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Topological Codes for Quantum Computation: the Toric Code

Tesi di Laurea Magistrale in
Computer Science and Engineering - Ingegneria Informatica

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Abstract

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Abstract in lingua italiana

Qui va l'Abstract in lingua italiana della tesi seguito dalla lista di parole chiave.

Parole chiave: qui, vanno, le parole chiave, della tesi

Contents

Abstract **i**

Abstract in lingua italiana **iii**

Contents **v**

Introduction **1**

1 Toric Code **3**
- 1.1 Description of the Model 3
- 1.2 Ground State 11
- 1.3 Excited States 11
- 1.4 Anyonic Excitations 11

2 Coding on the Toric Code **13**
- 2.1 Classical Error Correction 13
- 2.2 Quantum Error Correction (QEC) 13
- 2.3 QEC in the Toric Code 13

3 Conclusions and future developments **15**

Bibliography **17**

List of Figures **19**

List of Symbols **21**

Acknowledgements **23**

Introduction

This document is intended to be both an example of the Polimi LaTeX template for Master Theses, as well as a short introduction to its use. It is not intended to be a general introduction to LaTeX itself, and the reader is assumed to be familiar with the basics of creating and compiling LaTeX documents (see [4, 6]).

The cover page of the thesis must contain all the relevant information: title of the thesis, name of the Study Programme and School, name of the author, student ID number, name of the supervisor, name(s) of the co-supervisor(s) (if any), academic year. The above information are provided by filling all the entries in the command `\puttitle{}` in the title page section of this template.

Be sure to select a title that is meaningful. It should contain important keywords to be identified by indexer. Keep the title as concise as possible and comprehensible even to people who are not experts in your field. The title has to be chosen at the end of your work so that it accurately captures the main subject of the manuscript.

Since a thesis might be a substantial document, it is convenient to break it into chapters. You can create a new chapter as done in this template by simply using the following command

```
\chapter{Title of the chapter}
```

followed by the body text.

Especially for long manuscripts, it is recommended to give each chapter its own file. In this case, you write your chapter in a separated `chapter_n.tex` file and then include it in the main file with the following command

```
\input{chapter_n.tex}
```

It is recommended to give a label to each chapter by using the command

```
\label{ch:chapter_name}%
```

where the argument is just a text string that you'll use to reference that part as follows: *Chapter 1 contains* AN INTRODUCTION TO ....

If necessary, an unnumbered chapter can be created by

```
\chapter*{Title of the unnumbered chapter}
```

1 | Toric Code

1.1. Description of the Model

The Toric Code model is defined on a square lattice with periodic boundary conditions in both directions. These latter characteristics in topology are typical of what is known as a torus topology or simply a Torus, after which the model is named.

A square lattice, here labelled as L, is a particular lattice defined in a two dimensional Euclidian space. It is denoted as $\mathbb{Z}^2$ such that each lattice point is identified with a tuple of integers. Though, the above mentioned boundary conditions also specify that for any point (i, j) in the lattice, the neighboring points are going to be the following: $(i+1\ mod\ L, j)$, $(i-1\ mod\ L, j)$, $(i, j+1\ mod\ L)$ and $(i, j-1\ mod\ L)$.

Then, we are also going to consider a dual lattice, which will be labelled as L', and that will be positioned as represented in figure 1.1, where the continuous line represents the main lattice L, while the dashed line represents its dual L'. On each edge is located a spin-$\frac{1}{2}$ particle, i.e. a Fermion, represented in the image below with an empty circle. Circles shaded in gray represent the boundaary conditions.

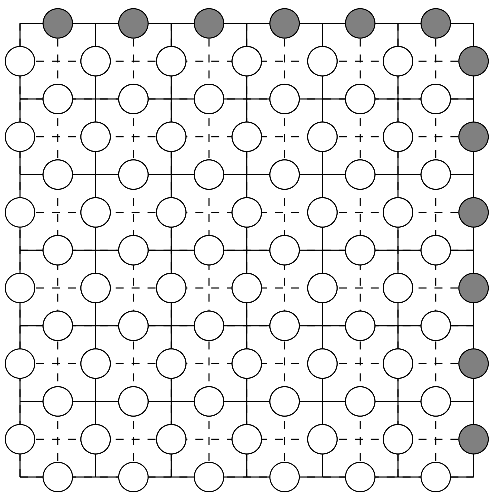

Figure 1.1: A square lattice with boundary conditions.

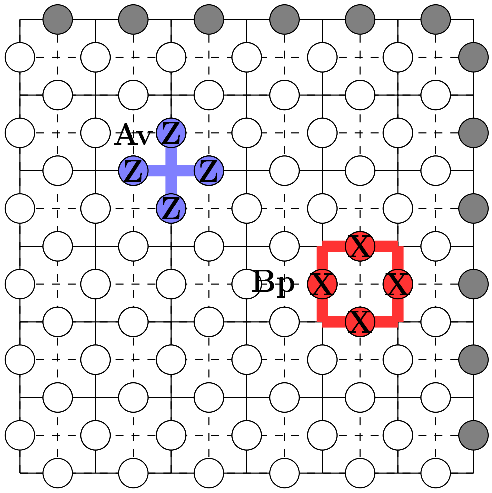

Figure 1.2: Veretx (red) and plaquette (blue) operators.

For each cell on L we are going to consider two spins, therefore the total number of spins will correspond to $2N$, where N represents both the total number of cells in L and the dimension of the lattice. Since all of these spins exhibit the same characteristics they are identified with identcial particles, which will be useful to know to study further properties of the model.

The key part of the model are the so called "vertex" and "plaquette" operators that are going to be placed, respectively, on the vertices v and cells p of L. Such operators can be defined formally as in definition 1.1.1 and implemented by means of Pauli matrices as exemplified in figure 1.2. Throughout the description, the Av operators are going to be shaded in blu, while the Bp operators are going to be shaded in red.

Definition 1.1.1 (Vertex and Plaquette operators) Given a vertex v and a plaquette p, we can define the following vertex and plaquette operators as tensor products over Pauli operators acting on four spins each located on the lattice and indicated by the indices $j \in star(v)$ and $j \in bdy(p)$

$$Av = \prod_{j \in star(v)} Z_j$$

$$Bp = \prod_{j \in bdy(v)} X_j.$$

All the elements that constitute the model of the toric code are brought together by its Hamiltonian, which is given by the following definition:

Definition 1.1.2 (Toric Code Hamiltonian) Given sites v and p of the lattice along with their respective vertex and plaquette operators and given the spin-$\frac{1}{2}$ particles located on each edge of the torus, we can define the following Hamiltonian for the system:

$$H = -\sum_v Av - \sum_p Bp.$$

Each one of the Av and Bp operators that appear in the Hamiltonian share some important properties, which are going to be covered in the following definitions and will later be used to further characterize the action of the operators.

Firstly, notice that Av and Bp operators are Hermitian and square to the identity.

Definition 1.1.3 (Hermitian operator) Let H be an Hilbert space and A: $H \to H$ be a linear operator. An operator A is called Hermitian if for all $\psi, \phi \in H$ we have that

$$\langle A\psi|\phi\rangle = \langle \psi|A^\dagger\phi\rangle = \langle \phi|A\psi\rangle^*.$$

where the daga symbol $\dagger$ represents the conjugate transpose and the $\langle\ \rangle$ the Hermitian inner product.

It is important for the following applications to specify that a square matrix A of complex numbers, representing a linear operator, is called hermitian if $A^\dagger = (A^*)^T = A$. One important consequence of Hermiticity is that the corresponding eigenvalues of the operator are real. Note also that for real matrices, as in the case of our operators, it is sufficient to compute only the transpose of the matrix to verify hermiticity.

Then, the involutory property comes from the unitarity of the matrices representing the vertex and plaquette operators, i.e. X and Z Pauli matrices.

Definition 1.1.4 (Unitary operator) Let H be an Hilbert space. Let U: $H \to U$ be a linear operator. We define U to be complex unitary if for all $\psi, \phi \in H$ we have that

$$\langle U\psi|U\phi\rangle = \langle \psi|\phi\rangle.$$

In terms of the conjugate transpose, we can define a complex matrix U to be unitary if $(U^*)^T = U^{-1}$. We note that it is possible to define an operator to be real unitary, with the only difference that $(U)^T = U^{-1}$. Then, we can also define that a real matrix U, representing a linear operator, is unitary if $(U)^T = U^{-1}$ or, equivalently, if $(U)^T U = I$. Note also that one important property of unitary operators is that their eigenvalues have modulus equal to one.

Given the considerations made in Definition 1.1.3. and 1.1.4. for ral matrices, we can easily prove the first following proposition:

Proposition 1.1.1 (Hermiticity and involutory property of Av and Bp operators) Av and Bp operators are Hermitian and satisfy the involutory property.

Proof.

We know that the operator $Av = \prod_{j \in star(v)} Z_j$ and that $Bp = \prod_{j \in bdy(v)} X_j$.

Firstly, recall the form of the σ_x and σ_z matrices representing the Pauli gates X and Z:

$$\mathrm{Z} = \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\mathrm{X} = \sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Prove that they are Hermitian:

$$(\sigma_z)^H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}^H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \sigma_z$$

$$(\sigma_x)^H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \sigma_x$$

Note that Pauli matrices are real matrices, i.e. $A^\dagger = A^T$. Then, knowing that $[\sigma_i, \sigma_j] =$

$0 \; for \; i = j$ we can write

$$(Av)^\dagger = (\sigma_{x_1}\sigma_{x_2}\sigma_{x_3}\sigma_{x_4})^\dagger = \sigma_{x_1}^\dagger\sigma_{x_2}^\dagger\sigma_{x_3}^\dagger\sigma_{x_4}^\dagger = \sigma_{x_1}\sigma_{x_2}\sigma_{x_3}\sigma_{x_4} = Av$$

$$(Bp)^\dagger = (\sigma_{z_1}\sigma_{z_2}\sigma_{z_3}\sigma_{z_4})^\dagger = \sigma_{z_1}^\dagger\sigma_{z_2}^\dagger\sigma_{z_3}^\dagger\sigma_{z_4}^\dagger = \sigma_{z_1}\sigma_{z_2}\sigma_{z_3}\sigma_{z_4} = Bp$$

Prove the involutory propertyy:

$$(\sigma_z)^2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} * \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$(\sigma_x)^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} * \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Then, again we can write

$$(Av)^2 = (\sigma_{x_1}\sigma_{x_2}\sigma_{x_3}\sigma_{x_4})^2 = \sigma_{x_1}^2\sigma_{x_2}^2\sigma_{x_3}^2\sigma_{x_4}^2 = I$$

$$(Bp)^2 = (\sigma_{z_1}\sigma_{z_2}\sigma_{z_3}\sigma_{z_4})^2 = \sigma_{z_1}^2\sigma_{z_2}^2\sigma_{z_3}^2\sigma_{z_4}^2 = I$$

□

Given Proposition 1.1.1, we can know further characterize the opeartors in terms of their eigenvalues and define their spectrum.

Proposition 1.1.2 (Eigenvalues of Av and Bp operators) Given Hermitian operators Av and Bp, which satisfying the involutory property, their eigenvalues are ± 1.

Proof.

Given the Hermiticity and involutory proof in 1.1.1, we firstly prove that Hermitian operators (and therefore our vertex and plaquette operators) have real eigenvalues:

Write the expression for the eigenvalues $Av|\xi\rangle = \lambda|\xi\rangle$ and take as hypothesis that $|\xi\rangle \neq 0$. Then, by means of the scalar product

$$\langle\xi|Av|\xi\rangle = \lambda\langle\xi|\xi\rangle$$

$$\lambda = \frac{\langle\xi|Av|\xi\rangle}{\langle\xi|\xi\langle} = \frac{\langle\xi|Av|\xi\rangle}{||\xi||^2} = \frac{\langle\xi|Av^\dagger|\xi\rangle}{||\xi||^2} = \frac{\langle\xi|Av|\xi\rangle^*}{||\xi||^2} = \lambda^*$$

Notice that we have applied antilinearity of the adjoint in the penultimate equality:

$$(\langle\xi|Av^\dagger|\xi\rangle)^\dagger = |\xi\rangle^\dagger(Av^\dagger)^\dagger\langle\xi|^\dagger = \langle\xi|Av|\xi\rangle^*.$$

Using hermiticity with the fact that $(Av)^2 = I$ we can derive the unitarity of Av and state that $AvAv^\dagger = Av^\dagger Av = (Av)^2 = I$, i.e. for a unitary operator U we have $U^\dagger = U^{-1}$.

Given the property of unitary operators that states that their eigenvalues have modulus equal to one:

By taking as hypothesis $|\xi\rangle \neq 0$, we write the expression $U|\xi\rangle = \lambda|\xi\rangle$ and its self-adjoint $\langle\xi|U^\dagger = \lambda^*\langle\xi|$. Then, knowing that $U^\dagger = U^{-1}$, by means of the scalar product we obtain

$$\langle\xi|UU^\dagger|\xi\rangle = \lambda\lambda^*\langle\xi|\xi\rangle$$

$$\langle\xi|I|\xi\rangle = \lambda\lambda^*\langle\xi|\xi\rangle$$

$$\langle\xi|\xi\rangle = \lambda\lambda^*\langle\xi|\xi\rangle$$

Finally, since we already know that Hermitian operators have real eigenvalues we can write $|\lambda|^2 = 1$.

Putting together the fact that Av has real eigenvalues with unitarity we obtain that the only two remaining possibilities for the eigenvalues of Av are ± 1.

The same reasoning can be carried out for the Bp operator.

$\square$

Given the result of proposition 1.1.2, the spectrum of the operators is easy to find:

Proposition 1.1.3 (Spectrum of Av and Bp operators) The spectrum of Av and Bp operators is equal to $\{-1, +1\}$.

We now analyze the relationship between vertex and plaquette operators. Such relationship is characterized by two fundamental mathematical concepts called "commutation" and "anticommutation". These relations serve as an important result in understanding the stability and fault-tolerance of the toric code.

Proposition 1.1.4 (Commutation of Av and Bp operators) The operator Av commutes with the operator Bp for an even number of edges.

Proposition 1.1.5 (Anticommutation of Av and Bp operators) The operator Av anticommutes with the operator Bp for an odd number of edges.

Proof.

Fix the origin of the coordinate system in the bottom left corner of the lattice as indicated in the picture below:

then, we can define the two vectors representing the site of application of the vertex and plaquette operator, respectively over the lattice L and dual lattice L'

$$\vec{v} = n\hat{e}_1 + m\hat{e}_2, \text{ where } n, m \in \mathbb{Z}$$

$$\vec{p} = (n + \tfrac{1}{2})\hat{e}_1 + (m + \tfrac{1}{2})\hat{e}_2, \text{ where } n, m \in \mathbb{Z}$$

Rewrite the operators as follows:

$$A_{\vec{v}} = \sigma^z_{\vec{v}+\frac{1}{2}\hat{e_1}} \sigma^z_{\vec{v}+\frac{1}{2}\hat{e_2}} \sigma^z_{\vec{v}-\frac{1}{2}\hat{e_1}} \sigma^z_{\vec{v}-\frac{1}{2}\hat{e_2}}$$

$$B_{\vec{p}} = \sigma^x_{\vec{p}+\frac{1}{2}\hat{e_1}} \sigma^x_{\vec{p}+\frac{1}{2}\hat{e_2}} \sigma^x_{\vec{p}-\frac{1}{2}\hat{e_1}} \sigma^x_{\vec{p}-\frac{1}{2}\hat{e_2}}$$

In order to simplify the calculations we rewrite Bp on the lattice L by rewriting the indeces in terms of vector $\vec{v}$

$$(n\hat{e_1} + m\hat{e_2}) + \tfrac{1}{2}\hat{e_2} = n\hat{e_1} + (m + \tfrac{1}{2}\hat{e_2})$$

$$(n\hat{e_1} + m\hat{e_2}) + \tfrac{1}{2}\hat{e_1} = (n + \tfrac{1}{2})\hat{e_1} + m\hat{e_2}$$

$$(n\hat{e_1} + m\hat{e_2}) + \tfrac{1}{2}\hat{e_1} + \hat{e_2} = (n + \tfrac{1}{2})\hat{e_1} + (m+1)\hat{e_2}$$

$$(n\hat{e_1} + m\hat{e_2}) + \tfrac{1}{2}\hat{e_2} + \hat{e_1} = (n+1)\hat{e_1} + (m + \tfrac{1}{2})\hat{e_2}$$

Then the $B_{\vec{p}}$ operator becomes:

$$B_{\vec{v}} = \sigma^x_{n\hat{e_1}+(m+\frac{1}{2}\hat{e_2})} \sigma^x_{(n+\frac{1}{2})\hat{e_1}+m\hat{e_2}} \sigma^x_{(n+\frac{1}{2})\hat{e_1}+(m+1)\hat{e_2}} \sigma^x_{(n+1)\hat{e_1}+(m+\frac{1}{2})\hat{e_2}}$$

and the Hamiltonian can be written by grouping the indices:

$$H = -\sum\nolimits_{m,n\in\mathbb{Z}} \{ \sigma^z_{(n+\frac{1}{2})\hat{e_1}+m\hat{e_2}} \sigma^z_{n\hat{e_1}+(m+\frac{1}{2})\hat{e_2}} \sigma^z_{(n-\frac{1}{2})\hat{e_1}+m\hat{e_2}} \sigma^z_{n\hat{e_1}+(m-\frac{1}{2})\hat{e_2}} + \sigma^x_{n\hat{e_1}+(m+\frac{1}{2}\hat{e_2})} \sigma^x_{(n+\frac{1}{2})\hat{e_1}+m\hat{e_2}} \sigma^x_{(n+\frac{1}{2})\hat{e_1}+(m+1)\hat{e_2}} \sigma^x_{(n+1)\hat{e_1}+(m+\frac{1}{2})\hat{e_2}} \}$$

Now calcuate the commutator $[A_{\vec{v}}, B_{\vec{v}}] = A_{\vec{v}}B_{\vec{v}} - B_{\vec{v}}A_{\vec{v}}$ by focusiing on the first term:

$$A_{\vec{v}}B_{\vec{v}} = \sigma^z_{(n+\frac{1}{2})\hat{e_1}+m\hat{e_2}} \sigma^z_{n\hat{e_1}+(m+\frac{1}{2})\hat{e_2}} \sigma^z_{(n-\frac{1}{2})\hat{e_1}+m\hat{e_2}} \sigma^z_{n\hat{e_1}+(m-\frac{1}{2})\hat{e_2}} * \sigma^x_{n\hat{e_1}+(m+\frac{1}{2}\hat{e_2})} \sigma^x_{(n+\frac{1}{2})\hat{e_1}+m\hat{e_2}} \sigma^x_{(n+\frac{1}{2})\hat{e_1}+(m+1)\hat{e_2}} \sigma^x_{(n+1)\hat{e_1}+(m+\frac{1}{2})\hat{e_2}}$$

Matrices do not commute but for Pauli matrices we have the following commutation relationship :

$$\sigma^x_{\vec{v}}\sigma^z_{\vec{v}'} = \sigma^z_{\vec{v}'}\sigma^x_{\vec{v}} + 2 * \sigma^x_{\vec{v}}\sigma^z_{\vec{v}'}\delta_{\vec{v}\vec{v}'}$$

where $$\delta_{\vec{v}\vec{v}'} = \begin{cases} 1, & if \quad \vec{v} = \vec{v}' \\ 0, & if \quad \vec{v} \neq \vec{v}' \end{cases}$$

which states that $\sigma^x_{\vec{v}}\sigma^z_{\vec{v}'}$ commutate for $\vec{v} \neq \vec{v}'$ but anticommutate for $\vec{v} = \vec{v}'$. This is known from the anticommutation relationship of Pauli matrices $\sigma^x\sigma^z = -\sigma^x\sigma^z$. Thus, for an even numer of overlapping edges, in our case 2 or 4, the commutator becomes:

$$[A_{\vec{v}}, B_{\vec{v}}] = 2 * \sigma^x_{n\hat{e_1}+(m+\frac{1}{2}\hat{e_2})}\sigma^x_{(n+\frac{1}{2})\hat{e_1}+m\hat{e_2}}\sigma^x_{(n+\frac{1}{2})\hat{e_1}+(m+1)\hat{e_2}}\sigma^x_{(n+1)\hat{e_1}+(m+\frac{1}{2})\hat{e_2}} * \\ \sigma^z_{(n+\frac{1}{2})\hat{e_1}+m\hat{e_2}}\sigma^z_{n\hat{e_1}+(m+\frac{1}{2})\hat{e_2}}\sigma^z_{(n-\frac{1}{2})\hat{e_1}+m\hat{e_2}}\sigma^z_{n\hat{e_1}+(m-\frac{1}{2})\hat{e_2}}$$

Instead, for an odd number of edges it becomes null $[A_{\vec{v}}, B_{\vec{v}}] = 0$.

This calculations conclude that $A_{\vec{v}}, B_{\vec{p}}$ commute for an even numer of edges but anticommute for an odd number of edges.

□

1.2. Ground State

1.3. Excited States

1.4. Anyonic Excitations

2 | Coding on the Toric Code

2.1. Classical Error Correction

2.2. Quantum Error Correction (QEC)

2.3. QEC in the Toric Code

3 | Conclusions and future developments

A final chapter containing the main conclusions of your research/study and possible future developments of your work have to be inserted in this chapter.

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List of Figures

1.1 A square lattice with boundary conditions. 3
1.2 Veretx (red) and plaquette (blue) operators. 4

List of Symbols

Variable	**Description**	**SI unit**
$\boldsymbol{u}$	solid displacement	m
$\boldsymbol{u}_f$	fluid displacement	m

Acknowledgements

Here you might want to acknowledge someone.