Математическая статистика.

Первое задание.

$\boxed{№1}$

$\xi \sim R(0;\theta)$, $\theta > 0$, $\theta \in \Theta = (0; +\infty)$

Оценки: $\tilde{\theta}_1 = 2\bar{x} = 2 \cdot \frac{1}{n}\sum\limits_{i=1}^{n} x_i$ $\qquad \vec{x}_n$ - выборка

$\tilde{\theta}_2 = x_{min}$ $\qquad P(x,\theta) = \frac{1}{\theta}\{(0,\theta)\}$

$\tilde{\theta}_3 = x_{max}$

$\tilde{\theta}_4 = x_1 + \frac{1}{n-1}\sum\limits_{k=2}^{n} x_k$

$$M\xi = \int_{-\infty}^{+\infty} x \, dF(x,\theta) = \int_0^{\theta} x \frac{1}{\theta - 0} dx = \frac{\theta}{2}$$

$$M\xi^2 = \int_{-\infty}^{+\infty} x^2 \, dF(x,\theta) = \int_0^{\theta} x^2 \frac{1}{\theta} dx = \frac{\theta^2}{3}$$

$$D\xi = M\xi^2 - (M\xi)^2 = \frac{\theta^2}{3} - \frac{\theta^2}{4} = \frac{\theta^2}{12}$$

а) Проверить на несмещённость и состоятельность

① $\tilde{\theta}_1 = 2\bar{x}$

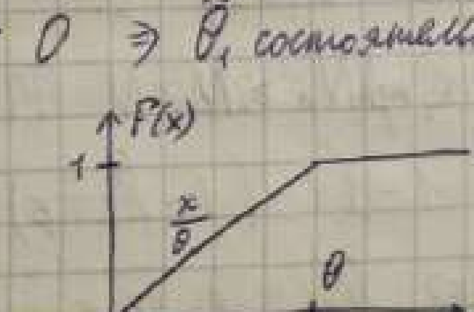
$$\forall \theta > 0 \quad M\tilde{\theta}_1 = M\left(2\frac{1}{n}\sum_{i=1}^{n} x_i\right) = \frac{2}{n}\sum_{i=1}^{n} Mx_i =$$

$$= \frac{2}{n} \cdot n \cdot M\xi = 2 \cdot \frac{\theta}{2} = \theta \quad \Rightarrow \tilde{\theta}_1 \text{ несмещённая}$$

$$\forall \theta > 0 \quad D\tilde{\theta}_1 = D\left(2\frac{1}{n}\sum_{i=1}^{n} x_i\right) = \frac{4}{n^2} D\left(\sum_{i=1}^{n} x_i\right) \overset{\text{незав.}}{=}$$

$$= \frac{4}{n^2}\sum_{i=1}^{n} Dx_i = \frac{4}{n^2} n D\xi = \frac{4\theta^2}{12n} \xrightarrow[n\to\infty]{} 0 \Rightarrow \tilde{\theta}_1 \text{ состоятельн.}$$

② $\tilde{\theta}_2 = x_{min}$



$$\Phi(x) = 1 - (1 - F(x))^n$$

$$\varphi(x) = n(1 - F(x))^{n-1} F'(x) =$$

$$= n\left(1-\frac{x}{\theta}\right)^{n-1}\cdot\frac{1}{\theta}\{(0;\theta)\}$$

$$\forall\theta>0\quad M\tilde{\theta}_2 = Mx_{min} = \int_0^\theta x\, n\left(1-\frac{x}{\theta}\right)^{n-1}\cdot\frac{1}{\theta}\,dx =$$

$$= \left\langle t = 1-\frac{x}{\theta},\ x=\theta(1-t)\right\rangle = \int_0^1 \theta(1-t)\,n\, t^{n-1}\,dt =$$

$$= n\theta\int_0^1 \left(t^{n-1}-t^n\right)dt = n\theta\left(\frac{1}{n}-\frac{1}{n+1}\right) = \theta\cdot\frac{1}{n+1}$$

$\Rightarrow \tilde{\theta}_2$ не является несмещ.

$$\tilde{\theta}_2' = (n+1)\,x_{min} \Rightarrow M\tilde{\theta}_2' = (n+1)\cdot M\tilde{\theta}_2 = \theta$$

$\Rightarrow \tilde{\theta}_2'$ является несмещённой

$$Mx^2_{min} = \int_0^\theta x^2 n\left(1-\frac{x}{\theta}\right)^{n-1}\frac{1}{\theta}\,dx = \left\langle t=1-\frac{x}{\theta}\right\rangle =$$

$$= \int_0^1 \theta^2(1-t)^2 n\, t^{n-1}\,dt = n\theta^2\int_0^1\left(t^{n-1}-2t^n+t^{n+1}\right)dt =$$

$$= n\theta^2\left(\frac{1}{n}-\frac{2}{n+1}+\frac{1}{n+2}\right) = \theta^2\,\frac{n^2+3n+2-2n^2-4n+n^2+n}{(n+1)(n+2)} =$$

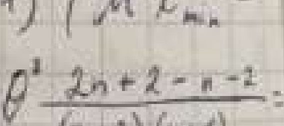
$$= \frac{2\theta^2}{(n+1)(n+2)}$$

$$\forall\theta>0\quad D\tilde{\theta}_2' = (n+1)^2 Dx_{min} = (n+1)^2\left(Mx^2_{min} - \left(Mx_{min}\right)^2\right) = (n+1)^2\left(\frac{2\theta^2}{(n+1)(n+2)} - \frac{\theta^2}{(n+1)^2}\right) = (n+1)\theta^2\,\frac{2n+2-n-2}{(n+2)(n+1)} =$$

$$= \theta^2\cdot\frac{n}{n+2} \not\xrightarrow[n\to\infty]{} 0$$ достаточное условие не вып.

$$\forall\theta>0\ \forall\varepsilon>0\quad P\left(|\tilde{\theta}_2'-\theta|\ge\varepsilon\right) \ge P\left(\tilde{\theta}_2'\ge\theta+\varepsilon\right) =$$

$$= P\left((n+1)x_{min}\ge\theta+\varepsilon\right) = P\left(x_{min}\ge\frac{\theta+\varepsilon}{n+1}\right) =$$



$$= 1-P\left(x_{min}<\frac{\theta+\varepsilon}{n+1}\right) = 1-\Phi\left(\frac{\theta+\varepsilon}{n+1}\right) =$$

$$= \left\langle \exists N: \forall n\ge N\ \ \Phi(x) = 1-(1-F(x))^n = 1-\left(1-\frac{x}{\theta}\right)^n\right\rangle =$$

$$= 1-\left(1-\left(1-\frac{\theta+\varepsilon}{(n+1)\theta}\right)^n\right) \xrightarrow[n\to\infty]{} e^{-\frac{\theta+\varepsilon}{\theta}} > 0$$

$\Rightarrow \tilde{\theta}_2'$ не является состоятельной

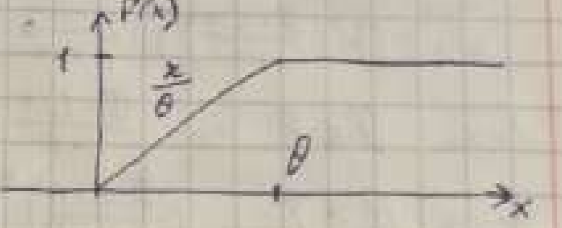
$$\forall \theta > 0 \;\; \forall \varepsilon > 0 \quad P(|\tilde{\theta}_2 - \theta| \geq \varepsilon) = P(|x_{min} - \theta| \geq \varepsilon) =$$

$$= \langle P(x_{min} > \theta) = 0 \rangle = P(x_{min} \leq \theta - \varepsilon) = \Phi(\theta - \varepsilon) =$$

$$= 1 - (1 - F(\theta - \varepsilon))^n \underset{0 < \theta - \varepsilon < \theta}{=} 1 - \left(1 - \frac{\theta - \varepsilon}{\theta}\right)^n = 1 - \left(\frac{\varepsilon}{\theta}\right)^n \xrightarrow[n \to \infty]{} 1$$

не является состоятельной $\tilde{\theta}_2$

③ $\tilde{\theta}_3 = x_{max}$



$$\Psi(x) = (F(x))^n$$

$$\psi(x) = \Psi'(x) = n\,(F(x))^{n-1} f(x)$$

$$\psi(x) = n \left(\frac{x}{\theta}\right)^{n-1} \frac{1}{\theta} \,\{(0, \theta)\}$$

$$\forall \theta > 0 \quad M\tilde{\theta}_3 = M x_{max} = \int_0^\theta x\, n \frac{x^{n-1}}{\theta^n}\, dx =$$

$$= \frac{n}{\theta^n} \int_0^\theta x^n\, dx = \frac{n}{\theta^n} \frac{\theta^{n+1}}{n+1} = \frac{n}{n+1}\, \theta$$

$\Rightarrow$ $\tilde{\theta}_3$ не является несмещённой

$$\tilde{\theta}_3' = \frac{n+1}{n} x_{max} \Rightarrow M\tilde{\theta}_3' = \frac{n+1}{n} M x_{max} = \theta$$

$\Rightarrow$ $\tilde{\theta}_3'$ является несмещённой

$$M x_{max}^2 = \int_0^\theta x^2 n \frac{x^{n-1}}{\theta^n}\, dx = \frac{n}{\theta^n} \int_0^\theta x^{n+1}\, dx = \frac{n}{n+2} \frac{\theta^{n+2}}{\theta^n} =$$

$$= \frac{n}{n+2}\, \theta^2$$

$$\forall \theta > 0 \quad D\tilde{\theta}_3' = \left(\frac{n+1}{n}\right)^2 D x_{max} = \left(\frac{n+1}{n}\right)^2 \left(M x_{max}^2 - (M x_{max})^2\right) =$$

$$= \left(\frac{n+1}{n}\right)^2 \left(\theta^2 \frac{n}{n+2} - \frac{n^2}{(n+1)^2} \theta^2\right) = \theta^2 \frac{1}{n} \frac{n^2 + 2n + 1 - n^2 - 2n}{n+2} = \frac{\theta^2}{n(n+2)} \xrightarrow[n \to \infty]{} 0$$

$\Rightarrow$ $\tilde{\theta}_3'$ является состоятельной (по дост. условию)

$$\forall \theta > 0 \;\; \forall \varepsilon > 0 \quad P(|\tilde{\theta}_3' - \theta| \geq \varepsilon) =$$

$$= P\left(\frac{n+1}{n} x_{max} \geq \theta + \varepsilon\right) + P\left(\frac{n+1}{n} x_{max} \leq \theta - \varepsilon\right) =$$

$= P\left(x_{max} \geq \frac{(\theta+\varepsilon)n}{n+1}\right) + P\left(x_{max} \leq \frac{(\theta-\varepsilon)n}{n+1}\right) = 1 - \Psi\left(\frac{(\theta+\varepsilon)n}{n+1}\right) +$

$+ \Psi\left(\frac{(\theta-\varepsilon)n}{n+1}\right) = 1 - \left(F\left(\frac{(\theta+\varepsilon)n}{n+1}\right)\right)^n + \left(F\left(\frac{(\theta-\varepsilon)n}{n+1}\right)\right)^n$

Если $\frac{(\theta+\varepsilon)n}{n+1} > \theta \Rightarrow \left(F\left(\frac{(\theta+\varepsilon)n}{n+1}\right)\right)^n = 1^n \xrightarrow[n\to\infty]{} 1$

Если $\frac{(\theta+\varepsilon)n}{n+1} \leq \theta \quad \exists N: \forall n \geq N \hookrightarrow \frac{(\theta+\varepsilon)n}{n+1} > \theta$

$\Rightarrow \left(F\left(\frac{(\theta+\varepsilon)n}{n+1}\right)\right)^n \xrightarrow[n\to\infty]{} 1$

Если $\theta \leq \varepsilon \Rightarrow \frac{(\theta-\varepsilon)n}{n+1} < 0 \Rightarrow \left(F\left(\frac{(\theta-\varepsilon)n}{n+1}\right)\right)^n = 0^n \xrightarrow[n\to\infty]{} 0$

Если $\theta > \varepsilon \quad \left(\frac{(\theta-\varepsilon)n}{(n+1)\theta}\right)^n = \left(\frac{\theta-\varepsilon}{\theta}\right)^n \cdot \left(1+\frac{1}{n}\right)^{-n} \xrightarrow[n\to\infty]{} 0$

Значит $P(|\tilde{\theta}_3' - \theta| \geq \varepsilon) = 1 - \left(F\left(\frac{(\theta+\varepsilon)n}{n+1}\right)\right)^n + \left(F\left(\frac{(\theta-\varepsilon)n}{n+1}\right)\right)^n \to$

$\xrightarrow[n\to\infty]{} 1 - 1 + 0 = 0 \Rightarrow \tilde{\theta}_3'$ является состоятельной (по определению)

$\forall \theta > 0 \ \forall \varepsilon > 0 \quad P(|\tilde{\theta}_3 - \theta| \geq \varepsilon) = P(x_{max} \geq \theta+\varepsilon) +$

$P(x_{max} \leq \theta - \varepsilon) = 1 - \Psi(\theta+\varepsilon) + \Psi(\theta-\varepsilon) = 1 - (F(\theta+\varepsilon))^n +$

$+ (F(\theta-\varepsilon))^n$

Если $\theta - \varepsilon > 0 \Rightarrow (F(\theta-\varepsilon))^n = \left(\frac{\theta-\varepsilon}{\theta}\right)^n \xrightarrow[n\to\infty]{} 0$

Если $\theta - \varepsilon \leq 0 \Rightarrow (F(\theta-\varepsilon))^n = 0^n \xrightarrow[n\to\infty]{} 0$

$\theta + \varepsilon > \theta \Rightarrow (F(\theta+\varepsilon))^n = 1^n \xrightarrow[n\to\infty]{} 1$

Значит $P(|\tilde{\theta}_3 - \theta| \geq \varepsilon) = 1 - (F(\theta+\varepsilon))^n + (F(\theta-\varepsilon))^n \to$

$\xrightarrow[n\to\infty]{} 1 - 1 + 0 = 0 \Rightarrow \tilde{\theta}_3$ является состоятельной (по определению)

④ $\tilde{\theta}_4 = x_1 + \frac{1}{n-1}\sum_{k=2}^{n} x_k$

$\forall \theta > 0 \quad M\tilde{\theta}_4 = Mx_1 + \frac{1}{n-1}\sum_{k=2}^{n} Mx_k = M\xi +$

$$+ \frac{1}{n-1}\sum_{k=2}^{n} M\xi = \frac{\theta}{2} + \frac{1}{n-1}(n-1)\frac{\theta}{2} = \theta$$

$\Rightarrow$ $\tilde{\theta}_4$ является несмещённой

$$\forall\theta>0 \quad D\tilde{\theta}_4 \underset{\text{независ.}}{=} Dx_1 + \frac{1}{(n-1)^2}\sum_{k=2}^{n} Dx_k = \frac{\theta^2}{12} +$$

$$+ \frac{1}{(n-1)^2}(n-1)\frac{\theta^2}{12} = \frac{\theta^2}{12}\left(1 + \frac{1}{n-1}\right) \underset{n\to\infty}{\not\longrightarrow} 0$$ не выпол. дост. ус.

$$x_1 \xrightarrow{P} \xi$$

$$\frac{1}{n-1}\sum_{i=2}^{n} x_i \xrightarrow{P} M\xi = \frac{\theta}{2}$$ (по ЗБЧ Хинчина)

$$\Rightarrow \quad x_1 + \frac{1}{n-1}\sum_{i=2}^{n} x_i \xrightarrow{P} \xi + \frac{\theta}{2}$$

$\Rightarrow$ $\tilde{\theta}_4$ не является состоятельной

b) Какая из исправленных оценок более эффективна?

$\tilde{\theta}_1 = 2\bar{x}$ $\qquad D\tilde{\theta}_1 = \frac{\theta^2}{3n}$

$\tilde{\theta}_3' = \frac{n+1}{n} x_{max}$ $\qquad D\tilde{\theta}_3' = \frac{\theta^2}{n(n+2)}$

Необходимо сравнить $D\tilde{\theta}_1$ и $D\tilde{\theta}_3'$

$$\frac{\theta^2}{3n} > \frac{\theta^2}{n(n+2)} \quad \Big| \cdot \frac{3n(n+2)}{\theta^2}$$

$$n+2 \underset{\text{при } n>1}{>} 3$$

$\Rightarrow$ $\tilde{\theta}_3'$ более эффективна, чем $\tilde{\theta}_1$