Tisserand Plot

Alessandro Zavoli

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1 Introduction

Tisserand plot can help understand the effects and benefits of (one or multiple) flybys.

This analysis assumes that the secondary body (pedix "s") flies in a circular orbit of radius a_s about the primary body (pedix "p").

Also, we denotes distance from the primary body with upper-case R, whereease distances in the Sphere of Influence (SOI) of the secondary body with lowercase r.

In this document we will only consider the (R_P, R_P) Tisserand plot, which is the simplest one to deal with elliptic orbits about the primary body. Note that hyperbolic orbits are not allowed in this plot, since the apocenter radius R_A does not exist. In that case, other kinds of Tisserand plots could be used such as the $R_P - \mathcal{E}$ (that is pericenter radius vs period) plot.

First, we show that we can cover the plot by iso-v_∞ contour lines. Indeed, each pair (R_P, R_A) corresponds to a unique pair (v_∞, α).

Let V_{sc} be the spacecraft velocity and V

$$V_{sc}^2 = V_s^2 + v_\infty^2 + 2V_s v_\infty \cos\alpha \tag{1}$$

with $V_s = \sqrt{\mu_p/r_s}$

$$a = (R_P + R_A)/2 \tag{2}$$
$$e = 1 - rp/a \tag{3}$$
$$\cos\nu = (p - r_s)/(er_s) \tag{4}$$
$$V_{sc,r} = \sqrt{\frac{\mu_p}{p}} e \sin\nu \tag{5}$$
$$V_{sc,t} = \sqrt{\frac{\mu_p}{p}}(1 + e\cos\nu) \tag{6}$$
$$V_{sc} = \sqrt{V_{sc,r}^2 + V_{sc,t}^2} \tag{7}$$
$$v_{\infty,r} = V_r \tag{8}$$
$$v_{\infty,t} = V_t - V_s \tag{9}$$
$$v_\infty = \sqrt{v_{\infty,r}^2 + v_{\infty,t}^2} \tag{10}$$
$$\alpha = \arccos\frac{V_{sc}^2 - V_s^2 - v_\infty^2}{2V_{sc}v_\infty} \tag{11}$$

The inverse transform follows the same reasoning

$$V_{sc}^2 = V_s^2 + v_\infty^2 + 2V_s v_\infty \cos\alpha \tag{12}$$
$$V_{sc,r} = v_\infty \sin\alpha \tag{13}$$
$$V_{sc,t} = V_s + v_\infty \cos\alpha \tag{14}$$
$$a = \frac{\mu_p/2}{\frac{V_{sc}^2}{2} - \frac{\mu_p}{R_s}} \tag{15}$$
$$h = R_s V_{sc,t} \tag{16}$$
$$e = \sqrt{1 - \frac{h^2}{\mu a}} \tag{17}$$
$$R_P = a(1 - e) \tag{18}$$
$$R_A = a(1 + e) \tag{19}$$

Using the inverse transform it is possible to draw lines (iso-contours) of constant v_∞, fixing $v_\infty = \bar{v}_\infty$ and varying $\alpha \in [0, \pi]$.

Iso-contours of *alpha* can also be plotted as dashed lines. These lines are obtained fixing the values of α (e.g., $\alpha = 0$, 15°, 30°, ..., 180°). and varying $v_\infty \in [0, v_{\infty_{\max}}]$.

FIGURA 1

As an alternative, one can draw lines of constant period (of the spacecraft orbit around the primary body), for a given $n : m$ resonance. These iso-contours appears a straight lines in the (R_A, R_P). In fact, $R_P + R_A = 2a$, wuth a constant if the period is constat, being $a =^3 \sqrt{\left(\frac{T}{2\pi}\right)^2 \mu_p}$.

For a resontant orbit, one hase $T_{sc} : T_s = n : m$, hence

$$a_{sc} = a_s(\frac{n}{m})^{2/3} \tag{20}$$

Thus, the iso-resonant lines are found for a specific choice of $n : m$ calculating the pairs (R_A, R_P) varying the eccentricity from 0 to 1 (or $e_{\max}$.

$$R_P = a_s(\frac{n}{m})^{2/3}\,(1 - e) \tag{21}$$

$$R_A = a_s(\frac{n}{m})^{2/3}\,(1 + e) \tag{22}$$

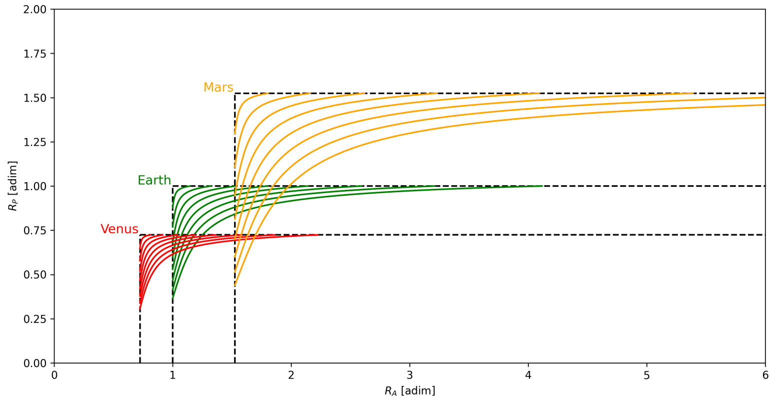

Figure 1: Tisserand plot for the Solar system.

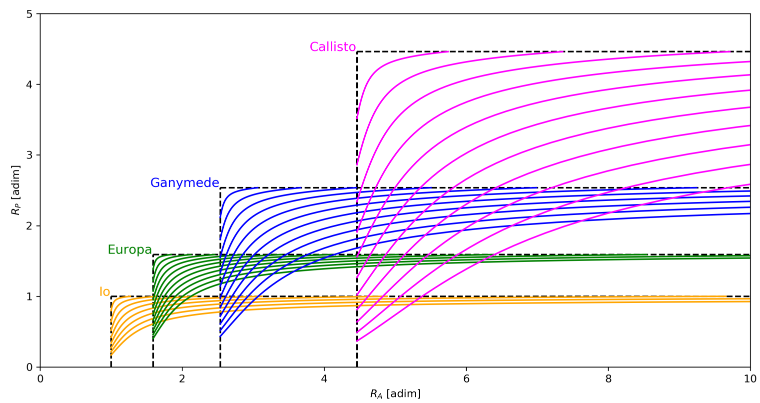

Figure 2: Tisserand plot for the Jovian system.

The effect of a flyby corrsponds to an instantaneous change of velocity. This means a jump on the Tisserand plot. Note that the v_∞ is not altered by a flyby, thus the points before and after the flyby must be located on the same iso-v_∞ contour line.

The amount of displacement along the v_∞ contour line depends on the change of the pump angle between the conditions after (α_+) and before (α_-) the flyby. Their difference is equal to the deflection angle δ

$$\boldsymbol{v}_{\infty-}^T \cdot \boldsymbol{v}_{\infty+} = v_\infty^2 \cos\delta \tag{23}$$

figura del triangolo di velocita'

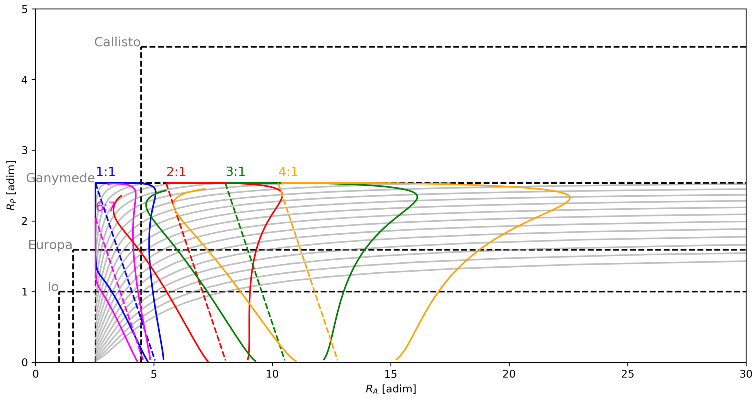

Figure 3: Tisserand plot for the Solar system.

The deflection angle δ, can be found as

$$\sin\frac{\delta}{2} = \frac{\mu_s/r_{flyby}}{v_\infty^2 + \mu_s/r_{flyby}} \tag{24}$$

where r_{flyby} is the pericenter radius of the planetocentric hyperbola, that is, the flyby radius.

$$\sin^2\frac{\delta}{2} = \frac{1-\cos\delta}{2} \tag{25}$$

$$\cos\delta = \cos(\pi - 2\Phi) = -\cos(2\Phi) = \tag{26}$$

$$\tag{27}$$