Probability & Statistics: Assignment 4

Due on October 28, 2018 at 8:30am

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Problem 1

We let n be the number of steps taken and draw the Pascal's Triangle:

$$
\begin{array}{lccccccccccccc}
n=0 & & & & & & & 1 & & & & & & \\
n=1 & & & & & & 1 & & 1 & & & & & \\
n=2 & & & & & 1 & & 2 & & 1 & & & & \\
n=3 & & & & 1 & & 3 & & 3 & & 1 & & & \\
n=4 & & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
n=5 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
n=6 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}
$$

a) There's no possible way with a random walk of odd length to return to the origin.

b) $\binom{6}{4} \cdot \frac{1}{2^6} = \frac{15}{64}$

c) There's no random walk which reaches a distance of 6 in only 4 steps.

d) $\binom{5}{3} \cdot \frac{1}{2^5} = \frac{5}{16}$

e) In the considered set of possible steps, our prime numbers are: $\{2, 3, 5\}$, thus our chance is:

$$P_{S_6}(\text{primes}) = P_{S_6}(\text{2D4U}) + P_{S_6}(\text{3D3U}) + P_{S_6}(\text{5D1U}) =$$

$$P_{S_6}(2) = P_{S_6}(2) + P_{S_6}(0) + P_{S_6}(-4) = \frac{1}{64} \cdot \binom{6}{1} + \frac{1}{64} \cdot \binom{6}{3} + \frac{1}{64} \cdot \binom{6}{4} = \frac{6}{64} + \frac{20}{64} + \frac{15}{64} = \frac{41}{64}$$

f) Out of all the possible 64 combinations, we want to calculate what is the chance of having two upward movements in a row and then compute its complement. If we have less than two moves down, then we know for sure that the path will contain at least two upward movement in a row, we compute how many combinations:

$$2^6 \cdot P(0D6U) = \binom{6}{6} = 1, \qquad 2^6 \cdot P(1D5U) = \binom{6}{5} = 6, \qquad 2^6 \cdot P(2D4U) = \binom{6}{4} = 15$$

In the case of $P(3D3U)$ we'll need to enumerate the cases in which we DON'T have two upward movements in a row:

- UDUDUD
- DUDUDU
- UDDUDU
- UDUDDU

Thus we take these four cases out of our calculations:

$$2^6 \cdot P(3D3U) - 4 = \binom{6}{3} - 4 = 20 - 4 = 16$$

In the last scenario of $P(4D2U)$ we only have five possible chances:

- UUDDDD
- DUUDDD
- DDUUDD
- DDDUUD

Problem 1 continued on next page...

- DDDDUU

We can finally sum all of the possible outcomes which involve a sequence of two upward movements in a row that we computed so far: $2^6 \cdot P(2U \text{ in a row}) = 1 + 6 + 15 + 16 + 5 = 43$ Now we calculate the complement of what we just computed: $1 - \frac{43}{64} = \frac{21}{64} = P(\text{not 2U in a row})$

Problem 2

Initially, out of all the possible scenarios, only $\frac{3}{4}$ will end with at least one of the two teams successfully completing their project. Thus we get the following table:

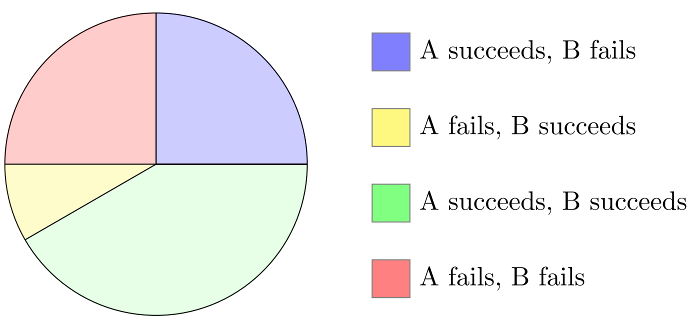
	A = S	A = F	P(B)
B = S			$\frac{1}{2}$
B = F		$\frac{1}{4}$	
P(A)	$\frac{2}{3}$		

With our partially filled table, we can still deduce some additional data, considering the fact that the marginals must sum up to 1: Moreover, we can deduce $P(A = F, B = S) = P(A = F) - P(A = F, B = F)$:

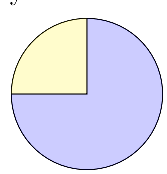
	A = S	A = F	P(B)
B = S			$\frac{1}{2}$
B = F		$\frac{1}{4}$	$\frac{1}{2}$
P(A)	$\frac{2}{3}$	$\frac{1}{3}$	

	A = S	A = F	P(B)
B = S	$\frac{5}{12}$	$\frac{1}{12}$	$\frac{1}{2}$
B = F	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$
P(A)	$\frac{2}{3}$	$\frac{1}{3}$	

This table now allows us to look up the probability of the considered event $P(A = F, B = S) = \frac{1}{12}$



$P(\text{only 1 team won}) = \frac{1}{3}$



But if we now know for a fact that only one team did succeed and we must compute the chance of team B being the one successful, then we need to consider how many scenarios were having only one team being successful (in the pie chart yellow+blue).

We can use the formula to compute the conditional probability:

$$P(B = S|1 \text{ team won}) = \frac{P((B = S) \cap (\text{only 1 team won}))}{P(\text{only 1 team won})} = \frac{P(A = F, B = S)}{P(A = F, B = S) + P(A = S, B = F)} = \frac{\frac{1}{12}}{\frac{1}{3}} = \frac{3}{12} = \frac{1}{4}$$

Which is the probability that, given that one team has won, B was the one being successful.

Problem 3

a) $P_Y(Y = 16) = 0.08 + 0.28 + 0.04 = 0.4$

b) $P_X = \{0.12 + 0.08, 0.42 + 0.28, 0.06 + 0.04\} \Rightarrow P_X = \{0.2, 0.7, 0.1\}$

c) According to **Definition 12**, $P_X(130)$ and $P_Y(16)$ are statistically independent because:

$$P(P(X = 130) \cap P(Y = 16)) = P_X(130) \cdot P_Y(16)$$

$$0.28 = 0.7 \cdot 0.4 \implies \textbf{true}$$

d) We can show that X and Y are independent if we show what we demonstrated at point **c)** for each event.

				X	
			0.2	0.7	0.1
			129	130	131
Y	0.6	15	$0.2 \cdot 0.6$	$0.7 \cdot 0.6$	$0.1 \cdot 0.6$
	0.4	16	$0.2 \cdot 0.4$	$0.7 \cdot 0.4$	$0.1 \cdot 0.4$

			X	
		129	130	131
Y	15	0.12	0.42	0.06
	16	0.08	0.28	0.04

Problem 4

We apply Bayes' Theorem to compute the following probabilities:

$$P(B|A) = \frac{P(B) \cdot P(A|B)}{P(A|B) \cdot P(B) + P(A|B^C)(1 - P(B))}$$

a)

$$P(\text{SPAM}|\text{shoes}) = \frac{P(\text{SPAM}) \cdot P(\text{shoes}|\text{SPAM})}{P(\text{shoes}|\text{SPAM}) \cdot P(\text{SPAM}) + P(\text{shoes}|\text{SPAM}^C)(1 - P(\text{SPAM}))} =$$

$$= \frac{\frac{1}{2} \cdot \frac{1}{16}}{\frac{1}{16} \cdot \frac{1}{2} + \frac{2}{14} \cdot \frac{1}{2}} = \frac{\frac{1}{32}}{\frac{1}{32} + \frac{1}{14}} = \frac{\frac{1}{32}}{\frac{23}{224}} = \frac{224}{736} = \frac{7}{23}$$

b)

$$P(\text{SPAM}|\text{walk} \cup \text{zoo}) = \frac{P(\text{SPAM}) \cdot P(\text{SPAM}|\text{walk} \cup \text{zoo})}{P(\text{walk} \cup \text{zoo}|\text{SPAM}) \cdot P(\text{SPAM}) + P(\text{walk} \cup \text{zoo}|\text{SPAM}^C)(1 - P(\text{SPAM}))} =$$

$$= \frac{\frac{1}{2} \cdot \frac{0}{16}}{\frac{0}{16} \cdot \frac{1}{2} + \frac{4}{14} \cdot \frac{1}{2}} = 0$$

c)

$$P(\text{SPAM}|\text{free} \cap \text{deal}) = \frac{P(\text{SPAM}) \cdot P(\text{SPAM}|\text{free} \cap \text{deal})}{P(\text{free} \cap \text{deal}|\text{SPAM}) \cdot P(\text{SPAM}) + P(\text{free} \cap \text{deal}|\text{SPAM}^C)(1 - P(\text{SPAM}))} =$$

$$= \frac{\frac{1}{2} \cdot \frac{2}{16}}{\frac{2}{16} \cdot \frac{1}{2} + \frac{0}{14} \cdot \frac{1}{2}} = \frac{\frac{1}{16}}{\frac{1}{16}} = 1$$

d)

$$P(\text{SPAM}|\text{puma}{\cup}\text{giraffe}) = \frac{P(\text{SPAM}) \cdot P(\text{SPAM}|\text{puma} \cup \text{giraffe})}{P(\text{puma} \cup \text{giraffe}|\text{SPAM}) \cdot P(\text{SPAM}) + P(\text{puma} \cup \text{giraffe}|\text{SPAM}^C)(1 - P(\text{SPAM}))} =$$

$$= \frac{\frac{1}{2} \cdot \frac{1}{16}}{\frac{1}{16} \cdot \frac{1}{2} + \frac{3}{14} \cdot \frac{1}{2}} = \frac{\frac{1}{32}}{\frac{1}{32} + \frac{3}{28}} = \frac{\frac{1}{32}}{\frac{31}{224}} = \frac{224}{992} = \frac{7}{31}$$

Optional Problem