1 Problem Solutions

1. Say whether the following is true or false and support your answer by a proof.

$$(\exists m \in N)(\exists m \in N)(3m + 5n = 12)$$

Claim. The statement $(\exists m \in N)(\exists m \in N)(3m + 5n = 12)$ is False.
Proof: We argue by contradiction. Suppose there are suchs m, n natural numbers such that

$$5n + 3m = 12$$

Then

$$\begin{aligned} 5n &= 12 - 3m \\ &= 3(4 - m), \text{ by algebra} \end{aligned}$$

Since $n > 0$, then $5n > 0$, So it happens that $5n = 3(4 - m) > 0$ and therefore $(4 - m) > 0$. Norice, there's an integer n such that $3(4 - m) = 5n$. Therefore $5 \mid 3(4 - m)$.

By Euclid's Lemma, if p is prime, whenever p divides a product ab, p divides at least one of a, b. (Check Assignment 9). Since 5 is a prime number, and $5 \mid 3(4 - m)$ then $5 \mid 3$ or $5 \mid (4 - m)$. Since 3 is prime, $5 \nmid 3$. So it must happens that $5 \mid (4 - m)$.
But $m > 0$, then $(4 - m) < 4$ and as shown above $0 < (4 - m)$. Then $0 < (4 - m) < 4$. Therefore $5 \nmid (4 - m)$. This is a contradiction to Euclid's Lemma, so there are not m, n natural numbers such that $5n + 3m = 12$. And that proves the claim.

2. Say whether the following is true or false and support your answer by a proof: The sum of any five consecutive integers is divisible by 5 (without remainder).

Theorem. For any integer n,

$$5 \mid n + (n + 1) + (n + 2) + (n + 3) + (n + 4)$$

Proof: Let n be a given integer. Then

$$\begin{aligned} n + (n + 1) + (n + 2) + (n + 3) + (n + 4) &= 5n + 1 + 2 + 3 + 4 \\ &= 5n + 10 \\ &= 5(n + 2), \text{ by algebra} \end{aligned}$$

Since $n + 2$ is an integer, and $n + (n + 1) + (n + 2) + (n + 3) + (n + 4) = 5(n + 2)$, then

$$5 \mid n + (n + 1) + (n + 2) + (n + 3) + (n + 4)$$

Since it was a given n, the proof holds for any integer n, which proves the theorem.

3. Say whether the following is true or false and support your answer by a proof: For any integer n, the number $n^2 + n + 1$ is odd.

Claim. For any integer n, the number $n^2 + n + 1$ is odd.
(i.e.) $n^2 + n + 1 = 2k + 1$ where k is an integer.

Proof: Let n be a given integer. Then

$$n^2 + n + 1 \quad = \quad n(n+1) + 1, \text{ by algebra}$$

Now let's shown that $n(n+1) = 2k$ for some integer k
<u>Case 1</u>
If n is an even number, then $n = 2h$ for some integer h. So

$$n(n+1) = (2h)(2h+1) = 2(h)(2h+1)$$

Since $h(2h+1)$ is an integer let $k = h(2h+1)$. Then $n(n+1) = 2(h)(2h+1) = 2k$.
<u>Case 2</u>
If n is an odd number, then $n = 2h+1$ for some integer h. So

$$n(n+1) = (2h+1)[(2h+1)+1] = (2h+1)(2h+2) = (2h+1)[2(h+1)] = 2(2h+1)(h+1)$$

Since $(2h+1)(h+1)$ is an integer let $k = (2h+1)(h+1)$. Then $n(n+1) = 2(2h+1)(h+1) = 2k$.
Therefore it holds that $n(n+1) = 2k$ for some integer k. Then

$$n^2 + n + 1 = n(n+1) + 1 = 2k + 1$$

Hence, for any integer n, the number $n^2 + n + 1$ is an odd number.

4. Prove that every odd natural number is of one of the forms $4n+1$ or $4n+3$, where n is an integer.

Claim. Every odd natural number is of one of the forms $4n + 1$ or $4n + 3$, where n is an integer.
Proof: By the Division Theorem, any number can be expressed in one of the form $4n, 4n+1, 4n+2, 4n+3$. Notice that

$$\begin{array}{rcccl} 4n & = & 2(2n) & = & 2k_0 \quad \text{where } k_0 = 2n \\ 4n+1 & = & 2(2n)+1 & = & 2k_1+1 \quad \text{where } k_1 = 2n \\ 4n+2 & = & 2(2n+1) & = & 2k_2 \quad \text{where } k_2 = 2n+1 \\ 4n+3 & = & 2(2n+1)+1 & = & 2k_3+1 \quad \text{where } k_3 = 2n+1 \end{array}$$

As shown, every number of the form $2k+1$, for some integer k, is one of the forms $4n+1, 4n+3$.
Hence every odd natural number is one of the forms $4n+1, 4n+3$. Which proves the Claim.

5. Prove that for any integer n at least one of the integers $n, n+2, n+4$ is divisible by 3.

Claim. For any integer n at least one of the integers $n, n+2, n+4$ is divisible by 3.

Proof: Let n be a given integer. By the Division Theorem, n can be expressed in one of the forms $3q, 3q+1, 3q+2$.

Case 1

If n is of the form $3q$, then $n = 3q$, for some integer q. So

$$3 \mid n$$

Case 2

If n is of the form $3q+1$, then $n+2 = (3q+1)+2 = 3q+3 = 3(q+1)$, for some integer $q+1$. So

$$3 \mid n+2$$

Case 3

If n is of the form $3q+2$, then $n+4 = (3q+2)+4 = 3q+6 = 3(q+2)$, for some integer $q+2$. Then

$$3 \mid n+4$$

In all three cases, at least one of the integers $n, n+2, n+4$ is divisible by 3. Since n was given, for any integer n, at least one of the integers $n, n+2, n+4$ is divisible by 3. This proves the theorem.

6. A classic unsolved problem in number theory asks if there are infinitely many pairs of 'twin primes', pairs of primes separated by 2, such as 3 and 5, 11 and 13, or 71 and 73. Prove that the only prime triple (i.e. three primes, each 2 from the next) is 3, 5, 7.

Theorem. The only prime triple (i.e. three primes, each 2 from the next) is 3, 5, 7.

Proof: We argue by contradiction. Suppose there exist another prime triple, notice that by definition, the prime triples are of the form $n, n+2, n+4$, each prime. Since we suppose there is another prime triple other than 3, 5, 7, it must happens that $n > 3$.

By the previous problem we proved that for any integer n at least one of the integers $n, n+2, n+4$ is divisible by 3. Therefore at least one of our primes is divisible by 3, but by the definition of prime number, primes can be divided evenly only by 1, or itself. Therefore, either

$$n = 3, \text{ or } n+2 = 3, \text{ or } n+4 = 3$$

But, as shown above

$$3 < n < n+2 < n+4$$

Therefore

$$n \neq 3, \text{ and } n+2 \neq 3, \text{ and } n+4 \neq 3$$

This is a contradiction, since it must happen that either $n = 3$, or $n+2 = 3$, or $n+4 = 3$. Therefore, does not exist another prime triple other than 3, 5, 7. This poves the theorem.

7. Prove that for any natural number n,

$$2 + 2^2 + 2^3 + \ldots + 2^n = 2^{n+1} - 2$$

Theorem. For any natural number n,

$$2 + 2^2 + 2^3 + \ldots + 2^n = 2^{n+1} - 2$$

Proof: By induction.

For $n = 1$, the left hand side is $2 = 2$ and the right-hand side is $2^2 - 2 = 4 - 2 = 2$, so the identity is valid for $n = 1$.

Assume the identity holds for n. Then:

$$\begin{aligned} 2 + 2^2 + 2^3 + \ldots + 2^{n+1} &= (2 + 2^2 + 2^3 + \ldots + 2^n) + 2^{n+1} \text{ , separate last term} \\ &= (2^{n+1} - 2) + 2^{n+1} \text{ , by the induction hypothesis} \\ &= 2(2^{n+1}) - 2 \text{ , by algebra} \\ &= 2^{n+2} - 2 \text{ , by algebra} \end{aligned}$$

wich is the identity for $n + 1$. Therefore, by Induction, the proof is complete.

8. Prove (from the definition of a limit of a sequence) that if the sequence $\{a_n\}_{n=1}^{\infty}$ tends to limit L as $n \to \infty$, then for any fixed number $M > 0$, the sequence $\{Ma_n\}_{n=1}^{\infty}$ tends to limit ML.

Theorem. If the sequence $\{a_n\}_{n=1}^{\infty}$ tends to limit L as $n \to \infty$, then for any fixed number $M > 0$, the sequence $\{Ma_n\}_{n=1}^{\infty}$ tends to limit ML as $n \to \infty$.

Proof: Let $M > 0$ and $\varepsilon > 0$ be given. We need to find an n such that for all $m \geq n$:

$$|Ma_m - ML| < \varepsilon$$

Notice that, since $\{a_n\}_{n=1}^{\infty}$ tends to limit L as $n \to \infty$, by definition

$$(\forall \varepsilon' > 0)(\exists n' \in N)(\forall m \geq n')[|a_m - L| < \varepsilon']$$

Since $M > 0$, take $\varepsilon' = \dfrac{\varepsilon}{M} > 0$, then, by definition there is an n' such that for all $m \geq n'$:

$$|a_m - L| < \varepsilon'$$

Take $n = n'$. Then for any $m \geq n$:

$$\begin{aligned} |Ma_m - ML| &= |M(a_m - L)| \\ &= |M||a_m - L| \\ &< M\varepsilon' \text{ , by definition} \\ &= M\frac{\varepsilon}{M} = \varepsilon \text{ , substituting} \end{aligned}$$

$$\therefore \quad |Ma_m - ML| < \varepsilon$$

Since ε were given, It holds for all $\varepsilon > 0$. So $\{Ma_n\}_{n=1}^{\infty}$ tends to limit ML as $n \to \infty$ for any fixed number $M > 0$. And that proves the theorem.

9. Given an infinite collection $A_n,\ n = 1, 2, \ldots$, of intervals of the real line, their intersection is defined to be

$$\bigcap_{n=1}^{\infty} A_n = \{x \mid (\forall n)(x \in A_n)\}$$

Give an example of a family of intervals $A_n,\ n = 1, 2, \ldots$, such that $A_{n+1} \subset A_n$ for all n, and $\bigcap_{n=1}^{\infty} A_n = \varnothing$. Prove that your example has the stated property.

Let $A_n,\ n = 1, 2, \ldots$ be a family of intervals such that for all n:

$$A_n = (n\ , \infty) = \{x \mid x > n\}$$

Theorem. This family of intervals, satisfy that $A_{n+1} \subset A_n$ for all n, and $\bigcap_{n=1}^{\infty} A_n = \varnothing$.

Proof: First we will show that $A_{n+1} \subset A_n$ for all n. Afterward, we will prove that $\bigcap_{n=1}^{\infty} A_n = \varnothing$.

1. To prove: $A_{n+1} \subset A_n$

Let n be given, and take $x \in A_{n+1}$. By our definition of the interval $A_{n+1},\ \ x > n+1$. Since $x > n+1 > n$ it happens that $x > n$. Therefore x satisfies the property of the interval A_n. Hence $x \in A_n$. Since we take arbitrary n, x, we conclude that for all n:

$$A_{n+1} \subset A_n$$

2. To prove: $\bigcap_{n=1}^{\infty} A_n = \emptyset$

We are going to prove that given any number x, there exist an interval A_n for some n, such that $x \notin A_n$, and therefore $(\exists n)(x \notin A_n)$, wich negates the property of the intersection, then

$$x \notin \bigcap_{n=1}^{\infty} A_n \Leftrightarrow \bigcap_{n=1}^{\infty} A_n = \emptyset$$

Let x be a given number
Case 1: $x \leq 1$
If $x \leq 1 \ \ \Rightarrow \ \ x \not> 1$, wich negates the property of A_1. Hence $x \notin A_1$, therefore $(\exists n)(x \notin A_n)$.
Case 2: $x > 1$
By the *Archimedean property* given $x, y \in R$ and $x, y > 0$, there is an $n \in N$ such that $nx > y$. Take $1, x \in R$, since $1, x > 0$, by the *Archimedean property* there is an $n \in N$ such that:

$$n = n(1) > x$$

Consider the interval A_n. Since $n > x \ \ \Rightarrow \ \ x \not> n$, wich negates the property of A_n. Hence $x \notin A_n$, therefore $(\exists n)(x \notin A_n)$.

And so we prove that $\displaystyle\bigcap_{n=1}^{\infty} A_n = \emptyset$.

Therefore, this family of intervals, satisfy that $A_{n+1} \subset A_n$ for all n, and $\bigcap_{n=1}^{\infty} A_n = \varnothing$. The proof is complete.

10. Give an example of a family of intervals $A_n,\ n = 1, 2, \ldots$, such that $A_{n+1} \subset A_n$ for all n, and $\bigcap_{n=1}^{\infty} A_n$ consists of a single real number. Prove that your example has the stated property.

Let $A_n,\ n = 1, 2, \ldots$ be a family of constant intervals such that for all n:

$$A_n = [k\ , k] = \{x \mid k \leq x \leq k\}\ = \{x \mid x = k\}$$

For some real number k.

Theorem. This family of intervals, satisfy that $A_{n+1} \subset A_n$ for all n, and $\bigcap_{n=1}^{\infty} A_n$ consists of a single real number.

Proof: First we will show that $A_{n+1} \subset A_n$ for all n. Afterward, we will prove that $\bigcap_{n=1}^{\infty} A_n$ consists of a single real number.

1. To prove: $A_{n+1} \subset A_n$

Let n be given, and take $x \in A_{n+1}$. By our definition of the interval $A_{n+1},\ \ x = k$. Therefore x satisfies the property of the interval A_n. Hence $x \in A_n$. Since we take arbitrary n, x, we conclude that for all n:

$$A_{n+1} \subset A_n$$

2. To prove: $\bigcap_{n=1}^{\infty} A_n$ consists of a single real number.

It's clearly that since $k = k$, by our definition of the family of intervals $(\forall n)(k \in A_n)$. Let's prove it's uniqueness.

We argue by contradiction. Suppose there is anoter real number z with $z \neq k$ such that $z \in \bigcap_{n=1}^{\infty} A_n$. Then, $(\forall n)(z \in A_n)$. In particular, $z \in A_1$, so z satisfies that $z = k$, but we pick $z \neq k$. This is a contradiction. Hence $\bigcap_{n=1}^{\infty} A_n$ consists of a single real number.

Therefore, this family of intervals, satisfy that $A_{n+1} \subset A_n$ for all n, and $\bigcap_{n=1}^{\infty} A_n$ consists of a single real number. The proof is complete.