Project Assignment 1

Dynamics in Systems and Networks: Network Dynamics
University of Trento
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- You can choose to solve the problems with any software and/or program languages (Matlab, Mathematica, Python or others) but note that only Matlab is "supported".
- Please submit your Assignment as a **typeset PDF file**.
- When something has to be **computed analytically**, only use pen and paper. **All answers should be clearly motivated:** providing end results of calculations only is not sufficient.
- Please also send your **code** written to solve the Problems.
  Comment your code well. Clarity is more important than efficiency.
  Your code should be written in a general way: if a question is slightly modified, getting the new correct answer should only require slight modifications in your code as well.
  All answers should be clearly motivated: providing end results of calculations only is not sufficient.
- Submit your report (and the code) via e-mail to `gianni.lunardi@unitn.it` (and to `giulia.giordano@unitn.it` in cc).
- **Submission deadline: the day before you wish to take the exam!**

1 Pressure sensors

The graph in Figure 1 represents a network of pressure sensors. Each node is a sensor, while the bidirectional links are the communication lines. Implement algorithms that compute the centrality of the nodes in this network according to different centrality measures:

a) The invariant distribution centrality;

b) The closeness and betweenness centrality;

c) The PageRank centrality, computed in an **iterative way**, in the following cases:

1. $\beta = 0.15$ and $\mu = \mathbb{1}$;
2. $\beta = 0.6$ and $\mu_i = 0 \ \ \forall i \neq 7, \ \mu_7 = 1$;
3. $\beta = 0.5$ and $\mu_i = 0 \ \ \forall i \neq 11, \ \mu_{11} = 1$;

d) Comment your results briefly.

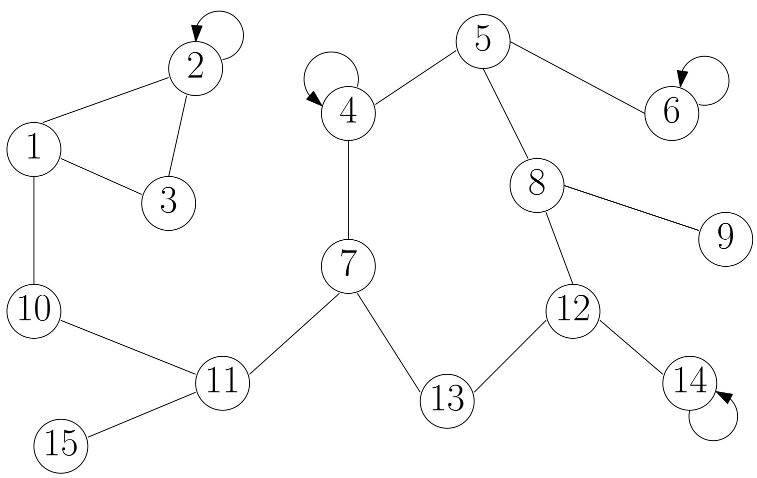

Figure 1: Graph representing the sensor network.

Then, study the French-DeGroot dynamics evolving over the same network in Figure 1. For each node, the initial state is $x_i(0) = \theta + \xi_i$, where $\theta = 101.325\,Pa$ is the atmospheric pressure, while ξ_i is the **Gaussian noise** on each sensor (zero mean and $\sigma = 0.5\,Pa$ as standard deviation). **Generate the random initial condition once and for all, and then keep it fixed**.

e) Compute analytically the limit of the distributed averaging dynamics on this network. Then, compare your analytical result with what you achieve through simulations.

f) How do the results of the previous step change if we add self-loops on all the nodes?

h) Compute the **exact** average state, using the consensus algorithm that calculates the average degree $\bar{w}$.

h) Write a script where you simulate the distributed averaging dynamics on this network, when the set of stubborn nodes is $\mathcal{S} = \{3, 12\}$ with the input values $u_3 = 12$ and $u_{12} = 6$. Plot the trajectories of all the nodes and determine the equilibrium vector.

i) Compute analytically the equilibrium vector for the distributed averaging dynamics with the stubborn node $\mathcal{S} = \{11\}$ with the input value $u_{11} = 15$. Then, compare your analytical result with what you achieve through simulations.

2 Robotic coordination

Consider a network of $n = 16$ robots with dynamics:

$$\dot{p}_i = u_i, \qquad i = 1, \ldots, n, \tag{1}$$

where $p_i \in \mathbb{R}^2$ is the position of robot i in Cartesian coordinates and $u_i \in \mathbb{R}^2$ is the steering control input. Assume that each robot starts from a random initial position inside a $10\,m \times 10\,m$ square. Let us consider the following distributed control law:

$$\begin{aligned} u_1 &= \frac{1}{2}(p_1 + p_2) - p_1, \\ u_i &= \frac{1}{3}(p_{i-1} + p_i + p_{i+1}) - p_i, \qquad i = 2, \ldots, n-1 \\ u_n &= \frac{1}{2}(p_{n-1} + p_n) - p_n, \end{aligned} \tag{2}$$

a) What is the behaviour of the control law? Show that the coordination law leads the robots to rendezvous (i.e. consensus).

b) Compute the Laplacian matrix L from the coordination law (2). Then derive also the degree D and adjacency W matrices.

c) Plot and characterize the graph associated with the robotic network.

d) Simulate the system for a certain time until it reaches a consensus. Then compare the analytical results (through the invariant centrality vector) with what you achieve through simulations.

e) Suppose that all the robots are contained in a walled box with corners $p_0 = (0,\ 0)$ and $p_{n+1} = (10,\ 10)$. The walls are stationary ($\dot{p}_0 = \dot{p}_{n+1} = 0$). Consider the following modified version of (2):

$$u_i = \frac{1}{3}(p_{i-1} + p_i + p_{i+1}) - p_i, \qquad i = 1, \ldots, n. \tag{3}$$

Does the system reach a rendezvous? Motivate your answer by repeating step d) for this case.

3 Compartmental vs. Networked Epidemics

Consider a SITR (Susceptible-Infectious-Treated-Recovered) compartmental model describing the dynamics of a non-lethal infectious disease. This model is similar to the SIR, but in this case a fraction α of infectives is selected for treatment, which reduces their infectivity by a coefficient $0 < \delta < 1$. The graph associated with the model is represented in Figure 2.

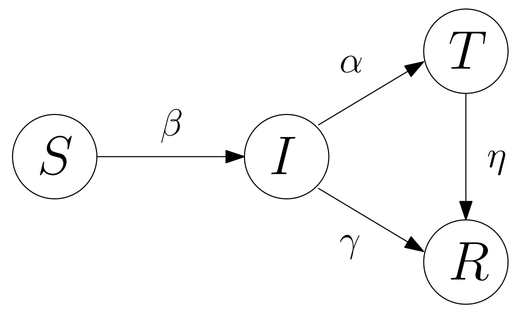

Figure 2: Flowchart of the SITR model.

- Write the ordinary differential equation systems corresponding to the SITR model and analyse its steady-state, equilibrium points (if existing) and dynamic behaviours.
- Simulate the system evolution over time, for different choices of the initial conditions, choosing the values of the system parameters so as to generate at least two meaningful scenarios with qualitatively different evolution (e.g., the number of infected individuals decreases since the beginning; or the number of infected individuals increases up to a peak value and then decreases). Also, study what happens when δ is close to 0 and 1.

Consider now a randomly generated network (undirected graph) with $n = 500$ nodes, representing individuals, where the links represent personal/physical interactions that may lead to contagion. Assume that each node can be either Susceptible, Infectious, Treated or Recovered. A Susceptible node can become Infectious, with a certain probability, if it has at least one Infectious neighbour. An Infectious node, after a certain time, becomes Treated (and then Recovered) or directly Recovered. The parameters that govern the evolution of the status of each node are related to the parameters in the compartmental version of the model.

- Initialise the network so that 30% of the nodes are Infectious, while the others are Susceptible. Then, simulate the time evolution of the fraction of nodes belonging to the four classes, depending on different choices of the initially Infectious nodes and of the parameters (consider different scenarios analogous to the ones you have considered for the corresponding compartmental version of the model).
- In each scenario, how do the results compare with the time evolution of the state variables of the corresponding compartmental model, with initial condition $S(0) = 0.7$, $I(0) = 0.3$, $T(0) = R(0) = 0$?