Group 15: Alessandro Caggia, Edoardo Calleri di Sala, Emanuele Cinti, Stefano De Bianchi

1. State-Level HMM

We remove the seasonal component using STL (seasonal window = 13) and retrieve the deseasonalized data. Subsequently, we estimate a state-dependent level model. The observations are modeled as

$$Y_t = \theta_{S_t} + \epsilon_t, \quad \epsilon_t \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0, \sigma^2_{S_t}),$$

that is, conditional on the latent state $S_t \in \{1, \ldots, K\}$, the Y_t's are i.i.d. $\sim \mathcal{N}(\theta_{S_t}, \sigma^2_{S_t})$, with both the mean and variance depending on the current regime. In order to select the required number of states, while keeping the results interpretable, we compare a two states and a three states model and select the 3 state model on the basis of the BIC and the AIC (AIC = -1262.947). Table 1 reports the transition matrix, while Table 2 shows the parameters both for the full sample analysis and for the required restricted sample analysis (1880-1975)

Table 1: Transition Matrix Across Samples

	Full			**1880–1975**		
	to S1	to S2	to S3	to S1	to S2	to S3
from S1	0.994	0.006	0.000	0.957	0.043	0.000
from S2	0.008	0.986	0.006	0.033	0.933	0.034
from S3	0.000	0.005	0.995	0.000	0.050	0.950

Table 2: Response Parameters Across Samples

	Full		**1880–1975**	
	Intercept	SD	Intercept	SD
State 1	0.617	0.261	-0.355	0.110
State 2	0.046	0.090	-0.146	0.088
State 3	-0.246	0.109	0.063	0.099

Figure 1 LHS assigns to each point in the time series the estimates state-dependent means estimated on the full sample, while Figure 1 RHS runs the same analysis on the reduced sample.
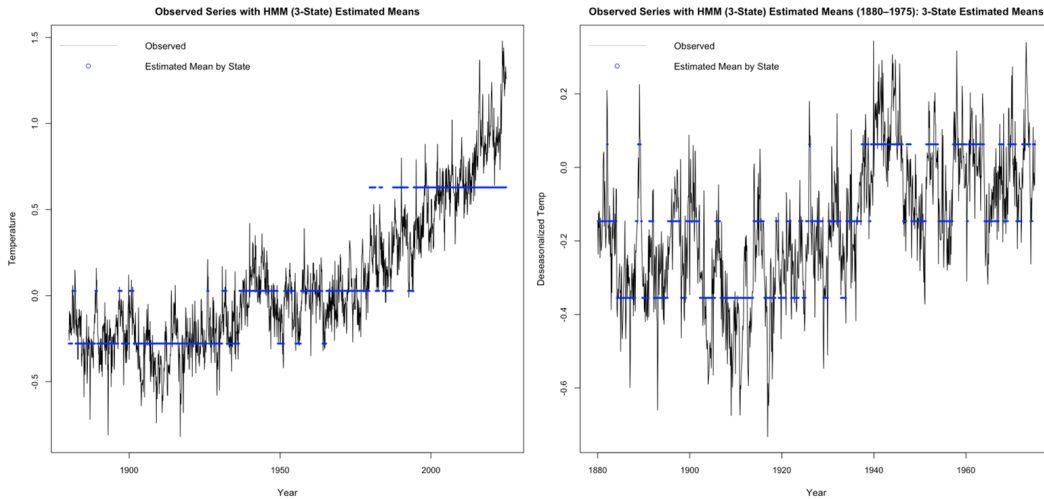

Figure 1: LHS: HMM 3-state model on full sample; RHS: Same model on 1880–1975

Figure 1 LHS shows the Observed Series with HMM (3-State) Estimated Means; RHS presents the Observed Series with HMM (3-State) Estimated Means from 1880 to 1975, 3-State Estimated Means

An HMM appears to be more suitable in the restricted sample scenario for the following reasons: intrinsically the time series has a trend component starting from the 70s. The HMM allowing only for a state-dependent level of the temperature is not structurally capable of modelling such a trend. Hence, it assigns observations to the three levels reflecting implicitly the trend component: it assigns observations belonging to the first period to state 1, observations belonging to the second period to state 2 and observations belonging to the last period to state 3. This is not wrong: indeed, as there is an underlying positive trend, later observations on average display higher values that are modeled through higher estimated means. The point is that a state-dependent trend is expected to be more appropriate for these data, while the state dependent level model still may be quite appropriate for describing the temperature data on the subsample going from 1880-1975, as no clear trend appears to be present. *Note: decompositions can be used to extract the trend component from the time series, thus revealing, as discussed in a previous assignment, the presence of a positive trend in the depertaure data starting form the 70s.*

2. HMM with State-Dependent Linear Trends

We now fit a state-dependent linear trend HMM. With respect to the state-level model the AIC and the BIC improve, suggesting a better fit. Figure 2 allows us to visualize the estimated trends. One state (s1) is associated with a relatively flat time trend, while the other (s2) exhibits a significantly steeper upward trend. Regarding the trend associated to this second state we can see that such trend is slightly overshooting the data in the 1900-1950 period while it is undershooting the data in the 1970 to 2025 subsample. This means that the trend estimated on the 1970 to 2025 subsample does not seem to be fitting the data properly, it being less steep than it should likely be. This aligns with how the HMM works: it is assigning a single trend coefficient to all the observations belonging to the given state (in this case s2). This aligns with what we expected: the HMM is estimating a common trend parameter on all the points that are assigned to state s2. Hence, the overall coefficient will be a weighted average of the trend parameters belonging to the assigned points. In this case, simplifying a bit the discussion, we have two subsamples assigned ot state 2: 1900-1950 and 1975+. A common beta is estimated, and so the trend fitted on 1975+ appears to be less steep than it should be because it is a weighted average of the two sub-samples.
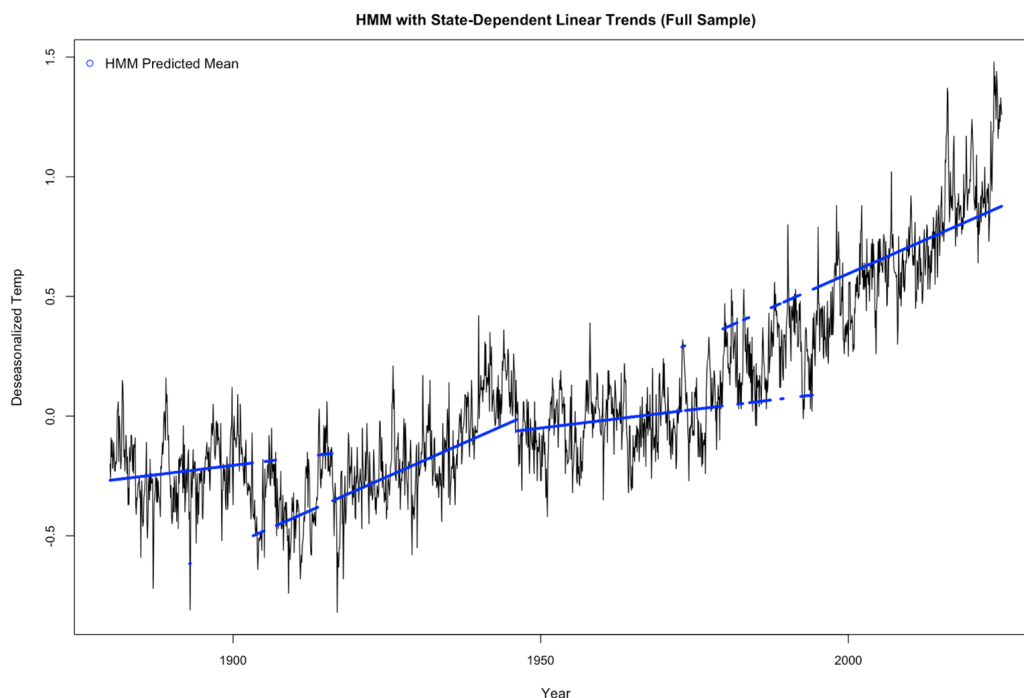

Figure 2: State-dependent linear trends (Full Sample)

3. Extensions: Quadratic Trend and TVTP

We expanded the model further by including time-varying transition probabilities (or TVTP), thus allowing for the probability of switching from one hidden state to another is no longer constant. It depends on time (or another covariate). However, this specification does not seem to improve results significantly. Additionally, we found out that the fit improved through the use of nonlinear trends, however, interpretation-wise, it does not appear obvious to justify the use of such non-linear trends (indeed, the HMM is modelling the non-linear trends as the upper and the lower bound of our time series, see figure 3). Lastly, using Covariate-Dependent Transitions (co2, economic activity etc.) could very likely enhance our analysis.
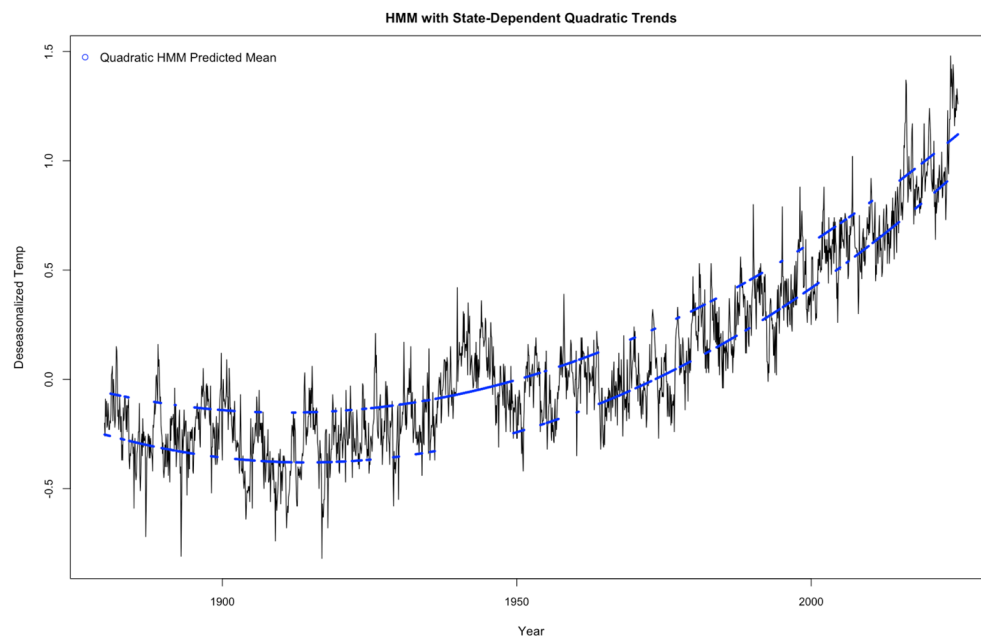

Figure 3: HMM with Quadratic Trends