Session 1

We begin our journey with the usual shyness that comes when "mathematicians" meet each other, a little awkwardness, and complex numbers. After a quick re-polishing of complex arithmetic, we end our review with the often-necessary mention of Euler's formula and Taylor series justification.

$$\begin{aligned}
e^{ix} &= 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} \dots \\
&= 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} \dots \\
&= 1 - \frac{x^2}{2!} + \dots i \left(x - \frac{x^3}{3!} + \dots \right) \\
&= \cos(x) + i\sin(x)
\end{aligned}$$

After, we swiftly move to the path and contour integration example, which will be handy in our future computations. We start with not super-involved, let $f(z) = z$, $\delta : [0,1] \to \mathbb{C}, \delta(t) = t$. Then we see that

$$\begin{aligned}
\int_\delta f(z)dz &= \int_0^1 f(\delta(t))\delta'(t)dt \\
&= \int_0^1 f(t)dt = \int_0^1 tdt = \frac{t^2}{2}\Big|_0^1 = \frac{1}{2}
\end{aligned}$$

Ok, that wasn't so bad! One more example and then we move to the more exciting stuff. Let $f(z) = z, \delta : [0, 2\pi] \to \mathbb{C}, \delta(t) = e^{it}$. (Note: δ traces the unit circle counter-clockwise in the complex plain!)

$$\begin{aligned}
\int_\delta f(z)dz &= \int_0^{2\pi} f(\delta(t))\delta'(t)dt \\
&= \int_0^{2\pi} f(e^{it})ie^{it}dt \\
&= i\int_0^{2\pi} e^{2it}dt \\
&= i\int_0^{2\pi} \cos(2t) + i\sin(2t)dt \\
&= i\left(\frac{\sin(2t)}{2} - i\frac{\cos(2t)}{2}\right)\Big|_0^{2\pi} = \frac{\cos(2t)}{2}\Big|_0^{2\pi} = 0
\end{aligned}$$

Now the fun begins! First, let's establish the outline of our goal. Taking a disk in the real plane, we consider the points with integer coordinates contained in such disk, and we attempt to find an approximation of the number of those points. We need some definitions. Let tD be the disk of radius t in the $\mathbb{R}^2$ plane (i.e. D would be the unit disk, etc.). Let

$$N(t) := \#\{tD \cap \mathbb{Z}^2\}$$

So $N(t)$ is the number of lattice points contained in the disk of radius t. Intuitively, $N(t) \approx \pi t^2$ or $N(t) = \pi t^2 + E(t)$, where $E(t)$ is the error we would like to bound. For simplicity, we call $P = \{tD \cap \mathbb{Z}^2\}$ the set of all lattice points in the disk.

As a first, we will attempt to justify the bound $|E(t)| \leq Ct$. It is convenient to consider the squares of unit length centred around each point, so let's do just that. For $p \in P$, let S_p be the square of side length one centred at p. Let

$$S = \bigcup_{p \in P} S_p$$

(i.e. S is the union of all points in $\mathbb{R}^2$ in all squares). A simple and convenient super-set of tD is $(t+1)D$, or the disk of radius $t+1$. Now,

$$|S| = {}^{[1]}|P| \cdot 1 = N(t)$$

We are then looking to prove $S \subseteq (t+1)D$

Proof. Let $s \in S$. Then there is a $p \in P$ s.t. $s \in S_p$. In other words, s is in the square of side length 1 centred at p, therefore $|s - p| \leq 1$.

Also, $|p| \leq t$, thus $|s - p + p| \leq {}^{[2]}|s - p| + p \leq t + 1$. But then, the distance from s to the origin is less than $t + 1$, so $s \in (t+1)D$. □

Therefore, $N(t) \leq \pi(t+1)^2$. And $E(t) \leq \pi(2t+1) \leq 7t$ for $t \geq 5$. We have an upper bound for $E(t)$! Even though a linear bound might not be that impressive, it is our first bound and is cause for celebration nevertheless.

Instead of stopping here, we push forward with newfound confidence and seek a lower bound. Thus start our search once again, but with the little expertise we have found, we make the educated guess that $(t-1)D$ will be a subset of S, showing this would allow us to get a lower bound s.t. $N(t) \geq \pi(t-1)^2 = |(t-1)D|$. Let's do it!

Proof. We need to show $(t-1)D \subseteq S$, we prove $S^c \subseteq (t-1)D^c$ instead [where $X^c = \{x \in \mathbb{R}^2 | x \notin X\}$]. Let $x \in S^c$, then there is a point $p \in \mathbb{Z}^2$ such that $s \in S_p$, but $p \notin P$. Thus $|p - s| \leq 1$ and $|p| > t$, so we have $|s| = |p - (p - s)| \geq |p| - |p - s| \geq t - 1$. □

[1] Notice that $|P|$ is the number of lattice points, and 1 is the area of each square

[2] Using the ever-helpful triangle inequality

Session 2

One week into our journey, we find ourselves in a jungle of disorientation and confusion. With trees as tall as buildings keeping the light from reaching the ground, we must set out to climb with the few skills we have and hope to reach the top.

Exercise 11.4: Estimate the integral

$$\int_0^\infty \cos(x^2)dx$$

After a few painful attempts at integration by parts, a decent group effort causes a glimmer of light to shine from the leaves, and we split the integral in two.

$$\int_0^\infty \cos(x^2)dx = \int_0^\delta \cos(x^2)dx + \int_\delta^\infty \cos(x^2)dx$$

Now, we can easily find a bound for the first integral on the right, just by inspecting the $\cos(x^2)$ function near 0. In fact, if we pick $\delta < \sqrt{\frac{\pi}{2}}$, then

$$\delta\cos(\delta^2) \leq \int_0^\delta \cos(x^2)dx \leq \delta$$

Now, for the second integral on the right, we must use a more tricky approach.

$$\begin{aligned}
\int_\delta^\infty \cos(x^2)dx &= \int_\delta^\infty \cos(x^2)\frac{2x}{2x}dx \\
&= \int_\delta^\infty \frac{d}{dx}\left(\sin(x^2)\right)\frac{1}{2x}dx \\
&= \frac{\sin(x^2)}{2x}\bigg|_\delta^\infty + \frac{1}{2}\int_\delta^\infty \frac{\sin(x^2)}{x^2}dx \\
&= -\frac{\sin(\delta^2)}{2\delta} + \frac{1}{2}\int_\delta^\infty \frac{\sin(x^2)}{x^2}dx
\end{aligned}$$

And thus

$$\begin{gathered}
-\frac{\sin(\delta^2)}{2\delta} - \frac{1}{2\delta} \leq \int_\delta^\infty \cos(x^2)dx \leq -\frac{\sin(\delta^2)}{2\delta} + \frac{1}{2\delta} \\
-\frac{1}{\delta} \leq \int_\delta^\infty \cos(x^2)dx \leq \frac{1}{\delta}
\end{gathered}$$

Finally,

$$\sqrt{\frac{2}{\pi}} < \delta\cos(\delta^2) - \frac{1}{\delta} \leq \int_0^\infty \cos(x^2)dx \leq \delta + \frac{1}{\delta} \leq \frac{\pi+1}{\sqrt{2\pi}}$$

With our established δ. even if not the prettiest solution, it is nonetheless A bound for the integral.

Exercise 11.6: Let D denote the unit disk in the plane and define $\chi_D(x) = 1$ if $x \in D$ and 0 if $x \notin D$. Define

$$\widehat{\chi}_D(\xi) = \int_D e^{-2\pi i x\xi}$$

We will prove that $\widehat{\chi}_D(\xi) = \widehat{\chi}_D(\mu)$ whenever $|\xi| = |\mu|$.

Proof. Let $\xi = R(\cos(\pi), \sin(\pi))$ and $x = t(\cos(\theta), \sin(\theta))$. Then $x \cdot \xi = Rt(\theta - \phi)$. Assume $|\xi| = |\mu|$. Then $\sqrt{\xi_1^2 + \xi_2^2} = \sqrt{\mu_1^2 + \mu_2^2}$. Let $\mu = C(\cos(w), \sin(w))$, then $R = C$. So $\mu = R(\cos(w), \sin(w))$, and $x\mu = Rt(\theta - w)$. Also, since we are integrating over the unit disk, $(\theta - \phi) \in [0, 2\pi]$ and $(\theta - w) \in [0, 2\pi]$, so $\widehat{\chi}_D(\xi) = \widehat{\chi}_D(\mu)$. □

Session 3

Now, past the midpoint in our learning, our intuition and grasp of the general structure of the proof are improved. Let us summarize the problem we are trying to face, and provide a little bit of background. In mathematics, the lattice point problem is the issue of discovering how many lattice points (i.e. points in $\mathbb{Z}^2$) are contained in the disk of radius t.

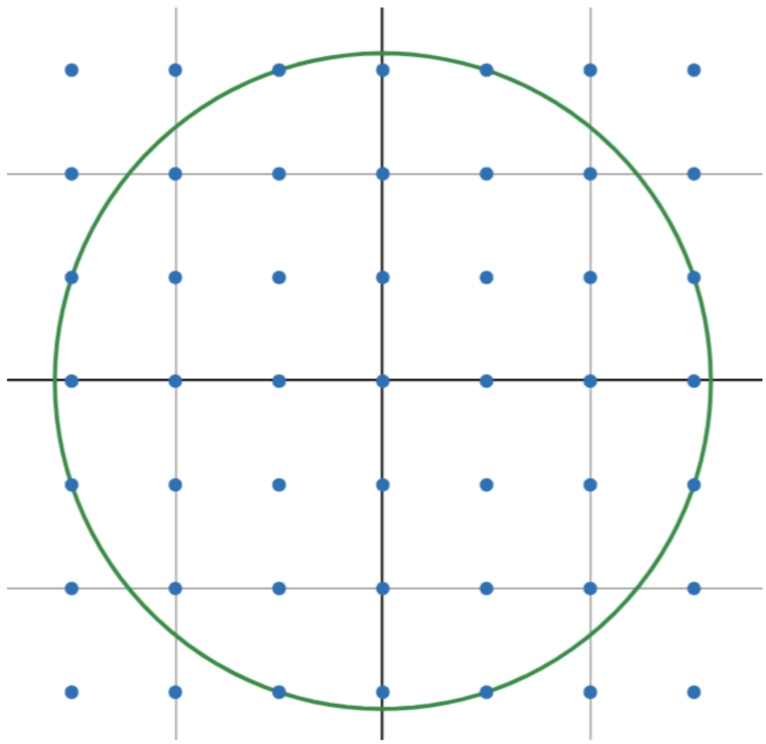

Figure 1: Lattice Points in Circle

Let $N(t)$ be the number of those points. Intuitively, we understand that $N(t) \approx \pi t^2 + E(t)$, hence the goal of modern research on the topic is to find a bound on $E(t)$. Godfrey Harold Hardy, a British mathematician of the 19-20th century, conjectured that the bound is

$$|E(t)| \leq C_\epsilon t^{\frac{1}{2}+\epsilon}$$

Currently, the best bound we know of is $E(t) \leq Ct^q$, with

$$\frac{1}{2} < q \leq \frac{131}{208}$$

where the lower bound was found by Hardy in 1915, and the upper bound by Martin Huxley in 2000.

The problem, when generalized to shapes other than a circle, finds application in many different applied fields, mostly in cryptography, integer programming, crystallography, quantum mechanics, and combinatorics. Meanwhile, for pure mathematics, it has obvious relevance in number theory, but also toric Hilbert functions and Kostant's partition function in representation theory.

In Alex Iosevich's "A View from the Top", chapter 12 is centred around a proof of a theorem established by Wacław Sierpiński in 1903:

$$|E(t)| \leq Ct^{\frac{2}{3}}$$

The methods modern mathematicians use to study the problem include the circle method (developed by Hardy and Littlewood), Fourier analysis, Minkowski's Theorem for convex sets in $\mathbb{R}^n$, and various lattice reduction techniques (such as Lenstra-Lenstra-Lovász algorithm).

In chapter 12, Iosevich uses Fourier transforms to establish the aforementioned bound. The Fourier transform

$$\widehat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-i2\pi\xi x}dx$$

is an incredibly flexible tool used in a variety of fields (including signal processing, data analysis, engineering, quantum mechanics and astronomy).

Now that we have an idea of the overarching goal, we continue with the math.

Exercise 11.3: Estimate the integral

$$\int_0^{\frac{\pi}{4}} e^{iR^2e^{2i\theta}}Re^{i\theta}$$

We attempt to find an upper bound to this integral.

$$\begin{aligned}
\left|\int_0^{\frac{\pi}{4}} e^{iR^2e^{2i\theta}}Re^{i\theta}\right| &= \left|\int_0^{\frac{\pi}{4}} e^{iR^2\cos(2\theta)+i\sin(2\theta)}Re^{i\theta}\right| \\
&\leq R\int_0^{\frac{\pi}{4}} e^{-R^2\sin(2\theta)}d\theta \\
&= R\int_0^{\delta} e^{-R^2\sin(2\theta)}d\theta + R\int_{\delta}^{\frac{\pi}{4}} e^{-R^2\sin(2\theta)}d\theta = I + II
\end{aligned}$$

We can convince ourselves that $I \leq R\delta$ and $II \leq \frac{\pi}{4}Re^{-R^2\sin(2\delta)}$, since $\sin(2\delta)$ is increasing on $[0, \frac{\pi}{4}]$. Our goal is to find a δ (in terms of R) s.t. I and II tend to 0 as $R \to \infty$.

We first tackle the first bound. Let $\delta = R^{-\alpha}$, with $\alpha > 1$. We see that

$$R\delta = RR^{-\alpha} = R^{1-\alpha} = \frac{1}{R^{\alpha-1}}$$

And since $\alpha - 1 > 0$:

$$\lim_{R\to\infty} \frac{1}{R^{\alpha-1}} = 0$$

So I approaches 0 as $R \to \infty$. Now to the second bound. We first should convince ourselves that for small values of δ, $\sin(2\delta) \geq \delta$ (maybe with Taylor series?). Having established this, let $\alpha < 2$, then

$$\begin{aligned}
\frac{\pi}{4} Re^{-R^2 \sin(2\delta)} &\leq \frac{\pi}{4} Re^{-R^2\delta} \\
&= \frac{\pi}{4} \frac{R}{e^{R^2\delta}} \\
&= \frac{\pi}{4} \frac{R}{e^{R^{2-\alpha}}} = A
\end{aligned}$$

And since $\alpha < 2$, $\lim_{R\to\infty} A = 0$, therefore $\lim_{R\to\infty} II = 0$. To conclude, for $R^-2 < \delta < R^-1$, the integral approaches 0 as $R \to \infty$.

Exercise 12.2: Let $\chi_t D(x) : \mathbb{Z}^2 \to \{0, 1\}$ s.t. $\chi_t D(x) = 1$ if x is in the disk of radius t centered at $(0, 0)$ in the $\mathbb{Z}^2$, and 0 otherwise. Define

$$N(t) = \sum_{n\in\mathbb{Z}^2} \chi_t D(n) \tag{1}$$

$$N_x(t) = \sum_{n\in\mathbb{Z}^2} \chi_t D(x - n) \tag{2}$$

For $x \in \mathbb{R}^2$. Our goal is to prove that (2) is a periodic function. More explicitly, $N(t)_x = N(t)_{x+m}$ for any $m \in \mathbb{Z}^2$.

$$N(t)_{x+m} = \sum_{n\in\mathbb{Z}^2} \chi_t D(x + m - n) \tag{3}$$

$$= \sum_{l\in\mathbb{Z}^2} \chi_t D(x - l) \qquad \text{Let } l = n - m \in \mathbb{Z}^2 \tag{4}$$

$$= \sum_{n\in\mathbb{Z}^2} \chi_t D(x - n) = N_x(t) \tag{5}$$

Intuitively, $N(t)$ measures the sum of all lattice points in the disk of radius t centred at the origin, while $N_x(t)$ measures the number of lattice points in the disk or radius t centred at x. What we just proved shows that if we change the coordinates of our center x (in $\mathbb{R}^2$) by integer values, the disk at the new center contains the same number of lattice points as the one before the move.

References

- All exercises are taken from "A View from the Top" by Alex Iosevich
- The UBC Mathematics SRP