 - 6x_2x_3 + 3x_3^2$

Eliminating Cross-Product Terms

- we can orthogonally diagonalize A so that $P^{-1}AP$ is a diagonal matrix, where P is an orthogonal matrix
  $\bar{x} = P\bar{y}$ $\quad Q(\bar{x}) = \bar{x}^T A \bar{x} = (P\bar{y})^T A (P\bar{y}) = (\bar{y}^T P^T) A (P\bar{y}) = \bar{y}^T (P^T A P)\bar{y} = \bar{y}^T (P^{-1}AP)\bar{y} =$
  $\bar{y}^T D \bar{y}$ where D is a diagonal matrix $= R(\bar{y})$ where R is the quadratic form whose matrix is $D = R(P^{-1}\bar{x}) = R(P^T\bar{x})$ (b/c $\bar{y} = P^{-1}\bar{x}$) so $Q(\bar{x}) = R(P^T\bar{x})$

ex - $A = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}$ is the matrix of the quadratic form Q. make a change of variable to transform Q into a quadratic form R that has no cross-product terms.

$\lambda I - A = \begin{bmatrix} \lambda+5 & -2 \\ -2 & \lambda+2 \end{bmatrix}$ $\quad \det(\lambda I - A) = (\lambda+5)(\lambda+2) - 4 = \lambda^2 + 7\lambda + 6 = (\lambda+6)(\lambda+1) \Leftrightarrow \lambda = -1, -6$

for $\underline{\lambda = -6} \Leftrightarrow A = \begin{bmatrix} -1 & -2 \\ -2 & -4 \end{bmatrix}$ $\quad -2R_1 + R_2 \to R_2 \Leftrightarrow \begin{bmatrix} -1 & -2 \\ 0 & 0 \end{bmatrix}$ let $v_2 = t$

$-v_1 - 2(t) = 0 \Leftrightarrow -v_1 = 2t \Leftrightarrow v_1 = -2t$

$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -2t \\ t \end{bmatrix} = t\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ $t \neq 0$ $= \bar{v}_1$

$\|\bar{v}_1\| = \sqrt{(-2)^2 + (1)^2} = \sqrt{5}$

$\bar{u}_1 = \frac{1}{\sqrt{5}}\begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$

for $\underline{\lambda = -1} \Leftrightarrow A = \begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix}$ $\quad 2R_2 + R_1 \to R_2 \Leftrightarrow \begin{bmatrix} 4 & -2 \\ 0 & 0 \end{bmatrix}$ let $v_2 = t$

$4v_1 - 2(t) = 0 \Leftrightarrow 4v_1 = 2t \Leftrightarrow v_1 = \frac{1}{2}t$

$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1/2\, t \\ t \end{bmatrix} = t\begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$ $t \neq 0$ $= \bar{v}_2$

$\|\bar{v}_2\| = \sqrt{(\frac{1}{2})^2 + (1)^2} = \sqrt{5/4} = \frac{\sqrt{5}}{2}$

$\bar{u}_2 = \frac{1}{\sqrt{5}/2}\begin{bmatrix} 1/2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$

form $P = \begin{bmatrix} -2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}$ $\quad P^{-1}AP = \begin{bmatrix} -6 & 0 \\ 0 & -1 \end{bmatrix}$

$\bar{x} = P\bar{y}$ $\quad Q(\bar{x}) = R(P^T\bar{x})$ where R is the quadratic form whose matrix is $\begin{bmatrix} -6 & 0 \\ 0 & -1 \end{bmatrix}$ for $\bar{x} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$

$Q(\bar{x}) = -5x_1^2 + 4x_1x_2 - 2x_2^2$

$Q(4,-2) = -5(4)^2 + 4(4)(-2) - 2(-2)^2 = -80 - 32 - 8 = \underline{-120}$

$P^T\bar{x} = \begin{bmatrix} -2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}\begin{bmatrix} 4 \\ -2 \end{bmatrix} = \begin{bmatrix} -10/\sqrt{5} \\ 0 \end{bmatrix}$ ✓

$R(P^T\bar{x}) = -6(\frac{-10}{\sqrt{5}})^2 - \cancel{1(0)^2} = -6(\frac{100}{5}) = \underline{-120}$

Principal Axes Theorem

- we can always transform a quadratic form Q into a quadratic form R that has no cross-product terms by orthogonally diagonalizing the matrix A of Q and making a change of variable $\bar{x} = P\bar{y}$
- the columns of P form an orthonormal basis for $\mathbb{R}^n$ and are called the principal axes of Q
- $\bar{x} = P\bar{y}$ where $\bar{y}$ is the coordinate matrix of $\bar{x}$ relative to the orthonormal basis formed by the columns of P

ex - suppose A is the matrix of the quadratic form Q. make a change of variable to transform Q into a quadratic form R that has no cross-product terms. then verify that $Q(\bar{x}) = R(P^T\bar{x})$ for the given vector $\bar{x}$. $\quad A = \begin{bmatrix} 9 & -4 \\ -4 & 3 \end{bmatrix}$ $\quad \bar{x} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$

$\lambda I - A = \begin{bmatrix} \lambda-9 & 4 \\ 4 & \lambda-3 \end{bmatrix}$ $\det(\lambda I - A) = (\lambda-9)(\lambda-3)-16 = \lambda^2-12\lambda+27-16 = (\lambda-11)(\lambda-1)$ $\lambda = 11, 1$

for $\lambda = 1 \Rightarrow A = \begin{bmatrix} -8 & 4 \\ 4 & -2 \end{bmatrix}$ $R_1+2R_2 \to R_2 \Rightarrow \begin{bmatrix} -8 & 4 \\ 0 & 0 \end{bmatrix}$ $-\frac{1}{8}R_1 \to R_1 \Rightarrow \begin{bmatrix} 1 & -1/2 \\ 0 & 0 \end{bmatrix}$ let $v_2 = t$

$v_1 - \frac{1}{2}(t) = 0 \Rightarrow v_1 = \frac{1}{2}t$

$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1/2\, t \\ t \end{bmatrix} = t\begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$ $t \neq 0$ let $t = 2 \Rightarrow \bar{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

for $\lambda = 11 \Rightarrow A = \begin{bmatrix} 2 & 4 \\ 4 & 8 \end{bmatrix}$ $2R_1 - R_2 \to R_2 \Rightarrow \begin{bmatrix} 2 & 4 \\ 0 & 0 \end{bmatrix}$ $\frac{1}{2}R_1 \to R_1 \Rightarrow \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$ let $v_2 = t$

$v_1 + 2(t) = 0 \Rightarrow v_1 = -2t$

$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -2t \\ t \end{bmatrix} = t\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ $t \neq 0$ let $t = 1 \Rightarrow \bar{v}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$\bar{v}_1$ ^ $\bar{v}_2$ are orthogonal to each other so just normalize

$\bar{u}_1 = \frac{1}{\|v_1\|}v_1 = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$ $\bar{u}_2 = \frac{1}{\|v_2\|}v_2 = \frac{1}{\sqrt{5}}\begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$

$\left\{\begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}, \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}\right\}$ is an orthonormal set of 2 eigenvectors $\Rightarrow$ form $P = \begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$

P is an orthogonal matrix that diagonalizes A, and $P^{-1}AP = \begin{bmatrix} 1 & 0 \\ 0 & 11 \end{bmatrix}$

check $Q(\bar{x}) = R(P^T\bar{x})$ for $\bar{x} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$

$Q(\bar{x}) = [x_1\ x_2]\begin{bmatrix} 9 & -4 \\ -4 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = [x_1\ x_2]\begin{bmatrix} 9x_1-4x_2 \\ -4x_1+3x_2 \end{bmatrix} = 9x_1^2-4x_1x_2-4x_1x_2+3x_2^2 = 9x_1^2-8x_1x_2+3x_2^2$

$Q(4,-1) = 9(4)^2-8(4)(-1)+3(-1)^2 = 144+32+3 = 179$

$R(P^T\bar{x}) \Rightarrow P^T\bar{x} = \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}\begin{bmatrix} 4 \\ -1 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{5} \\ -9/\sqrt{5} \end{bmatrix} \Rightarrow R(P^T\bar{x}) = 1\left(\frac{2}{\sqrt{5}}\right)^2 + 11\left(\frac{-9}{\sqrt{5}}\right)^2 = \frac{4}{5} + \frac{891}{5} = \frac{895}{5} = 179$ ✓

Singular Value Decomposition

- A is diagonalizable iff A has n linearly independent eigenvectors
- A is orthogonally diagonizable iff A is symmetric
- m×n matrix A $A = PDP^{-1}$ D is diagonal $A = U\Sigma V^{-1}$ where Σ kinda diagonal where $\begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}$ D diagonal matrix consisting of the square roots of the positive eigenvalues of A^TA

  U ^ V are orthogonal $V^{-1} = V^T \Rightarrow (A = U\Sigma V^T)$ singular value decomposition of A, singular values of A

ex - find a singular value decomposition of A $A = \begin{bmatrix} 7 & 1 \\ 0 & 0 \\ 5 & 5 \end{bmatrix}$

find eigenvalues/eigenvectors of $A^TA = \begin{bmatrix} 7 & 0 & 5 \\ 1 & 0 & 5 \end{bmatrix}\begin{bmatrix} 7 & 1 \\ 0 & 0 \\ 5 & 5 \end{bmatrix} = \begin{bmatrix} 49+25 & 7+25 \\ 7+25 & 1+25 \end{bmatrix} = \begin{bmatrix} 74 & 32 \\ 32 & 26 \end{bmatrix}$

$\lambda I - A^TA = \begin{bmatrix} \lambda-74 & -32 \\ -32 & \lambda-26 \end{bmatrix}$ $\det(\lambda I - A^TA) = (\lambda-74)(\lambda-26)-(-32)^2 = \ldots = (\lambda-90)(\lambda-10) \Rightarrow \lambda = 10, 90$

for $\lambda = 90 \Rightarrow A = \begin{bmatrix} 16 & -32 \\ -32 & 64 \end{bmatrix}$ $2R_1+R_2 \to R_2 \Rightarrow \begin{bmatrix} 16 & -32 \\ 0 & 0 \end{bmatrix}$ $\frac{1}{16}R_1 \to R_1 \Rightarrow \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$ $v_2 = t$

$v_1 - 2(t) = 0 \Rightarrow v_1 = 2t$

$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 2t \\ t \end{bmatrix} = t\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $t \neq 0$

normalize $\Rightarrow u_1 = \frac{1}{\|v_1\|}v_1 = \frac{1}{\sqrt{(2)^2+(1)^2}}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$

for $\lambda = 10 \Rightarrow A = \begin{bmatrix} -64 & -32 \\ -32 & -16 \end{bmatrix}$ $-2R_2+R_1 \to R_2 \Rightarrow \begin{bmatrix} -64 & -32 \\ 0 & 0 \end{bmatrix}$ $-\frac{1}{64}R_1 \to R_1 \Rightarrow \begin{bmatrix} 1 & 1/2 \\ 0 & 0 \end{bmatrix}$ let $v_2 = t$

$v_1 + \frac{1}{2}(t) = 0 \Rightarrow v_1 = -\frac{1}{2}t$

$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -1/2\, t \\ t \end{bmatrix} = t\begin{bmatrix} -1/2 \\ 1 \end{bmatrix}$ $t \neq 0$ let $t = 2 \Rightarrow \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

normalize $\Rightarrow u_2 = \frac{1}{\|v_2\|}v_2 = \frac{1}{\sqrt{(-1)^2+(2)^2}}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \frac{1}{\sqrt{5}}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$

$\{\bar{v}_1, \bar{v}_2\}$ is an orthonormal set of eigenvectors of A^TA

$V = \begin{bmatrix} 3/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 4/\sqrt{5} \end{bmatrix}$

$\lambda_1 = 90 \Rightarrow \sigma_1 = \sqrt{\lambda_1} = \sqrt{90} = 3\sqrt{10}$ $\quad \lambda_2 = 10 \Rightarrow \sigma_2 = \sqrt{\lambda_2} = \sqrt{10}$

$D = \begin{bmatrix} 3\sqrt{10} & \\ & \sqrt{10} \end{bmatrix}$ $\quad \Sigma$ 3x2 matrix with D in the upper left-hand corner and 0's everywhere else

$\Rightarrow \Sigma = \begin{bmatrix} 3\sqrt{10} & 0 \\ 0 & \sqrt{10} \\ 0 & 0 \end{bmatrix}$

- if A has r nonzero singular values, then $A\bar{v}_1, A\bar{v}_2, \ldots, A\bar{v}_r$ ($\Rightarrow$ orthogonal set), after being normalized, will be the first r columns of U

example - $\sigma_1 = 3\sqrt{10}$ $\quad \sigma_2 = \sqrt{10}$ $\quad A\bar{v}_1 = \begin{bmatrix} 7 & 1 \\ 0 & 0 \\ 5 & 5 \end{bmatrix} \begin{bmatrix} 3/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix} = \begin{bmatrix} 15/\sqrt{5} \\ 0 \\ -5/\sqrt{5} \end{bmatrix}$

$\|A\bar{v}_i\| = \sigma_i$ for each i : $\frac{1}{\|A\bar{v}_1\|} A\bar{v}_1 = \frac{1}{\sigma_1} A\bar{v}_1 = \frac{1}{3\sqrt{10}} \begin{bmatrix} 15/\sqrt{5} \\ 0 \\ 15/\sqrt{5} \end{bmatrix} = \begin{bmatrix} 5/\sqrt{50} \\ 0 \\ 5/\sqrt{50} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} = \bar{u}_1$

$A\bar{v}_2 = \begin{bmatrix} 7 & 1 \\ 0 & 0 \\ 5 & 5 \end{bmatrix} \begin{bmatrix} 1/\sqrt{5} \\ 0 \\ -1/\sqrt{5} \end{bmatrix} = \begin{bmatrix} 5/\sqrt{5} \\ 0 \\ -5/\sqrt{5} \end{bmatrix} \Rightarrow \frac{1}{\|A\bar{v}_2\|} A\bar{v}_2 = \frac{1}{\sqrt{10}} \begin{bmatrix} 5/\sqrt{5} \\ 0 \\ -5/\sqrt{5} \end{bmatrix} = \begin{bmatrix} 5/\sqrt{50} \\ 0 \\ -5/\sqrt{50} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix} = \bar{u}_2$

$\bar{x} = (x_1, x_2, x_3)$ such that $\bar{u}_1 \bar{x} = 0 \wedge \bar{u}_2 \bar{x} = 0 \Rightarrow$

$\frac{1}{\sqrt{2}} x_1 + \frac{1}{\sqrt{2}} x_3 = 0 \wedge \frac{1}{\sqrt{2}} x_1 - \frac{1}{\sqrt{2}} x_3 = 0$ $\quad$ eq1+eq2→eq2 $\Rightarrow \frac{2}{\sqrt{2}} x_1 = 0 \Rightarrow x_1 = 0$ $\quad \frac{1}{\sqrt{2}} x_3 = 0 \Rightarrow x_3 = 0$

$\bar{x} = \begin{bmatrix} 0 \\ x_2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $t \neq 0$ $= \bar{u}_3$ $\quad \Rightarrow \{\bar{u}_1, \bar{u}_2, \bar{u}_3\}$ orthonormal basis for $\mathbb{R}^3$

$U = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{bmatrix}$

$A = U \Sigma V^T = \begin{bmatrix} U \end{bmatrix} \begin{bmatrix} 3\sqrt{10} & 0 \\ 0 & \sqrt{10} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 3/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & -2/\sqrt{5} \end{bmatrix}$

ex - 2 skip

Principal Component Analysis

- mean deviation form ^ covariance matrix
- N objects, 1000 college graduates $\begin{bmatrix} 40{,}000 \\ 35{,}000 \end{bmatrix}$ observation vector where $\bar{x}_j$ jth college graduate
  $\begin{bmatrix} s_1 & s_2 & \cdots & s_{1000} \\ d_1 & d_2 & \cdots & d_{1000} \end{bmatrix}$ where $\bar{x}_j = \begin{bmatrix} s_j \\ d_j \end{bmatrix}$ matrix observations $p \times N$ matrix
- $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_N$ N observations vectors $[\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_N]$ $p \times N$ matrix
- sample mean of $\bar{x}_1 \ldots \bar{x}_N$ $\quad \bar{m} = \frac{1}{N}(\bar{x}_1 + \ldots + \bar{x}_N)$ $\quad \hat{x}_j = \bar{x}_j - \bar{m}$
- let $B = [\hat{x}_1 \ \hat{x}_2 \cdots \hat{x}_N]$ $p \times N$ matrix mean-deviation form
- let $S = \frac{1}{N-1} BB^T$ covariance matrix $p \times p$ matrix

Illustration - $\begin{bmatrix} s_1 & s_2 \\ d_1 & d_2 \end{bmatrix}$ $\quad \bar{x}_1 = \begin{bmatrix} s_1 \\ d_1 \end{bmatrix}$ $\quad \bar{x}_2 = \begin{bmatrix} s_2 \\ d_2 \end{bmatrix}$

$\bar{m} = \frac{1}{2}(\bar{x}_1 + \bar{x}_2) = \frac{1}{2}\left(\begin{bmatrix} s_1 \\ d_1 \end{bmatrix} + \begin{bmatrix} s_2 \\ d_2 \end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix} s_1 + s_2 \\ d_1 + d_2 \end{bmatrix} \Rightarrow \bar{s} = \frac{1}{2}(s_1 + s_2)$ $\quad \bar{d} = \frac{1}{2}(d_1 + d_2) \Rightarrow \bar{m} = \begin{bmatrix} \bar{s} \\ \bar{d} \end{bmatrix}$

$\hat{x}_1 = \bar{x}_1 - \bar{m} = \begin{bmatrix} s_1 - \bar{s} \\ d_1 - \bar{d} \end{bmatrix}$ $\quad \hat{x}_2 = \bar{x}_2 - \bar{m} = \begin{bmatrix} s_2 - \bar{s} \\ d_2 - \bar{d} \end{bmatrix}$

$B = \begin{bmatrix} s_1 - \bar{s} & s_2 - \bar{s} \\ d_1 - \bar{d} & d_2 - \bar{d} \end{bmatrix}$ $\quad B^T = \begin{bmatrix} s_1 - \bar{s} & d_1 - \bar{d} \\ s_2 - \bar{s} & d_2 - \bar{d} \end{bmatrix}$ $\quad BB^T = \ldots = [\ldots] \Rightarrow S = \frac{1}{2-1}[\ldots]$

s_{jj} of S variance of x_j $\quad$ total variance $\sum_{j=1}^{p} s_{jj}$

PCA - mean deviation form $\quad S = A^T A$ where $A = \frac{1}{\sqrt{N-1}} B^T$ where A is an $N \times p$ matrix

- if $\lambda_1, \lambda_2, \ldots \lambda_p$ are the p eigenvalues of S, we can order them from greatest to least as $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_p \geq 0$

  $S = A^T A$ where S is symmetric ^ orthogonally diagonalizable

  $\{\bar{v}_1, \bar{v}_2, \ldots \bar{v}_p\}$ orthonormal set of eigenvectors of S corresponding to the eigenvalues $\lambda_1, \ldots, \lambda_p$

  first principal component $\bar{v}_1$, second $\bar{v}_2$, etc

- $\bar{v}_1, \ldots \bar{v}_p$ principal components of data
- last notes: $A = \frac{1}{\sqrt{N-1}} B^T$ $N \times p$ matrix, $A = U \Sigma V^T$ where $\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_p$, $\sigma_1, \ldots, \sigma_p$ of A ^
  $\sigma_1 = \sqrt{\lambda_1}, \ldots, \sigma_p = \sqrt{\lambda_p}$