Color-Kinematic Numerators for Fermion Compton Amplitudes

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Abstract: We introduce a novel approach to compute Compton amplitudes involving a fermion pair inspired by Hopf algebra amplitude constructions. This approach features a recursive relation employing quasi-shuffle sets, directly verifiable by massive factorization properties. We derive results for minimal gauge invariant color-kinematic numerators with physical massive poles using this method. We have also deduced a graphical method for deriving numerators that simplifies the numerator generation and eliminates redundancies, thus providing several computational advantages.

Keywords: Scattering Amplitudes, Color-Kinematic Numerators

Contents

1 Introduction

The Kawai-Lewellen-Tye string theory relations [1] combine gravity and Yang-Mills theories [2] and have led to the important discovery of a color and kinematics duality [3–5]; a cornerstone for efficient computation of amplitudes and motivation for examining core principles of quantum particle scattering [6].

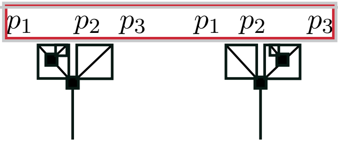
A considerable attraction over a decade has been finding a kinematic algebra that mimics that of the color group [7–9], and recently, there has been a breakthrough with the systematic framework developed by refs. [10–12]. This framework uses infinite-dimensional combinatorics [13, 14] and a direct numerator construction approach connected to quasi-shuffle Hopf algebras. The numerator technology developed this way is universal across many theories, including those with a spinless massive field, such as the heavy-mass effective field theory refs. [12, 15–20], scalar Yang-Mills theory with finite mass [21, 22], the $\alpha' F^3 + \alpha'^2 F^4$ higher-derivative interaction theory [23] (see also refs. [24–34]), and the $\mathrm{DF}^2 + \mathrm{YM}$ theory [35] (see as well refs. [36–38]).

This paper studies the kinematic algebra for the gauge invariant color-kinematic numerator associated with the generalized massive Compton amplitude with a fermion pair. Although this amplitude is considerably more complex than the spinless massive field case (see also refs. [39–42]), we show the realization of the algebra as a closed-form quasi-shuffle product. The amplitudes discussed are of theoretical interest as they allow for structural insight into fermion factorization limits and are useful when generating inspiration for bootstrap methods for general spin amplitudes [43, 44]. We thus hope that progress in this area could potentially help develop computational methods for spinning black hole merger observables, for instance, using the modern amplitude techniques pioneered in refs. such as [20, 45–51].

We present our work in this paper as follows. First, we introduce the kinematic Hopf algebra in section 2 and define a new evaluation map for massive Compton amplitudes with fermion pairs. Next, we demonstrate the validity of this formalism through computation and employ factorization properties to outline a proof of the construction. In section 3, we provide an evaluation map that simplifies numerators using the on-shell condition of the external massive fermion states. This graphical method streamlines numerator generation and thereby delivers computational benefits. Finally, in the final section, we provide an outlook for future research.

2 Color-kinematic fermion pair numerators

This section will deduce and demonstrate our algebraic construction for numerators. We will start by setting up a convenient and compact notation. A key element is an n-point massive fermion amplitude representation in terms of minimal color-kinematic numerators with $(n-3)!$ elements involving nested commutators. As an illustration of this, we consider ordered cubic vertex diagrams associated with gluon lines, $1, 2,$ and 3 connected through a propagator to gluon legs $4, 5, \ldots, n-2$ and two massive scalars $\overline{n-1}, \overline{n}$.



The two possible propagators respectively are $p_{12}^2 p_{123}^2$ and $p_{23}^2 p_{123}^2$, (defining $p_{ij} \equiv p_i + p_j$ and $p_{ij\ldots k} \equiv p_i + p_j + \ldots + p_k$) and it is easy to see that the color-kinematic duality holds writing out the Jacobi identity for numerators.

$$A_f(1,2,3,\overline{n-1},\overline{n}) = \frac{\mathcal{N}_f([[1,2],3],\overline{n-1},\overline{n})}{p_{12}^2 p_{123}^2} + \frac{\mathcal{N}_f([1,[2,3]],\overline{n-1},\overline{n})}{p_{23}^2 p_{123}^2} = \sum_{\Gamma\in\rho(3)} \frac{\mathcal{N}_f(\Gamma,\overline{n-1},\overline{n})}{d_\Gamma}, \tag{2.1}$$

where $\Gamma \in \rho(3) \equiv \{[[1,2],3],[1,[2,3]]\}$ and d_Γ are massless propagators associated with numerators. This notation readily extends to any number of legs as follows

$$A_f(1\ldots n-2,\overline{n-1},\overline{n}) = \sum_{\Gamma\in\rho(n-2)} \frac{\mathcal{N}_f(\Gamma,\overline{n-1},\overline{n})}{d_\Gamma}, \tag{2.2}$$

where now Γ sums over the possible $(n-3)!$ configurations of ordered nested numerators associated with the $n-2$ gluon legs. We can also compute graviton amplitudes using the double-copy. In this case, we write the n point amplitude as

$$M_f(1\ldots n-2,\overline{n-1},\overline{n}) = \sum_{\Gamma\in\tilde{\rho}(n-2)} \frac{\mathcal{N}_f(\Gamma,\overline{n-1},\overline{n})\mathcal{N}(\Gamma,\overline{n},\overline{n-1})}{d_\Gamma} . \tag{2.3}$$

We denote sums over configurations by $\tilde{\rho}(n-2)$, where the 'tilde' indicates that the sum includes all possible nested unordered commutator configurations associated with gravitons labels $\{1,\ldots\, n-2\}$.

Using the gravity momentum $\mathcal{S}$-kernel [52–56], we know that we can always generate generic color-kinematic numerators using the following prescription [9, 53]

$$\mathcal{N}_f([[[[\cdots[1,2],3],\ldots],n-2],\overline{n-1},\overline{n}) = \sum_{\beta\in S_{n-3}} \mathcal{S}[1\ldots n-2|1\beta]A(1\beta,\overline{n-1},\overline{n}) . \tag{2.4}$$

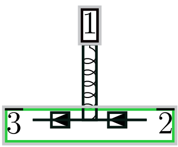
In the sum, S_{n-3} denotes permutations of gluon indices $\{2,3,\ldots,n-2\}$ and we work with the momentum $\mathcal{S}$-kernel definition employed in [13]. We will, for compactness in the following, denote the left-nested commutator of gluons as $[12\ldots n-2]$, *i.e.*, $[1234] \equiv [[[1,2],3],4]$. (For more details, see also refs. [57, 58]). When considering factorization, the following recursive property of the gravity momentum $\mathcal{S}$-kernel is ideal

$$\mathcal{S}[1\ldots j|1\beta_L j\beta_R] = 2(p_{\Theta_L(j)}\cdot p_j)\,\mathcal{S}[1\ldots j-1|1\beta_L\beta_R] . \tag{2.5}$$

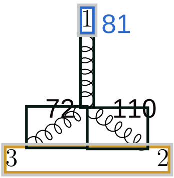
In this context, the symbol $\Theta_L(j)$ denotes the indices that come before the index j in the set $\{1,\beta_L\}$, with an index order that is lower than that of j. With these conventions, the expression $\mathcal{S}[1\alpha|1\beta]$ only depends on the massless momentum and cancels out all the massless propagators in the amplitude, although not manifestly.

2.1 Numerator examples at four and five points

Before diving into the kinematic algebra, we will explore a manifestly gauge-invariant double-copy numerator form of the quark-gluon Compton amplitude derived from Feynman rules. We have adopted the following convention for the three-point amplitudes



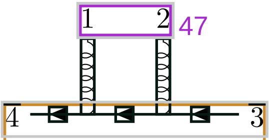
$= \bar{v}\not{\varepsilon}_1 u\,,$



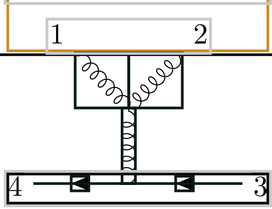
$$= (\varepsilon_1\cdot\varepsilon_3)(p_1\cdot\varepsilon_2) + (\varepsilon_2\cdot\varepsilon_3)(p_3\cdot\varepsilon_1) + (\varepsilon_1\cdot\varepsilon_2)(p_2\cdot\varepsilon_3)\,, \tag{2.6}$$

where $\not{\varepsilon} \equiv \gamma^\mu\varepsilon_\mu$, for a given gluon polarization vector, $\bar{v}$, and u denote the incoming and outgoing massive fermions states with on-shell conditions $\bar{v}(\not{p}_3 - m) = 0\,, (\not{p}_2 +$

$m)u = 0$, and mass m. The external momenta are considered outgoing. Our focus is on the first non-trivial case, the four-point quark-gluon amplitude, which is featured by two contributing color-ordered Feynman diagrams:



$$= \frac{1}{4(p_{14}^2 - m^2)} (\bar{v} \not{\varepsilon}_1 (\not{p}_{14} + m) \not{\varepsilon}_2 u) \,, \tag{2.7}$$



$$= -\frac{1}{2p_{12}^2} \Big[(p_2 \cdot \varepsilon_1) (\bar{v} \not{\varepsilon}_2 u) - (p_1 \cdot \varepsilon_2) (\bar{v} \not{\varepsilon}_1 u) + \frac{1}{2} (\varepsilon_1 \cdot \varepsilon_2) (\bar{v} (\not{p}_1 - \not{p}_2) u) \Big]$$

The four-point KLT momentum kernel matrix as indicated by eq. (2.5) provides a double-copy numerator of the form (we use $\mathcal{S}[12|12] = 2(p_1 \cdot p_2)$)

$$\mathcal{N}_f([12], \overline{3}, \overline{4}) = 2(p_1 \cdot p_2) A_f(12, \overline{3}, \overline{4}) = \frac{1}{2(p_{14}^2 - m^2)} \bar{v} \Big(p_1 \cdot p_2 \not{\varepsilon}_1 (\not{p}_{14} + m) \not{\varepsilon}_2 - (p_1 \cdot p_4) \Big[2\,(p_2 \cdot \varepsilon_1) \not{\varepsilon}_2 - 2\,(p_1 \cdot \varepsilon_2) \not{\varepsilon}_1 + (\varepsilon_1 \cdot \varepsilon_2)\,(\not{p}_1 - \not{p}_2) \Big] \Big) u \,. \tag{2.8}$$

After simplification, this leads to a compact formulation of the double-copy numerator, expressed in terms of the field strength tensor:

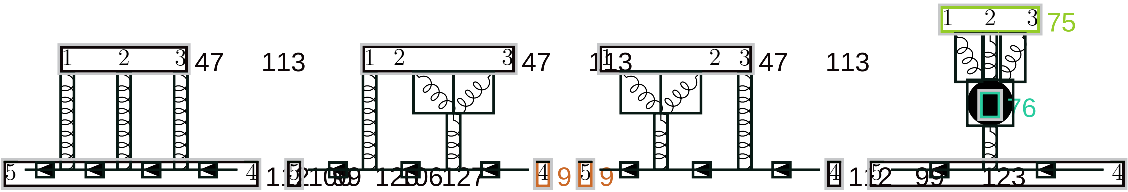
$$\mathcal{N}_f([12], \overline{3}, \overline{4}) = -\frac{1}{p_{41}^2 - m^2} \Big(\bar{v} \bullet (p_4 \cdot F_{12}) \bullet v + \frac{1}{4} \bar{u} \bullet F_1 \bullet (p_1 \cdot F_2) \bullet u \Big) \,. \tag{2.9}$$

In this context, we use $F_i^{\mu\nu}$ to denote the Abelian field strength tensor, so that $F_i^{\mu\nu} \equiv p_i^\mu \varepsilon_i^\nu - \varepsilon_i^\mu p_i^\nu$. We represent the Dirac matrix product with a bullet symbol $\bullet$, and we employ the simplified notation as follows:

$$\begin{aligned} \bar{v} \bullet X \bullet (p_1 \cdot F_2) \bullet Y \bullet u &\equiv (\bar{v} \bullet X \bullet \gamma_\mu \bullet Y \bullet u)\,(p_1 \cdot F_2)^\mu \,, \\ \bar{v} \bullet X \bullet F_{i_1 \ldots i_r} \bullet Y \bullet u &\equiv (\bar{v} \bullet X \bullet \gamma_\mu \bullet \gamma_\nu \bullet Y \bullet u)\, F_{i_1 \ldots i_r}^{\mu\nu} \,. \end{aligned} \tag{2.10}$$

Here $F_{i_1 i_2 \ldots i_r}^{\mu\nu} \equiv (F_{i_1} \cdot F_{i_2} \cdots F_{i_r})^{\mu\nu}$ denotes the contraction of a string of multiple field strength tensors.

Next, we turn our attention to the five-point quark-gluon amplitude, which we can compute from the four diagram topologies:



Building upon eq. (2.4), we derive the five-point double-copy numerator as follows:

$$\mathcal{N}_f([123], 4, 5) = \mathcal{S}[123|123]\, A_f(123, 4, 5) + \mathcal{S}[123|132]\, A_f(132, 4, 5) \,, \tag{2.11}$$

where the elements of the KLT momentum kernel, as outlined in eq. (2.5), are $\mathcal{S}[123|123] = 4(p_1{\cdot}p_2)\,(p_{12}{\cdot}p_3)$ and $\mathcal{S}[123|132] = 4(p_1{\cdot}p_2)\,(p_1{\cdot}p_3)$. This leads to manifestly gauge invariant and local double-copy five-point numerator

$$\mathcal{N}_f([123],\overline{4},\overline{5}) = \bar{v}\bullet\Bigg[\frac{(p_5{\cdot}F_{123}) + \frac{1}{4}F_1\bullet(p_1{\cdot}F_{23})}{p_{51}^2-m^2} - \frac{(p_5{\cdot}F_{12}) + \frac{1}{4}F_1\bullet(p_1{\cdot}F_2)}{p_{51}^2-m^2}\bullet\frac{2(p_{12}{\cdot}F_3{\cdot}p_5)+(p_{12}{\cdot}F_3)\bullet p_3}{p_{512}^2-m^2} - \frac{(p_5{\cdot}F_{13}) + \frac{1}{4}F_1\bullet(p_1{\cdot}F_3)}{p_{51}^2-m^2}\bullet\frac{2(p_{13}{\cdot}F_2{\cdot}p_5)+(p_{13}{\cdot}F_2)\bullet p_2}{p_{513}^2-m^2}\Bigg]\bullet u\,. \qquad (2.12)$$

Systematically obtaining such expressions for higher point amplitudes is challenging due to the intricate process of eliminating massless poles in the double-copy numerator and rewriting in terms of strings of field strength tensors. In the following section, we will work out an algebraic construction of numerators using Hopf algebra, which offers a direct and systematic universal approach.

2.2 Direct algebraic construction of numerators

We will now use Hopf algebra as an inspiration to simplify the construction of numerators and avoid the complexities of the direct computation approach. We will use pre-numerators, which follow the logic of [11]. We note that pre-numerators are not uniquely determined and still have residual freedom. However, we can eliminate it by working in a framework where we equate the left-nested numerator $\mathcal{N}_f([12\ldots n{-}2],\overline{n{-}1},\overline{n})$ with the pre-numerator $\mathcal{N}_f(12\ldots n{-}2,\overline{n{-}1},\overline{n})$ and fix all numerators $\mathcal{N}_f(\ldots 1\ldots,\overline{n{-}1},\overline{n})$ to be zero.[1]. We next compute the pre-numerator using the fusion product

$$\mathcal{N}_f(12\ldots n{-}2,\overline{n{-}1},\overline{n}) = \langle\!\langle T_{(1)}\star T_{(2)}\star\cdots\star T_{(n-2)}\rangle\!\rangle\,, \qquad (2.13)$$

where we label the generators *e.g.* $T_{(i)}$ and $T_{(\tau_1),(\tau_2),\ldots,(\tau_r)}$ and the subscripts $\tau_i\subset\{1,\ldots,n{-}2\}$ represent an ordered partition of a subset of the external gluon indices. The evaluation map denoted by $\langle\!\langle\ldots\rangle\!\rangle$ above provides the physical gauge-invariant object that appears in the color-kinematic numerator. We can work out the fusion products by introducing quasi-shuffle operator products given by:

$$T_{(1)}\star T_{(2)} = -T_{(12)} + T_{(1),(2)} + T_{(2),(1)}$$

$$\ldots$$

$$T_{(\tau_1),\ldots,(\tau_r)}\star T_{(j)} = -\sum_{i=1}^{r} T_{(\tau_1),\ldots,(\tau_i j),\ldots,(\tau_r)} + \sum_{\sigma\in\{\tau_1,\ldots,\tau_r\}\sqcup j}\Big(T_{(\sigma_1),\ldots,(\sigma_r),(\sigma_{r+1})}\Big)\,. \qquad (2.14)$$

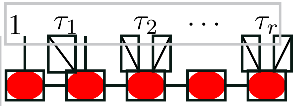
[1] We remark that working directly with crossing symmetric numerators will lead to the same result.

leading to the following explicit result for the four-point numerator:

$$\mathcal{N}_f(12,3,4) = \langle\!\langle T_{(1)} \star T_{(2)} \rangle\!\rangle = -\langle\!\langle T_{(12)} \rangle\!\rangle + \langle\!\langle T_{(1),(2)} \rangle\!\rangle + \langle\!\langle T_{(2),(1)} \rangle\!\rangle \,. \tag{2.15}$$

We now turn our focus to the evaluation that maps the abstract generators $T_{(\tau_1),(\tau_2),\ldots,(\tau_r)}$ to gauge invariant functions. For our massive fermion amplitudes, it reads

$$\langle\!\langle T_{(1\tau_1),(\tau_2),\ldots,(\tau_r)} \rangle\!\rangle = \Bigg\langle\!\!\Bigg\langle \quad \Bigg\rangle\!\!\Bigg\rangle \tag{2.16}$$



$$= \bar{v} \bullet \frac{H_{1\tau_1}}{p_{n1}^2 - m^2} \bullet \frac{p_{n1\tau_1} - m}{p_{n1\tau_1}^2 - m^2} \bullet (p_{\Theta_L(\tau_2)} \cdot F_{\tau_2}) \bullet \cdots \bullet \frac{p_{n1\tau_1\cdots\tau_{r-1}} - m}{p_{n1\tau_1\cdots\tau_{r-1}}^2 - m^2} \bullet (p_{\Theta_L(\tau_r)} \cdot F_{\tau_r}) \bullet u \,,$$

where we have $\langle\!\langle T_{(1),\ldots} \rangle\!\rangle = \langle\!\langle T_{\ldots,(1\tau_i),\ldots} \rangle\!\rangle = 0$ and

$$H_{1\tau} \equiv p_n \cdot F_{1\tau} + \frac{1}{4} F_1 \bullet (p_1 \cdot F_\tau) \,. \tag{2.17}$$

The symbol $\Theta_L(\tau_r)$ denotes all indices in $(1\tau_1) \cup (\tau_2) \cup \cdots \cup (\tau_{r-1})$ that are smaller than the first element in τ_r. We can write a closed-form color-kinematic numerator as follows:

$$\mathcal{N}_f(12\ldots n{-}2, \overline{n{-}1}, \overline{n}) = \sum_{\text{ordered partitions}} (-1)^{n+r} \langle\!\langle T_{(1\tau_1),(\tau_2),\ldots,(\tau_r)} \rangle\!\rangle \,, \tag{2.18}$$

where sum is over ordered partitions of the set $\{2,3,...,n{-}2\}$ into r nonempty subsets[2]. We can now expand this sum as follows.

$$\begin{aligned} \mathcal{N}_f(1\ldots n{-}2, \overline{n{-}1}, \overline{n}) &= (-1)^{n-3}\, \bar{v} \bullet \frac{H_{1\ldots n-2}}{p_{n1}^2 - m^2} \bullet u \\ &+ \sum_{\tau_R \subset \{2,\ldots,n-2\}} (-1)^{|\tau_R|}\, \bar{v} \bullet \left[\frac{\mathcal{X}\big(1\rho(\tau_R)\big) \bullet (p_{n1\rho(\tau_R)} - m) \bullet (p_{\Theta_L(\tau_R)} \cdot F_{\tau_R})}{p_{n1\rho(\tau_R)}^2 - m^2} \right] \bullet u \,, \end{aligned} \tag{2.19}$$

where we have defined $|\tau_R|$ as the number of elements in the set τ_R, and $\rho(\tau_R)$ as the complement legs in $\{2,\ldots,n{-}2\}$. $\mathcal{X}(1\rho(\tau_R))$ represents a particular Dirac matrix product of H functions (defined above) and massive propagators for the set of gluon legs $\{1\rho(\tau_R)\}$ that correspond to a color-kinematic numerator that is stripped of incoming and outgoing external fermion states. From this, we introduce a factorization on an internal massive leg denoted by I and give the expression,

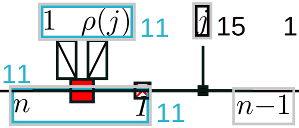
$$\begin{aligned} \mathcal{N}_f(1\ldots n{-}2, \overline{n{-}1}, \overline{n}) &= (-1)^{n-3}\, \bar{v} \bullet \frac{H_{1\ldots n-2}}{p_{n1}^2 - m^2} \bullet u \\ &+ \sum_{\tau_R \subset \{2,\ldots,n-2\}} \sum_{S_I} (-1)^{|\tau_R|}\, \mathcal{N}_f(1\rho(\tau_R), \overline{I}, \overline{n})\, \bar{v}_I \bullet \frac{(p_{\Theta_L(\tau_R)} \cdot F_{\tau_R})}{p_{n1\rho(\tau_R)}^2 - m^2} \bullet u \,. \end{aligned}$$

[2] More examples of color-kinematic numerators and a `Mathematica` package can be found at *KiHA5.0* GitHub repository [59].

where S_I denotes the possible spin states of I and where we have used in the last step that $\bar{v} \bullet \mathcal{X}(1\rho(\tau_R)) \bullet u_I \equiv \mathcal{N}_f(1\rho(\tau_R), \overline{I}, \overline{n})$.

To prove the recursive form shown in eq. (2.19), we start by considering the factorization behavior on the massive propagator that involves only one massless particle j on the right-hand side. We then take the collinear limit, in legs j and $n{-}1$,

$$\mathcal{N}_f(1\ldots n{-}2, \overline{n{-}1}, \overline{n}) \longrightarrow$$



$$= \sum_{S_I} \sum_{\beta_L \in S_{n-4}} \mathcal{S}[1\rho(j)|1\beta_L]\, A(1\beta_L, \overline{I}, \overline{n})\, (p_{\Theta_L(j)}{\cdot}p_j)\, A(j, \overline{n{-}1}, \overline{I})$$
$$= \sum_{S_I} \mathcal{N}_f(1\rho(j), \overline{I}, \overline{n})\, (p_{\Theta_L(j)}{\cdot}p_j)\, \mathcal{N}_f(j, \overline{n{-}1}, \overline{I})\,. \tag{2.20}$$

We have in the above equation utilized eq. (2.4) and eq. (2.5). Using the completeness relation for Dirac spinors and explicitly writing the expression for the fermionic propagator, the sum over intermediate states yields the following after evaluating the three-point numerator on the right side of the cut:

$$\mathcal{N}_f(1\ldots n{-}2, n{-}1, \overline{n}) \to (p_{\Theta_L(j)}{\cdot}p_j)\, \bar{v} \bullet \mathcal{X}(1\rho(j)) \bullet (p_{n1\rho(j)}{-}m) \bullet \varepsilon_j \bullet u\,. \tag{2.21}$$

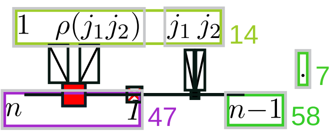
We can now deduce a manifest gauge invariant part of the numerator that replicates this factorization behavior.

$$\mathcal{N}^{(1)}(1\rho(j)|j, \overline{n{-}1}, \overline{n}) = -\frac{\bar{v} \bullet \mathcal{X}(1\rho(j)) \bullet (p_{n1\rho(j)} - m) \bullet (p_{\Theta_L(j)}{\cdot}F_j) \bullet u}{p^2_{n1\rho(j)} - m^2}\,, \tag{2.22}$$

where $\rho(j)|j$ indicates that the particle to the right of $\rho(j)$ is j. Subtracting all the possible one-leg contributions from the complete color-kinematic numerator, we obtain a remainder quantity,

$$\mathcal{N}_f(1\ldots n{-}2, \overline{n{-}1}, \overline{n}) - \sum_j \mathcal{N}^{(1)}(1\rho(j)|j, \overline{n{-}1}, \overline{n})\,. \tag{2.23}$$

By construction, it should also manifestly gauge invariant, free of any massive poles, and contain $n{-}3$ massless legs on the left-hand side (since the pieces we have subtracted contain the complete factorization behavior for those configurations). Next, we consider a massive cut containing two massless legs on the right-hand side



(2.24)

Then the remainder eq. (2.23) in this case tends to

$$(p_{\Theta_L(j_1)}{\cdot}p_{j_1})\bar{v} \bullet \mathcal{X}(1\rho(j_1j_2)) \bullet (p_{n1\rho(j_1j_2)}{-}m) \bullet X(j_1j_2) \bullet u\,. \tag{2.25}$$

where $\rho(j_1 j_2)$ is the complement of j_1, j_2 in the set $1, 2, \ldots, n-2$. The symbol $X(j_1 j_2)$, whose exact form is unimportant at this point, is a function of the polarizations and momenta of the external gluon legs associated with labelings j_1, j_2.

Now, subtracting the pieces that contribute to further poles in $X(j_1 j_2)$ yields a rank one tensor structure to the right of the propagator $(p_{n1\rho(j_1 j_2)} - m)$. Indeed, any higher rank tensor had to be generated by a Feynman diagram containing additional fermionic propagators, and the only way of canceling the pole would be by using the Dirac matrix algebra, which in turn lowers the tensor rank. To determine the gauge invariant tensor structure, one can fit the factorization behavior in the heavy-mass regime by parameterizing $p_n = m\, v$, $p_{n-1} = -m\, v - q$ and taking $m \to \infty$. Then the corresponding remainder quantity yields [13]

$$\mathcal{N}(1\rho(j_1 j_2), v) \times p_{\Theta_L(j_1)} \cdot F_{j_1 j_2} \cdot v\,. \tag{2.26}$$

Comparing the heavy mass limit of eq. (2.23) with eq. (2.26), the piece of the numerator that contributes exclusively to the two-particle cut is found to be:

$$\mathcal{N}^{(2)}(1\rho(j_1 j_2)|j_1 j_2, \overline{n-1}, \overline{n}) = \frac{\bar{v} \bullet \mathcal{X}(1\rho(j_1 j_2)) \bullet (p_{n1\rho(j_1 j_2)} - m) \bullet (p_{\Theta_L(j_1)} \cdot F_{j_1 j_2}) \bullet u}{p^2_{n1\rho(j_1 j_2)} - m^2}\,. \tag{2.27}$$

Subtracting these new contributions to the color-kinematic numerator results in an object that is free of one- and two-leg poles:

$$\mathcal{N}_f(1\ldots n-2, \overline{n-1}, \overline{n}) - \sum_j \mathcal{N}^{(1)}(1\rho(j)|j, \overline{n-1}, \overline{n}) - \sum_{j_1<j_2} \mathcal{N}^{(2)}(1\rho(j_1 j_2)|j_1 j_2, \overline{n-1}, \overline{n})\,. \tag{2.28}$$

The procedure can be repeated recursively for cuts containing any number $i < n-3$ of legs on the right-hand side:

$$\mathcal{N}^{(i)}(1\rho(\tau_R)|\tau_R, \overline{n-1}, \overline{n}) = (-1)^i \frac{\bar{v} \bullet \mathcal{X}(1\rho(\tau_R)) \bullet (p_{n1\rho(\tau_R)} - m) \bullet (p_{\Theta_L(\tau_R)} \cdot F_{\tau_R}) \bullet u}{p^2_{n1\rho(\tau_R)} - m^2}\,. \tag{2.29}$$

For the last cut with only leg one on the left, the expression contains an additional term,

$$\begin{aligned} \mathcal{N}^{(n-3)}(1|2\ldots n-2, \overline{n-1}, \overline{n}) &= (-1)^{n-3}\ \bar{v} \bullet \frac{(p_n \cdot F_{12\ldots n-2}) + \frac{1}{4} F_1 \bullet (p_1 \cdot F_{2\ldots n-2})}{p^2_{n1} - m^2} \bullet u \\ &= (-1)^{n-3}\ \bar{v} \bullet \frac{H_{12\ldots n-2}}{p^2_{n1} - m^2} \bullet u\,. \end{aligned} \tag{2.30}$$

By repeating our previous argument, we see that subtracting all these contributions,

$$\mathcal{N}_f(1\ldots n-2, \overline{n-1}, \overline{n}) - \mathcal{N}^{(1)} - \mathcal{N}^{(2)} - \cdots - \mathcal{N}^{(n-3)}\,, \tag{2.31}$$

leads to an object that contains no poles, *i.e.*, a polynomial piece. Through the use of dimensional analysis, it becomes apparent that a color-kinematic numerator must include the momenta p_i raised to an overall power of 2 and the polarization vector ε_i raised to an overall power of 3. It has been proven that no polynomial expression, which is manifestly gauge invariant, can satisfy these power requirements. This implies that the remainder must be equal to zero. Therefore, the recursive form presented in eq. (2.19) is valid, and as a result, the evaluation map in eq. (2.16) is correct.

3 An alternative evaluation map for generating simpler color-kinematic numerators

The previous section's evaluation map did not use the massive on-shell state condition. This section explores using this auxiliary constraint to provide an alternative graphical evaluation map. This map removes expression redundancies and further simplifies the intricate Dirac matrix product in the numerators. We will start by considering numerator examples at four, five, and six points, which will form a basis for universal and more straightforward graphical rules for evaluating numerator expressions.

3.1 Four-point color-kinematic numerator

Our first example is the four-point color-kinematic numerator. Here, we get directly from the algebraic construction

$$\mathcal{N}_f(12,\overline{3},\overline{4}) = \langle\!\langle T_{(12)} \rangle\!\rangle = \bar{v} \bullet \frac{H_{12}}{p_{41}^2 - m^2} \bullet u\,, \tag{3.1}$$

and an expression, which uses the on-shell condition and $H_{12} = p_4 \cdot F_{12} + \frac{1}{4} F_1 \bullet (p_1 \cdot F_2)$ from the general definition eq. (2.17).

3.2 Five-point color-kinematic numerator

At five-point, the algebraic construction yields:

$$\begin{aligned}\mathcal{N}_f(123,4,5) &= \langle\!\langle T_{(123)} - T_{(12),(3)} - T_{(13),(2)} \rangle\!\rangle \\ &= \bar{v} \bullet \Bigg[\frac{H_{123}}{p_{51}^2 - m^2} - \frac{H_{12}}{p_{51}^2 - m^2} \bullet \frac{p_{512} - m}{p_{512}^2 - m^2} \bullet (p_{12} \cdot F_3) \\ &\qquad - \frac{H_{13}}{p_{51}^2 - m^2} \bullet \frac{p_{513} - m}{p_{513}^2 - m^2} \bullet (p_1 \cdot F_2) \Bigg] \bullet u\end{aligned}$$

$$= \bar{v} \bullet \left[\frac{H_{123}}{p_{51}^2 - m^2} - \frac{H_{12}}{p_{51}^2 - m^2} \bullet \frac{2(p_{12}{\cdot}F_3{\cdot}p_5) + (p_{12}{\cdot}F_3) \bullet p_3}{p_{512}^2 - m^2} - \frac{H_{13}}{p_{51}^2 - m^2} \bullet \frac{2(p_{13}{\cdot}F_2{\cdot}p_5) + (p_{13}{\cdot}F_2) \bullet p_2}{p_{513}^2 - m^2} \right] \bullet u\,. \tag{3.2}$$

In the last step, we use the Clifford algebra and the on-shell condition to reduce the rank of the two tensors to a scalar product and obtain a simpler tensor rank. The H functions are

$$\begin{aligned} H_{12} &= p_5{\cdot}F_{12} + \frac{1}{4} F_1 \bullet (p_1{\cdot}F_2)\,, \\ H_{13} &= p_5{\cdot}F_{13} + \frac{1}{4} F_1 \bullet (p_1{\cdot}F_3)\,, \\ H_{123} &= p_5{\cdot}F_{123} + \frac{1}{4} F_1 \bullet (p_1{\cdot}F_{23})\,. \end{aligned} \tag{3.3}$$

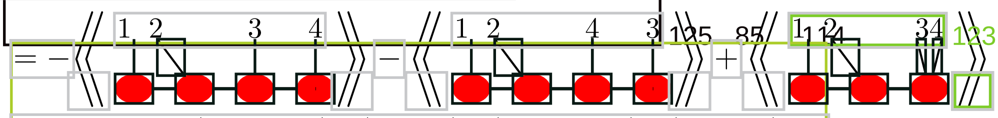
3.3 Six-point color-kinematic numerator

Finally, we consider the six-point color-kinematic numerator. Our starting point is the algebraic construction of the numerator, which reads,

$$\begin{aligned} \mathcal{N}_f(1234, \bar{5}, \bar{6}) = \langle\!\langle &-T_{(12),(3),(4)} - T_{(12),(4),(3)} + T_{(12),(34)} - T_{(13),(2),(4)} \\ &- T_{(13),(4),(2)} + T_{(13),(24)} - T_{(14),(2),(3)} - T_{(14),(3),(2)} \\ &+ T_{(14),(23)} + T_{(123),(4)} + T_{(124),(3)} + T_{(134),(2)} - T_{(1234)} \rangle\!\rangle\,. \end{aligned} \tag{3.4}$$

$T_{(1234)}$ is unchanged. $T_{(123),(4)}, T_{(124),(3)}, T_{(134),(2)}$ are simplified as in the five point case. There are new non-trivial cancelations for the terms associated with the evaluations of $\langle\!\langle -T_{(12),(3),(4)} - T_{(12),(4),(3)} + T_{(12),(34)} \rangle\!\rangle$, $\langle\!\langle -T_{(13),(2),(4)} - T_{(13),(4),(2)} + T_{(13),(24)} \rangle\!\rangle$ and $\langle\!\langle -T_{(14),(2),(3)} - T_{(14),(3),(2)} + T_{(14),(23)} \rangle\!\rangle$. Writing out the first one of these terms, we get:

$$\langle\!\langle -T_{(12),(3),(4)} - T_{(12),(4),(3)} + T_{(12),(34)} \rangle\!\rangle$$



$$\begin{aligned} = &- \frac{\bar{v} \bullet H_{12} \bullet (p_{612} - m) \bullet (p_{12}{\cdot}F_3) \bullet (p_{6123} - m) \bullet (p_{123}{\cdot}F_4) \bullet u}{(p_{61}^2 - m^2)(p_{612}^2 - m^2)(p_{6123}^2 - m^2)} \\ &- \frac{\bar{v} \bullet H_{12} \bullet (p_{612} - m) \bullet (p_{12}{\cdot}F_4) \bullet (p_{6124} - m) \bullet (p_{12}{\cdot}F_3) \bullet u}{(p_{61}^2 - m^2)(p_{612}^2 - m^2)(p_{6124}^2 - m^2)} \\ &+ \frac{\bar{v} \bullet H_{12} \bullet (p_{612} - m) \bullet (p_{12}{\cdot}F_3{\cdot}F_4) \bullet u}{(p_{61}^2 - m^2)(p_{612}^2 - m^2)}\,, \end{aligned} \tag{3.5}$$

where

$$H_{12} = p_6{\cdot}F_{12} + \frac{1}{4} F_1 \bullet (p_1{\cdot}F_2) \tag{3.6}$$

Terms involving two propagators yield

$$\bar{v}\bullet\frac{H_{12}}{p_{61}^2-m^2}\bullet\left[\frac{(p_{12}\cdot F_3)\bullet(p_{123}\cdot F_4)}{(p_{612}^2-m^2)}+\frac{(p_{12}\cdot F_4)\bullet(p_{12}\cdot F_3)}{(p_{612}^2-m^2)}+\frac{(p_{612}-m)\bullet(p_{12}\cdot F_{34})}{(p_{612}^2-m^2)}\right]\bullet u$$
$$=\bar{v}\bullet\frac{H_{12}}{p_{61}^2-m^2}\bullet\left[\frac{(p_{12}\cdot F_3)\bullet(p_{123}\cdot F_4)}{(p_{612}^2-m^2)}+\frac{(p_{12}\cdot F_4)\bullet(p_{12}\cdot F_3)}{(p_{612}^2-m^2)}+\frac{(p_{12}\cdot F_{34})\bullet p_{34}}{(p_{612}^2-m^2)}\right.$$
$$\left.+\frac{2\,(p_{12}\cdot F_{34}\cdot p_{612})}{(p_{612}^2-m^2)}\right]\bullet u\,. \tag{3.7}$$

The terms contained within the square brackets on the right-hand side we rephrase further as follows:

$$\frac{2\,(p_{12}\cdot F_3\cdot F_4\cdot p_{612})}{(p_{612}^2-m^2)}+\frac{(p_{12}\cdot F_3)\bullet(p_3\cdot F_4)}{(p_{612}^2-m^2)}+\frac{(p_{12}\cdot F_3\cdot F_4)\bullet p_{34}}{(p_{612}^2-m^2)}$$
$$+\frac{(p_{12}\cdot F_3)\bullet(p_{12}\cdot F_4)}{(p_{612}^2-m^2)}+\frac{(p_{12}\cdot F_4)\bullet(p_{12}\cdot F_3)}{(p_{612}^2-m^2)}\,, \tag{3.8}$$

and reexpress by employing the identity

$$\frac{(p_{12}\cdot F_3)\bullet(p_{12}\cdot F_4)}{(p_{612}^2-m^2)}+\frac{(p_{12}\cdot F_4)\bullet(p_{12}\cdot F_3)}{(p_{612}^2-m^2)}=-\frac{2\,(p_{12}\cdot F_3\cdot F_4\cdot p_{12})}{(p_{612}^2-m^2)}\,. \tag{3.9}$$

It leads to the following simplified expression:

$$\bar{v}\bullet\frac{H_{12}}{p_{61}^2-m^2}\bullet\left[\frac{2\,(p_{12}\cdot F_3\cdot F_4\cdot p_6)}{(p_{612}^2-m^2)}+\frac{(p_{12}\cdot F_3)\bullet(p_3\cdot F_4)}{(p_{612}^2-m^2)}+\frac{(p_{12}\cdot F_3\cdot F_4)\bullet p_{34}}{(p_{612}^2-m^2)}\right]\bullet u\,. \tag{3.10}$$

From this expression, we make the following observations. It is manifest that, in the simplified form, all the mass-dependent contributions are gone in the numerator, and the massive momentum-dependent terms only appear in the scalar part. Fully accounting for the on-shell conditions allows us a much-simplified expression.

It follows from above examples, that the function $H_{1\tau}$ in the evaluation map of the first component of the algebraic generator is unchanged. Other components are reduced to

$$H_{j\tau}(p_\rho,p_\omega)=2p_\rho\cdot F_{j\tau}\cdot p_\omega+\sum_{\sigma_1\in\tau}\Big(p_\rho\cdot F_{j\sigma_1}\Big)\bullet\Big(p_{\Theta'_L(\sigma_2)}\cdot F_{\sigma_2}\Big)\,,\ \text{if } j>1\,. \tag{3.11}$$

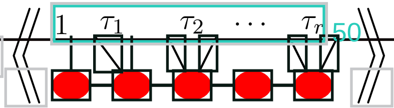
The expression involves the summation of all possible orderings of τ, denoted as $\sigma_1\sigma_2\equiv\tau$, here, $\Theta'_L(\sigma_2)$ represents all the indices in $j\sigma_1$ that come before the index of σ_2 in the canonical ordering. When σ_2 is an empty set, Θ'_L includes every index in $j\tau$. In this case, F_{σ_2} reduces to an identity matrix. For example

$$H_j(p_\rho,p_\omega)=2p_\rho\cdot F_{j\tau}\cdot p_\omega+\Big(p_\rho\cdot F_j\Big)\bullet p_j$$
$$H_{jk}(p_\rho,p_\omega)=2p_\rho\cdot F_{jk}\cdot p_\omega+\Big(p_\rho\cdot F_{jk}\Big)\bullet p_{jk}+\Big(p_\rho\cdot F_j\Big)\bullet\Big(p_j\cdot F_k\Big) \tag{3.12}$$

3.4 General color-kinematic numerator

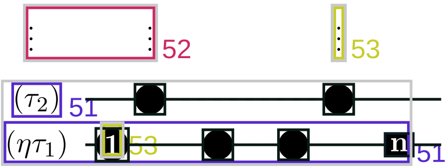
Armed with the understanding from the above examples, we will now introduce a more general and universal formalism. We have the following simplified evaluation map for numerators:

$$\langle\!\langle T_{(1\tau_1),(\tau_2),\ldots,(\tau_r)} \rangle\!\rangle \equiv$$



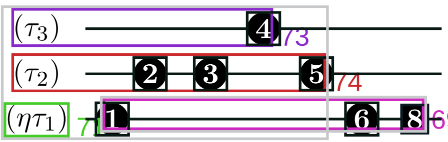
$$= \bar{v} \bullet \frac{H_{1\tau_1}}{p^2_{n1} - m^2} \bullet \frac{H_{\tau_2}\big(p_{\Theta_L(\tau_2)}, p_{\Theta_R(\tau_2)}\big)}{p^2_{n1\tau_1} - m^2} \bullet \cdots \bullet \frac{H_{\tau_r}\big(p_{\Theta_L(\tau_r)}, p_{\Theta_R(\tau_r)}\big)}{p^2_{n1\tau_1\ldots\tau_{r-1}} - m^2} \bullet u \,. \qquad (3.13)$$

We obtain this map by analyzing the color-kinematic numerator pattern observed in the above examples provided by eq. (2.16) using the additional on-shell conditions. We will also use the convenient musical diagrams [13] to ease the computation of numerators. The musical diagrams take the form



In a given diagram, the reference leg indices are located at the bottom line, and we sum all possible positions of the non-reference legs. For a given diagram, each horizontal line contains an argument of an H function required for the corresponding numerator. (We use the notation $\Theta_{L/R}(\tau_i)$ that refers to the set of all the left-lower/right-lower indices of τ_i. This definition of $\Theta_L(\tau_i)$ is consistent with the Θ_L function in the definition of gravity KLT momentum kernel used in eq. (2.5)).

Below, we illustrate using musical diagrams to evaluate abstract generator terms. As an example, we can consider $\langle\!\langle T_{(16),(235),(4)} \rangle\!\rangle$, which appears in the eight-point color-kinematic numerator.



Reading the arguments of the H functions (by following the indices of the various horizontal lines), we arrive at the contributions

$$\langle\!\langle T_{(16),(235),(4)} \rangle\!\rangle = \bar{v} \bullet \frac{H_{16}}{p^2_{81} - m^2} \bullet \frac{H_{235}(p_1, p_{68})}{p^2_{816} - m^2} \bullet \frac{H_4(p_{123}, p_{568})}{p^2_{816235} - m^2} \bullet u \,, \qquad (3.14)$$

where the H_{235} function reads:

$$H_{235}(p_1, p_{68}) = 2(p_1{\cdot}F_{235}{\cdot}p_{68}) + (p_1{\cdot}F_{25}) \bullet (p_2{\cdot}F_3)$$

$$+\,(p_1{\cdot}F_{235})\bullet p_{235}+(p_1{\cdot}F_{23})\bullet(p_{23}{\cdot}F_5)+(p_1{\cdot}F_2)\bullet(p_2{\cdot}F_{35})\,, \tag{3.15}$$

as well as the H_{16} function,

$$H_{16}=\big(p_8{\cdot}F_{16}\big)+\frac{1}{4}F_1\bullet\big(p_1{\cdot}F_6\big)\,, \tag{3.16}$$

and H_4,

$$H_4(p_{123},p_{568})=2\big(p_{123}{\cdot}F_4{\cdot}p_{568}\big)+\big(p_{123}{\cdot}F_4\big)\bullet p_4\,. \tag{3.17}$$

We conclude this section by presenting the following compact expression for numerators based on the above rules for evaluation:

$$\begin{aligned}\mathcal{N}_f(12\ldots n{-}2,\overline{n{-}1},\overline{n}) &= (-1)^{n-3}\;\bar{v}\bullet\Big[\frac{H_{1\ldots n-2}}{p_{n1}^2-m^2}\Big]\bullet u\\ &+\sum_{\tau_R\subset\{2,\ldots,n-2\}}(-1)^{|\tau_R|}\;\bar{v}\bullet\Big[\frac{\mathcal{X}(1\rho(\tau_R))\bullet H_{\tau_R}(p_{\Theta_L(\tau_R)},p_{\Theta_R(\tau_R)})}{p_{n1\tau_L}^2-m^2}\Big]\bullet u\,.\end{aligned} \tag{3.18}$$

We have explicitly checked the compact numerator expressions provided by this formula up to eight points. We also note that in all cases the fermion color-kinematic numerator tends to the scalar Yang-Mills case in [21] after the replacement $\bar{v}\gamma^\mu u\to p_n^\mu$ and $\bar{v}\gamma^{\mu_1}\gamma^{\mu_2}\cdots\gamma^{\mu_r}u\to 0$ as expected for consistency.

4 Conclusion and Outlook

Using a universal method, we have presented a new and compact construction for a color-kinematic numerator associated with the generalized massive Compton amplitude with a fermion pair. We have verified evaluation maps explicitly until eight points and also presented a more straightforward graphical method for computing such numerators.

As the formulation relies on the Lorentz group representations of the matter fields, one can apply similar techniques to miscellaneous Compton scattering amplitudes, for instance, those with two massive vectors or other massive particles. An exciting track is to investigate if it is possible to shed new light on gauge and gravity interactions in high-spin matter fields at finite spin order. Studying the classical limit for such amplitudes and resolving questions related to consistent factorization limits could potentially aid the development of more efficient computational methods for observing spinning black hole mergers. It is beyond this paper's scope, so we leave these and related questions for future studies.

5 Acknowledgements

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