

pyvar LLM Knowledge Base

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1 Introduction

This section outlines the mathematical foundations of our risk models. For simplicity, we group them into two categories: univariate and portfolio-level models. We begin with univariate models, which describe the return dynamics of a single asset or, equivalently, a portfolio treated as a single aggregated position. These models typically assume a specific distribution of returns and are used to compute risk measures for individual positions. We then introduce portfolio-level models, which consider the joint distribution of multiple asset returns. These multivariate approaches capture dependencies between assets and allow for a more accurate assessment of portfolio risk. Both analytical and simulation-based methods are discussed in this context. The section concludes with a presentation of the backtesting techniques used to evaluate the performance and reliability of the risk estimates produced by our models.

2 Basic VaR Models

Let $c \in (0, 1)$ denote the confidence level, and define $\alpha = 1 - c$ as the corresponding probability of a large loss. Value-at-Risk (VaR) at level α is defined as the worst loss L over a fixed horizon such that the probability of observing a larger loss is at most α :

$$\Pr(L > \text{VaR}_\alpha) \leq \alpha.$$

Intuitively, this means we are confident that with probability c , losses will not exceed VaR_α over the given horizon.

By convention, VaR is expressed as a positive monetary amount. To simplify the notation in single-asset models we omit the scaling by invested wealth W and use percentage-based notation. To simplify the calculations we assume that W is strictly positive and that returns follow a zero-mean process.

We define the loss variable as $\ell_t = -r_t$. We start with models that assume constant volatility over time. In the non-parametric (historical) approach:

$$\text{VaR}_\alpha^{\text{hist}} = Q_\alpha(\ell_t), \quad \text{VaR}_\alpha^{\text{normal}} = z_\alpha \sigma,$$

where $Q_\alpha(\ell_t)$ is the empirical α -quantile of losses, and $z_\alpha = \Phi^{-1}(1 - \alpha)$ is the positive quantile of the standard normal distribution.

Expected Shortfall (ES), also known as Conditional VaR, is computed as:

$$\text{ES}_{\text{hist}} = \mathbb{E}[\ell_t \mid \ell_t \geq \text{VaR}_\alpha], \quad \text{ES}_{\text{normal}} = \sigma \cdot \frac{\phi(z_\alpha)}{\alpha}, \quad \text{ES}_{t_\nu} = \sigma \cdot \frac{f_{t_\nu}(z_\alpha)}{\alpha} \cdot \frac{\nu + z_\alpha^2}{\nu - 1}$$

where ϕ is the standard normal density and f_{t_ν} is the density of the Student- t distribution with ν degrees of freedom.

Normal and Student- t VaR and ES estimates scale with the holding period as \sqrt{h} . For historical VaR, no scaling is needed—one can simply use weekly or monthly return data to estimate risk over longer horizons.

Despite their simplicity, these models have important drawbacks. The parametric normal VaR often underestimates risk, as daily financial returns typically exhibit fat tails. The Student- t model improves tail fit but cannot capture the asymmetry of return distributions.

Historical methods are the most widely used among basic VaR models due to their simplicity and minimal assumptions. Parametric models usually assume i.i.d. returns, which limits their realism—especially since financial markets exhibit time-varying volatility. Still, parametric normal VaR remains popular thanks to its closed-form ES expression and ease of scaling by time horizon.

3 Extreme Value Theory

Technically still a parametric model, EVT is more sophisticated than the methods discussed above. We implement it with the Peaks-Over-Threshold (POT) approach: once a high threshold u (e.g., the 99th-percentile loss) is chosen, a Generalized Pareto Distribution (GPD) is fitted to the excesses above u .

VaR and ES are then derived analytically from the fitted GPD parameters. This allows for more accurate estimation of rare, extreme losses. The formulas used are:

$$\text{VaR}_\alpha = u + \frac{\beta}{\xi} \left[\left(\frac{n}{n_u} (1 - c) \right)^{-\xi} - 1 \right], \quad \text{ES}_\alpha = \frac{\text{VaR}_\alpha + \beta - \xi u}{1 - \xi},$$

where ξ and β are the estimated shape and scale parameters, n is the sample size, and n_u is the number of exceedances.

Among parametric methods, EVT offers significantly greater precision in the estimation of tail risk. It naturally accommodates very high confidence levels and is based on solid statistical theory, unlike the normal distribution which is unrealistic for financial losses.

However, EVT models are more complex to estimate and rely heavily on sufficient data in the tail. The results are highly sensitive to the chosen threshold u , and poor

threshold selection can compromise accuracy. As such, EVT is most suitable when modeling extreme risks with a sufficiently large dataset.

4 Volatility Models

We have implemented the most popular univariate volatility models.

The GARCH(1,1) model is specified as:

$$\sigma_t^2 = \omega + a r_{t-1}^2 + b \sigma_{t-1}^2, \quad r_t = \sigma_t z_t,$$

where z_t are i.i.d. standardized innovations. VaR is then computed as:

$$\text{VaR}_t = \sigma_t z_\alpha.$$

Here, $z_\alpha = -Q_\alpha(z_t)$ is the positive α -quantile from the empirical distribution of the standardized innovations.

Future volatility can be forecasted analytically using GARCH(1,1).

The τ -step ahead forecast and cumulative T -step forecast are:

$$\begin{aligned} \mathbb{E}[\sigma_{t+\tau}^2] &= \text{VL} + (a + b)^\tau (\sigma_t^2 - \text{VL}), \\ \mathbb{E}[\sigma_{t,T}^2] &= \text{VL} \left(T - 1 - \frac{(a + b)(1 - (a + b)^{T-1})}{1 - (a + b)} \right) + \sigma_t^2 \frac{1 - (a + b)^T}{1 - (a + b)}, \end{aligned}$$

which naturally allow multi-period VaR estimation.

These models—especially GARCH and its extensions—are among the best performing in practice for financial data. They are designed to capture the volatility clustering, leptokurtosis, and persistence seen in asset returns.

We also support:

- EGARCH, which models the log of variance and captures asymmetric responses (leverage effects) of volatility to shocks.
- GJR-GARCH, which adds an indicator for negative returns to allow higher volatility responses to losses than to gains.
- APARCH, which introduces a power term in the volatility equation, allowing flexible tail behavior and better control over asymmetry.

All models are estimated by maximum likelihood using different innovation distributions: Normal, Student- t , Generalized Error Distribution (GED), and Skew- t . The choice of innovation distribution can improve tail risk estimates.

For simpler benchmarks we include ARCH(p) and two rolling estimators, Moving

Average (MA) and Exponentially Weighted Moving Average (EWMA):

$$\sigma_t^{2,\text{ARCH}} = \omega + \sum_{i=1}^p \alpha_i r_{t-i}^2, \quad \sigma_t^{2,\text{MA}} = \frac{1}{n} \sum_{i=1}^n r_{t-i}^2, \quad \sigma_t^{2,\text{EWMA}} = \lambda \sigma_{t-1}^2 + (1 - \lambda) r_{t-1}^2.$$

ES for volatility models becomes:

$$\text{ES}_t = \sigma_t \mathbb{E}[-z_t \mid z_t < -z_\alpha].$$

All our volatility models use a semi-empirical approach for VaR computation: we first estimate the empirical distribution of standardized innovations z_t , then compute the required quantile z_α .

Despite being slightly more complex than basic models, volatility models offer a major improvement by capturing the time-varying nature of financial risk.

5 Time-Varying Correlation Models

We now introduce models designed to capture time-varying correlations at the portfolio level, extending beyond single assets. Specifically, we implement two simple approaches: moving average (MA) and exponentially weighted moving average (EWMA), which mirror their univariate volatility counterparts.

Let $x_t = (x_{1,t}, \dots, x_{N,t})^\top$ denote the vector of monetary positions, and let $w_t = x_t / \sum_{i=1}^N x_{i,t}$ be the corresponding vector of portfolio weights. We allow monetary positions in an asset to be negative, representing short positions. However, for simplicity, we assume that the total portfolio value on any day is strictly positive.

In this setting, we adopt both a parametric approach—assuming normally distributed returns—and a semi-empirical method based on standardized innovations.

VaR is computed as:

$$\text{VaR}_t = z_\alpha \cdot \sqrt{w_t^\top \Sigma_t w_t},$$

where Σ_t is the time-varying covariance matrix estimated by the model.

Expected Shortfall is computed either using the analytical formula under the normal distribution or with the semi-empirical method used for univariate volatility models.

These models are widely used and particularly popular in risk management due to their ability to reflect changes in market structure over time. The semi-empirical variant, using past standardized innovations, is often more accurate than the parametric one, which tends to underestimate risk under the assumption of normally distributed returns.

However, modeling the full variance-covariance (VCV) matrix Σ_t is more delicate than

modeling individual volatilities. The presence of unstable or poorly conditioned weights can significantly distort risk measures, making this approach more sensitive to estimation noise. This adds considerable computational cost—especially for large portfolios.

More advanced multivariate models, such as DCC-GARCH or VEC(1,1), are not included due to limited support in Python and their increased estimation complexity.

6 Factor Models

We implement the Sharpe model and Fama-French three factor models to estimate portfolio VaR and ES. These models use asset betas with respect to systematic risk factors, allowing us to infer portfolio-level risk from exposures to common sources of variation.

For the Sharpe Model, we assume that each asset's return r_i follows the linear factor structure:

$$r_i = \alpha_i + \beta_i r_m + \varepsilon_i,$$

where r_m is the return on the market portfolio, β_i is the asset's sensitivity to the market factor, α_i is the asset's abnormal return, and ε_i is a zero-mean idiosyncratic shock uncorrelated with r_m and other residuals. The market return r_m has variance σ_m^2 , and $\text{Var}(\varepsilon_i) = \sigma_{\varepsilon_i}^2$.

The total portfolio variance is:

$$\sigma_p^2 = \left(\sum_{i=1}^N w_i \beta_i \right)^2 \sigma_m^2 + \sum_{i=1}^N w_i^2 \sigma_{\varepsilon_i}^2.$$

VaR and ES are computed as follows under the normality assumption:

$$\text{VaR}_t = z_\alpha \cdot \sigma_p, \quad \text{ES}_t = \sigma_p \cdot \frac{\phi(z_\alpha)}{1 - \alpha}.$$

We also extend the previous model by considering three risk factors: market excess return (Mkt-RF), size (SMB), and value (HML). Each asset's excess return is modeled as:

$$r_i - r_f = \alpha_i + \beta_{i,1} \cdot \text{Mkt-RF} + \beta_{i,2} \cdot \text{SMB} + \beta_{i,3} \cdot \text{HML} + \varepsilon_i,$$

where r_f is the risk-free rate, and ε_i is the idiosyncratic error. We estimate factor loadings $\beta_{i,k}$ through OLS regression.

Let B be the $N \times 3$ matrix of estimated factor loadings, Σ_f the 3×3 sample covariance matrix of the factors, and Σ_ε the diagonal matrix of residual variances $\sigma_{\varepsilon_i}^2$. The total portfolio variance is:

$$\sigma_p^2 = w^\top (B \Sigma_f B^\top + \Sigma_\varepsilon) w.$$

Once the portfolio variance is known, we compute VaR and ES as we did for the single factor model.

Factor models are particularly powerful for dimensionality reduction in large portfolios, as they allow one to explain risk using a few systematic sources rather than modeling each asset directly. This is intuitive in finance, where most risk arises from exposure to common factors like the overall market or macroeconomic themes.

However, this approach requires additional data—specifically factor returns and proper asset mapping. The interpretation of portfolio risk also shifts: factor VaR can be viewed as the risk equivalent of holding a leveraged position in the market or factors, rather than in the assets themselves. Like other models assuming normality, it tends to underestimate tail risk.

7 Analytic VaR

We compute several Analytic VaR measures using the previously defined monetary position vector $x_t = (x_{1,t}, \dots, x_{N,t})^\top$ and the covariance matrix Σ .

The quantile z_α corresponds to the desired tail probability level and is always derived from the standard normal distribution, as the underlying assumption is that asset returns are normally distributed.

The asset-normal parametric portfolio VaR is defined as:

$$\text{VaR}_t = z_\alpha \cdot \sqrt{x_t^\top \Sigma x_t} \cdot \sqrt{h}.$$

Assuming all asset returns are perfectly correlated ($\rho = 1$) and there are no short positions, the undiversified portfolio VaR is simply the sum of the individual asset VaRs:

$$\text{UVaR}_t = \sum_{i=1}^N \text{VaR}_{i,t}.$$

The marginal VaR, representing the sensitivity of portfolio VaR to a small change in position $x_{i,t}$, is given by:

$$\Delta \text{VaR}_{i,t} = \text{VaR}_t \cdot \frac{(\Sigma x_t)_i}{x_t^\top \Sigma x_t}.$$

It measures the change in total VaR from adding one extra unit of asset i .

The component VaR is:

$$\text{CVaR}_{i,t} = x_{i,t} \cdot \Delta \text{VaR}_{i,t},$$

which captures the absolute contribution of asset i to total portfolio VaR.

The relative component VaR is:

$$\text{RCVaR}_{i,t} = \frac{\text{CVaR}_{i,t}}{\text{VaR}_t},$$

interpreted as the percentage share of total VaR attributable to asset i .

For a vector a representing changes in the portfolio allocation, the incremental VaR is:

$$\text{IVaR}_t = \Delta \text{VaR}_t^\top \cdot a,$$

which estimates the change in total VaR resulting from shifting the portfolio by a .

Each of these VaR metrics can be complemented with a corresponding Expected Shortfall (ES), computed under the normality assumption.

Since most of these portfolio risk metrics lack closed-form expressions under more realistic distributions, we are effectively constrained to the normal assumption—despite its well-known shortcomings in capturing heavy tails and asymmetry in financial returns.

8 Simulation Methods

Simulation methods are powerful tools for computing VaR and ES in portfolios that include complex instruments such as options, which exhibit non-linear payoffs.

We implement a Monte Carlo simulation approach to estimate the one-day ahead distribution of portfolio returns under the assumption of multivariate normality.

Future returns are simulated as:

$$\mu = \mathbb{E}[r_t], \quad \Sigma = \text{Cov}(r_t), \quad L = \text{Cholesky}(\Sigma),$$

$$r^{(i)} = \mu + Lz^{(i)}, \quad z^{(i)} \sim \mathcal{N}(0, I),$$

$$S^{(i)} = S_0 \circ (1 + r^{(i)}),$$

where \circ denotes element-wise multiplication.

The simulated portfolio profit and loss is calculated as:

$$\Delta P^{(i)} = w^\top (S^{(i)} - S_0) + \sum_{j=1}^{N_{\text{opt}}} q_j \left(C(S_j^{(i)}, \tau_j') - C_{j,0} \right),$$

where q_j is the number of option contracts, $C(\cdot)$ is the Black-Scholes option price, and $\tau_j' = \max(T_j - \frac{1}{252}, 0)$ is the adjusted time to maturity.

We define simulated portfolio losses as $\ell^{(i)} = -\Delta P^{(i)}$. From the empirical distribution

$\{\ell^{(i)}\}_{i=1}^N$, we compute:

$$\text{VaR}_\alpha = Q_\alpha(\ell), \quad \text{ES}_\alpha = \mathbb{E}[\ell \mid \ell \geq \text{VaR}_\alpha].$$

We also implement a multi-day version of Monte Carlo, which forecasts VaR and ES over a longer horizon. This version is limited to pure equity portfolios.

As a non-parametric alternative, we implement Historical Simulation and its bootstrap variant (with replacement). Instead of generating returns from a parametric distribution, we simulate alternative return paths by re-applying historical or resampled return vectors to the current portfolio.

Losses $\ell^{(i)}$ are computed as in the Monte Carlo method. VaR and ES are then estimated from their empirical distribution in the same way.

This approach is fully non-parametric and captures fat tails and nonlinear dependence in historical returns. When bootstrap is enabled, repeated resampling improves robustness in small-sample settings.

It is important to note that this is a simplified simulation framework: we do not account for interest rate risk, exchange rate risk, or volatility risk. Additionally, the Black-Scholes-Merton pricing model used for options is itself a stylized and limited approximation, especially under non-normal dynamics. While the normality assumption in Monte Carlo is a known limitation, relaxing it would lead to significantly more complex mathematical and computational challenges. Finally, we do not backtest simulation-based methods, as doing so would require a full rolling re-estimation of simulated distributions at each time step—a process that is extremely computationally intensive, and in many practical contexts, intractable.

9 Backtesting

Backtesting assesses whether the observed losses are consistent with the VaR predictions by checking the frequency and structure of violations (exceptions). The objective is to test whether the model is correctly calibrated.

We implement three standard tests.

The Kupiec test (unconditional coverage) evaluates if the number of observed exceptions N over T days matches the expected number under the model. If p is the failure probability, then under the null N follows a binomial distribution. The likelihood ratio test statistic is:

$$\text{LR}_{\text{uc}} = -2 \left\{ \ln \left[(1-p)^{T-N} p^N \right] - \ln \left[\left(1 - \frac{N}{T} \right)^{T-N} \left(\frac{N}{T} \right)^N \right] \right\}, \quad \text{LR}_{\text{uc}} \sim \chi_1^2.$$

The Christoffersen test (conditional coverage) tests the independence of exceptions by modeling their dynamics as a first-order Markov chain. It compares the likelihoods of transition counts under the null (independence) and alternative. The statistic is:

$$\text{LR}_{\text{c}} = -2(\ln L_0 - \ln L_1), \quad \text{LR}_{\text{c}} \sim \chi_1^2,$$

where L_0 is the likelihood under independence and L_1 under the alternative.

The joint test combines both:

$$\text{LR} = \text{LR}_{\text{uc}} + \text{LR}_{\text{c}} \sim \chi_2^2.$$

Rejecting the null in any of the tests suggests that the VaR model is either miscalibrated (too many or too few violations) or fails to capture time dependence in risk (e.g., volatility clustering).

In practice, the Kupiec test is often passed by well-calibrated models such as the Student- t parametric model, historical simulation, or EVT. However, the Christoffersen and joint tests are frequently failed by models that ignore time-varying volatility—such as the normal parametric model—since they tend to underestimate clustering in violations. Being statistical tests, these results are also subject to randomness: even a basic normal VaR model may occasionally pass all tests over short, quiet periods or for low-volatility assets. We generally recommend using at least one year of data for meaningful inference. Finally, while the joint test aggregates both components, it is important to examine each test individually: passing one and failing the other may still result in a non-rejection overall, but masks which specific assumption (coverage or independence) is violated.

10 Conclusion

In this work, we have introduced and analyzed a range of VaR models, encompassing parametric, non-parametric, and simulation-based approaches. These models can also be classified by their scope—either univariate, focusing on a single asset, or designed for the portfolio level.

Each method provides a different balance between theoretical rigor, computational efficiency, and empirical realism. From a practical standpoint, the choice of an appropriate VaR model depends on the portfolio characteristics and the specific use case. For small

portfolios, simple asset-level or portfolio-level normal VaR models may suffice. When accounting for time-varying volatility and correlations, models incorporating exponentially weighted moving averages (EWMA) or moving-average-based correlations offer a more responsive alternative. For large portfolios of equity positions, single-factor or multi-factor models provide a scalable solution. In portfolios with significant derivative exposure or long-term horizons, simulation-based methods such as Monte Carlo or historical simulation are more appropriate.

When forecasting single positions under changing volatility conditions, GARCH-type models offer a useful framework. For very high confidence level estimates, EVT is generally best. For the most immediate and straightforward estimates, standard parametric or historical VaR methods remain widely used.

Ultimately, the most effective VaR model is often the one that balances theoretical soundness with practical usability. The ideal method is the one that passes a comprehensive set of backtesting procedures and is based on assumptions most aligned with the structure and dynamics of the portfolio under consideration.

However, in many real-world applications, a trade-off is often made—favoring computational simplicity and speed over precision and full realism.