

# Nucleon–Nucleon Scattering: $R$ - and $S$ -Matrix Formalism

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## Contents

<b>1</b>	<b>Introduction</b>	<b>1</b>
<b>2</b>	<b>The <math>S</math>-Matrix and <math>R</math>-Matrix: Definitions and Interpretations</b>	<b>1</b>
2.1	The Scattering Matrix ( $S$ -Matrix) . . . . .	1
2.2	The Reactance Matrix ( $R$ -Matrix) . . . . .	2
<b>3</b>	<b>From Schrödinger Equation to Scattering Matrices</b>	<b>2</b>
3.1	Radial Schrödinger Equation . . . . .	2
3.2	Definition of the $S$ -Matrix . . . . .	2
3.3	Definition of the $R$ -Matrix . . . . .	3
<b>4</b>	<b>Matrix Structure: Uncoupled and Coupled Channels</b>	<b>3</b>
4.1	Summary Table: Key Characteristics . . . . .	3
4.2	Remarks . . . . .	3
<b>5</b>	<b>Variational code</b>	<b>3</b>
5.1	BB phase shifts and mixing angle . . . . .	3
5.2	Stapp phase shifts and mixing angle . . . . .	4
<b>6</b>	<b>Conclusion</b>	<b>4</b>
<b>A</b>	<b>Asymptotic expansion and scattering length</b>	<b>4</b>
A.1	The scattering length . . . . .	5
A.1.1	Eliminating the energy dependence from the spherical Bessel functions . . . . .	5
A.2	BB scattering lengths $a_i$ and mixing constant $e_J$ . . . . .	7
A.3	Stapp scattering lengths $\tilde{a}_i$ and mixing constant $\tilde{e}_J$ . . . . .	7

## 1 Introduction

In quantum scattering theory, the evolution of an interacting two-body system is elegantly encoded in the  $S$ -matrix, while the  $R$ -matrix offers a numerically convenient alternative based on boundary matching. This document reviews the definitions, physical interpretations, and mathematical derivations of both quantities, particularly within the context of nucleon–nucleon scattering.

## 2 The $S$ -Matrix and $R$ -Matrix: Definitions and Interpretations

### 2.1 The Scattering Matrix ( $S$ -Matrix)

**Definition:** The  $S$ -matrix connects asymptotic incoming and outgoing states:

$$|\text{out}\rangle = S|\text{in}\rangle$$

**Physical Role:**

- Encodes all observable aspects of scattering: phase shifts, cross sections, and mixing.
- Ensures conservation of probability via unitarity:  $S^\dagger S = I$ .
- For uncoupled channels:  $S_\ell = e^{2i\delta_\ell}$ , where  $\delta_\ell$  is the phase shift.

**2.2 The Reactance Matrix ( $R$ -Matrix)**

**Definition:** Defined via the logarithmic derivative of the wavefunction at the boundary of the interaction region:

$$R_\ell(E) = \left. \frac{a u'_\ell(a)}{u_\ell(a)} \right|_{\text{internal}}$$

**Physical Role:**

- Arises from dividing configuration space into internal ( $r < a$ ) and external ( $r > a$ ) regions.
- Useful for resonance physics and numerical stability.
- Related to the  $S$ -matrix via:

$$S = \frac{1 + iR}{1 - iR}$$

**3 From Schrödinger Equation to Scattering Matrices****3.1 Radial Schrödinger Equation**

Consider two nucleons interacting via a central potential  $V(r)$ . The time-independent Schrödinger equation reads:

$$\left[ -\frac{\hbar^2}{2\mu} \nabla^2 + V(r) \right] \psi(\mathbf{r}) = E\psi(\mathbf{r})$$

**Partial Wave Expansion:** Using spherical symmetry:

$$\psi(\mathbf{r}) = \sum_{\ell m} \frac{u_\ell(r)}{r} Y_{\ell m}(\hat{r})$$

The radial equation becomes:

$$\left[ -\frac{\hbar^2}{2\mu} \frac{d^2}{dr^2} + \frac{\hbar^2 \ell(\ell+1)}{2\mu r^2} + V(r) \right] u_\ell(r) = E u_\ell(r)$$

**Asymptotic Behavior:** For  $r \rightarrow \infty$  (free motion), define the wave number  $k = \sqrt{2\mu E}/\hbar$ . Then:

$$u_\ell(r) \xrightarrow{r \rightarrow \infty} \sin \left( kr - \frac{\ell\pi}{2} + \delta_\ell \right)$$

**3.2 Definition of the  $S$ -Matrix**

Rewriting the asymptotic form as a combination of incoming and outgoing spherical waves:

$$u_\ell(r) \sim \frac{1}{2i} \left[ e^{-i(kr - \ell\pi/2)} - S_\ell e^{i(kr - \ell\pi/2)} \right]$$

This identifies  $S_\ell = e^{2i\delta_\ell}$ .

### 3.3 Definition of the $R$ -Matrix

In the  $R$ -matrix framework:

- **Internal region:**  $r < a$ , where interactions occur.
- **External region:**  $r > a$ , free-particle motion.

In the external region, the general solution is:

$$\psi_\ell(r) = C_\ell [F_\ell(kr) \cos \delta_\ell + G_\ell(kr) \sin \delta_\ell] , \quad (1)$$

where  $C_\ell$  is a normalization constant. Matching this with the internal solution at  $r = a$ , one derives [1]:

$$S_\ell = \frac{1 + iR_\ell}{1 - iR_\ell}$$

## 4 Matrix Structure: Uncoupled and Coupled Channels

### 4.1 Summary Table: Key Characteristics

Quantity	Uncoupled	Coupled (Stapp)	Coupled (BB)
$S$	$e^{2i\delta}$	$\text{diag}(e^{2i\delta_i}) O(\epsilon) \text{diag}(e^{2i\delta_i})$	$O^T(\epsilon) \text{diag}(e^{2i\delta_i}) O(\epsilon)$
$R$	$\tan \delta$	$\text{diag}(\tan \delta_i) O(\epsilon) \text{diag}(\tan \delta_i)$	$O^T(\epsilon) \text{diag}(\tan \delta_i) O(\epsilon)$
$O$	–	$O(\epsilon) = \begin{pmatrix} \cos 2\epsilon & i \sin 2\epsilon \\ i \sin 2\epsilon & \cos 2\epsilon \end{pmatrix}$	$O(\epsilon) = \begin{pmatrix} \cos \epsilon & \sin \epsilon \\ -\sin \epsilon & \cos \epsilon \end{pmatrix}$

Table 1: Summary of  $S$ - and  $R$ -matrix structures in uncoupled and coupled cases, using Stapp [2] and Blatt–Biedenharn [3] conventions.

### 4.2 Remarks

- In coupled channels, the phase shifts  $\delta_i$  and mixing angles  $\epsilon$  fully characterize the scattering process.
- The two conventions differ in the placement of rotation matrices but yield the same observables.

## 5 Variational code

In the variational code the  $R$ -matrix is evaluated using Koön principle to second order. Then through the following steps one recovers the phase-shifts and mixing angles for both the Stapp and the Blatt–Biedenharn (BB) conventions.

### 5.1 BB phase shifts and mixing angle

Using Tab. 1 the  $R$  matrix can be written as

$$R = \begin{pmatrix} \tan \delta_1 \cos^2 \epsilon + \tan \delta_2 \sin^2 \epsilon & (\tan \delta_1 - \tan \delta_2) \sin \epsilon \cos \epsilon \\ (\tan \delta_1 - \tan \delta_2) \sin \epsilon \cos \epsilon & \tan \delta_1 \sin^2 \epsilon + \tan \delta_2 \cos^2 \epsilon \end{pmatrix} . \quad (2)$$

The combination  $R_{11} - R_{22}$  is

$$R_{11} - R_{22} = \cos(2\epsilon) (\tan \delta_1 - \tan \delta_2) .$$

Therefore

$$\tan(4\epsilon) = \frac{2 R_{12}}{R_{11} - R_{22}} \quad \rightarrow \quad \epsilon = \frac{1}{2} \text{atan} \left( \frac{2 R_{12}}{R_{11} - R_{22}} \right) .$$

Once  $\epsilon$  is known, one can evaluate

$$\tan \delta_1 = \cos^2 \epsilon R_{11} + \sin^2 \epsilon R_{22} + 2 \cos \epsilon \sin \epsilon R_{12}$$

and

$$\tan \delta_2 = \sin^2 \epsilon R_{11} + \cos^2 \epsilon R_{22} - 2 \cos \epsilon \sin \epsilon R_{12}.$$

## 5.2 Stapp phase shifts and mixing angle

One can then use  $\delta_1$ ,  $\delta_2$  and  $\epsilon$  to evaluate the  $S$ -matrix, which is independent from the parametrization,

$$S = S_{\text{BB}} = \begin{pmatrix} e^{2i\delta_1} \cos^2 \epsilon + e^{2i\delta_2} \sin^2 \epsilon & (e^{2i\delta_1} - e^{2i\delta_2}) \sin \epsilon \cos \epsilon \\ (e^{2i\delta_1} - e^{2i\delta_2}) \sin \epsilon \cos \epsilon & e^{2i\delta_1} \sin^2 \epsilon + e^{2i\delta_2} \cos^2 \epsilon \end{pmatrix}. \quad (3)$$

It is possible now to extract the phase shifts and mixing angle in the Stapp parametrization. In this parametrization

$$S = S_{\text{Stapp}} = \begin{pmatrix} e^{2i\delta_1} \cos(2\epsilon) & i e^{i(\delta_1+\delta_2)} \sin(2\epsilon) \\ i e^{i(\delta_1+\delta_2)} \sin(2\epsilon) & e^{2i\delta_2} \cos(2\epsilon) \end{pmatrix}.$$

The determinant in this case is

$$\det S_{\text{Stapp}} = e^{2i(\delta_1+\delta_2)}$$

and therefore

$$\sin(2\epsilon) = \sqrt{-\frac{S_{12}^2}{\det S}} \quad (4)$$

and

$$\cos(2\epsilon) = \sqrt{1 - \sin^2(2\epsilon)}.$$

It is possible to evaluate

$$e^{i\delta_k} = \sqrt{\frac{S_{kk}}{\cos(2\epsilon)}} = \sqrt{e^{2i\delta_k}}.$$

Therefore

$$\delta_1 = \text{acos} \left[ \text{Re} \left( \sqrt{\frac{S_{11}}{\cos(2\epsilon)}} \right) \right] \times \begin{cases} 1 & \text{if } \text{Im} \left( \sqrt{\frac{S_{11}}{\cos(2\epsilon)}} \right) \geq 0 \\ -1 & \text{if } \text{Im} \left( \sqrt{\frac{S_{11}}{\cos(2\epsilon)}} \right) < 0 \end{cases}$$

and

$$\delta_2 = \text{acos} \left[ \text{Re} \left( \frac{S_{22}}{\cos(2\epsilon)} \right) \right] \times \begin{cases} 1 & \text{if } \text{Im} \left( \frac{S_{22}}{\cos(2\epsilon)} \right) \geq 0 \\ -1 & \text{if } \text{Im} \left( \frac{S_{22}}{\cos(2\epsilon)} \right) < 0 \end{cases}.$$

## 6 Conclusion

The  $S$ -matrix and  $R$ -matrix are central tools in analyzing nucleon–nucleon scattering. While the  $S$ -matrix encapsulates the observable content of the interaction, the  $R$ -matrix provides a convenient and often more numerically robust intermediate object, especially in resonance or coupled-channel analyses. Their connection through a Möbius transformation reflects deep structural links in scattering theory.

## A Asymptotic expansion and scattering length

As seen in Eq. (1) it is possible to write for non-coupled channels

$$\psi_\ell(r) = A_\ell (F_\ell + \tan \delta_\ell G_\ell),$$

where  $A_\ell \equiv C_\ell \cos \delta_\ell$ . In this case  $R_{\ell\ell} = \tan \delta_\ell$  and therefore

$$\psi_\ell(r) = A_\ell (F_\ell + R_{\ell\ell} G_\ell).$$

Here  $R_\ell$  ( $G_\ell$ ) are the regular (irregular) solution to the Schrödinger equation and in the case if the potential is short-range

$$\begin{cases} F_\ell(kr) & \rightarrow j_\ell(kr), \\ G_\ell(kr) & \rightarrow y_\ell(kr), \end{cases}$$

where  $j_\ell$  ( $y_\ell$ ) is the regular (irregular) spherical Bessel function.

In case of a long range potential of the type  $\propto 1/r$

$$\begin{cases} F_\ell & \rightarrow F_\ell(\eta, kr), \\ G_\ell & \rightarrow G_\ell(\eta, kr), \end{cases}$$

where  $F_\ell(\eta, kr)$  ( $G_\ell(\eta, kr)$ ) is the regular (irregular) Coulomb function.

## A.1 The scattering length

For small energies there are two observables of importance, which depend linearly on the energy, the scattering length  $a_\ell$  and the effective range  $r_{e,\ell}$ , they are connected to the momentum  $k$  and the phase shift  $\delta_\ell$  by

$$k^{2\ell+1} \cot \delta_\ell = -\frac{1}{a_\ell} + \frac{1}{2} r_{e,\ell} k^2.$$

Specifically, for  $E \rightarrow 0$

$$k^{2\ell+1} \cot \delta_\ell \rightarrow -\frac{1}{a_\ell}.$$

Since for uncoupled channels  $R_{\ell\ell} = \tan \delta_\ell$  it is possible to write

$$R_{\ell\ell} \rightarrow -a_\ell k^{2\ell+1}.$$

### A.1.1 Eliminating the energy dependence from the spherical Bessel functions

Focusing on the spherical Bessel function, the asymptotic limit for their argument going to zero is [4]

$$\begin{cases} F_\ell(kr) = j_\ell(x) & \rightarrow \frac{x^\ell}{(2\ell+1)!!}, \\ G_\ell(kr) = y_\ell(x) & \rightarrow -\frac{(2\ell-1)!!}{x^{\ell+1}}. \end{cases}$$

Using for  $E \rightarrow 0$  the functions [code]

$$\begin{cases} \tilde{F}_\ell(r) = r^\ell, \\ \tilde{G}_\ell(r) = -\frac{1}{(2\ell+1)r^{\ell+1}} (1 - e^{-\epsilon r})^{2\ell+1}, \end{cases}$$

for  $r \rightarrow \infty$  it is possible to write

$$\tilde{A}_\ell (\tilde{F}_\ell + R \tilde{G}_\ell) = \tilde{A}_\ell \left( r^\ell - \frac{\tilde{R}_{\ell\ell}}{2\ell+1} \frac{1}{r^{\ell+1}} \right)$$

and confronting it with the actual behavior

$$A_\ell (F_\ell(kr) + R_{\ell\ell} G_\ell(kr)) \simeq A_\ell \left( \frac{k^\ell r^\ell}{(2\ell+1)!!} - R_{\ell\ell} \frac{(2\ell-1)!!}{k^{\ell+1} r^{\ell+1}} \right)$$

it is possible to map

$$\frac{\tilde{R}_{\ell\ell}}{2\ell+1} = \frac{(2\ell+1)!! (2\ell-1)!!}{k^{2\ell+1}} R_{\ell\ell}$$

and therefore

$$\tilde{R}_{\ell\ell} = -[(2\ell+1)!!]^2 a_\ell.$$

**Coupled case** In the case of a coupled channel

$$\begin{cases} A_{\ell_1} (F_{\ell_1}(kr) + R_{\ell_1 \ell_1} G_{\ell_1}(kr) + R_{\ell_1(\ell_2)} G_{\ell_2}(kr)) , \\ A_{\ell_2} (F_{\ell_2}(kr) + R_{(\ell_2) \ell_1} G_{\ell_1}(kr) + R_{(\ell_2)(\ell_2)} G_{\ell_2}(kr)) \end{cases}$$

the choice for small energies brings to

$$\begin{cases} \tilde{A}_{\ell_1} \left( r^{\ell_1} - \frac{\tilde{R}_{\ell_1 \ell_1}}{(2\ell_1 + 1)} \frac{1}{r^{\ell_1 + 1}} - \frac{\tilde{R}_{\ell_1 \ell_2}}{(2\ell_2 + 1)} \frac{1}{r^{\ell_2 + 1}} \right) , \\ \tilde{A}_{\ell_2} \left( r^{\ell_2} - \frac{\tilde{R}_{\ell_2 \ell_1}}{(2\ell_1 + 1)} \frac{1}{r^{\ell_1 + 1}} - \frac{\tilde{R}_{\ell_2 \ell_2}}{(2\ell_2 + 1)} \frac{1}{r^{\ell_2 + 1}} \right) . \end{cases}$$

Comparing with the actual behavior

$$\begin{cases} A_{\ell_1} \left( \frac{k^{\ell_1} r^{\ell_1}}{(2\ell_1 + 1)!!} - \tilde{R}_{\ell_1 \ell_1} \frac{(2\ell_1 - 1)!!}{k^{\ell_1 + 1} r^{\ell_1 + 1}} - \tilde{R}_{\ell_1 \ell_2} \frac{(2\ell_2 - 1)!!}{k^{\ell_2 + 1} r^{\ell_2 + 1}} \right) , \\ A_{\ell_2} \left( \frac{k^{\ell_2} r^{\ell_2}}{(2\ell_2 + 1)!!} - \tilde{R}_{\ell_2 \ell_1} \frac{(2\ell_1 - 1)!!}{k^{\ell_1 + 1} r^{\ell_1 + 1}} - \tilde{R}_{\ell_2 \ell_2} \frac{(2\ell_2 - 1)!!}{k^{\ell_2 + 1} r^{\ell_2 + 1}} \right) , \end{cases}$$

it is possible to extract

$$\begin{cases} \tilde{R}_{\ell_1 \ell_1} = [(2\ell_1 + 1)!!]^2 \frac{R_{\ell_1 \ell_1}}{k^{2\ell_1 + 1}} , \\ \tilde{R}_{\ell_1 \ell_2} = (2\ell_1 + 1)!! (2\ell_2 + 1)!! \frac{R_{\ell_1 \ell_2}}{k^{\ell_1 + \ell_2 + 1}} , \\ \tilde{R}_{\ell_2 \ell_1} = (2\ell_1 + 1)!! (2\ell_2 + 1)!! \frac{R_{\ell_2 \ell_1}}{k^{\ell_1 + \ell_2 + 1}} , \\ \tilde{R}_{\ell_2 \ell_2} = [(2\ell_2 + 1)!!]^2 \frac{R_{\ell_2 \ell_2}}{k^{2\ell_2 + 1}} \end{cases}$$

Since for coupled channels  $\ell_2 = \ell + 2$  where  $\ell \equiv \ell_1$  it is possible to use Eq. (2) and write

$$\begin{pmatrix} \tilde{R}_{\ell_1 \ell_1} & \tilde{R}_{\ell_1 \ell_2} \\ \tilde{R}_{\ell_2 \ell_1} & \tilde{R}_{\ell_2 \ell_2} \end{pmatrix} = \begin{pmatrix} [(2\ell + 1)!!]^2 \left( \frac{\tan \delta_1}{k^{2\ell + 1}} \cos^2 \epsilon + \frac{\tan \delta_2}{k^{2\ell + 1}} \sin^2 \epsilon \right) & (2\ell + 1)!! (2\ell + 5)!! \frac{\tan \delta_1 - \tan \delta_2}{k^{2\ell + 3}} \sin \epsilon \cos \epsilon \\ (2\ell + 1)!! (2\ell + 5)!! \frac{\tan \delta_1 - \tan \delta_2}{k^{2\ell + 3}} \sin \epsilon \cos \epsilon & [(2\ell + 5)!!]^2 \left( \frac{\tan \delta_1}{k^{2\ell + 5}} \sin^2 \epsilon + \frac{\tan \delta_2}{k^{2\ell + 5}} \cos^2 \epsilon \right) \end{pmatrix}.$$

Simplifying

$$\begin{pmatrix} \tilde{R}_{\ell_1 \ell_1} & \tilde{R}_{\ell_1 \ell_2} \\ \tilde{R}_{\ell_2 \ell_1} & \tilde{R}_{\ell_2 \ell_2} \end{pmatrix} = - \begin{pmatrix} [(2\ell + 1)!!]^2 (a_1 \cos^2 \epsilon + a_2 k^4 \sin^2 \epsilon) & (2\ell + 1)!! (2\ell + 5)!! (a_1 - k^4 a_2) \frac{\sin 2\epsilon}{2k^2} \\ (2\ell + 1)!! (2\ell + 5)!! (a_1 - k^4 a_2) \frac{\sin 2\epsilon}{2k^2} & [(2\ell + 5)!!]^2 \left( a_1 \frac{\sin^2 \epsilon}{k^4} + a_2 \cos^2 \epsilon \right) \end{pmatrix}.$$

From this it is possible to infer that, in order for the  $R$  matrix to not diverge  $\epsilon \simeq k^2$  at least, this is proven in Ref. [3]. Defining

$$e_J = \frac{\epsilon}{k^2}$$

the  $R$  matrix becomes

$$\begin{pmatrix} \tilde{R}_{\ell_1 \ell_1} & \tilde{R}_{\ell_1 \ell_2} \\ \tilde{R}_{\ell_2 \ell_1} & \tilde{R}_{\ell_2 \ell_2} \end{pmatrix} = - \begin{pmatrix} [(2\ell + 1)!!]^2 a_1 & (2\ell + 1)!! (2\ell + 5)!! a_1 e_J \\ (2\ell + 1)!! (2\ell + 5)!! a_1 e_J & [(2\ell + 5)!!]^2 (a_1 e_J^2 + a_2) \end{pmatrix}.$$

**Notice** that these results are in the BB parametrization!

## A.2 BB scattering lengths $a_i$ and mixing constant $e_J$

In this case it is possible to solve for  $a_1$ ,  $a_2$  and  $e_J$

$$\begin{cases} a_1 &= -\frac{\tilde{R}_{\ell_1 \ell_1}}{[(2\ell+1)!!]^2}, \\ e_J &= \frac{(2\ell+1)!! \tilde{R}_{\ell_1 \ell_2}}{(2\ell+5)!! \tilde{R}_{\ell_1 \ell_1}}, \\ a_2 &= \frac{\tilde{R}_{\ell_1 \ell_2}^2 - \tilde{R}_{\ell_1 \ell_1} \tilde{R}_{\ell_2 \ell_2}}{(2\ell+5)!! \tilde{R}_{\ell_1 \ell_1}}, \end{cases} \quad \text{and} \quad \begin{cases} \delta_1 &\simeq -a_1 k^{2\ell_1+1}, \\ \epsilon &\simeq e_J k^2 \\ \delta_2 &\simeq -a_2 k^{2\ell_2+1}. \end{cases}$$

## A.3 Stapp scattering lengths $\tilde{a}_i$ and mixing constant $\tilde{e}_J$

Inserting Eq. (3) into Eq. (4)

$$\sin 2\tilde{\epsilon} = \frac{1}{2} \sqrt{-e^{-2i(\delta_1+\delta_2)} (e^{2i\delta_1} - e^{2i\delta_2})^2 \sin^2(2\epsilon)},$$

where  $\delta_1$ ,  $\delta_2$  and  $\epsilon$  are respectively the scattering length of the first and second channel, and the mixing angle in the BB parametrization. From now on the quantities in the Stapp parametrization will have a tilde on top of them. To extract  $\tilde{a}_i$  and  $\tilde{e}_J$  it is necessary to expand this, obtaining

$$\tilde{\epsilon} \simeq -\frac{a_1 \sqrt{a_2^2 e_J^2}}{a_2} k^{\ell_1+\ell_2+1}.$$

Defining  $\ell_2 \equiv \ell_1 + \Delta\ell$  and  $\ell \equiv \ell_1$

$$\tilde{\epsilon} \simeq -\frac{a_1 \sqrt{a_2^2 e_J^2}}{a_2} k^{2\ell+\Delta\ell+1}.$$

or, using  $\Delta\ell = 2$

$$\tilde{\epsilon} \simeq -\frac{a_1 \sqrt{a_2^2 e_J^2}}{a_2} k^{2\ell+3}.$$

Therefore

$$\tilde{e}_J = \frac{\tilde{\epsilon}}{k^{2\ell+3}}$$

giving

$$\tilde{e}_J = -a_1 |e_J| \text{sign}(a_2).$$

Doing the same procedure for

$$\tilde{\delta}_1 = -\frac{i}{2} \log \left( \frac{S_{11}}{\cos \tilde{\epsilon}} \right)$$

leads to

$$\tilde{\delta}_1 = -a_1 k^{2\ell_1+1}$$

and therefore

$$\tilde{a}_1 = a_1.$$

Finally, repeating the procedure for  $\tilde{\delta}_2$

$$\tilde{\delta}_2 = -\frac{i}{2} \sqrt{\frac{S_{22}}{\cos \tilde{\epsilon}}}$$

one gets

$$\tilde{\delta}_2 \simeq -(a_2 + a_1 e_J^2) k^{2\ell_2+1}$$

and therefore

$$\tilde{a}_2 = a_2 + a_1 e_J^2.$$

To summarize

$$\begin{cases} \tilde{a}_1 &= a_1, \\ \tilde{e}_J &= -a_1 |e_J| \text{sign}(a_2), \\ \tilde{a}_2 &= a_2 + a_1 e_J^2, \end{cases} \quad \text{and} \quad \begin{cases} \tilde{\delta}_1 &\simeq -\tilde{a}_1 k^{2\ell_1+1}, \\ \tilde{\epsilon} &\simeq \tilde{e}_J k^{\ell_1+\ell_2+1}, \\ \tilde{\delta}_2 &\simeq -\tilde{a}_2 k^{2\ell_2+1}. \end{cases}$$

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