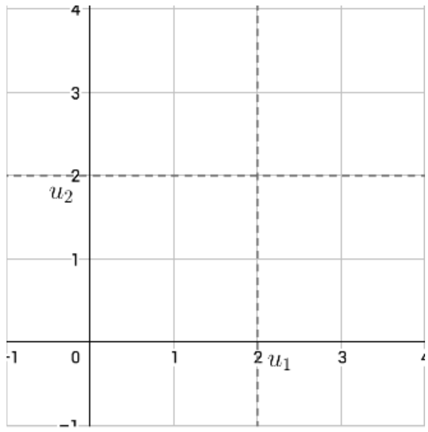
Monday, 19 February 2018

# Linear Algebra

## Vectors

- Every real number can be used to represent a point on a line which is a 1-Dimensional space (1D), R1
- A pair of real numbers can be used to represent a point in a plane, which is a 2D space, R2
- A triplet of real numbers can be used to represent a point in the 3D space, R3




Definition: A scalar is just another name for a real number

Definition: a vector of dimension n is a **ordered** collection of elements, called *vector components*

Notation: Vectors are usually represented by letters. In a typewritten word letters are usually written in **boldface**. In handwriting, the right arrow written above sometimes used to represent vectors. **We will underline vectors.**

Let $\underline{u} \in \mathbb{R}^n$, the i-th component of the vector will be written as $u_i, i = 1,\ldots,n$

## Vector Representation

Let $\underline{u} \in \mathbb{R}^2, \underline{u} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

Representation 1: Two ordered numbers

Representation 2: an arrow starting from the origin point going 3 units right and 1 up

Representation 3: A vector can also be a point in $\mathbb{R}^2$ with the coordinates $x_1 = 3, x_2 = 1$, the end point of the arrow starting from the origin.

Definition: Two vectors are equal if and only if the corresponding components are equal. Let $\underline{u}, \underline{v} \in \mathbb{R}^n$ $\underline{u} = \underline{v} \iff u_i = v_i$, for any $i = 1, \ldots, n$

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## Vectorial Addition

Definition: Let $\underline{u}$ and $\underline{v} \in \mathbb{R}^n$. The vector $\underline{w} \in \mathbb{R}^n$, is the sum of $\underline{u}$ and $\underline{v}$.

$$\underline{w} = \underline{u} + \underline{v}, \text{ if } \underline{u}_i + \underline{v}_i, \forall i = 1, \dots, n$$

*($\underline{u}$ and $\underline{v}$ must belong to the same space)*

Definition: The product of a scalar $\alpha \in \mathbb{R}$ and a vector $\underline{u} \in \mathbb{R}^n$ is defined as:

$$\alpha \underline{u} = \begin{pmatrix} \alpha u_1 \\ \alpha u_2 \end{pmatrix}$$

Definition: Let $\alpha, \beta \in \mathbb{R}$ and $\underline{u}$ and $\underline{v} \in \mathbb{R}^n$. The sum of the $\alpha \underline{u}$ and $\beta \underline{v}$ is called a **linear combination** of $\underline{u}$ and $\underline{v}$:

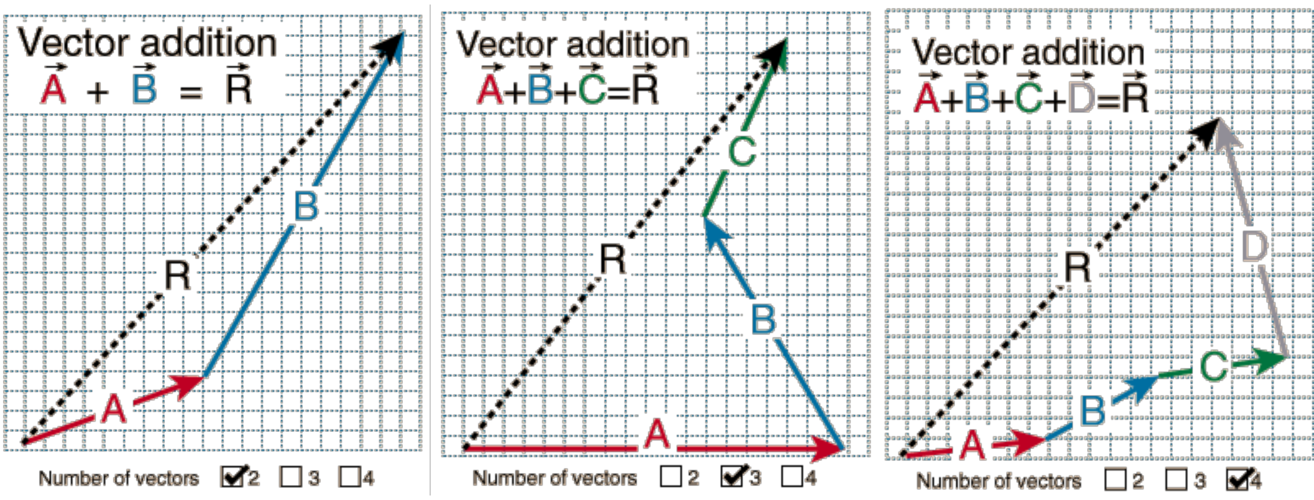
$$\alpha \underline{u} + \beta \underline{v} = \begin{pmatrix} \alpha u_1 + \beta v_1 \\ \alpha u_2 + \beta v_2 \end{pmatrix}$$

Definition: The vector $\underline{u} \in \mathbb{R}^n$ with all of its components $= 0$, is called the **zero vector** $\underline{0}$

## Graphical Representation of Vectorial Operations

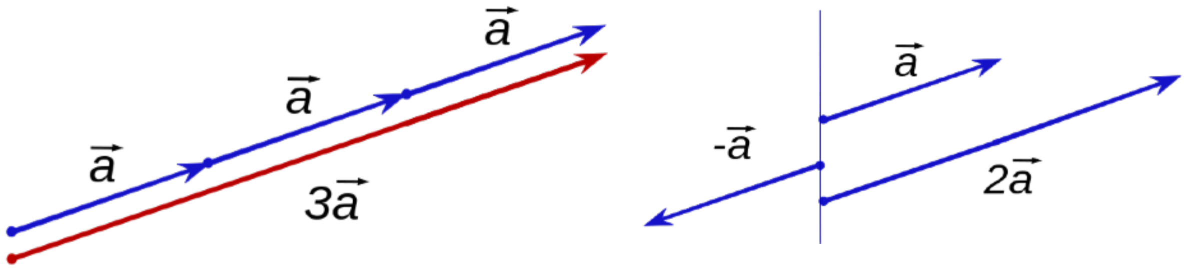
### Vectorial Addition

1. Draw a vector $\underline{A}$ to scale;
2. Draw a vector $\underline{B}$ to scale, starting at the end of $\underline{A}$;
3. Draw a line from the beginning of $\underline{A}$ to the end of $\underline{B}$. The Angele can be measured and the size of $\underline{R}$ determined by scaling



**Vectorial Multiplication by a scalar** (*scaling)*

4. Draw a vector A to scale;
5. Scale it by n times, where n is the scalar, if n is negative, the process is called ***negating***;
6. Conserve the direction of the initial vector A.



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Consider u, v $\in \mathbb{R}^n$ are on the same line, if and only if there exists two scalars $\alpha$ and $\beta \in \mathbb{R}$ such that $\alpha \underline{u} + \beta \underline{v} = \underline{0}$ with $\alpha \neq 0$ or $\beta \neq 0$

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## Vectorial Dot Product

Definition: Let us consider the vectors $\underline{u}, \underline{v}$ Dot or Scalar Product is defined as:

$$< \underline{u}, \underline{v} > = u_1 v_1 + u_2 v_2 + \ldots + u_n v_n = \sum_{i=1}^{n} u_i v_i$$

If the Dot Product of two vectors is zero it means they are ortogonal

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## Properties of the dot product

$\underline{u}, \underline{v} \in \mathbb{R}^n, \forall \alpha \in \mathbb{R}$

$$< \alpha \underline{u}, \underline{v} > = \alpha < \underline{u}, \underline{v} >$$

Proof:

$$< \alpha \underline{u}, \underline{v} > = (\alpha u_1) v_1 + (\alpha u_2) v_2 + \ldots (\alpha u_n) v_n =$$

$$\alpha (u_1 v_1 + u_2 v_2 + \ldots + u_n v_n) = \alpha < \underline{u}, \underline{v} >$$

Bilinearity:

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## Length of a Vector

Let us consider $\underline{u} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}, \qquad \underline{v} = \underline{u}, \qquad \underline{u}, \underline{v} \in \mathbb{R}^2$

$$< \underline{u}, \underline{v} > = u_1 \cdot v_1 + u_2 \cdot v_2 + \ldots + u_n \cdot v_n = 9 + 16 = 5^2$$

Definition: The length of a vector $\underline{u} \in \mathbb{R}^n$ (also called the Euclidean Norm) is defined as:

$$||\underline{u}|| = \sqrt{< \underline{u}, \underline{v} >}$$

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Definition: A unit vector is a vector where the length equals 1. Let us consider $\underline{u} \in \mathbb{R}^n, ||\underline{u} \in \mathbb{R}^n, ||\underline{u}|| = 1$

Notation: Sometimes unit vectors are represented using a "hat" on tops, $\underline{\hat{u}}$

Question: How do we transform a vector $\underline{u} \in \mathbb{R}^n, \underline{u} \neq \underline{0}$, whose $||\underline{u}|| \neq 1$ and $\neq 0$ into a unit vector?

Answer: We divide it by its length: $\underline{\hat{u}} = \dfrac{\underline{u}}{||\underline{u}||} \implies \underline{u} = \hat{u}||\underline{u}||$

Question: if the end points of <u>all</u> unit vectors in a plane ($\mathbb{R}^2$ space, 2D). What do we get?

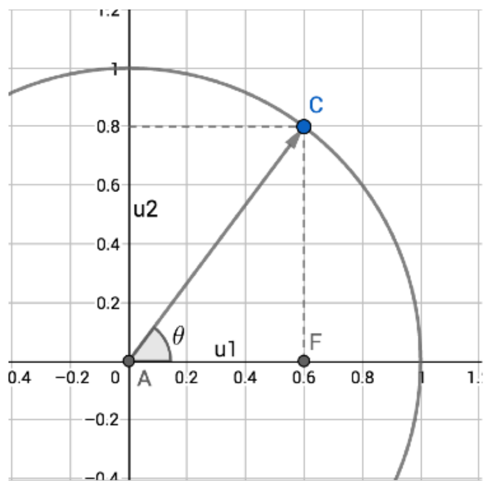
Answer: A circle of radius 1, which is called the *Trigonometric Circle*

Let us consider $\underline{u} \in \mathbb{R}^2$, $||\underline{u}|| = 1$

$$\cos\theta = \frac{u_1}{||\underline{u}||}$$

$$\sin\theta = \frac{u_2}{||\underline{u}||} \qquad \implies \underline{u} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

$$||\underline{u}|| = 1$$




$$< \underline{u}, \underline{v} > = \cos\theta\cos\gamma + \sin\theta\sin\gamma = \cos(\gamma - \theta) = \cos\Psi = \cos(\angle(\underline{u}, \underline{v}))$$

$$\implies \cos( < (\underline{u}, \underline{v})) = < \underline{u}, \underline{v} >$$

We did this demonstration for 2D space, but it is valid for all he other spaces, $\mathbb{R}^n$ i.e, $< \underline{u}, \underline{v} > = \cos(\angle(\underline{u}, \underline{v}))$, for: $\forall \underline{u}, \underline{v} \in \mathbb{R}^n$, that have $||\underline{u}|| = ||\underline{v}|| = 1$

How do we calculate the angle between two arbitrary vectors $\underline{u}, \underline{v} \in \mathbb{R}^n$, the lengths are $\neq 1$?

Let us consider $\underline{u}, \underline{v} \in \mathbb{R}^n$, $||\underline{u}|| \neq 1$, $||\underline{v}|| \neq 1, \underline{u}, \underline{v} \neq \underline{0}$

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$$< \underline{u}, \underline{v} > = \left\langle \frac{||\underline{u}||}{||\underline{u}||} \cdot \underline{u}, \frac{||\underline{v}||}{||\underline{v}||} \cdot \underline{v} \right\rangle = \left\langle ||\underline{u}|| \cdot \frac{1}{||\underline{u}||} \cdot \underline{u}, ||\underline{v}|| \cdot \frac{1}{||\underline{v}||} \cdot \underline{v} \right\rangle =$$

$$= \left\langle ||\underline{u}|| \cdot \underline{\hat{u}}, ||\underline{v}|| \cdot \underline{\hat{v}} \right\rangle = ||\underline{u}|| \cdot ||\underline{v}|| < \hat{u}, \hat{v} > = ||\underline{u}|| \cdot ||\underline{v}|| \cos(\angle(\underline{u}, \underline{v}))$$

Lemma:

Let us consider $\underline{u}, \underline{v} \in \mathbb{R}^n$, two arbitrary vectors $\underline{u}, \underline{v} \neq \underline{0}$, $||\underline{u}|| \neq 1$, $||\underline{v}|| \neq 1$

$$\cos(\angle(\underline{u}, \underline{v})) = \frac{< \underline{u}, \underline{v} >}{||\underline{u}|| \cdot ||\underline{v}||}$$

Lemma (Cauchy Schwarz Inequality):

$$| < \underline{u}, \underline{v} > \leq ||\underline{u}|| \cdot ||\underline{v}||$$