Exercise 5: Variance reduction methods

1. Estimate the integral $\int_0^1 e^x dx$ by simulation (the crude Monte Carlo estimator). Use eg. an estimator based on 100 samples and present the result as the point estimator and a confidence interval.
2. Estimate the integral $\int_0^1 e^x dx$ using antithetic variables, with comparable computer ressources.
3. Estimate the integral $\int_0^1 e^x dx$ using a control variable, with comparable computer ressources.
4. Estimate the integral $\int_0^1 e^x dx$ using stratified sampling, with comparable computer ressources.
5. Use control variates to reduce the variance of the estimator in exercise 4 (Poisson arrivals).
6. Demonstrate the effect of using common random numbers in exercise 4 for the difference between Poisson arrivals (Part 1) and a renewal process with hyperexponential interarrival times. **Remark:** You might need to do some thinking and some re-programming.

7. For a standard normal random variable $Z \sim \mathsf{N}(0, 1)$ using the crude Monte Carlo estimator estimate the probability $Z > a$. Then try importance sampling with a normal density with mean a and variance σ^2. For the expirements start using $\sigma^2 = 1$, use different values of a (e.g. 2 and 4), and different sample sizes. If time permits experiment with other values for σ^2. Finally discuss the efficiency of the methods.
8. Use importance sampling with $g(x) = \lambda \exp(-\lambda * x)$ to calculate the integral $\int_0^1 e^x \mathrm{d}x$ of Question 1. Try to find the optimal value of λ by calculating the variance of $h(X)f(X)/g(X)$ and verify by simulation. Note that importance sampling with the exponential distribution will not reduce the variance.
9. For the Pareto case derive the IS estimator for the mean using the first moment distribution as sampling distribution. Is the approach meaningful? and could this be done in general? With this insight could you change the choice of $g(x)$ in the previous question