# Homework #3
# Numerical Methods for ODEs

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Let's consider the following Cauchy problem with only Dirichlet boundary conditions

$$\begin{cases} -\dfrac{\partial^2 u}{\partial x^2} - \dfrac{\partial^2 u}{\partial y^2} = f \text{ in } \Omega \\ u = 0 \text{ on } \Gamma \end{cases} \tag{1}$$

Where the domain $\Omega$ is the rectangular shape defined as

$$\Omega = \{(x, y) : -1 \leq x \leq 1, -1 \leq y \leq 1\}.$$

The boundary domain is $\Gamma = \partial\Omega$ and the source term $f$ is the funcion

$$f(x, y) = -4 + 2x^2 + 2y^2.$$

In order to solve (1) using FEM on a given mesh, we can use the `Exercize` script. It uses the function `stiffmatrix` which generates the stiffness matrix $H$ and the right hand side $f_h$ related to the linear system

$$Hu = f_h. \tag{2}$$

First of all this function generates the pattern of the stiffness matrix using the adjacency matrix related to a given mesh, then it computes the local stiffness matrix $H^{(e)}$ for every triangular element $e$ of the mesh. At the end it assemblies all of these contrubutions and it finally compute the $H$ matrix.

This function also returns the two vectors `deltai` and $f_h$. The first one contains the $\sum_e \Delta_e/3$ (the surface measure relative to every node $i$) while the elements $(f_h)_i$ of the second one are an approximation of the integral

$$\int_\Omega f(x, y)\varphi_i \, d\Omega.$$

After all of this computations we have to impose the boundary conditions on (2). Since we don't want to destroy the symmetry of H, we apply the *penalty* method: for the nodes on $\Gamma$, we mutiply the diagonal term by $R_{max} = 10^{15}$ and we replace the r.h.s. with 0 (because (1) has homogeneous b.c.).

We can finally solve (2) using PCG with Jacobi and incomplete Cholesky preconditioning. We set the tollerance as $tol = 10^{-8} \cdot ||f_h||_2$ and the initial guess $x_0$ as the null vector.

In order to check the correctness of this code we apply the FEM on the five different meshes. We know, by the theory, that the Euclidian norm of the error decreases as $\ell^2$ , where $\ell$ represent the average mesh size (i.e. the characteristic length of the edges). Thanks to the kind of refinement, the total number of triangles increases by a factor 4 and the characteristic length decreases by a factor 2 at each level. By this, in order to compute the average mesh size of all the meshes, it's sufficient to compute the first one and then we can obtain all the others halving it. (To compute the first one we use the function `averagemeshsize`: first it sum the perimeter of all the triangle, then the function divides it by 3 times the number of triangles). Anyway we don't need to compute the average mesh size since it's sufficient the trend of the refinement (i.e. the vector $v = [1, 1/2, 1/4, 1/8, 1/16]$).

Since we know that the exact solution of (1) is

$$u(x, y) = x^2 + y^2 - x^2y^2 - 1,$$

we can also compute the error norm as

$$\begin{aligned} \varepsilon = ||u_h - u||_2 &= \sqrt{\int_\Omega (u_h - u)^2 \, d\Omega} \\ &\approx \left\{ \sum_{i=1}^{n} \left[ (u_i - u(x_i, y_i))^2 \frac{\sum_e \Delta_e}{3} \right] \right\}^{1/2} \end{aligned}$$

where $u_h$ is the numerical solution.

By all these informations we obtain the convergence profile in Figure 1, 2 respectively for the Jacobi and the incomplete Cholesky preconditiong. We also report in the following table some useful values related to the decay of the error, it's also reported the ratio between the error of two following refinement for each level.

By all these results we can conclude that in our code the Euclidian norm of the error really decays as expected by the theoretical result.

| | $\varepsilon_{jac}$ | ratio | $\varepsilon_{chol}$ | ratio |
|---|---|---|---|---|
| mesh0 | $6.9120 * 10^{-2}$ | 4.2372 | $6.9120 * 10^{-2}$ | 4.2372 |
| mesh1 | $1.6313 * 10^{-2}$ | 4.0942 | $1.6313 * 10^{-2}$ | 4.0942 |
| mesh2 | $3.9844 * 10^{-3}$ | 4.0316 | $3.9844 * 10^{-3}$ | 4.0316 |
| mesh3 | $9.8830 * 10^{-4}$ | 4.0098 | $9.8830 * 10^{-4}$ | 4.0098 |
| mesh4 | $2.4647 * 10^{-4}$ | – | $2.4647 * 10^{-4}$ | – |

Since we are using PCG, we also report the convergence plots for the residual norm over the iterations. In Figure 3, 4, 5, 6, 7 are reported the plots related to Jacobi preconditioning while in Figure 8, 9, 10, 11, 12 are reported the plots related to incomplete Cholesky preconditioning.

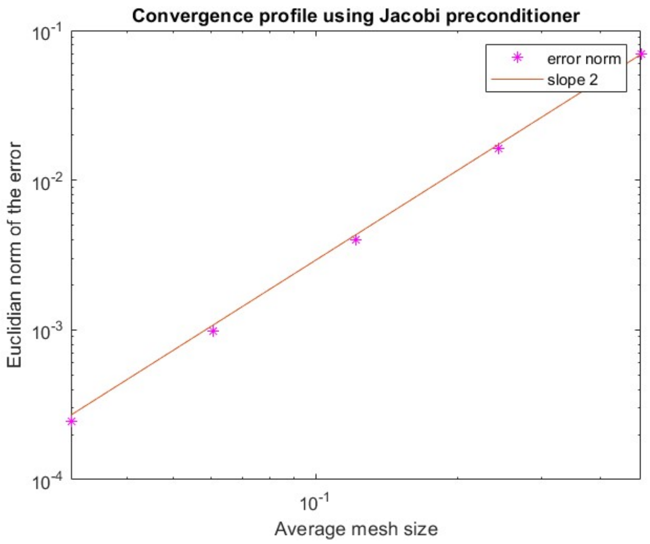


Figure 1: Convergence decay using Jacobi preconditioner.

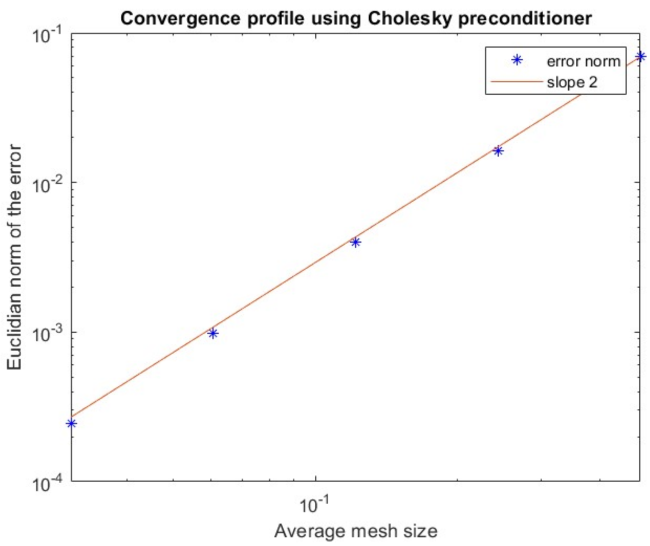


Figure 2: Convergence decay using incomplete preconditioner.

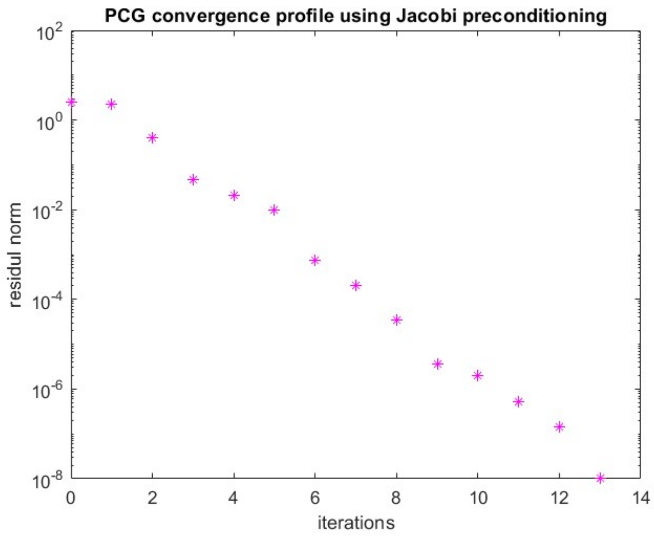


Figure 3: Convergence profile using Jacobi preconditioner for mesh0.

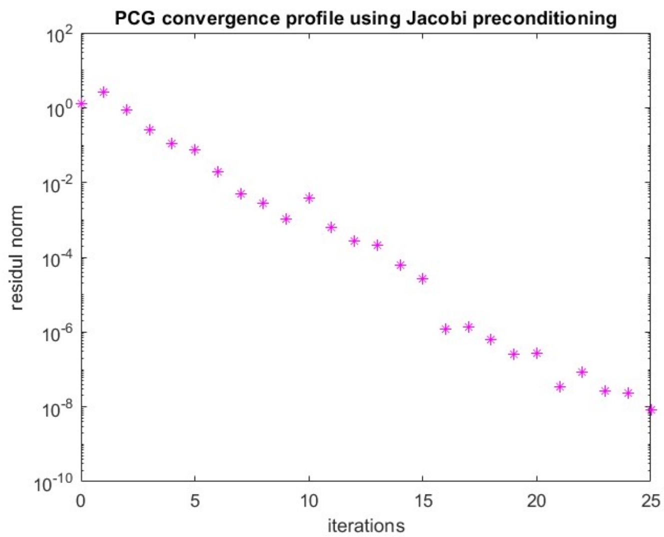


Figure 4: Convergence profile using Jacobi preconditioner for mesh1.

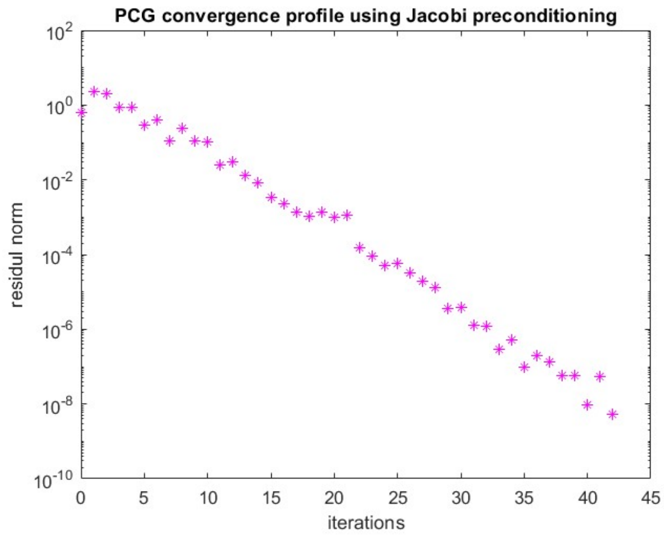


Figure 5: Convergence profile using Jacobi preconditioner for mesh2.

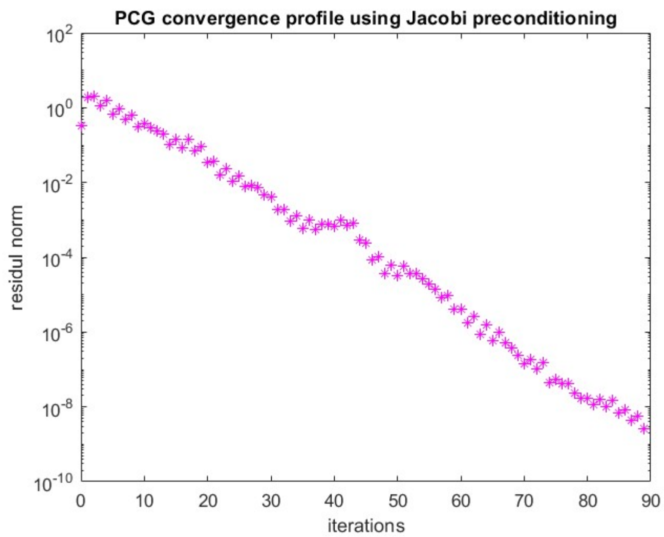


Figure 6: Convergence profile using Jacobi preconditioner for mesh3.

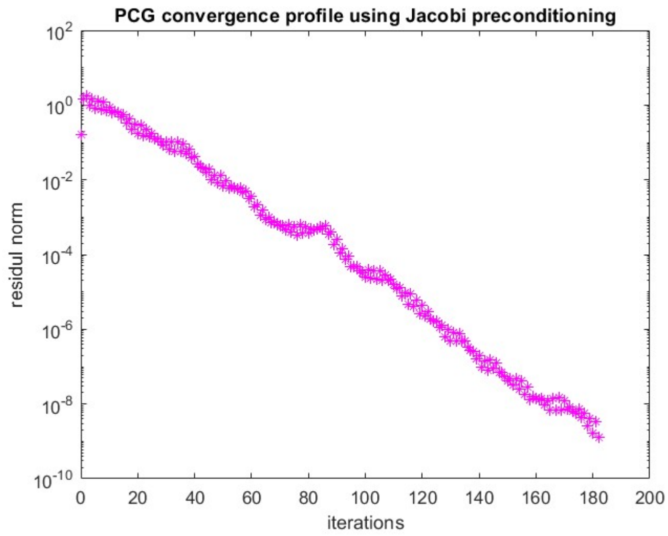


Figure 7: Convergence profile using Jacobi preconditioner for mesh4.

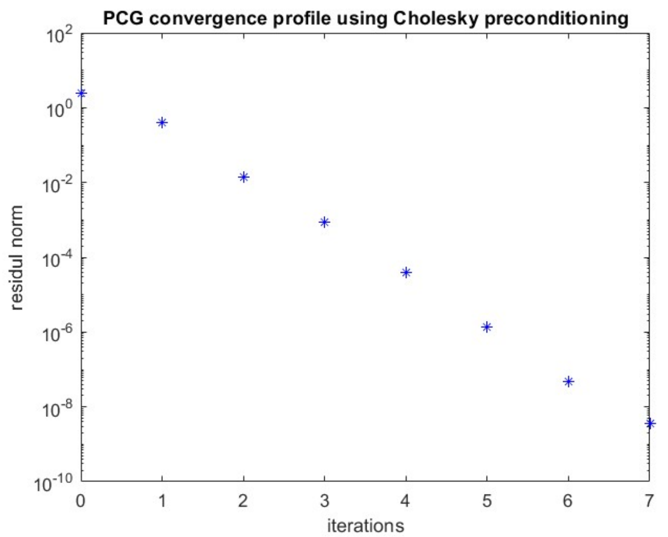


Figure 8: Convergence profile using incomplete Cholesky preconditioner for mesh0.

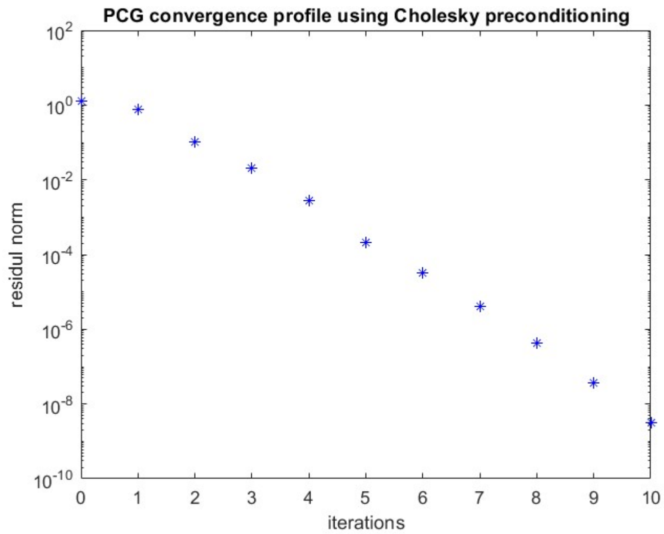


Figure 9: Convergence profile using incomplete Cholesky preconditioner for mesh1.

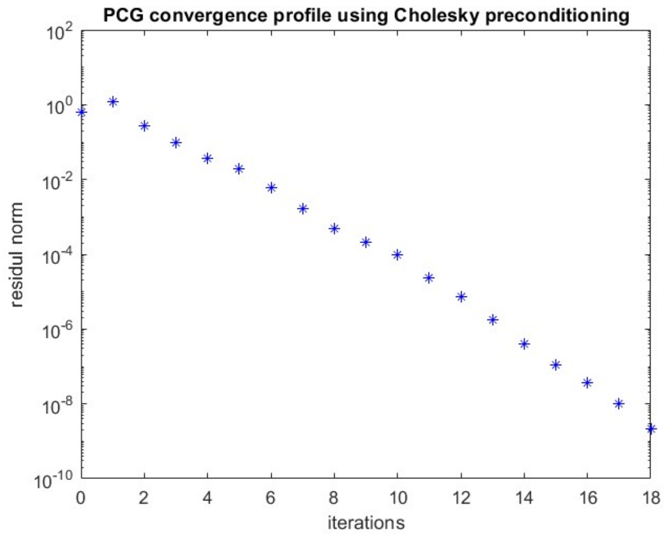


Figure 10: Convergence profile using incomplete Cholesky preconditioner for mesh2.

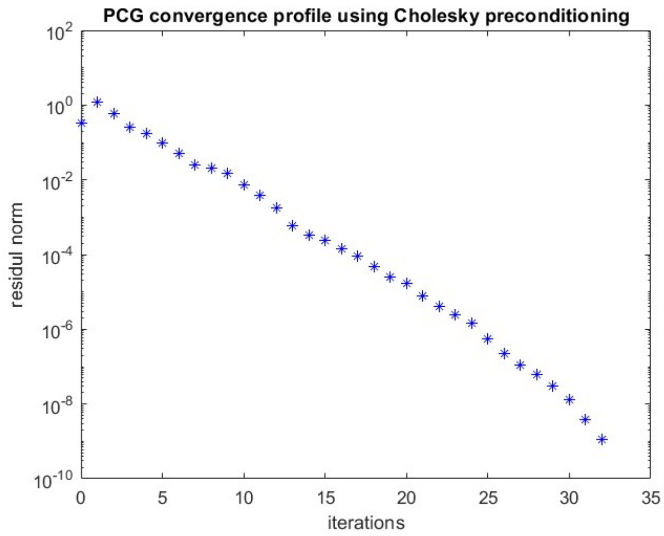


Figure 11: Convergence profile using incomplete Cholesky preconditioner for mesh3.

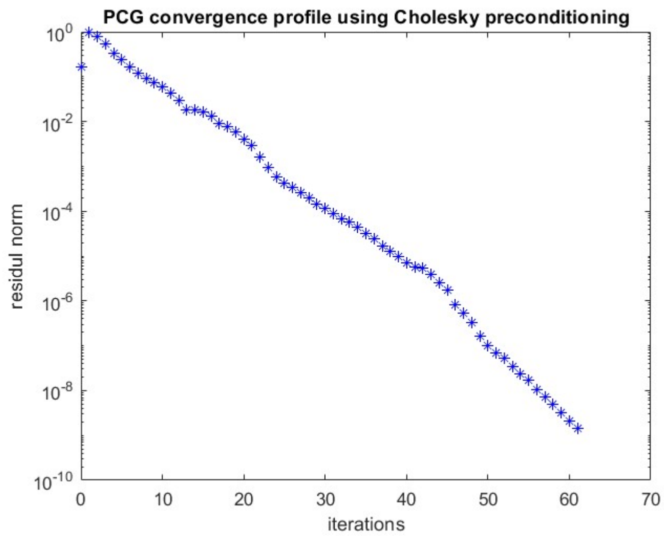


Figure 12: Convergence profile using incomplete Cholesky preconditioner for mesh4.