Automata and Formal Languages

The Regular Operations

- Let A and B be languages, we defined union, concatenation and star as follows:
  - **Union**: $A \cup B = \{x \mid x \in A \vee x \in B\}$
  - **Concatenation**: $A \cdot B = \{xy \mid x \in A \wedge y \in B\}$
  - **Star**: $A^* = \{x_1x_2 \ldots x_k \mid k \geq 0 \wedge \forall x_i \in A\}$

Example

$A = \{\text{nice, bad}\}$ $B = \{\text{dog, cat}\}$

Union: $A \cup B = \{$nice, bad, dog, cat$\}$

Concatenation: $AB = \{$nicedog, nicecat, baddog, badcat$\}$

$A \cdot B \neq B \cdot A$

Star: $A^* = \{\epsilon$, nice, bad, nicenice, badbad, nicebad, badnice, nicenicenice, $\ldots\}$

$A = \{w \mid w$ contains at least a 1$\}$ $B = \{w \mid w$ contains at least a 0$\}$

$A \cup B = \{w \mid w$is not the empty string$\}$

$A \cdot B = \{w \mid w$ contains the substring 10$\}$

$A^* = \{w \mid w$ does not contain a 0$\}$

Properties

Closure: $\mathbb{N}$ (set of natural numbers) is closed under multiplication. It means that:

- If x and y are in $\mathbb{N}$, the product xy must also be in $\mathbb{N}$

The collection of Regular Languages is closed under all three of the regular operators.

Therefore it exists an FA that recognises them.

The regular operators are useful tools for understanding the power of FAs.

Proof by construction

Thesis: Regular languages are closed under the **Union** operator.

To prove that the union of two regular languages is a regular language, we **construct** the automaton that recognises the union of the languages:

If A_1 and A_2 are regular

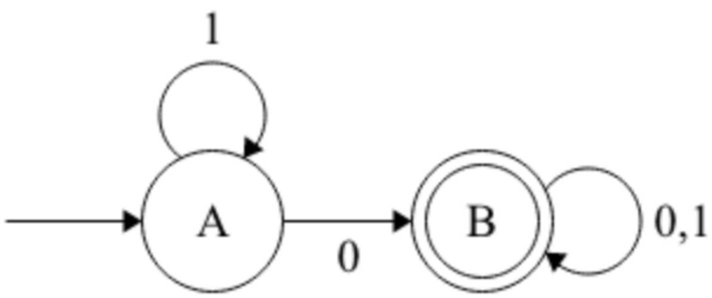
With $A_1 = L(M_1)$ and $A_2 = L(M_2)$
We construct a finite automaton such that L(M) = A1 U A2

M should accept a string if M_1 accepts it or M_2 accepts it.

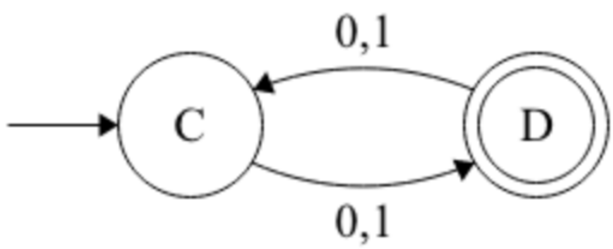
M needs to emulate both M_1 and M_2

Example

$A_1 = \{w \mid w \text{ contains at least a } 0\}$



$A_2 = \{w \mid w \text{ contains an odd number of symbols}\}$



$Q_1 = \{A, B\}, Q_2 = \{C, D\}$