Artificial Intelligence

11. Predicate Logic Reasoning, Part I: Basics

Do *You* Think About the World in Terms of "Propositions"?

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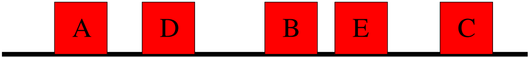
Agenda

Let's Talk About Blocks, Baby . . .

Dear students: What do you see here?



You say: "All blocks are red"; "All blocks are on the table"; "A is a block".

And now: Say it in propositional logic!

→ "isRedA","isRedB", . . . , "onTableA", "onTableB", . . . , "isBlockA", . . .

Wait a sec! Why don't we just say, e.g., "AllBlocksAreRed" and "isBlockA"?

→ Could we conclude that A is red? No. These statements are atomic (just strings); their inner structure ("all blocks", "is a block") is not captured.

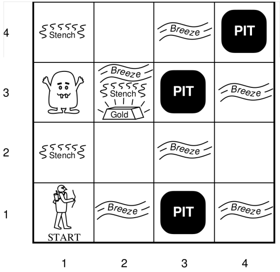
→ Predicate Logic extends propositional logic with the ability to explicitly speak about objects and their properties.

→ Variables ranging over objects, predicates describing object properties, . . .

→ "$\forall x[Block(x) \rightarrow Red(x)]$"; "$Block(A)$"

→ We consider first-order logic, and will abbreviate FOL.

Let's Talk About the Wumpus Instead?



Percepts: $[Stench, Breeze, Glitter, Bump, Scream]$

- Cell adjacent to Wumpus: $Stench$ (else: $None$).
- Cell adjacent to Pit: $Breeze$ (else: $None$).
- Cell that contains gold: $Glitter$ (else: $None$).
- You walk into a wall: $Bump$ (else: $None$).
- Wumpus shot by arrow: $Scream$ (else: $None$).

Say, in propositional logic: "Cell adjacent to Wumpus: $Stench$."

- $W_{1,1} \rightarrow S_{1,2} \wedge S_{2,1}$
- $W_{1,2} \rightarrow S_{2,2} \wedge S_{1,1} \wedge S_{1,3}$
- $W_{1,3} \rightarrow S_{2,3} \wedge S_{1,2} \wedge S_{1,4}$
- ...

$\rightarrow$ Even when we *can* describe the problem suitably, for the desired reasoning, the propositional formulation typically is way too large to write (by hand).

$\rightarrow$ FOL solution: "$\forall x[Wumpus(x) \rightarrow \forall y[Adjacent(x, y) \rightarrow Stench(y)]]$"

Blocks/Wumpus, Who Cares? Let's Talk About Numbers!

→ Even worse!

Example "Integers": (A limited vocabulary to talk about these)

- The objects: $\{1, 2, 3, \dots\}$.
- Predicate 1: "$Even(x)$" should be true iff x is even.
- Predicate 2: "$Equals(x, y)$" should be true iff $x = y$.
- Function: $Succ(x)$ maps x to $x + 1$.

Old problem: Say, in propositional logic, that "$1 + 1 = 2$".

→ Inner structure of vocabulary is ignored (cf. "AllBlocksAreRed").

→ FOL solution: "$Equals(Succ(1), 2)$".

New problem: Say, in propositional logic, "if x is even, so is $x + 2$".

→ It is impossible to speak about infinite sets of objects!

→ FOL solution: "$\forall x[Even(x) \rightarrow Even(Succ(Succ(x)))]$".

Now We're Talking . . .

$$\forall y, x_1, x_2, x_3 \; [Equals(Plus(PowerOf(x_1, y), PowerOf(x_2, y)), PowerOf(x_3, y)) \rightarrow (Equals(y, 1) \vee Equals(y, 2))]$$

Theorem proving in FOL! Arbitrary theorems, in principle.

Fermat's last theorem is of course infeasible, but interesting theorems can and have been proved automatically.
See `http://en.wikipedia.org/wiki/Automated_theorem_proving`.

Note: Need to axiomatize "Plus", "PowerOf", "Equals".
See `http://en.wikipedia.org/wiki/Peano_axioms`

What Are the Practical Relevance/Applications?

. . . **even asking this question is a sacrilege:** (Quotes from Wikipedia)

"In Europe, logic was first developed by Aristotle. Aristotelian logic became widely accepted in science and mathematics."

"The development of logic since Frege, Russell, and Wittgenstein had a profound influence on the practice of philosophy and the perceived nature of philosophical problems, and philosophy of mathematics."

What Are the Practical Relevance/Applications?

You're asking it anyhow?

- Logic programming. Prolog et al.
- Databases. Deductive databases where elements of logic allow to conclude additional facts. Logic is tied deeply with database theory.
- Semantic technology. Mega-trend since > a decade. Use FOL fragments to annotate data sets, facilitating their use and analysis.
  → Prominent FOL fragment: Web Ontology Language OWL.
  → Prominent data set: The WWW. (→ Semantic Web)

Assorted quotes on Semantic Web and OWL:

"The brain of humanity."

"The Semantic Web will never work."

"A TRULY meaningful way of interacting with the Web may finally be here: the Semantic Web. The idea was proposed 10 years ago. A triumvirate of internet heavyweights – Google, Twitter, and Facebook – are making it real."

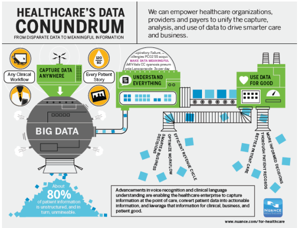
(A Few) Semantic Technology Applications

Web Queries

Jeopardy (IBM Watson)

Context-Aware Apps

Healthcare



Our Agenda for This Topic

→ Our treatment of the topic "Predicate Logic Reasoning" consists of Chapters 11 and 12.

- **This Chapter:** Basic definitions and concepts; normal forms.

  → Sets up the framework and basic operations.

- **Chapter 12:** Compilation to propositional reasoning; unification; lifted resolution.

  → Algorithmic principles for reasoning about predicate logic.

Our Agenda for This Chapter

- **Syntax:** How to write FOL formulas?

  → Obviously required.

- **Semantics:** What is the meaning of FOL formulas?

  → Obviously required.

- **Normal Forms:** What are the basic normal forms, and how to obtain them?

  → Needed for algorithms, which are defined on these normal forms.

The Alphabet of FOL

Common symbols:

- Variables: $x, x_1, x_2, \dots, x', x'', \dots, y, \dots, z, \dots$
- Truth/Falseness: $\top$, $\bot$. (As in propositional logic)
- Operators: $\neg$, $\vee$, $\wedge$, $\rightarrow$, $\leftrightarrow$. (As in propositional logic)
- Quantifiers: $\forall$, $\exists$.
  $\rightarrow$ Precedence: $\neg > \forall, \exists > \dots$ (we'll be using brackets).

Application-specific symbols:

- Constant symbols ("object", e.g., $BlockA$, $BlockB, a, b, c, \dots$)
- Predicate symbols, arity ≥ 1 (e.g., $Block(.)$, $Above(.,.)$)
- Function symbols, arity ≥ 1 (e.g., $WeightOf(.)$, $Succ(.)$)

Definition (Signature). *A signature Σ in predicate logic is a finite set of constant symbols, predicate symbols, and function symbols.*

$\rightarrow$ In mathematics, Σ can be infinite; not considered here.

Our "Silly Running Example": Lassie & Garfield

Constant symbols: $Lassie$, $Garfield$, $Bello$, $Lasagna$, $\dots$

Predicate symbols: $Dog(.)$, $Cat(.)$, $Eats(.,.)$, $Chases(.,.)$, $\dots$

Function symbols: $FoodOf(.)$, $\dots$

Example: $\forall x[Dog(x) \to \exists y\, Chases(x, y)]$, which in words means "Every dog chases something".

[We'll be showing the Lassie & Garfield example in this color and square brackets all over the lecture.]

Syntax of FOL

→ Terms represent objects:

Definition (Term). *Let Σ be a signature. Then:*

1. *Every variable and every constant symbol is a Σ-term.* $[x,\ Garfield]$
2. *If $t_1, t_2, \dots, t_n$ are Σ-terms and $f \in \Sigma$ is an n-ary function symbol, then $f(t_1, t_2, \dots, t_n)$ also is a Σ-term.* $[FoodOf(x)]$

Terms without variables are ground terms. $[FoodOf(Garfield)]$

→ For simplicity, we usually don't write the "Σ-".

→ Atoms represent atomic statements about objects:

Definition (Atom). *Let Σ be a signature. Then:*

1. *$\top$ and $\bot$ are Σ-atoms.*
2. *If $t_1, t_2, \dots, t_n$ are terms and $P \in \Sigma$ is an n-ary predicate symbol, then $P(t_1, t_2, \dots, t_n)$ is a Σ-atom.* $[Chases(Lassie, y)]$

Atoms without variables are ground atoms. $[Chases(Lassie, Garfield)]$

Syntax of FOL, ctd.

→ Formulas represent complex statements about objects:

Definition (Formula). *Let Σ be a signature. Then:*

1. *Each Σ-atom is a Σ-formula.*
2. *If φ is a Σ-formula, then so is $\neg\varphi$.*

The formulas that can be constructed by rules 1. and 2. are literals.

If φ and ψ are Σ-formulas, then so are:

4. $\varphi \wedge \psi$, $\varphi \vee \psi$, $\varphi \rightarrow \psi$, *and* $\varphi \leftrightarrow \psi$.

If φ is a Σ-formula and x is a variable, then

5. $\forall x \varphi$ *is a Σ-formula* ("Universal Quantification").
6. $\exists x \varphi$ *is a Σ-formula* ("Existential Quantification").

→ [E.g., $Cat(Garfield) \vee \neg Cat(Garfield)$; and $\exists x[Eats(Garfield, x)]$]

Alternative Notation

Here	Elsewhere
$\neg\varphi$	$\sim\varphi$ $\quad\overline{\varphi}$
$\varphi \wedge \psi$	$\varphi \& \psi$ $\quad \varphi \bullet \psi$ $\quad \varphi, \psi$
$\varphi \vee \psi$	$\varphi \vert \psi$ $\quad \varphi ; \psi$ $\quad \varphi + \psi$
$\varphi \rightarrow \psi$	$\varphi \Rightarrow \psi$ $\quad \varphi \supset \psi$
$\varphi \leftrightarrow \psi$	$\varphi \Leftrightarrow \psi$ $\quad \varphi \equiv \psi$
$\forall x \varphi$	$(\forall x)\varphi \bigwedge x\varphi$
$\exists x \varphi$	$(\exists x)\varphi \bigvee x\varphi$

Questionnaire

Example "Animals" Σ**:** Constant symbols $\{Lassie, Garfield, Bello, Lasagna\}$; predicate symbols $\{Dog(.), Cat(.), Eats(.,.), Chases(.,.)\}$; funtion symbols $\{FoodOf(.)\}$.

Question!

Which of these are Σ**-formulas?**

(A): $\forall x[Chases(x, Garfield) \rightarrow Chases(Lassie, x)]$

(B): $Eats(Bello, Cat(Garfield))$

(C): $\forall x[(Dog(x) \wedge Eats(x, Lasagna)) \rightarrow \exists y(Cat(y) \wedge Chases(y, x))]$

(D): $\exists x[Dog(x) \wedge Eats(x, Lasagna) \; \forall y(Cat(y) \rightarrow Chases(y, x))]$

$\rightarrow$ (A), (C): Yes.

$\rightarrow$ (B): No, we can't nest predicates.

$\rightarrow$ (D): No, missing a connective between "$Eats(x, Lasagna)$" and "$\forall y(Cat(y) \rightarrow Chases(y, x))$".

Questionnaire, ctd.

Example "Integers" Σ: Constant symbols $\{1, 2, 3, \dots\}$; predicate symbols $\{Even(.), Equals(., .)\}$; funtion symbols $\{Succ(.)\}$.

Question!

Which of these are Σ-formulas?

(A): $\exists x[Even(x) \to Even(Succ(Succ(x)))]$.

(B): $\exists x[Even(x) \to Succ(Even(Succ(x)))]$.

(C): $Even(1) \to \forall x Equals(x, Succ(x))$.

(D): $Even(1) \to \forall 2 Equals(2, 2)$.

$\to$ (A): Yes.

$\to$ (B): No, we can't apply a function to a predicate.

$\to$ (C): Yes.

$\to$ (D): No, we can't quantify over constants.

The Meaning of FOL Formulas

Example: $\forall x[Block(x) \rightarrow Red(x)]$, $Block(A)$

$\rightarrow$ For all objects x, if x is a block, then x is red. A is a block.

More generally: (Intuition)

- Terms represent objects. $[FoodOf(Garfield) = Lasagna]$
- Predicates represent relations on the universe. $[Dog = \{Lassie, Bello\}]$
- Universally-quantified variables: "for all objects in the universe".
- Existentially-quantified variables: "at least one object in the universe".

$\rightarrow$ Similar to propositional logic, we define interpretations, models, satisfiability, validity, . . .

Semantics of FOL: Interpretations

Definition (Interpretation). *Let Σ be a signature. A Σ-interpretation is a pair (U, I) where U, the universe, is an arbitrary non-empty set $[U = \{o_1, o_2, \dots\}]$, and I is a function, notated as superscript, so that*

1. *I maps constant symbols to elements of U: $c^I \in U$* $\quad [Lassie^I = o_1]$
2. *I maps n-ary predicate symbols to n-ary relations over U:*
   $$P^I \subseteq U^n \quad [Dog^I = \{o_1, o_3\}]$$
3. *I maps n-ary function symbols to n-ary functions over U:*
   $$f^I \in [U^n \mapsto U] \quad [FoodOf^I = \{(o_1 \mapsto o_4), (o_2 \mapsto o_5), \dots\}]$$

$\rightarrow$ We will often refer to I as the interpretation, omitting U. Note that U may be infinite.

Definition (Ground Term Interpretation). *The interpretation of a ground term under I is $(f(t_1, \dots, t_n))^I = f^I(t_1^I, \dots, t_n^I)$.* $[(FoodOf(Lassie))^I = FoodOf^I(Lassie^I) = FoodOf^I(o_1) = o_4]$

Definition (Ground Atom Satisfaction). *Let Σ be a signature and I a Σ-interpretation. We say that I satisfies a ground atom $P(t_1, \dots, t_n)$, written $I \models P(t_1, \dots, t_n)$, iff $(t_1^I, \dots, t_n^I) \in P^I$. We also call I a model of $P(t_1, \dots, t_n)$.* $\quad [I \models Dog(Lassie)$ because $Lassie^I = o_1 \in Dog^I]$

Interpretations: Example

Example "Integers": $U = \{1, 2, 3, \dots\}$; $1^I = 1$, $2^I = 2$, $3^I = 3$, $\dots$; $Even^I = \{2, 3, 4, 6, \dots\}$, $Equals^I = \{\langle 1, 1\rangle, \langle 2, 2\rangle, \dots\}$; $Succ^I = \{(1 \mapsto 2), (2 \mapsto 3), \dots\}$.

Question 1: $I \models Even(2)$? Yes.

Question 2: $I \models Even(Succ(2))$? Yes! $Succ(2)^I = 3 \in Even^I$.

Note: Nobody forces us to design I in accordance with the standard meaning of the predicates. Need to axiomatize them. [Remember:

$$\forall y, x_1, x_2, x_3\ [Equals(Plus(PowerOf(x_1, y), PowerOf(x_2, y)), PowerOf(x_3, y)) \rightarrow (Equals(y, 1) \vee Equals(y, 2))]$$

$\rightarrow$ Details: `http://en.wikipedia.org/wiki/Peano_axioms`]

Question 3: $I \models Equals(x, Succ(2))$? Interpretations do not handle variables. We must fix a variable assignment first.

Semantics of FOL: Variable Assignments

Definition (Variable Assignment). *Let Σ be a signature and (U, I) a Σ-interpretation. Let X be the set of all variables. A variable assignment α is a function $\alpha : X \mapsto U$.* $[\alpha(x) = o_1]$

Definition (Term Interpretation). *The interpretation of a term under I and α is:*

1. $x^{I,\alpha} = \alpha(x)$ $[x^{I,\alpha} = o_1]$
2. $c^{I,\alpha} = c^I$ $[Lassie^{I,\alpha} = Lassie^I]$
3. $(f(t_1, \dots, t_n))^{I,\alpha} = f^I(t_1^{I,\alpha}, \dots, t_n^{I,\alpha})$
   $[(FoodOf(x))^{I,\alpha} = FoodOf^I(x^{I,\alpha}) = FoodOf^I(o_1) = o_4]$

Definition (Atom Satisfaction). *Let Σ be a signature, I a Σ-interpretation, and α a variable assignment. We say that I and α satisfy an atom $P(t_1, \dots, t_n)$, written $I, \alpha \models P(t_1, \dots, t_n)$, iff $(t_1^{I,\alpha}, \dots, t_n^{I,\alpha}) \in P^I$. We also call I and α a model of $P(t_1, \dots, t_n)$.*

$[I, \alpha \not\models Dog(FoodOf(x))$: $(FoodOf(x))^{I,\alpha} = o_4 \notin Dog^I]$

Semantics of FOL: Formula Satisfaction

Notation: In $\alpha\frac{x}{o}$ we overwrite x with o in α: for $\alpha = \{(x \mapsto o_1), (y \mapsto o_2), \dots\}$, $\alpha\frac{x}{o} = \{(x \mapsto o), (y \mapsto o_2), \dots\}$.

Definition (Formula Satisfaction). *Let Σ be a signature, I a Σ-interpretation, and α a variable assignment. We set:*

$I, \alpha \models \top$ *and*	$I, \alpha \not\models \bot$	
$I, \alpha \models \neg\varphi$	*iff*	$I, \alpha \not\models \varphi$
$I, \alpha \models \varphi \wedge \psi$	*iff*	$I, \alpha \models \varphi$ *and* $I, \alpha \models \psi$
$I, \alpha \models \varphi \vee \psi$	*iff*	$I, \alpha \models \varphi$ *or* $I, \alpha \models \psi$
$I, \alpha \models \varphi \rightarrow \psi$	*iff*	*if* $I, \alpha \models \varphi$, *then* $I, \alpha \models \psi$
$I, \alpha \models \varphi \leftrightarrow \psi$	*iff*	*if* $I, \alpha \models \varphi$ *if and only if* $I, \alpha \models \psi$
$I, \alpha \models \forall x\varphi$	*iff*	*for all* $o \in U$ *we have* $I, \alpha\frac{x}{o} \models \varphi$
$I, \alpha \models \exists x\varphi$	*iff*	*there exists* $o \in U$ *so that* $I, \alpha\frac{x}{o} \models \varphi$

If $I, \alpha \models \varphi$, we say that I and α satisfy φ (are a model of φ).

[$\varphi = \forall x[Dog(x) \rightarrow \exists y Chases(x, y)]$, $Dog^{I,\alpha} = \{Lassie^{I,\alpha}, Bello^{I,\alpha}\}$, $Chases^{I,\alpha} = \{(Lassie^{I,\alpha}, Garfield^{I,\alpha})\}$. Then $I, \alpha \not\models \varphi$ because Bello does not chase anything.]

FOL Satisfiability etc.

Satisfiability

A FOL formula φ is:

- satisfiable if there exist I, α that satisfy φ.
- unsatisfiable if φ is not satisfiable.
- falsifiable if there exist I, α that do not satisfy φ.
- valid if $I, \alpha \models \varphi$ holds for all I and α. We also call φ a tautology.

Entailment and Equivalence

φ entails ψ, $\varphi \models \psi$, if every model of φ is a model of ψ.
φ and ψ are equivalent, $\varphi \equiv \psi$, if $\varphi \models \psi$ and $\psi \models \varphi$.

Attention: In presence of free variables!

$\rightarrow$ Do we have $Dog(x) \models Dog(y)$? No. Example: $Dog^I = \{o_1\}$, $\alpha = \{(x \mapsto o_1), (y \mapsto o_2)\}$. Then $I, \alpha \models Dog(x)$ but $I, \alpha \not\models Dog(y)$.

Free and Bound Variables

$$\varphi := \forall x[R(\boxed{y}, \boxed{z}) \wedge \exists y(\neg P(y,x) \vee R(y, \boxed{z}))]$$

Definition (Free Variables). *By $vars(e)$, where e is either a term or a formula, we denote the set of variables occuring in e. We set:*

$$\begin{array}{lcl}
free(P(t_1, \dots, t_n)) & := & vars(t_1) \cup \dots \cup vars(t_n) \\
free(\neg\varphi) & := & free(\varphi) \\
free(\varphi * \psi) & := & free(\varphi) \cup free(\psi) \text{ for } * \in \{\wedge, \vee, \rightarrow, \leftrightarrow\} \\
free(\forall x\varphi) & := & free(\varphi) \setminus \{x\} \\
free(\exists x\varphi) & := & free(\varphi) \setminus \{x\}
\end{array}$$

$free(\varphi)$ are the free variables of φ. φ is closed if $free(\varphi) = \emptyset$.

→ In the above φ, which variable appearances are free? The boxed ones.

→ Knowledge Base (aka logical theory) = set of closed formulas. From now on, we asume that φ is closed.

→ We can ignore α, and will write $I \models \varphi$ instead of $I, \alpha \models \varphi$.

Questionnaire

Example "Animals": $U = \{o_1, o_2, o_3, o_4, o_5\}$; $Lassie^I = o_1$, $Garfield^I = o_2$, $Bello^I = o_3$, $Lasagna^I = o_4$, $Chappi^I = o_5$; $Dog^I = \{o_1, o_3\}$, $Cat^I = \{o_2\}$, $Eats^I = \{(o_1, o_4), (o_2, o_4), (o_3, o_5)\}$, $Chases^I = \{(o_1, o_3), (o_3, o_2), (o_2, o_1)\}$.

Question!

For which of these φ do we have $I \models \varphi$?

(A): $\forall x[Chases(x, Garfield) \rightarrow Chases(Lassie, x)]$

(B): $Eats(Bello, Cat(Garfield))$

(C): $\forall x[(Dog(x) \wedge Eats(x, Lasagna)) \rightarrow \exists y(Cat(y) \wedge Chases(y, x))]$

(D): $\exists x[Dog(x) \wedge Eats(x, Lasagna) \wedge \forall y(Cat(y) \rightarrow Chases(y, x))]$

→ (A): Yes. (Only Bello chases Garfield; Lassie chases Bello.)

→ (B): Not a formula (cf. slide 17).

→ (C): Yes. (The only dog eating Lasagna is Lassie; Garfield chases Lassie.)

→ (D): Yes. (Lassie is a dog eating Lasagna; it is chased by the only cat, Garfield.)

Questionnaire, ctd.

Example "Integers": $U = \{1, 2, 3, \ldots\}$; $1^I = 1$, $2^I = 2$, $\ldots$;
$Even^I = \{2, 4, 6, \ldots\}$, $Equals^I = \{\langle 1, 1\rangle, \langle 2, 2\rangle, \ldots\}$;
$Succ^I = \{(1 \mapsto 2), (2 \mapsto 3), \ldots\}$.

Question!

For which of these φ do we have $I \models \varphi$?

(A): $\exists x[Even(x) \rightarrow Even(Succ(Succ(x)))]$.

(B): $\exists x[Even(x) \rightarrow Succ(Even(Succ(x)))]$.

(C): $Even(1) \rightarrow \forall x Equals(x, Succ(x))$.

(D): $Even(1) \rightarrow \forall 2 Equals(2, Succ(2))$.

$\rightarrow$ (A): Yes: $x = 2$ does the job. Actually we can strengthen the formula to $\forall x[Even(x) \rightarrow Even(Succ(Succ(x)))]$.

$\rightarrow$ (B): Not a formula (cf. slide 18).

$\rightarrow$ (C): Yes: While $\forall x Equals(x, Succ(x))$ is false, $Even(1)$ is false as well and thus the overall implication is true.

$\rightarrow$ (D): Not a formula (cf. slide 18).

Before We Begin

Why normal forms?

- Convenient: full syntax when *describing the problem* at hand.
- Not convenient: full syntax when *solving the problem*.

"Solving the problem"? Decide satisfiability!

→ Uniform decision problem to tackle deduction as well as other applications. (Same as in propositional logic, cf. **Chapters 09 and 10**.)

Which normal forms?

- Prenex normal form: Move all quantifiers up front.
- Skolem normal form: Prenex, + remove all existential quantifiers while preserving satisfiability.
- Clausal normal form: Skolem, + CNF transformation while preserving satisfiability.

Prenex Normal Form: Step 1

quantifier prefix + (quantifier-free) matrix

$Qx_1Qx_2Qx_3 \dots Qx_n \; \varphi$

Step 1: Eliminate $\rightarrow$ and $\leftrightarrow$, move $\neg$ inwards

1. $(\varphi \leftrightarrow \psi) \equiv [(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)]$ (Eliminate "$\leftrightarrow$")
2. $(\varphi \rightarrow \psi) \equiv (\neg\varphi \vee \psi)$ (Eliminate "$\rightarrow$")
3. $\neg(\varphi \wedge \psi) \equiv (\neg\varphi \vee \neg\psi)$ and $\neg(\varphi \vee \psi) \equiv (\neg\varphi \wedge \neg\psi)$
   $\neg\forall x\varphi \equiv \exists x\neg\varphi$ and $\neg\exists x\varphi \equiv \forall x\neg\varphi$ (Move "$\neg$" inwards)

Example: $\neg\forall x[(\forall x P(x)) \rightarrow R(x)]$

$\rightarrow$ Eliminate $\rightarrow$ and $\leftrightarrow$: $\neg\forall x[\neg(\forall x P(x)) \vee R(x)]$.

$\rightarrow$ Move $\neg$ across first quantifier: $\exists x\neg[\neg(\forall x P(x)) \vee R(x)]$.

$\rightarrow$ Move $\neg$ inwards: $\exists x[(\forall x P(x)) \wedge \neg R(x)]$.

Prenex Normal Form: Step 2

quantifier prefix + (quantifier-free) matrix

$$\underbrace{Qx_1Qx_2Qx_3\dots Qx_n}_{\text{quantifier prefix}}\ \underbrace{\varphi}_{\text{matrix}}$$

Step 2: Move quantifiers outwards

- $(\forall x\varphi) \wedge \psi \equiv \forall x(\varphi \wedge \psi)$, if x not free in ψ.
- $(\forall x\varphi) \vee \psi \equiv \forall x(\varphi \vee \psi)$, if x not free in ψ.
- $(\exists x\varphi) \wedge \psi \equiv \exists x(\varphi \wedge \psi)$, if x not free in ψ.
- $(\exists x\varphi) \vee \psi \equiv \exists x(\varphi \vee \psi)$, if x not free in ψ.

Example "Animals": $\forall x[\neg Dog(x) \vee \exists y\, Chases(x, y)]$
$\rightarrow$ Move "$\exists y$" outwards: $\forall x\exists y[\neg Dog(x) \vee Chases(x, y)]$.

Example: $\exists x[(\forall x P(x)) \wedge \neg R(x)]$
$\rightarrow$ We can't move "$\forall x$" outwards because x is free in "$\neg R(x)$"!

Prenex Normal Form: Variable Renaming

Notation: If x is a variable, t a term, and φ a formula, then the instantiation of x with t in φ, written $\varphi\frac{x}{t}$, replaces all free appearances of x in φ by t. If $t = y$ is a variable, then $\varphi\frac{x}{y}$ renames x to y in φ.

Lemma. *If $y \notin vars(\varphi)$, then $\forall x\varphi \equiv \forall y\varphi\frac{x}{y}$ and $\exists x\varphi \equiv \exists y\varphi\frac{x}{y}$.*

Step 2 Addition: Rename variables if needed

For each Step 2 rule: If x is free in ψ, then rename x in $(\forall x\varphi)$ respectively $(\exists x\varphi)$ to some new variable y. Then, the rule can be applied.

Example: $\exists x[(\forall xP(x)) \wedge \neg R(x)]$

$\rightarrow$ Rename $\frac{x}{y}$ in $(\forall xP(x))$: $\exists x[(\forall yP(y)) \wedge \neg R(x)]$.

$\rightarrow$ Move $\forall y$ outwards: $\exists x\forall y[P(y) \wedge \neg R(x)]$.

Theorem. *There exists an algorithm that, for any FOL formula φ, efficiently (i.e., in polynomial time) calculates an equivalent formula in prenex normal form.* (Proof: We just outlined that algorithm.)

Skolem Normal Form

$$\underbrace{\forall x_1 \forall x_2 \forall x_3 \dots \forall x_n}_{\text{universal prefix}} \; + \; \underbrace{\varphi}_{\text{(quantifier-free) matrix}}$$

Theorem (Skolem). *Let* $\varphi = \forall x_1 \dots \forall x_k \exists y \psi$ *be a closed FOL formula in prenex normal form, such that all quantified variables are pairwise distinct, and the* k*-ary function symbol* f *does not appear in* φ*. Then* φ *is satisfiable if and only if* $\forall x_1 \cdots \forall x_k \psi \frac{y}{f(x_1,\dots,x_k)}$ *is satisfiable.* (Proof omitted.)

Note: Here, "0-ary function symbol" = constant symbol.

Transformation to Skolem normal form

Rename quantified variables until distinct. Then iteratively remove the outmost existential quantifier, using Skolem's theorem.

Example. $\exists x \forall y \exists z R(x, y, z)$ is transformed to:

→ Remove "$\exists x$": $\forall y \exists z R(f, y, z)$. Remove "$\exists z$": $\forall y R(f, y, g(y))$.

→ Note the arity/arguments of f vs. g: "$x_1 \dots x_k$" in the above!

Skolem Normal Form, ctd.

Notation: A formula is in Skolem normal form (SNF) if it is in prenex normal form and has no existential quantifiers.

Theorem. *There exists an algorithm that, for any closed FOL formula φ, efficiently calculates an SNF formula that is satisfiable iff φ is. We denote that formula φ^*.* (Proof: We just outlined that algorithm.)

Example 1: (a) $\varphi_1 = \exists y \forall x[\neg Dog(x) \vee Chases(x, y)]$: "There exists a y chased by every dog x". (b) $\varphi_1^* = \forall x[\neg Dog(x) \vee Chases(x, f)]$: "The object named f is chased by every dog x".

Example 2: (a) $\varphi_2 = \forall x \exists y[\neg Dog(x) \vee Chases(x, y)]$: "For every dog x, there exists y chased by x". (b) $\varphi_2^* = \forall x[\neg Dog(x) \vee Chases(x, f(x))]$: "For every dog x, we can interprete $f(x)$ with a y chased by x".

$\rightarrow$ Satisfying existential quantifier for universally quantified variables $x_1, \dots, x_k$ $=$ choosing a value for a function of $x_1, \dots, x_k$.

Note: φ^* is *not* equivalent to φ. φ^* implies φ, but not vice versa. Example for $I \models \varphi$ but $I \not\models \varphi^*$: $\varphi = \exists x\ Dog(x)$, $\varphi^* = Dog(f)$, $Dog^I = \{Lassie\}$, $f^I = \{Garfield\}$.

Questionnaire

Question!

Which are Skolem normal forms of
$\forall x \exists y[\neg Dog(x) \vee \neg Eats(x, Lasagna) \vee (Cat(y) \wedge Chases(y, x))]$**?**

(A): $\forall x \exists y[\neg Dog(x) \vee \neg Eats(x, Lasagna) \vee (Cat(f(x)) \wedge Chases(f(x), x))]$

(B): $\forall x[\neg Dog(x) \vee \neg Eats(x, Lasagna) \vee (Cat(f) \wedge Chases(f, x))]$

(C): $\forall x[\neg Dog(x) \vee \neg Eats(x, Lasagna) \vee (Cat(f(x)) \wedge Chases(f(x), x))]$

(D): $\forall x[\neg Dog(x) \vee \neg Eats(x, Lasagna) \vee (Cat(g(x)) \wedge Chases(g(x), x))]$

$\rightarrow$ (A): No, we need to remove the existential quantifier over y. (B): No, f needs x as an argument. (C): Yes: "$\exists y$" is removed, and "y" is replaced by a new function symbol with argument x. (D): Same as (C).

$\rightarrow$ Note the different function symbols in (C) and (D): The Skolem normal form is unique up to renaming of function symbols.

Clausal Normal Form

universal prefix + disjunction of literals

$$\forall x_1 \forall x_2 \forall x_3 \dots \forall x_n (l_1 \vee \dots \vee l_n)$$

$\rightarrow$ Written $\{l_1, \dots, l_n\}$.

Transformation to clausal normal form

1. Transform to SNF: $\forall x_1 \forall x_2 \forall x_3 \cdots \forall x_n \; \varphi$.
2. Transform φ to satisfiability-equivalent CNF ψ. (Same as in propositional logic.)
3. Write as set of clauses: One for each disjunction in ψ.
4. Standardize variables apart: Rename variables so that each occurs in at most one clause. (Needed for FOL resolution, **Chapter 12**.)

Theorem. *There exists an algorithm that, for any closed FOL formula φ, efficiently calculates a formula in clausal normal form that is satisfiable iff φ is satisfiable.* (Proof: We just outlined that algorithm.)

All 3 Transformations: Example

$$\forall x[\forall y(Animal(y) \rightarrow Loves(x,y)) \rightarrow \exists y Loves(y,x)]$$

$\rightarrow$ Means what? "Everyone who loves all animals is loved by someone."

1. **Eliminate equivalences and implications:**
   $\forall x[\neg\forall y(\neg Animal(y) \vee Loves(x,y)) \vee \exists y Loves(y,x)]$
2. **Move negation inwards:**
   $\forall x[\exists y\neg(\neg Animal(y) \vee Loves(x,y)) \vee \exists y Loves(y,x)]$
   $\forall x[\exists y(\neg\neg Animal(y) \wedge \neg Loves(x,y)) \vee \exists y Loves(y,x)]$
   $\forall x[\exists y(Animal(y) \wedge \neg Loves(x,y)) \vee \exists y Loves(y,x)]$
3. **Move quantifiers outwards:** $\rightarrow$ Prenex normal form.
   $\rightarrow$ Note: y is **not** free in "$\exists y Loves(y,x)$": (Make variables distinct).
   $\forall x\exists y[(Animal(y) \wedge \neg Loves(x,y)) \vee \exists y Loves(y,x)]$
   $\rightarrow$ Note: y **is** free in "$(Animal(y) \wedge \neg Loves(x,y))$".
   $\forall x\exists y\exists z[(Animal(y) \wedge \neg Loves(x,y)) \vee Loves(z,x)]$

All 3 Transformations: Example, ctd.

$$\forall x \exists y \exists z [(Animal(y) \wedge \neg Loves(x, y)) \vee Loves(z, x)]$$

4. **Remove existential quantifiers:** $\rightarrow$ Skolem normal form.

   $\forall x \exists z [(Animal(f(x)) \wedge \neg Loves(x, f(x))) \vee Loves(z, x)]$

   $\forall x [(Animal(f(x)) \wedge \neg Loves(x, f(x))) \vee Loves(g(x), x)]$

5. **Transform to CNF:**

   $\forall x [(Animal(f(x)) \vee Loves(g(x), x)) \wedge (\neg Loves(x, f(x)) \vee Loves(g(x), x))]$

6. **Write as set of clauses:**

   $\{\{Animal(f(x)), Loves(g(x), x)\}, \{\neg Loves(x, f(x)), Loves(g(x), x)\}\}$

7. **Standardize variables apart:** $\rightarrow$ Clausal normal form.

   $\{\{Animal(f(x)), Loves(g(x), x)\}, \{\neg Loves(y, f(y)), Loves(g(y), y)\}\}$

Questionnaire

Example "Animals" (simplified): $U = \{o_1, o_2, o_3\}$; $Lassie^I = o_1$, $Garfield^I = o_2$, $Bello^I = o_3$; $Dog^I = \{o_1, o_3\}$, $Chases^I = \{(o_3, o_2), (o_1, o_3)\}$.

Question!

Which of these φ (1) have $I \models \varphi$, or (2) are satisfiable by extending I with an interpretation of f?

(A): $\forall x \exists y [Dog(x) \rightarrow Chases(x, y)]$

(B): $\exists y \forall x [Dog(x) \rightarrow Chases(x, y)]$

(C): $\forall x [Dog(x) \rightarrow Chases(x, f(x))]$

(D): $\forall x [Dog(x) \rightarrow Chases(x, f)]$

$\rightarrow$ (A): Yes, (1) because Bello chases Garfield and Lassie chases Bello. (B): No, because Bello respectively Lassie chase *different* y. (C): Yes, (2) by choosing $f(o_3) := o_2$ and $f(o_1) := o_3$ (cf. (A)). (D): No, because f has no argument (cf. (B)).

$\rightarrow$ Note that (C) is a SNF for (A), and (D) is a SNF for (B). Note also that (D) is a "flawed SNF" for (A) where we forgot to give f the argument x. (Compare slide 35)

Summary

- Predicate logic allows to explicitly speak about objects and their properties. It is thus a more natural and compact representation language than propositional logic; it also enables us to speak about infinite sets of objects.
- Logic has thousands of years of history. A major current application in AI is Semantic Technology.
- First-order predicate logic (FOL) allows universal and existential quantification over objects.
- A FOL interpretation consists of a universe U and a function I mapping constant symbols/predicate symbols/function symbols to elements/relations/functions on U.
- In prenex normal form, all quantifiers are up front. In Skolem normal form, additionally there are no existential quantifiers. In clausal normal form, additionally the formula is in CNF.
- Any FOL formula can efficiently be brought into a satisfiability-equivalent clausal normal form.

Reading

- *Chapter 8: First-Order Logic*, Sections 8.1 and 8.2 [Russell and Norvig (2010)].

  Content: A less formal account of what I cover in "Syntax" and "Semantics". Contains different examples, and complementary explanations. Nice as additional background reading.

  Sections 8.3 and 8.4 provide additional material on using FOL, and on modeling in FOL, that I don't cover in this lecture. Nice reading, not required for exam.

- *Chapter 9: Inference in First-Order Logic*, Section 9.5.1 [Russell and Norvig (2010)].

  Content: A very brief (2 pages) description of what I cover in "Normal Forms". Much less formal; I couldn't find where (if at all) RN cover transformation into prenex normal form. Can serve as additional reading, can't replace the lecture.

References I

Stuart Russell and Peter Norvig. *Artificial Intelligence: A Modern Approach (Third Edition)*. Prentice-Hall, Englewood Cliffs, NJ, 2010.