Integrală triplă

i) Dacă $A = [a,b]\times[c,d]\times[k,p]$ și $f: A \to \mathbb{R}$ e cont atunci A e mult camp și măs J și
$$\iiint_A f(x,y,z)\,dx\,dy\,dz = \int_a^b \left(\int_c^d \left(\int_k^p f(x,y,z)\,dz\right)dy\right)dx$$

ii) Fie $B \subset \mathbb{R}^2$ o mult camp și măs J și $A = \{(x,y,z)\in\mathbb{R}^3 \mid (x,y)\in B$ și $\varphi(x,y) \le z \le \psi(x,y)\}$ unde $\varphi, \psi: B \to \mathbb{R}$ cont

Fie $f: A \to \mathbb{R}$ cont. Atunci A e mult camp și măs J
$$\iiint_A f(x,y,z)\,dx\,dy\,dz = \iint_B \left(\int_{\varphi(x,y)}^{\psi(x,y)} f(x,y,z)\,dz\right)dx\,dy$$

Exemple:

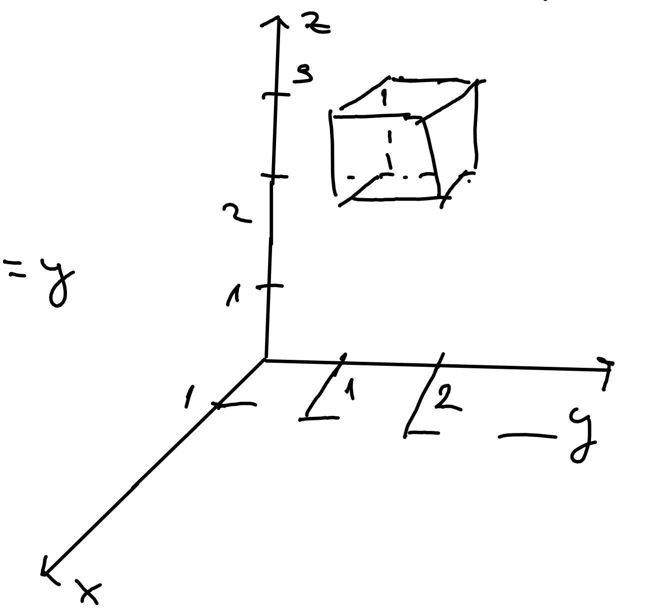
a) $\iiint_A y\,dx\,dy\,dz$, unde $A = [0,1]\times[1,2]\times[2,3]$

$A = [0,1]\times[1,2]\times[2,3] \Rightarrow$

$\Rightarrow A \in J(\mathbb{R}^3)$ și compactă

Fie $f: A \to \mathbb{R}$, $f(x,y,z) = y$

f cont



$$\iiint_A f(x,y,z)\,dx\,dy\,dz = \int_0^1\left(\int_1^2\left(\int_2^3 y\,dz\right)dy\right)dx$$
$$= \int_0^1 \int_1^2 zy\Big|_{z=2}^{z=3} dy\,dx = \dots = \frac{3}{2}$$

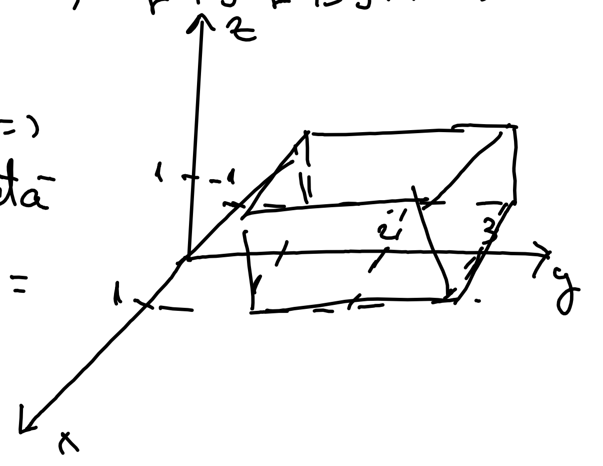
b) $\iiint_A (xyz + y^2)\,dx\,dy\,dz$, $A = [-1,1]\times[2,3]\times[0,1]$

$A = [-1,1]\times[2,3]\times[0,1] \Rightarrow$

$\Rightarrow A \in J(\mathbb{R}^3)$ și compactă

Fie $f: A \to \mathbb{R}$, $f(x,y,z) = xyz + y^2$

f cont



$$\iiint_A f(x,y,z)\,dx\,dy\,dz = \int_{-1}^1\left(\int_2^3\left(\int_0^1 xyz + y^2\,dz\right)dy\right)dx$$
$$= \int_{-1}^1\left(\int_2^3 \frac{xyz^2}{2} + y^2 z\Big|_0^2\right)dy\,dx$$
$$= \int_{-1}^1\left(\int_2^3 2xy + 2y^2\,dy\right)dx$$
$$= \int_{-1}^1 xy^2 + \frac{2}{3}y^3\Big|_2^3 dx$$
$$= \int_{-1}^1 5x + \frac{2}{3}\cdot 19\,dx = \frac{38}{3}$$

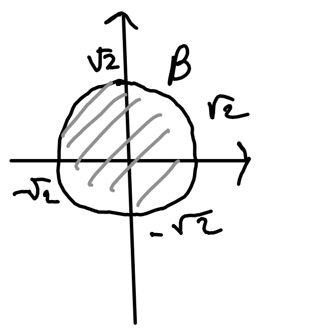
c) $\iiint_A (x^2+y^2)z\,dx\,dy\,dz$, $A = \{(x,y,z)\in\mathbb{R}^3 \mid (x,y)\in B, x^2+y^2 \le z \le \sqrt{6-x^2-y^2}\}$

și $B = \{(x,y)\in\mathbb{R}^2 \mid x^2+y^2 \le 2\}$

$$\iiint_A (x^2+y^2)z\,dx\,dy\,dz = \iint_B \left(\int_{x^2+y^2}^{\sqrt{6-x^2-y^2}} (x^2+y^2)z\,dz\right)dx\,dy$$
$$= \iint_B \left(\frac{(x^2+y^2)}{2} z^2\Big|_{x^2+y^2}^{\sqrt{6-x^2-y^2}}\right)dx\,dy = \iint_B \frac{x^2+y^2}{2}\left(6-x^2-y^2-(x^2+y^2)^2\right)dx\,dy$$

B convexă și mărg $\Rightarrow B \in J(\mathbb{R}^2)$

B camp



Fie $f: B \to \mathbb{R}$, $f(x,y) = \frac{x^2+y^2}{2}\left(6-x^2-y^2-(x^2+y^2)^2\right)$

f cont

Facem schimbare de variabilă
$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \quad r\in[0,\infty),\ \theta\in[0,2\pi]$$

$(x,y)\in B \Rightarrow x^2+y^2 \le 2 \Rightarrow r \le 2 \Rightarrow r \in [0,\sqrt{2}]$

$\theta \in [0,2\pi]$

Fie $C = [0,\sqrt{2}]\times[0,2\pi] \Rightarrow C \in J(\mathbb{R}^2)$ și C camp

$$\iint_B f(x,y)\,dx\,dy = \iint_C r\, f(r\cos\theta, r\sin\theta)\,d\theta\,dr$$
$$= \int_0^{\sqrt{2}}\int_0^{2\pi} r\cdot\frac{r^2}{2}(6-r^2-r^4)\,d\theta\,dr$$
$$= \int_0^{\sqrt{2}} \frac{r^3}{2}(6-r^2-r^4)\,\theta\Big|_0^{2\pi} dr$$
$$= 2\pi\int_0^{\sqrt{2}} \frac{r^3}{2}(6-r^2-r^4)\,dr$$
$$= 2\pi\int 3r^3 - \frac{r^5}{2} - \frac{r^7}{2}\,)dr =$$
$$= 2\pi\left(\frac{3}{4}r^4\Big|_0^{\sqrt{2}} - \frac{1}{12}r^6\Big|_0^{\sqrt{2}} - \frac{1}{16}r^8\Big|_0^{\sqrt{2}}\right)$$
$$= 6\pi - \frac{4}{3}\pi - 2\pi = \frac{8}{3}\pi$$

Schimbarea de variabilă în integrala triplă

Fie $A\in \mathcal{J}(\mathbb{R}^3)$ şi $f:A\to\mathbb{R}$ continuă.
Vom schimba variabila astfel:
$$\begin{cases} x = h_1(u,v,w) \\ y = h_2(u,v,w) \\ z = h_3(u,v,w). \end{cases}$$

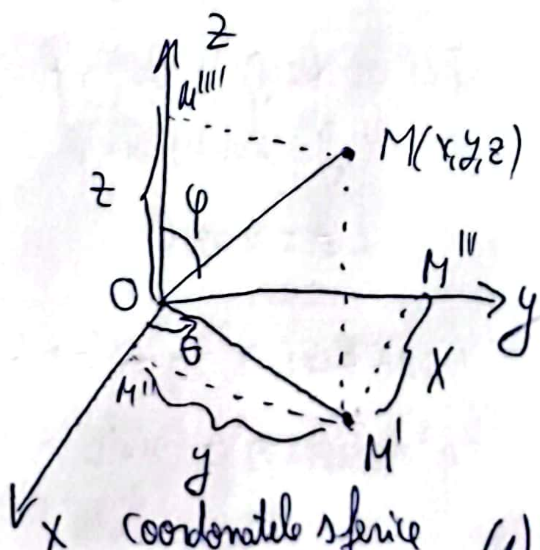
Jacobianul este
$$\begin{vmatrix} \frac{\partial h_1}{\partial u} & \frac{\partial h_1}{\partial v} & \frac{\partial h_1}{\partial w} \\ \frac{\partial h_2}{\partial u} & \frac{\partial h_2}{\partial v} & \frac{\partial h_3}{\partial w} \\ \frac{\partial h_3}{\partial u} & \frac{\partial h_3}{\partial v} & \frac{\partial h_3}{\partial w} \end{vmatrix} \overset{not}{=} J.$$

Ca la integrala dublă:
$$\iiint_A f(x,y,z)\,dx\,dy\,dz = \iiint_B |J|\cdot f(h_1(u,v,w), h_2(u,v,w), h_3(u,v,w))\,du\,dv\,dw,$$

unde ~~B~~ mulţimea B se găseşte din condiţia $(x,y,z)\in A$.

1) Trecerea la coordonate sferice:
$$\begin{cases} x = \alpha + r\cos\theta\sin\varphi \\ y = \beta + r\sin\theta\sin\varphi \\ z = \gamma + r\cos\varphi \end{cases}, \quad r\in[0,\infty),\ \theta\in[0,2\pi],\ \varphi\in[0,\pi],$$
$\alpha,\beta,\gamma\in\mathbb{R}$

Jacobianul acestei transformări este $r^2\sin\varphi$.



coordonatele sferice (1)

2) Trecerea la coordonate sferice generalizate

$$\begin{cases} x=\alpha+a\,r\cos\theta\sin\varphi \\ y=\beta+b\,r\sin\theta\sin\varphi \\ z=\gamma+c\,r\cos\varphi \end{cases} \quad r\in[0,\infty),\ \theta\in[0,2\pi],\ \varphi\in[0,\pi],$$

$a,b,c>0,\ \alpha,\beta,\gamma\in\mathbb{R}$.

Jacobianul este $a\,b\,c\,r^2\sin\varphi$.

3) Trecerea la coordonate cilindrice

$$\begin{cases} x=r\cos\theta \\ y=r\sin\theta \\ z=z \end{cases} ,\ r>0,\ \theta\in[0,2\pi],\ z\in\mathbb{R}$$

(echivalentul coordonatelor polare, dar pentru integrala triplă)

Jacobianul este r.

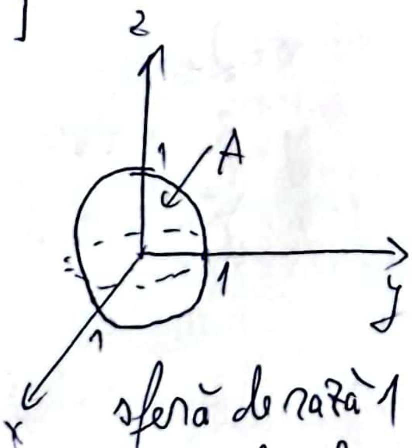
Exerciții:

1) $\iiint_A \sqrt{x^2+y^2+z^2}\,dx\,dy\,dz$, unde $A=\{(x,y,z)\in\mathbb{R}^3 / x^2+y^2+z^2\le 1\}$

A convexă și mărginită $\Rightarrow A\in J(\mathbb{R}^3)$ compactă.

Fie $f:A\to\mathbb{R}$, $f(x,y,z)=\sqrt{x^2+y^2+z^2}$ cont.

Fie $\begin{cases} x=r\cos\theta\sin\varphi \\ y=r\sin\theta\sin\varphi \\ z=r\cos\varphi \end{cases}$, $r\in[0,\infty)$, $\theta\in[0,2\pi]$, $\varphi\in[0,\pi]$



sferă de rază 1
desenați talentul

$(x,y,z)\in A \Leftrightarrow x^2+y^2+z^2\le 1 \Leftrightarrow r^2\cos^2\theta\sin^2\varphi+r^2\sin^2\theta\sin^2\varphi+r^2\cos^2\varphi\le 1 \Leftrightarrow$

$\Leftrightarrow r^2\sin^2\varphi(\cos^2\theta+\sin^2\theta)+r^2\cos^2\varphi\le 1 \Leftrightarrow r^2\le 1 \Leftrightarrow r\in[0,1],\ \theta\in[0,2\pi],\ \varphi\in[0,\pi]$.

Fie $B=[0,1]\times[0,2\pi]\times[0,\pi]$. Avem că

$$\iiint_A f(x,y,z)\,dx\,dy\,dz=\iiint_B r^2\sin\varphi\, f(r\cos\theta\sin\varphi, r\sin\theta\sin\varphi, r\cos\varphi)\,dr\,d\theta\,d\varphi$$

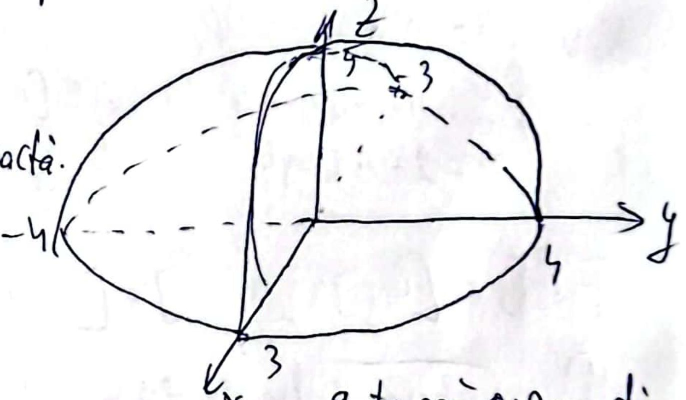
$$=\int_0^1\left(\int_0^{2\pi}\left(\int_0^{\pi} r^2\sin\varphi\sqrt{r^2}\,d\varphi\right)d\theta\right)dr=\int_0^1\left(\int_0^{2\pi} r^3(-\cos\varphi)\Big|_{\varphi=0}^{\varphi=\pi}d\theta\right)dr=$$

$$=\int_0^1 2r^3\theta\Big|_{\theta=0}^{\theta=2\pi}dr=4\pi\cdot\frac{1}{4}=\pi.$$

2) $\iiint_A 1\,dx\,dy\,dz$, unde $A=\left\{(x,y,z)\in\mathbb{R}^3 \,/\, \frac{x^2}{9}+\frac{y^2}{16}+\frac{z^2}{25}\le 1,\ z\ge 0\right\}$.

A este jumătatea de sus a unui elipsoid de semiaxe 3, 4 și 5.

~~Fie~~



A închisă și mărginită $\Rightarrow A\in\mathcal{J}(\mathbb{R}^3)$ compactă.

Fie $f:A\to\mathbb{R}$, $f(x,y,z)=1$ const

gata mă opresc din desenat cartofi

Fie $\begin{cases} x=3r\cos\theta\sin\varphi\\ y=4r\sin\theta\sin\varphi\\ z=5r\cos\varphi\end{cases}$, $r\in[0,\infty)$, $\theta\in[0,2\pi]$, $\varphi\in[0,\pi]$

$(x,y,z)\in A \Leftrightarrow \begin{cases}\frac{9r^2\cos^2\theta\sin^2\varphi}{9}+\frac{16r^2\sin^2\theta\sin^2\varphi}{16}+\frac{25r^2\cos^2\varphi}{25}\le 1\\ 5r\cos\varphi\ge 0\end{cases} \Leftrightarrow$

$\Leftrightarrow r\in[0,1]$, $\theta\in[0,2\pi]$, $\varphi\in\left[0,\frac{\pi}{2}\right]$.

Fie $B=[0,1]\times[0,2\pi]\times\left[0,\frac{\pi}{2}\right]$.

Avem că $\iiint_A f(x,y,z)\,dx\,dy\,dz = \cancel{\iiint_B 60 r^2 \sin\varphi \, dr\, d\theta\, d\varphi}$

$$= \iiint_B 60 r^2 \sin\varphi \, f(3r\cos\theta\sin\varphi, 4r\sin\theta\sin\varphi, 5r\cos\varphi)\, dr\, d\theta\, d\varphi =$$

$$= \iiint_B 60 r^2 \sin\varphi \, dr\, d\theta\, d\varphi = 60 \int_0^1 r^2 dr \cdot \int_0^{\frac{\pi}{2}} \sin\varphi\, d\varphi \cdot \int_0^{2\pi} 1\, d\theta =$$

$$= 60 \left.\frac{r^3}{3}\right|_0^1 \cdot \left.(-\cos\varphi)\right|_0^{\frac{\pi}{2}} \cdot \left.\theta\right|_0^{2\pi} = 60 \cdot \frac{1}{3} \cdot 1 \cdot 2\pi = 40\pi$$

3) $\iiint_A \sqrt{x^2+y^2+z^2}\, dx\,dy\,dz$, $A = \{(x,y,z) \in \mathbb{R}^3 / x^2+y^2+z^2 \le 4z\}$.

$$x^2+y^2+z^2 \le 4z \Leftrightarrow x^2+y^2+z^2-4z+4 \le 4 \Leftrightarrow x^2+y^2+(z-2)^2 \le 2^2.$$

S.V: $\begin{cases} x = r\cos\theta\sin\varphi \\ y = r\sin\theta\sin\varphi \\ z = 2 + r\cos\varphi \end{cases}$, $r \in [0,2]$, $\varphi \in [0,\pi]$, $\theta \in [0,2\pi]$
(asta reiese din $(x,y,z) \in A$).

Fie $B = [0,2] \times [0,\pi] \times [0,2\pi]$.

$$\iiint_A \sqrt{x^2+y^2+z^2}\, dx\,dy\,dz = \iiint_B r^2 \sin\varphi \sqrt{r^2+4r\cos\varphi+4}\, dr\, d\theta\, d\varphi =$$

$$= \iint_{[0,2]\times[0,2\pi]} \left(\int_0^{\pi} \sqrt{r^2+4r\cos\varphi+4} \cdot r^2 \sin\varphi\, d\varphi \right) dr\, d\theta.$$

Calculăm $I = \int_0^{\pi} \sqrt{r^2+4r\cos\varphi+4} \cdot r^2\sin\varphi\, d\varphi$. Fie $u = r^2+4r\cos\varphi+4$;
$du = -4r\sin\varphi\, d\varphi$

$$I = \int_{r^2+4r+4}^{r^2-4r+4} \sqrt{u} \cdot \left(-\frac{r}{4}\right) du = \frac{r}{4} \int_{(r-2)^2}^{(r+2)^2} \sqrt{u}\, du = \frac{r}{4} \cdot \left.\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right|_{(r-2)^2}^{(r+2)^2} = \frac{r}{6} \left.\sqrt{u^3}\right|_{(r-2)^2}^{(r+2)^2} =$$

$$= \frac{r}{6}\left(\sqrt{(r+2)^6} - \sqrt{(r-2)^6}\right) = \frac{r}{6}\left(\left|(r+2)^3\right| - \left|(r-2)^3\right|\right).$$

Dar $r \in [0,2] \Rightarrow r-2<0$, iar $\left|(r-2)^3\right| = (2-r)^3$. De aici

$$i = \frac{r}{6}\left((r+2)^3 - (2-r)^3\right) = \frac{r}{6}(24r + 2r^3) = 4r^2 + \frac{r^4}{3}.$$

Atunci, $\iiint_A \sqrt{x^2+y^2+z^2}\,dx\,dy\,dz = \iint_{[0,2]\times[0,\pi]} \left(4r^2 + \frac{r^4}{3}\right) dr\,d\theta =$

$$= \left(\int_0^2 4r^2 + \frac{r^4}{3}\,dr\right)\left(\int_0^{2\pi} 1\,d\theta\right) = \left(\frac{4r^3}{3} + \frac{r^5}{15}\right)\Big|_0^2 \cdot 2\pi = \frac{128\pi}{5}.$$

Sper că nu am greșit la calcule...

Obs. Schimbarea lui z similară se poate face și la coord sferice generalizate, adică cu un număr random în față pe care îl adun la ce am eu.

De ex: $\begin{cases} x = 3 + 2r\cos\theta\sin\varphi \\ y = 2 + 3r\sin\theta\sin\varphi \\ z = 1 + 5r\cos\varphi \end{cases}$, pt $\left\{ \frac{(x-3)^2}{4} + \frac{(y-2)^2}{9} + \frac{(z-1)^2}{25} \leq 1 \right\}$.

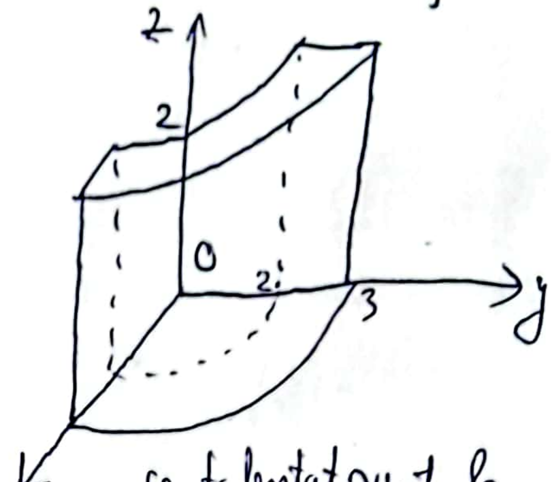
4) $\iiint_A xyz\,dx\,dy\,dz$, unde $A = \left\{(x,y,z) \in \mathbb{R}^3 \,/\, \frac{(x-2)^2}{4} + \frac{(y-3)^2}{9} + \frac{(z-5)^2}{36} \leq 1\right\}$.

Rezolvarea ~~acestui exercițiu~~ acestei integrale triple este lăsată ca exercițiu pentru cititor :).

(Adică îmi e mie lene, dar se face cu observația de mai sus).

5) $\iiint_A xz\,dx\,dy\,dz$, unde $A=\{(x,y,z) \mid 4\le x^2+y^2\le 9,\ x\ge 0,\ y\ge 0,\ 0\le z\le 2\}$.

A este un cilindru cu raza bazei 3, generatoarea paralelă cu Oz și înălțime 2, din care se extrage un cilindru de același tip, dar cu raza bazei de 2.



ce talentat sunt la desenat cilindre

Metoda 1: Dacă vrem să ne complicăm viața putem spune că

$$\iiint_A f(x,y,z)\,dx\,dy\,dz = \iiint_{A_1} f(x,y,z)\,dx\,dy\,dz - \iiint_{A_2} f(x,y,z)\,dx\,dy\,dz,$$

unde $A_1=\{(x,y,z)\in\mathbb{R}^3 \mid x^2+y^2\le 9,\ x\ge 0,\ y\ge 0,\ 0\le z\le 2\}$ și

$A_2=\{(x,y,z)\in\mathbb{R}^3 \mid x^2+y^2\le 4,\ x\ge 0\ y\ge 0,\ 0\le z\le 2\}$.

Nu mulțumesc. Nu fac două integrale degeaba.

Metoda 2:

A închisă și mărginită $\Rightarrow A\in J(\mathbb{R}^3)$ compactă.

Fie $f: A\to\mathbb{R}$, $f(x,y,z)=xz$ continuă.

Fie $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$, $r\in[2,3]$, $\theta\in[0,\frac{\pi}{2}]$, $z\in[0,2]$,

$dx\,dy\,dz = r\,dr\,d\theta\,dz$, $B=[2,3]\times[0,\frac{\pi}{2}]\times[0,2]$

Deci $\iiint_A xz\,dx\,dy\,dz = \iiint_B r(\cos\theta)z\,dr\,d\theta\,dz = \int_2^3 r^2\,dr\cdot\int_0^{\frac{\pi}{2}}\cos\theta\,d\theta\cdot\int_0^2 z\,dz =$

$= \frac{r^3}{3}\Big|_2^3 \cdot \sin\theta\Big|_0^{\frac{\pi}{2}}\cdot\frac{z^2}{2}\Big|_0^2 = \frac{19}{3}\cdot 1\cdot 2 = \frac{38}{3}$.