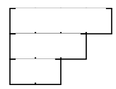
X

$\gamma\colon \mathrm{Hilb}$

Hilb

Sag01].



$$(x_{i_j} - x_{i_k}).$$
$$j<k\in[l]$$

$$\Delta_{T_i}.$$

$$\begin{aligned}\mathrm{sp}_T(x) &= \Delta_{(9,2,5)}(x)\Delta_{(3,1,7)}(x)\Delta_{(6,8)}(x)\Delta_{(4)}(x)\\ &= (x_9 - x_2)(x_9 - x_5)(x_2 - x_5)(x_3 - x_1)(x_3 - x_7)(x_1 - x_7)(x_6 - x_8).\end{aligned}$$

x

$$\mathrm{sgn}(\sigma)\sigma\circ\tau\cdot x$$

,

$$\begin{aligned}\mathbb{C}[x_1,x_2,x_3]/(p_1,p_2,p_3) &\cong S^{(3)}\oplus 2S^{(2,1)}\oplus S^{(1,1,1)}\\ &= \langle 1\rangle\oplus\langle x_i-x_j\rangle\oplus\langle (x_i-x_j)x_k\rangle\oplus\langle (x_1-x_2)(x_1-x_3)(x_2-x_3)\rangle.\end{aligned}$$

□

$$T_{[Z]}\,\mathrm{Hilb}^G_\rho(W) \cong \mathrm{Hom}^G_A(I, A/I),$$

$$\mathrm{GL}_n^{S_n} = \{a\mathbb{1}_n + b\mathbf{1}_n \in \mathrm{GL}_n : a, b \in \mathbb{C}^*\},$$

$$(\mathbb{C}^*)^2 \xrightarrow{\cong} \mathrm{GL}_n^{S_n},\ (s,t) \mapsto t\mathbb{1}_n + \frac{(s-1)t}{}$$

$$H_{\rho,h} \longrightarrow \mathrm{Hilb}$$

$$\rho \cong M^{\lambda^1} \oplus \cdots \oplus M^{\lambda^N},$$

Proof. Let $I \subseteq P = \mathbb{C}[x_1, \ldots, x_n]$ be a radical symmetric ideal such that $P/I \cong \rho$ as S_n-

number N of disjoint S_n-orbits with vanishing ideals $I_1, \ldots, I_N$. The Chinese remainder theorem provides a canonical algebra isomorphism $P/I \cong \bigoplus_{i=1}^N P/I_i$ which is S_n-equivariant, so it suffices to prove the case of only one orbit. Let $q_1, \ldots, q_k \in \mathbb{C}^n$ be the distinct points of this orbit and let $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m \geq 1) \vdash n$ be the corresponding partition, i.e., each q_i has m distinct coordinates which occur with multiplicities $\lambda_1, \ldots, \lambda_m$. Again by the Chinese remainder theorem we obtain a canonical isomorphism $P/I \cong \bigoplus_{i=1}^k P/\mathfrak{m}_{q_i}$ of algebras, in particular of vector spaces. We define an S_n-module structure on the direct

$$\overline{f} \in P/\mathfrak{m}_{q_i} \quad \Rightarrow \quad \sigma(\overline{f}) \coloneqq \overline{\sigma(f)} \in P/\mathfrak{m}_{\sigma(q_i)}.$$

inclusion $\mathbb{C} \subseteq P/\mathfrak{m}_{q_i}$ is an isomorphism, so $\bigoplus_{i=1}^k P/\mathfrak{m}_{q_i} \cong M^\lambda$ as S_n-modules. □

containing only radical ideals, so the same is true for the closed subscheme $\mathrm{Hilb}^{S_n}_\rho(\mathbb{C}^n)$.

orem 5.2].

scheme $\mathcal{H}_{M^\lambda}$ is connected and has dimension m. Moreover, the Hilbert–Chow morphism $\gamma\colon \mathcal{H}_{M^\lambda} \longrightarrow \mathbb{C}^n/S_n$ is finite and all fibers over points in $\overline{O(\lambda)}/S_n$ are singletons. Finally,

$$\overline{O(\lambda)}/S_n \subseteq \mathrm{im}(\gamma) \subseteq \bigcup O(\mu)/S_n.$$

$$\begin{tikzcd}
\mathbb{C}^m \arrow[r, "\varphi_\lambda"] \arrow[d, two heads] & \overline{O(\lambda)}/S_n \\
{} \arrow[ur, dotted] &
\end{tikzcd}$$

Evidence for a positive answer to Question 3.6 is provided by Theorem 3.4, Corollary 3.5 in the cases $m = 1$ and $m = n$ where the proof is much simpler. In fact, for $m = 1$ and $m = n$ intersection $(p_1, p_2, \ldots, p_n)$.

Lemma 3.7. Let F_T^S be a higher Specht polynomial with T any Young tableau of shape

$$\Big(\sum_{\tau \in C(T)} \operatorname{sgn}(\tau)(\alpha \circ \tau) \circ \sigma \Big)(x$$

$$\Big(\sum_{\tau \in C(T)} \operatorname{sgn}(\alpha \circ \tau)(\alpha \circ \tau) \circ \sigma \Big)(x$$

$$\Big(\sum_{\tau \in C(T)} \operatorname{sgn}(\tau) \tau \circ \sigma \Big)(x$$

so $\operatorname{ev}(F_T^S) = \operatorname{ev}(\alpha(F_T^S)) = -\operatorname{ev}(F_T^S)$, hence $\operatorname{ev}(F_T^S) = 0$. □

. Instead, applying the Young symmetrizer $\sum_{\sigma \in R(T)} \sum_{\tau \in C(T)} \operatorname{sgn}(\tau) \tau \circ \sigma$ to *any* polynomial results in a multiple

$d(\lambda) \coloneqq \sum_{i=1}^n (i-1)\lambda_i$. The multiplicity of S^λ in $P_{d(\lambda)}$ is 1, and it is generated by the Specht

$$P \times P \longrightarrow P, \quad \langle f, g \rangle \coloneqq f(\partial_1, \ldots, \partial_n)(g(x_1, \ldots, x_n)).$$

filled in such a way that the integers $l_1 + \ldots + l_{i-1} + 1, \ldots, l_1 + \ldots + l_i$ all appear exactly once in T_i. We write $F_{\mathbf{T}} \coloneqq \prod_{i=1}^{k} F_{T_i}$. Lemma 3.7 generalizes to this setting, i.e., every higher

similarly as Lemma 3.7.

words, there is a surjective map $l : [m] \to [k]$ such that $l_i = \sum_{j \in l^{-1}(i)} \lambda_j$. Denote by $\mu^i \vdash l_i$ the partition given by those parts of λ indexed by $l^{-1}(i)$. We now denote by $q = q_1, q_2, \ldots, q_s$ the

$$V \coloneqq \left(S^{\mu^1} \otimes 1 \otimes \cdots \otimes 1\right) \oplus \cdots \oplus \left(1 \otimes 1 \otimes \cdots \otimes S^{\mu^k}\right) \subseteq P,$$

all direct summands embedded in the smallest possible degrees $d(\mu^1), \ldots, d(\mu^k)$, i.e., V is

summands into V.) For any $f \in P$ we can write $f = f^1 + \ldots + f^k + g$, where f^i is the sum

and only if $\langle f^i, S^{\mu^i} \rangle = 0$ for all $i = 1, \ldots, k$, where S^{μ^i} is understood again as the span of all

$$\big) \otimes \cdots \otimes \big(\mathbb{C}[x_{n-l_k+1}, \ldots, x_n]/I$$

ideals corresponding to the partition $\mu^i \vdash l_i$. In particular, $P/I' \cong M^{\mu^1} \otimes \cdots \otimes M^{\mu^k}$ as G-

be the translate of I' whose only support point is q; this translation is G-equivariant. We claim $I_q \subseteq I'_q$. For this, we first observe that the canonical map $V \to P/I_q$ is injective, i.e., no direct summand of V is contained in I_q. The reason is that the restriction of the Chinese remainder isomorphism above to $S^\lambda \subseteq P_{d(\lambda)}$ is injective. Now, $S^\lambda \subseteq P_{d(\lambda)}$ is

of the corresponding Specht polynomials of the μ^i. In particular, any Specht polynomial for μ^i is non-zero in P/I_{q_i}, hence indeed V injects into P/I_q. To see that $I_q \subseteq I$
first translate both ideals G-equivariantly into the origin. Note that V still injects into P/I_0 because the Specht polynomials generating V are not affected by this translation. Now

without loss of generality that $\langle f^1, S^{\mu^1} \rangle \neq 0$. Multiplying f by some monomial h appearing

of f^1 does not annihilate S^{μ^1}. Hence the G-submodule of I_0 generated by f contains an element f' which spans a G-submodule isomorphic to $S^{\mu^1} \otimes 1 \otimes \cdots \otimes 1$ such that the degree $d(\mu^1)$ part of f' is a Specht polynomial in S^{μ^1}. Moreover, f' does not have any non-zero homogeneous components of smaller degrees because $P_{<d(\mu^1)}$ does not contain any copy of $S^{\mu^1} \otimes 1 \otimes \cdots \otimes 1$. Therefore, we may assume $f' = a + b$ with $a \in S^{\mu^1} \subseteq P_{d(\mu^1)}$ a Specht polynomial and $b \in P_{>d(\mu^1)}$. As f' spans a G-module isomorphic to $S^{\mu^1} \otimes 1 \otimes \cdots \otimes 1$, the polynomial b lies in the corresponding isotypic component of P and is divisible by a by Lemma 3.7. Thus inductively, since $\mathfrak{m}^N \subseteq I_0$ for some N, we obtain $a \in I_0$. Therefore, the canonical map $V \to P/I_0$ has a non-trivial kernel, contradicting what we said above. So indeed, $I_q \subseteq I'_q$.

$$\binom{l_1}{\mu^1} \cdots \Big($$

$$= \qquad \Big) \cdots \binom{l_k}{\mu^k},$$

□

Theorem 3.11 ([Tan82, GP92]). Let $\lambda \vdash n$ and let λ' be its transpose. For $S \subseteq [n]$ we

$r_\lambda(|S|) \coloneqq |S| - n + 1 + \sum_{i=1}^{n-|S|} \lambda$

A short computation shows that $|S| \geq r_\lambda(|S|)$ is equivalent to $|S| \geq n - \lambda_1 + 1$. Under this

$e_r(x_S) - x_i e_{r-1}(x_{S \setminus i})$ for $i \in S$ shows that I_λ is in fact generated already by the full elementary

the vector space of all S_n-equivariant P-module homomorphisms $I_\lambda \to P/I_\lambda$. Since I_λ lies

$r_\lambda(n - \lambda_2 - 1) = \cdots = r_\lambda(n - \lambda_1 + 1)$. We abbreviate $e_r = e_r(x_{[n]})$. We will now repeatedly use the relation $e_r(x_{S \setminus i}) = e_r(x_S) - x_i e_{r-1}(x_{S \setminus i})$ for all $i \in S$. First we can observe that

$$e_2(x_{[n-1]}) = e_2 - x_n e_1(x_{[n-1]}) = e_2 - x_n e_1 + x_n^2,$$

can see that $e_2(x_{[n-1]}) \in I$. Similarly, $e_r(x_{[n-1]}) \in I$ for all $r \geq 2 = r_\lambda(n-1)$. Inductively,

$$\begin{aligned} e_r(x_{[n-l]}) &= e_r(x_{[n-l] \cup n}) - x_n e_{r-1}(x_{[n-l]}) \\ &= e_r(x_{[n-l] \cup n}) - x_n e_{r-1}(x_{[n-l] \cup n}) + x_n^2 e_{r-2}(x_{[n-l]}), \end{aligned}$$

and for $r \geq r_\lambda(n-l) = l+1 = r_\lambda(n-l+1)+1$ all three terms lie in I. If $\lambda_1 \leq \lambda_2 + 1$, this implies $I_\lambda = I$. If $\lambda_1 \geq \lambda_2 + 2$, we continue as follows: For all l in the range $\lambda_1 - 1 \geq l \geq \lambda_2 + 1$, the partial elementary symmetric polynomials $e_{r_\lambda(|S|)}(x_S)$, $|S| = n - l$, all have the same degree $\lambda_2 + 1$,

generators in this range are redundant. In the end we obtain $I_\lambda = (p_1, x_i^2, S_n(x_1 x_2 \cdots x_{\lambda_2+1}))$. On the other hand, $P/I_\lambda \cong M^\lambda = \oplus_{0 \leq k \leq \lambda_2} S^{(n-k,k)}$ as follows from the combinatorial formula for the Kostka numbers. These two facts together give even more: If $\lambda_1 \geq \lambda_2 + 2$, what we have proven so far applied to the partition $\mu \coloneqq (\lambda_1 - 1, \lambda_2 + 1)$ shows further that I_λ is

$$\oplus_{0 \leq k \leq \lambda_2+1} S^{(n-k,k)} \cong P/I_\mu \twoheadrightarrow P/I_\lambda \cong \oplus_{0 \leq k \leq \lambda_2} S^{(n-k,k)}$$

has kernel S^μ, embedded in degree $d(\mu) = \lambda_2 + 1$. Moreover, each Specht module $S^{(n-k,k)}$ in P/I_λ is embedded in degree k, hence in degree $< \lambda_2 + 1$. Therefore, if I' denotes the

$P/I' \twoheadrightarrow P/I_\lambda$ which is an isomorphism in degrees $\leq \lambda_2 + 1$. But I_λ is generated in degrees $\leq \lambda_2 + 1$, so $I' = I_\lambda$.

Since M^λ does not contain any copy of S^μ, we deduce that a homomorphism $f \in \mathrm{Hom}_P^{S_n}(I_\lambda, P/I_\lambda)$ is determined already by its values $f(p_1) = \alpha \cdot 1$, $f(p_2) = \beta \cdot 1$ and $f(x_i^2 - x_j^2) = \gamma \cdot (x_i - x_j)$ for some $\alpha, \beta, \gamma \in \mathbb{C}$. Hence it suffices to show that α, β, γ must satisfy some non-trivial linear relation. If $\lambda_1 > \lambda_2$ we will show that necessarily $\beta = 0$

γ, finishing the proof. We start with $\lambda_1 > \lambda_2$. Since $x_1 x_2 \cdots x_{\lambda_2+1} \in I_\lambda$, we can consider its image under f. Write $f(x_1 x_2 \cdots x_{\lambda_2+1}) = f_0 + f_1 + \ldots + f_{\lambda_2}$

$x_1 x_2 \cdots x_{\lambda_2+1}$ and therefore maps to both $\frac{\beta}{\alpha} x_2 \cdots x_{\lambda_2+1}$ and $x_1 f(x_1 x_2 \cdots x_{\lambda_2+1})$ at the same

, we have the following relation: Define $g \coloneqq (x_1 - x_2)(x_3 - x_4) \cdots (x_{n-1} - x_n)$.

$$\frac{}{x_i - x_{i+1}} (x_i^2 - x_{i+1}^2).$$

$\square$

3.3. **Two ideal inclusions.** While exploring Tanisaki ideals we encountered the contain-

Hilbert schemes, we do include them as we have not found them in the literature.

$$\tilde{I}_\mu = (p_1, \ldots, p_{m-1}, S_n \cdot x_1^m, S_n \cdot (x_1 x_2 \cdots x_{R_k(\mu)})^k : 1 \le k \le m-1).$$

and any subset $I \subset [n]$ of cardinality $R_k(\mu)$ we have $\{a_i | i \in I\} \cap \{a_1 \ldots, a_k\} \neq \emptyset$ by the

Corollary 1].

Lemma 3.13. We have $(p_1, \ldots, p_n, \mathrm{sp}_T : \mathrm{sh}(T) \ntrianglerighteq \mu) \subseteq \tilde{I}_\mu \subseteq I_\mu$.

$$(\mathrm{sp}_T \,|\, \mathrm{sh}(T) \ntrianglerighteq \mu) \subseteq (S_n \cdot x_1^m, S_n \cdot (x_1 x_2 \cdots x_{R_k(\mu)})^k : 1 \le k \le m-1).$$

$$\prod_{\substack{i<j\\ i,j\in C_k}} (x_i - x_j).$$

$$\cdots x^{1}_{|C_1|+|C_2|-2} \cdots x^{|C_l|-1}_{|C_1|+\ldots+|C_{l-1}|-l+2} \cdots x^{1}_{|C_1|+\ldots+|C_l|-l}.$$

$$\sum^{l}$$

$$\sum^{l}$$

$$(\lambda_{k+r} - \lambda_{k+r+1})r$$
$$= \lambda_{k+1} - \lambda_{k+2} + 2\lambda_{k+2} - 2\lambda_{k+3} + \ldots$$
$$= \lambda_{k+1} + \lambda_{k+2} + \lambda_{k+3} + \ldots$$

Since $\#\{j : (\lambda')_j \geq k\} = \lambda_k$ and because $\mu \ntrianglelefteq \lambda$, there is an integer s with $\mu_1 + \ldots + \mu_s > \lambda_1 + \ldots + \lambda_s$. However, this is equivalent to $n - (\mu_1 + \ldots + \mu_s) < n - (\lambda_1 + \ldots + \lambda_s)$ which in turn is equivalent to $\mu_m + \ldots + \mu_{s+1} + 1 < \lambda_{\mathrm{len}(\lambda)} + \ldots + \lambda_{s+1} + 1$. The left hand side equals $R_s(\mu)$

assigned different values.

$$
\begin{aligned}
&\oplus \langle p_1(x_i - x_j)\rangle \oplus \langle x_i^2 - x_j^2\rangle \\
&\oplus \langle (x_i - x_j)(x_k - x_l)\rangle \\
&= 2S^{(n)} \oplus 2S^{(n-1,1)} \oplus S^{(n-2,2)},
\end{aligned}
$$

$$
\oplus \langle p_1(x_i - x_j)\rangle \oplus \langle x_i^2 - x_j^2\rangle
$$

$$
\begin{aligned}
&\oplus \langle p_1^2(x_i - x_j)\rangle \oplus \langle p_2(x_i - x_j)\rangle \oplus \langle p_1(x_i^2 - x_j^2)\rangle \oplus \langle x_i^3 - x_j^3\rangle \\
&\oplus \langle p_1(x_i - x_j)(x_k - x_l)\rangle \oplus \langle (x_i + x_j + x_k + x_l)(x_i - x_j)(x_k - x_l)\rangle \\
&\oplus \langle (x_i - x_j)(x_i - x_k)(x_j - x_k)\rangle \\
&\oplus \langle (x_i - x_j)(x_k - x_l)(x_s - x_t)\rangle \\
&= 3S^{(n)} \oplus 4S^{(n-1,1)} \oplus 2S^{(n-2,2)} \oplus S^{(n-2,1,1)} \oplus S^{(n-3,3)},
\end{aligned}
$$

$$
\begin{aligned}
&\oplus \langle p_1^2(x_i - x_j)\rangle \oplus \langle p_2(x_i - x_j)\rangle \oplus \langle p_1(x_i^2 - x_j^2)\rangle \oplus \langle x_i^3 - x_j^3\rangle \\
&\oplus \langle p_1(x_i - x_j)(x_k - x_l)\rangle \oplus \langle (x_i + x_j + x_k + x_l)(x_i - x_j)(x_k - x_l)\rangle \\
&\oplus \langle (x_i - x_j)(x_i - x_k)(x_j - x_k)\rangle \\
&= 3S^{(5)} \oplus 4S^{(4,1)} \oplus 2S^{(3,2)} \oplus S^{(3,1,1)},
\end{aligned}
$$

$$
\begin{aligned}
&\oplus \langle p_1^2(x_i - x_j)\rangle \oplus \langle p_2(x_i - x_j)\rangle \oplus \langle p_1(x_i^2 - x_j^2)\rangle \oplus \langle x_i^3 - x_j^3\rangle \\
&\oplus \langle (x_i - x_j)(x_i - x_k)(x_j - x_k)\rangle \\
&= 3S^{(4)} \oplus 4S^{(3,1)} \oplus S^{(2,2)} \oplus S^{(2,1,1)},
\end{aligned}
$$

$$
\begin{aligned}
&\oplus \langle p_1^2(x_i - x_j)\rangle \oplus \langle p_2(x_i - x_j)\rangle \oplus \langle p_1(x_i^2 - x_j^2)\rangle \\
&\oplus \langle (x_1 - x_2)(x_1 - x_3)(x_2 - x_3)\rangle
\end{aligned}
$$

2	$(r-n+1)S^{(n)} \oplus S^{(n-1,1)}$	$(x_i - x_j)_{\geq 2} + \mathfrak{m}^{r-n+1}$ $(p_1(x_i - x_j), x_i^2 - x_j^2, (x_i - x_j)(x_k - x_l)) + \mathfrak{m}^{r-n}$	
3	$S^{(n)} \oplus S^{(n-1,1)}$		
4	$2S^{(n)} \oplus S^{(n-1,1)}$	$\mathfrak{m}^2$ $(p_1, x_i^2 - x_j^2, (x_i - x_j)(x_k - x_l)) + \mathfrak{m}^3$	
5	$3S^{(n)} \oplus S^{(n-1,1)}$	$(ap_1^2 + bp_2, p_1(x_i - x_j), x_i^2 - x_j^2, (x_i - x_j)(x_k - x_l)) + \mathfrak{m}^3$	$[-1:n]$
6	$S^{(n)} \oplus 2S^{(n-1,1)}$	$(p_1, p_2, (x_i - x_j)(x_k - x_l)) + \mathfrak{m}^3$	smooth
		$(p_1, (x_i - x_j)(x_k - x_l)) + \mathfrak{m}^3$ $(p_1, p_2, (x_i - x_j)(x_k - x_l)) + \mathfrak{m}^4$ $(p_1^2, p_2, ap_1(x_i - x_j) + b(x_i^2 - x_j^2), (x_i - x_j)(x_k - x_l)) + \mathfrak{m}^3$	
8	$S^{(5)} \oplus S^{(4,1)} \oplus S^{(3,2)}$		
9	$S^{(4)} \oplus S^{(3,1)} \oplus S^{(2,2)}$		
10	$2S^{(4)} \oplus S^{(3,1)} \oplus S^{(2,2)}$	$(p_1^2, p_2, p_1(x_i - x_j), x_i^2 - x_j^2) + \mathfrak{m}^3$	
11	$3S^{(4)} \oplus S^{(3,1)} \oplus S^{(2,2)}$	$(ap_1^2 + bp_2, p_1(x_i - x_j), x_i^2 - x_j^2) + \mathfrak{m}^3$	smooth
12	$S^{(3)} \oplus 2S^{(2,1)} \oplus S^{(1,1,1)}$	(p_1, p_2, p_3)	
13	$2S^{(3)} \oplus 2S^{(2,1)}$	$(p_1, p_2, (x_1 - x_2)(x_1 - x_3)(x_2 - x_3))$	smooth

metric ideals $I \subseteq P = \mathbb{C}[x_1, \ldots, x_n]$ for $n \geq 3$ and such that $r = \dim(P/I) \leq 2n$. Note that all unspecified indices in each appearing generator are meant to be *distinct* and run through

only occurring for $n \geq 4$ but not for $n = 3$. To avoid redundancy in the table, for row 2 one should take $r \geq n+3$. For example, for $r = n+2$ the first ideal of row 2 is the same as that of

Theorem 4.2. Let $n \geq 3$. Table 1 lists all homogeneous symmetric ideals I with $\dim_{\mathbb{C}} P/I =$

If both of p_2 and $x_i^2 - x_j^2$ do not lie in I, then necessarily $I = (p_1, (x_i - x_j)(x_k - x_l)) + \mathfrak{m}^3$

using Lemma 4.4 and observing that $(x_i - x_j)(x_i - x_k)(x_j - x_k) \in ((x_i - x_j)(x_k - x_l)) \subseteq I$ for

$$(x_1 - x_2)(x_1 - x_3)(x_2 - x_3) = (x_3 - x_1) \cdot (x_1 - x_2)(x_3 - x_4) + (x_1 - x_2) \cdot (x_1 - x_3)(x_2 - x_4).$$

For $n = 3$ there are two more possibilities here, namely rows 12 and 13. If both $p_2 \in I$ and $x_i^2 - x_j^2 \in I$, then $I = (p_1) + \mathfrak{m}^2$, row 3. Otherwise, $x_i^2 - x_j^2 \in I$ but $p_2 \notin I$. Applying the

$(P/I)_3$ does not contain any trivial representation. Moreover, $\dim(P/I)_{\le 2} = n+1$. However, $(x_1 - x_2)(x_1 - x_3)(x_2 - x_3) \in (x_i^2 - x_j^2)$ for all $n \ge 3$. Indeed, the formula

$$(x_1 - x_2)(x_1 - x_3)(x_2 - x_3) = x_1(x_3^2 - x_2^2) + x_2(x_1^2 - x_3^2) + x_3(x_2^2 - x_1^2)$$

$(p_1(x_i - x_j), x_i^2 - x_j^2, (x_i - x_j)(x_k - x_l)) \subseteq I$, implying $I_3 = P_3$, so we obtain the second ideal

Otherwise, still assuming $I_1 = \langle p_1 \rangle$, we have $n \in \{4, 5\}$ and $(x_i - x_j)(x_k - x_l) \notin I$. If $n = 5$,

again the Reynolds operator to $(x_1^2 - x_2^2)x_1$, we get $p_3 \in I$. Hence, if $p_2 \in I$, we get row 9. If

Secondly, we assume $I_1 = 0$, so $\dim(P/I)_{\le 1} = n + 1$. For $n \ge 5$ and $n = 3$, all irreducible S_n-representations except for the trivial and the alternating one have dimension $\ge n - 1$. Moreover, the least degree k such that P_k contains an alternating representation is $k = ($ which is $> n$ for all $n \ge 4$. Hence, for all $n \ge 5$, this forces that $(P/I)_{\ge 2}$ is either 0, a direct sum of only trivial representations or $S^{(n-1,1)}$, and in the last case (even for $n = 3, 4$) this forces $I_3 = P_3$ for dimension reasons, so we obtain the third ideal of row 7 for some $(a, b) \ne (0, 0)$. If $(P/I)_2 = 0$, then $I = \mathfrak{m}^2$, so we get the first ideal of row 4 (even for $n = 3, 4$). Otherwise, assume we have any $n \ge 3$ and that $(P/I)_{\ge 2}$ only consists of trivial two ideals of row 2 or the ideal of row 5 for some $(a,b) \neq (0,0)$ (the case $(a,b) = \lambda(-1,n)$

generated by all non-trivial representations in all degrees $2 \leq k \leq n-1$ and by $\mathfrak{m}^n$. Then,

dimension at most 1 for all $k \geq 3$. For this, observe that $p_1(x_i - x_j) \in I$ and $x_i^k - x_j^k \in I$ for all $k \geq 2$ for dimension reasons. Applying again the Reynolds operator to $p_d(x_1 - x_2)x_1$, $d \in \{1,2\}$, and to $(x_1^k - x_2^k)x_1$ gives scalar multiples of $np_dp_2 - p_dp_1^2$ and $np_{k+1} - p_kp_1$, respectively. The claim $\dim(P/J)_k \leq 1$ for all $k \geq 3$ easily follows from this. For dimension reasons, then, $J_k = (x_i - x_j)_k$ or $I_k = P_k$ for all $k \geq 3$. Hence, the same is true for I_k for all $k \geq 3$. This shows that we indeed get one of the three ideals in rows 2 and 5.

representation which is also different from $S^{(n-1,1)}$. For $n = 4$, then, we must have $S^{(2,2)}$ in $(P/I)_2$, i.e., $(x_i - x_j)(x_k - x_l) \notin I$. Hence, $\dim(P/I)_{\leq 2} \geq 7$. Moreover, both $p_1(x_i - x_j)$

alternating representation $S^{(1,1,1)}$ in degree 3, generated by $(x_1 - x_2)(x_1 - x_3)(x_2 - x_3)$. Then I contains $(p_1(x_i - x_j), x_i^2 - x_j^2)$ for dimension reasons. But we have already seen that then automatically $(x_1 - x_2)(x_1 - x_3)(x_2 - x_3) \in I$ as well, so this is impossible. $\square$

$$x_1^3 - x_2^3 \in (x_np_1(x_1 - x_2), p_1(x_1^2 - x_2^2), p_2(x_1 - x_2), (x_i - x_j)(x_k - x_l)),$$
$$p_2(x_1 - x_2), x_1^3 - x_2^3 \in (x_np_1(x_1 - x_2), x_i^2 - x_j^2, (x_i - x_j)(x_k - x_l)).$$

$$\begin{aligned} f &\coloneqq n(n-1)(x_1^3 - x_2^3) - (n^2 - 3n + 3)p_2(x_1 - x_2) \\ &\quad - (2n-3)p_1(x_1^2 - x_2^2) + n(n-2)x_np_1(x_1 - x_2), \\ g &\coloneqq (n-2)p_2(x_1 - x_2) - nx_np_1(x_1 - x_2) + nx_n(x_1^2 - x_2^2) \end{aligned}$$

computations. $\square$

Table 1, i.e., how to decide which ideals correspond to smooth points of $\mathcal{H}_\rho$. For all ideals

$$\begin{tikzcd} {} \arrow[dr, "\beta^*"'] & & {} \arrow[d, hook] \\ & {} & {} \end{tikzcd}$$

$$\operatorname{Hom}_P^{S_n}(I, P/I) \cong \ker(\beta^*) \cap \operatorname{Hom}_P^{S_n}(P \otimes N_1, P/I).$$

$$\ker(\beta^*) \cap \operatorname{Hom}_{S_n}(N_1, \rho),$$

$$N_1 = \langle x_i^2 - x_j^2 \rangle \oplus \langle p_1(x_i - x_j) \rangle \oplus \langle (x_i - x_j)(x_k - x_l) \rangle \oplus \langle p_1^2 - np_2 \rangle \oplus \langle p_1^d \rangle.$$

Moreover, $P/I = dS^{(n)} \oplus S^{(n-1,1)} = \langle 1 \rangle \oplus \langle p_1 \rangle \oplus \cdots \oplus \langle p_1^{d-1} \rangle \oplus \langle x_i - x_j \rangle$. Hence, any $f \in$

$$f((x_i - x_j)(x_k - x_l)) = 0,$$
$$f(x_i^2 - x_j^2) = \alpha(x_i - x_j),$$
$$f(p_1(x_i - x_j)) = \beta(x_i - x_j),$$
$$\gamma_i p_1^i, \quad {}_{i=0}$$
$$\delta_i p_1^i. \quad {}_{i=0}$$

If this f lies in $\mathrm{Hom}_P^{S_n}(I, P/I)$, then necessarily $\gamma_0 = \gamma_1 = \cdots = \gamma_{d-2} = 0$. Indeed, applying

$x_2) \cdot p_1^{d-1} = 0 \pmod I$ and to $(x_1 - x_2) \sum_{i=0}^{d-1} \delta_i p_1^i = \delta_0(x_1 - x_2) \pmod I$, hence $\delta_0 = 0$.

are among $\delta_1, \ldots, \delta_{d-1}$. By what we have explained above, such a relation arises from a

which is non-zero in P/I. If $\deg(g) \geq 2$, then after rescaling we have $g = p_1^k \pmod I$ for some $2 \leq k \leq d-1$. But such a syzygy does not exist since no power p_1^k (including $k = 0$

this gives precisely the relation we have already used above to show $\delta_0 = 0$. This proves $\dim_{\mathbb{C}}(\mathrm{Hom}_P^{S_n}(I, P/I)) = d + 2$.

dimension $d+1$. For this, let $a, b, c_2, c_3, \ldots, c_d \in \mathbb{C}$ such that $a \neq b$ and $c_2, c_3, \ldots, c_d$ are all

of $(b, a, a, \ldots, a)$ and the $d-1$ diagonal points $(c_2, \ldots, c_2), \ldots, (c_d, \ldots, c_d)$. The product

polynomials $(x_i - x_j)(x_i + x_j - (a+b))$, $(x_i - x_j)(p_1 - ((n-1)a + b))$ and $(x_i - x_j)(x_k - x_l)$. Fixing $c_2, \ldots, c_d$ and letting (a, b) tend to $(0, 0)$ linearly, i.e., along the family (ta, tb), we obtain the ideal $J_{(c_2,\ldots,c_d)} = ((p_1) + \mathfrak{m}^2) \cap \mathfrak{m}_{(c_2,\ldots,c_2)} \cap \cdots \cap \mathfrak{m}_{(c_d,\ldots,c_d)}$. For all distinct $c_2, \ldots, c_d$,

$$f(x_i^2 - x_j^2) = \gamma(x_i - x_j),$$

But necessarily $\alpha = 0$ since f maps $p_1^2(x_i - x_j)$ to both $\alpha(x_i - x_j)$ and $\delta p_1(x_i - x_j) = 0 \pmod{I}$.

the vanishing ideal J of two distinct diagonal points $(c, \dots, c), (d, \dots, d)$ and the S_n-orbit of $(b, a, a, \dots, a)$ for $a \neq b$ where we assume $(n-1)a + b = nc$. Then, clearly, $(p_1 - nc)(p_1 - nd) \in J$

$$p \coloneqq x_1 x_2 (x_1 - x_2) p_1 - (x_1^2 - x_2^2) p_2 + (x_1 - x_2) p_3 \in ((x_i - x_j)(x_k - x_l)),$$

$$p \coloneqq n(n-1)(x_1^3 - x_2^3) - (n^2 - 3n + 3)p_2(x_1 - x_2)$$
$$- (2n-3)p_1(x_1^2 - x_2^2) + n(n-2)x_n p_1(x_1 - x_2) \in ((x_i - x_j)(x_k - x_l)).$$

$$+ n(n-2)\alpha x_n(x_1 - x_2) - (2n-3)(x_1^2 - x_2^2)\alpha.$$

$$g \coloneqq n(n-2)x_n(x_1 - x_2) - (2n-3)(x_1^2 - x_2^2) \in I.$$

Row 7(b). $I = (p_1, p_2, (x_i - x_j)(x_k - x_l)) + \mathfrak{m}^4$ for $n \geq 4$. Here, $\mathfrak{m}^4$ is already contained in $(p_1, p_2, (x_i - x_j)(x_k - x_l))$, so these generators form a minimal S_n-stable basis of N_1. Moreover, $\rho = P/I = \langle 1 \rangle \oplus \langle p_3 \rangle \oplus \langle x_i - x_j \rangle \oplus \langle x_i^2 - x_j^2 \rangle = 2S^{(n)} \oplus 2S^{(n-1,1)}$. Hence, any

Row 7(c). $I = (p_1^2, p_2, ap_1(x_i - x_j) + b(x_i^2 - x_j^2), (x_i - x_j)(x_k - x_l)) + \mathfrak{m}^3$. As for *Row 5* above,

$$p_3(x_1-x_2)-p_2(x_1^2-x_2^2)+p_1x_1x_2(x_1-x_2)\in((x_i-x_j)(x_k-x_l)).$$

$$((n-1)a+b)^2g'(a,a,b,a,\dots,a)+((n-1)a^2+b^2)g''(a,a,b,a,\dots,a)=0$$

for all $a,b\in\mathbb{C}$, and all analogous equations where b is at the i-th position with $3\le i\le n$. Hence, in all these $n-2$ equations all coefficients of a^3,a^2b,ab^2,b^3, which are linear forms in the coefficients of g' and g'', must vanish. A computation shows that the only way in which this is possible is that $g'=\mu(x_1-x_2)$ and $g''=\nu(x_1-x_2)$ for some $\mu,\nu\in\mathbb{C}$. Coming back to $p_1(x_1-x_2)g+p_1^2g'+p_2g''$, applying the transposition $(1,2)$ to this polynomial, adding the result to it and evaluating on $(b,a,a,\dots,a)$ shows that $g=\lambda(x_1+x_2)$ for some $\lambda\in\mathbb{C}$.

must vanish on $(b,a,a,\dots,a)$ for all $a,b\in\mathbb{C}$. It can be checked that this is only possible if $\lambda=\mu=\nu=0$. In particular, we obtain $g=0$, so a g with the desired properties does not exist.

$$N_1 = \langle p_1^2, p_2, -2p_1(x_i - x_j) + n(x_i^2 - x_j^2), (x_i - x_j)(x_k - x_l)\rangle.$$

Moreover, $\rho = P/I = \langle 1\rangle \oplus \langle p_1\rangle \oplus \langle x_i - x_j\rangle \oplus \langle p_1(x_i - x_j)\rangle = 2S^{(n)} \oplus 2S^{(n-1,1)}$. Hence, any

$$f(-2p_1(x_i - x_j) + n(x_i^2 - x_j^2)) = \gamma_0(x_i - x_j) + \gamma_1 p_1(x_i - x_j),$$

$$\begin{aligned}
p &\coloneqq (n-2)p_1^2(x_1 - x_2) - n(n-2)p_2(x_1 - x_2) \\
&\quad - (n-1)(-2p_1(x_1 - x_2) + n(x_1^2 - x_2^2))(x_1 - x_2) \\
&\quad + 2(n-1)(-2p_1(x_1 - x_3) + n(x_1^2 - x_3^2))(x_1 - x_2) \in ((x_i - x_j)(x_k - x_l)).
\end{aligned}$$

$$\begin{aligned}
0 = f(p) &= ((n-2)\alpha_0 - n(n-2)\beta_0)(x_1 - x_2) + ((n-2)\alpha_1 - n(n-2)\beta_1)p_1(x_1 - x_2) \\
&\quad - (n-1)\gamma_0(x_1 - x_2)^2 + 2(n-1)\gamma_0(x_1 - x_2)(x_1 - x_3).
\end{aligned}$$

$$-2p_1(x_1 - x_2) + n(x_1^2 - x_2^2) - (n-2)(b-a)(x_1 - x_2)$$

$\square$

$$N_1 = \langle x_i^2 - x_j^2 \rangle \oplus \langle p_1(x_i - x_j) \rangle \oplus \langle (x_i - x_j)(x_k - x_l) \rangle \oplus \langle p_1^2 - np_2 \rangle \oplus \langle p_1^d \rangle$$

$$(x_i - x_j)(x_k - x_l),$$

$$c_i p_1^i,$$

$$c_k p_1^k \Big) (x_1 - x_2) - (p_1(x_1 - x_2) + a_1(x_1 - x_2)) \left(\sum^{d-1} c_{k+1} p_1^k \right)$$

$$\sum^{d-1} c_{k+1} p_1^k (x_1 - x_2)$$

$$\equiv -b_1 p_1^2 - (a_1 b_1 + b_2) p_1 - a_1 b_2 \pmod{J}.$$

ideal (over $\mathbb{Q}$). They are $\mathfrak{p}_1 \coloneqq (a_1, b_1, a_2 - b_2)$ and $\mathfrak{p}_2 \coloneqq (b_2 + a_1 + (n-1)a_2, a_1^2 + (n-2)a_1 a_2 + a_2 b_1)$.

$$N_1 = \langle p_1^2 \rangle \oplus \langle p_2 \rangle \oplus \langle p_1(x_i - x_j) \rangle \oplus \langle (x_i - x_j)(x_k - x_l) \rangle \oplus \langle p_3 \rangle$$

$$(x_i - x_j)(x_k - x_l),$$

$$(Z \cap H)_{\mathrm{red}} \hookrightarrow H \xrightarrow{\varphi} \mathrm{Hilb}$$

$\square$

$$\gamma\colon \mathcal{H}_\rho \longrightarrow \mathrm{Hilb}_l(\mathbb{C}^n/S_n)$$

$$\mathcal{H}ilb^{S_n}_{\mathbb{C}[S_n]^{\oplus l}}(\mathbb{C}^n)(T) \to \mathcal{H}ilb_l(\mathbb{C}^n/S_n)(T)$$

$$\mathcal{J} \otimes_{\mathcal{O}_T[p_1,\dots,p_n]} \mathcal{O}_T[x_1,\dots,x_n] \hookrightarrow \mathcal{O}_T[x_1,\dots,x_n].$$

$$(\mathcal{O}_T[p_1,\dots,p_n]/\mathcal{J}) \otimes_{\mathcal{O}_T[p_1,\dots,p_n]} \mathcal{O}_T[x_1,\dots,x_n]$$

$$\mathcal{I}^{S_n} \otimes_{\mathcal{O}_T[p_1,\dots,p_n]} \mathcal{O}_T[x_1,\dots,x_n] \subseteq \mathcal{I}.$$

Question 5.4. Is it true that for every S_n-representation ρ there exists k_0 such that for all

etry. 2