Quantum synthesis of arbitrary unitary operators

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Nature provides us with a restricted set of microscopic interactions. The question is whether we can synthesize out of these fundamental interactions an arbitrary unitary operator. In this paper we present a constructive algorithm for realization of any unitary operator which acts on a (truncated) Hilbert space of a single bosonic mode. The algorithm itself is not unitary because it involves a conditional measurement. However, it does yield a constant probability of the conditional measurement which does not depend on the input state of the bosonic system. We consider a physical implementation of unitary transformations acting on one-dimensional vibrational states of a trapped ion. As an example we present an algorithm which realizes the discrete Fourier transform.

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I. INTRODUCTION

Controlled manipulations with individual quantum systems, such as trapped ions or cold atoms in atomic physics, molecules irradiated by laser fields, and Rydberg atoms interacting with quantized micromaser fields, provide us with a deeper understanding of fundamental principles of physics. Simultaneously, the possibility to control individual quantum systems opens new perspectives in the application of quantum physics. Specifically, coherent control over dynamics of quantum systems is of vital importance for quantum computing and information processing [1,2].

Quantum information processing can be schematically divided into three stages. The first stage is the encoding of information into quantum systems, i.e., this corresponds to a preparation of states of quantum systems. The second stage is the information processing which in general is equivalent to a specific unitary evolution of the quantum system, i.e., this is an application of a given quantum algorithm. The third stage is the reading of output states of quantum registers (i.e., the ''decoding'' of information from quantum systems). Obviously this final stage is the measurement of a quantum system and the reconstruction of relevant information.

There are several physical systems which are believed to be candidates for quantum processors. In particular, Cirac and Zoller [3] have shown that a system of trapped ions can be utilized as a prototype of a quantum computer. Therefore, it is of great interest to understand how the three stages of the information processing as specified above can be implemented in this system.

(i) State preparation. Recently, several methods for *deterministic* synthesis (preparation) of vibrational states of trapped ions have been proposed. In particular, a scheme for preparation of quantum states of one- and two-mode bosonic fields [e.g., one-dimensional (1D) and 2D quantum states of vibrational motion of trapped ions] has been proposed by Law and Eberly [4] and Kneer and Law [5] (see also [6], and for a more general discussion on the state preparation, see [7]).

(ii) State measurement. There are various experimental techniques which allow us to measure and reconstruct quantum states of trapped ions (for a review, see [8]).

(iii) Arbitrary unitary evolution. One of the most important tasks in information processing is to design processors which take an *arbitrary* input and process it according to a specific prescription. Nature provides us with a restricted set of microscopic interactions. The question is whether we can synthesize out of these fundamental interactions an arbitrary unitary operator. In this paper we present a constructive algorithm for realization of any unitary operator which acts on a finite-dimensional Hilbert space. An algorithmic proof that any discrete finite-dimensional unitary matrix can be factorized into a sequence of two-dimensional beam splitter transformations was given by Reck *et al.* [9]. The problem of controlled dynamics of quantum systems has been addressed recently by Harel and Akulin [10] and by Lloyd and Braunstein [11]. Harel and Akulin [10] have proposed a method to attain any desired unitary evolution of quantum systems by switching on and off alternatively two distinct constant perturbations. The power of the method was shown in controlling the 1D translational motion of a cold atom.

Our aim is to find a constructive algorithm to realize an arbitrary unitary operator $\hat{V}$ which transforms any state $|\psi\rangle$ of a single bosonic mode, e.g., a 1D vibrational state of a trapped ion in the x direction to another state $|\psi'\rangle$, i.e., $|\psi'\rangle=\hat{V}|\psi\rangle$. In particular, we consider a truncated $(N+1)$-dimensional Hilbert space of the bosonic mode. Within this truncated Hilbert space, the desired unitary operator $\hat{V}$ in the number-state basis reads

$$\hat{V}=\sum_{m,n=0}^{N,N} V_{n,m}|n\rangle\langle m|. \tag{1}$$

Under action of the operator $\hat{V}$, a given input state transforms as

$$|\psi\rangle=\sum_{m=0}^{N} c_m|m\rangle \overset{\hat{V}}{\rightarrow} |\psi'\rangle=\hat{V}|\psi\rangle=\sum_{n=0} c'_n|n\rangle, \tag{2}$$

where $c_n' = \Sigma_{m=0}^{N} V_{n,m} c_m$. Our task is to represent by a feasible physical process any operator $\hat{V}$ (1). The synthesis of the operators thus enables us to realize universal quantum gates for qubits which can be encoded into vibrational levels. We note that the synthesis itself should not necessarily be represented by a unitary operation (e.g., one stage of the synthesis presented in this paper is realized by a conditional measurement), nevertheless when the condition is fulfilled the desired unitary transformation on an arbitrary (*a priori* unknown) state is performed.

The paper is organized as follows. In Sec. II we briefly introduce physical tools which we use to realize arbitrary unitary operators for a trapped ion. The synthesis algorithm is described in Sec. III. The method is illustrated in Sec. IV, where a realization of quantum gates which perform the Fourier transform is considered. We also discuss stability of the algorithm and possibilities to realize also nonunitary operators. We finish our paper with conclusions.

II. TOOLS FOR SYNTHESIS: LASER-STIMULATED PROCESSES

Our realization of the unitary transformation $\hat{V}$ given by Eq. (1) is based on an enlargement of the Hilbert subspace of the given system. Namely, the transformed bosonic mode corresponds to one vibrational mode (in the x direction, for concreteness) of a quantized center-of-mass motion of an ion confined in the 2D trapping potential. Within our synthesis procedure, the vibrational x mode becomes entangled with the auxiliary degrees of freedom (*ancilla*) which are represented by the second vibrational mode (e.g., quantized vibrational motion in the y direction) and three internal electronic levels $|a\rangle, |b\rangle, |c\rangle$ of the ion. The particular choice of the physical system is motivated by a feasibility of highly coherent control over motional degrees of freedom as demonstrated in recent experiments [8], which causes trapped ions to be candidates for quantum processors.

A physical realization of the desired operator $\hat{V}$ for an ion confined in a 2D trapping potential consists of a sequential switching (on/off) of laser fields which irradiate the ion. Namely, we utilize four types of laser-stimulated interactions which are associated with the following (effective) interaction Hamiltonians:

$$\begin{aligned}
\hat{H}^{(1,m)} &= (\Delta_y + \hat{s}_x)(|a\rangle\langle a| - |b\rangle\langle b|) + g_1 |a\rangle\langle b| + g_1^{\star} |b\rangle\langle a|, \\
\hat{H}^{(2)} &= g_2 |b\rangle\langle c| \hat{a}_y^{\dagger} \mathcal{F}(\hat{a}_y^{\dagger}\hat{a}_y) + g_2^{\star} |c\rangle\langle b| \mathcal{F}(\hat{a}_y^{\dagger}\hat{a}_y) \hat{a}_y, \\
\hat{H}^{(3)} &= g_3 |b\rangle\langle c| + g_3^{\star} |c\rangle\langle b|, \\
\hat{H}^{(4)} &= g_4 |b\rangle\langle c| \hat{a}_x^{\dagger} \mathcal{F}(\hat{a}_x^{\dagger}\hat{a}_x) + g_4^{\star} |c\rangle\langle b| \mathcal{F}(\hat{a}_x^{\dagger}\hat{a}_x) \hat{a}_x,
\end{aligned} \tag{3}$$

where $\hat{s}_x = \Sigma_m s_m |m\rangle_x\langle m|$ with $s_m = \chi[1 + e^{-2\eta_x^2} L_m^0(4\eta_x^2)]$ and $\mathcal{F}(\hat{a}_q \hat{a}_q) = e^{-\eta_q^2/2} \Sigma_{k=0} [(-1)^k \eta_q^{2k}/(k+1)!k!] \hat{a}_q^{\dagger k} \hat{a}_q^k$. The Lamb-Dicke parameters are defined as $\eta_q = \omega_q/(c\sqrt{2m_a\nu_q})$ (assuming units such that $\hbar = 1$), where ω_q, ν_q are frequencies of the laser and vibrational mode in direction $q (q = x,y)$, respectively. Further, L_m^0 denote the Laguerre polynomial; m_a is the mass of the ion.

The dynamical Stark shift operator $\hat{s}_x = \Sigma_m s_m |m\rangle_x\langle m|$ is induced by a detuned standing-wave laser field applied in the x direction [5]. For $\chi \gg |g_1|$, the effective Hamiltonian $H^{(1,m)}$ addresses the states with fixed number m of excitations (indicated by the superscript) in the mode x setting the detuning Δ_y of the laser field, applied in the orthogonal y direction, equal to the dynamical Stark shift $-s_m$. In other words, when the interaction is governed by the Hamiltonian $H^{(1,m)}$ with $\Delta_y = -s_m$, there is an exchange of the population only between the states $|m,n\rangle \otimes |a\rangle \Leftrightarrow |m,n\rangle \otimes |b\rangle$ with the given number of quanta m in the vibrational mode x and any number of phonons n in the mode y. The populations of the other number states (with the number of excitations in the mode x different from m) effectively do not change due to a large detuning. However, there are significant phase shifts of the amplitudes of the off-resonant states. It should be stressed that the addressing of states with a given number m of excitations in the mode x is effective only for large ratios $\chi/|g_1| \gg 1$. Moreover, the approximate Hamiltonian $H^{(1,m)}$ is itself justified only when $\chi \gg |g_1|$ (for details, see [5]).

The considered interactions (3) are quite typical for a trapped ion. The effective interaction Hamiltonians represent a classical driving of the ion [see $H^{(3)}$] and a nonlinear Jaynes-Cummings model [12] [see $H^{(2),(4)}$] when the applied lasers are tuned to appropriate vibrational sidebands. These Hamiltonians are thoroughly discussed in [5,13].

The dynamics governed by the interaction Hamiltonians (3) can be separated into independent 2D subspaces. Switching on a particular interaction ''channel'' associated with one of the interaction Hamiltonians $\hat{H}^{(p)}$ (3) for a time τ is described as the action of the corresponding unitary time-evolution operator $\hat{U}^{(p)} = \exp[-i\hat{H}^{(p)}\tau]$ on the state vector of the system under consideration.

III. SYNTHESIS OF TRANSFORMATIONS

To implement the desired transformation $\hat{V}$ (1) for an ion confined in the 2D trapping potential, we realize a mapping $|\psi_x, 0_y\rangle \overset{\hat{V}}{\rightarrow} |0_x, \psi_y'\rangle = |0_x, \hat{V}\psi_y\rangle$ of two-mode bosonic states (here subscripts indicate particular vibrational modes; in what follows the subscripts will be omitted for a given ordering of modes). To be more explicit, the realization of the transformation $\hat{V}$ can be expressed as the mapping in the extended Hilbert space of two bosonic modes and internal electronic levels in the following form:

$$|0,\psi'\rangle \otimes |b\rangle = \mathcal{N} \hat{P}_{|b\rangle} \hat{B} \hat{A} |\psi,0\rangle \otimes |a\rangle, \tag{4}$$

where the operators $\hat{A}$ and $\hat{B}$ represent a sequence of four types of unitary operations. The final projection $\hat{P}_{|b\rangle}$ on the state $|b\rangle$ selects *conditionally* the right outcome ($\mathcal{N}$ is a proper normalization constant). In the subsequent step one could use the two-mode linear coupler based on laser-stimulated Raman transitions [14] to swap the states of the

vibrational modes, i.e., $|0,\psi'\rangle \rightarrow |\psi',0\rangle$. The additional π pulse can be used to flip the electronic state from the level $|b\rangle$ into the initial level $|a\rangle$.

The operators $\hat{A}$, $\hat{B}$, $P_{|b\rangle}$ appearing in Eq. (4) indicate three essential steps which lead us subsequently to the desired transformations.

Step A. The operator $\hat{A}$ ''spreads'' the amplitudes c_m's of the component number states $|n_x,0\rangle \otimes |a\rangle$ over the whole Hilbert space $\mathcal{H}_x \otimes \mathcal{H}_y \otimes \mathcal{H}_{in}$ so that the entangled state of the composed system becomes[1]

$$|\psi^A\rangle = \sum_{m,n=0}^{N,N} V_{n,m} c_m e^{i\varphi_m^A} |m,n\rangle \otimes |a\rangle. \tag{5}$$

This task can be done by the method of 1D quantum state synthesis proposed by Law and Eberly [4]. An important tool represents also the photon-number-dependent interaction $\hat{H}^{(1,m)}$ considered by Kneer and Law [5], which enables us to address individually the subspaces with a fixed number m of phonons in the x direction. The operator $\hat{A}$ can be written as

$$\hat{A} = \prod_{m=0}^{N} \hat{U}^{(1,m)}_{-\pi/2\exp(i\phi_m)} \hat{A}(m) \hat{U}^{(1,m)}_{\pi/2}, \tag{6}$$

where

$$\hat{A}(m) = \hat{U}^{(2)}_{|m,N;b\rangle} \hat{U}^{(3)}_{|m,N-1;c\rangle} \cdots \hat{U}^{(2)}_{|m,1;b\rangle} \hat{U}^{(3)}_{|m,0;c\rangle}. \tag{7}$$

The subscript of the unitary transformation $\hat{U}^{(1,m)}_{g_1\tau}$ indicates the required setting of the corresponding interaction parameter $g_1\tau = |g_1\tau| e^{i\phi_m}$. In other transformations $\hat{U}^{(p)}_{|\Phi\rangle}$ $(p=2,3)$ the subscripts denote steps in 1D quantum-state synthesis as explained below. In other words, within a particular subspace with a fixed number m of phonons in the x direction, we flip from the electronic level $|a\rangle$ to $|b\rangle$ by means of $\hat{U}^{(1,m)}_{\pi/2}$. Then to ''spread'' $|m,0\rangle \otimes |b\rangle$ we apply 1D quantum state synthesis associated with the action of the operator $\hat{A}(m)$ to ''prepare'' the superposition $\Sigma_{n=0}^{N} V_{n,m} |m,n\rangle \otimes |b\rangle$. After that the electronic level $|b\rangle$ is flipped back to $|a\rangle$ via the action of $\hat{U}^{(1,m)}_{-\pi/2\exp(i\phi_m)}$ [here ϕ_m's represent proper phase factors which will be discussed later, see Eq. (11)].

The appropriate interaction parameters for our 1D quantum-state synthesis can be found when we solve the inverse task which is given by the inverse transformation: $|m,0\rangle \otimes |b\rangle = \hat{A}^\dagger(m) \Sigma_n V_{n,m} |m,n\rangle \otimes |b\rangle$. The inverse task is based on ''sweeping'' down the probability from the component states of the given superposition in the y mode into the vacuum. Therefore, the subscripts of the unitary operators $\hat{U}^{(p)}_{|\Phi\rangle}$ $(p=2,3)$ in Eq. (7) indicate that interaction parameters $g_p\tau = e^{i\varphi} |g_p\tau|$ have to be chosen in such way that after the action of the $\hat{U}^{\dagger(p)}_{|\Phi\rangle}$ the amplitude (population) of the component state $|\Phi\rangle$ becomes equal to zero. The action of the operator $\hat{A}^\dagger(m)$ is shown schematically in Fig. 1. We have applied the procedure introduced by Law and Eberly for synthesis of 1D bosonic states in a straightforward way. Therefore, we refer readers to the original paper [4] for other details (generalized Hamiltonians to operate beyond the Lamb-Dicke limit can be found in [13]). Note that in our case we apply the state-synthesis procedure only in the y direction and the resulting state (5) remains *unknown*.

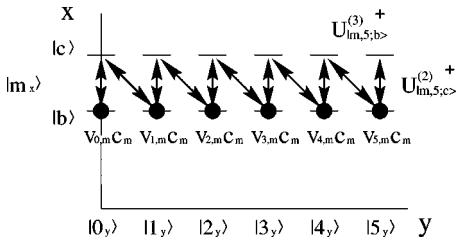

FIG. 1. The action of the operator $\hat{A}^\dagger(m)$ on the row with a fixed number m of phonons in the mode x. The population from the superposition $\Sigma_n V_{n,m} |n\rangle_y$ is ''swept'' down into the vacuum $|0\rangle_y$ in the mode y.

To resume, the action of the operator $\hat{A}$ encodes the matrix elements $V_{n,m}$ (multiplied with the amplitudes c_m's of the unknown state) into rows of the 2D vibrational ''lattice'' of number states $|m,n\rangle$. In the next step an appropriate superposing of columns within the 2D vibrational number-state ''lattice'' is required.

Step B. In the second step the operator $\hat{B}$ creates the state in which the amplitudes of the states $|0,n\rangle \otimes |b\rangle$ $(n=0,\ldots,N)$ are proportional to $c_n' = \Sigma_{m=0}^{N} V_{n,m} c_m$. The state of the system $|\psi^B\rangle = \hat{B}|\psi^A\rangle$ after this synthesis step reads

$$|\psi^B\rangle = \frac{1}{\sqrt{N+1}} |0,\psi'\rangle \otimes |b\rangle + \sum_{m=0}^{N} \sum_{n=0}^{N} z_{m,n} |m,n\rangle \otimes |a\rangle. \tag{8}$$

The operator $\hat{B}$ can be written in the form

$$\hat{B} = \left[\prod_{m=0}^{N-1} \hat{B}(m) \right] \hat{U}^{(1,N)}_{-i\pi/2}, \tag{9}$$

where

$$\hat{B}(m) = \hat{U}^{(1,m)}_{-i \arctan(1/\sqrt{N+1-m})} \hat{U}^{(3)}_{-i\pi/2} \hat{U}^{(4)}_{-i\pi/(2\sqrt{m+1})}. \tag{10}$$

Here the subscripts of the unitary operators $U^{(p)}_{g_p\tau}$ indicate again the proper choice of interaction parameters $g_p\tau$.[2]

[1] In our synthesis algorithm we neglect off-resonant transitions between internal levels in $\hat{H}^{(1,m)}$. Strictly speaking, the equality sign applies only in the limit $\chi/|g_1| \rightarrow \infty$.

[2] Here and in Fig. 2 the explicit expressions for required settings, e.g., $g_4\tau = -i\pi/(2\sqrt{m+1})$, refer for clarity to the Lamb-Dicke regime $\eta_x \ll 1$. Outside of the Lamb-Dicke regime, the Rabi frequency $\sqrt{m+1}$ is simply replaced by the nonlinear Rabi frequency $e^{-\eta_x^2/2} L_m^1(\eta_x^2)/\sqrt{m+1}$, where L_m^1 denotes the associated Laguerre polynomial.

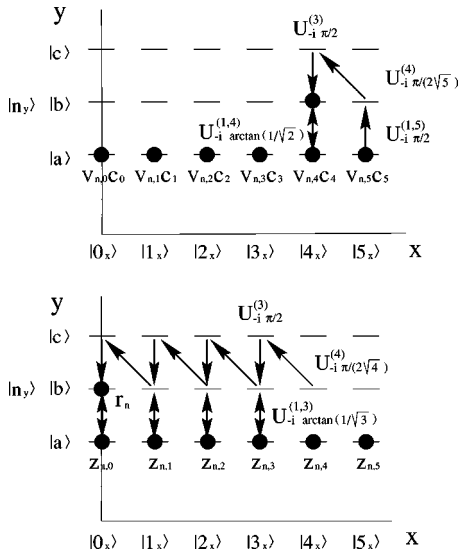

FIG. 2. The action of the operator $\hat{B}$ on the column with n vibrational quanta in the mode y. The same changes occur simultaneously on ''parallel'' columns with different n. The upper figure shows the action of $\hat{U}^{(1,N)}_{-i\pi/2}$ (for $N=5$) followed by the basic sequence of operations $\hat{B}(m)$ (for $m=4$). Decreasing the number of quanta m in the x direction, the basic sequence $\hat{B}(m)$ is recursively repeated as shown in the lower figure. After the action of the operator $\hat{B}$, the value of the amplitude of the component state $|0,n\rangle \otimes |b\rangle$ is $r_n=(\Sigma_{m=0}^{N} V_{n,m}c_m)/\sqrt{N+1}$.

The action of the operator $\hat{B}$ on a particular column with a given number n of phonons in the y mode is shown schematically in Fig. 2. Simultaneously, the operator $\hat{B}$ acts in the same way on ''parallel'' columns with different n. As illustrated in the upper part of Fig. 2, the operator $\hat{U}^{(1,N)}_{-i\pi/2}$ (for $N=5$) transfers the population from the internal level $|a\rangle$ to $|b\rangle$ only in the row with the fixed number of quanta $m=5$ in the mode x. Further, this population is transfered to the internal level $|b\rangle$ in the neighboring row with the number of quanta $m=4$ performing transformations $\hat{U}^{(3)}_{-i\pi/2}\hat{U}^{(4)}_{-i\pi/(2\sqrt{m+1})}$.

The equal superposition of the amplitudes $V_{n,4}c_4$ and $V_{n,5}c_5$ in the row with $m=4$ is obtained after action of $\hat{U}^{(1,4)}_{-i\ \arctan(1/\sqrt{2})}$, i.e., undergoing one-half of the Rabi flipping. Decreasing m (the number of quanta in the mode x), the basic sequence $\hat{B}(m)$ is recursively repeated as indicated in the lower part of Fig. 2. Note that $\hat{U}^{(1,m)}_{-i\ \arctan(1/\sqrt{N-m+1})}$ is responsible for adding the amplitude $V_{n,m}c_m$ (with a proper weight) to previously superposed amplitudes $\Sigma_{k=m+1}^{N} V_{n,k}c_k$ performing an adequate part of the Rabi flipping. After the action of the whole operator $\hat{B}$, the amplitude r_n of the component state $|0,n\rangle\otimes|b\rangle$ takes the value $r_n=(1/\sqrt{N+1})\Sigma_{m=0}^{N} V_{n,m}c_m$, i.e., $r_n \sim c_n'$. The transformed state $|\psi^B\rangle$ is given in Eq. (8).

At this step we should notice that *each* action of the ''elementary'' unitary operator $\hat{U}^{(1,m)}$, associated with the interaction Hamiltonian $\hat{H}^{(1,m)}$ with $\Delta_y=-s_m$, causes significant phase shifts on off-resonant rows with the number of quanta in the mode x different from m [5]. These phase shifts have to be compensated *in advance* in order to ''superpose'' the amplitudes via the action of $\hat{B}(m)$ as described above (see the role of $\hat{U}^{(1,m)}_{-i\ \arctan(1/\sqrt{N-m+1})}$). This compensation can be done when we include appropriate phase shifts ϕ_m's directly in the operator $\hat{A}$ [see Eq. (6)]. The explicit expression reads

$$\phi_m=-\frac{\pi}{2}-\sum_{k=0}^{m-1} f_m^{(k)}(\pi)-\sum_{k=m+1}^{N} f_m^{(k)}\left(\arctan\frac{1}{\sqrt{N-k+1}}\right), \tag{11}$$

where $f_m^{(k)}(|g_1|\tau)=\arg[\cos(\Omega_m^{(k)}\tau)+i(s_k-s_m)/2\Omega_m^{(k)}\sin(\Omega_m^{(k)}\tau)]$ with $\Omega_m^{(k)}=\sqrt{(s_k-s_m)^2/4+|g_1|^2}$. The origin of the expression (11) can be traced back to the operators $\hat{U}^{(1,m)}$ in the steps $\hat{A}$ and $\hat{B}$. The aim is to cancel the phase shifts of the amplitudes in order to ''superpose'' them on the mth row by means of $\hat{B}(m)$. Therefore, the first sum in Eq. (11) compensates (in advance) for the subsequent shifts in $\hat{A}$ due to $\hat{U}^{(1,k)}$ for $k=0,\ldots,m-1$. The second sum compensates for the shifts which will take place during ''superposing'' operations $\hat{B}(k)$ for $k=m+1,\ldots,N$, which precede $\hat{B}(m)$. As seen from Eq. (11), these phases depend on the parameters s_n which are related to Stark shifts given explicitly by the formula below Eq. (3). We stress that the phases ϕ_m do not depend on the input state, i.e., they are independent of the complex amplitudes c_n [see Eq. (2)].

Step C. Comparing the state $|\psi^B\rangle$ [Eq. (8)] with the desired one [Eq. (4)], we see that the target state is entangled to the internal level $|b\rangle$. On the other hand, also undesired component states $|m\neq 0,n\rangle\otimes|a\rangle$ are now contributing to Eq. (8) with nonvanishing amplitudes. However (fortunately), all the undesired components are entangled with the internal level $|a\rangle$. Therefore, we can perform a conditional measurement to project the state vector (8) on the internal level $|b\rangle$. To be more specific, the internal state of the ion can be determined by driving the transition from the level $|a\rangle$ to an auxiliary level $|r\rangle$ and observing the fluorescence signal.[3] No signal (no interaction with probing field) means that the undisturbed ion is occupying the level $|b\rangle$ being in the motional state $|0,\psi'\rangle$. This means that after the conditional measurement [indicated in Eq. (4) by the projector $\hat{P}_{|b\rangle}$], the state vector (8) is reduced to the desired state vector (4). The probability to find the right outcome for the unitary transformations is equal to $1/N+1$.

In spite of the involved *conditional selection* of the right outcomes, our algorithm is *universal* as the sequence of the ''elementary'' operations (with appropriate interaction parameters) which represents the desired transformation is always *independent* of input states. Moreover, contrary to conditional measurement schemes known from quantum state

[3]As a check, one could drive also the transition from the level $|c\rangle$ to another auxiliary level $|r'\rangle$. Errors in the synthesis procedure are thus indicated by the presence of the fluorescence signal.

preparation, in our case the probability of the right outcome is constant, being also *independent* of input states.

IV. DISCUSSION

One of the important applications of the operator synthesis is a realization of *universal* quantum gates for qubits which are encoded in vibrational levels. The number states of the vibrational mode can represent a quantum register.

To illustrate our synthesis procedure, we considered a realization of the operator which ''rotates'' the population between N vibrational Fock states under consideration,

$$\hat{V}_R=\frac{1}{\sqrt{N+1}}\left(\sum_{j=0}^{N-1}|j+1\rangle\langle j|+|0\rangle\langle N|\right). \qquad (12)$$

It corresponds to a cyclic ''rotation'' of the quantum register. The operator $\hat{V}_R$ represents the unitary exponential phase operator of the Pegg-Barnett formalism [15].

In the second example, the synthesis procedure is applied for the unitary operator of the quantum Fourier transform defined as [16]

$$\hat{V}_{\mathrm{QFT}}=\frac{1}{\sqrt{N+1}}\sum_{m,n=0}^{N-1}\exp\left(2\pi i\frac{mn}{N+1}\right)|m\rangle\langle n|. \qquad (13)$$

The operator of the quantum Fourier transform represents an important tool in quantum computing [16].

In the presented synthesis procedure, we have neglected transitions between internal levels on off-resonant ''rows'' when the interaction Hamiltonian $\hat{H}^{(1,m)}$ is applied. Strictly speaking, the off-resonant transitions in $\hat{H}^{(1,m)}$ can be neglected only in the limit $\chi/|g_1|\to\infty$. Our estimation of the error due to the finite values of the ratio $\chi/|g_1|$ (feasible in practice) is based on the fidelity of the outputs to the ideally transformed states (4). The fidelity of two states $|\Phi\rangle,|\Phi'\rangle$ is defined as their squared scalar product $|\langle\Phi'|\Phi\rangle|^2$. As a testing input state of the x mode, we can take a uniform superposition of involved number states, i.e., $|\psi\rangle=(1/\sqrt{N+1})\Sigma_{m=0}^{N}|m\rangle_x$ (the initial internal state of the ion is $|a\rangle$ and the vibrational y mode is in the vacuum). Figure 3 shows that the fidelity is close to 1 for large ratios $\chi/|g_1|\gg 1$ for both operators $\hat{V}_R$ and $\hat{V}_{\mathrm{QFT}}$.[4] In such a case both the validity of the approximate Hamiltonian $\hat{H}^{(1,m)}$ is justified [5] and the level flipping ($|a\rangle\leftrightarrow|b\rangle$) on off-resonant ''rows'' (as the source of nonideal fidelity) can be neglected.

Let us note that for nonunitary transformations the probability depends on initial states and can be from the interval (0,1). However, the realization of the transformation (given by the sequence of the ''elementary'' operations with appropriate interaction parameters) is always *independent* of input states.

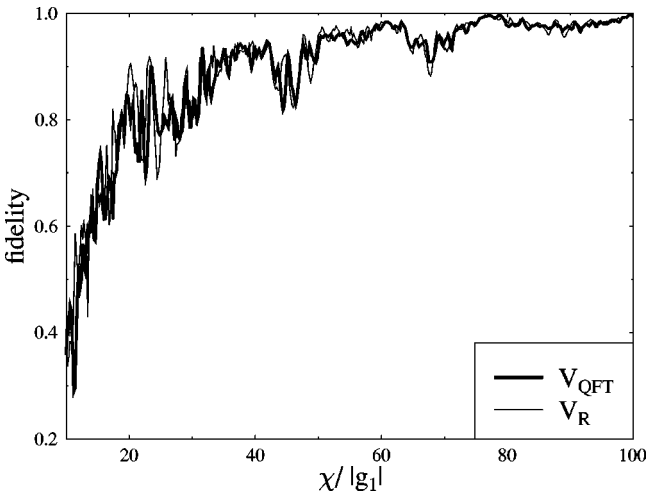

FIG. 3. The fidelity of outputs to the ideally transformed states as a function of the ratio $\chi/|g_1|$ for the quantum Fourier transform $\hat{V}_{\mathrm{QFT}}$ (thick solid line) and the exponential phase operator $\hat{V}_R$ (thin solid line). Lamb-Dicke parameters are $\eta_x=\eta_y=0.4$.

V. CONCLUSIONS

In this paper we have proposed a constructive algorithm for the synthesis of operators, a superior task to the synthesis of quantum states. Our method allows us to find *analytical* expressions for switching times and interaction parameters of the utilized laser-stimulated processes by which an arbitrary unitary dynamics can be realized. One of the important applications of the operator synthesis is a realization of *universal* quantum gates for qubits which are encoded in vibrational levels. As an example, we consider a realization of the discrete Fourier transform.

The solution of the problem we present in our paper is neither unique nor optimal (in the sense of a number of elementary operations used for a construction of the given unitary operator). The optimization of the procedure is the problem which has to be solved. The other problem which deserves attention is the stability of the algorithm with respect to noise inherent in the system. In fact, one can consider two types of uncertainties which might play an important role. First, it is the noise induced by the environment, i.e., the elementary gates are not unitary. The second source of noise (a kind of technical noise) is due to the fact that it is not possible to keep the interaction times and parameters fixed as given by the theory. Fluctuations in these parameters might reduce the fidelity of the realization of the desired unitary evolution. We will address these questions elsewhere.

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[4] In Fig. 3 we consider parameters outside of the Lamb-Dicke regime. To operate in the Lamb-Dicke regime requires a further increase of the ratio $\chi/|g_1|\gg 1$ to reach a fidelity close to 1.

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