Asset Returns: Stylized Facts and ARCH/GARCH models

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Financial Econometrics course

Asset Returns

- Many asset classes
  - Equities, bonds, commodities, currencies, etc.
  - Significantly different *economic* features

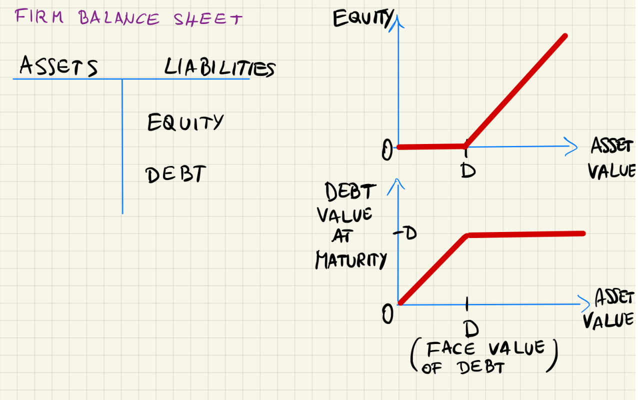


Figure: Merton's view of the firm.

Asset Returns

- Many asset classes
  - Equities, bonds, commodities, currencies, etc.
  - Significantly different *economic* features
- *Empirical* regularities of asset returns or stylized facts
  - Surprisingly similar across all asset classes!
  - Model-free phenomena (essentially)
  - Impose constraints on models of asset returns
  - Relevant for asset and risk management, derivative pricing, etc.

Market Index: S&P500

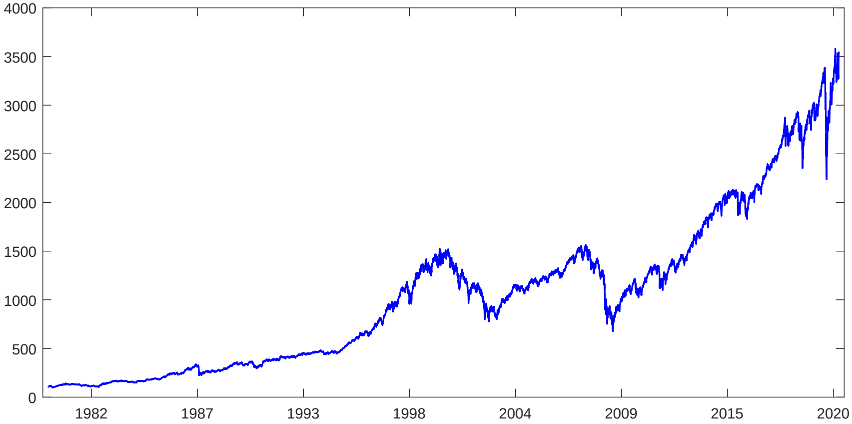


Figure: Daily closing prices of the S&P500 index S_t from 1980 to 2020.

Market Returns: S&P500 Index

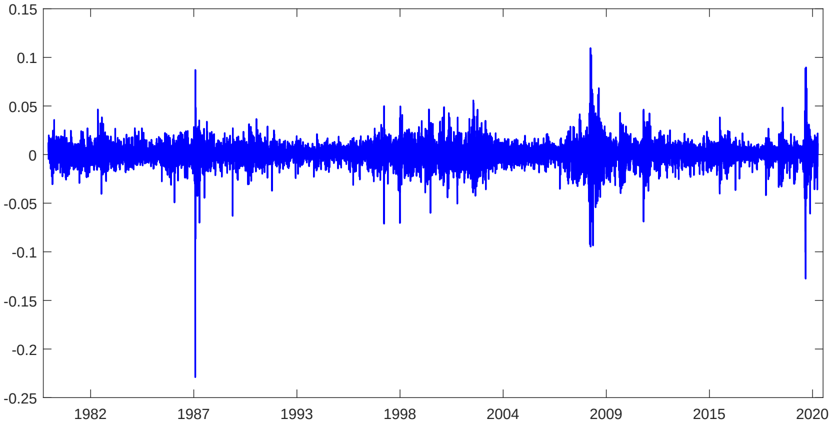


Figure: Daily log-returns, $\log(S_t/S_{t-1})$, of the S&P500 index S_t. Note: Stochastic volatility, volatility clustering and large movements of market returns are evident.

Asset Returns: Stylized facts

- Time variation of asset returns or volatility
  - Stochastic (i.e., changes randomly over time)
  - Persistent (i.e., temporal dependence or volatility clustering)
- Price discontinuities (i.e., jumps)
- Heavy tails of return distributions
- Leverage effect, i.e., $\text{Cov}[\text{asset returns}, \text{volatility changes}] < 0$
- ...

Temporal Dependence of Market Returns

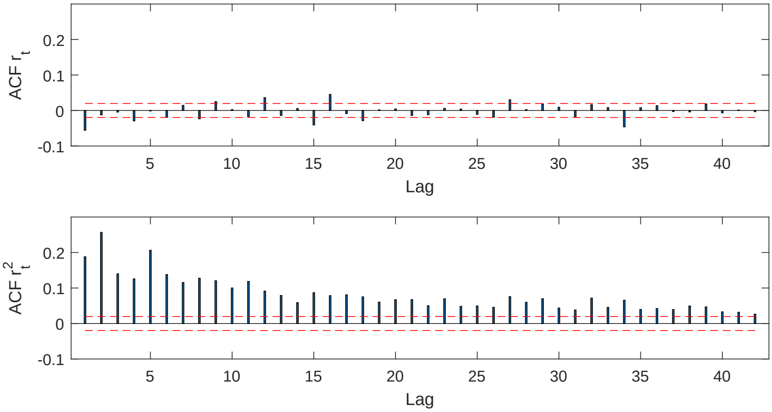


Figure: Upper graph: Sample autocorrelations of daily S&P500 log-returns. Lower graph: Sample autocorrelations of daily squared S&P500 log-returns. Note: Market returns do not exhibit any temporal dependence, squared returns do (volatility clustering).

Sample Autocorrelation

Given T observations of a time series process, $y_t, t = 1, \ldots, T$, suppose, for simplicity $\mathbb{E}[y_t] = 0$ and $\mathbb{V}[y_t] = 1$, the sample autocorrelation function at lag ℓ is

$$ACF(\ell) = \frac{1}{T-\ell} \sum_{t=1}^{T-\ell} y_t \, y_{t+\ell}$$

which is the sample covariance between y_t and $y_{t+\ell}$.

For example, $ACF(0)$ is just the sample variance of y_t, and

$$ACF(1) = \frac{1}{T-1} \sum_{t=1}^{T-1} y_t \, y_{t+1}$$

which is the sample covariance between y_t and y_{t+1}.

Heavy Tails of Market Return Distribution

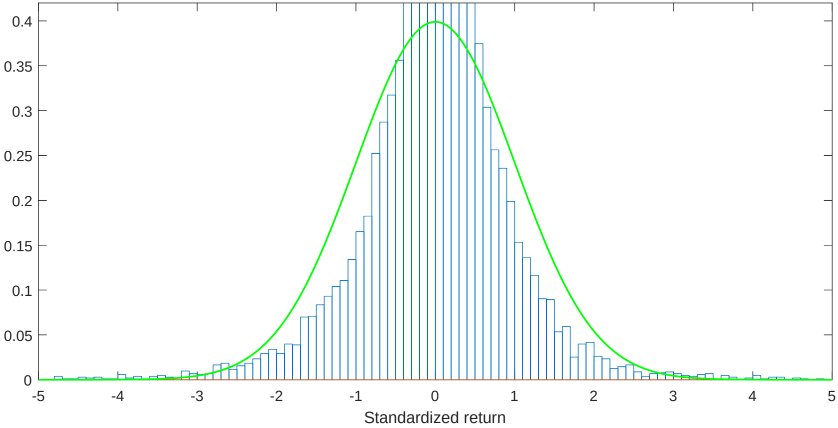


Figure: Histogram of standardized daily S&P500 log-returns (bars). Standard normal pdf (green line). Note: Empirical market return distribution has more probability mass in the tails (and around zero) relative to a normal distribution. 1987 market crash (standardized -20.5) not in the figure.

Leverage Effect: Cov[market returns, volatility changes]<0

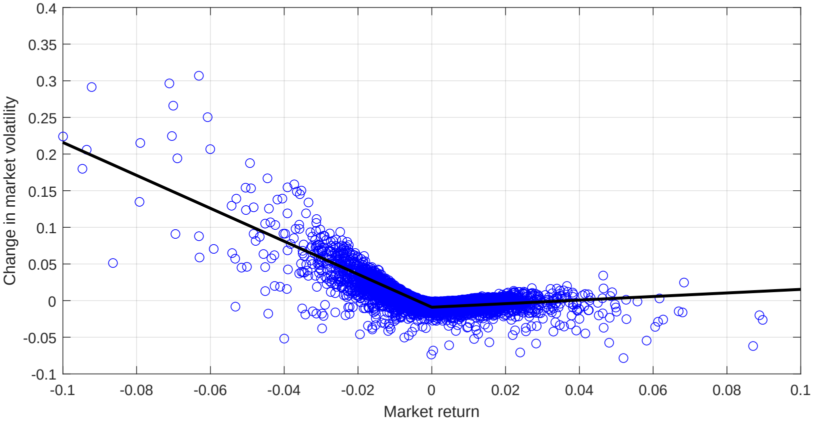


Figure: Scatter plot of changes in annualized daily market volatility (y-axis) versus daily market returns (x-axis). Note: Correlation between market return and volatility change is -0.63. Strong asymmetric impact of market returns on volatility changes: slopes are -2.25 versus 0.24. 1987 market crash (-0.23) not in the figure.

ARCH/GARCH Models for Asset Returns

- Primarily models for (daily) stochastic volatility
  - Engle (1982), Bollerslev (1986)
- Very popular
- Easy to use
- Reproduce (at least qualitatively, many) return stylized facts

ARCH model (for teaching purposes)

- Asset, daily log-return $r_t = \log(S_t/S_{t-1})$

$$r_t = \sigma_t \, z_t$$

where z_t is an i.i.d. shock $\mathcal{D}(0,1)$. Note $\mathbb{E}[r_t] = 0$

- ARCH(1) conditional variance $\sigma_t^2 = \mathbb{V}[r_t|\mathcal{F}_{t-1}]$

$$\sigma_t^2 = \alpha_0 + \alpha_1 r_{t-1}^2$$

where $\mathcal{F}_{t-1}$ is the information set at time $t-1$

- Plus other conditions to ensure $\sigma_t^2 > 0$, stationary, etc.

$\Rightarrow$ σ_t^2 is stochastic, exhibits (some) persistence when $\alpha_1 > 0$, known at time $t-1$

GARCH model (largely used in practice)

- Asset, daily log-return $r_t = \log(S_t/S_{t-1})$

$$r_t = \sigma_t \, z_t$$

  where z_t is an i.i.d. shock $\mathcal{D}(0,1)$. Note $\mathbb{E}[r_t] = 0$

- GARCH(1,1) conditional variance $\sigma_t^2 = \mathbb{V}[r_t|\mathcal{F}_{t-1}]$

$$\sigma_t^2 = \beta_0 + \beta_1 \sigma_{t-1}^2 + \beta_2 r_{t-1}^2$$

  where $\mathcal{F}_{t-1}$ is the information set at time $t-1$

- Plus other conditions to ensure $\sigma_t^2 > 0$, stationary, etc.

$\Rightarrow$ σ_t^2 is stochastic, can be quite persistent when $\beta_1 \approx 0.9$, known at time $t-1$

GARCH model: Maximum Likelihood Estimation

- Joint density of log-returns $r_1, \ldots, r_T$ from p.d.f. $f(r_t|\mathcal{F}_{t-1})$

- Log-likelihood function, assume $r_t|\mathcal{F}_{t-1} \sim \mathcal{N}(0, \sigma_t^2)$

$$
\begin{aligned}
\log f(r_1, \ldots, r_T) &= \sum_{t=1}^{T} \log f(r_t|\mathcal{F}_{t-1}) \\
&= -\frac{1}{2} \sum_{t=1}^{T} \left[\log(2\pi) + \log \sigma_t^2 + r_t^2/\sigma_t^2\right]
\end{aligned}
$$

- Maximize log-likelihood w.r.t. $\beta_0, \beta_1, \beta_2$ that enters σ_t^2, using a numerical search (no closed form solution of $\hat{\beta}_0, \hat{\beta}_1, \hat{\beta}_2$, in general)

- Once $\beta_0, \beta_1, \beta_2$ are estimated, σ_t^2 can be computed recursively (using some starting values of σ_1^2 and r_1^2 such as sample variance of log-returns)

Market Returns and GARCH Volatility

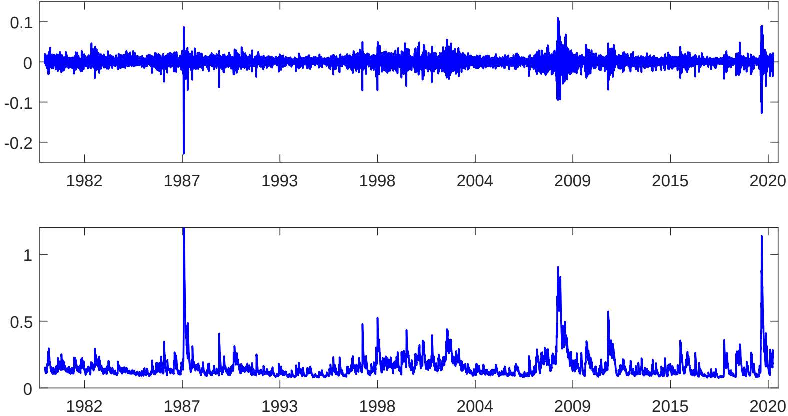


Figure: Upper graph: Daily S&P500 log-returns. Lower graph: Estimated GARCH volatility $\hat{\sigma}_t$ annualized. Note: GARCH volatility tracks market return variability.

Market Returns and GARCH Innovations

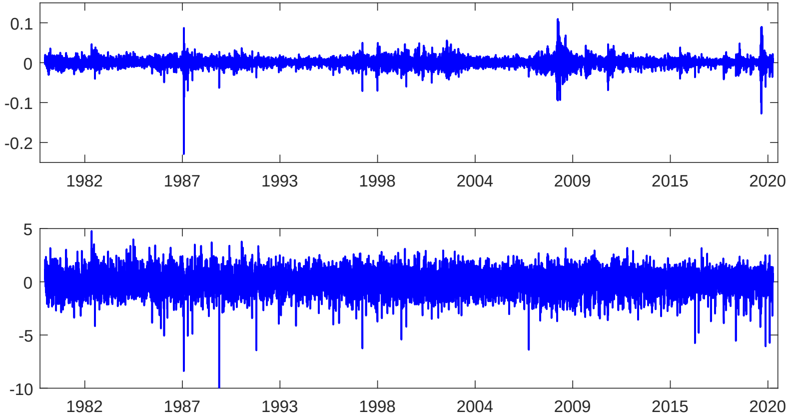


Figure: Upper graph: Daily S&P500 log-returns. Lower graph: Estimated GARCH innovations $\hat{z}_t = r_t/\hat{\sigma}_t$. Note: GARCH innovations exhibit much less volatility clustering than market returns.

Temporal Dependence

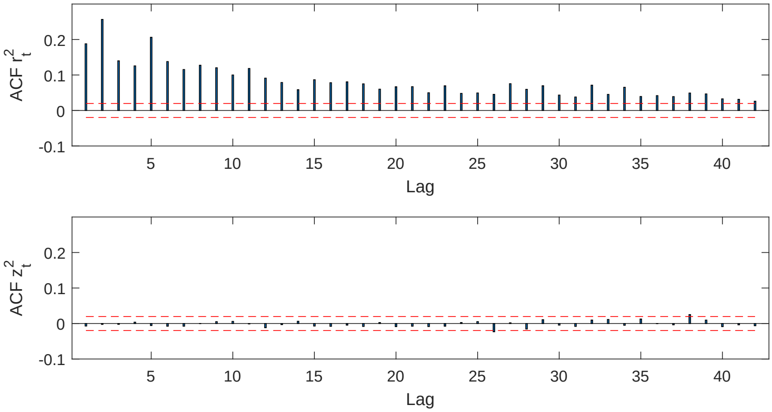


Figure: Upper graph: Sample autocorrelations of squared daily S&P500 log-returns. Lower graph: Sample autocorrelations of squared GARCH innovations, $\hat{z}_t^2$ where $\hat{z}_t = r_t/\hat{\sigma}_t$.
Note: Squared market returns exhibit strong temporal dependence (volatility clustering), squared GARCH innovations do not.

Value-at-Risk: Application of GARCH Models

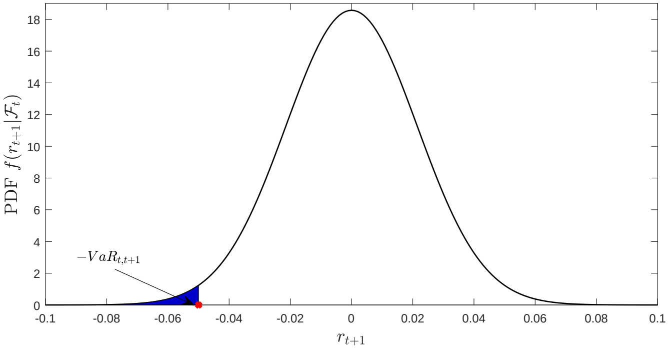
- Trading activity is subject to risk (and other) constraints[1]
- Value-at-Risk (VaR) sets the risk constraint and is imposed by the regulator (Basel Committee, 1996)
- GARCH models are often used to compute VaR in banks, etc.

[1] Such as leverage and liquidity constraints.

Value-at-Risk

- VaR is (essentially) a left-tail quantile at a given level of the profit and loss (P&L) distribution over a given time horizon
- At day t, in the daily P&L distribution below, the 1% level VaR is equal to 0.05, with $r_{t+1} = \log(S_{t+1}/S_t)$

$$0.01 = P(r_{t+1} < -VaR_{t,t+1}|\mathcal{F}_t) = P(r_{t+1} < -0.05|\mathcal{F}_t)$$



Computing Value-at-Risk using GARCH-type models

- In GARCH models, $r_{t+1} = \sigma_{t+1}\, z_{t+1}$, where z is i.i.d. $\mathcal{D}(0,1)$
- At day t, because $\sigma_{t+1} \in \mathcal{F}_t$, the only random variable is z_{t+1}

$$
\begin{aligned}
0.01 &= P(r_{t+1} < -VaR_{t,t+1} | \mathcal{F}_t) \\
&= P(\sigma_{t+1}\, z_{t+1} < -VaR_{t,t+1} | \mathcal{F}_t) \\
&= P(\, z_{t+1} < -VaR_{t,t+1}/\sigma_{t+1} | \mathcal{F}_t)
\end{aligned}
$$

$\Rightarrow -VaR_{t,t+1}/\sigma_{t+1} = z_{0.01}$, with $z_{0.01}$ the 1% quantile of the distribution of z.
For e.g., if $z \sim \mathcal{N}(0,1)$, $z_{0.01} = -2.3$

- At day t, the 1% level VaR over the horizon $[t, t+1]$ is

$$
VaR_{t,t+1} = -\sigma_{t+1}\, z_{0.01}
$$

- These calculations hold for any $\sigma_{t+1} \in \mathcal{F}_t$, not just GARCH models

Backtesting Value-at-Risk

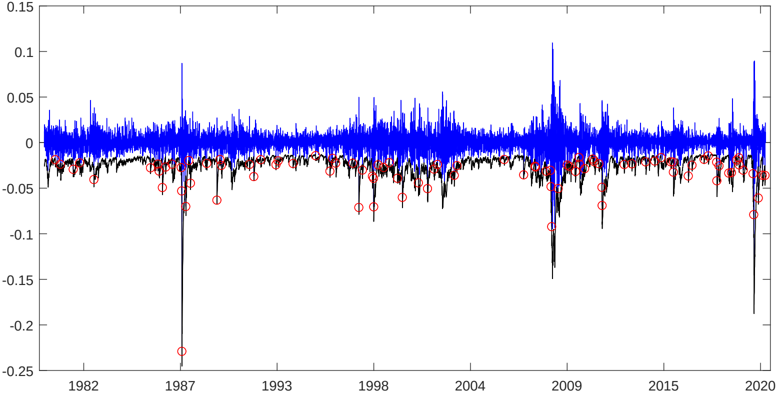


Figure: Daily S&P500 log-returns (blue line), minus VaR at 1% level over one day (black line), $-VaR_{t,t+1} = \sigma_{t+1}\, z_{0.01}$, where σ_{t+1} follows a GARCH model, and $z_{0.01} = -2.6$ from empirical distribution of z. VaR violations, $r_{t+1} < -VaR_{t,t+1}$ (red circle). Note: VaR violations occurred 94 times out of 10,303 daily estimates, i.e., $94/10303 \times 100 = 0.9\%$ of the times, which is "close" to the expected 1% (by definition of quantile).

Backtesting Value-at-Risk: Nov 2019 – Nov 2020

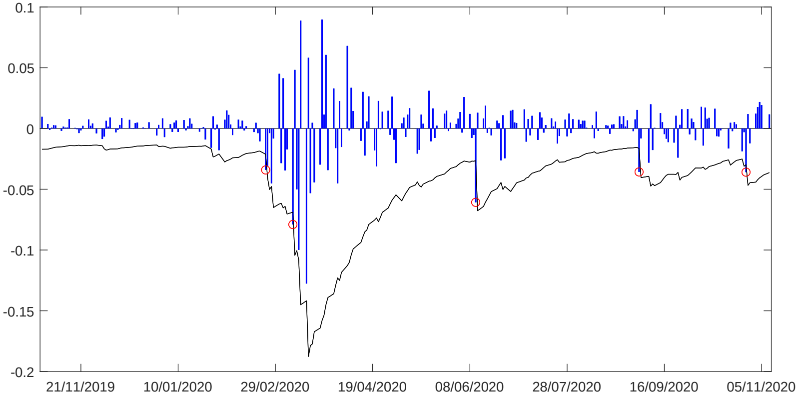


Figure: Daily S&P500 log-returns (blue bars), minus VaR at 1% level over one day (black line). VaR violations, $r_{t+1} < -VaR_{t,t+1}$ (red circle). Note: VaR violations occurred 5 times out of 258 daily estimates from Nov 2019 to Nov 2020, i.e., $5/258 \times 100 = 1.9\%$ of the times, which is "close" to the expected 1% (by definition of quantile).

Expected shortfall (ES) or conditional VaR

- There is one major problem with VaR. What is it?
- Other measure of risk required by regulators (e.g. SST)
- If $r_{t+1} < -VaR_{t,t+1}$, what is the expected loss?
- At day t, given that $VaR_{t,t+1} = -\sigma_{t+1}\, z_{0.01}$

$$\begin{aligned}
-ES_{t,t+1} &= \mathbb{E}[r_{t+1} \,|\, r_{t+1} < -VaR_{t,t+1},\, \mathcal{F}_t] \\
&= \mathbb{E}[\sigma_{t+1}\, z_{t+1} \,|\, \sigma_{t+1}\, z_{t+1} < \sigma_{t+1}\, z_{0.01}] \\
&= \sigma_{t+1}\, \mathbb{E}[z_{t+1} \,|\, z_{t+1} < z_{0.01}]
\end{aligned}$$

  where $\mathbb{E}[z_{t+1} \,|\, z_{t+1} < z_{0.01}]$ is the ES at 1% level of the shocks z (estimated via its empirical counterpart or models)

- Hold for any $\sigma_{t+1} \in \mathcal{F}_t$, not just GARCH models

Expected shortfall: Nov 2019 – Nov 2020

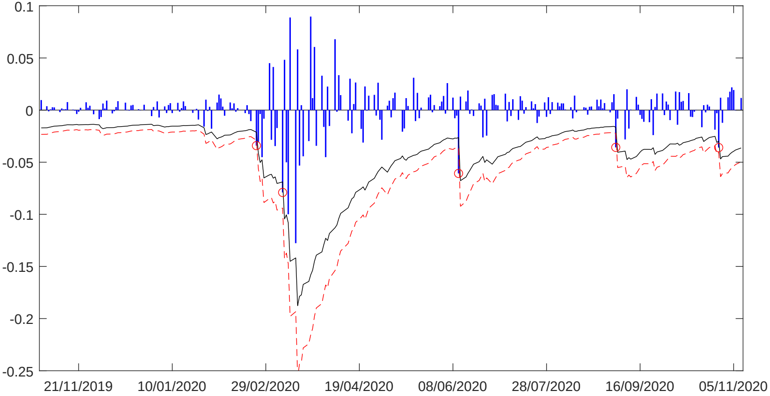


Figure: Daily S&P500 log-returns (blue line); minus VaR (black line) $-VaR_{t,t+1} = \sigma_{t+1}\, z_{0.01}$; minus expected shortfall (dashed red line) $-ES_{t,t+1} = \sigma_{t+1}\, \mathbb{E}[z_{t+1} \,|\, z_{t+1} < z_{0.01}]$ at 1% level over one day based on GARCH model and empirical distribution of z.

GARCH Models: Some Common Specifications

- Return innovation z_t (driving $r_t = \sigma_t \, z_t$) *non-normal*
  - e.g., Bollerslev (1987), Engle and Gonzalez-Rivera (1991)

- Asymmetric GARCH (to capture leverage effects)

$$\sigma_t^2 = \beta_0 + \beta_1 \sigma_{t-1}^2 + \beta_2 r_{t-1}^2 + \beta_3 I_{t-1} r_{t-1}^2$$

  where $I_{t-1} = 1$ when $r_{t-1} < 0$, and $I_{t-1} = 0$ otherwise
  - Glosten, Jagannathan, and Runkle (1993)

- Long-run and short-run component GARCH

$$\begin{aligned} \sigma_t^2 &= q_t + \tilde{\beta}_1 (\sigma_{t-1}^2 - q_{t-1}) + \tilde{\beta}_2 \, \eta_{1,t-1} \\ q_t &= \gamma_0 + \gamma_1 q_{t-1} + \gamma_2 \, \eta_{2,t-1} \end{aligned}$$

  where $\eta_{1,t-1}$ and $\eta_{2,t-1}$ are zero-mean innovations
  - Christoffersen et al. (2008)

GARCH Models Nowadays

- Widely used to estimate and forecast conditional variances
- Typically estimated with non-normal return innovations
- Largely applied for risk management and derivative pricing
  - e.g., Barone-Adesi, Engle, and M. (2008)

References

- Barone-Adesi, Engle, and Mancini (2008), "A GARCH Option Pricing Model with Filtered Historical Simulation," *Review of Financial Studies*, 21, 1223–1258.
- Bollerslev (1986), "Generalized Autoregressive Conditional Heteroskedasticity," *Journal of Econometrics*, 31, 307–327.
- Bollerslev (1987), "A conditionally Heteroskedastic Time Series Model for Speculative Prices and Rates of Return," *Review of Economics and Statistics*, 69, 542–547.
- Christoffersen, Jacobs, Ornthanalai, and Wang (2008), "Option Valuation with Long-Run and Short-Run Volatility Components," *Journal of Financial Economics*, 90, 272–297.
- Engle (1982), "Autoregressive Conditional Heteroscedasticity with Estimates of the Variance of United Kingdom Inflation," *Econometrica*, 50, 987–1007.
- Engle and Gonzalez-Rivera (1991), "Semiparametric ARCH models," *Journal of Business and Economic Statistics*, 9, 345–359.
- Glosten, Jagannathan, and Runkle (1993), "On the Relation Between the Expected Value and the Volatility of the Nominal Excess Returns on Stocks," *Journal of Finance*, 48, 1779–1801.