4.5 Closures

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Exercise 3

Proof. From the definition of asymmetry, we know that $\forall x \in A \forall y \in A[(x,y) \in R \rightarrow (y,x) \in R]$. Thus, $\neg\exists(x,y) \in R(y,x) \in R$. This means that the set $\{(x,y) \in R \wedge (y,x) \in R\} = \varnothing$. Considering the definition of antisymmetry, then it is *vacuously true* that $\forall x \in A \forall y \in A[(xRy \wedge yRx) \rightarrow x = y]$.

$\square$

Proof. Suppose that R is a strict partial order on A, so it is irreflexive and transitive. Let $x, y \in A$ and suppose that $(x,y) \in R$. Since R is irreflexive and $x \in A$, then $(x,x) \notin R$. Since R is transitive and $(x,x) \notin R$, using the contrapositive we get that $(x,y) \notin R$ or $(y,x) \notin r$. It cannot be the case that $(x,y) \notin R$, because we assume that $(x,y) \in R$. Thus, it must be the case that $(y,x) \notin R$. Since x, y were arbitrary elements of A, it follows that R is asymmetric.

$\square$

Exercise 4

Proof. To prove that S is a partial order on A, we need to prove that S is reflexive, transitive, and antisymmetric on A. To prove that S is reflexive, we know that S is the reflexive closure of R, thus by definition S is reflexive.
To prove that S is transitive, let $x, y, z \in A$ and suppose that $(x,y) \in S$ and $(y,z) \in S$. Since S is the reflexive closure of R, then $S = R \cup i_A$. Thus, $(x,y) \in S$ means that $(x,y) \in R$ or $(x,y) \in i_A$. Similarly, since $(y,z) \in S$, then $(y,z) \in i_A$. The last two statements are equivalent to saying that either $(x,y) \in i_A$ or $(x,y) \in R$ and $(y,z) \in R$. **Case 1:** $(x,y) \in i_A$. Since $x = y$ and $y = z$, then $(x,z) \in i_A$ so $(x,z) \in S$. **Case 2:** $(x,y) \in R$ and $(y,z) \in R$. Since $x = y$, then we have that $(x,x) = (y,y) \in R$. But R is asymmetric, so this cannot happen. Similarly, since $y = z$, then $(y,y) = (z,z) \in R$, but since R is asymmetric, this cannot happen and we rule out this case. Since x, y, z were arbitrary elements of A, it follows that S is transitive on A.
To prove that S is antisymmetric, let $x, y \in A$ and suppose that $(x,y) \in S$ and $(y,x) \in S$. Since S is the reflexive closure of R, then $S = R \cup i_A$. Thus, $(x,y) \in S$ means that $(x,y) \in R$ or $(x,y) \in i_A$. Similarly, $(y,x) \in S$ means that $(y,x) \in R$ or $(y,x) \in i_A$. The last two are statements are equivalent to saying that either $(x,y) \in i_A$ or $(x,y) \in R$ and $(y,x) \in R$. **Case 1:** $(x,y) \in i_A$. Then, $x = y$. **Case 2:** $(x,y) \in R$ and $(y,x) \in R$. Since R is asymmetric, then we rule out this case. Given that x, y were arbitrary elements of A, then S is antisymmetric. Hence, S is a partial order. $\square$

Proof. Suppose that R is a strict total order and let $x, y \in A$. Given that R is a strict partial order and $x, y \in A$, then we have that either $(x,y) \in R$, $(y,x) \in R$, or $x = y$. Recall that $S = R \cup i_A$. **Case 1:** $(x,y) \in R$. Then $(x,y) \in S \vee (y,x) \in S$. **Case 2:** $(y,x) \in R$. Then $(y,x) \in S \vee (x,y) \in S$. **Case 3:** $x = y$. Then $(x,x) = (y,y) = (x,y) \in i_A$, so $(x,y) \in S$. Since x, y were arbitrary elements of A, then $\forall x \in A \forall y \in A(xSy \vee ySx)$. Given that S is a strict partial order and that the latter property is satisfied, then S is a total order.

$\square$

Exercise 5

Proof. To prove that S is the largest element of the set, we need to prove that $\forall T \in \mathscr{F}(T, S) \in R$ and $S \in \mathscr{F}$. We will start with the second goal. Let $(x, y) \in S$. This means that $(x, y) \in R$ and $(x, y) \notin i_A$. Since (x, y) were arbitrary elements of S, then $S \subseteq R$. Since $(x, y) \notin i_A$, this means that S is irreflexive. Thus, $S \in \mathscr{F}$. To prove the first goal, let $T \in \mathscr{F}$ and suppose $(x, y) \in T$, Since $T \in \mathscr{F}$, then $T \subseteq R$ and T is irreflexive. Since $(x, y) \in T$ and $T \subseteq R$, then $(x, y) \in R$. Also, since T is irreflexive and $(x, y) \in T$, then $x \neq y$. Thus, $(x, y) \in S$. Since (x, y) was an arbitrary element of T, then $T \subset S$. Also, since T was an arbitrary element of $\mathscr{F}$, then $\forall T \in \mathscr{F} T \subset S$. Hence, S is the largest element of the set $\mathscr{F}$.

□

Proof. To prove that S is a strict partial order on A, we need to to prove that S is irreflexive and transitive. Let's prove transitivity first. Let $x, y, z \in A$ and suppose that $(x, y) \in S$ and $(y, z) \in S$. This means that $(x, y) \in R$ and $(y, z) \in R$ respectively. Since R is a partial order on A, then R is transitive. Hence, $(x, z) \in R$. Since $(x, y) \in S$, this also means that $x \neq y$. Similarly, $y \neq z$. Thus, $x \neq z$, so $(x, z) \not{ini}_A$. Since $(x, z) \in R$ and $(x, z) \notin i_A$, then $(x, z) \in S$. Given that x, y, z were arbitrary elements of A, then S is transitive on A. We already proved that S is irreflexive. However, we can take a different approach now. Let $x \in A$. Since R is reflexive on A, then $(x, x) \in R$. Also, $(x, x) \in i_A$, so $(x, x) \notin S$. Since x was an arbitrary element of A, then S is irreflexive.

□

Exercise 7

Proof. ($\rightarrow$). Suppose R is reflexive on A. To prove that R is its own reflexive closure we need to prove three conditions: 1) $R \subseteq R$, which is trivial; 2) R is reflexive, which is what we assumed; 3) $\forall T \in \mathscr{F} R \subseteq T$. Let $T \in \mathscr{F}$ and suppose $(x, y) \in R$. Since $T \in \mathscr{F}$, then $R \subseteq T$. Since $(x, y) \in R$, then $(x, y) \in T$. Given that (x, y) were arbitrary elements of R, then $R \subseteq T$. Since T was an arbitrary element of $\mathscr{F}$, then $\forall T \in \mathscr{F} R \subseteq T$.

($\leftarrow$). Suppose R is its own reflexive closure. Thus, R is reflexive.

□

It should be clear that similar theorems hold for symmetric and transitive closures.

Exercise 8

Exercise 9

Proof. Let us first prove that $Dom(S) = Dom(R)$. We start by proving that $Dom(R) \subseteq Dom(S)$. Let $x \in Dom(R)$. Thus there is some element $y \in A$ so that $(x, y) \in R$. Since S is the transitive closure of R, then $R \subseteq S$, so $(x, y) \in S$. Hence, $x \in Dom(S)$. Since x was an arbitrary element of $Dom(R)$, then $Dom(R) \subseteq Dom(S)$. We now prove that $Dom(S) \subseteq Dom(R)$. Let $x \in Dom(S)$. Thus, there is some element $y \in A$ so that $(x, y) \in S$. Let $T = \{(x, y) \in S | x \in Dom(R) \quad \text{and} \quad y \in Ran(R)\}$. Since $\forall T \in \mathscr{F} S \subseteq T$, then $S \subseteq T$. Given that $(x, y) \in S$, then $(x, y) \in T$. This means that $x \in Dom(R)$. Since x was an arbitrary element of $Dom(S)$, then $Dom(S) \subseteq Dom(R)$. Thus, $Dom(S) = Dom(R)$.

To prove that $Ran(S) = Ran(R)$, we follow the same approach. □

Some comments: This is not trivial. The key is to write down all the facts we know. For example, since S is the transitive closure of R under the subset partial order, then $\forall T \in \mathscr{F} S \subseteq T$. In our givens, it appears that $(x, y) \in S$, so this should tell us we need to use these two assumptions. That is why we look for some set $T \in \mathscr{F}$ that will satisfy our goal.

Exercise 10

Proof. To prove that $\mathscr{F} \neq \varnothing$, we need to prove that there is some element in $\mathscr{F}$. Suppose $A \times A \in \mathscr{F}$. This means that $R \subseteq A \times A$, which we know is true because R is a relation on A. Now, let $x, y \in A$ and suppose that $(x, y) \in A \times A$. Thus, $x \in A$ and $y \in A$, which we already knew. Then it is also true that $(y, x) \in A \times A$. Since x, y were arbitrary elements of A, then $A \times A$ is symmetric. Thus, $\mathscr{F} \neq \varnothing$.

$\square$

Proof. We will prove this by verifying the three conditions we know must be satisfied. Let $(x, y) \in R$ and suppose that $M \in \mathscr{F}$. Thus, $R \subseteq M$. Since $(x, y) \in R$, then $(x, y) \in M$. Given that M was an arbitrary element of $\mathscr{F}$, then $(x, y) \in \cap\mathscr{F}$. Since (x, y) were arbitrary elements, then $R \subseteq S$. To prove the second conditions, suppose that $x, y \in A$ and let $(x, y) \in \cap\mathscr{F}$. Also, suppose that $M \in \mathscr{F}$. Thus, $(x, y) \in M$. Since $M \in \mathscr{F}$, then M is symmetric, and since $(x, y) \in M$ then $(y, x) \in M$. Given that M was an arbitrary element of $\mathscr{F}$, then $(y, x) \in \cap\mathscr{F}$.Since x, y were arbitrary elements of A, then $\cap\mathscr{F}$ is symmetric. To prove the third condition, let $T \in \mathscr{F}$ and suppose that $(x, y) \in \cap\mathscr{F}$. Thus, $(x, y) \in T$. Since (x, y) was an arbitrary element of A, then $\cap\mathscr{F} \subseteq T$. Given that T was an arbitrary element of $\mathscr{F}$, then $\forall T \in \mathscr{F} \cap \mathscr{F} \subseteq T$.

$\square$

Exercise 11

Proof. Let $(x, y) \in S_1$. Since S_1 is the reflexive closure of R_1, we know that $S_1 = R_1 \cup i_A$. Thus, $(x, y) \in R_1$ or $(x, y) \in i_A$. **Case 1:** $(x, y) \in R_1$. Since $R_1 \subseteq R_2$, then $(x, y) \in R_2$ and so $(x, y) \in R_2 \cup i_A$. **Case 2:** $(x, y) \in i_A$. Then $(x, y) \in R_2 \cup i_A$. Since (x, y) was an arbitrary element of S_1, then $S_1 \subseteq S_2$.

$\square$

Proof. Let $(x, y) \in S_1$. Since S_1 is the symmetric closure of R_1, then $S_1 = R_1 \cup R_1^{-1}$. Thus, $(x, y) \in R_1$ or $(x, y) \in R_1^{-1}$. **Case 1:** $(x, y) \in R_1$. Since $R_1 \subseteq R_2$, then $(x, y) \in R_2 \cup R_2^{-1}$. **Case 2:** $(x, y) \in R_1^{-1}$. Then $(y, x) \in R_1$, and so $(y, x) \in R_2$. Thus, $(x, y) \in R_2^{-1}$ and so $(x, y) \in R_2 \cup R_2^{-1}$. Since (x, y) was an arbitrary element of S_1, then $S_1 \subseteq S_2$.

$\square$

Proof. Let $(x, y) \in S_1$ and suppose that $M \in \mathscr{F}_2$. We know that S_1 is the transitive closure of R_1, so $S_1 = \cap\mathscr{F}_1$. Since $M \in \mathscr{F}_2$, then $R_2 \subseteq M$ and M is transitive. Since $R_1 \subseteq R_2$ and $R_2 \subseteq M$, then $R_1 \subseteq M$. Thus, $M \in \mathscr{F}_1$. Given that $(x, y) \in S_1$ and $M \in \mathscr{F}_1$, then $(x, y) \in M$. Since M was an arbitrary element of $\mathscr{F}_2$, then $(x, y) \in S_2$. Given that (x, y) was an arbitrary element of S_1, then $S_1 \subseteq S_2$.

$\square$

Exercise 12

Proof. $S_1 \cup S_2 = (R_1 \cup i_A) \cup (R_2 \cup i_A) = (R_1 \cup R_2) \cup i_A = R \cup i_A = S$.

$\square$

Proof. Let us prove first that $S_1 \cup S_2 \subseteq S$. Let $(x, y) \in S_1 \cup S_2$. Thus, $(x, y) \in S_1$ or $(x, y) \in S_2$. **Case 1:** $(x, y) \in S_1$. Then $(x, y) \in R_1$ or $(x, y) \in R_1^{-1}$. If $(x, y) \in R_1$, then $(x, y) \in R$. Thus, $(x, y) \in R \cup R^{-1}$. If $(x, y) \in R_1^{-1}$, then $(x, y) \in R_1^{-1} \cup R_2^{-1}$. Hence, $(x, y) \in (R_1 \cup R_2)^{-1}$. **Case 2:** $(x, y) \in S_2$. Then $(x, y) \in R_2$ or $(x, y) \in R_2^{-1}$. If $(x, y) \in R_2$, then $(x, y) \in R$. Thus, $(x, y) \in R \cup R^{-1}$. If $(x, y) \in R_2^{-1}$, then $(x, y) \in R_1^{-1} \cup R_2^{-1}$. Hence, $(x, y) \in (R_1 \cup R_2)^{-1}$. To prove that $S \subseteq S_1 \cup S_2$, suppose $(x, y) \in S$. Then $(x, y) \in R$ or $(x, y) \in R^{-1}$. **Case 1:** $(x, y) \in R$. Thus, $(x, y) \in R_1$ or $(x, y) \in R_2$. If $(x, y) \in R_1$, then $(x, y) \in S_1$, since $R_1 \subseteq S_1$. Hence, $(x, y) \in S_1 \cup S_2$. A similar argument follows for $(x, y) \in R_2$. **Case 2:** $(x, y) \in R^{-1}$. Then, $(y, x) \in R$ so $(y, x) \in R_1$ or $(y, x) \in R_2$. If $(y, x) \in R_1$, then $(y, x) \in S_1$ because $R_1 \subseteq S_1$. Thus, $(x, y) \in S_1$ since S_1 is symmetric. Hence, $(x, y) \in S_1 \cup S_2$. A similar argument follows for $(y, x) \in R_2$.

$\square$

Proof. Let $(x, y) \in S_1 \cup S_2$ and suppose that $T \in \mathscr{F}$, where $\mathscr{F} = \{T \subseteq A \times A | R \subseteq T \quad \text{and} \quad T \quad \text{transitive}\}$. Since $(x, y) \in S_1 \cup S_2$, then $(x, y) \in S_1$ or $(x, y) \in S_2$. **Case 1:** $(x, y) \in S_1$. We know that $R_1 \subseteq R_1 \cup R_2 = R \subseteq T$, since $T \in \mathscr{F}$. Given that T is transitive, then $T \in \mathscr{F}_1$. Thus, $(x, y) \in T$. **Case 2:** A similar argument follows as in Case 1. Since T was an arbitrary element of $\mathscr{F}$, then $(x, y) \in S$. Given that (x, y) was an arbitrary element of $S_1 \cup S_2$, then $S_1 \cup S_2 \subseteq S$. □

Exercise 13

For (a) we want to prove that $S_1 \cap S_2 = S$.

Proof. $S_1 \cap S_2 = (R_1 \cup i_A) \cap (R_2 \cup i_A) = (R_1 \cap R_2) \cup i_A = R \cup i_A = S$. □

For (b) we want to prove that $S_1 \cap S_2 = S$.

$$\begin{aligned}
S &= R \cup R^{-1} \\
&= (R_1 \cap R_2) \cup (R_1 \cap R_2)^{-1} \\
&= [(R_1 \cap R_2)^{-1} \cup R_1] \cap [(R_1 \cap R_2)^{-1} \cup R_2] \\
&= [(R_1^{-1} \cap R_2^{-1}) \cup R_1] \cap [(R_1^{-1} \cap R_2^{-1}) \cup R_2] \\
&= (R_1 \cup R_1^{-1}) \cap (R_1 \cup R_2^{-1}) \cap (R_2 \cup R_1^{-1}) \cap (R_2 \cup R_2^{-1}) \\
&= (S_1 \cap S_2) \cap (R_1 \cup R_2^{-1}) \cap (R_2 \cup R_1^{-1})
\end{aligned}$$

which clearly shows us that $S \neq S_1 \cap S_2$.

For (c) we want to prove that $S_1 \cap S_2 \subseteq S$. We can write down our scratch work:

- We start by assuming that $(x, y) \in S_1 \cap S_2$ and that $T \in \mathscr{F}$
- Since $T \in \mathscr{F}$, then we know that $R \subseteq T$ and T is transitive. We would like to know if T is an element of both $\mathscr{F}_1$ and $\mathscr{F}_2$, just like we did in the previous exercise.
- Since T is transitive, we would like to know whether $R_1 \subseteq T$ and $R_2 \subseteq T$. Since we know that $R \subseteq T$, we would like to know if $R_1 \subseteq R$. However this could or could not happen. If $R_1 = R_2$, then $R_1, R_2 \subseteq R$. If $R_1 \neq R_2$, we would need more information. Thus it is not possible to say that $S_1 \cap S_2 \subseteq S$

However, we can prove that $S \subseteq S_1 \cap S_2$.

Proof. Let $(x, y) \in S$ and suppose that $T_1 \in \mathscr{F}_1$ and $T_2 \in \mathscr{F}_2$. Since S_1 is the transitive closure of R_1, then $\forall T_1 \in \mathscr{F}_1 S_1 \subseteq T_1$. Given that $T_1 \in \mathscr{F}_1$, then $S_1 \subseteq T_1$. Similarly for $T_2 \in \mathscr{F}_2$, then $S_2 \subseteq T_2$. Given that S_1 and S_2 are the transitive closure of R_1 and R_2 respectively, then $R_1 \subseteq S_1$ and $R_2 \subseteq S_2$. Thus, $R_1 \cap R_2 \subseteq S_1 \cap S_2$. We know that $S_1 \cap S_2$ is transitive (prove it), so $S_1 \cap S_2 \in \mathscr{F}$. Given that $(x, y) \in S$, then $(x, y) \in S_1 \cap S_2$. Since $S_1 \subseteq T_1$ and $S_2 \subseteq T_2$, then $(x, y) \in T_1$ and $(x, y) \in T_2$. Since T_1 and T_2 were arbitary elements of $\mathscr{F}_1$ and $\mathscr{F}_2$ respectively, then $(x, y) \in S_1 \cap S_2$. Given that (x, y) was an arbitrary element of S, then $S \subseteq S_1 \cap S_2$. □

Exercise 15

By now we should be able to have some "mathematical hunches". If we are looking for a closure that is both reflexive and symmetric, we might want to think of $S = (R \cup i_A) \cap (R \cup R^{-1}$ or $S = (R \cup i_A) \cup (R \cup R^{-1}$. While working on the proof, we should be able to notice which one of our candidates is correct.

Proof. Let $S = R \cup (i_A \cup R^{-1})$. To prove the first clause, it is quite obvious that $R \subseteq R \cup (i_A \cup R^{-1})$.
For the second clause, we will first prove reflexiveness. Let $x \in A$. We know that $i_A = \{(x, x)|x \in A\}$. Thus, $(x, x) \in i_A$ and so $(x, x) \in i_A \cup (R \cup R^{-1})$. Since x was an arbitrary element of A, then S is reflexive. To prove symmetry, let $x, y \in A$ and suppose that $x \in S$. **Case 1:** $(x, y) \in R$, Then $(y, x) \in R^{-1})$, so $(y, x) \in S$. **Case 2:** $(x, y) \in i_A$. Then $x = y$, so $(y, x) \in i_A$. Hence $(y, x) \in S$. **Case 3:** $(x, y) \in R^{-1}$. Then $(y, x) \in R$, so $(y, x) \in S$. Given that x, y were arbitrary elements of A then S is symmetric.
To prove the third clause, let $T \in \mathscr{F}$ and suppose $(x, y) \in S$. Given that $T \in \mathscr{F}$, then $R \subseteq T$ and T is both symmetric and transitive. **Case 1:** $(x, y) \in R$. Since $R \subseteq T$, then $(x, y) \in T$.**Case 2:** $(x, y) \in i_A$. Given that T is symmetric, then $i_A \subseteq T$. Since $x = y$, then $(x, y) \in T$. **Case 3:** $(x, y) \in R^{-1}$. Then $(y, x) \in R$. Since $R \subseteq T$, then $(y, x) \in T$. Given that T is symmetric, then $(x, y) \in T$. Since (x, y) was an arbitrary element of S, then $S \subseteq T$. Sine T was an arbitrary element of $\mathscr{F}$, then $\forall T \in \mathscr{F} S \subseteq T$

□

Exercise 16

Proof. Suppose R is symmetric, and let $x, y \in A$ and $(x, y) \in S$. Since S is the reflexive closure of R, then $(x, y) \in R$ or $(x, y) \in i_A$. **Case 1:** $(x, y) \in R$. R is symmetric, so $(y, x) \in R$. Since $R \subseteq S$, then $(y, x) \in S$. **Case 2:** $(x, y) \in i_A$. Then, $x = y$. Given that S is the reflexive closure of R, then $(x, x) = (y, y) = (y, x) \in S$. Since x, y were arbitrary elements of A, then S is symmetric.

□

Proof. Suppose R is transitive, and let $x, y, z \in A$, xSy and ySz. **Case 1:** xRy and yRz. Since R is transitive, then xRz. Given that $R \subseteq S$, so xSz. **Case 2:** xRy and $(y, z) \in i_A$. Then, $y = z$ so xRz. Since $R \subseteq S$, then xSz. **Case 3:** $(x, y) \in i_A$ and $(y, z) \in R$. Then, $x = y$ so xRz. Since $R \subseteq S$, then xSz. **Case 4:** $(x, y) \in i_A$ and $(y, z) \in i_A$. Then, $x = y = z$. Since S is reflexive, we know that $(x, x) = (y, y) = (z, z) = (x, z) \in S$.
Now, given that x, y, z were arbitrary elements of A, then S is transitive.

□

Exercise 17

Proof. Suppose R is symmetric, and let $x, y \in A$ and $(x, y) \in S$. We would like to know if $S^{-1} \in \mathscr{F}$, so we will prove this. Let $(x, y) \in R$. Since R is symmetric, then $(y, x) \in R$. Given that $R \subseteq S$, then $(y, x) \in S$ iff $(x, y) \in S^{-1}$. Since (x, y) was an arbitrary element of $A \times A$, then $R \subseteq S^{-1}$. Now, let $x, y, z \in A$ and suppose that $(x, y) \in S^{-1}$ and $(y, z) \in S^{-1}$. Then, $(x, y) \in S$ and $(y, z) \in S$. Given that S is transitive, it follows that $(z, y) \in S$ iff $(y, z) \in S^{-1}$. Thus, given that x, y, z were arbitrary elements of A, then S^{-1} is transitive. Now we can proceed with our original proof. Since $(x, y) \in S$ and $S^{-1} \in \mathscr{F}$, then $(x, y) \in S^{-1}$ iff $(y, x) \in S$. Given that x, y were arbitrary elements of A, then S is symmetric.

□

Exercise 18

Proof. To prove the first clause, we know hat Q is the symmetric closure of R, so $R \subseteq Q$. Also, S is the transitive closure of Q, so $Q \subseteq S$. Thus, $R \subseteq S$. To prove the second clause, we know that S is the transitive closure of Q, so S is transitive. Since Q is symmetric and $Q \subseteq A \times A$, by Exercise 17 we know that S is symmetric too. To prove the third clause, let $T \in \mathscr{F}$. Thus, $R \subseteq T$ and T is both transitive and symmetric. Since T is transitive, then $T \in \mathscr{F}_1$, where $\mathscr{F}_1 = \{T_1 \subseteq A \times A | Q \subseteq T_1 \quad \text{and} \quad T_1 \quad \text{transitive}\}$, then $S \subseteq T$. Since T was an arbitrary element of $\mathscr{F}$, then $\forall T \in \mathscr{F} S \subseteq T$.

□

Proof. Let $\mathscr{F} = \{T \subseteq A \times A | Q' \subseteq T \quad \text{and} \quad T \quad \text{symmetric}\}$. We know that $R \subseteq Q$, where Q' is the transitive closure of R and S is the transitive closure of Q. By Exercise 11, we then know that $Q' \subseteq S$. Also, we know that S is symmetric, so $S \in \mathscr{F}$. Since S' is the symmetric closure of Q', then $\forall T \in \mathscr{F} S' \subseteq T$, so $S' \subseteq S$.

$\square$