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TILINGS BY REGULAR POLYGONS—II

A CATALOG OF TILINGS

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Abstract—Several classification theorems involving highly symmetric tilings by regular polygons have been established recently. This paper surveys that work and gives drawings of these tilings—many of which were not shown in the original papers. Included are all tilings with at most three symmetry classes (orbits) of tiles, vertices or edges and those tilings which satisfy certain homogeneity criteria; i.e. tilings where locally congruent portions of the tiling are always equivalent under a global symmetry of the tiling.

INTRODUCTION

Tilings of the plane which use only regular polygons include many of the most symmetric tilings, and many of the most beautiful tilings. Such tilings have been known since antiquity, but most of the classification results involving them have appeared only in the last two decades. These classification results are scattered across several papers, and some of the results, unfortunately, appear without the pictures of the tilings which provide their greatest appeal. The purpose of this work is to combine these results into a single catalog of highly regular tilings; to describe the various natures of their regularities; and to provide pictures of these tilings. It should be noted that many of these tilings also appear in Grünbaum and Shephard [1], who go into much greater depth discussing regularity of more general types of tilings. The reader is also referred to this work for other types of tilings and for precise definitions of the terminology associated with tilings.

In addition to using only regular polygons, all tilings in this paper are assumed to be "edge-to-edge", i.e. whenever two polygons intersect at more than one point, they share an edge. This means that tiles cannot meet at edges that are offset with respect to each other (such as the triangles in Fig. 1), and means that all of the polygons must have the same edge length (unlike the polygons in Fig. 1). Although this restriction eliminates many fascinating tilings, it makes the classification theorems much more tractable. Very few classification theorems have been established for non edge-to-edge tilings by regular polygons. The reader is referred to Section 2.4 of Grünbaum and Shephard [1] for a survey of most of the available results and pictures of many of the relevant tilings. The reader is warned that some of the statements made later in the present paper are not correct for tilings more general than the edge-to-edge tilings by regular polygons.

REGULARITY PROPERTIES

A tiling has associated with it in a natural way not only the constituent *tiles*, but also the *edges* and *vertices* where these tiles meet. Collectively the tiles, vertices and edges of a tiling are called the *elements* of that tiling. One measure of the regularity of a tiling is how many equivalence classes of these elements the tiling has under the symmetry group of the tiling. In the language of group theory, these equivalence classes are the *orbits* of the elements under the symmetry group. We say that two elements of the tiling are *symmetric with* each other if they are in the same orbit; i.e. one can be carried into the other by a symmetry of the tiling. The number of orbits of the tiling elements is finite if and only if the tiling is *periodic*, i.e. it has translational symmetries in at least two different

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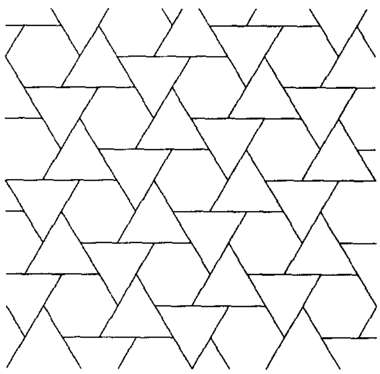


Fig. 1. A non edge-to-edge tiling by regular polygons. This tiling is homogeneous with respect to tiles, vertices and edges; and is strongly edge-homogeneous.

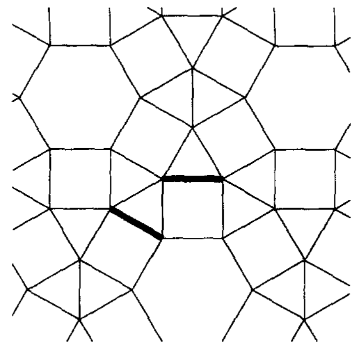


Fig. 2. The two bold edges have the same simple edge type, but different edge types.

directions. Those tilings with only a few different orbits of one or more of the tiling elements are the most regular or most symmetric tilings. The tilings (edge-to-edge tilings by regular polygons) which have at most three orbits of one of the tiling elements have all been classified, and the number of these tilings is listed in Table 1, along with references to the original results. The author [2] has shown that the number of orbits of vertices is always bounded by the number of orbits of edges; consequently the results of the last row of Table 1 can be established from examining the tilings corresponding to the results of the middle row of this table.

The discovery of the tilings with a single orbit of tiles, listed in Table 1 as "ancient" is lost in antiquity, but they were almost certainly known to the early Greek geometers. The Pythagoreans were least aware that the only ways to combine regular polygons of one type at a vertex were three hexagons, four squares, or six triangles (see Ref. [3]), and hence must have known of these tilings. Pappus of Alexandria in the third century A.D. writes of the classification of these tilings in the preface of his *Collection* (quoted in full in Ref. [4]) as if it was an old, well-known fact. In addition, Pappus is aware of the need for the assumption that the tilings are edge-to-edge, and is careful to state this.

We should note that we give credit to Kepler for the classification of the tilings with only one orbit of vertices. Although he found all of the tilings, his proof that there are no others is incorrect. In classifying the possible vertex types which can be constructed from polygons (regardless of whether they can be extended to tilings), he misses the vertex types 3.7.42, 3.8.24 and 3.9.18 (using the notation to be defined later). This is a fairly minor point, since his Theorem 17 can be used to show that these vertex types cannot occur in a tiling. Nevertheless, the first fully correct proof appears to be due to Sommerville [5].

Another way of viewing a tiling as "regular" is what we describe as *homogeneous*: any two tiling elements of some type which "might" be symmetric with each other actually are. For example, a triangle obviously cannot be symmetric with a square, but two different triangles could, conceivably, be symmetric with each other. Thus, a tiling would be described as *tile-homogeneous* if any two congruent polygons were in the same orbit; i.e. were symmetric with each other. Figure 1, although not an edge-to-edge tiling, is an example of a tile-homogeneous tiling; all triangles are equivalent, and all hexagons are equivalent. Notice that homogeneity can be thought

Table 1. Classification results

	One orbit	Two orbits	Three orbits
Tiles	3 [Ancient]	13 [6]	25 [7]
Vertices	11 [8]	20 [9]	61 [10, 11]
Edges	4 [12]	4 [10, 13]	10 [10, 13]

of in terms of local/global properties: whenever two areas of the tiling (e.g. tiles) locally look the same, there is a global symmetry which also recognizes that "sameness".

To define vertex-homogeneity, we first define the *vertex figure* of a vertex in a tiling to be the union of all edges incident to that vertex. This is the obvious "local region" for a vertex. A tiling is then *vertex-homogeneous* if any two vertices with congruent vertex figures are symmetric with each other.†

The correct definition of edge-homogeneous is not quite as obvious. If one views an edge as connecting the two tiles which contain it, then an "edge figure" consists of an edge and its two incident tiles. With this viewpoint one would think of the two bold edges of Fig. 2 as being "locally" the same. On the other hand, if we think of an edge as connecting the two vertices at its ends, then an "edge figure" consists of an edge together with all edges which meet it. Under this viewpoint, the two bold edges of Fig. 2 are "locally" different. For this survey we refer to an edge and its two incident tiles as a "simple edge figure", and call an edge with its two incident vertex figures an "edge figure". A tiling in which any two congruent edge figures are symmetric is called *edge-homogeneous* while a tiling in which any two congruent simple edge figures are symmetric is called *strongly edge-homogeneous*. This terminology reflects the author's belief in what the "correct" definition of edge-homogeneous should be. It is easily verified that a tiling which is strongly edge-homogeneous is also edge-homogeneous (the converse is false).

Most of the homogeneous tilings have been classified. The 135 vertex-homogeneous tilings were classified by Krötenheerdt [9, 14, 15]. The 22 tile-homogeneous tilings were classified by DeBroey and Landuyt [6]. The 22 strongly edge-homogeneous tilings were classified by the author [10]. The edge-homogeneous tilings have not been classified, and it is expected that there would be a very large number of such tilings. It is worth noting that there are some relationships between the various types of homogeneity. All of the strongly edge-homogeneous tilings are also vertex-homogeneous (a direct proof of this fact is the basis of the classification of such tilings). All but one of the tile-homogeneous tilings are also vertex-homogeneous. (This exceptional tiling is also not edge-homogeneous; it is the tiling of Fig. 9.)

It is the author's feeling that part of the success of a classification theorem in the theory of tilings can be measured by the beauty of the new tilings discovered. From this viewpoint it would seem that the two most successful classification theorems for edge-to-edge tilings by regular polygons are the tilings with a single vertex orbit [8] and the vertex-homogeneous tilings [9, 14, 15]. This latter work included the classification of all tilings with two vertex orbits. The author suspects that several other beautiful tilings could be found among the edge-homogeneous tilings.

Another measure of the success of a classification theorem might be its completeness. It is always possible to imagine further classification theorems which would extend Table 1 a little further. The classification of the homogeneous tilings is more of a "final" classification, it is difficult (and rather artificial) to try to extend these definitions to broader classes of tilings.

THE TILINGS

On the following pages are drawings of all the tilings which arise from the classification theorems mentioned in the previous section. In general, the number of vertex orbits seems to be a good measure of the complexity of the tiling. Consequently, the tilings are arranged according to the number of vertex orbits in the tiling, labelled according to the vertex types which appear, and a representative of each vertex orbit is marked by a bold circle. The *vertex type* of a vertex, as used to label the tilings, is a listing, in either clockwise or counter-clockwise order, of all the polygons which meet the vertex. For example, a vertex of type 3.4.6.4 meets, in order, a 3-gon (triangle), a 4-gon (square), a 6-gon (hexagon), and another 4-gon. Of all possible such labellings of a vertex, the vertex type is that one which precedes all others lexicographically; e.g. we use 3.4.6.4 in preference to 4.6.4.3, even though they describe the same vertex. For conciseness, we list a vertex of type 3.3.3.3.6 as $3^4.6$, and similarly for other types. In addition to the labelling of tilings by the

†Grünbaum and Shephard [1] use the term "homogeneous" for vertex-homogeneous and "equitransitive" for tile-homogeneous. They do not define a notion of edge-homogeneous.

vertex types of each orbit, we also use subscripts when necessary to distinguish two different tilings with otherwise similar labellings; e.g. $(3^6; 3^3.4^2)_1$ vs $(3^6; 3^3.4^2)_2$. Within a figure, the tilings are arranged lexicographically according to this labelling.

The number of orbits of each tile element is listed with each tiling; e.g. a tiling with two vertex orbits, three tile orbits, and four edge orbits would have this fact noted by the legend "$v = 2$; $t = 3$; $e = 4$" In the terminology of Ref. [1], such a tiling would be called "2-isogonal, 3-isohedral, and 4-isotoxal". For readers familiar with this terminology, the captions of the figures also use these words. Although representatives of the tile orbits and edge orbits are not noted (as with the vertices), the former is usually easy to identify, and the latter can usually be found with only a little work once you note that representatives of all edge orbits must be incident to one (or more) of the marked vertices.

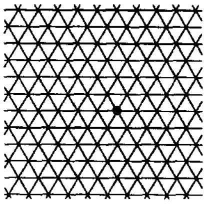
Figure 3 shows the so-called "Platonic" tilings, those whose symmetry group is transitive on the tiles (i.e. those with $t = 1$). Figure 4 shows the other "Archimedean" tilings, those whose symmetry group is transitive on the vertices ($v = 1$, but $t > 1$). Normally, "Archimedean" only means that all of the vertex figures are congruent. For the tilings considered here, this can be shown to imply that the vertex figures are all symmetric with each other; i.e. tilings with only one vertex figure are necessarily vertex-homogeneous. Figure 5 shows the tilings with $v = 2$, all of which are vertex-homogeneous. The tilings with $v = 3$ are split into two groups: those in Fig. 6 are vertex-homogeneous; those in Fig. 7 are not (hence those in Fig. 7 have two types of vertex figures, but three vertex orbits). Figures 3–7 also include all tilings with $e \leqslant 3$, since (see Ref. [2]) $e \geqslant v$. Figure 8 shows the vertex-homogeneous tilings which have four or more vertex orbits. Figure 9 shows the unique tiling which is tile-homogeneous but not vertex-homogeneous. This is also the only tiling with two orbits of tiles which is not pictured previously. Finally Fig. 10 shows the remaining nine tilings which have three orbits of tiles.

If a tiling satisfies any of the homogeneity criteria, this is also noted, except where all (or nearly all) of the tilings in some figure satisfy one of the homogeneity criteria. In this case, the homogeneity properties may be just noted in the caption to the figure (and not individually). Further, since strongly edge-homogeneous tilings are necessarily edge-homogeneous, the latter fact is not mentioned when both are true. For conciseness, "strongly edge-homogeneous" is abbreviated as "S edge-homogeneous".

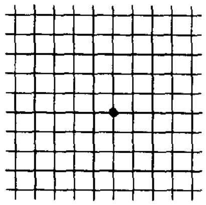
Finally, the symmetry group of each tiling is listed, using the standard crystallographic notation. Further details on the groups corresponding to this notation can be found in several places; e.g. Ref. [1, Section 1.4]. Of the 17 two-dimensional symmetry groups, all but four of them occur as the symmetry group of a tiling included here. Each of the other four groups also can arise as the symmetry group of an edge-to-edge tiling by regular polygons, but we cannot say for sure what the "simplest" of such tilings are. It is worth noting that some of the tilings in the following pictures occur in "enantiomorphic" forms; i.e. the mirror image of the tiling cannot be superimposed on the original. One example is the first tiling in Fig. 4. For the purposes of the drawings in this paper and for the numbers listed in Table 1, we do not view two enantiomorphic forms of a tiling as different tilings. Only nine of the 165 tilings included here have different enantiomorphic forms. These can easily be found by noting that a tiling has two such forms if and only if its symmetry group contains no reflections or glide-reflections; i.e. if and only if the crystallographic name of the symmetry group contains neither an "m" nor a "g". The number of "enantiomorphically different" tilings with at most three orbits of some tile element is given in Table 2.

Table 2. Classification results for enantiomorphically different tilings

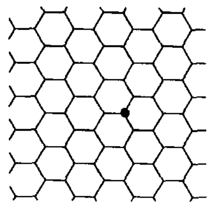
	One orbit	Two orbits	Three orbits
Tiles	3	14	26
Vertices	12	21	65
Edges	4	4	11



(3^6) p6m
$v = 1,\ t = 1,\ e = 1$

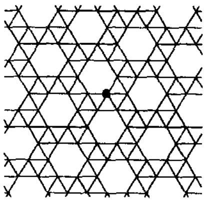


(4^4) p4m
$v = 1,\ t = 1,\ e = 1$

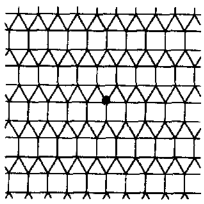


(6^3) p6m
$v = 1,\ t = 1,\ e = 1$

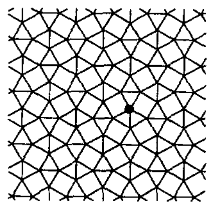
Fig. 3. The three "Platonic" tilings, i.e. those with $t = 1$ (also called "isohedral" tilings). These tilings are all homogeneous with respect to vertices, tiles and edges, and are strongly edge-homogeneous.



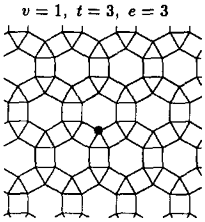
($3^4.6$) p6
$v = 1,\ t = 3,\ e = 3$



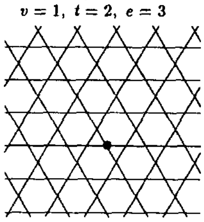
($3^3.4^2$) cmm
$v = 1,\ t = 2,\ e = 3$



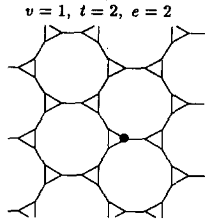
($3^2.4.3.4$) p4g
$v = 1,\ t = 2,\ e = 2$



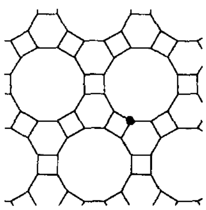
(3.4.6.4) p6m
$v = 1,\ t = 3,\ e = 2$



(3.6.3.6) p6m
$v = 1,\ t = 2,\ e = 1$



(3.12^2) p6m
$v = 1,\ t = 2,\ e = 2$



(4.6.12) p6m
$v = 1,\ t = 3,\ e = 3$

(4.8^2) p4m
$v = 1,\ t = 2,\ e = 2$

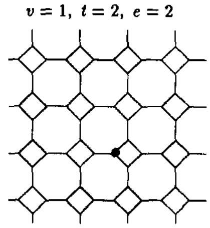
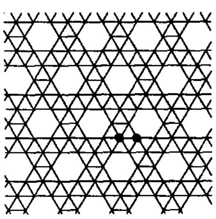
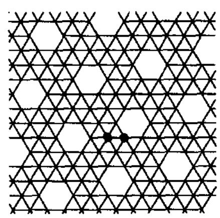


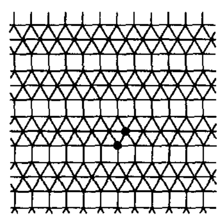
Fig. 4. The eight "Archimedean" tilings which are not Platonic; i.e. those with $v = 1$ (also called "isogonal" tilings) but where $t > 1$. These tilings are all homogeneous with respect to both vertices and edges. All but the first tiling are tile-homogeneous and strongly edge-homogeneous.



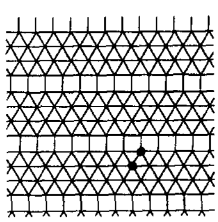
(3^6; $3^4.6$)$_1$ p6m
$v = 2,\ t = 3,\ e = 3$
edge-homogeneous



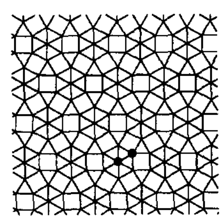
(3^6; $3^4.6$)$_2$ p6
$v = 2,\ t = 5,\ e = 7$



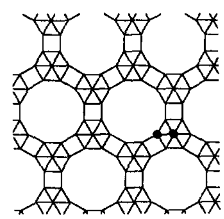
(3^6; $3^3.4^2$)$_1$ pmm
$v = 2,\ t = 3,\ e = 4$
edge-homogeneous



(3^6; $3^3.4^2$)$_2$ cmm
$v = 2,\ t = 4,\ e = 5$

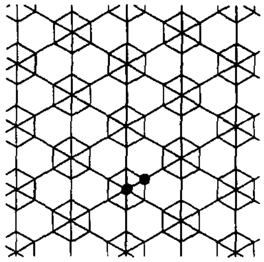


(3^6; $3^2.4.3.4$) p6m
$v = 2,\ t = 3,\ e = 3$
edge-homogeneous

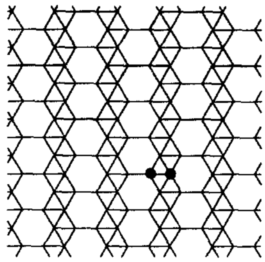


(3^6; $3^2.4.12$) p6m
$v = 2,\ t = 4,\ e = 4$
S edge-homogeneous

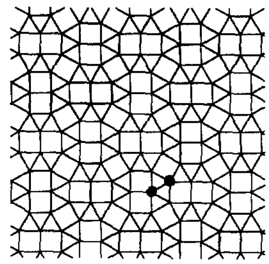
Fig. 5—*continued overleaf.*



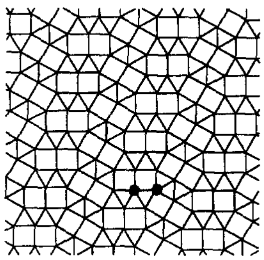
$(3^6;\ 3^2.6^2)$ p6m
$v = 2,\ t = 2,\ e = 3$
tile-, S edge-homogeneous



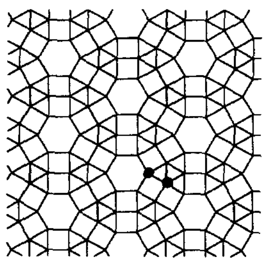
$(3^4.6;\ 3^2.6^2)$ cmm
$v = 2,\ t = 2,\ e = 4$
tile-, edge-homogeneous



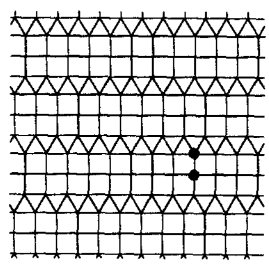
$(3^3.4^2;\ 3^2.4.3.4)_1$ p4g
$v = 2,\ t = 4,\ e = 5$
edge-homogeneous



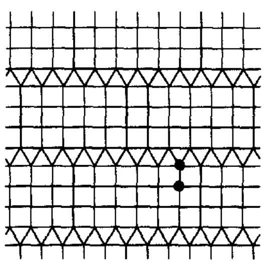
$(3^3.4^2;\ 3^2.4.3.4)_2$ pgg
$v = 2,\ t = 6,\ e = 6$
edge-homogeneous



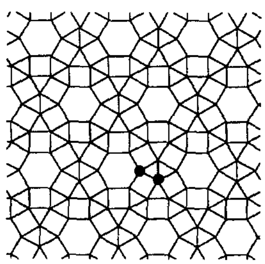
$(3^3.4^2;\ 3.4.6.4)$ p6m
$v = 2,\ t = 4,\ e = 4$
S edge-homogeneous



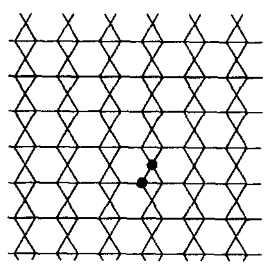
$(3^3.4^2;\ 4^4)_1$ cmm
$v = 2,\ t = 2,\ e = 4$
tile-, edge-homogeneous



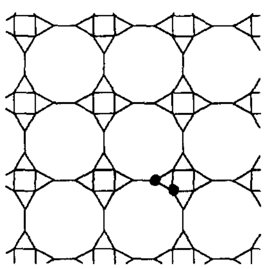
$(3^3.4^2;\ 4^4)_2$ cmm
$v = 2,\ t = 3,\ e = 5$



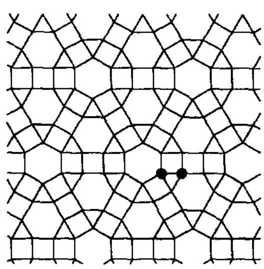
$(3^2.4.3.4;\ 3.4.6.4)$ p6m
$v = 2,\ t = 4,\ e = 4$
edge-homogeneous



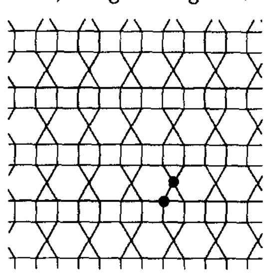
$(3^2.6^2;\ 3.6.3.6)$ pmm
$v = 2,\ t = 2,\ e = 3$
tile-, S edge-homogeneous



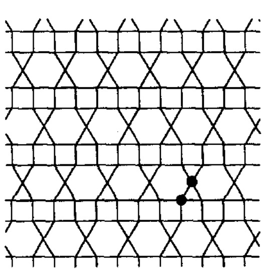
$(3.4.3.12;\ 3.12^2)$ p4m
$v = 2,\ t = 3,\ e = 3$
tile-, S edge-homogeneous



$(3.4^2.6;\ 3.4.6.4)$ p6m
$v = 2,\ t = 5,\ e = 5$
edge-homogeneous



$(3.4^2.6;\ 3.6.3.6)_1$ pmm
$v = 2,\ t = 4,\ e = 4$
S edge-homogeneous



$(3.4^2.6;\ 3.6.3.6)_2$ cmm
$v = 2,\ t = 3,\ e = 4$
tile-, S edge-homogeneous

$(3.4.6.4;\ 4.6.12)$ p6m
$v = 2,\ t = 4,\ e = 4$
tile-, S edge-homogeneous

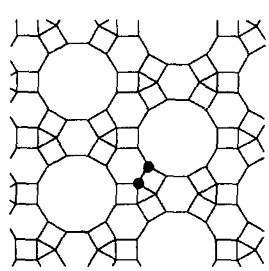
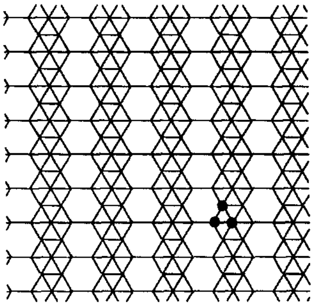
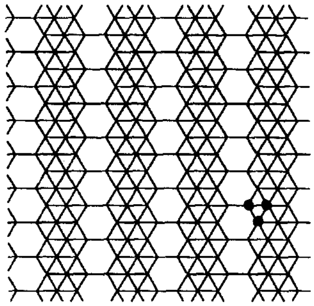


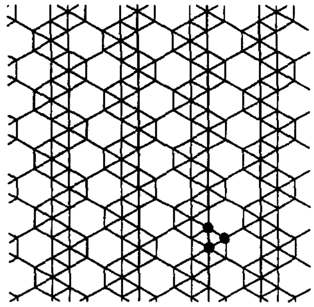
Fig. 5. The 20 2-isogonal tilings; i.e. those with $v = 2$. All of these tilings are vertex-homogeneous.



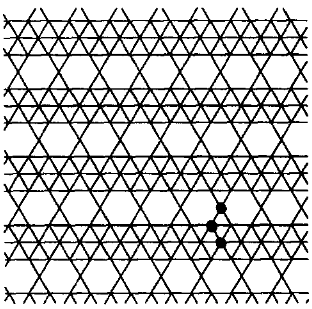
$(3^6;\ 3^4.6;\ 3^2.6^2)_1$
pmm $v = 3,\ t = 3,\ e = 5$
edge-homogeneous



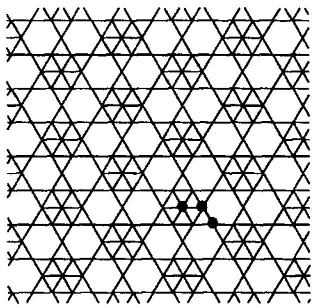
$(3^6;\ 3^4.6;\ 3^2.6^2)_2$
cmm $v = 3,\ t = 3,\ e = 6$
edge-homogeneous



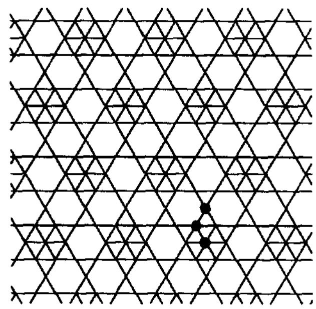
$(3^6;\ 3^4.6;\ 3^2.6^2)_3$
pmg $v = 3,\ t = 5,\ e = 8$



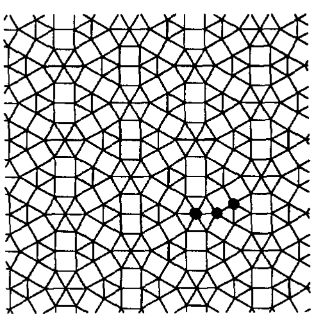
$(3^6;\ 3^4.6;\ 3.6.3.6)_1$
cmm $v = 3,\ t = 5,\ e = 6$
edge-homogeneous



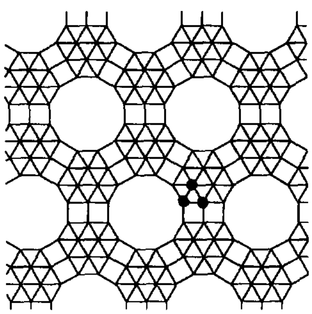
$(3^6;\ 3^4.6;\ 3.6.3.6)_2$
p6m $v = 3,\ t = 3,\ e = 3$
edge-homogeneous



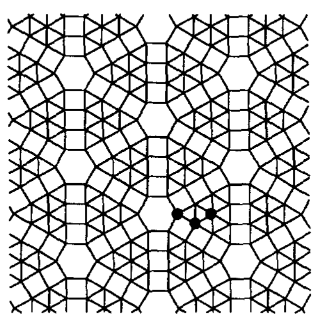
$(3^6;\ 3^4.6;\ 3.6.3.6)_3$
p6m $v = 3,\ t = 4,\ e = 4$
edge-homogeneous



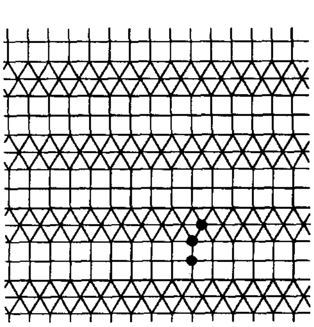
$(3^6;\ 3^3.4^2;\ 3^2.4.3.4)$
p6m $v = 3,\ t = 4,\ e = 5$
edge-homogeneous



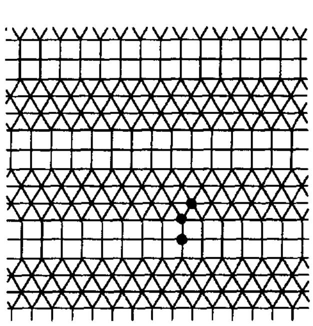
$(3^6;\ 3^3.4^2;\ 3^2.4.12)$
p6m $v = 3,\ t = 6,\ e = 7$
edge-homogeneous



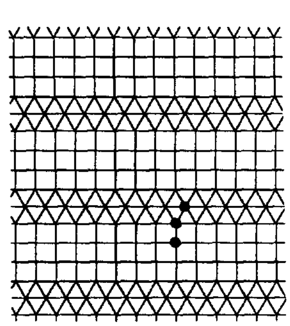
$(3^6;\ 3^3.4^2;\ 3.4.6.4)$
p6m $v = 3,\ t = 6,\ e = 6$
edge-homogeneous



$(3^6;\ 3^3.4^2;\ 4^4)_1$
pmm $v = 3,\ t = 3,\ e = 5$
edge-homogeneous

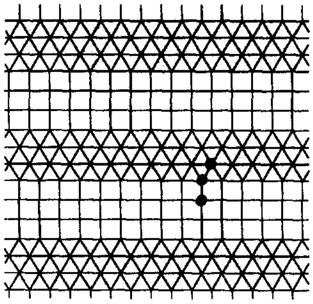


$(3^6;\ 3^3.4^2;\ 4^4)_2$
cmm $v = 3,\ t = 4,\ e = 6$

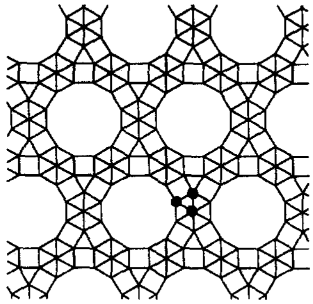


$(3^6;\ 3^3.4^2;\ 4^4)_3$
pmm $v = 3,\ t = 4,\ e = 6$

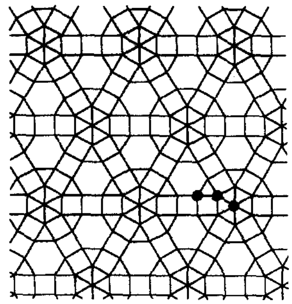
Fig. 6—*continued overleaf.*



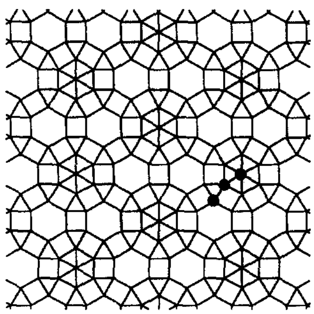
$(3^6;\ 3^3.4^2;\ 4^4)_4$
cmm $v = 3,\ t = 5,\ e = 7$



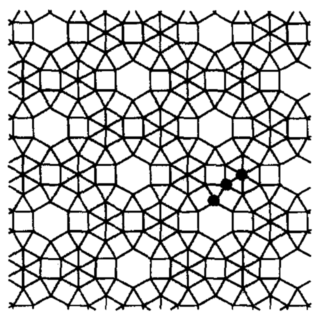
$(3^6;\ 3^2.4.3.4;\ 3^2.4.12)$
p6m $v = 3,\ t = 5,\ e = 6$
edge-homogeneous



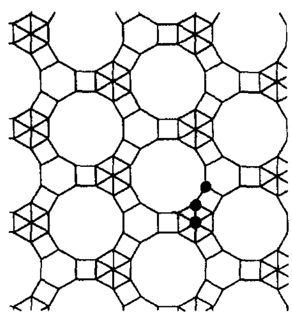
$(3^6;\ 3^2.4.3.4;\ 3.4^2.6)$
p6m $v = 3,\ t = 5,\ e = 6$
edge-homogeneous



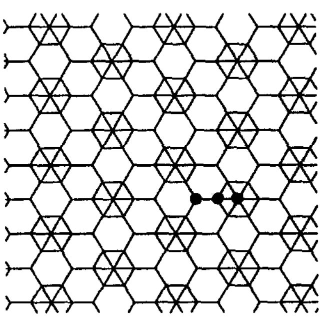
$(3^6;\ 3^2.4.3.4;\ 3.4.6.4)_1$
p6m $v = 3,\ t = 5,\ e = 6$



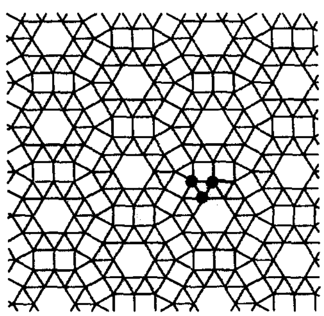
$(3^6;\ 3^2.4.3.4;\ 3.4.6.4)_2$
p6m $v = 3,\ t = 6,\ e = 6$



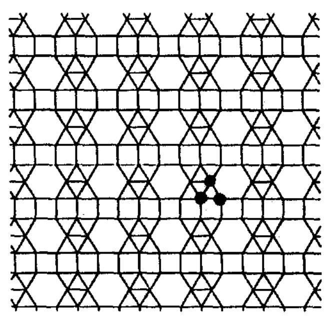
$(3^6;\ 3^2.4.12;\ 4.6.12)$
p3m1 $v = 3,\ t = 5,\ e = 6$
S edge-homogeneous



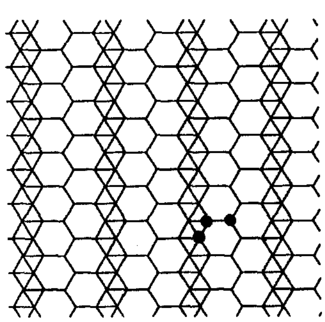
$(3^6;\ 3^2.6^2;\ 6^3)$
p6m $v = 3,\ t = 2,\ e = 3$
S edge-, tile-homogeneous



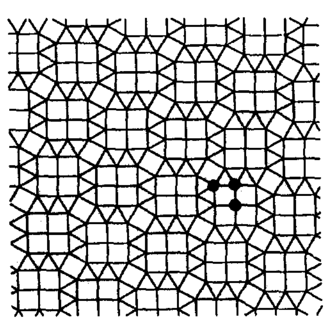
$(3^4.6;\ 3^3.4^2;\ 3^2.4.3.4)$
p6m $v = 3,\ t = 5,\ e = 6$
edge-homogeneous



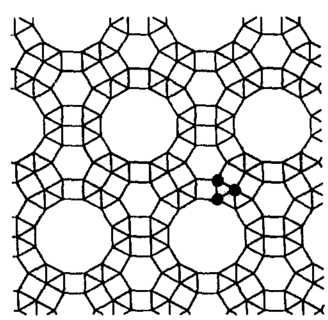
$(3^4.6;\ 3^3.4^2;\ 3.4^2.6)$
pmm $v = 3,\ t = 5,\ e = 7$
edge-homogeneous



$(3^4.6;\ 3^2.6^2;\ 6^3)$
pmg $v = 3,\ t = 2,\ e = 5$
edge-, tile-homogeneous

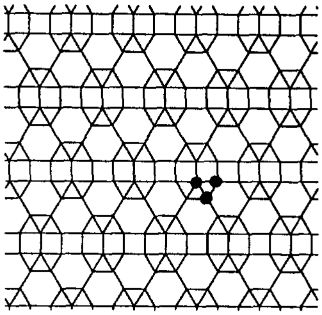


$(3^3.4^2;\ 3^2.4.3.4;\ 4^4)$
p4 $v = 3,\ t = 4,\ e = 6$
edge-homogeneous

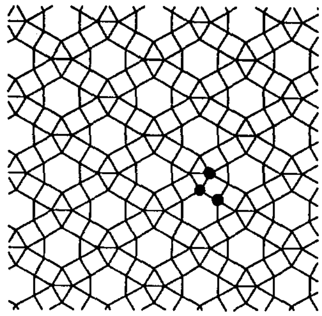


$(3^3.4^2;\ 3^2.4.12;\ 3.4.6.4)$
p6m $v = 3,\ t = 6,\ e = 8$

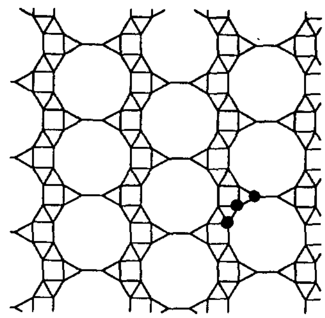
Fig. 6—*continued opposite.*



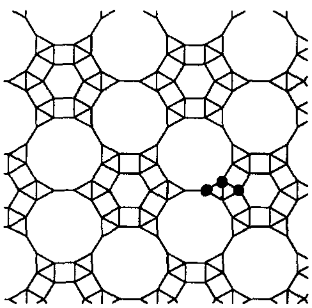
$(3^3.4^2;\ 3^2.6^2;\ 3.4^2.6)$
cmm $v = 3,\ t = 5,\ e = 8$
edge-homogeneous



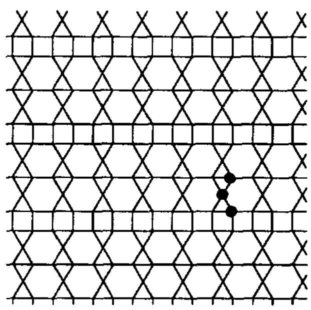
$(3^2.4.3.4;\ 3.4^2.6;\ 3.4.6.4)$
cmm $v = 3,\ t = 4,\ e = 6$
edge-homogeneous



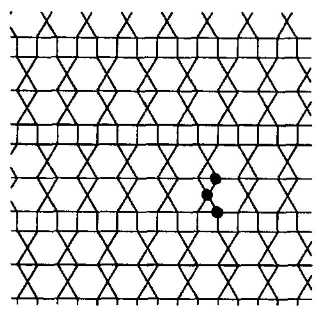
$(3^2.4.12;\ 3.4.3.12;\ 3.12^2)$
cmm $v = 3,\ t = 4,\ e = 7$
edge-homogeneous



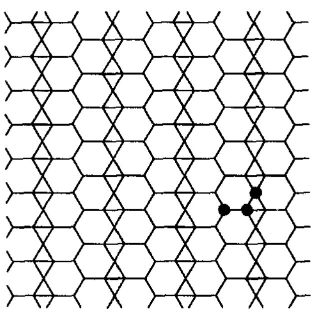
$(3^2.4.12;\ 3.4.6.4;\ 3.12^2)$
p6m $v = 3,\ t = 5,\ e = 6$
S edge-homogeneous



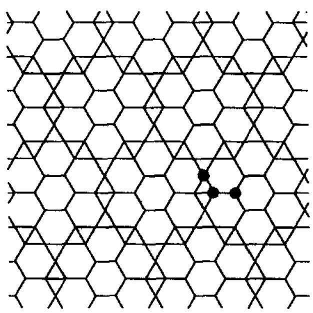
$(3^2.6^2;\ 3.4^2.6;\ 3.6.3.6)_1$
pmm $v = 3,\ t = 5,\ e = 7$
edge-homogeneous



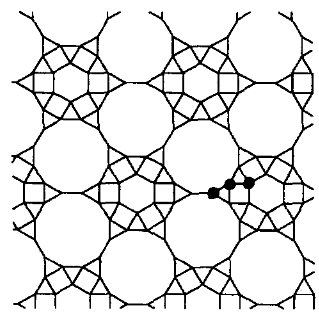
$(3^2.6^2;\ 3.4^2.6;\ 3.6.3.6)_2$
cmm $v = 3,\ t = 4,\ e = 7$
edge-homogeneous



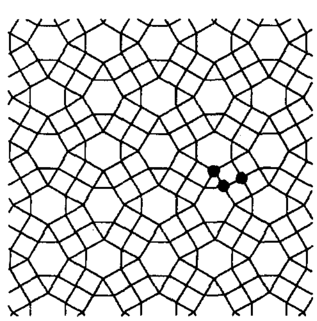
$(3^2.6^2;\ 3.6.3.6;\ 6^3)_1$
cmm $v = 3,\ t = 2,\ e = 4$
edge-, tile-homogeneous



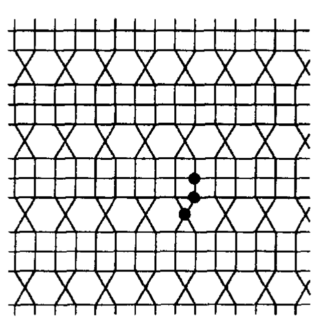
$(3^2.6^2;\ 3.6.3.6;\ 6^3)_2$
p6m $v = 3,\ t = 4,\ e = 5$
edge-homogeneous



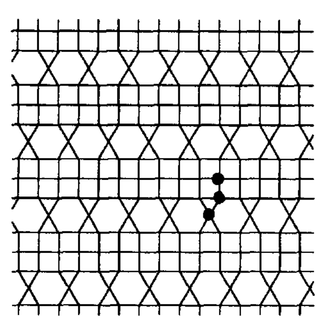
$(3.4.3.12;\ 3.4.6.4;\ 3.12^2)$
p6m $v = 3,\ t = 5,\ e = 6$
edge-homogeneous



$(3.4^2.6;\ 3.4.6.4;\ 4^4)$
p4g $v = 3,\ t = 3,\ e = 4$
S edge-, tile-homogeneous

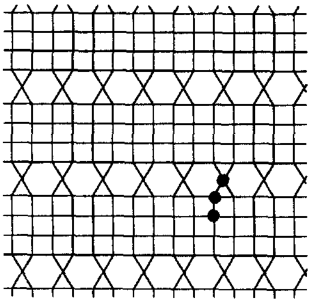


$(3.4^2.6;\ 3.6.3.6;\ 4^4)_1$
pmm $v = 3,\ t = 4,\ e = 6$

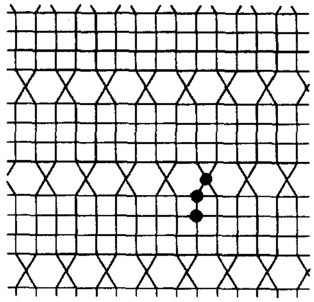


$(3.4^2.6;\ 3.6.3.6;\ 4^4)_2$
cmm $v = 3,\ t = 4,\ e = 5$
edge-homogeneous

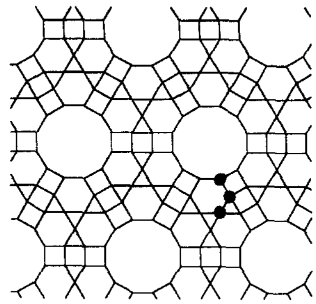
Fig. 6—*continued overleaf.*



$(3.4^2.6;\ 3.6.3.6;\ 4^4)_3$
pmm $v = 3,\ t = 6,\ e = 7$

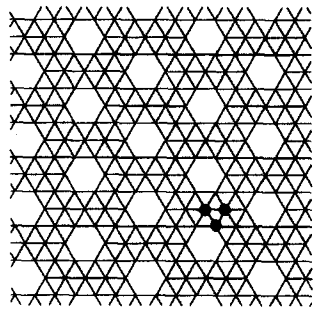


$(3.4^2.6;\ 3.6.3.6;\ 4^4)_4$
cmm $v = 3,\ t = 5,\ e = 7$

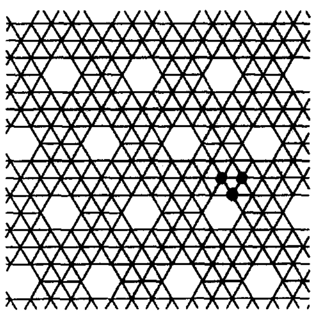


$(3.4^2.6;\ 3.6.3.6;\ 4.6.12)$
p6m $v = 3,\ t = 6,\ e = 7$
edge-homogeneous

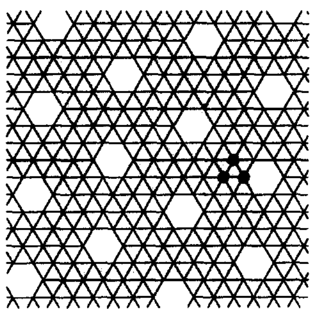
Fig. 6. The 39 3-isogonal tilings ($v = 3$) which are vertex-homogeneous.



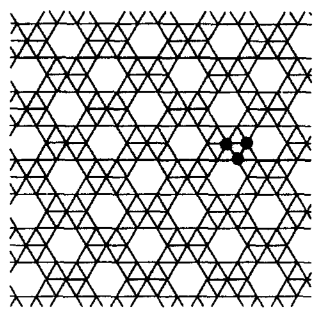
$(3^6;\ 3^6;\ 3^4.6)_1$
p6m $v = 3,\ t = 3,\ e = 4$
edge-homogeneous



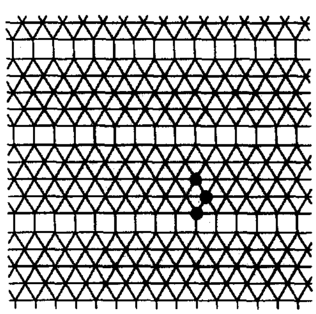
$(3^6;\ 3^6;\ 3^4.6)_2$
p6m $v = 3,\ t = 5,\ e = 5$



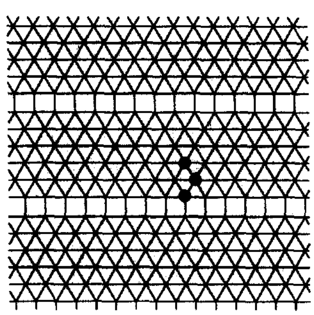
$(3^6;\ 3^6;\ 3^4.6)_3$
p6 $v = 3,\ t = 7,\ e = 9$



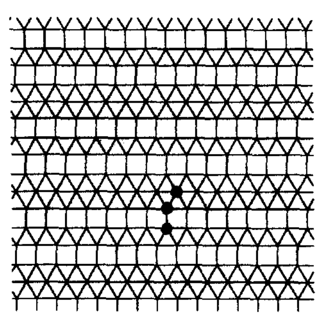
$(3^6;\ 3^4.6;\ 3^4.6)$
cmm $v = 3,\ t = 4,\ e = 6$



$(3^6;\ 3^6;\ 3^3.4^2)_1$
pmm $v = 3,\ t = 5,\ e = 6$

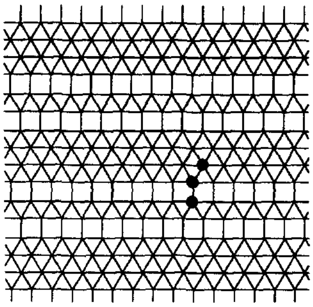


$(3^6;\ 3^6;\ 3^3.4^2)_2$
cmm $v = 3,\ t = 6,\ e = 7$

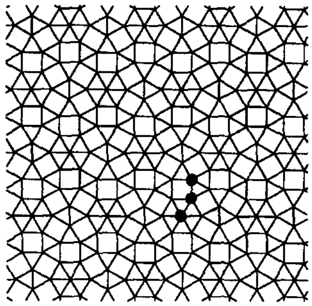


$(3^6;\ 3^3.4^2;\ 3^3.4^2)_1$
cmm $v = 3,\ t = 4,\ e = 6$

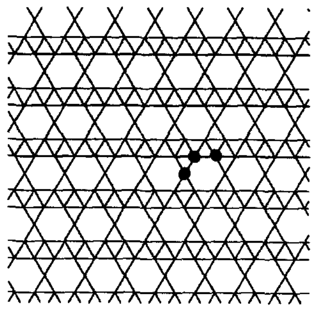
Fig. 7—*continued opposite.*



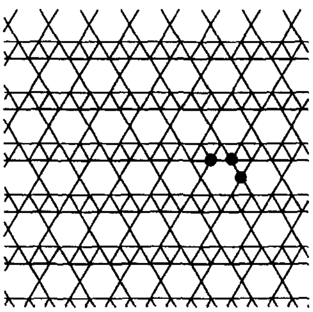
$(3^6; 3^3.4^2; 3^3.4^2)_2$
pmg $v = 3,\ t = 5,\ e = 7$



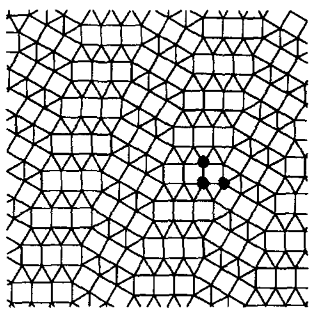
$(3^6; 3^2.4.3.4; 3^2.4.3.4)$
p6m $v = 3,\ t = 8,\ e = 5$
edge-homogeneous



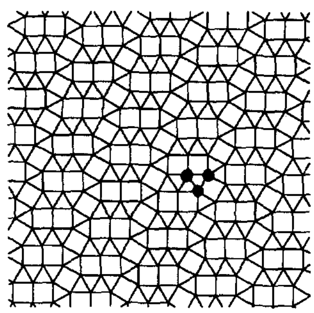
$(3^4.6; 3^4.6; 3.6.3.6)_1$
p2 $v = 3,\ t = 4,\ e = 7$



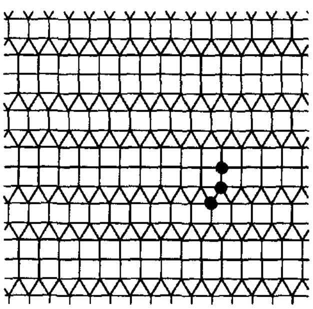
$(3^4.6; 3^4.6; 3.6.3.6)_2$
pmg $v = 3,\ t = 4,\ e = 7$



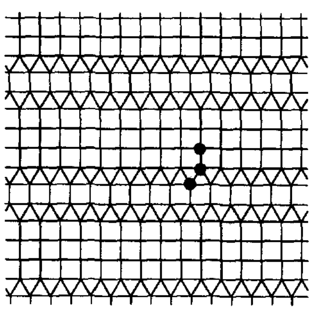
$(3^3.4^2; 3^3.4^2; 3^2.4.3.4)$
pgg $v = 3,\ t = 5,\ e = 8$



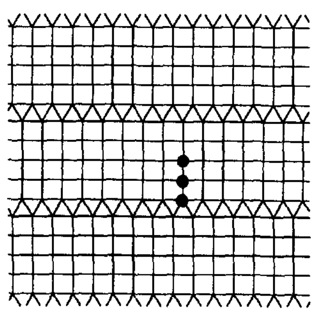
$(3^3.4^2; 3^2.4.3.4; 3^2.4.3.4)$
p2 $v = 3,\ t = 6,\ e = 9$



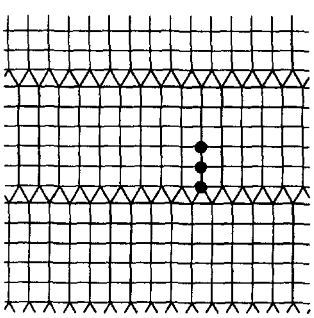
$(3^3.4^2; 3^3.4^2; 4^4)_1$
pmm $v = 3,\ t = 4,\ e = 6$



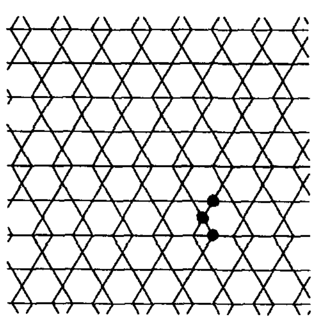
$(3^3.4^2; 3^3.4^2; 4^4)_2$
pmm $v = 3,\ t = 5,\ e = 7$



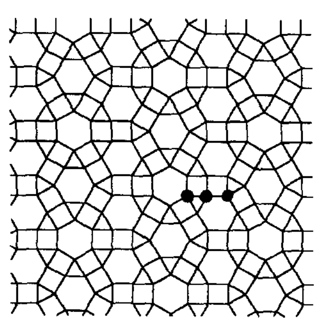
$(3^3.4^2; 4^4; 4^4)_1$
cmm $v = 3,\ t = 3,\ e = 6$



$(3^3.4^2; 4^4; 4^4)_2$
cmm $v = 3,\ t = 4,\ e = 7$

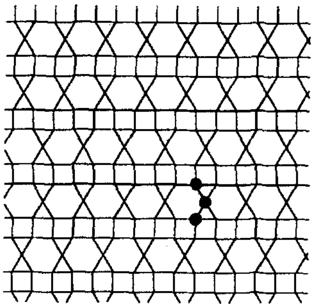


$(3^2.6^2; 3.6.3.6; 3.6.3.6)$
cmm $v = 3,\ t = 3,\ e = 5$

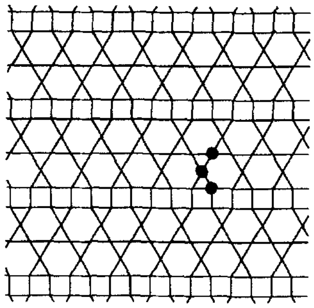


$(3.4^2.6; 3.4.6.4; 3.4.6.4)$
p6m $v = 3,\ t = 6,\ e = 6$
edge-homogeneous

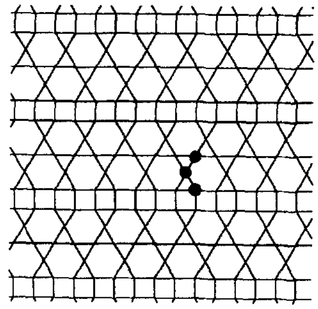
Fig. 7—*continued overleaf.*



$(3.4^2.6;\ 3.4^2.6;\ 3.6.3.6)$
cmm $v = 3,\ t = 6,\ e = 8$

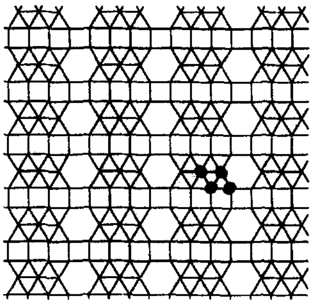


$(3.4^2.6;\ 3.6.3.6;\ 3.6.3.6)_1$
pmg $v = 3,\ t = 4,\ e = 6$

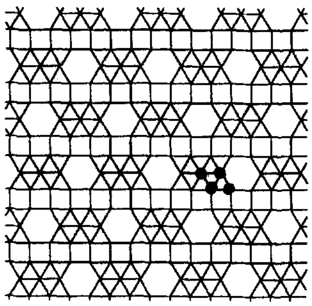


$(3.4^2.6;\ 3.6.3.6;\ 3.6.3.6)_2$
cmm $v = 3,\ t = 5,\ e = 6$

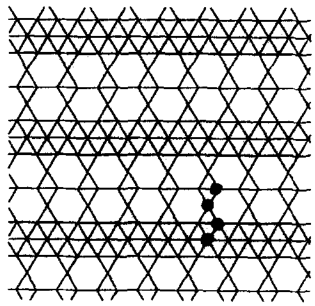
Fig. 7. The 22 3-isogonal tilings ($v = 3$) which are not vertex-homogeneous. These tilings all have two distinct vertex figures, but three orbits of vertices.



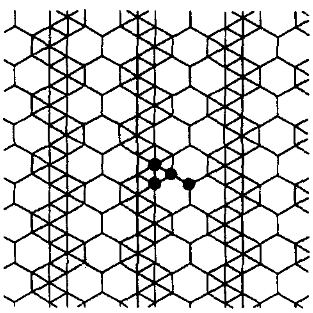
$(3^6;\ 3^4.6;\ 3^3.4^2;\ 3.4^2.6)_1$
pmm $v = 4,\ t = 7,\ e = 9$
edge-homogeneous



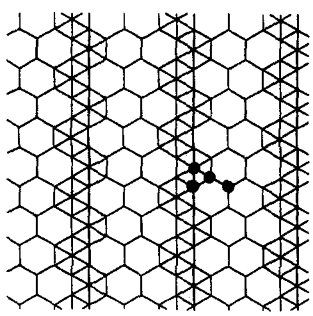
$(3^6;\ 3^4.6;\ 3^3.4^2;\ 3.4^2.6)_2$
cmm $v = 4,\ t = 6,\ e = 8$
edge-homogeneous



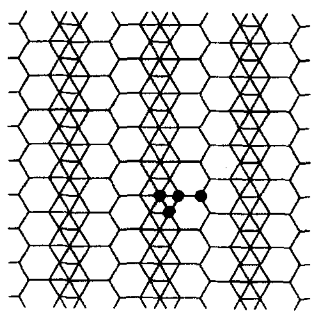
$(3^6;\ 3^4.6;\ 3^2.6^2;\ 3.6.3.6)$
cmm $v = 4,\ t = 6,\ e = 9$
edge-homogeneous



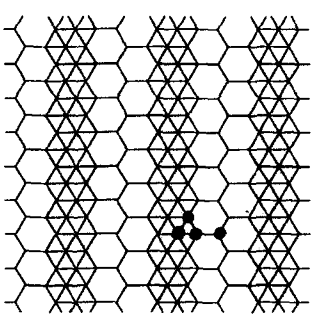
$(3^6;\ 3^4.6;\ 3^2.6^2;\ 6^3)_1$
cmm $v = 4,\ t = 6,\ e = 9$



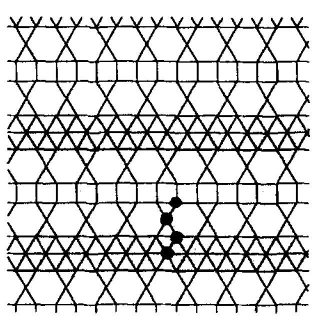
$(3^6;\ 3^4.6;\ 3^2.6^2;\ 6^3)_2$
pmg $v = 4,\ t = 6,\ e = 10$



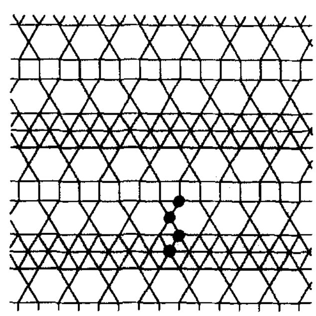
$(3^6;\ 3^4.6;\ 3^2.6^2;\ 6^3)_3$
cmm $v = 4,\ t = 3,\ e = 6$
edge-homogeneous



$(3^6;\ 3^4.6;\ 3^2.6^2;\ 6^3)_4$
pmg $v = 4,\ t = 3,\ e = 7$
edge-homogeneous

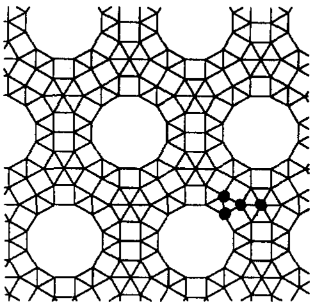


$(3^6;\ 3^4.6;\ 3.4^2.6;\ 3.6.3.6)_1$
cmm $v = 4,\ t = 8,\ e = 10$
edge-homogeneous

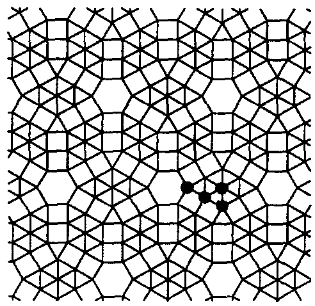


$(3^6;\ 3^4.6;\ 3.4^2.6;\ 3.6.3.6)_2$
pmg $v = 4,\ t = 7,\ e = 10$
edge-homogeneous

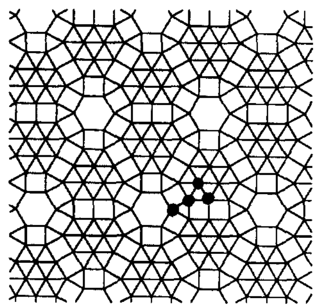
Fig. 8—*continued opposite*.



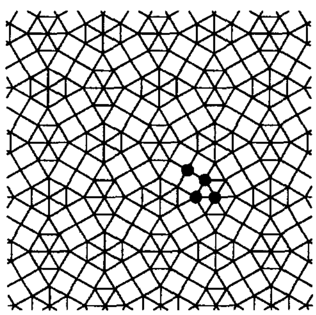
$(3^6;\ 3^3.4^2;\ 3^2.4.3.4;\ 3^2.4.12)$
p6m $v = 4,\ t = 7,\ e = 9$



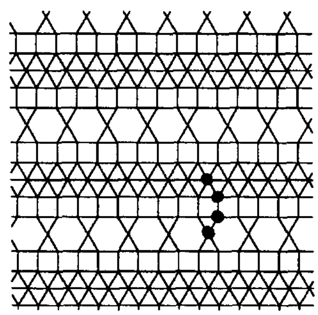
$(3^6;\ 3^3.4^2;\ 3^2.4.3.4;\ 3.4.6.4)_1$
p6m $v = 4,\ t = 7,\ e = 8$
edge-homogeneous



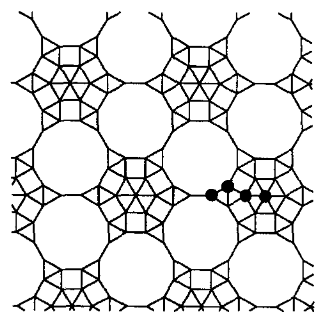
$(3^6;\ 3^3.4^2;\ 3^2.4.3.4;\ 3.4.6.4)_2$
p6m $v = 4,\ t = 8,\ e = 9$
edge-homogeneous



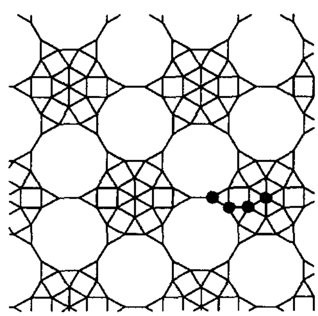
$(3^6;\ 3^3.4^2;\ 3^2.4.3.4;\ 4^4)$
p4g $v = 4,\ t = 4,\ e = 6$
edge-homogeneous



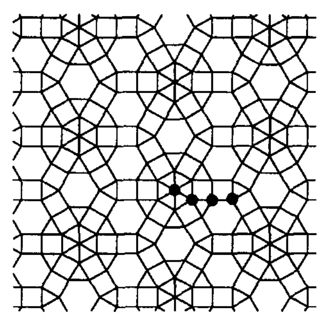
$(3^6;\ 3^3.4^2;\ 3.4^2.6;\ 3.6.3.6)$
cmm $v = 4,\ t = 7,\ e = 9$



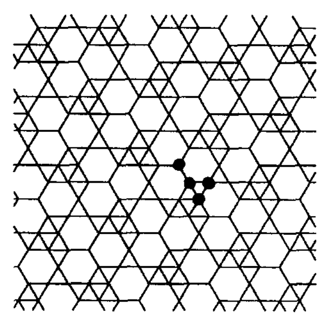
$(3^6;\ 3^2.4.3.4;\ 3^2.4.12;\ 3.12^2)$
p6m $v = 4,\ t = 5,\ e = 7$
edge-homogeneous



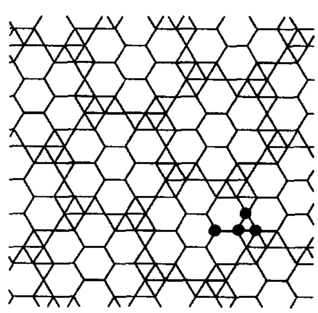
$(3^6;\ 3^2.4.3.4;\ 3.4.3.12;\ 3.12^2)$
p6m $v = 4,\ t = 5,\ e = 7$
edge-homogeneous



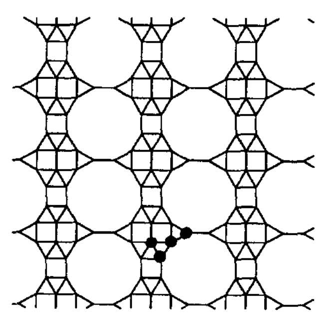
$(3^6;\ 3^2.4.3.4;\ 3.4^2.6;\ 3.4.6.4)$
p6m $v = 4,\ t = 6,\ e = 7$
edge-homogeneous



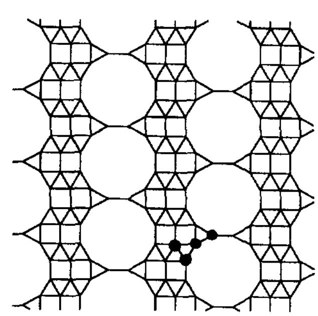
$(3^4.6;\ 3^2.6^2;\ 3.6.3.6;\ 6^3)_1$
p3 $v = 4,\ t = 5,\ e = 7$
edge-homogeneous



$(3^4.6;\ 3^2.6^2;\ 3.6.3.6;\ 6^3)_2$
p6 $v = 4,\ t = 5,\ e = 7$
edge-homogeneous

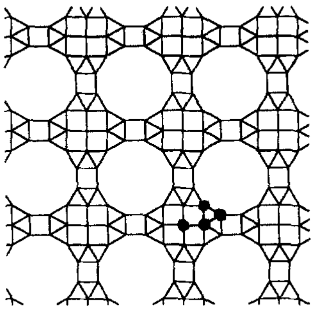


$(3^3.4^2;\ 3^2.4.12;\ 3.4.3.12;\ 3.12^2)_1$
pmm $v = 4,\ t = 6,\ e = 9$
edge-homogeneous

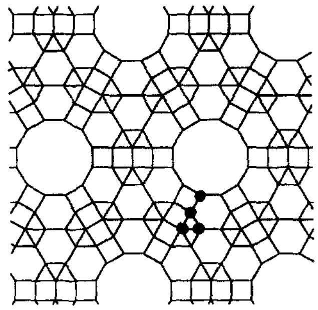


$(3^3.4^2;\ 3^2.4.12;\ 3.4.3.12;\ 3.12^2)_2$
cmm $v = 4,\ t = 6,\ e = 10$
edge-homogeneous

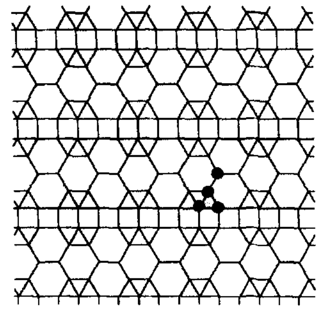
Fig. 8—*continued overleaf.*



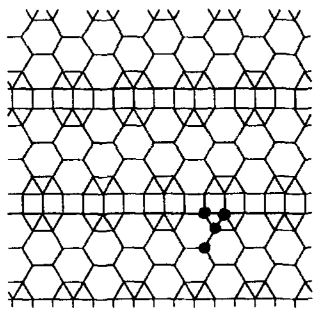
$(3^3.4^2;\ 3^2.4.12;\ 3.4.3.12;\ 4^4)$
p4m $v = 4,\ t = 5,\ e = 6$
edge-homogeneous



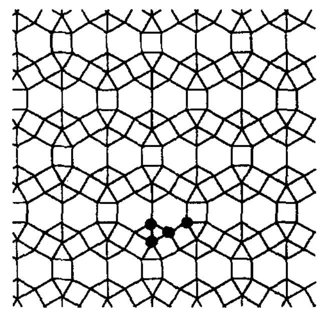
$(3^3.4^2;\ 3^2.6^2;\ 3.4^2.6;\ 4.6.12)$
p6m $v = 4,\ t = 7,\ e = 10$
edge-homogeneous



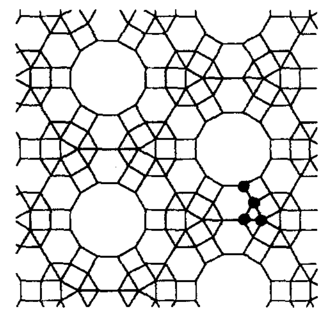
$(3^3.4^2;\ 3^2.6^2;\ 3.4^2.6;\ 6^3)_1$
pmm $v = 4,\ t = 6,\ e = 9$
edge-homogeneous



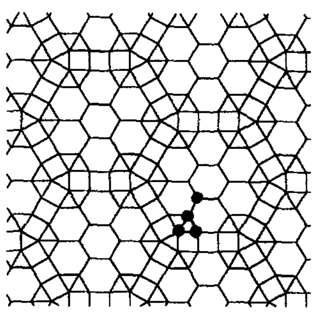
$(3^3.4^2;\ 3^2.6^2;\ 3.4^2.6;\ 6^3)_2$
cmm $v = 4,\ t = 6,\ e = 10$



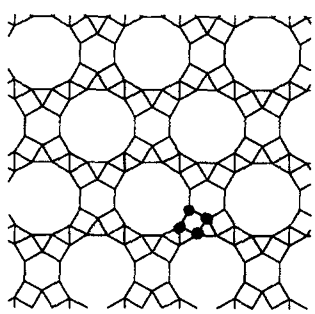
$(3^2.4.3.4;\ 3^2.6^2;\ 3.4^2.6;\ 3.4.6.4)$
cmm $v = 4,\ t = 5,\ e = 9$
edge-homogeneous



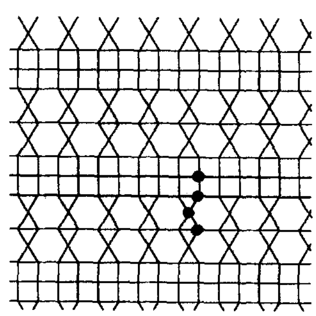
$(3^2.4.3.4;\ 3^2.6^2;\ 3.4^2.6;\ 4.6.12)$
p6m $v = 4,\ t = 6,\ e = 9$
edge-homogeneous



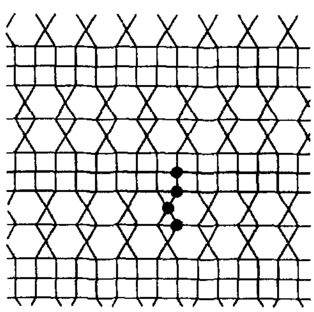
$(3^2.4.3.4;\ 3^2.6^2;\ 3.4^2.6;\ 6^3)$
p6m $v = 4,\ t = 6,\ e = 8$
edge-homogeneous



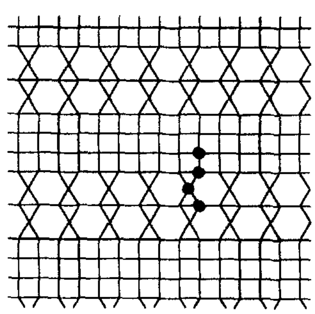
$(3^2.4.12;\ 3.4.3.12;\ 3.4.6.4;\ 4.6.12)$
cmm $v = 4,\ t = 5,\ e = 8$
edge-homogeneous



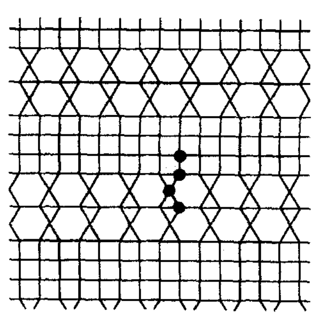
$(3^2.6^2;\ 3.4^2.6;\ 3.6.3.6;\ 4^4)_1$
pmm $v = 4,\ t = 5,\ e = 9$



$(3^2.6^2;\ 3.4^2.6;\ 3.6.3.6;\ 4^4)_2$
cmm $v = 4,\ t = 5,\ e = 8$
edge-homogeneous

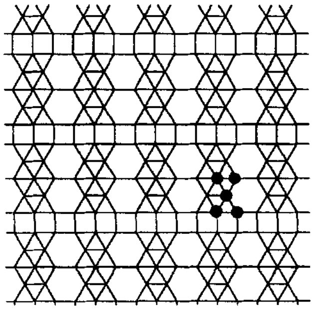


$(3^2.6^2;\ 3.4^2.6;\ 3.6.3.6;\ 4^4)_3$
pmm $v = 4,\ t = 7,\ e = 10$

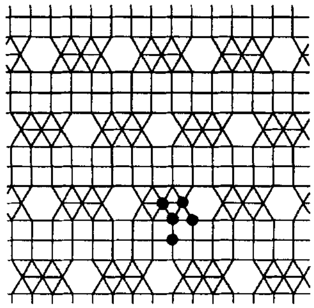


$(3^2.6^2;\ 3.4^2.6;\ 3.6.3.6;\ 4^4)_4$
cmm $v = 4,\ t = 6,\ e = 10$

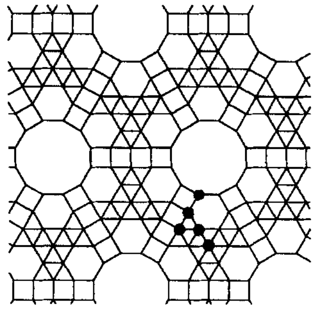
Fig. 8—*continued opposite.*



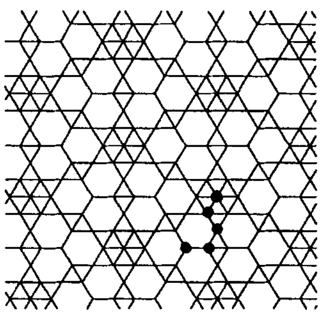
$(3^6;\ 3^4.6;\ 3^3.4^2;\ 3^2.6^2;\ 3.4^2.6)$
pmm $v = 5,\ t = 7,\ e = 11$
edge-homogeneous



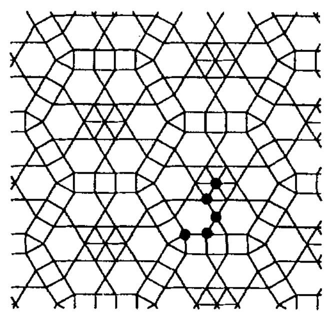
$(3^6;\ 3^4.6;\ 3^3.4^2;\ 3.4^2.6;\ 4^4)$
cmm $v = 5,\ t = 7,\ e = 11$



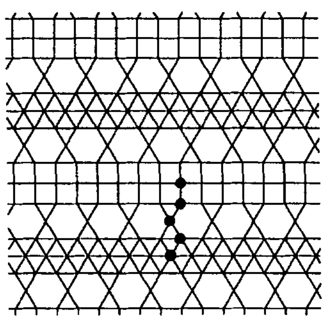
$(3^6;\ 3^4.6;\ 3^3.4^2;\ 3.4^2.6;\ 4.6.12)$
p6m $v = 5,\ t = 8,\ e = 11$
edge-homogeneous



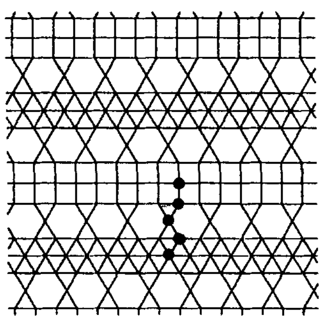
$(3^6;\ 3^4.6;\ 3^2.6^2;\ 3.6.3.6;\ 6^3)$
p6m $v = 5,\ t = 4,\ e = 6$
edge-homogeneous



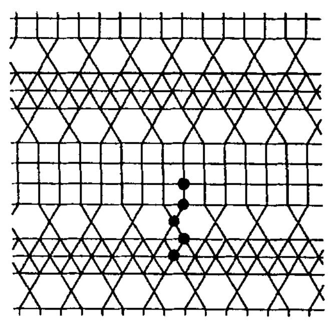
$(3^6;\ 3^4.6;\ 3.4^2.6;\ 3.4.6.4;\ 3.6.3.6)$
p6m $v = 5,\ t = 7,\ e = 8$
edge-homogeneous



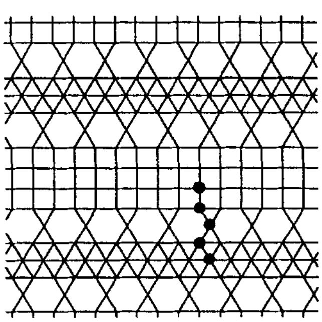
$(3^6;\ 3^4.6;\ 3.4^2.6;\ 3.6.3.6;\ 4^4)_1$
pmg $v = 5,\ t = 8,\ e = 11$
edge-homogeneous



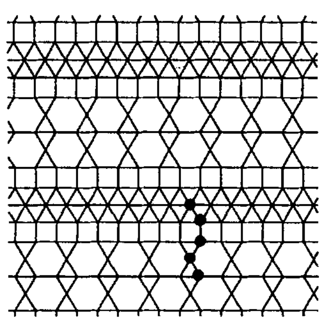
$(3^6;\ 3^4.6;\ 3.4^2.6;\ 3.6.3.6;\ 4^4)_2$
cmm $v = 5,\ t = 8,\ e = 12$



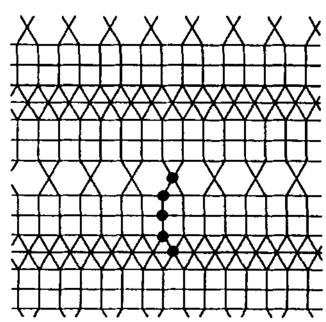
$(3^6;\ 3^4.6;\ 3.4^2.6;\ 3.6.3.6;\ 4^4)_3$
pmg $v = 5,\ t = 9,\ e = 13$



$(3^6;\ 3^4.6;\ 3.4^2.6;\ 3.6.3.6;\ 4^4)_4$
cmm $v = 5,\ t = 10,\ e = 13$

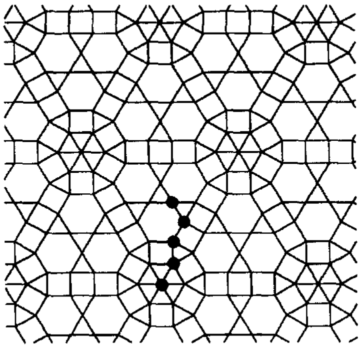


$(3^6;\ 3^3.4^2;\ 3^2.6^2;\ 3.4^2.6;\ 3.6.3.6)$
cmm $v = 5,\ t = 8,\ e = 12$

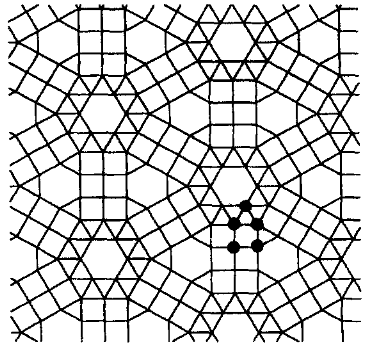


$(3^6;\ 3^3.4^2;\ 3.4^2.6;\ 3.6.3.6;\ 4^4)$
cmm $v = 5,\ t = 9,\ e = 12$

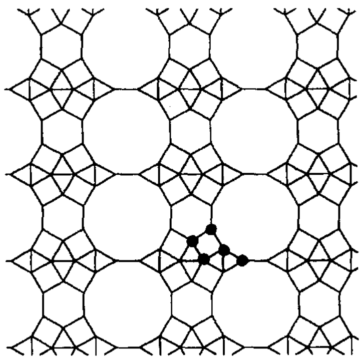
Fig 8—*continued overleaf.*



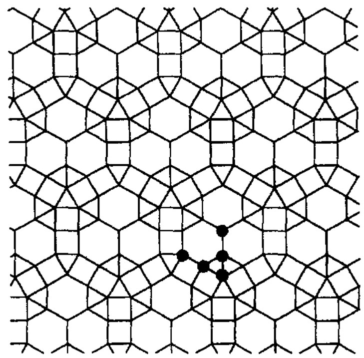
(3^6; $3^2.4.3.4$; $3.4^2.6$; $3.4.6.4$; $3.6.3.6$)
p6m $v = 5$, $t = 8$, $e = 10$
edge-homogeneous



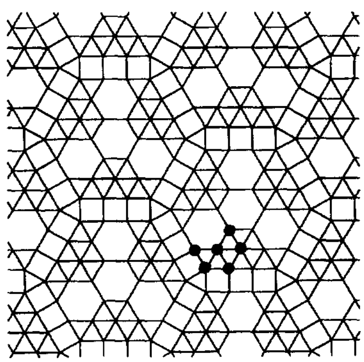
($3^4.6$; $3^3.4^2$; $3^2.4.3.4$; $3.4^2.6$; 4^4)
p6m $v = 5$, $t = 7$, $e = 10$
edge-homogeneous



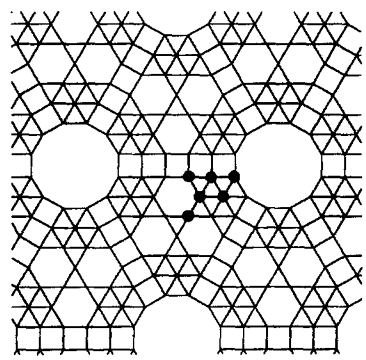
($3^2.4.3.4$; $3^2.4.12$; $3.4.6.4$; 3.12^2; $4.6.12$)
pmm $v = 5$, $t = 6$, $e = 9$
edge-homogeneous



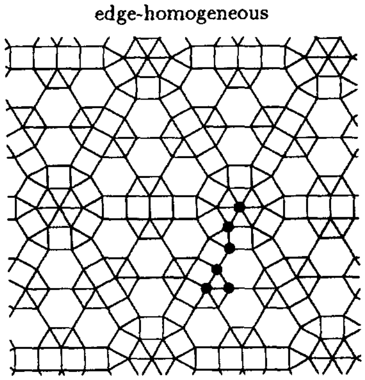
($3^2.4.3.4$; $3^2.6^2$; $3.4^2.6$; $3.4.6.4$; 6^3)
p3m1 $v = 5$, $t = 6$, $e = 8$
edge-homogeneous



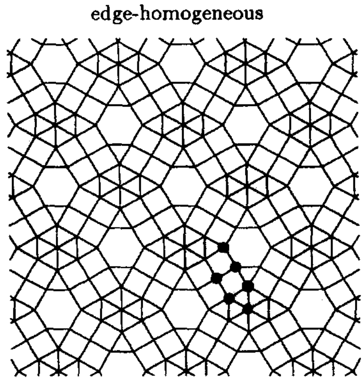
(3^6; $3^4.6$; $3^3.4^2$; $3^2.4.3.4$; $3^2.6^2$; $3.4^2.6$)
p31m $v = 6$, $t = 11$, $e = 15$
edge-homogeneous



(3^6; $3^4.6$; $3^3.4^2$; $3^2.4.12$; $3.4^2.6$; $3.6.3.6$)
p6m $v = 6$, $t = 12$, $e = 15$
edge-homogeneous

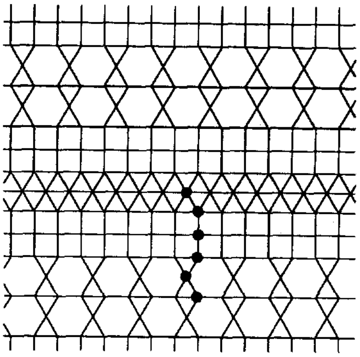


(3^6; $3^3.4^2$; $3^2.4.3.4$; $3^2.6^2$; $3.4^2.6$; $3.4.6.4$)
p6m $v = 6$, $t = 9$, $e = 13$
edge-homogeneous

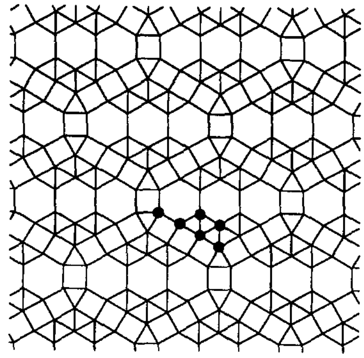


(3^6; $3^3.4^2$; $3^2.4.3.4$; $3.4^2.6$; $3.4.6.4$; 4^4)
cmm $v = 6$, $t = 7$, $e = 10$
edge-homogeneous

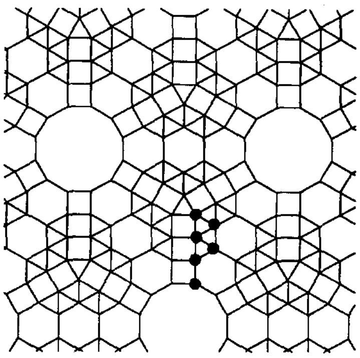
Fig. 8—*continued opposite.*



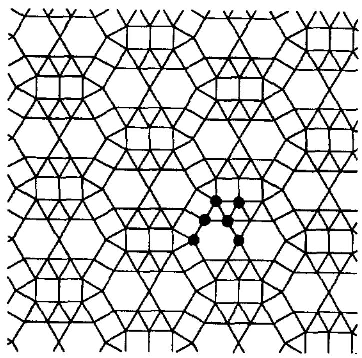
$(3^6;\ 3^3.4^2;\ 3^2.6^2;\ 3.4^2.6;\ 3.6.3.6;\ 4^4)$
cmm $v = 6,\ t = 10,\ e = 15$



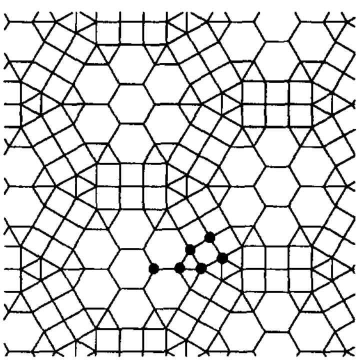
$(3^4.6;\ 3^3.4^2;\ 3^2.4.3.4;\ 3^2.6^2;\ 3.4^2.6;\ 3.4.6.4)$
cmm $v = 6,\ t = 9,\ e = 13$
edge-homogeneous



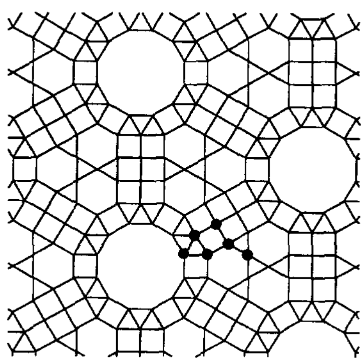
$(3^4.6;\ 3^3.4^2;\ 3^2.4.3.4;\ 3^2.6^2;\ 3.4^2.6;\ 4.6.12)$
p6m $v = 6,\ t = 10,\ e = 14$
edge-homogeneous



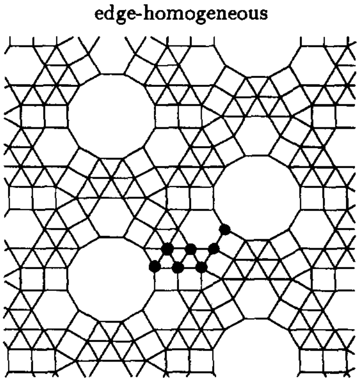
$(3^4.6;\ 3^3.4^2;\ 3^2.4.3.4;\ 3.4^2.6;\ 3.4.6.4;\ 3.6.3.6)$
cmm $v = 6,\ t = 8,\ e = 12$
edge-homogeneous



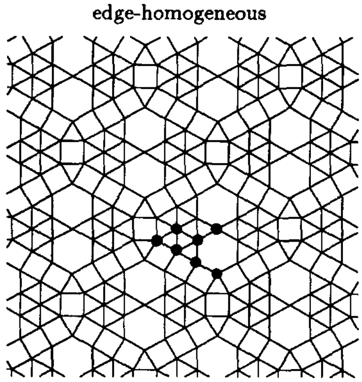
$(3^3.4^2;\ 3^2.4.3.4;\ 3^2.6^2;\ 3.4^2.6;\ 4^4;\ 6^3)$
p6m $v = 6,\ t = 7,\ e = 11$
edge-homogeneous



$(3^3.4^2;\ 3^2.4.12;\ 3.4^2.6;\ 3.4.6.4;\ 3.6.3.6;\ 4^4)$
p6m $v = 6,\ t = 9,\ e = 13$
edge-homogeneous

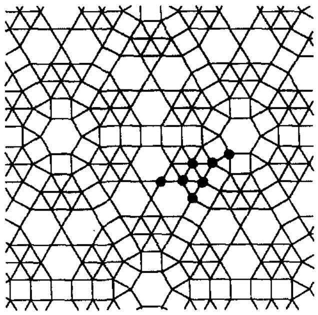


$(3^6;\ 3^4.6;\ 3^3.4^2;\ 3^2.4.3.4;\ 3^2.4.12;\ 3.4^2.6;\ 4.6.12)$
p31m $v = 7,\ t = 11,\ e = 16$
edge-homogeneous

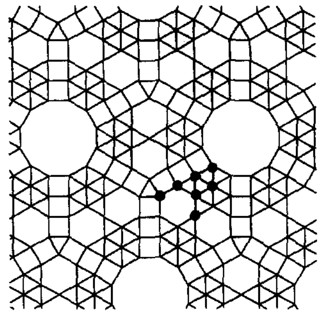


$(3^6;\ 3^4.6;\ 3^3.4^2;\ 3^2.4.3.4;\ 3.4^2.6;\ 3.4.6.4;\ 3.6.3.6)_1$
cmm $v = 7,\ t = 11,\ e = 14$
edge-homogeneous

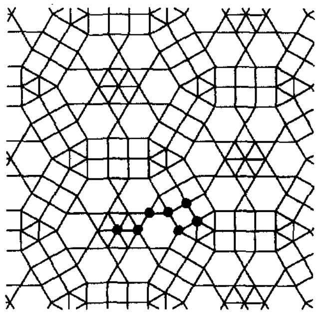
Fig. 8—*continued overleaf.*



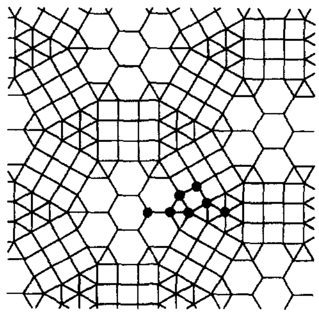
(3^6; $3^4.6$; $3^3.4^2$; $3^2.4.3.4$; $3.4^2.6$; $3.4.6.4$; $3.6.3.6)_2$
p6m $v = 7$, $t = 14$, $e = 17$
edge-homogeneous



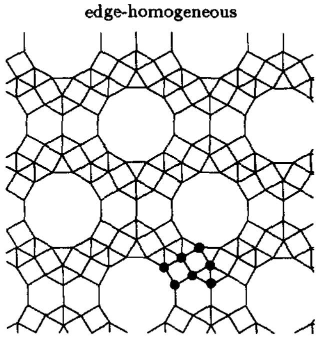
(3^6; $3^4.6$; $3^3.4^2$; $3^2.4.12$; $3.4^2.6$; $3.4.6.4$; $3.6.3.6$)
p6m $v = 7$, $t = 12$, $e = 15$
edge-homogeneous



(3^6; $3^4.6$; $3^3.4^2$; $3.4^2.6$; $3.4.6.4$; $3.6.3.6$; 4^4)
p6m $v = 7$, $t = 8$, $e = 11$
edge-homogeneous

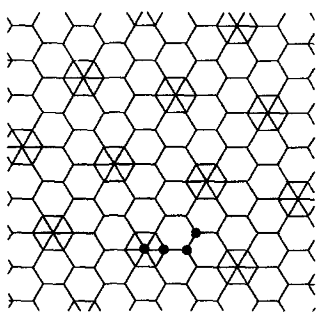


(3^6; $3^3.4^2$; $3^2.4.3.4$; $3^2.6^2$; $3.4^2.6$; 4^4; 6^3)
p6m $v = 7$, $t = 10$, $e = 14$



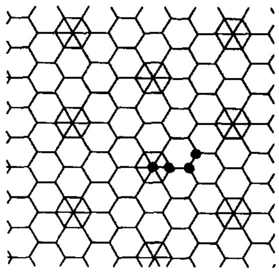
($3^3.4^2$; $3^2.4.3.4$; $3^2.4.12$; $3^2.6^2$; $3.4.3.12$; $3.4^2.6$; $4.6.12$)
cmm $v = 7$, $t = 8$, $e = 15$
edge-homogeneous

Fig. 8. The vertex-homogeneous tilings with $v \geqslant 4$.

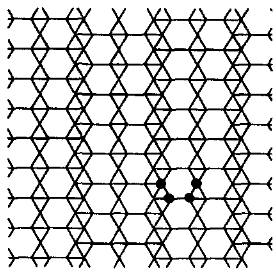


(3^6; $3^2.6^2$; 6^3; $6^3)_1$
p6 $v = 4$, $t = 2$, $e = 5$
tile-homogeneous

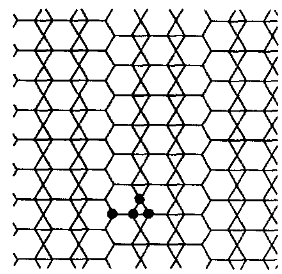
Fig. 9. The unique tile-homogeneous tiling which is not vertex-homogeneous. This is also the unique tiling with $t = 2$ which is not included in Figs 3–5.



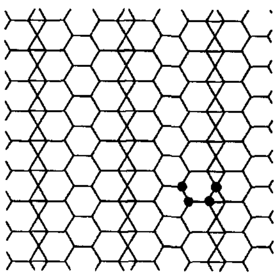
$(3^6; 3^2.6^2; 6^3; 6^3)_2$
p6m $v = 4$, $t = 3$, $e = 5$



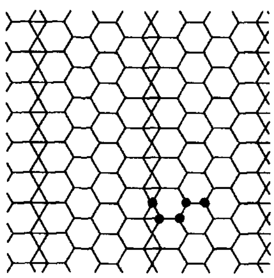
$(3^4.6; 3^2.6^2; 3^2.6^2; 3.6.3.6)$
cmm $v = 4$, $t = 3$, $e = 6$
edge-homogeneous



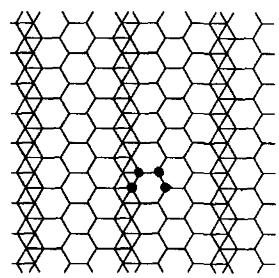
$(3^2.6^2; 3^2.6^2; 3.6.3.6; 6^3)$
cmm $v = 4$, $t = 3$, $e = 6$



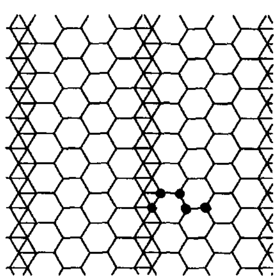
$(3^2.6^2; 3.6.3.6; 6^3; 6^3)$
pmm $v = 4$, $t = 3$, $e = 5$



$(3^2.6^2; 3.6.3.6; 6^3; 6^3; 6^3)$
cmm $v = 5$, $t = 3$, $e = 6$



$(3^4.6; 3^2.6^2; 6^3; 6^3)$
cmm $v = 4$, $t = 3$, $e = 6$



$(3^4.6; 3^2.6^2; 6^3; 6^3; 6^3)$
pmg $v = 5$, $t = 3$, $e = 7$

Fig. 10. The seven tilings with $t = 3$ which are not included in the earlier figures.

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