Maths TD # 1 — 24.09.2012

Exercice 1

1) $z_1 = (1-3i)(1+3i)$
$= 1 + 3i - 3i - 9i^2$
$= 10$

2) $z_2 = 3 + \frac{2}{i} = 3 + \frac{2}{i} \cdot \frac{i}{i}$
$= 3 + \frac{2i}{i^2} = 3 - 2i$

$\frac{1}{i} = -1$

3) $z_3 = (1+i)(2-i)(3+i)$
$= (2 - i + 2i - i^2)(3+i)$
$= (3+i)(3+i)$
$= 9 + 3i + 3i + i^2$
$= 8 + 6i$

4) $z_4 = \frac{2+5i}{1-i} + \frac{1-i}{2-5i}$
$= \frac{2+5i}{1-i} \cdot \frac{1+i}{1+i} + \frac{1-i}{2-5i} \cdot \frac{2+5i}{2+5i}$
$= \frac{(2+5i)(1+i)}{2} + \frac{(1-i)(2+5i)}{29}$
$= \frac{-3+7i}{2} + \frac{7+3i}{29}$

$$e^{iu} + e^{iv} = e^{i\frac{u+v}{2}}\left(e^{i\frac{u-v}{2}} + e^{\frac{-iu-v}{2}}\right)$$
$$= e^{i\frac{u+v}{2}}\left(2\cos\frac{u-v}{2}\right).$$

$2\cos a = e^{ia} + e^{-ia}$
$2i\sin a = e^{ia} - e^{ia}$

5) $z_5 = \frac{e^{2i\theta} + e^{4i\theta}}{1 - e^{6i\theta}}$ où θ est un angle non-multiple de $\frac{\pi}{3}$

$z_5 = \frac{e^{2i\theta} + e^{4i\theta}}{1 - e^{6i\theta}} \cdot \frac{1 + e^{6i\theta}}{1 + e^{6i\theta}}$

$e^{2i\theta} + e^{4i\theta} = e^{3i\theta}\, 2\cos\theta$
$1 - e^{6i\theta} = e^{i\theta} - e^{6i\theta} = e^{3i\theta}(-2\sin 3\theta)$
car $= e^{3i\theta}\left(e^{-3i\theta} - e^{-3i\theta}\right)$

$z = \left|\frac{\cos\theta}{i\sin\theta}\right|$

Exercice 2 :

a. $2+2i \qquad |2+2i| = \sqrt{2^2 + 2^2} = \sqrt{8} = 2\sqrt{2}$

$\sqrt{8}\left(\frac{2}{2\sqrt{2}} + \frac{2}{2\sqrt{2}} i\right) = 2\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$
$= 2\sqrt{2}\, e^{i\frac{\pi}{4}}$

b. $i^{95} = i^{3}$ car $i^{4} = 1$ et $95 = 4 \times 23 + 3$

$i^{95} = -i \iff \cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2} = e^{i\frac{3\pi}{2}}$

c. $\sqrt{3} + 3i$ $\quad |\sqrt{3} + 3i| = \sqrt{3+9} = \sqrt{12} = 2\sqrt{3}$

$\Rightarrow 2\sqrt{3}\left(\frac{\sqrt{3}}{2\sqrt{3}} + \frac{3}{2\sqrt{3}} i\right) = 2\sqrt{3}\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

$= 2\sqrt{3}\, e^{i\frac{\pi}{3}}$

d. $(1+i)^{5}$ $\quad |1+i| = \sqrt{2}$

$= \left(\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right)^{5}$

$= \sqrt{2}^{5}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^{5}$

$= 4\sqrt{2}\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right)$

e. $\left(\frac{1+i}{1-i}\right)^{3} = \left(\frac{1+i}{1-i} \cdot \frac{1+i}{1+i}\right)^{3}$

$= \left(\frac{(1+i)^{2}}{2}\right)^{3} = \left(\frac{\cancel{2}i}{\cancel{2}}\right)^{3}$

$= i^{3} = -i$

$\hookrightarrow \cos -\frac{\pi}{2} + i\sin\left(-\frac{\pi}{2}\right) = e^{-i\pi/2}$

Exercice 3:

1) $z = 0 + i$ $\quad |z| = 1$ $\quad z = \frac{0}{1} + i\frac{1}{1}$

Find exponential form: $= 0 + 1i$

$= \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = e^{\frac{i\pi}{2}}$ $\quad$ Find roots: $e^{\frac{i\pi}{2}} = a^{2}$

$\pm e^{\frac{i\pi}{4}} = a$

$$\left.\begin{array}{ll} r_1 = e^{\frac{i\pi}{4}} & r_1 = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} \\ r_2 = -e^{\frac{i\pi}{4}} & r_2 = -\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) \end{array}\right\} \text{trigonométrique.}$$

Algébrique: $r_1 = 1\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)$ $\quad r_2 = -\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$

$= \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$

2) $z = 0 - i$ $|z| = 1$ $z = \frac{0}{1} - \frac{1}{1}i$

$\theta = \frac{3\pi}{2}$ $z = 0 - 1i$

Donc $e^{i\frac{3\pi}{2}} = a^2$

$r_1 = e^{i\frac{3\pi}{4}} = \cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}$

$r_2 = -e^{i\frac{3\pi}{4}} = -\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$

3) $i = a$ $|z| = 1$

$z = 0 + i$ $z = 0 + 1i$

$z = e^{i\frac{\pi}{2}}$

racine evidente : $r_0 = e^{i\frac{\pi}{8}}$

$r_1 = r_0 \cdot e^{\frac{i\pi 2}{4}} = e^{i\frac{\pi}{8} + \frac{i\pi 2}{4}} = e$

$r_2 = r_1 \cdot e^{i\frac{\pi}{4}2} = e^{\frac{i3\pi}{8} + \frac{i\pi 2}{4}} = e$

$r_3 = r_2 \cdot e^{i\frac{\pi}{4}2} = e^{\frac{i5\pi}{8} + \frac{i\pi 2}{4}} = e$

$\theta =$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$
$\sin\theta$	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$
$\cos\theta$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$
$\tan\theta$	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$

Si $z = e^{i\theta}$

$\Rightarrow \bar{z} = e^{-i\theta}$

25.09.2012

Exercice 6:

(E_3) $z^2 + \sqrt{3}z - i = 0$ Quand on a $a, b, c \in \mathbb{C}$

$az^2 + bz + c = 0$

$\Delta = b^2 - 4ac \in \mathbb{C}$

on pose $\delta^2 = \Delta$

$z_1 = \frac{-b + \delta_1}{2a}$ et $z_2 = \frac{-b + \delta_2}{2a}$, δ_1, δ_2 2 racines carrées de Δ.

Dans ce cas, $\Delta = \sqrt{3}^2 - 4(-i)$ on pose $\delta^2 = \Delta$ et $\delta = x + iy$, $x, y \in \mathbb{R}$

$\Delta = 3 + 4i$

$z_1 = \frac{-\sqrt{3} - \sqrt{3+4i}}{2}$ $z_2 = \frac{-\sqrt{3} + \sqrt{3+4i}}{2}$

to solve

$\Leftrightarrow \begin{cases} x^2 - y^2 = 3 & \text{partie réelle} \\ 2xy = 4 & \text{partie imaginaire} \\ x^2 + y^2 = \sqrt{3^2 + 4^2} = 5 & \text{module} \end{cases}$ $|\delta|^2 = |\Delta|$

$\Leftrightarrow \begin{cases} x^2 - y^2 = 3 \\ xy = 2 \\ 2x^2 = 8 \end{cases} \Leftrightarrow \begin{cases} y^2 = x^2 - 3 = 1 \\ xy = 2 \\ x = \pm 2 \end{cases} \Leftrightarrow \begin{cases} y = \pm 1 \\ x \text{ et } y \text{ de même signe} \\ x = \pm 2 \end{cases}$

$\Leftrightarrow \delta = 2 + i$ ou $\delta = -2 - i$

plug in: $z_1 = \frac{-\sqrt{3} + 2 + i}{2}$ $z_2 = \frac{-\sqrt{3} + (-2 - i)}{2}$

$= 1 - \frac{\sqrt{3}}{2} + i\frac{1}{2}$ $= \frac{-\sqrt{3} - 2 - i}{2}$

$= -1 - \frac{\sqrt{3}}{2} - i\frac{1}{2}$

(E_4) $z^4 + z^3 - 2z = 0$ Trouver les racines evidente d'abord:

$z(z^3 + z^2 - 2) = 0$ 0 et 1

$z(z-1)(z^2 + 2z + 2) = 0$

$z = 0$ $z = 1$ and $z^2 + 2z + 2 = 0$

1 est une racine evidente pq $z(z^3 + z^2 - 2) = 0$ $z^3 + z^2 - z = 0$

$1^3 + 1^2 - 2 = 0$ $2 - 2 = 0$ $0 = 0$

$z(z-1)(az^2 + bz + c) = 0$

$z^3 + z^2 - 2 = (z-1)(az^2 + bz + c)$

$z^3 + z^2 - 2 = az^3 + bz^2 + cz - az^2 - bz - c$

$z^3 + z^2 - z = az^3 + (b-a)z^2 + (c-b)z - c \quad \forall z \in \mathbb{C}$

$a = 1$ $(b-a) = 1$ $c - b = 0$ $-c = -2$

$b = a + 1$ $c = b$ $c = 2$

$2 = 1 + 1$ $b = 2$

$2 = 2$

Donc $z(z-1)(z^2 + 2z + 2) = 0$

Maintenant, on cherche à resoudre $z^2 + 2z + 2 = 0$.

Calcule des racines.

$\Delta = b^2 - 4ac$ $\Delta = 4 - 4(2)$

$= 4 - 8$

$= -4$ ou $= (2i)^2 = (-2i)^2$

$z_1 = \dfrac{-2 + \sqrt{(2i)^2}}{2}$ $z_2 = \dfrac{-2 - \sqrt{(2i)^2}}{2}$

$= -1 + i$ $= -1 - i$

Donc, les racines de $z^4 + z^3 - 2z = 0$ sont $0, 1, -1+i$ et $-1-i$.

Exercice 7 :

1) $S = \sum_{k=0}^{32} \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^k = \dfrac{1 - \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^{32+1}}{1 - \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)}$, however, the power 33 is a tad too high.

We will try to find the Δ and change $-\frac{1}{2} + i\frac{\sqrt{3}}{2}$ into its trigonometric form.

If $z = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$, then $\left|-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right| = \left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 = 1$

$$\begin{cases} \cos(\text{Arg}) = -\frac{1}{2} \\ \sin(\text{Arg}) = \frac{\sqrt{3}}{2} \end{cases} \Longrightarrow \text{Arg} = \frac{2\pi}{3}$$

$22i\pi$ périodique, donc, $2i\pi$

$$S = \frac{1 - \left(e^{\frac{2i\pi}{3}}\right)^{33}}{1 - e^{\frac{2i\pi}{3}}} = \frac{1 - e^{\frac{2i\,33\pi}{3}}}{1 - e^{\frac{2i\pi}{3}}} = \frac{1 - e^{2i\pi}}{1 - e^{\frac{2i\pi}{3}}} = 0$$

Because $S = \sum_{k=0}^{32} j^k = \underbrace{1 + j + j^2}_{0} + \underbrace{j^3 + j^4 + j^5}_{j^3\underbrace{(1+j+j^2)}_{0}} + \dots + j^{32} = 0$

2) $S_n = \sum_{k=0}^{n} \cos(k\theta)$

$\cos(k\theta) = \dfrac{e^{ik\theta} + e^{-ik\theta}}{2}$ $\qquad S_n = \sum_{k=0}^{n} \left(\dfrac{(e^{i\theta})^k + (e^{-i\theta})^k}{2}\right)$

$= \dfrac{(e^{i\theta})^k + (e^{-i\theta})^k}{2}$ $\qquad S_n = \frac{1}{2}\left[\sum_{k=0}^{n} (e^{i\theta})^k + \sum_{k=0}^{n} (e^{-i\theta})^k\right]$

deux suites géométrique

On peut pas faire la somme d'une suite géometrique qui est égale à 1.

- $\cos\theta = 0 \Rightarrow e^{i\theta} = 1 = e^{-i\theta}$

* Pas de somme !! *

$$S_n = \frac{1}{2}\big((n+1) + (n+1)\big) = n+1$$

- $\cos\theta \neq 0\ [2\pi]$, ou $\theta \neq 2k\pi$, $k \in \mathbb{Z}$

$$S_n = \frac{1}{2}\left(\frac{1-e^{i(n+1)\theta}}{1-e^{i\theta}} + \frac{1-e^{-i(n+1)\theta}}{1-e^{-i\theta}}\right)$$

$$S_n = \frac{1}{2}\left(\frac{(1-e^{-i\theta})(1-e^{i(n+1)\theta}) + (1-e^{-i(n+1)\theta})(1-e^{i\theta})}{(1-e^{i\theta})(1-e^{-i\theta})}\right)$$

$$\boxed{\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}}$$

$$S_n = \frac{1}{2}\left(\frac{1-e^{i(n+1)\theta} - e^{-i\theta} + e^{in\theta} + 1 - e^{i\theta} - e^{-i(n+1)\theta} + e^{-in\theta}}{1-e^{-i\theta} - e^{i\theta} + e^{0}}\right)$$

$$S_n = \frac{1}{2}\left(\frac{\text{"} \qquad \text{"}}{2-2\cos\theta}\right)$$

$$S_n = \left(\frac{1}{2}\right) \cdot \frac{2-(e^{i\theta}+e^{-i\theta}) - (e^{i(n+1)\theta} + e^{-i(n+1)\theta}) + (e^{in\theta} + e^{-in\theta})}{2-2\cos\theta}$$

$$S_n = \frac{1-\cos\theta - \cos((n+1)\theta) + \cos(n\theta)}{2-2\cos\theta}$$

$$S_n = \frac{1}{2}\left(\frac{1+\cos(n\theta) - \cos((n+1)\theta)}{1-\cos\theta}\right)$$

3) $T_n = \sum_{k=0}^{n} \binom{n}{k} \sin(k\theta)$

Exercice 8 :

1) $\theta \in \mathbb{R}$, exprimer $\cos(3\theta)$ en fonction de $\cos(\theta)$ et $\sin(\theta)$

$$(e^{i\theta})^3 = (\cos\theta + i\sin\theta)^3 = e^{3i\theta}$$
$$= \cos 3\theta + i\sin 3\theta$$

$$= \cos(\theta)^3 + 3(\cos\theta)^2 i\sin\theta + 3\cos\theta(i\sin\theta)^2 + (i\sin(\theta))^3$$
$$= (\cos^3\theta - 3\cos\theta\sin^2\theta) + i(3\cos^2\theta\sin\theta - \sin^3\theta)$$

Donc $\cos(3\theta) = \cos^3\theta - 3\cos\theta\sin^2\theta$

2) $\theta \in \mathbb{R}$, exprimer $\cos(5\theta)$ en fonction de $\cos\theta$.

Calculer $\cos\left(\frac{\pi}{5}\right)$ et $\cos\left(\frac{2\pi}{5}\right)$

$$\cos(5\theta) + i\sin(5\theta) = (\cos\theta + i\sin\theta)^5$$

$$\begin{array}{cccccc} 1 \\ 1 & 1 \\ 1 & 2 & 1 \\ 1 & 3 & 3 & 1 \\ 1 & 4 & 6 & 4 & 1 \\ 1 & 5 & 10 & 10 & 5 & 1 \end{array}$$

$2° \rightarrow$ (row $1\ 2\ 1$), $4° \rightarrow$ (row $1\ 4\ 6\ 4\ 1$)

$*$ $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$ $\qquad \binom{n}{k} = \frac{n!}{k!(n-k)!}$

$= \cos^5\theta + 5\cos^4\theta\,(i\sin\theta) + 10\cos^3\theta\,(i\sin\theta)^2 + 10\cos^2\theta\,(i\sin\theta)^3 + 5\cos\theta\,(i\sin\theta)^4 + (i\sin\theta)^5$

$\Rightarrow \cos(5\theta) = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$

2) $\theta \in \mathbb{R}$ $\quad \cos 5\theta$ en fonction de $\cos\theta$

$\cos(5\theta) = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$

or $\sin^2\theta = 1 - \cos^2\theta$

$\cos(5\theta) + i\sin(5\theta) = (\cos\theta + i\sin\theta)^5$

$= \cos^5\theta + 5\cos^4\theta\,(i\sin\theta) + 10\cos^3\theta\,(i\sin\theta)^2 + 10\cos^2\theta\,(i\sin\theta)^3 + 5\cos\theta\,(i\sin\theta)^4 + (i\sin\theta)^5$

- partie réelle.

$\cos(5\theta) = \cos^5\theta + 10\cos^3\theta\,(-1)\sin^2\theta + 5\cos\theta\,(+1)\sin^4\theta$

$\cos(5\theta) = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$

Now we need to get rid of the sins in the equation.

$\sin^2\theta = 1 - \cos^2\theta$

$\cos(5\theta) = \cos^5\theta - 10\cos^3\theta\,(1-\cos^2\theta) + 5\cos\theta\,(1-\cos^2\theta)^2$

$= \cos^5\theta - 10\cos^3\theta + 10\cos^5\theta + 5\cos\theta\,(1-\cos^2\theta)^2$

$= \cos^5\theta - 10\cos^3\theta + 10\cos^5\theta + 5\cos\theta\,(1 - 2\cos^2\theta + \cos^4\theta)$

$= \cos^5\theta - 10\cos^3\theta + 10\cos^5\theta + 5\cos\theta - 10\cos^3\theta + 5\cos^5\theta$

$= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$ FINAL ANSWER

Calculer $\cos\frac{\pi}{5}$: $\quad \theta = \frac{\pi}{5}$ $\quad$ si $x = \cos\left(\frac{\pi}{5}\right)$

$\cos\left(\frac{5\pi}{5}\right) = 16x^5 - 20x^3 + 5x = -1 \quad (\cos\pi = -1)$

Donc $\quad 0 = 16x^5 - 20x^3 + 5x + 1 \longrightarrow$ UNSOLVABLE

$$\boxed{\cos(2\theta) = 2\cos^2\theta - 1}$$

So, we try to find an equation where $\cos\theta = 0$ so that we have an obvious root.

$\theta = \frac{\pi}{10}$ $\quad x = \cos\left(\frac{\pi}{10}\right)$ $\quad \left[\cos(2\theta) = 2\cos^2\theta - 1\right]$

$$0 = \cos\left(\frac{5\pi}{10}\right) = 16x^5 - 20x^3 + 5x$$

racine evidente $x = 0$.

or $x \neq 0$ car $\cos\frac{\pi}{10} > 0$. $\quad 0 = 16x^4 - 20x^2 + 5$ (*)

on pose $x^2 = y$ $\quad 0 = 16y^2 - 20y + 5$

(*) equation bicarrée.

$\Delta = 20^2 - (4)(5)(16)$
$= 80$

$$y = \frac{20 \pm \sqrt{80}}{32} = \frac{20 \pm 4\sqrt{5}}{32} = \frac{5 \pm \sqrt{5}}{8}$$

or $\cos^2\left(\frac{\pi}{4}\right) \leq y = x^2 = \cos^2\left(\frac{\pi}{10}\right) \leq 1$ car $\cos \searrow$ sur $\left[0, \frac{\pi}{2}\right]$.

$\frac{1}{\sqrt{2}} = \cos\left(\frac{\pi}{4}\right) \leq \cos\left(\frac{\pi}{10}\right) \leq \cos(0) = 1$ or $y \longmapsto y^2$ est croissante sur $\mathbb{R}_+$.

$\frac{1}{2} \leq y = x^2 = \cos^2\left(\frac{\pi}{10}\right) \leq 1$ or $\frac{5 - \sqrt{5}}{8} < \frac{1}{2}$

donc $y = \frac{5 + \sqrt{5}}{8}$ $\quad y = \cos^2\left(\frac{\pi}{10}\right) = \frac{5 + \sqrt{5}}{8}$

or $\cos\left(\frac{\pi}{10}\right) > 0$ donc $\cos\left(\frac{\pi}{10}\right) = \sqrt{\frac{5 + \sqrt{5}}{8}}$

$$\cos\left(\frac{\pi}{5}\right) = 2\cos^2\left(\frac{\pi}{10}\right) - 1 = 2 \cdot \frac{5 + \sqrt{5}}{8} - 1 = \frac{5 + \sqrt{5} - 4}{4}$$

$$= \frac{1 + \sqrt{5}}{4}$$

Exercice 9 :

1) (E_1) $\cos(x) = \frac{\sqrt{3}}{2}$ quand $x = \frac{\pi}{6}$ ou $-\frac{\pi}{6}$

donc $x = \left[\begin{array}{ll} \frac{\pi}{6} + 2k\pi & \text{avec } k \in \mathbb{Z} \\ -\frac{\pi}{6} + 2k\pi & \text{avec } k \in \mathbb{Z} \end{array}\right.$

(E_2) $\sin\left(\frac{x}{2}\right) = -\frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}$ donc $\frac{x}{2} = \begin{cases} -\frac{\pi}{4} - \frac{\pi}{2} + 2k\pi & \text{avec } k \in \mathbb{Z} \\ -\frac{\pi}{4} + 2k\pi & \text{avec } k \in \mathbb{Z}. \end{cases}$

$x = \begin{cases} -\frac{\pi}{2} - \pi + 4k\pi & \text{avec } k \in \mathbb{Z} \\ -\frac{\pi}{2} + 4k\pi & \text{avec } k \in \mathbb{Z}. \end{cases}$

(E_3) $\sqrt{3}\sin(x) - \cos(x) = 1$ Utiliser $\cos^2(x) + \sin^2(x) = 1$

$\sqrt{3}\sin(x) = 1 + \cos(x)$, $\cos(x) \in [-1, 1]$

or $\sin(x) = \pm\sqrt{1 - \cos^2(x)}$, $\sin(x) \geq 0$

donc $\sqrt{3}\sqrt{1 - \cos^2(x)} = 1 + \cos(x) \geq 0$

or $x \longmapsto x^2$ est un bijection sur $\mathbb{R}_+$.

$3(1 - \cos^2(x)) = (1 + \cos(x))^2 = 1 + 2\cos(x) + \cos^2(x)$

$4\cos^2(x) + 2\cos(x) - 2 = 0$ on pose : $X = \cos(x)$

$4X^2 + 2X - 2 = 0$

$\Delta = 2^2 - 4 \times 4 \times (-2) = 4 + 32 = 36.$

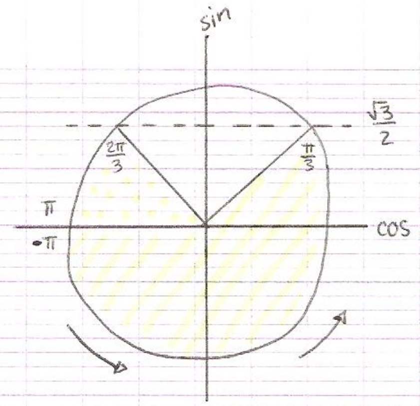
$X = \frac{-2 \pm \sqrt{36}}{4 \cdot 2} = -\frac{1}{4} \pm \frac{3}{4}$

donc $\cos(x) = -1$ ou $\frac{1}{2}$. donc $\sqrt{3}\sin x = 1 + \cos x$

$\begin{cases} \cos(x) = -1 \\ \sin(x) = 0 \end{cases}$ ou $\begin{cases} \cos(x) = \frac{1}{2} \\ \sin(x) = \frac{\sqrt{3}}{2} \end{cases}$ $\Longleftrightarrow$ $x = -\pi + 2k\pi$ $k \in \mathbb{Z}$

ou $x = \frac{\pi}{3} + 2k\pi$, $k \in \mathbb{Z}$.

2) $[-\pi, \pi]$ $\quad (I_1)\ \sin(x) < \frac{\sqrt{3}}{2}$



$$x \in \left[-\pi, \frac{\pi}{3}\right[\cup \left]\frac{2\pi}{3}, \pi\right]$$

$$S = \left[-\pi, \frac{\pi}{3}\right[\cup \left]\frac{2\pi}{3}, \pi\right]$$

(I_2) $\cos^2(x) \geq \cos(2x) + \frac{3}{4}$ $\quad$ utiliser $\cos(2x) = 2\cos^2 x - 1$ (1)

ou $\cos^2(x) = \frac{\cos(2x)+1}{2}$ (2)

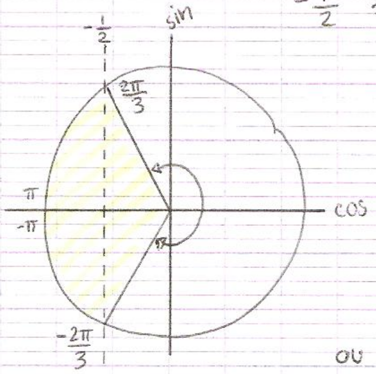
$$\cos^2(x) \geq 2\cos^2 x - 1 + \frac{3}{4}$$

$$\cos^2(x) - 2\cos^2 x$$

En utilisant (2): $\frac{\cos(2x)+1}{2} \geq \cos(2x) + \frac{3}{4}$

$$1 - \frac{6}{4} \geq \cos(2x)$$

$$-\frac{1}{2} \geq \cos(2x)$$



donc $-\pi + 2k\pi \leq 2x \leq -\frac{2\pi}{3} + 2k\pi$

ou $\frac{2\pi}{3} + 2k\pi \leq 2x \leq \pi + 2k\pi$

① $-\frac{\pi}{2} + k\pi \leq x \leq -\frac{\pi}{3} + k\pi, \ k \in \mathbb{Z}$

② $\pi/3 + k\pi \leq x \leq \frac{\pi}{2} + k\pi, \ k \in \mathbb{Z}$

① $\begin{cases} x \in \left[-\frac{\pi}{2}, -\frac{\pi}{3}\right] & k=0 \\ x \in \left[\frac{\pi}{2}, \frac{2\pi}{3}\right] & k=1 \end{cases}$ $\quad$ ② $x \in \left[\frac{\pi}{3}, \frac{\pi}{2}\right]$ $k=0$
$\quad x \in \left[-\frac{2\pi}{3}, -\frac{\pi}{2}\right]$ $k=-1$

$$S = \left[-\frac{2\pi}{3}, -\frac{\pi}{3}\right] \cup \left[\frac{\pi}{3}, \frac{2\pi}{3}\right]$$

Exercice 10:

1) $S = 1 + e^{\frac{2i\pi}{5}} + e^{\frac{4i\pi}{5}} + e^{\frac{6i\pi}{5}} + e^{\frac{8i\pi}{5}}$

$= 1 + e^{\frac{2i\pi}{5}} + \left(e^{\frac{2i\pi}{5}}\right)^2 + \left(e^{\frac{2i\pi}{5}}\right)^3 + \left(e^{\frac{2i\pi}{5}}\right)^4$ or $e^{\frac{2i\pi}{5}} \neq 1$.

$S = \dfrac{1-\left(e^{\frac{2i\pi}{5}}\right)^5}{1-e^{2i\pi/5}} = \dfrac{1-e^{2i\pi}}{1-e^{\frac{2i\pi}{5}}}$ donc $S = 0$

(the sum of all powers of a complex number $= 0$).

2) $R = 1 + \cos\left(\frac{2\pi}{5}\right) + \cos\left(\frac{4\pi}{5}\right) + \cos\left(\frac{6\pi}{5}\right) + \cos\left(\frac{8\pi}{5}\right)$

↳ partie réelle

$= R_e(1) + R_e\left(e^{\frac{2i\pi}{5}}\right) + R_e\left(e^{\frac{4i\pi}{5}}\right) + R_e\left(e^{\frac{6i\pi}{5}}\right) + R_e\left(e^{\frac{8i\pi}{5}}\right)$

$= R_e(\text{Somme})$

$= 0.$

$\cos\left(\frac{2\pi}{5}\right)$? $\quad \cos\left(\frac{8\pi}{5}\right) = \cos\left(2\pi - \frac{2\pi}{5}\right) = \cos\left(-\frac{2\pi}{5}\right) = \cos\left(\frac{2\pi}{5}\right)$

$\cos 2\theta = 2\cos^2\theta - 1$

$\cos\left(\frac{6\pi}{5}\right) = \cos\left(2\pi - \frac{4\pi}{5}\right) = \cos\left(\frac{4\pi}{5}\right)$ donc $S = 1 + 2\cos\left(\frac{2\pi}{5}\right) + 2\cos\left(\frac{4\pi}{5}\right)$.

☆ or $\cos\left(\frac{4\pi}{5}\right) = \cos\left(2 \cdot \frac{2\pi}{5}\right) = 2\cos^2\left(\frac{2\pi}{5}\right) - 1$

$R = 4\cos^2\left(\frac{2\pi}{5}\right) + 2\cos\left(\frac{2\pi}{5}\right) - 1 = 0$

On pose : $X = \cos\left(\frac{2\pi}{5}\right) > 0$ $\qquad 0 = 4X^2 + 2X - 1$

$\Delta = 2^2 - (4)(4)(-1) = 4 + 16$

$= 20$

$X = \dfrac{-2 \pm \sqrt{20}}{8}$ $\qquad X = -\frac{1}{4} \pm \frac{\sqrt{5}}{4}$ or $X = \cos\left(\frac{2\pi}{5}\right) > 0$

$\cos\left(\frac{2\pi}{5}\right) = \dfrac{\sqrt{5}-1}{4}$

Exercice 8 : continued

3) $\cos(4\theta) + i\sin(4\theta) = (\cos\theta + i\sin\theta)^4$

$$= \cos^4\theta + 4\cos^3\theta\,(i\sin\theta) + 6\cos^2\theta\,(i\sin\theta)^2 + 4\cos\theta\,(i\sin\theta)^3 + (i\sin\theta)^4$$

$$= (\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta) + i\,(4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta)$$

partie réelle — partie imaginaire

Identification :

linéarisation :

$\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta = \cos(4\theta)$ or $\sin^2\theta = 1 - \cos^2\theta$

donc $\cos^4\theta - 6\cos^2\theta\,(1-\cos^2\theta) + (1-\cos^2\theta)^2 = \cos(4\theta)$

$\cos^4\theta - 6\cos^2\theta + 6\cos^4\theta + 1 - 2\cos^2\theta + \cos^4\theta = \cos(4\theta)$

$8\cos^4\theta - 8\cos^2\theta + 1 = \cos(4\theta)$ or $\boxed{\cos^2\theta = \frac{\cos(2\theta)+1}{2}}$

donc $8\cos^4\theta - 4(\cos(2\theta)+1) + 1 = \cos(4\theta)$

$\cos^4\theta$:

$\Longrightarrow \cos^4\theta = \frac{1}{8}\cos(4\theta) + \frac{1}{2}\cos(2\theta) + \frac{3}{8}$

$\sin^4\theta$:

donc $\sin^4\theta = \cos(4\theta) - \cos^4\theta + 6\cos^2\theta\sin^2\theta$

$= \cos(4\theta) - \cos^4\theta + 6\cos^2\theta\,(1-\cos^2\theta)$.

$= \cos(4\theta) - 7\cos^4\theta + \underbrace{6\cos^2\theta}$

or $\cos^2\theta = \frac{\cos(2\theta)+1}{2}$

$= \cos(4\theta) - 7\left(\frac{1}{8}\cos(4\theta) + \frac{1}{2}\cos(2\theta) + \frac{3}{8}\right) + 3(\cos 2\theta + 1)$

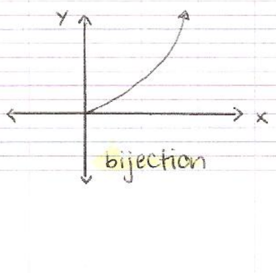
$\sin^4\theta = \frac{1}{8}\cos(4\theta) - \frac{1}{2}\cos(2\theta) + \frac{3}{8}$

Exercice 11 :

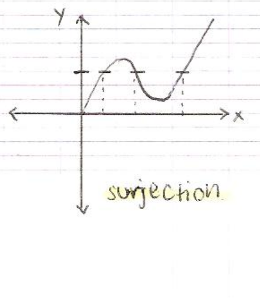
1) $n \in \mathbb{N}^*$, $z \in \mathbb{C}$, $z^n = 1$ $|z| = ?$

$|z|^n = |z^n| = 1$ or $|z| \in \mathbb{R}_+$

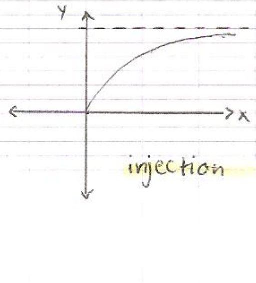
dans $\mathbb{R}_+$, $x \longmapsto x^n$ est un bijection, donc $|z| = 1$



bijection



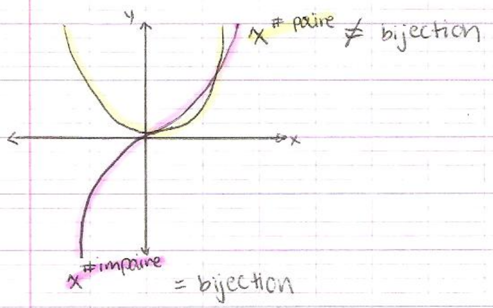
surjection



injection

Par contre, $x \longmapsto x^2$ n'est pas une bijection de $\mathbb{R}$ dans $\mathbb{R}$.

2) $z^{11} = -1$ $\quad P(z) = z^{11} + 1$, P a 11 racines dans $\mathbb{C}$.



$\mathbb{R} \longrightarrow \mathbb{R}$

$x \longmapsto x^{11}$ est une bijection.

donc $z^{11} = -1$ a une seule solution dans $\mathbb{R}$.

3) $\mathbb{U}_3 = \{z \in \mathbb{C},\ z^3 = 1\} = \{1, e^{\frac{2i\pi}{3}}, e^{\frac{4i\pi}{3}}\} = \{1, j, j^2\}$

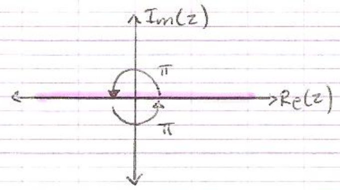
$\mathbb{U}_6 = \{z \in \mathbb{C},\ z^6 = 1\} = \{1, e^{\frac{2i\pi}{6}}, e^{\frac{4i\pi}{6}}, e^{\frac{6i\pi}{6}} = -1, e^{\frac{8i\pi}{6}}, e^{\frac{10i\pi}{6}}\}$

Exercice 12 :

1) $z_1, z_2 \in \mathbb{C}^*$. M_1, M_2 d'affixe z_1, z_2.

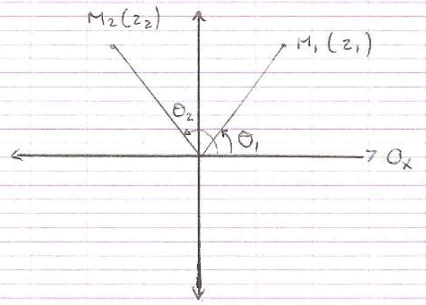
$\frac{z_1}{z_2} \in \mathbb{R}$ $\quad z_1 = |z_1| e^{i\theta_1}$ et $z_2 = |z_2| e^{i\theta_2}$

$$\frac{z_1}{z_2} = \frac{|z_1| e^{i\theta_1}}{|z_2| e^{i\theta_2}} = \underbrace{\boxed{\frac{|z_1|}{|z_2|}}}_{\in \mathbb{R}_+} \cdot e^{i(\theta_1 - \theta_2)} \in \mathbb{R}.$$



we want to stay on the $\mathbb{R}e$ axis.

$z \in \mathbb{R} \iff Arg(z) = k\pi$ avec $k \in \mathbb{Z}$



donc $\theta_1 - \theta_2 = k\pi$ avec $k \in \mathbb{Z}$

$\theta_1 = (\widehat{\vec{O}_x, \overrightarrow{OM_1}})$ angle.

$\theta_2 = (\widehat{\vec{O}_x, \overrightarrow{OM_2}})$ angle

$\theta_1 - \theta_2 = (\widehat{\overrightarrow{OM_1}, \overrightarrow{OM_2}})$, O, M_1, M_2 sont alignés.

Exercice 13:

1) $(O, \vec{i}, \vec{j})$ $E : |(1-i)z + 2i| = 9$

on pose : $z = x + iy$

Cercle de centre (x_0, y_0) et de rayon R. $(x-x_0)^2 + (y-y_0)^2 = R^2$

$$(1-i)z + 2i = (1-i)(x+iy) + 2i$$
$$= x + iy - ix - i^2y + 2i$$
$$= (x+y) + i(y - x + 2)$$

$$|\underbrace{(x+y)}_{\in \mathbb{R}} + i\underbrace{(y-x+2)}_{\in \mathbb{R}}| = 9$$

$$(x+y)^2 + (y-x+2)^2 = 9^2$$
$$x^2 + 2xy + y^2 + y^2 + x^2 + 2^2 - 2xy + 2\times 2y - 2\times 2x$$
$$= 9^2$$

$2x^2 + 2y^2 + 4y - 4x + 4 = 81$ // $(x-x_0)^2 + (y-y_0)^2 = R^2$

identity remarkable : $2(x^2 - 2x + \underbrace{1 - 1}_{=0}) + 2(y^2 + 2y + \underbrace{1 - 1}_{=0}) + 4 = 81$

$2((x-1)^2 - 1) + 2((y+1)^2 - 1) + 4 = 81$

$(x-1)^2 + (y+1)^2 = \frac{81}{2} = \left(\frac{9}{\sqrt{2}}\right)^2$

on le veut sur cette forme.

cercle de centre $z_0 = 1 - i$ et de rayon $\frac{9}{\sqrt{2}}$

2) $F : |((z+1)/(z-1+i\sqrt{3})| = 1$ on pose $z = x + iy$

$$F : \left|\frac{z+1}{z-1+i\sqrt{3}}\right| = 1$$

$$|z+1| = |z - 1 + i\sqrt{3}|$$
$$|(1+x) + iy| = |(x-1) + i(\sqrt{3} + y)|$$
$$(1+x)^2 + y^2 = (x-1)^2 + (\sqrt{3}+y)^2$$
$$\cancel{x^2} + 2x + \cancel{1} + \cancel{y^2} = \cancel{x^2} - 2x + \cancel{1} + 3 + 2\sqrt{3}y + \cancel{y^2}$$
$$\frac{4x-3}{2\sqrt{3}} = y = \frac{2x}{\sqrt{3}} - \frac{\sqrt{3}}{2}$$ adroite.

3) $G : \left| \dfrac{1+iz}{1-iz} \right| = 1$ on pose : $z = x+iy \quad x, y \in \mathbb{R}$

$\dfrac{|1+iz|}{|1-iz|} = 1 \iff |1+iz| = |1-iz|$

et $z \neq \dfrac{1}{i} = -i$

or $1+iz = 1+i(x+iy) = (1-y) + ix$

$1-iz = 1-i(x+iy) = (1+y) - ix$

$\iff (1-y)^2 + x^2 = (1+y)^2 + (-x)^2$ et $z \neq -i \quad (x,y) \neq (0,-1)$

$\iff \cancel{1} - 2y + \cancel{y^2} + \cancel{x^2} = 1 + 2y + \cancel{y^2} + \cancel{x^2}$ et $(x,y) \neq (0,-1)$

$\iff y = 0$ et $(x,y) \neq (0,-1)$