2a. We have a black-box that samples with probabilities $q_1, \dots, q_n$. How to sample with probabilities $p_1, \dots, p_n$, if we know that:

$$\forall_{i=1,\dots,n} \quad (1-\varepsilon) q_i \le p_i \le (1+\varepsilon) q_i$$

Expected run time should be $O(1)$, for a fixed ε.

Algorithm: put $\overline{p_i} = \frac{p_i}{(1+\varepsilon)}$. We have $\overline{p_i} \le q_i$.

1) Now sample from $q_1, \dots, q_n$

2) Say k is sampled

3) With probability $\frac{\overline{p_k}}{q_k}$ pick k, else repeat from 1).

$$\boxed{\begin{array}{ll} q_i = \frac{1}{n} & (1+\varepsilon) = n \cdot p_{max} \\ \overline{p_i} = \frac{p_i}{n \cdot p_{max}} & \frac{\overline{p_i}}{q_i} = \frac{p_i}{p_{max}} \end{array}}$$

Proof: Let Y - value sampled from 1)
X - " - from 3)

$$
\begin{aligned}
P(X=k) &= \sum_{i=1}^{n} P(X=k \mid Y=i) P(Y=i) \\
&= \frac{\overline{p_k}}{q_k} q_k + \sum_{i=1}^{n} \left(1 - \frac{\overline{p_i}}{q_i}\right) q_i P(X=k) \\
&= \frac{p_k}{(1+\varepsilon)} + \sum_{i=1}^{n} (q_i - \overline{p_i}) P(X=k) \\
&= \frac{p_k}{(1+\varepsilon)} + P(X=k)\left(1 - \sum_i \overline{p_i}\right) \\
&= \frac{p_k}{(1+\varepsilon)} + P(X=k)\left(1 - \frac{1}{(1+\varepsilon)}\right) \Rightarrow P(X=k) = p_k
\end{aligned}
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$\square$