4.4 Ordering Relations

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Exercise 5

Proof. To prove that $R \cap (B \times B)$ is a partial order on B we need to prove that $R \cap (B \times B)$ is reflexive, transitive, and antisymmetric. To prove reflexiveness, let $x \in B$. Since $B \subseteq A$, Then $x \in A$. Since R is a partial order on A, then R is reflexive on A. Thus, xRx. Since $x \in B$, then $(x, x) \in (B \times B)$. Since x was an arbitrary element of B, then $R \cap (B \times B)$ is reflexive on B.

To prove transitivity, let $x, y, z \in B$, and suppose that $(x, y) \in R \cap (B \times B)$ and $(y, z) \in R \cap (B \times B)$. Then, $(x, y) \in R$ and $(y, z) \in R$. Since R is a partial order on A, then R is transitive on A. Thus, since xRy and yRz, then xRz. Since $x, z \in B$, then $(x, z) \in B \times B$. Hence, $(x, z) \in R \cap (B \times B)$ and so $R \cap (B \times B)$ is transitive on B.

To prove antisymmetry, let $x, y \in B$, and suppose that $(x, y) \in R \cap (B \times B)$ and $(y, x) \in R \cap (B \times B)$. Then, $(x, y) \in R$ and $(y, x) \in R$. Since R is a partial order on A, then R is antisymmetric on A, and given that $(x, y) \in R$ and $(y, x) \in R$, we conclude that $y = x$. Thus, $R \cap (B \times B)$ is antisymmetric on B.

□

Exercise 8

Proof. To prove that T is a partial order on $A \times B$ we need to prove that T is reflexive, transitive, and antisymmetric. To prove reflexiveness, let $(x, y) \in A \times B$. Thus, $x \in A$ and $y \in B$. Since R is a partial order on A, then R is reflexive on A, and since $x \in A$, then xRx. Similarly, since S is a partial order on B, then S is reflexive on B, and since $y \in B$, then ySy. Since xRx and ySy, then $(x, y)T(x, y)$. Since (x, y) was an arbitrary element of $A \times B$, then T is reflexive on $A \times B$.

To prove that T is transitive, let $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in A \times B$, and suppose that $(x_1, y_1)T(x_2, y_2)$ and $(x_2, y_2)T(x_3, y_3)$. Since $(x_1, y_1)T(x_2, y_2)$, then x_1Rx_2 and y_1Sy_2. Similarly, since $(x_2, y_2)T(x_3, y_3)$, then x_2Rx_3 and y_2Sy_3. Given that R is a partial order on A, then R is transitive on A, so x_1Rx_3. Similarly, since S is a partial order on B, then S is transitive on B, so y_1Sy_3. Since x_1Rx_3 and y_1Sy_3, then $(x_1, y_1)T(x_3, y_3)$, since $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ were arbitrary elements of $A \times B$, then T is transitive on $A \times B$.

To prove that T is antisymmetric, let $(x_1, y_1), (x_2, y_2) \in A \times B$, and suppose that $(x_1, y_1)T(x_2, y_2)$ and $(x_2, y_2)T(x_1, y_1)$. Thus, x_1Rx_2 and y_1Sy_2, and x_2Rx_1 and y_2Sy_1, respectively. Since R is a partial order on A, then R is antisymmetric so $x_1 = x_2$. Similarly, since S is a partial order on B, then S is antisymmetric so $y_1 = y_2$. Thus, $(x_1, y_1) = (x_2, y_2)$. Given that $(x_1, y_1), (x_2, y_2)$ were arbitrary elements of $A \times B$, then T is antisymmetric on $A \times B$.

□

To answer the second question, try proving it. Doing the cases that will appear, there will be a contradiction. Thus, T does not have to be a total order even if both R and S are.

Exercise 12

Proof. Suppose B has a minimal element B_1. By definition, then $B_1 \in B$, so that $B_1 \neq \varnothing$ and $\forall x \in \mathbb{R} \forall y \in \mathbb{R}[(x \in B_1 \wedge x < y) \rightarrow y \in B_1]$. Since $B_1 \neq \varnothing$, then there is some element $b \in B_1$. Since $\forall x \in \mathbb{R} \forall y \in \mathbb{R}[(x \in B_1 \wedge x < y) \rightarrow y \in B_1]$ and $b \in B_1$, then $\forall y \in \mathbb{R}[(b \in B_1 \wedge x < y) \rightarrow y \in B_1]$. Let

$Y = \{y \in \mathbb{R} | y > b\}$, so $Y \subseteq B_1$. Note that since $b \in B_1$ and $b \notin Y$, then $B_1 \neq Y$. Given that $Y \subset B_1$ and $B_1 \subseteq B$, then $Y \subseteq B$. Thus, there is some set $Y \in B$ so that $Y \subseteq B_1$, but $Y \neq B_1$, which is a contradiction of our definition of the S-minimal element of B. Thus, B has no minimal element. $\square$

Exercise 14

Proof. ($\rightarrow$). Suppose that b is the R-largest element of B and let $x \in B$ be an arbitrary element. It follows that xRb, so it is true that $(b, x) \in R^{-1}$.

($\leftarrow$). Suppose that b is the R^{-1}-smallest element of B and let $x \in B$ be an arbitrary element. It follows that $bR^{-1}x$, so it is true that $(x, b) \in R$.

$\square$

Proof. ($\rightarrow$). Suppose b is the R-maximal element of B. Also, let $x \in B$ and suppose that $xR^{-1}b$, so it follows that bRx. Since $x \in B$ and bRx, and given that b is the R-maximal element of B, it follows that $x = b$.

($\leftarrow$). Suppose that b is an R^{-1}-minimal element of B. Furthermore, let $x \in B$ and suppose that bRx, so it is true that $xR^{-1}b$. Since $x \in B$ and $xR^{-1}b$, and given that b is an R^{-1}-minimal element of B, it follows that $x = b$.

$\square$

Exercise 16

Proof. Suppose that b is the R-largest element of B. First, we prove that b is also the R-maximal element of B. Let $x \in B$ and suppose that bRx. Thus, since b is the R-largest element of B, it follows that xRb. Since $b, x \in B$ and $B \subseteq A$, then $b, x \in A$, Since R is a partial order, then it is antisymmetric, so we conclude that $x = b$. Thus, b is a largest element of B. To prove that b is the only R-maximal element of B, suppose c is an R-maximal element of B, so $c \in B$. Thus, since $c \in B$ and b is an R-maximal element of B, it follows that cRb. Since $b \in B$ and cRb, and given that c is R-maximal, it follows that $c = b$. Thus, b is the only maximal element of B.

$\square$

Exercise 17

If I have any mistakes, please let me know. I am solving it in the way I understand the question that is being posed. We can try proving this statement, so let's you see how our scratch work would look like. The goal would be that c is R-smallest element of C, which we know that it is represented as $\forall x \in CcRx$. Our list of givens would be the following:

- R is a partial order on A
- $B \subseteq A$
- $C \subseteq R : \forall c_1 \in C \forall c_2 \in C[(c_1, c_2) \in C \rightarrow (c_1, c_2) \in R]$
- $C \subseteq A$
- c is R-minimal element of $C : \forall x \in C(xRc \rightarrow x = c)$ and $c \in C$
- $x \in C$, where x is arbitrary (this comes from the logical form of the goal we are trying to prove)

Based on this list, we can get to the following:

1. $x, c \in C$. Since $C \subseteq A$, then $x, c \in A$
2. Since $C \subseteq R$, then xRc or cRx
    - It is true that xRc. However, since c is R-minimal, then $x = c$. Thus, $(x, x) = (c, c) = (x, c) \in R$ and we would be stuck

Thus, our reasoning tells us that the statement is not necessarily true

Exercise 18

Proof. $(\rightarrow)$. Let $x \in A$, and suppose that x is an upper bound of B_1. Also, suppose that $z \in B_2$. Since $\forall x \in B_2 \exists y \in B_1(xRy)$ and $z \in B_2$, then there is some element $y_1 \in B_1$ so that zRy_1. Since $B_2 \subseteq A$ and $z \in B_2$, then $z \in A$. Similarly, since $B_1 \subseteq A$ and $y_1 \in B_1$, then $y_1 \in A$.Given that R is a partial order, R is transitive. Hence, since zRy_1 and $y_1 Rx$, it follows that zRx. Since z was an arbitrary element of B_2, if x is an upper bound of B_1, then x is an upper bound of B_2.

$(\leftarrow)$. Let $x \in A$, and suppose that x is an upper bound of B_2. Also, suppose that $x \in A$. Since $\forall x \in B_1 \exists y \in B_2(xRy)$ and $z \in B_1$, then there is some element $y_2 \in B_2$ so that $y_2 \in B_2$ and zRy_2. Given that x is an upper bound of B_2 and $y_2 \in B_2$, then $y_2 Rx$. Note that $B_1 \subseteq A$ and $B_2 \subseteq A$, so $z, y_2 \in A$. Since R is a partial order, it is transitive, and given that zRy_2 and $y_2 Rx$, it follows that zRx. Given that z was an arbitrary element of B_1, if x is an upper bound of B_2, then x is an upper bound of B_1.

□

To prove (b), we can actually use the contrapositive or contradiction strategies, since they share something in common

Proof. Suppose that $B_1 \cap B_2 = \varnothing$. We will prove by contradiction, so suppose that either B_1 or B_2 has a maximal element.
Case 1: B_1 has a maximal element. Let $b_1 \in B_1$ be the minimal element of B_1. Since $\forall x \in B_1 \exists m \in B_2(xRM)$ and $b_1 \in B_1$, then there is some element $m_1 \in B_2$ and $b_1 Rm_1$. Since $\forall x \in B_2 \exists n \in B_1(xRn)$ and $m_1 \in B_2$, there is some $n_1 \in B_1$ and $m_1 Rn_1$. Since $B_1 \subseteq A$ and $B_2 \subseteq A$, and $b_1, n_1 \in B_1$ and $m_1 \in B_2$, then $b_1, m_1, n_1 \in A$. Since R is a partial order, then it is transitive and since $b_1 Rm_1$ and $m_1 Rn_1$, then $b_1 Rn_1$. Since b_1 is a maximal element and $n_1 \in B_1$, then $n_1 = b_1$, so $m_1 Rn_1 = m_1 Rb_1$. Given that $m_1 Rb_1$ and $b_1 Rm_1$, and since R is antisymmetric on A, then $m_1 = b_1$. Since $B_1 \cap B_2 = \varnothing$ and $b_1 \in B_1$, then $b_1 \notin B_2$. But $m_1 = b_1$ and $m_1 \in B_2$, so this is a contradiction. Thus, neither B_1 nor B_2 has a maximal element.
Case 2: The logic is similar to Case 1.

□

Exercise 20

Proof. Suppose b is the smallest element of B. This implies that since $B \subseteq A$ and $b \in B$, then $b \in A$, and so b is a lower bound of B. Let L be the set of all lower bounds of B, so we can conclude that $L \neq \varnothing$. Now, let $l \in L$ be an arbitrary element. In particular, we know that $L \subseteq A$. We can think of cases:
Case 1: $L = \{b\}$. Since $b \in B$, and $b \in A$, and since R is reflexive on A, then bRb and b is the g.l.b.
Case 2: L has more than one element. All other elements of L must only be in A and be smaller than b, otherwise there would be no smallest element in B. Thus, b is the g.l.b.

□

The proof for part (b) follows a similar reasoning.

Exercise 21

Proof. Suppose $x \in U$ and xRy. We need to prove that $y \in U$, so let $z \in B$ be an arbitrary element. Since $x \in U$, then $\forall m \in BmRx$ and $x \in A$. Using the latter and since $z \in B$, then zRx. Since $B \subseteq A$ and $z \in B$, then $z \in A$. Given that R is a partial order, then it is transitive, so since zRx and xRy, it follows that zRy. Since z was an arbitrary element of B, we conclude that $y \in U$.

□

Proof. Let $x \in B$ an arbitrary element. We now have to prove that x is a lower bound for U. Then, suppose that $u \in U$ so $\forall m \in BmRu$. Since $x \in B$, then xRu. Given that u was an arbitrary element of U, then x is a lower bound for U. Since x was an arbitrary element of B, then $\forall x \in B(xisalowerboundforU)$.

□

Part (c) is kind of tricky. We have to be able to distinguish that if x is a l.u.b. of B, we are saying that x is the smallest element of U, not B!

Proof. Suppose that x is g.l.b. of U. Suppose $u_2 \in U$, and so we have to prove that xR_2 and $x \in U$. Suppose $b \in B$, so our new goals are xRu_2, bRx, and $x \in A$. Let L_U be the set of l.b. of U. Since x is g.l.b. of U, then it is the largest element of L_U. Thus, $\forall m_1 \in U(xRm_1)$ and $x \in A$. Given that $\forall m_1 \in U(xRm_1)$ and $u_2 \in U$, then xRu_2. Also, since $u_2 \in U$, it follows that $\forall b \in B(bRu_2)$ and $u_2 \in A$. Since $b \in B$ and $\forall b \in B(bRu_2)$, then bRu_2. Given that $x, u_2 \in A$, then u_2Rx. Since R is a partial order, then it is transitive on A. Given that bRu_2 and u_2Rx, then bRx. Since u_2 was an arbitrary element of U, then $x \in U$. Since u_2 was an arbitrary element of U, then x is a l.u.b. of B.

□

Exercise 23

This proof is quite tricky. We really need to know our definitions. I will first prove the l.u.b. and then the g.u.b.

Proof. Suppose $\mathscr{F} \subseteq \mathscr{P}(A)$ and $\mathscr{F} \neq \varnothing$. Let $M \in Up$, where Up is the set of all upper bounds of $\mathscr{F}$. Suppose that $x \in \cup\mathscr{F}$. Thus, there is some set $N \in \mathscr{F}$ so that $x \in N$. Since $M \in Up$, this means that M is an upper bound for $\mathscr{F}$, and given that $N \in \mathscr{F}$, it follows that $N \subseteq M$. We know that $x \in N$, so $x \in M$. Since x was an arbitrary element of $\cup\mathscr{F}$, then $\cup\mathscr{F} \subseteq M$. We would like to prove that $\cup\mathscr{F} \in Up$. Suppose that $T \in \mathscr{F}$ and let $y \in T$. Also, suppose that $z \in \cup\mathscr{F}$. Thus, there is some set $N \in \mathscr{F}$ so that $z \in N$. Since $\mathscr{F} \subseteq \mathscr{P}(A)$, and $N \in \mathscr{F}$, it follows that $N \subseteq A$. Since $z \in N$, then $z \in A$. Also, since $T \in \mathscr{F}$ and $y \in T$, then $y \in \cup\mathscr{F}$. Given that z was an arbitrary element of $\cup\mathscr{F}$, then $\cup\mathscr{F} \subseteq A$. Also, given that y was an arbitrary element of T, then $T \subseteq \cup\mathscr{F}$. Furthermore, since T was an arbitrary element of $\mathscr{F}$, then $\forall T \in \mathscr{F}(T \subseteq \cup\mathscr{F}$. And given that $\cup\mathscr{F} \subseteq A$, which means that $\cup\mathscr{F} \in \mathscr{P}(A)$, we conclude that $\cup\mathscr{F}$ is an upper bound. Hence, $\cup\mathscr{F}$ is the l.u.b. for $\mathscr{F}$.

□

Proof. Suppose $\mathscr{F} \subseteq \mathscr{P}(A)$ and $\mathscr{F} \neq \varnothing$. We want to prove that $\cap\mathscr{F}$ is a the g.l.b. of $\mathscr{F}$. Thus, two conditions must be satisfied: 1) $\forall M \in L(M \subseteq \cap\mathscr{F}$ and $\cap\mathscr{F} \in L$, where L is the set of lower bounds for $\mathscr{F}$. To prove the second condition, $F \in \mathscr{F}$ and suppose that $x \in \cap\mathscr{F}$. Thus, since $F \in \mathscr{F}$ and $x \in \cap\mathscr{F}$, then $x \in F$. Since $F \subseteq \mathscr{P}(A)$ and $x \in F$, then $x \in A$. This also shows that $\cap\mathscr{F} \subseteq A$. To prove the second condition, suppose $M \in L$ and let $x \in M$. Also, suppose that $F \in \mathscr{F}$. Thus, since $M \in L$ means that M is a lower bound for $\mathscr{F}$, then $M \subseteq F$. Since $x \in M$, then $x \in F$. Given that x was an arbitrary element of M, then $M \subseteq \cap\mathscr{F}$. Since F was an arbitrary element of $\mathscr{F}$, then $\forall M \in L(M \subseteq \cap\mathscr{F}$. Thus, we conclude that $\cap\mathscr{F}$ is the g.l.b. of $\mathscr{F}$.

□