

# Nucleon–Nucleon Scattering: $R$ - and $S$ -Matrix Formalism

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## 1 Introduction

In quantum scattering theory, the evolution of an interacting two-body system is elegantly encoded in the  $S$ -matrix, while the  $R$ -matrix offers a numerically convenient alternative based on boundary matching. This document reviews the definitions, physical interpretations, and mathematical derivations of both quantities, particularly within the context of nucleon–nucleon scattering.

## 2 The $S$ -Matrix and $R$ -Matrix: Definitions and Interpretations

### 2.1 The Scattering Matrix ( $S$ -Matrix)

**Definition:** The  $S$ -matrix connects asymptotic incoming and outgoing states:

$$|\text{out}\rangle = S|\text{in}\rangle$$

**Physical Role:**

- Encodes all observable aspects of scattering: phase shifts, cross sections, and mixing.
- Ensures conservation of probability via unitarity:  $S^\dagger S = I$ .
- For uncoupled channels:  $S_\ell = e^{2i\delta_\ell}$ , where  $\delta_\ell$  is the phase shift.

## 2.2 The Reactance Matrix ( $R$ -Matrix)

**Definition:** Defined via the logarithmic derivative of the wavefunction at the boundary of the interaction region:

$$R_\ell(E) = \left. \frac{a u'_\ell(a)}{u_\ell(a)} \right|_{\text{internal}}$$

**Physical Role:**

- Arises from dividing configuration space into internal ( $r < a$ ) and external ( $r > a$ ) regions.
- Useful for resonance physics and numerical stability.
- Related to the  $S$ -matrix via:

$$S = \frac{1 + iR}{1 - iR}$$

## 3 From Schrödinger Equation to Scattering Matrices

### 3.1 Radial Schrödinger Equation

Consider two nucleons interacting via a central potential  $V(r)$ . The time-independent Schrödinger equation reads:

$$\left[ -\frac{\hbar^2}{2\mu} \nabla^2 + V(r) \right] \psi(\mathbf{r}) = E \psi(\mathbf{r})$$

**Partial Wave Expansion:** Using spherical symmetry:

$$\psi(\mathbf{r}) = \sum_{\ell m} \frac{u_\ell(r)}{r} Y_{\ell m}(\hat{r})$$

The radial equation becomes:

$$\left[ -\frac{\hbar^2}{2\mu} \frac{d^2}{dr^2} + \frac{\hbar^2 \ell(\ell+1)}{2\mu r^2} + V(r) \right] u_\ell(r) = E u_\ell(r)$$

**Asymptotic Behavior:** For  $r \rightarrow \infty$  (free motion), define the wave number  $k = \sqrt{2\mu E}/\hbar$ . Then:

$$u_\ell(r) \xrightarrow{r \rightarrow \infty} \sin \left( kr - \frac{\ell\pi}{2} + \delta_\ell \right)$$

### 3.2 Definition of the $S$ -Matrix

Rewriting the asymptotic form as a combination of incoming and outgoing spherical waves:

$$u_\ell(r) \sim \frac{1}{2i} \left[ e^{-i(kr - \ell\pi/2)} - S_\ell e^{i(kr - \ell\pi/2)} \right]$$

This identifies  $S_\ell = e^{2i\delta_\ell}$ .

### 3.3 Definition of the $R$ -Matrix

In the  $R$ -matrix framework:

- **Internal region:**  $r < a$ , where interactions occur.
- **External region:**  $r > a$ , free-particle motion.

In the external region, the general solution is:

$$u_\ell(r) = A_\ell [F_\ell(kr) \cos \delta_\ell + G_\ell(kr) \sin \delta_\ell]$$

Matching this with the internal solution at  $r = a$ , one derives [1]:

$$S_\ell = \frac{1 + iR_\ell}{1 - iR_\ell}$$

## 4 Matrix Structure: Uncoupled and Coupled Channels

### 4.1 Summary Table: Key Characteristics

Quantity	Uncoupled	Coupled (Stapp)	Coupled (BB)
$S$	$e^{2i\delta}$	$\text{diag}(e^{2i\delta_i}) O(\epsilon) \text{diag}(e^{2i\delta_i})$	$O^T(\epsilon) \text{diag}(e^{2i\delta_i}) O(\epsilon)$
$R$	$\tan \delta$	$\text{diag}(\tan \delta_i) O(\epsilon) \text{diag}(\tan \delta_i)$	$O^T(\epsilon) \text{diag}(\tan \delta_i) O(\epsilon)$
$O$	–	$O(\epsilon) = \begin{pmatrix} \cos 2\epsilon & i \sin 2\epsilon \\ i \sin 2\epsilon & \cos 2\epsilon \end{pmatrix}$	$O(\epsilon) = \begin{pmatrix} \cos \epsilon & \sin \epsilon \\ -\sin \epsilon & \cos \epsilon \end{pmatrix}$

Table 1: Summary of  $S$ - and  $R$ -matrix structures in uncoupled and coupled cases, using Stapp [2] and Blatt–Biedenharn [3] conventions.

### 4.2 Remarks

- In coupled channels, the phase shifts  $\delta_i$  and mixing angles  $\epsilon$  fully characterize the scattering process.
- The two conventions differ in the placement of rotation matrices but yield the same observables.

## 5 Variational code

In the variational code the  $R$ -matrix is evaluated using Koön principle to second order. Then through the following steps one recovers the phase-shifts and mixing angles for both the Stapp and the Blatt–Biedenharn (BB) conventions.

### 5.1 BB phase shifts and mixing angle

Using Tab. 1 the  $R$  matrix can be written as

$$R = \begin{pmatrix} \tan \delta_1 \cos^2 \epsilon + \tan \delta_2 \sin^2 \epsilon & (\tan \delta_1 - \tan \delta_2) \sin \epsilon \cos \epsilon \\ (\tan \delta_1 - \tan \delta_2) \sin \epsilon \cos \epsilon & \tan \delta_1 \sin^2 \epsilon + \tan \delta_2 \cos^2 \epsilon \end{pmatrix}.$$

The combination  $R_{11} - R_{22}$  is

$$R_{11} - R_{22} = \cos(2\epsilon) (\tan \delta_1 - \tan \delta_2).$$

Therefore

$$\tan(4\epsilon) = \frac{2 R_{12}}{R_{11} - R_{22}} \quad \rightarrow \quad \epsilon = \frac{1}{2} \text{atan} \left( \frac{2 R_{12}}{R_{11} - R_{22}} \right).$$

Once  $\epsilon$  is known, one can evaluate

$$\tan \delta_1 = \cos^2 \epsilon R_{11} + \sin^2 \epsilon R_{22} + 2 \cos \epsilon \sin \epsilon R_{12}$$

and

$$\tan \delta_2 = \sin^2 \epsilon R_{11} + \cos^2 \epsilon R_{22} - 2 \cos \epsilon \sin \epsilon R_{12}.$$

### 5.2 Stapp phase shifts and mixing angle

One can then use  $\delta_1$ ,  $\delta_2$  and  $\epsilon$  to evaluate the  $S$ -matrix, which is independent from the parametrization,

$$S = S_{\text{BB}} = \begin{pmatrix} e^{2i\delta_1} \cos^2 \epsilon + e^{2i\delta_2} \sin^2 \epsilon & (e^{2i\delta_1} - e^{2i\delta_2}) \sin \epsilon \cos \epsilon \\ (e^{2i\delta_1} - e^{2i\delta_2}) \sin \epsilon \cos \epsilon & e^{2i\delta_1} \sin^2 \epsilon + e^{2i\delta_2} \cos^2 \epsilon \end{pmatrix}.$$

It is possible now to extract the phase shifts and mixing angle in the Stapp parametrization. In this parametrization

$$S = S_{\text{Stapp}} = \begin{pmatrix} e^{2i\delta_1} \cos(2\epsilon) & i e^{i(\delta_1 + \delta_2)} \sin(2\epsilon) \\ i e^{i(\delta_1 + \delta_2)} \sin(2\epsilon) & e^{2i\delta_2} \cos(2\epsilon) \end{pmatrix}.$$

The determinant in this case is

$$\det S_{\text{Stapp}} = e^{2i(\delta_1 + \delta_2)}$$

and therefore

$$\sin(2\epsilon) = \sqrt{-\frac{S_{12}^2}{\det S}}$$

and

$$\cos(2\epsilon) = \sqrt{1 - \sin^2(2\epsilon)}.$$

It is possible to evaluate

$$e^{i\delta_k} = \sqrt{\frac{S_{kk}}{\cos(2\epsilon)}} = \sqrt{e^{2i\delta_k}}.$$

Therefore

$$\delta_1 = \text{acos} \left[ \text{Re} \left( \sqrt{\frac{S_{11}}{\cos(2\epsilon)}} \right) \right] \times \begin{cases} 1 & \text{if } \text{Im} \left( \sqrt{\frac{S_{11}}{\cos(2\epsilon)}} \right) \geq 0 \\ -1 & \text{if } \text{Im} \left( \sqrt{\frac{S_{11}}{\cos(2\epsilon)}} \right) < 0 \end{cases}$$

and

$$\delta_2 = \text{acos} \left[ \text{Re} \left( \frac{S_{22}}{\cos(2\epsilon)} \right) \right] \times \begin{cases} 1 & \text{if } \text{Im} \left( \frac{S_{22}}{\cos(2\epsilon)} \right) \geq 0 \\ -1 & \text{if } \text{Im} \left( \frac{S_{22}}{\cos(2\epsilon)} \right) < 0 \end{cases}.$$

## 6 Conclusion

The  $S$ -matrix and  $R$ -matrix are central tools in analyzing nucleon–nucleon scattering. While the  $S$ -matrix encapsulates the observable content of the interaction, the  $R$ -matrix provides a convenient and often more numerically robust intermediate object, especially in resonance or coupled-channel analyses. Their connection through a Möbius transformation reflects deep structural links in scattering theory.

## References

- [1] L. M. Delves, *Advances in Nuclear Physics*, vol. 5 (1972), Eds. M. Baranger, E. Vögt (Plenum Press, London, New York).
- [2] H. P. Stapp, T. Ypsilantis and N. Metropolis, *Phys. Rev.* **105**, 302 (1957).
- [3] J. M. Blatt and L. C. Biedenharn, *Phys. Rev.* **86**, 399 (1952).