Probability & Statistics: Assignment 6

Due on 1 December, 2018 at 8:30am

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Problem 1

a) In order to find $P(X > 22.07)$ we need to find the corresponding value in the Φ distribution, that is, we need to normalise the considered value:

$$P(X > 22.07) = P\left(X^* > \frac{22.07 - \mu}{\sigma}\right)$$

Knowing that $\mathrm{Var}(X) = \sigma^2 = 0.16$, we can write:

$$P\left(X^* > \frac{22.07 - 21.37}{0.4}\right)$$

$$P\left(X^* > \frac{0.7}{0.4}\right) = P\left(X^* > 1.75\right)$$

Since this last inequality is about the normalised random variable X^* we can look up the cumulative probability for $X^* < 1.75$ and subtract it from the total probability to get its complement, which is what we are looking for.

$$\Phi(1.75) = 0.9599$$

$$P(X^* > 1.75) = 1 - \Phi(1.75) = 1 - 0.9599 = 0.0401 \approx 4\%$$

b) We know want to compute $P(Y \leq 2)$, which is the probability that out of 15 mints, at most 2 weigh more than 20.857g. This means that we are dealing with a binomial distribution, since a mint can only weigh less or more. But what is the probability of such events?

We need to compute then $P(X < 20.857)$, by normalising again we get:

$$P\left(X^* < \frac{20.857 - 21.37}{0.4}\right) = P\left(X^* < \frac{-0.513}{0.4}\right) = P(X^* < -1.2825) = \Phi(-1.2825) \approx 0.1003 \approx 10\%$$

Thus we can now compute our probability as a binomial distribution:

$$P(k) = \sum_{i=0}^{k} \binom{n}{i} p^i (1-p)^{n-i}$$

$$P(2) = \binom{15}{0}(0.9)^{15} + \binom{15}{1}(0.1)(0.9)^{14} + \binom{15}{2}(0.001)(0.9)^{13} \approx 0.8159 \approx 82\%$$

Problem 2

a) If we want to find σ of a known distribution $N(\mu = 12.1, \sigma^2)$ in order for $P(X < 12) = 0.01$ to be true. Thus we want to find for which value of Φ must coincide to the same value.

$$P\left(X^* < \frac{12 - 12.1}{\sigma}\right) = 0.01$$

By looking at the table of the standard normal CDF $\Phi(z)$, we can see that:

$$\Phi(-2.33) = 0.01$$

Which implies:

$$\frac{-0.1}{\sigma} = -2.33$$
$$\sigma = \frac{0.1}{2.33} \approx 0.043$$

b) Just like we did before, we can derive μ by fixing $\sigma = 0.05$ so that $P(X < 12) = 0.01$:

$$\frac{12 - \mu}{0.05} = -2.33$$
$$12 - \mu = -\frac{2.33}{20}$$
$$\mu = 12.1165$$

Problem 3

The times required to produce an item are independent random variables between 1 and 5 with the same probability:

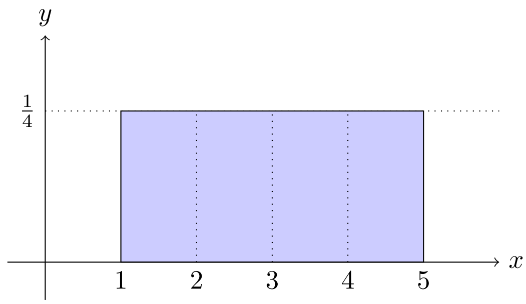

Figure 1: Uniform distribution: $p(x) = \frac{1}{4}, \quad x \in \{1, \dots, 5\}$

We can thus compute the mean μ, that is the average time for a single piece:

$$\mathrm{E}[X] = \int_1^5 \frac{1}{4} x \; dx = \frac{1}{4} \int_1^5 x \; dx = \frac{1}{4} \left[\frac{1}{2} x^2 \right]_1^5 = \frac{1}{4} \left[\frac{25}{2} - \frac{1}{2} \right] = \frac{12}{4} = 3$$

$$\mu = 3$$

We now want to compute $\mathrm{Var}(X)$:

$$\mathrm{Var}(X) = \mathrm{E}[X^2] - \mathrm{E}[X]^2$$

$$\mathrm{E}[X^2] = \frac{1}{4} \int_1^5 x^2 \; dx = \frac{1}{4} \left[\frac{1}{3} x^3 \right]_1^5 = \frac{1}{4} \left[\frac{125}{3} - \frac{1}{3} \right] = \frac{124}{12} = \frac{31}{3}$$

$$\mathrm{Var}(X) = \frac{31}{3} - 3^2 = \frac{31}{3} - \frac{27}{3} = \frac{4}{3}$$

$$\mathrm{Var}(X) = \sigma^2 = \frac{4}{3}$$

$$\sigma = \sqrt{\frac{4}{3}} = \frac{2}{\sqrt{3}}$$

We should know define a new random variable S_n which is associated to the sum of the times for n different items being produced:

$$S_n = X_1 + \dots + X_n = \sum_{i=1}^{n} X_i, \quad 0 < i \leqslant n$$

$$\mathrm{E}[S_n] = n \cdot \mathrm{E}[X_i] = 3n = \mu$$

$$\mathrm{Var}(S_n) = n \cdot \mathrm{Var}(X_i) = \frac{4}{3} n$$

$$\sigma = \sqrt{\mathrm{Var}(S_n)} = 2\sqrt{\frac{n}{3}}$$

Now that we have a normal distribution for a random variable S_n, we go back to our original question and to our hint that:

$$P(N_{320} \geq 100) = P(S_{100} < 320)$$

Problem 3 continued on next page. . .

Where S_{100} is exactly our random variable S_n with $n = 100$.
Let us normalise the right side of this equation and substitute in:

$$P\left(S_n^* \leq \frac{320 - \mu}{\sigma}\right) = P\left(S_n^* \leq \frac{320 - 3n}{2\sqrt{\frac{n}{3}}}\right)$$

$$P\left(S_n^* \leq \frac{320 - 300}{\frac{20}{\sqrt{3}}}\right) = P\left(S_n^* \leq \frac{1}{\frac{1}{\sqrt{3}}}\right) = P\left(S_n^* \leq \sqrt{3}\right) = \Phi(\sqrt{3}) \approx 0.9582 \approx 96\%$$

Problem 4

We'll use the Bell-curve approximation to give bound to the binomial distribution for the Casino games. Just like we did for the previous assignment, we can define again the variable $X_n = \frac{1}{n}\sum_{i=1}^{n} X_i$, with the already known $\mu = \frac{1}{2}, \sigma^2 = \frac{1}{4n}$

The next step will consist in normalising the current distribution of X_n into the standard normal distribution X_n^*:

$$X_n^* = \frac{X_n - \mu}{\sigma}$$

a) With $n = 40$ and $p = 0.7$:
This means that $\sigma = \sqrt{\frac{1}{4n}} = \frac{1}{2\sqrt{n}} = \frac{1}{2\sqrt{40}} = 0.079$ and that $\epsilon = p - \mathrm{E}[X_n] = p - \mu = 0.7 - 0.5 = 0.2$

$$1 - \Phi\left(\frac{\frac{1}{5}}{\frac{1}{\sqrt{160}}}\right) = 1 - \Phi\left(\frac{4\sqrt{10}}{5}\right) \approx 1 - \Phi(2.53) \approx 1 - 0.9943 = 0.0057$$

b) Given $n = 40, p = 0.501$:
Consequently, we get: σ stays the same and $\epsilon = 0.501 - 0.5 = 0.001$

$$1 - \Phi\left(\frac{\frac{1}{100}}{\frac{1}{\sqrt{160}}}\right) = 1 - \Phi\left(\frac{\sqrt{10}}{25}\right) = 1 - \Phi(0.01) \approx 1 - 0.5040 = 0.496$$

c) Given $n = 1500$ and $p = 0.6$ we get $\sigma = \frac{1}{2\sqrt{1500}} = 0.013, \epsilon = p - \mathrm{E}[X_n] = 0.6 - 0.5 = 0.1$

$$1 - \Phi\left(\frac{\frac{1}{10}}{\frac{1}{20\sqrt{15}}}\right) = 1 - \Phi\left(\frac{20\sqrt{15}}{10}\right) = 1 - \Phi(2\sqrt{15}) \approx 0$$

d) Given $n = 1500$ and $p = 0.501$ we get the same σ and $\epsilon = 0.501 - 0.5 = 0.001$

$$1 - \Phi\left(\frac{20\sqrt{15}}{1000}\right) = 1 - \Phi\left(\frac{\sqrt{15}}{50}\right) \approx 1 - 0.530871 = 0.469129$$

e) Given $n = 1000000, p = 0.7$ we get $\sigma = \frac{1}{2 \cdot 1000}, \epsilon = 0.7 - 0.5 = 0.2$

$$1 - \Phi\left(\frac{2000}{5}\right) = 1 - \Phi(400) \approx 0$$

f) Given $n = 1000000, p = 0.501$ thus we have the same σ and $\epsilon = 0.001$

$$1 - \Phi\left(\frac{\frac{1}{1000}}{\frac{1}{2000}}\right) = 1 - \Phi\left(\frac{2000}{1000}\right) = 1 - \Phi(2) \approx 1 - 0.97725 = 0.02275$$

For all of the previous points, the Chebyshev inequality was always respected and always provided a valid upper bound.