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k-uniform tilings by regular polygons

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2 Abstract

k-uniform tilings by regular polygons are tilings with k equivalence classes of vertices with respect to the symmetries of the tiling. The enumeration of all k-uniform tilings for specific values of k is a far from trivial problem. In this paper, I review the most important steps in the investigation of k-uniform tilings: Kepler's enumeration of the 1-uniform tilings in 1619, the rediscovery of the 1-uniform tilings by Sommerville in 1905, the complete enumeration of the k-homogeneous tilings (a set of tilings which includes the 1-uniform and 2-uniform tilings) by Krötenheerdt in 1969–1970, Chavey's enumeration of the 3-uniform tilings in 1984, and Galebach's enumeration of the 4-uniform, 5-uniform and 6-uniform tilings in 2002–2003. I also discuss some approaches that might be used in future work on k-uniform tilings.

3 Sammanfattning på svenska

k-uniforma tesselleringar av regelbundna polygoner är tesselleringar med k vertexekvivalensklasser med avseende på tesselleringens symmetrier. Uppräkning av alla k-uniforma tesselleringar för specifika värden på k är ett långtifrån trivialt problem. I denna översikt beskriver jag de viktigaste stegen i utforskandet av k-uniforma tesselleringar: Keplers uppräkning av de 1-uniforma tesselleringarna 1619, återupptäckten av de 1-uniforma tesselleringarna av Sommerville 1905, den fullständiga uppräkningen av de k-homogena tesselleringarna (en mängd av tesselleringar som bland annat omfattar de 1- och 2-uniforma tesselleringarna) av Krötenheerdt 1969–1970, Chaveys uppräkning av de 3-uniforma tesselleringarna 1984, och Galebachs uppräkning av de 4-, 5- och 6-uniforma tesselleringarna 2002–2003. Jag diskuterar också några angreppssätt som kan användas i det fortsatta studiet av k-uniforma tesselleringar.

4 Introduction

Tilings are geometrical patterns that appear in many everyday situations. In practice, tilings are always finite, but mathematical tilings are, in general, in-finite. The act of tiling—that is filling a plane, space etc. with units of a few defined shapes—creates pavements, brick walls, mosaics, efficient storage of milk cartons, the familiar pattern on footballs and of course glazed tilings in bathrooms (see Figure 1). In science, tilings are important for the study of crystal structures (Conrad, Krumeich, Reich, and Harbrecht, 2000), and provide inspiration for synthetic organic chemistry (Bunz, Rubin, and Tobe, 1999). Complicated tilings are common in Islamic architecture, and in art, they form the foundation of the woodcuts of M.C. Escher (Grünbaum and Shephard, 1987, pp. 1–11).

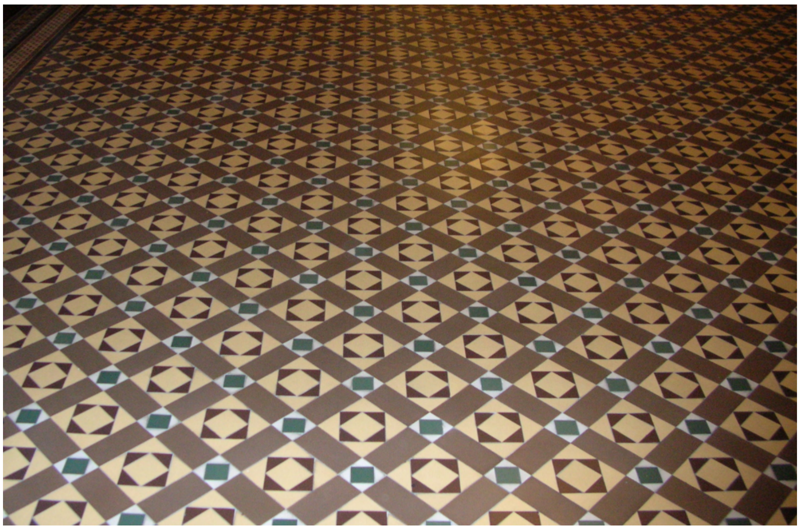

Figure 1: Tiling of a floor in a hallway in the Parliament Building of Quebec. Photo by the author.

In this review, I will describe the historical development of a specific class of tilings, namely k-uniform tilings by regular polygons. These are tilings of the Euclidean plane. The tiles, that is the basic units, of these tilings are equilateral triangles, squares, regular pentagons and so forth (we will see later that only a few of these shapes actually occur in k-uniform tilings). A precise definition of k-uniform is given in chapter 6, but roughly, the k-uniform tilings can be described as those tilings where there are k possible surroundings (in a global sense) around a vertex.

5 Historical overview

The systematic study of tilings of the plane by has a long history, stretching back at least to the ancient Greeks. Part of that history is the study of k-uniform tilings by regular polygons.

The enumeration of k-uniform tilings has been somewhat sporadic. Ancient Greek mathematicians such as Pappus of Alexandria (4th century AD) were aware of the *regular* or *Platonic* tilings, where all tiles are identical regular polygons (Chavey, 1989). There are three such tilings: by equilateral triangles, by squares and by regular hexagons. They form a subset of the 1-uniform tilings, which were listed for the first time by Kepler (1619/1997) in his work Harmonice Mundi (Grünbaum et al., 1987, p. 110; Chavey, 1989). However, his accomplishment was forgotten, and not completely rediscovered until the work of Sommerville in 1905 (Grünbaum et al., 1987, p. 110). Harmonice Mundi was even printed without diagrams in one edition (Grünbaum et al., p. 13), showing how little emphasis was put on this aspect of Kepler's work.

After the 1-uniform tilings had been completely investigated by Sommerville, not much of interest happened until Krötenheerdt listed the 2-uniform tilings (1969). Chavey (1984a) extended Krötenheerdt's work to list the 3-uniform tilings, and Grünbaum and Shephard provided a comprehensive survey of the field of tilings in their book Tilings and Patterns (1987). The 4-, 5- and 6-uniform tilings have been found by computer and published on the Web by Galebach (2009), but his method is not described, and no explicit proof that his lists are complete seems to be available.

Since the mid-1980's, a new approach to the study of tilings can be found in the work of Dress and associates (see for example Dress, Huson, and Molnár, 1993; Huson, 1993; and Delgado-Friedrichs, 2003) who have developed a graph-theoretical way of describing tilings, the Delaney–Dress symbol. This approach has yet to be applied to k-uniform tilings by regular polygons (Delgado-Friedrichs, personal communication, September 22, 2009) but will probably enable new perspectives when it does.

6 Some basic notions

There is little standardized terminology in the field of tilings, so I will go into some detail in defining some basic notions of tiling theory. The definitions are stated for the Euclidean plane, but can be easily generalized to other manifolds (the sphere, hyperbolic planes) or dimensions. They are in some aspects more restrictive than the most general definitions that appear in the literature (for example Grünbaum et al., 1987), but I will not consider more general tilings in this paper.

- A *tile* is a simply connected topological disk in the Euclidean plane. See Figure 2.
- An *edge* is a subset of the intersection of two tiles that is connected, contains more than one point, and cannot be enlarged by the addition of new points in the intersection. See Figure 2.
- A *vertex* is a point that is common to three (or more) tiles. See Figure 2.
- A *tiling* is a subdivision of the Euclidean plane such that all points belong to at least one tile, the number of congruence classes of tiles are finite, and the interior points of tiles are pairwise disjoint.

The tilings I consider will be *periodic*, meaning that they have translation symmetries in at least two different directions (Chavey 1984a, p. 4).

This paper is mostly concerned with tilings where the tiles are polygons. To avoid confusion, I will follow Grünbaum and Shephard (1987) in calling the vertices of polygons *corners* and the edges of polygons *sides* when there is a

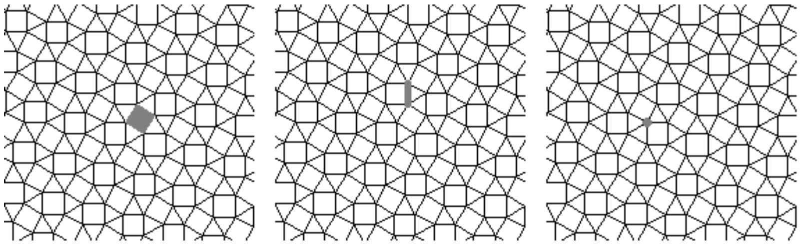

Figure 2: Left to right: A tile, an edge and a vertex marked in a tiling.

need to distinguish them from the vertices and edges of tilings. When dealing with non-convex polygons, I also consider the points where sides intersect to be corners. Tilings by polygons are said to be *edge-to-edge* when the vertices are corners of all the incident polygons (see Figure 3).

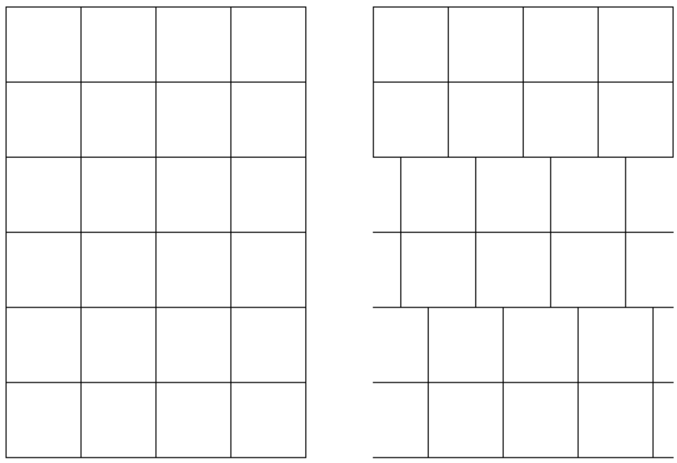

Figure 3: An edge-to-edge tiling by squares (left) and a tiling by squares that is not edge-to-edge.

I furthermore restrict my attention to tilings of *regular polygons*, that is polygons where all angles have equal size and all sides have the same length. As commonly done in the study of tilings by regular polygons, I will call triangles *trigons* and quadrilaterals *tetragons*, and assume that polygons are regular unless otherwise stated. The reader can also assume that the tilings discussed are tilings by regular, non-convex polygons, unless explicitly stated otherwise. It can be noted that in edge-to-edge tilings by regular polygons, all tiles will have the same edge length. When the tiles are regular polygons, the definition of edge can be simplified considerably:

- An *edge* of a tiling by regular polygons is an intersection of two tiles that contains more than one point.

The surroundings of a vertex can be described by its *vertex figure* (see Figure 4), a term that is defined slightly differently by different authors. Krötenheerdt (1969) defines vertex figure (or actually *Eckgebilde*) as the union of a vertex and its surrounding tiles, but Chavey (1984a) defines it as the union of a vertex and its surrounding edges. As long as we are dealing with tilings by regular polygons, this distinction is of little practical importance.

Vertex figures give rise to the related concepts of vertex species and vertex types (see Figure 4). The *species* of a vertex is the number and shape of the polygons in the corresponding vertex figure. The *type* of a vertex also takes the order of polygons into account.

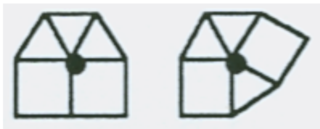

Figure 4: Two vertices with their vertex figures, according to Krötenheerdt's definition. The vertices have the same species, but different types. Image from Borgefors (2008).

To describe vertex types and the tilings they form, we need some notation. Following Grünbaum and Shephard (1987, p. 59), I will denote vertex types by listing the number of sides of each incident polygon, in cyclic order, separated by full stops (for example 3.7.42). The starting point is chosen to give the lexicographically smallest listing (for example 3.4.6.4 rather than 4.6.4.3). If several (say, k) congruent polygons (with, say, n sides) are adjacent to each other, the notation is abbreviated by replacing the k ns by n^k (for example 4^4 rather than 4.4.4.4). A tiling formed by a specific vertex type is simply denoted by putting the symbol of the vertex type in parentheses.

7 1-Archimedean or 1-uniform tilings

1-Archimedean tilings are tilings where only one vertex figure is used, which means that there is a *local symmetry* between the vertices. Later in this paper, I will sketch a proof that all vertex figures in 1-Archimedean tilings can be mapped into each other by symmetries of the tiling, which means that all 1-Archimedean tilings also fulfill the stronger condition of being *1-uniform*, that is, having vertices subject to *global symmetry*. Therefore, I will prefer to call these tilings 1-uniform, unless I want to emphasize their local symmetry properties. It is common to call these tilings Archimedean or uniform, but I will use the longer forms for consistency.

7.1 Vertex figures

The starting point of an exploration of the edge-to-edge tilings by regular polygons is finding which combinations of polygons can fit around a point, that is which vertex figures are possible. It is quite easy to derive an equation for this:

A polygon with n sides has an angular sum of $(n-2)\pi$, so if the polygon is regular, each angle is

$$(n-2)\pi/n = \pi(1-\frac{2}{n})$$

and around each vertex, the sum of angles is

$$2\pi = \sum_{i=1}^{k} \pi(1-\frac{2}{n_i}) \Leftrightarrow 1 = \sum_{i=1}^{k}(\frac{1}{2}-\frac{1}{n_i}) \Leftrightarrow \frac{k}{2}-1 = \sum_{i=1}^{k}\frac{1}{n_i}$$

where k is the number of polygons around the vertex: at least 3 (by definition) and at most 6 (as only six regular trigons can fit around a point, and there are no regular polygons with smaller angles). Going through all possible values of k and n_i, taking into account that the same polygons sometimes can be ordered in different ways around a point, shows that there are 21 possible vertex figures, shown in Figure 5.

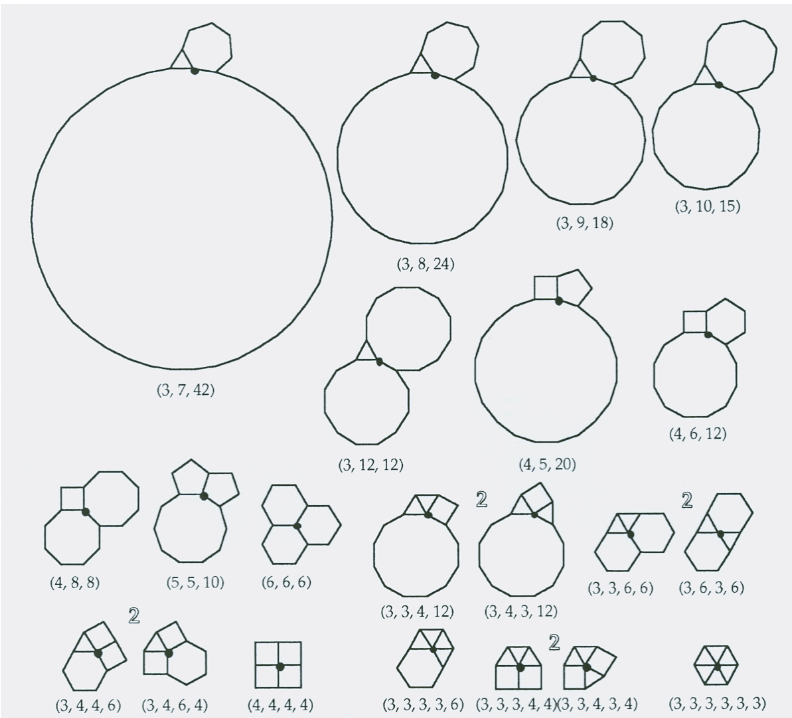

Figure 5: The 21 possible vertex figures. Image from Borgefors (2008).

Of these, the six shown in Figure 6 cannot occur in edge-to-edge tilings by regular polygons, as the extension of the pattern necessarily leads to angles that do not correspond to any regular polygon (Sommerville 1905). Four cannot form tilings by themselves (so they do not occur in 1-uniform tilings), namely $3^2.4.12$, $3^2.6^2$, $3.4.3.12$ and $3.4^2.6$, but they *can* occur in tilings together with other vertex figures. The vertex figure 4.8^2 can only occur in 1-uniform tilings (Krötenheerdt, 1969).

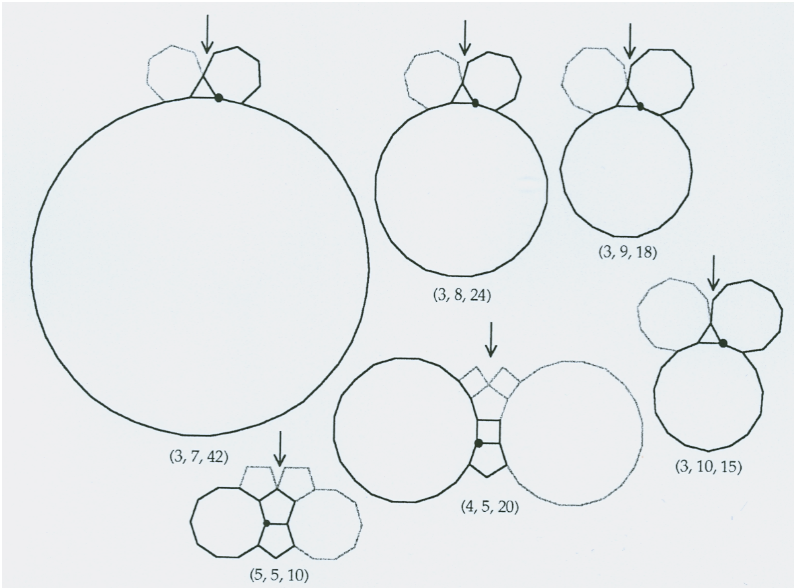

Figure 6: The six vertex figures that cannot occur in edge-to-edge tilings by regular polygons, with impossible angles marked by arrows. Image from Borgefors (2008).

7.2 Kepler

The first systematic treatment of the 1-uniform tilings was done by Johannes Kepler (1619/1997). His work, Harmonices Mundi, is available in a recent English translation by Aiton, Duncan and Field, published in 1997. The work is divided into six books, where the second concerns tilings of the Euclidean plane and the sphere. Kepler called 1-uniform tilings "perfect congruence" (Kepler, 1619/1997, p. 99). "Congruence" is Kepler's term for an edge-to-edge tiling (p. 99) by regular or equilateral polygons (p. 112). He calls polygons, convex or not, *plane figures*, and equilateral polygons *semiregular plane figures* (p. 17).

When a vertex figure cannot be used to tile the entire plane by itself, Kepler calls it "imperfect congruence" (p. 99). He also allows star polygons to be used in tilings (p. 102), but regards them as a somewhat special case (when mentioned in the theorems listing the 1-uniform tilings, it is usually with the phrase "Here we must consider. . . "). In this report, I will denote star pentagons in vertex figures and tilings by the number of sides followed by an asterisk (such as 5* for the star pentagon). Polygons that cannot tile either the Euclidean plane or the sphere (except by forming prisms or antiprisms) are called "incongruent regular plane figures" (p. 102).

After stating definitions, Kepler makes some basic observations. He notes that at least three polygons have to meet at a vertex, and that the sum of their angles is always exactly four right angles (2π) (pp. 102–103). He also studies the situation when three different polygons meet at a vertex and at least one of them has an odd number of edges. Then no 1-Archimedean tiling with vertices of this type is possible, as the two other polygons need to alternate around the polygon with an odd number of edges (p. 103).

Kepler then enumerates the possible vertex figures and the 1-uniform tilings they can form (see Figure 7). He first divides the problem into cases based on the number of kinds of tiles around a vertex (only 1, 2 or 3 are possible), then the number of tiles. The tilings based on star polygons that he finds are $(4.6{*}.4.6{*}.4.6{*})$, denoted K; $(3^3.12{*}.3^2.12{*})$, denoted T; $(4^2.8{*}.4.8{*};4^4)$, denoted X; $(4.8.4.8{*},4.8^2)$, denoted Y; and $(3.4.6.3.12{*})$, denoted Nn. (pp. 103–112) He also seems to regard the tiling $(3.6.3.6;3^2.6^2)$, denoted R (pp. 104–105) as a "perfect congruence", which is a bit surprising. Of the three tilings he discusses that can be "continued indefinitely" although they are not 1-Archimedean tilings by regular, convex or non-convex polygons, $(4^2.8*.4.8*;4^4)$ and $(4.8.4.8{*},4.8^2)$ are said to be of "mixed form", but $(3.6.3.6,3^2.6^2)$ is not. Another anomaly is the pattern that Kepler calls Kk (see Figure 8), which he claims cannot be extended uniformly (pp. 107, 109–110). As shown in the figure, the pattern can be extended uniformly, but unlike the other tilings that Kepler discusses, it is not edge-to-edge. However, Chavey (1984a, pp. 5, 10–11) argues that the figure is wrong, and that Kepler meant to discuss the $3.4^2.6$-vertex, which indeed cannot be extended uniformly (one has to include 3.4.3.6-vertices as well). One possible source of this confusion might be that Kepler did not draw the figures published in the book himself; they were drawn by Wilhelm Schickard (Aiton, Duncan, and Field, 1997, p. xxiv). Chavey (1984a, p. 12) remarks that Kepler incorrectly rules out three possible vertex figures. Kepler states (p. 110) that vertex figures of the form $3.7.x_1, 3.8.x_2$ and $3.9.x_3$ are impossible, but in fact $x_1 = 42$, $x_2 = 24$, $x_3 = 18$ give valid vertex figures. However, they do not form tilings.

7.3 Sommerville

After Kepler, the 1-uniform tilings were largely forgotten. Some attempts to rediscover them were made in the late 19th century, but the first complete discussion was not made until Sommerville (1905). As Chavey (1984a, p. 12) points out, Sommerville is also the first author to rule out all vertex types that cannot form tilings explicitly. In a late addition to his article, Sommerville credits Kepler with the discovery of the 1-uniform tilings, thereby overlooking Kepler's omission of the 3.7.42, 3.8.24 and 3.9.18 vertex figures.

Unlike Kepler, Sommerville uses algebra to describe his reasoning, making his argument somewhat easier to follow for a modern reader. However, his terminology is very different from that used by later authors. He calls tilings "networks of the plane", regular tilings "regular networks of the plane", other edge-to-edge tilings by regular polygons "semi-regular networks of the plane", vertices "nodes", edges "lines", tiles "meshes", vertex species "species of node" and vertex types "varieties". He also uses a distinct notation, with letters instead of numbers for polygons (T for regular trigon, S for regular tetragon (square), H for regular hexagon, O for regular octagon, D for regular dodecagon) and subscripts instead of superscripts. For example, the (4.8^2) tiling is denoted SO_2.

Apart from enumerating the 1-uniform tilings, Sommerville notes that the $(3^4.6)$ tiling is enantiomorphic, which means that it has two mirror images that cannot be mapped into each other by translations and rotations alone.

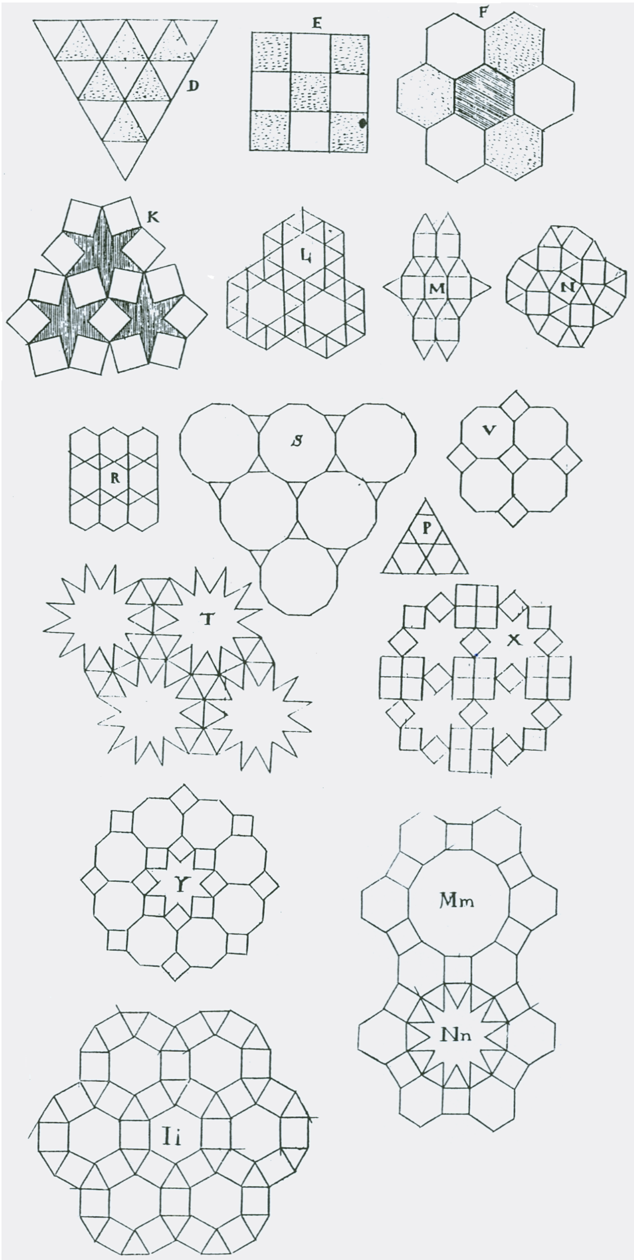

Figure 7: Some of Kepler's tilings. The 1-uniform tilings by regular, non-convex polygons are D (3^6), E (4^4), F (6^3), L ($3^4.6$), M ($3^3.4^2$), N ($3^2.4.3.4$), P (3.6.3.6), S (3.12^2), V (4.8^2), I (3.4.6.4) and Mm (4.6.12). Note the trigons on the left and right sides of M, which should not be there, and that the Mm pattern is not extended to a full tiling. Image from Kepler (1619/1997, pp. 104, 106–107, 111).

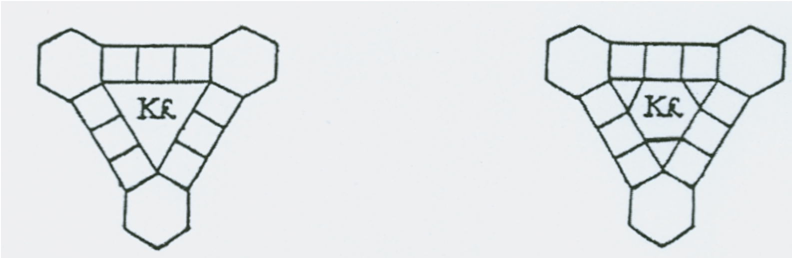

Figure 8: Kepler's Kk tiling, and Chavey's suggestion to correct it. Image from Chavey (1984a, p. 5).

8 Complexity of tilings by regular polygons

As I have already hinted at, there are several concepts that are closely related to 1-uniform or 1-Archimedean tilings by regular polygons. For example, there are two ways to loosen the restriction that all vertices should look the same: having k equivalence classes of vertices with regard to local symmetry, where two vertices are equivalent if they are of the same type, and having k equivalence classes of vertices with regard to global symmetry, where two vertices are equivalent if they can be mapped to each other by symmetries of the tilings. With the former condition, we get the k-Archimedean tilings, with the later, the k-uniform tilings.

When studying tilings with more than one vertex type, there are two common notations. Both separate the vertex types by semicolons and surround the list of vertex types with parentheses in analogy with the notation for 1-uniform tilings, but one notation lists each occuring type once, while the other lists each vertex type once for each transitivity class of vertices of this type. As I will only discuss which vertex types occur in a specific vertex, the first notation is most appropriate for this paper. In analogy with the concept of type of a vertex, the *type of a tiling* is the vertex types that occur in the tiling.

Instead of looking at the vertices, one can also look at the edges or the tiles of a tiling. To emphasize the analogy between the three, k-uniform tilings can also be called *vertex-k-transitive* edge-to-edge tilings by regular polygons, which makes the connection to *tile-k-transitive* and *edge-k-transitive* tilings clear (Dress et al., 1993). In general, these terms do not carry the assumption that the tilings are edge-to-edge tilings by regular polygons. The edge-to-edge tile-1-transitive tilings by regular polygons are the regular tilings: (3^6), (4^4) and (6^3), which have been known since antiquity (Chavey, 1989). A concept that connects these ideas is the *dual* of a tiling, which is formed by replacing each tile by a vertex at the centre of the tile, each vertex by a tile centred at the vertex, and each edge by an edge crossing it at right angles at its midpoint. For a specific value of k, the dual of a tile-k-transitive tiling is a vertex-k-transitive tiling, and the dual of an edge-k-transitive tiling is another edge-k-transitive tiling. However, the dual of a tiling by regular polygons is not in general a tiling by regular polygons.

When going from the 1-uniform tilings to k-uniform tilings with higher values of k, the number of tilings increases rapidly. In fact, there are several easy ways to construct infinitely many 2-Archimedean tilings, see Figures 9 and 10. However, the tilings created by these methods are not very interesting, often consisting of strips of tetragons or similar. But is it possible to create a rigorous definition of interesting tilings? It might at least be possible. One way would be to consider all tilings that contain trigons boring, but as shown in the following theorem, that would only leave four interesting tilings, so this criterion is overly restrictive.

Theorem. All tilings by regular polygons that are not 1-uniform contain trigons.

Proof. The only vertex types that do not contain trigons are 4^4, 4.5.20, 4.6.12, 4.8^2, $5^2.10$ and 6^3. Of these, 4.5.20 and $5^2.10$ cannot form tilings at all, and 4.8^2 can only form the 1-uniform tiling (4.8^2). It remains to show that construction of tilings from 4^4, 4.6.12 and 6^3 leads to either 1-uniform tilings or introduction of trigons.

Assume that the sought tiling contains a 4^4-vertex. The neighbouring vertex figures will have two adjacent tetragons. The possible vertex types are then $3^3.4^2$ (introduces trigons), $3.4^2.6$ (introduces trigons) and 4^4 (1-uniform tiling if no other vertex type occurs).

Now assume that the sought tiling contains a 6^3-vertex. The neighbouring vertex figures will have two adjacent hexagons. The possible vertex types are then $3^2.6^2$ (introduces trigons) and 6^3 (1-uniform tiling if no other vertex type occurs).

Now assume that the sought tiling contains a 4.6.12-vertex. But the neighbouring vertices cannot be of any other trigon-less type (then they would have appeared earlier in the proof), so they have to contain a trigon or be of the type 4.6.12, which leads to a 1-uniform tiling. □

9 2-uniform and other homogeneous tilings

In a series of papers in German, Krötenheerdt (1969, 1970a, 1970b) studied the *homogene Mosaike n:ter Ordnung* (homogeneous mosaics of n'th order). "Mosaik" is simply Krötenheerdt's term for tiling. "Homogeneous of n'th order" is a condition that with our definitions can be stated as *both n-uniform and n-Archimedean*. For simplicity, we will call "homogeneous tilings of n'th order" *n-homogeneous tilings* in the following. (Other authors, for example Grünbaum and Shephard (1987, p. 69), have used the name *Krötenheerdt tilings*.) Krötenheerdt does not define either n-uniform or n-Archimedean. Rather, he defines n-homogeneous tilings using three criteria that can be stated as follows:

- The tiling is edge-to-edge.
- Every vertex belongs to one of n non-empty classes, such that two vertices are in the same class if they are of the same type, and in different classes if they are of different types.
- All vertices in a class can be transformed into one another using global symmetries of the tiling.

The third criterion is what makes n-homogeneous different from n-uniform. It is worth noting that Krötenheerdt by "homogen" specifically means 1-homo-

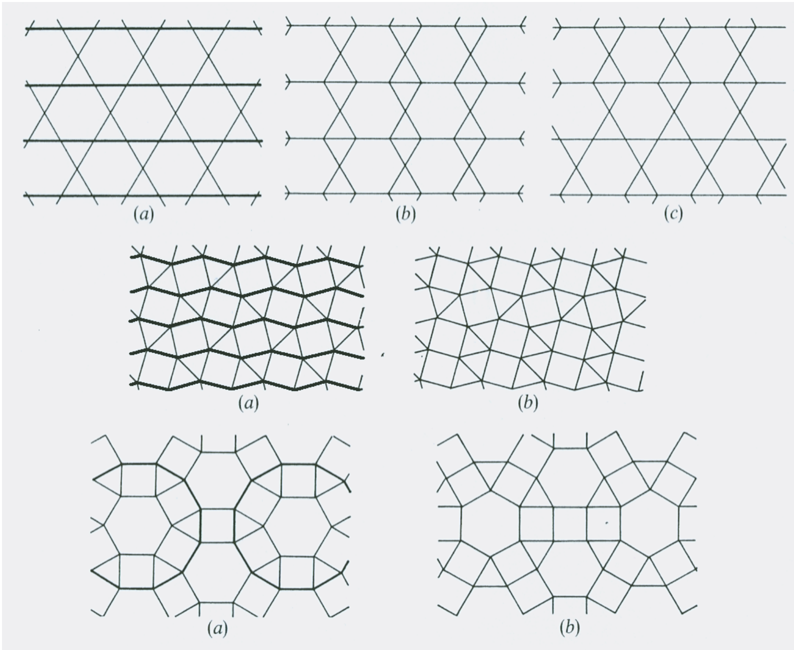

Figure 9: Three ways to create infinitely many 2-Archimedean tilings. Top row: The (3.6.3.6)-tiling may be cut along the lines indicated on the left, and the resulting strips slid independently of each other, resulting in infinitely many $(3^2.6^2; 3.6.3.6)$-tilings. Middle row: The $(3^2.4.3.4)$-tiling may be cut along the lines indicated on the left, and the resulting strips either reflected or not in a vertical line, yielding infinitely many $(3^3.4^2; 3^2.4.3.4)$-tilings. Bottom row: In the (3.4.6.4)-tiling, "dodecagons" may be cut out and rotated $\pi/6$. Rotating different dodecagons yields different $(3.4^2.6; 3.4.6.4)$-tilings. Image and tiling generation algorithms from Grünbaum et al. (1987, p. 62).

geneous, and by "Archimedisch" (or "halbregulär") 1-uniform but not regular. Other definitions are easily recognisable: "Eck" means vertex, "Kante" edge, "Flächenstück" tile, "Eckgebilde" vertex figure, and "reguläre Mosaik" regular tiling. "Reguläres Eckgebilde" is a vertex figure where all incident tiles are regular polygons. The notation for specific tilings is similar to the one used in this paper, except that abbreviations with superscripts are not used.

In his first paper (1969), Krötenheerdt shows that the number of n-homogeneous tilings is finite, and enumerates all homogeneous tilings of first and second order (see example in Figure 11). Implicitly, he thereby also enumerates all 1- and 2-uniform tilings, because there are no 1-uniform tilings which are not also 1-Archimedean, and no 2-uniform tilings which are not also 2-Archimedean. The reasons are outlined in the following theorems.

Theorem. There are no n-uniform, m-Archimedean tilings where $n < m$.

Proof. A vertex that belongs to a certain class with regard to the symmetry of the tiling must be of the same type as all other vertices in this class. □

Theorem. There are no m-Archimedean tilings with $m > 14$.

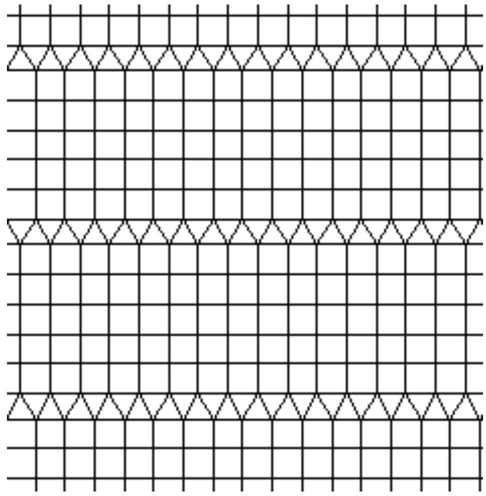

Figure 10: A 3-uniform, 2-Archimedean $(3^3.4^2; 4^4)$-tiling. It can generate infinitely many other $(3^3.4^2; 4^4)$-tilings by adding more rows of tetragons. Image from Galebach (2009, 3-uniform tiling number 10).

Proof. There are only 15 possible vertex figures (Kepler, 1619/1997, p. 103–112; Sommerville, 1905). Of these, 4.8^2 can only occur in 1-Archimedean tilings (Krötenheerdt, 1969). Using only the other 14, no more than 14-Archimedean tilings are possible. □

Theorem. There are no 1-Archimedean, n-uniform tilings for $n > 1$.

Sketch of proof. It is well known that there are exactly 11 1-Archimedean tilings (Kepler, 1619/1997; Sommerville, 1905). Thus it has to be proven that these 11 tilings are 1-uniform. Chavey (1984a, pp. 19–30) shows that tilings with a finite number of congruence classes of tiles (such as all tilings considered in this paper) are connected. As all vertices have locally identical surroundings in 1-Archimedean tilings, it suffices to show that a vertex figure can be mapped to all *adjacent* vertex figures by symmetries of the tiling. The rest of the proof is by exhaustion. □

Krötenheerdt's proof that there are only finitely many n-homogeneous tilings uses the concept "regulärer Vieleckskomplex", which is defined inductively from regular vertex figure (in this context called "regulärer Vieleckskomplex 1. Ordnung") by the addition of new regular vertex figures whose vertex has an edge to a vertex already in the "regulärer Vieleckskomplex". It is interesting to note that the definition makes use of the edges that are incident to a vertex, although the edges do not belong to the vertex figure according to Krötenheerdt's definition.

In the following papers (1970a, 1970b), Krötenheerdt enumerates the n-homogeneous tilings for $n = 3, 4, 5, 6, 7$, using a rather intricate division into various cases. He also proves that there are no n-homogeneous tilings for $n > 7$.

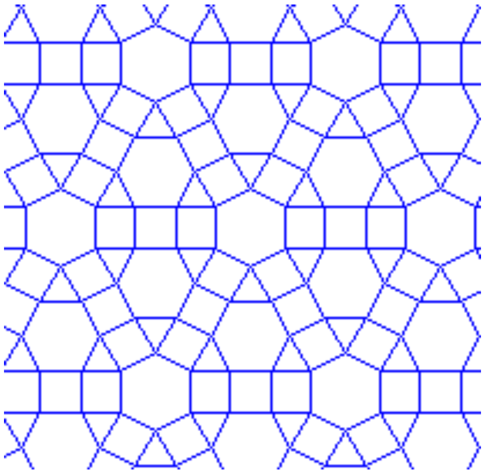

Figure 11: The $(3.4^2.6; 3.4.6.4)$-tiling, which is a 2-uniform tiling. If Kepler's Kk pattern were extended according to Chavey, this would be the result. Image from Galebach (2009, 2-uniform tiling number 5).

10 3-uniform tilings

The 3-uniform tilings were enumerated by Chavey (1984a) in his PhD thesis. The thesis is partly written as a commentary to Grünbaum and Shephard (1987), which Chavey read in draft form. Some of the results from the thesis are also presented in two articles (1984b, 1989).

Chavey studies transitivity of both tiles, vertices and edges, what he collectively calls the *elements* of the tiling (1984a, p. 2). In a group-theoretical fashion, he calls the equivalence classes of elements *orbits*. The number of orbits of vertices, tiles and edges, respectively, is denoted by v, t and e. A tiling with v vertex figure orbits (t tile orbits, e edge figure orbits) is denoted v-*isogonal* (t-isohedral, e-isotoxal) (pp. 4, 6). Thus, v-isogonal (or sometimes k-isogonal) is simply Chavey's term for what we call k-uniform.

When studying local symmetry, Chavey defines vertex figures as the union of a vertex and its surrounding edges (p. 6) (denoting vertex types in the same way as here), edge figures as the union of an edge with its adjacent edges (p. 6), and simple edge figures as the union of an edge with the tiles it separates (1989). He calls k-Archimedean tilings k-*gonal* (1984a, p. 7), and restricts his use of *Archimedean* to the 1-Archimedean tilings (pp. 9–10).

Chavey's concept of homogenity deserves special mention. He defines both vertex-homogeneous, tile-homogeneous and edge-homogeneous tilings. What they have in common is that all elements that could conceivably be symmetric (that is, on the same orbit) in fact are. The concept does not specify how many orbits there are (p. 7).

In a series of lemmata (pp. 83–93), Chavey characterizes 10 of the 16 types of 2-Archimedean tilings completely (four of these were also characterized by Sommerville (1905) or Krötenheerdt (1969)). The lemmata are based on a theorem which derives the possible edge types in 2-Archimedean tilings (see Figure 12). One of the conceptual tools he uses is the *dissection* of a hexagon into six trigons and, reversely, the *fusion* of six trigons around a 3^6-vertex into a hexagon. The characterized types are $(3^6; 3^2.4.12)$, $(3^6; 3^2.6^2)$, $(3^3.4^2; 3.4.6.4)$, $(3^6; 3^3.4^2)$, $(3^3.4^2; 4^4)$, $(3^2.6^2; 3.6.3.6)$, $(3.4^2.6; 3.6.3.6)$, $(3^6; 3^4.6)$, $(3.4.6.4; 4.6.12)$ and $(3.4^2.6; 3.4.6.4)$. Of these, the first three are shown to be unique and 2-uniform. $(3^2.4.3.4; 3.4.6.4)$ is also shown to be unique and 2-uniform if there are no edges of the type $3.4.3^2.4/4.6.4.3$ (where the listing of polygons should be understood to start at the edge). If this restriction is lifted, there are infinitely many tilings of this type, as can be seen from Figure 13. He also shows (pp. 77–83) that there are only five other possible types of 2-Archimedean tilings, namely $(3^6; 3^2.4.3.4)$, $(3^4.6; 3^2.6^2)$, $(3^4.6; 3.6.3.6)$, $(3^3.4^2; 3^2.4.3.4)$ and $(3.4.3.12; 3.12^2)$. In a manuscript that at the time of writing was unpublished, Chavey also characterizes $(3.4.3.12; 3.12^2)$ completely (Chavey, personal communication, November 5, 2009).

Chavey's lemmata might be one way of distinguishing boring from interesting tilings. The types that are not unique are characterized in terms of strips, or fused or dissected hexagons and dodecagons. Consequently, one might assume that tiling types with only one (or a few) possible tiling are more interesting.

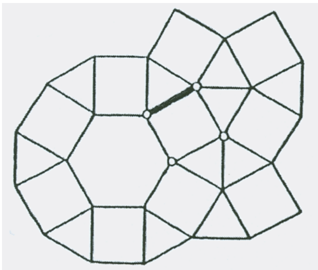

Figure 12: The $3.4.3^2.4/4.6.4.3$ edge type forces a specific 2-Archimedean configuration. Image from Chavey (1984a, p. 82).

Chavey completes the actual enumeration of the 3-uniform tilings by enumerating the 2-Archimedean, 3-uniform tilings (pp. 109–164). Together with the 3-homogeneous tilings enumerated by Krötenheerdt (1970a, 1970b), the 3-uniform tilings are thus completely enumerated, as there are no 1-Archimedean, 3-uniform tilings.

11 Combinatorial and computational tiling theory

In the 1980's, a new approach to tilings was developed, providing a general description of tilings. As noted by Huson (1993), a general approach to tiling classification has not yet been found, instead many specific methods have had to be adopted. To unify them, Dress and associates started developing Delaney–Dress symbols in the mid 1980's. Delaney–Dress symbols consist of coloured graphs and functions on the nodes of the graphs, and are to my knowledge best described by Huson (1993) and Delgado-Friedrichs (2003). They are sometimes also called Delaney symbols or D-symbols. Delaney–Dress symbols have given

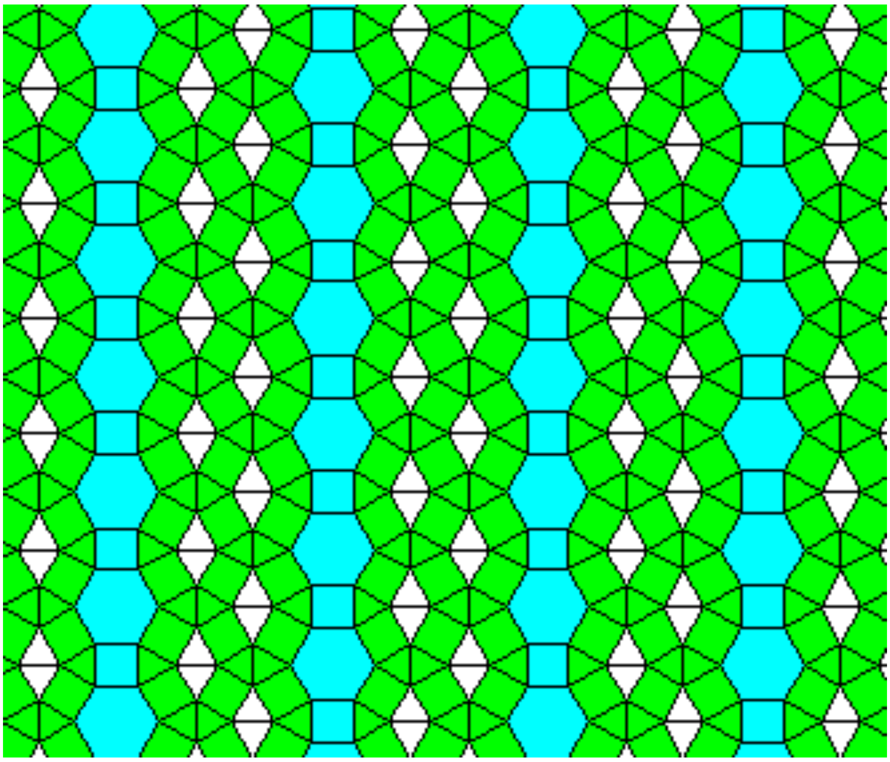

Figure 13: Between each strip of tetragons and hexagons (blue) in a $(3^2.4.3.4;3.4.6.4)$-tiling, there might be an arbitrary number of strips (at least one) of trigons and tetragons. The remaining space is filled by pairs of trigons. Image based on Galebach (2009, 5-uniform tiling number 219).

rise to combinatorial tiling theory and computational tiling theory. The former is the systematical study of the combinatorial structure of tilings, and the latter is the construction of data structures and algorithms based on Delaney–Dress symbols (Delgado-Friedrichs, 2003).

To understand Delaney–Dress symbols, we first need some background information. Delaney–Dress symbols describe *equivariant tilings*, which are pairs consisting of a tiling of some manifold (for example the Euclidean plane) and a discrete group of homeomorphisms on this manifold that is compatible with the tiling. The group does not, however, need to be the complete symmetry group of the tiling, as is usually assumed. The tilings described by Delaney–Dress symbols are usually *normal*, meaning that the intersection of each pair of tiles is connected, and *proper*, meaning that each tile has at least three vertices. (Huson, 1993)

The Delaney–Dress symbol is based on the *chamber system* or *formal barycentric subdivision* of the tiling. The definition of chamber system that follows is based on Huson (1993), but Delgado-Friedrichs (2003) has a similar definition. For all vertices, edges and tiles, choose an interior point, called a 0-centre, 1-centre or 2-centre, respectively. Connect the 2-centre in each tile with each

0-centre and 1-centre on the boundary of the tile, with pairwise non-intersecting edges. The non-regular trigons that result are called *chambers*. The edge opposite the i-centre in a chamber is called an i*-edge*. According to Delgado-Friedrichs (2003), "Ideally, the barycentric subdivision should be constructed as to retain the symmetry of the tiling. This is always possible."

Now we start to construct the Delaney–Dress symbol. Its one major part, the Delaney–Dress graph, is a graph where the nodes correspond to equivalence classes of chambers. The nodes are connected by i-coloured edges when there is an i-edge between the corresponding chambers. Put another way, if the tiling has dimension n, there are functions $s_0 \ldots s_n$ such that s_i exchanges all nodes that are connected by i-coloured edges. (Delgado-Friedrichs, 2003) Huson (1993) calls these functions σ_i.

The Delaney–Dress graph does not define the tiling unambiguously, so we have to add functions that map the nodes $\mathscr{D}$ to positive integers, which will disambiguate the tilings. However, different authors use different functions, and their correspondance is not entirely straightforward. Huson (1993) defines functions $m_{ij} : \mathscr{D} \to \mathbb{N}, 0 \leq i < j \leq 2$ such that

$$m_{ij}(D) := \min \{m \in \mathbb{N} | C(\sigma_i \sigma_j)^m = C \text{ for any } C \in D\}$$

but Delgado-Friedrichs (2003) defines only two functions in the 2-dimensional case: $m_0(C)$, the number of vertices of the tiles containing chambers of the class C, and $m_1(C)$, the degree of the vertices adjacent to chambers of the class C.

Together, the Delaney–Dress graph and the m functions constitute the Delaney–Dress symbol (see Figure 14). Two tilings are *equivariantly equivalent* if and only if their Delaney–Dress symbols are isomorphic. Delaney–Dress symbols are usually assumed to be finite. In fact, all periodic tilings have finite Delaney–Dress symbols. (Delgado-Friedrichs, 2003)

Due in part to the different m functions, the formal definition of Delaney–Dress symbols is stated quite differently by Huson (1993) and Delgado-Friedrichs (2003). According to Huson,

> A system $(\mathscr{D}; m) := ((\mathscr{D}, \mathscr{E}); m_{01}, m_{12}, m_{02})$ consisting of a finite, connected Delaney–Dress graph $\mathscr{D} := (\mathscr{D}, \mathscr{E})$ and functions m_{01}, m_{12}, $m_{02} : \mathscr{D} \to \mathbb{N}$, is called a (2-dimensional) Delaney–Dress symbol if and only if for all $D \in \mathscr{D}$ and $0 \leq i \leq j \leq 2$ the following hold:
>
> - (DS1) $m_{ij}(D) = m_{ij}(D\sigma_i) = m_{ij}(D\sigma_j)$
> - (DS2) $D(\sigma_i\sigma_j)^{m_{ij}(D)} = D(\sigma_j\sigma_i)^{m_{ij}(D)} = D$
> - (DS3) $m_{02}(D) = 2$
> - (DS4) $m_{01}(D) \leq 2$
> - (DS5) $m_{12}(D) \leq 3$
>
> Conditions (DS1) and (DS2) follow directly from the definition of the functions m_{ij}, whereas (DS3) reflects the fact that any edge in a tiling lies between precisely two tiles. Property (DS4) corresponds to the fact that the tiles we are considering all contain at least two vertices, whereas (DS5) indicates that, by definition, every vertex is incident to at least three tiles.

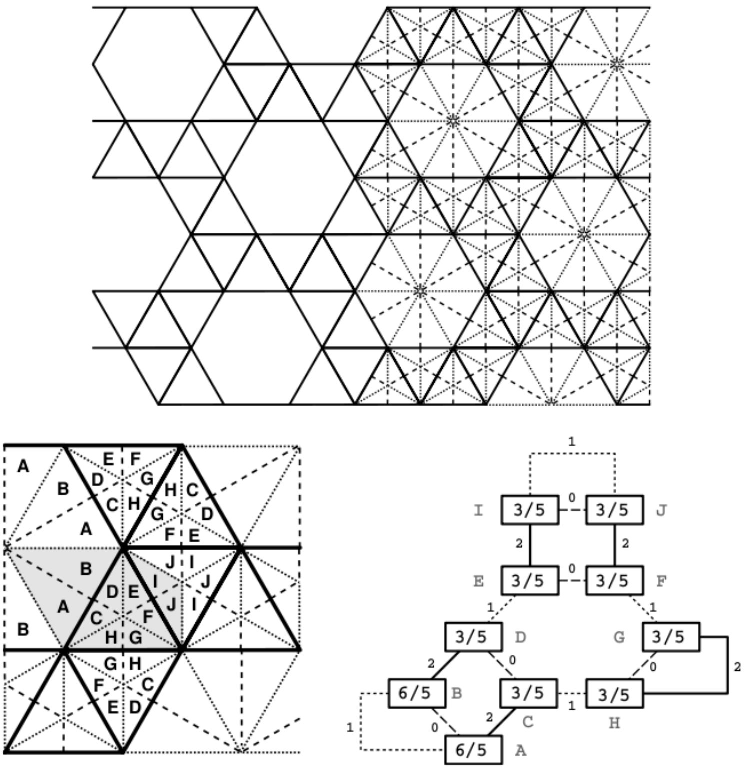

Figure 14: The $(3^4.6)$-tiling with its barycentric subdivision (top), chamber classes (left) and Delaney–Dress symbol (right) according to Delgado-Friedrichs' definition. Solid lines correspond to 2-edges, dashed lines with short dashes to 1-edges, and dashed lines with long dashes to 0-edges. The nodes are labelled with their m_0 and m_1 values. Image from Delgado-Friedrichs (2003).

Delgado-Friedrichs defines Delaney–Dress symbols (Delaney symbols in his terminology)

> A Delaney symbol of dimension n is a set $\mathbf{C}$ together with functions $s_0, \ldots, s_n$ from $\mathbf{C}$ into $\mathbf{C}$ and functions $m_0, \ldots, m_{n-1}$ from $\mathbf{C}$ into the positive integers, such that the following is true for all $C \in \mathbf{C}$ and all applicable i and j:
>
> - (DS0) The underlying Delaney graph is connected, i.e., each element can be mapped onto any other by repeatedly applying functions from the set $s_0, \ldots, s_n$.
> - (DS1) $s_i(s_i(C)) = C$.
> - (DS2) $s_i(s_j(C)) = s_j(s_i(C))$ whenever $j > i + 1$.
> - (DS3) $m_i(C) = m_i(s_i(C)) = m_i(s_{i+1}(C))$.
> - (DS4) $f_i^{m_i(C)}(C) = C$, where $f_i^0(C) := C$ and $f_i^{k+1}(C) := s_i(s_{i+1}(f_i^k(C)))$.

Computational tiling theory makes use of the concept of *fundamental* tiles and tilings. In an equivariant tiling $(\mathscr{T}, \Gamma)$, a tile $A \in \mathscr{T}$ is fundamental if the (induced) stabilizer $\Gamma_A := \{\gamma \in \Gamma | \gamma A = A\}$ is trivial. A fundamental tiling is a tiling of fundamental tiles only. (Huson, 1993) This should not be confused with the concept of *fundamental domain*, which is a connected region that contains a chamber of each class (Delgado-Friedrichs, 2003). Huson (1993) describes methods to generate all fundamental tile-1-transitive tilings, fundamental tile-k-transitive tilings, and non-fundamental tile-k-transitive tilings. The description is, however, quite abstract.

Despite their usefulness, Delaney–Dress symbols have not seen much application to k-uniform tilings. According to Delgado-Friedrichs (personal communication, September 22, 2009), vertex-k-transitive tilings (including k-uniform tilings) have not been studied much because they are duals of the more completely investigated tile-k-transitive tilings.

12 4-uniform, 5-uniform and 6-uniform tilings

At the beginning of the 21st century, the 4-uniform, 5-uniform and 6-uniform tilings were enumerated by Galebach. The enumeration was done by a computer program, and does not appear to have been published outside Galebach's website (Galebach, 2009) and the On-Line Encyclopedia of Integer Sequences (Sloane, 2003). The program is capable of determining the number of m-Archimedean, n-uniform tilings for given m and n, as shown in Table 1. Unfortunately, the program does not seem to be described in more detail anywhere.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	> 14	Total
1	11	0	0	0	0	0	0	0	0	0	0	0	0	0	0	11
2	0	20	0	0	0	0	0	0	0	0	0	0	0	0	0	20
3	0	22	39	0	0	0	0	0	0	0	0	0	0	0	0	61
4	0	33	85	33	0	0	0	0	0	0	0	0	0	0	0	151
5	0	74	149	94	15	0	0	0	0	0	0	0	0	0	0	332
6	0	100	284	187	92	10	0	0	0	0	0	0	0	0	0	673
7	0	?	?	?	?	?	7	0	0	0	0	0	0	0	0	?
8	0	?	?	?	?	?	20	0	0	0	0	0	0	0	0	?
9	0	?	?	?	?	?	?	8	0	0	0	0	0	0	0	?
10	0	?	?	?	?	?	?	27	0	0	0	0	0	0	0	?
11	0	?	?	?	?	?	?	?	1	0	0	0	0	0	0	?
12	0	?	?	?	?	?	?	?	?	0	0	0	0	0	0	?
13	0	?	?	?	?	?	?	?	?	?	?	?	0	0	0	?
14	0	?	?	?	?	?	?	?	?	?	?	?	?	0	0	?
> 14	0	?	?	?	?	?	?	?	?	?	?	?	?	?	0	?

Table 1: The number of m-Archimedean, n-uniform tilings, according to Galebach (2009). In the leftmost column, the values of n are given, in the topmost row the values of m. Compare this table to the theorems in Chapter 9.

To see if there are any patterns to be found among the types of 2-Archimedean tilings that were not described by Chavey or others, I have identified them among the tiling images generated by Galebach. Their numbers, according to

Galebach's numbering, is given in Table 2. The listing inspired me to formulate the theorem in Chapter 13, and might lead to further advances as outlined in Chapter 14.

k	2	3	4	5	6
$(3^6; 3^2.4.3.4)$	18	57		318	664
				319	
$(3^4.6; 3^2.6^2)$	12			262	560
				263	
$(3^4.6; 3.6.3.6)$		45	125	267	585
		46	129	275	590
				276	591
				282	592
					597
					600
					606
$(3^3.4^2; 3^2.4.3.4)$	16	53	142	308	646
	17	55	143	309	647
			144	310	648
				311	649
				312	652
				315	657
					660
					661
$(3^2.4.3.4; 3.4.6.4)$			106	216	488
			107	219	497

Table 2: The numbers used to denote the 2-Archimedean, k-uniform tilings in Galebach's lists of tilings, for indicated values of k and tiling types.

13 Connecting two classes of 2-Archimedean tilings

I have made a small observation regarding the description of the tilings of type $(3^6; 3^2.4.3.4)$ and $(3.4.3.6, 3^2.4.3.4)$. The observation, which means that a complete description of one of the types will lead to a complete description of the other, can be stated formally as the following theorem.

Theorem. Tilings of type $(3^6; 3^2.4.3.4)$ are formed from tilings of type $(3.4.3.6; 3^2.4.3.4)$ by dissection of hexagons (except a dissection of the tiling $(3.4.3.6)$ which will also yield a tiling of type $(3^6, 3^2.4.3.4)$). See Figure 15).

To prove the theorem, I make the following

Definition. A *block of order 1* is a set containing of a vertex, its incident tiles, and all edges and vertices that are adjacent to these tiles. A *block* in general is a maximal union of blocks of order 1 that all vertices that belong to the interior of the block are of the same type, and the block forms a connected set of tiles. The type of the block is the type of the interior vertices.

Remark. Where two blocks meet, they will overlap such that there is an edge from each vertex on the boundary of one block to a vertex on the boundary of the other block.

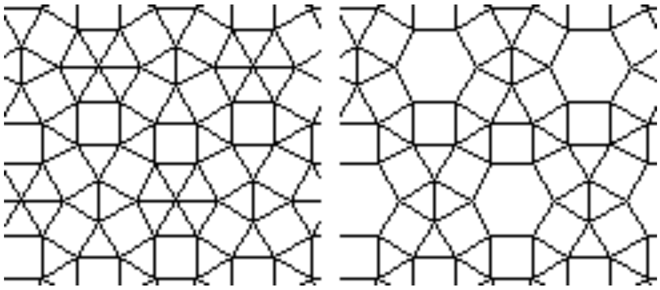

Figure 15: The dissection of hexagons described in the theorem in Chapter 13. Image based on Galebach (2009, 2-uniform tiling number 9).

Proof of theorem. Consider a 3^6-block in a $(3^6, 3^2.4.3.4)$-tiling. Somewhere, it has to meet a $3^2.4.3.4$-block (else the tiling will be of type (3^6)). Consider a non-interior vertex in the 3^6-block. As it lies on the boundary, it has to be of type $3^2.4.3.4$, so the boundary of the 3^6-block turns $\pi/6$ at this vertex. Now consider another vertex on the boundary of the 3^6-block, which is a neighbour of the first. It is also of type $3^2.4.3.4$. If the trigon that is shared by the two vertex figures has a tetragon on each side, the third vertex of this trigon cannot be of type 3^6. But then the first vertex has no neighbour of type 3^6, so it cannot belong to a 3^6-block, which is a contradiction (see Figure 16). So the neighbours of the trigon are a tetragon (also shared between the first two vertices) and another trigon. The boundary of the 3^6-block thus turns another $\pi/6$ at this vertex, *in the same direction* as at the first. Repeating this process, we see that the 3^6-block has order 1. All 3^6-blocks can then be replaced by hexagons by fusion of the trigons. At the fusion, the $3^2.4.3.4$-vertices on the boundary of the 3^6-block become 3.4.3.6-vertices, but other $3^2.4.3.4$-vertices that might occur in the tiling will remain unchanged. □

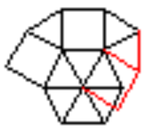

Figure 16: The contradiction of the proof in Chapter 13. The required properties of the "second vertex" contradict the initial assumption that the "third vertex" is of type 3^6. Image based on Galebach (2009, 2-uniform tiling number 9).

14 Some ideas for future research

As the number of possible tilings increases as k increases, the future study of k-uniform tilings will probably need the help of computers. Chavey (personal communication, September 22, 2009) sees the work of Galebach as an indication of the need for computer programs. Ideally, these programs should be provably correct and able to execute in a reasonable amount of time. The algorithms devised for tile-k-transitive tilings, using Delaney–Dress symbols, could probably be adapted to find k-uniform tilings by using the duality property, but it is not entirely straightforward as only a few duals of tilings by regular polygons are tilings by regular polygons themselves—namely the duals of the Platonic tilings.

To make the algorithms more efficient, knowledge of the m-Archimedean tilings is probably helpful. The easiest step in this direction seems to be to complete the analysis of the five types of 2-Archimedean tilings that have not been done by Chavey.

A characterization of the $(3.4.3.6, 3^2.4.3.4)$ tilings should probably make use of the fact that 3.4.3.6 forces a dodecagon-like structure, with the difference that "dodecagons" in these tilings might overlap. There are three possible ways for "dodecagons" to form strips: by meeting tetragon-to-trigon, meeting trigon-to-trigon or overlapping (by two trigons and a tetragon). These strips can then meet each other in several ways. At overlaps, strips can meet by overlapping "dodecagons" in a new direction or by having (possibly zero) strips consisting of alternating trigons and tetragons inbetween, with pairs of trigons filling the gaps. Non-overlapping "dodecagon" strips can meet either by arranging three "dodecagons" around a trigon or by arranging four "dodecagons" around a four-pointed star consisting of a tetragon surrounded by trigons. The later case forces each "dodecagon" trigon that neighbours another "dodecagon" to neighbour a tetragon in that "dodecagon".

The connection between $(3.4.3.6, 3^2.4.3.4)$ and $(3.4.3.12, 3.12^2)$ is presumably not trivial, as only $(3.4.3.6, 3^2.4.3.4)$ tilings that do not have "overlapping" dodecagons (i.e. only using three-"dodecagons"-around-a-trigon or four-"dodecagons"-around-a-four-pointed-star patterns) might be transformed into $(3.4.3.12, 3.12^2)$ tilings. Conversely, some restrictions stemming from the need to avoid tetragons from two "dodecagons" meeting disappear when going from $(3.4.3.6, 3^2.4.3.4)$ to $(3.4.3.12, 3.12^2)$ tilings.

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