Heavy Separability of the Second Kind

Subhajit Das (19082)

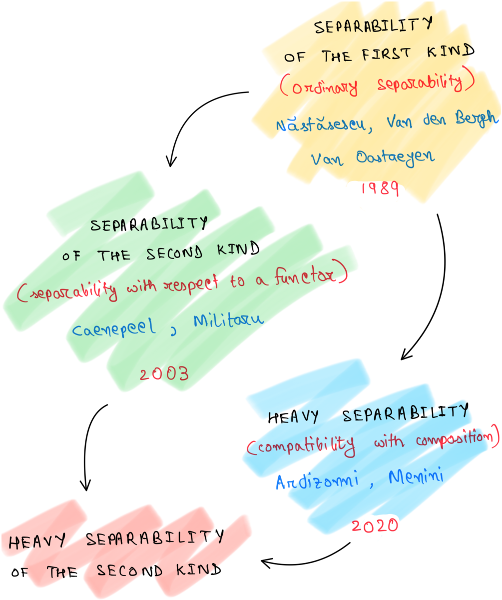
Advisor : Dr. Abhishek Banerjee

Department of Mathematics,
Indian Institute of Science

INDIAN INSTITUTE OF SCIENCE

भारतीय विज्ञान संस्थान

Introduction



Introduction

Notation : If $G : \mathcal{A} \longrightarrow \mathcal{B}$ is a functor, then G induces a natural transformation

$$G_{\_,\,\_} : \mathcal{A}(\_,\_) \to \mathcal{B}(G(\_), G(\_)) \quad \text{in } [\mathcal{A}^{op} \times \mathcal{A}, Set].$$

Introduction

Notation : If $G : \mathcal{A} \longrightarrow \mathcal{B}$ is a functor, then G induces a natural transformation

$$G_{\_,\_} : \mathcal{A}(\_,\_) \to \mathcal{B}(G(\_), G(\_)) \quad \text{in } [\mathcal{A}^{op} \times \mathcal{A}, Set].$$

We fix a functor $F : \mathcal{C} \to \mathcal{D}$ from a category $\mathcal{C}$ to a category $\mathcal{D}$.

Introduction

Notation : If $G : \mathcal{A} \longrightarrow \mathcal{B}$ is a functor, then G induces a natural transformation

$$G_{\_,\,\_} : \mathcal{A}(\_,\_) \to \mathcal{B}(G(\_), G(\_)) \quad \text{in } [\mathcal{A}^{op} \times \mathcal{A}, Set].$$

We fix a functor $F : \mathcal{C} \to \mathcal{D}$ from a category $\mathcal{C}$ to a category $\mathcal{D}$.

1. (C. Năstăsescu, M. Van den Bergh, F. Van Oystaeyen) : The functor F is said to be **separable** if the natural transformation $F_{\_,\,\_}$ is a split-monomorphism in the functor category $[\mathcal{C}^{op} \times \mathcal{C}, \mathrm{Set}]$. The terminology is motivated from the notion of separability of ring extensions in algebra : We recall that a ring extension $R \to S$ is said to be separable if the map $s \otimes s' \mapsto ss' : S \otimes_R S \to S$ is a split-epimorphism of $S - S$ bimodules. They proved that $R \to S$ is separable iff the restriction of scalars functor $\mathbb{M}_S \longrightarrow \mathbb{M}_R$ is separable.

Introduction

Notation : If $G : \mathcal{A} \longrightarrow \mathcal{B}$ is a functor, then G induces a natural transformation

$$G_{\_,\,\_} : \mathcal{A}(\_,\_) \to \mathcal{B}(G(\_), G(\_)) \quad \text{in } [\mathcal{A}^{op} \times \mathcal{A}, Set].$$

We fix a functor $F : \mathcal{C} \to \mathcal{D}$ from a category $\mathcal{C}$ to a category $\mathcal{D}$.

1. (C. Năstăsescu, M. Van den Bergh, F. Van Oystaeyen) : The functor F is said to be **separable** if the natural transformation $F_{\_,\,\_}$ is a split-monomorphism in the functor category $[\mathcal{C}^{op} \times \mathcal{C}, \mathrm{Set}]$. The terminology is motivated from the notion of separability of ring extensions in algebra : We recall that a ring extension $R \to S$ is said to be separable if the map $s \otimes s' \mapsto ss'$: $S \otimes_R S \to S$ is a split-epimorphism of $S - S$ bimodules. They proved that $R \to S$ is separable iff the restriction of scalars functor $\mathbb{M}_S \longrightarrow \mathbb{M}_R$ is separable.
2. (S. Caenepeel, G. Militaru) : Given a functor $I : \mathcal{C} \longrightarrow \mathcal{X}$, the functor F is said to be ***I*-separable** if the natural transformation $I_{\_,\,\_}$ factors through $F_{\_,\,\_}$.

Introduction

Notation : If $G : \mathcal{A} \longrightarrow \mathcal{B}$ is a functor, then G induces a natural transformation

$$G_{\_,\_} : \mathcal{A}(\_,\_) \to \mathcal{B}(G(\_), G(\_)) \quad \text{in } [\mathcal{A}^{op} \times \mathcal{A}, Set].$$

We fix a functor $F : \mathcal{C} \to \mathcal{D}$ from a category $\mathcal{C}$ to a category $\mathcal{D}$.

1. (C. Năstăsescu, M. Van den Bergh, F. Van Oystaeyen) : The functor F is said to be **separable** if the natural transformation $F_{\_,\_}$ is a split-monomorphism in the functor category $[\mathcal{C}^{op} \times \mathcal{C}, \mathrm{Set}]$. The terminology is motivated from the notion of separability of ring extensions in algebra : We recall that a ring extension $R \to S$ is said to be separable if the map $s \otimes s' \mapsto ss' : S \otimes_R S \to S$ is a split-epimorphism of $S - S$ bimodules. They proved that $R \to S$ is separable iff the restriction of scalars functor $\mathbb{M}_S \longrightarrow \mathbb{M}_R$ is separable.
2. (S. Caenepeel, G. Militaru) : Given a functor $I : \mathcal{C} \longrightarrow \mathcal{X}$, the functor F is said to be I-**separable** if the natural transformation $I_{\_,\_}$ factors through $F_{\_,\_}$.
3. (A. Ardizzoni, C. Menini) : The functor F is said to be **heavily separable** if the natural transformation $F_{\_,\_}$ has a left inverse P which is compatible with composition in the sense that for all objects $a, b, c \in \mathcal{C}$, for all morphisms $f \in \mathcal{D}(Fa, Fb)$, $g \in \mathcal{D}(Fb, Fc)$, $P_{a,c}(g \circ f) = P_{b,c}(g) \circ P_{a,b}(f)$.

Introduction

Notation : If $G : \mathcal{A} \longrightarrow \mathcal{B}$ is a functor, then G induces a natural transformation

$$G_{\_,\_} : \mathcal{A}(\_,\_) \to \mathcal{B}(G(\_), G(\_)) \quad \text{in } [\mathcal{A}^{op} \times \mathcal{A}, Set].$$

We fix a functor $F : \mathcal{C} \to \mathcal{D}$ from a category $\mathcal{C}$ to a category $\mathcal{D}$.

1. (C. Năstăsescu, M. Van den Bergh, F. Van Oystaeyen) : The functor F is said to be **separable** if the natural transformation $F_{\_,\_}$ is a split-monomorphism in the functor category $[\mathcal{C}^{op} \times \mathcal{C}, \text{Set}]$. The terminology is motivated from the notion of separability of ring extensions in algebra : We recall that a ring extension $R \to S$ is said to be separable if the map $s \otimes s' \mapsto ss' : S \otimes_R S \to S$ is a split-epimorphism of $S - S$ bimodules. They proved that $R \to S$ is separable iff the restriction of scalars functor $\mathbb{M}_S \longrightarrow \mathbb{M}_R$ is separable.
2. (S. Caenepeel, G. Militaru) : Given a functor $I : \mathcal{C} \longrightarrow \mathcal{X}$, the functor F is said to be I-**separable** if the natural transformation $I_{\_,\_}$ factors through $F_{\_,\_}$.
3. (A. Ardizzoni, C. Menini) : The functor F is said to be **heavily separable** if the natural transformation $F_{\_,\_}$ has a left inverse P which is compatible with composition in the sense that for all objects $a, b, c \in \mathcal{C}$, for all morphisms $f \in \mathcal{D}(Fa, Fb)$, $g \in \mathcal{D}(Fb, Fc)$, $P_{a,c}(g \circ f) = P_{b,c}(g) \circ P_{a,b}(f)$.

In this work, we amalgamate these two notions to get a new notion, which we call **Heavy Separability of the Second kind**. We define it as follows :

Definition

Let $F : \mathcal{C} \longrightarrow \mathcal{D}$ be a functor. Given a functor $I : \mathcal{C} \longrightarrow \mathcal{X}$, we say that F is heavily-I-separable if there is a natural transformation $P : \mathcal{D}(F(\_), F(\_)) \longrightarrow \mathcal{X}(I(\_), I(\_))$ such that

1. *The following diagram commutes in the functor category $[\mathcal{C}^{op} \times \mathcal{C}, Set]$ (the condition of I-separability)*

$$\begin{tikzcd} \mathcal{C}(\_,\_) \arrow[r, "F_{\_,\_}"] \arrow[d, "I_{\_,\_}"'] & \mathcal{D}(F(\_), F(\_)) \arrow[dl, "P"] \\ \mathcal{X}(I(\_), I(\_)) & \end{tikzcd}$$

2. *for all objects a, b, c in $\mathcal{C}$, the following diagram commutes (the "heavy" part) :*

$$\begin{tikzcd} \mathcal{D}(Fa, Fb) \times \mathcal{D}(Fb, Fc) \arrow[r, "P_{a,b} \times P_{b,c}"] \arrow[d, "\circ"'] & \mathcal{X}(Ia, Ib) \times \mathcal{X}(Ib, Ic) \arrow[d, "\circ"] \\ \mathcal{D}(Fa, Fc) \arrow[r, "P_{a,c}"] & \mathcal{X}(Ia, Ic) \end{tikzcd}$$

As a simple example, the identity functor $1_{\mathcal{C}}$ is heavily-I-separable for any $I : \mathcal{C} \longrightarrow \mathcal{X}$.

We first collect an important property of this notion :

Proposition

Let $F : \mathcal{C} \longrightarrow \mathcal{D}, G : \mathcal{D} \longrightarrow \mathcal{E}$ and $I : \mathcal{C} \to \mathcal{X}$ be functors.

1. *If F is heavily-I-separable and G is heavily separable, then $G \circ F$ is heavily-I-separable.*
2. *If $G \circ F$ is heavily-I-separable, then so is F.*

Remark : It follows from part 1 of the above proposition that any heavily separable functor F is heavily-I-separable for any functor I such that $Dom(I) = Dom(F)$.

We first collect an important property of this notion :

Proposition

Let $F : \mathcal{C} \longrightarrow \mathcal{D}, G : \mathcal{D} \longrightarrow \mathcal{E}$ and $I : \mathcal{C} \to \mathcal{X}$ be functors.

1. *If F is heavily-I-separable and G is heavily separable, then $G \circ F$ is heavily-I-separable.*
2. *If $G \circ F$ is heavily-I-separable, then so is F.*

Remark : It follows from part 1 of the above proposition that any heavily separable functor F is heavily-I-separable for any functor I such that $Dom(I) = Dom(F)$.

The most important theorem in the theory of separability of functors is the classical Rafael's Theorem (1990) which gives equivalent (and easier to work with) conditions for the ordinary separability of adjoints.

Rafael's Theorem

Given an adjoint pair $(F \dashv G : \mathcal{D} \longrightarrow \mathcal{C})$, the left adjoint F is separable iff the unit $\eta : 1_{\mathcal{C}} \longrightarrow GF$ of the adjunction is a split monomorphism in the functor category $End(\mathcal{C})$. Dually, the right adjoint G is separable iff the counit $\varepsilon : FG \longrightarrow 1_{\mathcal{D}}$ is a split epimorphism in $End(\mathcal{D})$.

We have suitably generalized this to a Rafael-type Theorem in the context of heavy separability of the second kind. This is our main result.

A Rafael-type Theorem

Theorem

Consider an adjunction :

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "G"] \end{tikzcd}$$

Let $I : \mathcal{C} \longrightarrow \mathcal{X}$ *and* $J : \mathcal{D} \longrightarrow \mathcal{Y}$ *be functors. Then,*

(1) The left adjoint F *is heavily-*I*-separable if and only if there is a natural transformation* $\gamma : IGF \longrightarrow I$ *such that* $\gamma \circ I\eta = id$ *and*

$$\gamma \circ (\gamma GF) = \gamma \circ (IG\varepsilon F)$$

(2) The right adjoint G *is heavily-*J*-separable if and only if there is a natural transformation* $\delta : J \longrightarrow JFG$ *such that* $J\varepsilon \circ \delta = id$ *and*

$$(\delta FG) \circ \delta = (JF\eta G) \circ \delta$$

We consider some applications of our Rafael-type theorem.

I. Ring homomorphisms

For a ring homomorphism $f : R \to S$, we denote the associated induction (extension of scalars) functor by $f^* : \mathcal{M}_R \longrightarrow \mathcal{M}_S$ and the restriction of scalars functor by $f_* : \mathcal{M}_S \longrightarrow \mathcal{M}_R$. Throughout this section, we fix a ring morphism $R \xrightarrow{f} S$. There is an adjunction $f^* \dashv f_*$. Ardizonni and Menini show in [1] that the induction functor f^* is heavily separable iff f is a split monomorphism of rings. We generalize this in the context of heavy separability of the second kind. Using the Rafael-type theorem, we obtain :

Theorem

Let $Q \xrightarrow{g} R$ be a ring morphism. Then, f^* is heavily-g_*-separable if and only if there exists a morphism $E : S \to R$ of right Q-modules such that : (i) $E \circ f = 1_R$ and (ii) $E(s_1.E(s_2)) = E(s_1 s_2)$ for all $s_1, s_2 \in S$.

I. Ring homomorphisms

For a ring homomorphism $f : R \to S$, we denote the associated induction (extension of scalars) functor by $f^* : \mathcal{M}_R \longrightarrow \mathcal{M}_S$ and the restriction of scalars functor by $f_* : \mathcal{M}_S \longrightarrow \mathcal{M}_R$. Throughout this section, we fix a ring morphism $R \xrightarrow{f} S$. There is an adjunction $f^* \dashv f_*$. Ardizonni and Menini show in [1] that the induction functor f^* is heavily separable iff f is a split monomorphism of rings. We generalize this in the context of heavy separability of the second kind. Using the Rafael-type theorem, we obtain :

Theorem

Let $Q \xrightarrow{g} R$ be a ring morphism. Then, f^* is heavily-g_*-separable if and only if there exists a morphism $E : S \to R$ of right Q-modules such that : (i) $E \circ f = 1_R$ and (ii) $E(s_1.E(s_2)) = E(s_1 s_2)$ for all $s_1, s_2 \in S$.

A better way to interpret condition (ii) is as follows :

Corollary

f^* is heavily-g_*-separable if and only if there exists a morphism $E : S \to R$ of right Q-modules such that $E \circ f = 1_R$ and ker E is a left ideal of S.

We now use the above corollary to obtain a non-trivial example of the fact that heavy separability of the second kind need not imply ordinary heavy separability.

Consider the ring map $\phi : \mathbb{C} \to M_2(\mathbb{R})$, given by , $\phi(a+ib) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ for all $a+ib \in \mathbb{C}$

We note that $M_2(\mathbb{R})$ is a simple ring so that, if $\psi : M_2(\mathbb{R}) \to \mathbb{C}$ is a (unital) ring homomorphism, then ψ is injective, which is absurd as $\mathbb{C}$ is an integral domain while $M_2(\mathbb{R})$ is not. Thus, there are no (unital) ring homomorphisms of $M_2(\mathbb{R})$ into $\mathbb{C}$. Consequently, ϕ^* is not heavily-separable.

Now, consider the ring inclusion $\mathbb{R} \xrightarrow{i} \mathbb{C}$. We note that the map $E : M_2(\mathbb{R}) \to \mathbb{C}$ given by,

$\mathrm{E}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = d - ib$, is right $\mathbb{R}$-linear and satisfies $E \circ \phi = 1_{\mathbb{C}}$.

Further, the kernel of E is $\left\{ \begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix} : a, c \in \mathbb{R} \right\}$ which is a left ideal of $M_2(\mathbb{R})$.

Therefore, by the above corollary ϕ^* is heavily-i_*-separable.

We also consider the heavy separability of the second kind for restriction of scalars. Let $h : T \to S$ be a ring map. Then,

Theorem

f_* is heavily-h_*-separable iff there exists an element $\sum_i a_i \otimes b_i \in S \otimes_R S$ such that

$$\sum_i t a_i \otimes b_i = \sum_i a_i \otimes b_i t, \quad \textit{for all } t \in T$$

$$\sum_i a_i b_i = 1$$

$$\sum_{i,j} a_i \otimes b_i a_j \otimes b_j = \sum_i a_i \otimes 1 \otimes b_i$$

II. The free-(co)algebra functor associated to a (co)monad

We recall that a monad on a category $\mathcal{C}$ is a monoid object in the strict monoidal category $(End(\mathcal{C}), \circ, 1_{\mathcal{C}})$. Let $\mathbf{T} = (T, \mu, \eta)$ be a monad on a category $\mathcal{C}$. An adjunction $(F \dashv G : \mathcal{D} \to \mathcal{C})$ is said to be a $\mathbf{T}$-adjunction if the monad $(GF, \eta, G\varepsilon F)$ defined by the adjunction is $\mathbf{T}$. A similar terminology is used for comonads.

II. The free-(co)algebra functor associated to a (co)monad

We recall that a monad on a category $\mathcal{C}$ is a monoid object in the strict monoidal category $(End(\mathcal{C}), \circ, 1_{\mathcal{C}})$. Let $\mathbf{T} = (T, \mu, \eta)$ be a monad on a category $\mathcal{C}$. An adjunction $(F \dashv G : \mathcal{D} \to \mathcal{C})$ is said to be a $\mathbf{T}$-adjunction if the monad $(GF, \eta, G\varepsilon F)$ defined by the adjunction is $\mathbf{T}$. A similar terminology is used for comonads.

We first consider a corollary of our Rafael-type theorem :

Corollary

Let $I : \mathcal{C} \to \mathcal{X}$ be a functor. Then, for any two $\mathbf{T}$-adjunctions, the left adjoint of one is heavily-I-separable iff the left adjoint of the other is heavily-I-separable.

The above corollary suggests the well-definedness of the following definition :

II. The free-(co)algebra functor associated to a (co)monad

We recall that a monad on a category $\mathcal{C}$ is a monoid object in the strict monoidal category $(End(\mathcal{C}), \circ, 1_{\mathcal{C}})$. Let $\mathbf{T} = (T, \mu, \eta)$ be a monad on a category $\mathcal{C}$. An adjunction $(F \dashv G : \mathcal{D} \to \mathcal{C})$ is said to be a $\mathbf{T}$-adjunction if the monad $(GF, \eta, G\varepsilon F)$ defined by the adjunction is $\mathbf{T}$. A similar terminology is used for comonads.

We first consider a corollary of our Rafael-type theorem :

Corollary

Let $I : \mathcal{C} \to \mathcal{X}$ be a functor. Then, for any two $\mathbf{T}$-adjunctions, the left adjoint of one is heavily-I-separable iff the left adjoint of the other is heavily-I-separable.

The above corollary suggests the well-definedness of the following definition :

Definition

Let $\mathbf{T} = (T, \mu, \eta)$ be a monad on a category $\mathcal{C}$ and let $(F \dashv G)$ be a $\mathbf{T}$-adjunction. Let $I : \mathcal{C} \longrightarrow \mathcal{X}$. We say that the monad $\mathbf{T}$ is heavily-I-separable if F is heavily-I-separable.

We have the analogous result and definition for the dual case of comonads.

In 1965, Eilenberg and Moore proved that given an adjoint pair of endofunctors $(L \dashv R)$ on a category $\mathcal{C}$, a comonad structure $(d : L \to L \circ L, e : L \to 1_{\mathcal{C}})$ on L induces a "corresponding" monad structure $(m = \overline{d} : R \circ R \to R, i = \overline{e} : 1_{\mathcal{C}} \to R)$ on R and conversely.

In 1965, Eilenberg and Moore proved that given an adjoint pair of endofunctors $(L \dashv R)$ on a category $\mathcal{C}$, a comonad structure $(d : L \to L \circ L, e : L \to 1_{\mathcal{C}})$ on L induces a "corresponding" monad structure $(m = \overline{d} : R \circ R \to R, i = \overline{e} : 1_{\mathcal{C}} \to R)$ on R and conversely. Now, there are Eilenberg-Moore adjunctions :

$$\begin{tikzcd} \mathcal{C}^L \arrow[r, bend left, "U^L"] \arrow[r, phantom, "\perp"] & \mathcal{C} \arrow[l, bend left, "F^L"] \end{tikzcd} \qquad \begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F_R"] \arrow[r, phantom, "\perp"] & \mathcal{C}_R \arrow[l, bend left, "U_R"] \end{tikzcd}$$

where $\mathcal{C}^L$ and $\mathcal{C}_R$ are the Eilenberg-Moore categories of L-coalgebras and R-algebras respectively. $\mathrm{F}^L : x \mapsto (Lx, d_x)$ and $F_R : x \mapsto (Rx, m_x)$ are the free L-coalgebra and the free R-algebra functors respectively, and $U^L : (x, g) \mapsto x$ and $U_R : (x, h) \mapsto x$ are the associated forgetful functors.

In 1965, Eilenberg and Moore proved that given an adjoint pair of endofunctors $(L \dashv R)$ on a category $\mathcal{C}$, a comonad structure $(d : L \to L \circ L, e : L \to 1_{\mathcal{C}})$ on L induces a "corresponding" monad structure $(m = \overline{d} : R \circ R \to R, i = \overline{e} : 1_{\mathcal{C}} \to R)$ on R and conversely. Now, there are Eilenberg-Moore adjunctions :

$$\begin{tikzcd} \mathcal{C}^L \arrow[r, bend left, "U^L"] \arrow[r, phantom, "\perp"] & \mathcal{C} \arrow[l, bend left, "F^L"] \end{tikzcd} \qquad \begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F_R"] \arrow[r, phantom, "\perp"] & \mathcal{C}_R \arrow[l, bend left, "U_R"] \end{tikzcd}$$

where $\mathcal{C}^L$ and $\mathcal{C}_R$ are the Eilenberg-Moore categories of L-coalgebras and R-algebras respectively. $\mathrm{F}^L : x \mapsto (Lx, d_x)$ and $F_R : x \mapsto (Rx, m_x)$ are the free L-coalgebra and the free R-algebra functors respectively, and $U^L : (x, g) \mapsto x$ and $U_R : (x, h) \mapsto x$ are the associated forgetful functors.

In this setup, Böhm, Brzeziński and Wisbauer showed that the free L-coalgebra functor F^L is separable iff the free R-algebra functor F_R is separable. Using our Rafael-type theorem, we generalize this in the context of heavy-separability of the second kind.

Theorem

Let $(L \dashv R)$ be an adjoint pair of endofunctors on $\mathcal{C}$. Assume that (L, d, e) is a comonad with corresponding monad structure $(R,\ m := \overline{d},\ i := \overline{e})$ on R. Let $(I \dashv J : \mathcal{C} \to \mathcal{C})$ be any adjoint pair such that I commutes with the comonad structure on L, i.e.

$$IL = LI, \qquad Id = dI, \qquad Ie = eI$$

Then,

1. *The free L-coalgebra functor $F^L : \mathcal{C} \to \mathcal{C}^L$ is I-separable if and only if the free R-algebra functor $F_R : \mathcal{C} \to \mathcal{C}_R$ is J-separable.*
2. *If I is a full functor, then the comonad (L, d, e) is heavily-I-separable if and only if the monad (R, m, i) is heavily-J-separable. In particular, F^L is heavily-I-separable if and only if F_R is heavily-J-separable.*

Theorem

Let $(L \dashv R)$ be an adjoint pair of endofunctors on $\mathcal{C}$. Assume that (L, d, e) is a comonad with corresponding monad structure $(R,\ m := \overline{d},\ i := \overline{e})$ on R. Let $(I \dashv J : \mathcal{C} \to \mathcal{C})$ be any adjoint pair such that I commutes with the comonad structure on L, i.e.

$$IL = LI, \qquad Id = dI, \qquad Ie = eI$$

Then,

1. *The free L-coalgebra functor $F^L : \mathcal{C} \to \mathcal{C}^L$ is I-separable if and only if the free R-algebra functor $F_R : \mathcal{C} \to \mathcal{C}_R$ is J-separable.*
2. *If I is a full functor, then the comonad (L, d, e) is heavily-I-separable if and only if the monad (R, m, i) is heavily-J-separable. In particular, F^L is heavily-I-separable if and only if F_R is heavily-J-separable.*

Taking $I, J = 1_{\mathcal{C}}$, it follows that :

Corollary

If $(L \dashv R)$ is an adjoint pair of endofunctors on a category $\mathcal{C}$, with (L, d, e) being a comonad and (R, m, i) being the corresponding monad, then F^L is heavily separable if and only if F_R is heavily separable.

We now consider a concrete application of this.

The heavy separability of the free contramodule functor associated to a coring

Given a ring A, we recall that an A-coring is a comonoid object $(C, \Delta : C \to C \otimes_A C, \epsilon : C \to A)$ in the monoidal category of $A - A$ bimodules.

The heavy separability of the free contramodule functor associated to a coring

Given a ring A, we recall that an A-coring is a comonoid object $(C, \Delta : C \to C \otimes_A C, \epsilon : C \to A)$ in the monoidal category of $A - A$ bimodules.

We consider the Tensor-Hom adjunction :
$$\begin{tikzcd} \mathbb{M}_A \arrow[r, bend left, "\_\otimes_A C"] \arrow[r, phantom, "\perp"] & \mathbb{M}_A \arrow[l, bend left, "\mathrm{Hom}_A(C, \_)"] \end{tikzcd}$$

The heavy separability of the free contramodule functor associated to a coring

Given a ring A, we recall that an A-coring is a comonoid object $(C, \Delta : C \to C \otimes_A C, \epsilon : C \to A)$ in the monoidal category of $A - A$ bimodules.

We consider the Tensor-Hom adjunction :
$$\begin{tikzcd} \mathbb{M}_A \arrow[r, bend left, "\_\otimes_A C"] \arrow[r, phantom, "\perp"] & \mathbb{M}_A \arrow[l, bend left, "\mathrm{Hom}_A(C,\_)"] \end{tikzcd}$$

It is well-known that this induces a canonical monad structure (m, i) on the functor $Hom_A(C, \_) =: [C, \_] : \mathbb{M}_A \to \mathbb{M}_A$ via :

$m = Hom_A(C, Hom_A(C, \_)) \xrightarrow{\sim} Hom_A(C \otimes_A C, \_) \xrightarrow{\Delta^*} Hom_A(C, \_)$

$i = 1_{\mathbb{M}_A} \xrightarrow{\sim} Hom_A(A, \_) \xrightarrow{\epsilon^*} Hom_A(C, \_)$.

The heavy separability of the free contramodule functor associated to a coring

Given a ring A, we recall that an A-coring is a comonoid object $(C, \Delta : C \to C \otimes_A C, \epsilon : C \to A)$ in the monoidal category of $A - A$ bimodules.

We consider the Tensor-Hom adjunction :
$$\begin{tikzcd} \mathbb{M}_A \arrow[r, bend left, "\_\otimes_A C"] \arrow[r, phantom, "\perp"] & \mathbb{M}_A \arrow[l, bend left, "\mathrm{Hom}_A(C,\_)"] \end{tikzcd}$$

It is well-known that this induces a canonical monad structure (m, i) on the functor $Hom_A(C, \_) =: [C, \_] : \mathbb{M}_A \to \mathbb{M}_A$ via :

$$m = Hom_A(C, Hom_A(C, \_)) \xrightarrow{\sim} Hom_A(C \otimes_A C, \_) \xrightarrow{\Delta^*} Hom_A(C, \_)$$
$$i = 1_{\mathbb{M}_A} \xrightarrow{\sim} Hom_A(A, \_) \xrightarrow{\epsilon^*} Hom_A(C, \_).$$

Further, this induces a corresponding comonad structure $(d = \overline{m}, e = \overline{i})$ on the left adjoint $\_ \otimes_A C$ which is easily computed to be :

$$d = 1_{\_} \otimes_A \Delta : \_ \otimes_A C \to (\_ \otimes_A C) \otimes_A C, \quad e = 1_{\_} \otimes_A \epsilon : \_ \otimes_A C \to \_ \otimes_A A \xrightarrow{\sim} 1_{\mathbb{M}_A}.$$

The Eilenberg-Moore category of $(\_ \otimes_A C)$-coalgebras is computed to be the **category** $\mathbb{M}^C$ **of right** C**-comodules** and the associated free $(\_ \otimes_A C)$-coalgebra functor is the **induction functor** $M \mapsto (M \otimes_A C, 1_M \otimes_A \Delta)$ **from** $\mathbb{M}_A$ **to** $\mathbb{M}^C$.

The heavy separability of the free contramodule functor associated to a coring

Given a ring A, we recall that an A-coring is a comonoid object $(C, \Delta : C \to C \otimes_A C, \epsilon : C \to A)$ in the monoidal category of $A-A$ bimodules.

We consider the Tensor-Hom adjunction :
$$\begin{tikzcd} \mathbb{M}_A \arrow[r, bend left, "\_\otimes_A C"] \arrow[r, phantom, "\perp"] & \mathbb{M}_A \arrow[l, bend left, "\mathrm{Hom}_A(C,\_)"] \end{tikzcd}$$

It is well-known that this induces a canonical monad structure (m, i) on the functor $Hom_A(C, \_) =: [C, \_] : \mathbb{M}_A \to \mathbb{M}_A$ via :

$$m = Hom_A(C, Hom_A(C, \_)) \xrightarrow{\sim} Hom_A(C \otimes_A C, \_) \xrightarrow{\Delta^*} Hom_A(C, \_)$$
$$i = 1_{\mathbb{M}_A} \xrightarrow{\sim} Hom_A(A, \_) \xrightarrow{\epsilon^*} Hom_A(C, \_).$$

Further, this induces a corresponding comonad structure $(d = \overline{m}, e = \overline{i})$ on the left adjoint $\_ \otimes_A C$ which is easily computed to be :

$$d = 1_{\_} \otimes_A \Delta : \_ \otimes_A C \to (\_ \otimes_A C) \otimes_A C, \quad e = 1_{\_} \otimes_A \epsilon : \_ \otimes_A C \to \_ \otimes_A A \xrightarrow{\sim} 1_{\mathbb{M}_A}.$$

The Eilenberg-Moore category of $(\_ \otimes_A C)$-coalgebras is computed to be the **category** $\mathbb{M}^C$ **of right** C**-comodules** and the associated free $(\_ \otimes_A C)$-coalgebra functor is the **induction functor** $M \mapsto (M \otimes_A C, 1_M \otimes_A \Delta)$ **from** $\mathbb{M}_A$ **to** $\mathbb{M}^C$.

On the other hand, the Eilenberg-Moore category of $Hom_A(C, \_)$-algebras is the **category** $\mathbb{M}_{[C,\_]}$ **of right** C**-contramodules**. The associated free $(Hom_A(C, \_))$-algebra functor is the **free** C**-contramodule functor** $M \mapsto (Hom_A(C, M), m_M)$ **from** $\mathbb{M}_A$ **into** $\mathbb{M}_{[C,\_]}$.

Now, Ardizonni and Menini had shown that the induction functor $\_ \otimes_A C : \mathbb{M}_A \to \mathbb{M}^C$ is heavily separable iff the coring C has an invariant group-like element, i.e. an element $e \in C$ such that $ae = ea$ for all $a \in A$, $\epsilon(e) = 1$ and $\Delta(e) = e \otimes e$.

Now, Ardizonni and Menini had shown that the induction functor $\_ \otimes_A C : \mathbb{M}_A \to \mathbb{M}^C$ is heavily separable iff the coring C has an invariant group-like element, i.e. an element $e \in C$ such that $ae = ea$ for all $a \in A$, $\epsilon(e) = 1$ and $\Delta(e) = e \otimes e$.

Further, by the previous corollary, the free C-contramodule functor is heavily separable iff the induction functor $\_ \otimes_A C$ is so. Thus, we have the following theorem :

Theorem

The free C-contramodule functor $\mathbb{M}_A \to \mathbb{M}_{[C,\_]}$, $M \mapsto (Hom_A(C, M), m_M)$ is heavily separable iff C has an invariant group-like element, that is, an element $e \in C$ such that (1) $ae = ea$ for all $a \in A$ (2) $\epsilon(e) = 1$ (3) $\Delta(e) = e \otimes e$.

In particular, if A is a commutative ring, then an A-coalgebra is an A-coring, so that the above theorem holds for coalgebras.

In particular, we can apply these results to the Sweedler coring associated to a ring map. Given a ring map $R \to S$, the $S - S$-bimodule $S \otimes_R S$ has an S-coring structure given by, $\Delta : S \otimes_R S \to (S \otimes_R S) \otimes_S (S \otimes_R S) \cong S \otimes_R S \otimes_R S, : s_1 \otimes s_2 \mapsto s_1 \otimes 1 \otimes s_2$ and $\epsilon : S \otimes_R S \to S, : s_1 \otimes s_2 \mapsto s_1 s_2$. This is called the Sweedler coring associated to the ring map.

It is easy to see that that an invariant group like element of the Sweedler coring $S \otimes_R S$ is the same as a heavy separability idempotent for the ring map $R \to S$. Thus, combining all these, we therefore have

Proposition

Let $R \to S$ be a ring homomorphism. Then, the following are equivalent :

1. *The restriction of scalars functor $\mathbb{M}_S \to \mathbb{M}_R$ is heavily separable.*
2. *The free-$S \otimes_R S$ contramodule functor $\mathbb{M}_S \to \mathbb{M}_{[S \otimes_R S, \text{-}]}$ is heavily separable.*

III. Forgetful functors associated to Entwined Modules

Let k be commutative ring. Let (A, C, ψ) be a (right-right) entwining structure over k where $(A, \nabla_A : A \otimes A \to A, i_A : k \to A)$ is the k-algebra, $(C, \Delta_C : C \to C \otimes C, \epsilon_C : C \to k)$ is the k-coalgebra and $\psi : C \otimes A \to A \otimes C$, $c \otimes a \mapsto \sum a_\psi \otimes c^\psi$ is the entwining map of the structure.

III. Forgetful functors associated to Entwined Modules

Let k be commutative ring. Let (A, C, ψ) be a (right-right) entwining structure over k where $(A, \nabla_A : A \otimes A \to A, i_A : k \to A)$ is the k-algebra, $(C, \Delta_C : C \to C \otimes C, \epsilon_C : C \to k)$ is the k-coalgebra and $\psi : C \otimes A \to A \otimes C$, $c \otimes a \mapsto \sum a_\psi \otimes c^\psi$ is the entwining map of the structure. An (A, C, ψ)-entwined module is a triple (M, ρ_M, ρ^M) where M is a k-module, $\rho_M : M \otimes_k A \to M$ is a right A-module structure on M and $\rho^M : M \to M \otimes C$ is a right C-comodule structure on M such that they are compatible with the entwining map ψ.

III. Forgetful functors associated to Entwined Modules

Let k be commutative ring. Let (A, C, ψ) be a (right-right) entwining structure over k where $(A, \nabla_A : A \otimes A \to A, i_A : k \to A)$ is the k-algebra, $(C, \Delta_C : C \to C \otimes C, \epsilon_C : C \to k)$ is the k-coalgebra and $\psi : C \otimes A \to A \otimes C$, $c \otimes a \mapsto \sum a_\psi \otimes c^\psi$ is the entwining map of the structure. An (A, C, ψ)-entwined module is a triple (M, ρ_M, ρ^M) where M is a k-module, $\rho_M : M \otimes_k A \to M$ is a right A-module structure on M and $\rho^M : M \to M \otimes C$ is a right C-comodule structure on M such that they are compatible with the entwining map ψ. We denote the category of (A, C, ψ)-entwined modules and A-linear C-colinear morphisms by $\mathbb{M}(\psi)_A^C$. This in particular covers all kinds of commonly studied modules like algebra modules, coalgebra comodules, Hopf modules, etc.

III. Forgetful functors associated to Entwined Modules

Let k be commutative ring. Let (A, C, ψ) be a (right-right) entwining structure over k where $(A, \nabla_A : A \otimes A \to A, i_A : k \to A)$ is the k-algebra, $(C, \Delta_C : C \to C \otimes C, \epsilon_C : C \to k)$ is the k-coalgebra and $\psi : C \otimes A \to A \otimes C$, $c \otimes a \mapsto \sum a_\psi \otimes c^\psi$ is the entwining map of the structure. An (A, C, ψ)-entwined module is a triple (M, ρ_M, ρ^M) where M is a k-module, $\rho_M : M \otimes_k A \to M$ is a right A-module structure on M and $\rho^M : M \to M \otimes C$ is a right C-comodule structure on M such that they are compatible with the entwining map ψ. We denote the category of (A, C, ψ)-entwined modules and A-linear C-colinear morphisms by $\mathbb{M}(\psi)_A^C$. This in particular covers all kinds of commonly studied modules like algebra modules, coalgebra comodules, Hopf modules, etc.

Now, there are canonical adjunctions :

$$\begin{tikzcd} \mathbb{M}(\psi)_A^C \arrow[r, bend left, "U^C"] \arrow[r, phantom, "\perp"] & \mathbb{M}_A \arrow[l, bend left, "F^C"] \end{tikzcd} \qquad \begin{tikzcd} \mathbb{M}^C \arrow[r, bend left, "F_A"] \arrow[r, phantom, "\perp"] & \mathbb{M}(\psi)_A^C \arrow[l, bend left, "U_A"] \end{tikzcd}$$

where U^C and U_A are the forgetful functors forgetting the C-coaction and the A-action respectively and F^C, F_A are the associated free functors :

$$F^C : (M, \rho_M) \mapsto (M \otimes C,\ \rho_{M\otimes C} = (\rho_M \otimes 1_C) \circ (1_M \otimes \psi),\ \rho^{M\otimes C} = 1_M \otimes \Delta_C)$$
$$F_A : (M, \rho^M) \mapsto (M \otimes A,\ \rho_{M\otimes A} = 1_M \otimes \nabla_A,\ \rho^{M\otimes A} = (1_M \otimes \psi) \circ (\rho^M \otimes 1_A))$$

Caenepeel and Militaru considered the classical problem of finding concrete conditions for the separability of one forgetful functor with respect to the other. First, we consider the U_A-separability of U^C.

Proposition (Caenepeel and Militaru)

U^C is U_A-separable iff there exists a k-linear map $\theta : C \otimes C \to A$ such that for all $x, y \in C$,

$$\theta(x \otimes y_{[1]}) \otimes y_{[2]} = \theta(x_{[2]} \otimes y)_\psi \otimes x_{[1]}^\psi \quad \textit{and} \quad \theta \circ \Delta_C = i_A \circ \epsilon_C$$

Caenepeel and Militaru considered the classical problem of finding concrete conditions for the separability of one forgetful functor with respect to the other. First, we consider the U_A-separability of U^C.

Proposition (Caenepeel and Militaru)

U^C is U_A-separable iff there exists a k-linear map $\theta : C \otimes C \to A$ such that for all $x, y \in C$,

$$\theta(x \otimes y_{[1]}) \otimes y_{[2]} = \theta(x_{[2]} \otimes y)_\psi \otimes x_{[1]}^\psi \quad \text{and} \quad \theta \circ \Delta_C = i_A \circ \epsilon_C$$

Using the Rafael-type theorem, we have considered the problem in the context of heavy separability of the second kind.

Theorem

U^C is heavily-U_A-separable iff there exists a k-linear map $\theta : C \otimes C \to A$ such that for all $x, y, z \in C$

$$\theta(x \otimes y_{[1]}) \otimes y_{[2]} = \theta(x_{[2]} \otimes y)_\psi \otimes x_{[1]}^\psi, \qquad \theta \circ \Delta_C = i_A \circ \epsilon_C \quad \text{and}$$

$$\theta(y \otimes z_{[1]})_\psi . \theta(x^\psi \otimes z_{[2][1]}) \otimes z_{[2][2]} = \epsilon_C(y)\theta(x \otimes z_{[1]}) \otimes z_{[2]}$$

Now, we consider the U^C-separability of U_A.

Proposition (Caenepeel and Militaru)

U_A is U^C-separable iff there exists a k-linear map $e : C \to A \otimes A$, $e : c \mapsto e^1(c) \otimes e^2(c)$ (summation understood) such that for all $c \in C, a \in A$

$$e^1(c) \otimes e^2(c)a = a_\psi e^1(c^\psi) \otimes e^2(c^\psi) \quad \text{and} \quad \nabla_A \circ e = i_A \circ \epsilon_C$$

Now, we consider the U^C-separability of U_A.

Proposition (Caenepeel and Militaru)

U_A is U^C-separable iff there exists a k-linear map $e: C \to A \otimes A$, $e: c \mapsto e^1(c) \otimes e^2(c)$ (summation understood) such that for all $c \in C, a \in A$

$$e^1(c) \otimes e^2(c)a = a_\psi e^1(c^\psi) \otimes e^2(c^\psi) \quad \text{and} \quad \nabla_A \circ e = i_A \circ \epsilon_C$$

Using the Rafael-type Theorem, we have :

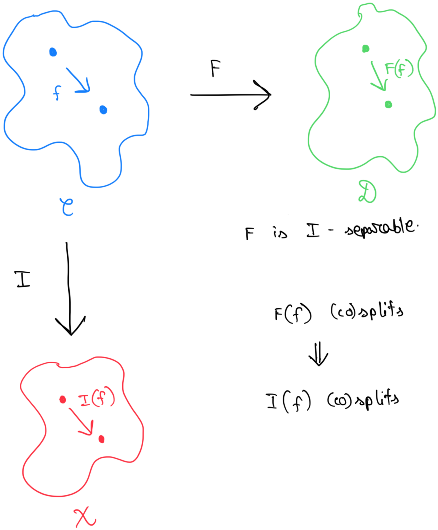
Theorem

U_A is heavily-U^C-separable iff there exists a k-linear map $e: C \to A \otimes A$, $e: c \mapsto e^1(c) \otimes e^2(c)$ (summation understood) such that for all $c \in C, a \in A$

$$e^1(c) \otimes e^2(c)a = a_\psi e^1(c^\psi) \otimes e^2(c^\psi), \qquad \nabla_A \circ e = i_A \circ \epsilon_C \quad \text{and}$$

$$\epsilon_C\left(c_{[1][1]}\right) e^1\left(c_{[2]}\right)_\psi \otimes e^1\left(\left(c_{[1][2]}\right)^\psi\right) \otimes e^2\left(\left(c_{[1][2]}\right)^\psi\right) . e^2\left(c_{[2]}\right) = e^1\left(c_{[2]}\right) \otimes \epsilon_C\left(c_{[1]}\right) . 1 \otimes e^2\left(c_{[2]}\right)$$

Separable functors come equipped with Maschke-type Theorems



F is I - separable.

F(f) (co)splits

$\Downarrow$

I(f) (co)splits

Maschke Functors of the second kind

Consider functors $F : \mathcal{C} \to \mathcal{D}, I : \mathcal{C} \to \mathcal{X}$. F is I-Maschke if for every $f : c \to c'$ in $\mathcal{C}$ such that $F(f)$ is a split monomorphism, $I(f)$ is a split monomorphism. Dualizing this, gives the notion of a dual I-Maschke functor. For $I = 1_{\mathcal{C}}$, an I-Maschke (dual I-Maschke) functor is simply called a Maschke (dual Maschke) functor. For a quick example, we consider a field extension L/K. Then, the associated restriction of scalars functor is Maschke and dual Maschke. Caenepeel and Militaru proved the following Maschke-type theorem for I-separable functors :

Proposition (Maschke-type theorem for I-separable functors)

Any I-separable functor is I-Maschke and dual I-Maschke.

The converse is not true : If L/K is a non-separable field extension, then the associated restriction of scalars functor is not separable.

I have extended the "Maschke functor" notion to a "heavy"-version which corresponds to the heaviness-axiom in our definition of heavy separability of the second kind.

Heavily Maschke functors of the second kind

We first need a notation :
For any $g : d \to d'$ in a category $\mathcal{D}$, we denote by $ret(g)$, the set of all retractions (left inverses) of g in $\mathcal{D}$. We note that $ret(g) = \emptyset$ if g is not a split monomorphism.

Heavily Maschke functors of the second kind

We first need a notation :
For any $g : d \to d'$ in a category $\mathcal{D}$, we denote by $ret(g)$, the set of all retractions (left inverses) of g in $\mathcal{D}$. We note that $ret(g) = \emptyset$ if g is not a split monomorphism.

Now, let $F : \mathcal{C} \to \mathcal{D}$, $I : \mathcal{C} \to \mathcal{X}$ be functors. We consider the following definition :

Definition

We define F to be heavily-I-Maschke if there is a collection $\{R_f : ret(F(f)) \to ret(I(f)) \mid f \text{ is an arrow of } \mathcal{C}\}$ of set maps satisfying the following axiom : Heaviness of the splitting : For every composite $c \xrightarrow{f} c' \xrightarrow{f'} c''$ in $\mathcal{C}$,

$$R_{f' \circ f}(g \circ g') = R_f(g) \circ R_{f'}(g') \text{ for all } g \in ret(F(f)), g' \in ret(F(f'))$$

Dualizing this gives the notion of a dual heavily-I-Maschke functor.

Heavily Maschke functors of the second kind

We first need a notation :
For any $g : d \to d'$ in a category $\mathcal{D}$, we denote by $ret(g)$, the set of all retractions (left inverses) of g in $\mathcal{D}$. We note that $ret(g) = \emptyset$ if g is not a split monomorphism.

Now, let $F : \mathcal{C} \to \mathcal{D}$, $I : \mathcal{C} \to \mathcal{X}$ be functors. We consider the following definition :

Definition

We define F to be heavily-I-Maschke if there is a collection $\{R_f : ret(F(f)) \to ret(I(f)) \mid f \text{ is an arrow of } \mathcal{C}\}$ of set maps satisfying the following axiom : Heaviness of the splitting : For every composite $c \xrightarrow{f} c' \xrightarrow{f'} c''$ in $\mathcal{C}$,

$$R_{f' \circ f}(g \circ g') = R_f(g) \circ R_{f'}(g') \text{ for all } g \in ret(F(f)), g' \in ret(F(f'))$$

Dualizing this gives the notion of a dual heavily-I-Maschke functor. We now have that :

Theorem (Maschke-type Theorem for Heavily-I-separable functors)

Any heavily-I-separable functor is a heavily-I-Maschke functor and a dual heavily-I-Maschke functor.

Future Work

Theorem (Caenepeel and Militaru)

Let $(G \vdash F : \mathcal{C} \to \mathcal{D})$ be an adjoint pair and let $I : \mathcal{C} \to \mathcal{X}$, $J : \mathcal{D} \to \mathcal{Y}$ be functors. Then,

1. F is I-Maschke iff each component of $I\eta$ is a split monomorphism in $\mathcal{X}$.

2. G is dual-J-Maschke iff each component of $J\varepsilon$ is a split epimorphism in $\mathcal{Y}$.

Future Work

Theorem (Caenepeel and Militaru)

Let $(G \vdash F : \mathcal{C} \to \mathcal{D})$ be an adjoint pair and let $I : \mathcal{C} \to \mathcal{X}$, $J : \mathcal{D} \to \mathcal{Y}$ be functors. Then,

1. F is I-Maschke iff each component of $I\eta$ is a split monomorphism in $\mathcal{X}$.

2. G is dual-J-Maschke iff each component of $J\varepsilon$ is a split epimorphism in $\mathcal{Y}$.

I am currently looking for a suitable version of this theorem for heavily Maschke functors of the second kind. This will augment our understanding of the relationship between heavily separable functors of the second kind and heavily Maschke functors of the second kind, and in particular, help obtain better instances of such functors.

References

[1] A. Ardizzoni and C. Menini Heavily Separable Functors
J. Algebra, Vol. 543 (2020), 170-197

[2] G. Böhm, T. Brzeziński and Robert Wisbauer
Monads and comonads on module categories
J. Algebra, Vol. 322 (2009), no.5, 1719-1747

[3] T. Brzeziński, S. Caenepeel, G. Militaru and S. Zhu
Frobenius and Maschke type theorems for Doi-Hopf modules and entwined modules revisited: a unified approach
Lecture Notes in Pure and Appl. Math., Vol. 221 (2001), 1-31

[4] S. Caenepeel and G. Militaru
Maschke functors, semisimple functors and separable functors of the second kind: applications
J. Pure Appl. Algebra, Vol. 178 (2003), no.2, 131-157

[5] S. Eilenberg and J. C. Moore
Adjoint functors and triples
Illinois J. Math., Vol. 9 (1965), 381-398

[6] C. Năstăsescu, M. Van den Bergh and F. Van Oystaeyen
Separable functors applied to graded rings
J. Algebra, Vol. 123 (1989), no.2, 397-413

[7] M. D. Rafael
Separable functors revisited
Comm. Algebra, Vol. 18 (1990), no.5, 1445-1459

Thank You