Basic VaR models

This lecture agenda

- Nonparametric VaR
- Parametric VaR
- Analytic VaR

- Relevant readings: Hull Ch. 12, 13, 14

VaR

- We define VaR as **the worst loss L over a target horizon such that there is a low, prespecified probability p that the actual loss will be larger**. In short,

$$Prob[L > VaR] \leq p$$

where we follow the convention of reporting VaR as a positive (monetary) number.

- "We are confident that with probability $1 - p$ our portfolio won't suffer losses larger than VaR in the next T days".
- p is something like 1% or 5%. Define the confidence level $c = 1 - p$.
- VaR assumes that the position is "frozen" over the horizon

Choice of VaR inputs

- The choice of *horizon* and *confidence level* depends on the use (purpose) of the VaR number.
- If used for internal purposes as a *Benchmark Measure* to compare risks across different investments, many banks use now a 99 percent level and a daily horizon.
- As a *Potential Loss Measure*, the horizon depends on the *liquidation period* (the period needed for the assets in order to be liquidated and serve as a hedge against markets risks) and on the period over which the portfolio remains relatively *constant*, which is a function of the frequency of the trading activity.
- As a measure of *Equity capital*, VaR represents a capital cushion against risks.
  - confidence level should reflect the cost of a loss exceeding the VaR
  - time horizon should reflect the time required to take corrective actions (e.g. raising new capital) as losses materialize.
- VaR only considers what happens AT the horizon only, ignoring intervening losses. In reality, positions are evaluated every day and brokers may require posting additional collateral if the portfolio experiences severe losses => liquidity risk.

VaR as quantile

- Assume that (future) returns to our position have mean μ and quantile R^*.
- A **relative VaR** is calculated as

$$\boxed{VaR(mean) = W(\mu - R^*)}$$

It captures deviations from a target, or budget. If the mean is positive, it leads to a more conservative approach. It reflects the concept of *unexpected* losses.

- An **absolute VaR** is calculated as

$$\boxed{VaR(zero) = -W\,R^*}$$

where a mean of zero is assumed.

- In practice, over small horizon the mean effect is small. It may, however, make a difference when looking at longer (yearly) horizons.

VaR as quantile – cont'd

- Ignoring the mean effect, you can think of (date 0 being today and date T being the VaR horizon):

$$\begin{aligned} VaR_0 &= Quant\left(\underbrace{(P/L)_T}_{\text{CHF}}\right) = Quant\left(\underbrace{W_T - W_0}_{\text{CHF}}\right) = Quant\left(\underbrace{\Delta W}_{\text{CHF}}\right) \\ &= Quant\left(\underbrace{W_0}_{\text{CHF}}\underbrace{\frac{\Delta W}{W_0}}_{\%}\right) = W_0\,Quant\left(\underbrace{\frac{\Delta W}{W_0}}_{\%}\right) = W_0\,Quant\left(\underbrace{R_{1\to T}}_{\%}\right) \end{aligned}$$

where $R_{1\to T}$ is the total (cumulative) return from tomorrow until T

Nonparametric vs Parametric VaR

- Two strategies:
  - estimate $\mathit{Quant}\,(R_{1\to T})$ on historical data from definition (note: must be computed on T-day returns!) $\to$ Nonparametric VaR
  - link quantile $\mathit{Quant}\,(R_{1\to T})$ to volatility $\sigma(R_t)$, $t = \{1, ..., T\}$ by assuming a density plus a model for volatility and dependency $\to$ Parametric VaR

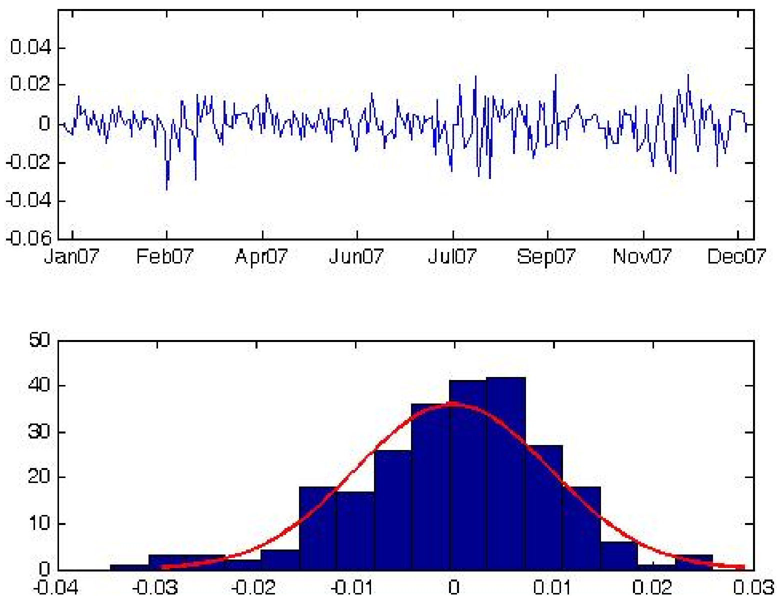
Part I

Nonparametric VaR

Nonparametric VaR - I

- In nonparametric VaR, we make no assumptions about the shape of the return distribution.
- Call R^* the predicted p-th quantile of the return distribution, and let's ignore the expectation. This could be for example the historical quantile over a time window.
- Assume that you hold a position of 100 million CHF invested in an ETF that closely tracks the SMI.
- It's Jan 1,2008 and you want to calculate the 1-day VaR at the 99% level based on returns of the index in the past 250 trading days.
- This is the time series you would face:

SMI - 2007



Nonparametric VaR - Example 1

- You sort this series, and see that the third worst return equals -2.89% while the fourth one is -2.80%. Interpolating, yields -2.85%.
- Thus, you conclude that

$$VaR(99\%;1) = 100mCHF \times 2.85\% = 2.85mCHF$$

- There is a 99% probability that you won't lose more than 2.89 million of CHF over the next day.

Nonparametric VaR - Example 2

- Assume now that it's Jan 1, 2009 and that you hold again a 100 million CHF position in the same ETF.
- Again, you are interested in the 1-day VaR of your position, but the time-series you face is quite different.
- In fact, following the same steps as above you find out that the third and fourth most negative returns are now -6.32% and -5.74%. Interpolating, yields -6.02%.
- The corresponding VaR is now

$$VaR(99\%;1) = 100mCHF \times 6.02\% = 6.02mCHF$$

Part II

Parametric VaR

Parametric VaR

- Suppose we are willing to assume that the density of returns belongs to a parametric family.
- As long as the selected pdf is summarized entirely by (μ, σ), the quantile R^* can be expressed as

$$\boxed{R^* = \mu - \alpha\sigma}$$

  for a suitable α depending on our confidence level.
- The relative T-day VaR if returns are i.i.d. equals

$$VaR(mean) = \alpha\, \sigma \sqrt{T}\, W$$

- Different distributions imply different α for the same confidence level.
- We look at Portfolio-Normal vs Asset-Normal approaches.

Portfolio-Normal VaR

- We consider directly the portfolio Π worth W today, without decomposing the portfolio into its underlying components.
- We assume that daily portfolio returns are independent and normally distributed with volatility σ_Π.
- Under this assumption, VaR can be calculated as

$$VaR(1-p;T) = \alpha\,\sigma_\Pi \sqrt{T}\,W$$

  where α is taken from the quantiles of the standard normal r.v.
- For example, the VaR at 99% c.I. and horizon T equals

$$VaR_\Pi(99\%;T) = 2.33\,\sigma_\Pi \sqrt{T}\,W$$

Example Portfolio-Normal VaR - I

- You hold a portfolio of stocks worth 15M of CHF.
- You estimate the daily volatility of returns to your portfolio to be about 1% (or 16% on annual basis).
- You want to determine the VaR over 10 days at the 99% confidence level: "there exists a 99% probability that in the next 10 days, due to market fluctuations, you portfolio will experience a loss no larger than an amount of VaR CHF".

Example Portfolio-Normal VaR - II

- Daily volatility of your position:

$$1\% \times \text{CHF } 15,000,000 = \text{CHF } 150,000$$

- 10-day volatility of your position (here we assume independence):

$$\text{CHF } 150,000 \times \sqrt{10} = \text{CHF } 474,342$$

- VaR of your position (here we assume normality):

$$VaR_{\Pi} = 2.33 \times \text{CHF } 474,342 = \text{CHF } 1,105,216$$

Asset approach

- Suppose we break down the portfolio Π into N assets
- Call x_i the \$ invested in asset i, so that $\sum_{i=1}^{N} x_i = W$
  - define the $N \times 1$ vector $x = [x_1, x_2, ..., x_N]'$
- Call $w_i = x_i/W$ asset i's relative weight, so that $\sum_{i=1}^{N} w_i = 1$
  - define the $N \times 1$ vector $w = [w_1, w_2, ..., w_N]'$
- The portfolio return R_{Π} is a weighted avg of the assets' returns, $R_{\Pi} = \sum_{i=1}^{N} w_i R_i$

EX You hold a portfolio worth 5M CHF in stock A and 2M CHF in stock B

- $W = 7M\,CHF$, $x_A = 5M\,CHF$, $x_B = 2M\,CHF$
- $w_A = 5/7$, $w_B = 2/7$
- $R_{\Pi} = w_A R_A + w_B R_B$

Asset approach, cont'd

- The variance of returns to Π equals:

$$V(R_\Pi) = \sum_{i=1}^{N} w_i^2\sigma_i^2 + 2\sum_{i=1}^{N}\sum_{j=1,j>i}^{N} w_i w_j \sigma_{ij} = w'\Sigma w = \sigma_\Pi^2$$

- The variance of $\Delta\Pi$ (P/L) equals:

$$V(R_\Pi \times W) = \sigma_\Pi^2 W^2 = x'\Sigma x$$

(!!! for long/short positions, the definition of w is tricky, better use this formula!)

EX You estimated $\sigma_A = 6.4\%$, $\sigma_B = 2.3\%$, and $\rho \equiv \text{corr}(A, B) = 0.40$

$$\begin{aligned} V(R_\Pi) &= \left(\frac{5}{7}\right)^2 0.064^2 + \left(\frac{2}{7}\right)^2 0.023^2 + 2\left(\frac{5}{7}\right)\left(\frac{2}{7}\right)(0.064 \times 0.023 \times 0.40) \\ &= 0.00237 = 0.0488^2 = \sigma_\Pi^2 \end{aligned}$$

$$\begin{aligned} V(R_\Pi \times W) &= \sigma_\Pi^2 W^2 = 0.00237 \times (7M\,CHF)^2 = 0.116M\,CHF^2 \\ &= 0.064^2(5M\,CHF)^2 + 0.023^2(2M\,CHF)^2 \\ &\quad +2(5M\,CHF)(2M\,CHF)(0.064 \times 0.023 \times 0.40) \end{aligned}$$

Asset-Normal VaR

- Suppose **now** that asset returns are normally distributed
- R_Π will also then be normally distributed, as linear combination of normals
- Therefore, the portfolio VaR can be calculated as:

$$VaR(1-p;T) = \alpha\sigma_\Pi W\sqrt{T} = \alpha\sqrt{w'\Sigma w}W\sqrt{T} = \alpha\sqrt{x'\Sigma x}\sqrt{T}$$

- Only in the case of *long-only* positions, we have that:

$$VaR_\Pi = \sqrt{\sum_{i=1}^{N}\sum_{j=1}^{N} VaR_i VaR_j \rho_{ij}}$$

where $\rho_{ii} = 1$

Example Asset-Normal VaR - I

EX The individual 99% level, 1-day VaR to stock A and B are:

$$VaR_A(99\%;1) = 2.33 \times 0.064 \times 5M\,CHF \times \sqrt{1} = 0.746M\,CHF$$
$$VaR_B(99\%;1) = 2.33 \times 0.023 \times 2M\,CHF \times \sqrt{1} = 0.107M\,CHF$$

- The portfolio VaR is:

$$VaR_\Pi(99\%;1) = 2.33 \times \sqrt{0.116M\,CHF^2} \times \sqrt{1} = 0.795M\,CHF$$

- Since the portfolio is long-only, this equals:

$$VaR_\Pi(99\%;1) = \sqrt{(0.746M\,CHF)^2 + (0.107M\,CHF)^2 + 2\,(0.746M\,CHF)\,(0.107M\,CHF)\,0.40}$$

- The sum of the individuals VaRs is above the VaR of our portfolio as the stocks are imperfectly correlated $\rightsquigarrow$ benefits of diversification

Portfolio vs Asset Normal Approach

- Consider a Buy-and -Hold strategy: you form your portfolio today (i.e. buy no. shares) and don't touch it afterwards.
- Question: What does it imply for the weights w_i? do they change over time?
- Yes, they do! The weights of stocks with the highest return increase at the expense of the others
- This implies that the VaR from Portfolio Normal approach should be recomputed with the new weights => computationally intense
- The Portfolio Normal approach is way easier in the case of constantly rebalanced portfolios that leave the w_i unchanged

Statistical Accuracy

- Like any other statistic, VaR is affected by *sampling error* : there is uncertainty surrounding the calculation of as we have a limited set of data from which we calculate its main inputs. The true VaR is unknown.
- The confidence interval from parametric VaR tends to be more narrow than that of non-parametric
- Since σ is an estimate of a moment, it uses information about the whole distribution.
- The empirical quantile is based on the ordering of observations. Higher uncertainty plagues quantiles in the tail as they involve rare events.
- Lesson: sometimes imposing even a wrong distribution may be preferable than assuming none!

Part III

Analytic VaR

Managing Risk with VaR

- The use of parametric VaR allows us to measure directly the benefits of reduction in risk obtained by combining several assets in portfolios obtaining measures of *marginal risk*.
- These measure try to quantify the effect on the overall VaR of changing the current positions by a small or big amount.
- We can then use VaR not only to *measure*, but also to *manage* risk. That is, we can set limit to the positions of traders, decide which assets to add or delete from our portfolio, ... based on VaR-based measures.
- For example, we can identify which position we should close in order to reduce our VaR the most. Given that the answer depends on the overall variance-covariance structure of the assets, it is all but obvious at first.

Undiversified VaR

- Consider a *long-only* portfolio
- If all positions are perfectly correlated with each other, there are no diversification benefits
- If $\rho_{i,j} = 1$, from slide 20 it follows portfolio VaR is the sum of individual VaRs
- **Undiversified VaR**: the VaR when there are no short positions (?) and all correlations are unity

$$\boxed{UVaR_{\Pi} = \sum_{i=1}^{N} VaR_i}$$

- The difference $UVaR_{\Pi} - VaR_{\Pi}$ reflects the benefits from diversification

EX Continuing with the ex above, we have

$$UVaR = 0.746M\,CHF + 0.107M\,CHF = 0.853M\,CHF$$

- Benefits from diversification amount to 0.853M CHF - 0.795M CHF = 0.058M CHF

Marginal VaR

- Individual VaRs measure the riskiness of an asset taken in isolation. However, if we hold a portfolio, what matters is contribution to the overall portfolio risk.
- **Marginal VaR**: how does the portfolio VaR change if I add/subtract a *tiny* amount of \$ to asset i? Concept of **derivative**: we want to know

$$\Delta VaR_i = \frac{\partial VaR}{\partial x_i} = \frac{\partial(\alpha\sigma_\Pi W)}{\partial(w_i W)} = \alpha\frac{\partial\sigma_\Pi}{\partial w_i}$$

- Some steps of math get you:

$$\frac{\partial\sigma_\Pi}{\partial w_i} = \frac{cov(R_i, R_\Pi)}{\sigma_\Pi}$$

- Thus,

$$\boxed{\Delta VaR_i = \alpha\frac{cov(R_i, R_\Pi)}{\sigma_\Pi}}$$

Marginal VaR and Beta

- Recall the definition of the *beta* of a security with respect to a portfolio Π:

$$\beta_i = \frac{cov(R_i, R_\Pi)}{\sigma^2_\Pi}$$

or, in matrix notation, the vector of betas:

$$\beta = \frac{\Sigma w}{w'\Sigma w}$$

- Thus, we have that

$$\boxed{\Delta VaR = \alpha\sigma_\Pi\beta = \frac{VaR_\Pi}{W}\beta}$$

- This tells us that, to change the VaR of our position, we need to act on those assets having the largest ΔVaR_i as they will have the greatest impact on the overall risk of the position.

Incremental VaR

- In calculating ΔVaR, we have taken a derivative. This gives the impact of a "marginal" (tiny) change on VaR_{Π}. What if I change my portfolio by a possibly large amount a \$, from x to $x + a$? What is my new VaR?

- In principle, we should re-evaluate VaR_{Π}, thus obtaining $VaR_{\Pi+a}$:

$$VaR_{\Pi+a} = \alpha\sqrt{(x+a)'\Sigma(x+a)}$$

  ... but this may be computationally long for very large portfolios...

- We can, however, take a first-order approximation of VaR_{p+a} around a:

$$VaR_{\Pi+a} \cong VaR_{\Pi} + (\Delta VaR)' \times a$$

  so that the **Incremental VaR** is:

$$\boxed{IVaR \equiv (\Delta VaR)' \times a}$$

Example

EX Let's assume the trader wants to increase the position in stock B by CHF 100,000. What is the resulting effect on VaR$_{\Pi}$?

- The long way is to modify the vector of positions x to

$$x = \begin{bmatrix} 5M \\ 2.100M \end{bmatrix}$$

  so that, following the same steps as above, we obtain VaR$_{\Pi}$ = 0.797M CHF. The actual increase in VaR is CHF 2'748.

- Alternatively, the marginal VaR equals:

$$\Delta VaR = \begin{bmatrix} 0.14798 \\ 0.02734 \end{bmatrix}$$

  leading to a predicted change of

$$\Delta VaR_{\Pi}' \times [0; CHF\,100,000] = 0.14798 \times 0 + 0.02734 \times 100,000 = CHF\,2'734$$

Component VaR

- If correlations are not unity, the sum of individual VaRs is above portfolio VaR
- **Component VaR**: a measure that adds up to VaR_Π and reflects the contribution of each asset to VaR_Π

$$\boxed{CVaR_i = \Delta VaR_i \cdot x_i}$$

  which tells us (approximately) how much would VaR_Π change if we were to fully delete asset i from Π
- For CVaR it is the case that

$$\sum_{i=1}^{N} CVaR_i = VaR_\Pi$$

- **Relative Component VaR**: the relative (%) contribution of asset i to overall risk

$$\boxed{RCVaR_i = \frac{CVaR_i}{VaR_\Pi}}$$

Example, cont'd

- The CVaRs equal:

$$\begin{aligned} CVaR_A &= 0.14798 \times 5M\,CHF = 0.740M\,CHF \\ CVaR_B &= 0.02734 \times 2M\,CHF = 0.055M\,CHF \end{aligned}$$

  The sum equals the portfolio VaR.
- The RCVAR are:

$$RVaR = CVaR / VaR_\Pi = \begin{bmatrix} 0.93 \\ 0.07 \end{bmatrix}$$

- Thus, 93% of risk is in stock A, and both positions increase risk.

$$
\begin{aligned}
\text{Individual VaR } VaR_i &= \alpha\sigma_i W_i \\
\text{Undiversified VaR } UVaR_\Pi &= \sum_i VaR_i \\
\text{Portfolio VaR } VaR_\Pi &= \alpha\sigma_\Pi W = \alpha\sqrt{x'\Sigma x} \\
\text{Marginal VaR } \Delta VaR &= \alpha\sigma_\Pi \beta = \frac{VaR}{W}\beta \\
\text{Incremental VaR} &= (\Delta VaR)' \times a \\
\text{Component VaR } CVaR_i &= (\Delta VaR)' x_i = VaR_\Pi \beta_i w_i = VaR_i \rho_{i\Pi} \\
\text{Relative Component VaR } RCVaR_i &= \frac{CVaR_i}{VaR_\Pi} = w_i\beta_i
\end{aligned}
$$

Useful results:

$$
\begin{aligned}
\sigma_\Pi^2 &= \sum_i w_i \sigma_{i\Pi} \\
\beta &= W \times \frac{\Sigma x}{x'\Sigma x}
\end{aligned}
$$