Network Dynamics Homework II

Alessia Leclercq
Politecnico di Torino
s291871@studenti.polito.it

I. EXERCISE 1

A) What is, according to the simulations, the average time it takes a particle that starts in node a to leave the node and then return to it?

A continuous Markov chain has been simulated using an iterative algorithm in which the probability of jumping from node i to a neighbor j is modelled by P_{ij}, where:

$$\omega_i = \sum_{j \neq i} \Lambda_{ij} \quad and \quad P_{ij} = \frac{\Lambda_{ij}}{\omega_i} \tag{1}$$

The time between two Poisson clock ticks, t_{next} is modelled using an uniformly distributed random variable $u \sim \mathcal{U}(0,1)$ in the following way:

$$t_{next} = -\frac{ln(u)}{\omega_i} \tag{2}$$

Where ω_i is the rate of the clock associated to node i. The algorithm requires a starting and an ending node to be specified, as well as the number of simulations to be performed: in this case, both starting and ending positions coincide with node a. Each simulation time is computed as the sum of the transitions times (Eq. 2) of the walk and a list containing all simulation times is returned by the algorithm. After 1000 simulations of the Markov Chain, the resulting average is 6.66.

B) How does the result in A) compare to the theoretical return-time $\mathbb{E}_a[T_a^+]$?

The theoretical return time for a strongly connected graph, as the one of the exercise, can be easily computed as:

$$\mathbb{E}_i[\bar{T}_i^+] = \frac{1}{\omega_i \bar{\pi}_i}, \quad i \in \mathcal{X} \tag{3}$$

Where $\bar{\pi}$ is the stationary probability vector of the continuous time Markov Chain, $\bar{\pi} = \bar{P}'\bar{\pi}$, and $\bar{P}$ is defined as follows:

$$\bar{P}_{ii} = 1 - \sum_{j \neq i} \bar{P}_{ij}, \quad \bar{P}_{ij} = \frac{\Lambda_{ij}}{\omega^*} \tag{4}$$

Where $\omega^* = max_i(\omega_i)$.

Since the graph is strongly connected $\bar{\pi}$ is unique. In our case, the theoretical expected return time to a is 6.75. The average return time in point A) is close to the theoretical expected return time, and reaches it when the number of simulations increases.

C) What is, according to the simulations, the average time it takes to move from node o to node d?

The same algorithm used in point A) is applied, with starting position set to node o and ending position set to node d. After 1000 simulations, the resulting average time is 8.34.

D) How does the result in C) compare to the theoretical hitting-time $\mathbb{E}_o[T_d]$?

The theoretical hitting-time is computed by solving the following linear system of equations:

$$\begin{cases} \mathbb{E}_i[T_d] = 0, & i = d \\ \mathbb{E}_i[T_d] = \frac{1}{\omega_i} + \sum_{j \in \mathcal{X}} P_{ij} \mathbb{E}_j[T_d], & i \neq d \end{cases} \tag{5}$$

To solve the linear system, first I deleted from P and ω the rows, columns and values related to node d, then I used *numpy.lingalg.solve* to ocmpute the result. The theoretical expected hitting time is 8.79. Also in this case, one can see that the average hitting time of point C) is close to the theoretical one, and it reaches it when the number of iterations is incremented.

E) Interpret the matrix Λ as the weight matrix of a graph $G = (\mathcal{V}, \varepsilon, \Lambda)$ and simulate the French DeGroot dynamics on G with an arbitrary initial condition $x(0)$. Does the dynamics converge to a consensus state for every initial condition $x(0)$? Motivate your answer

Since the graph is strongly connected (hence, there is a single connected component) and aperiodic, every node is influenced by the opinion of the other nodes in the graph, either directly when an edge is present, or indirectly through the neighbors. Therefore, the consensus is guaranteed to converge in the long run to a single value α, given any initial condition $x(0)$:

$$\lim_{t \to +\infty} x(t) = \alpha \mathbf{1}, \quad \alpha = \pi' x(0) \tag{6}$$

Where π is the invariant distribution centrality of G.

The initial condition $x(0)$ is set by driving values from an uniformly distributed random variable $x \sim U(0,1)$. After simulating the French DeGroot model $x(t) = P^t x(0)$ for 100 iterations, all nodes converge to a consensus $\alpha = 0.435$.

F) Assume that the initial state of the dynamics for each node $i \in \mathcal{V}$ is given by $x_i(0) = \epsilon_i$, where ϵ_i, $i \in \mathcal{V}$ are i.i.d random variables with variance $\sigma = 2$. Compute the variance of the consensus value, and compare your results with numerical simulations.

Given a strongly connected and aperiodic graph and an underlying state of the world $\theta = 0$, suppose that every agent

Fig. 1. Condensation graph for Exercise 1G and Exercise 1H

$i \in \mathcal{V}$ observes a noisy version of it, so that the initial opinion distribution is $x_i(0) = \epsilon_i$, where ϵ_i are zero-mean i.i.d. random variables with variance $\sigma = 2$. The linear averaging dynamics converges to a consensus value:

$$\bar{\pi} = \sum_{k \in G} \pi_k x_k(0) = \theta + \sum_{k \in G} \pi_k \epsilon_k \tag{7}$$

Where $\pi = [0.13043478, 0.17391304, 0.26086957, 0.26086957, 0.17391304]$. The theoretical variance of the consensus value, then, can be computed as follows:

$\mathbb{V}[\bar{\pi}] = \mathbb{V}[\theta + \sum_{k \in G} \pi_k \epsilon_k] = \mathbb{V}[\theta] + \sum_{k \in G} \mathbb{V}[\pi_k \epsilon_k] = \sigma \sum_{k \in G} \pi_k^2 = 0.43.$

After the French DeGroot model $x(t) = P^t x(0)$ is run for 100 times, the variance of the obtained consensus values is 0.36. If one increases the number of simulations, then the obtained variance reaches the theoretical one.

G) Remove the edges (d, a) and (d, c). Describe and motivate the asymptotic behaviour of the dynamics. If the dynamics converges to a consensus state, how is the consensus value related to the initial condition $x(0)$? Assume that the initial state of the dynamics for each node $i \in \mathcal{V}$ is given by $x_i(0) = \epsilon_i$, where ϵ_i, $i \in \mathcal{V}$ are i.i.d random variables with variance $\sigma = 2$. Compute the variance of the consensus value. Motivate your answer.

As one can see from Figure 1, now the graph is not strongly connected anymore, but there is a sink component in the condensation graph: it corresponds to node d in G. Consequently, node d is not influences by the opinion of the other nodes in the graph and it does not change its initial opinion in the long run.
To compute the consensus variance, Eq. 7 still holds, however, the unique distribution centrality π is supported only on the sink node d, i.e., $\pi = [0, 0, 0, 0, 1]$. Now, the consensus value α coincides with the initial opinion of node d (Eq. 6) and the theoretical consensus variance is: $\mathbb{V}[\bar{\pi}] = \sigma \sum_{k \in G} \pi_k^2 = \sigma \pi_d^2 = 2$.

H) Consider the graph $G = (\mathcal{V}, \varepsilon, \Lambda)$, and remove the edges (c, b) and (d, a). Analyse the French DeGroot dynamics on the new graph. In particular, describe and motivate the asymptotic behaviour of the dynamics in terms of the initial condition $x(0)$.
The condensation graph of G is shown in Figure 1, where the sink component is made of nodes d and c and it is aperiodic. Then, according to the French DeGroot model, the consensus follows Eq. 6. However, the unique invariant distribution centrality is supported on the nodes belonging to the sink component ($\pi = [0, 0, 0, .5, .5]$), therefore, α is the average of the initial condition of node d and c.
The initial condition $x(0)$ values are driven by a standard normal distribution and equals 0.64 and -0.62 for, respectively, node c and node d. In both cases, either by computing α according to Eq. 6, or by computing the average between $x_c(0)$ and $x_d(0)$, the obtained value is 0.01.

II. EXERCISE 2

A. Particle perspective

To simulate the continuous time Markov Chain from the particle perspective, one should focus on the single particles and move them individually. Therefore, once a particle enters in node i, it stays there for an average time which is modelled according to Eq. 2 and then it moves to a neighbor node according to matrix P.

A) If 100 particles all start in node a, what is the average time for a particle to return to node a?

Each particle moves in the network independently from the other particles. Therefore, each particle is simulated 100 times as in Exercise 1 point A), for a total of $100 * 100$ simulations. The expected return time to node a is computed as the average over all simulations and it is 6.78.

B) How does this compare to the answer in $Problem1$, why?
It is the same. This is expected since the particles move independently from each other in the network, i.e., each particle moves in the network as it was alone.

B. System Perspective

In this case, the focus is put on the number of particles that are in each node over time. A single clock for the whole network is defined with rate 100, and the transition times are computed as:

$$t_{next} = -\frac{ln(u)}{100} \tag{8}$$

Then, at every clock tick, a node is randomly selected based on the particles distribution, and one particle is moved to a neighbor according to $\bar{P}$ (Eq. 4).

C) If 100 particles start in node o, and the system is simulated for 60 time units, what is the average number of particles in the different nodes at the end of the simulation?
The Markov Chain has been evolved for 60 times units. The average number of particles in the network nodes is [19.96619304, 14.61228156, 21.77641679, 21.88028744, 21.76482117], with a global average of 20.

D) Illustrate the simulation above with a plot showing

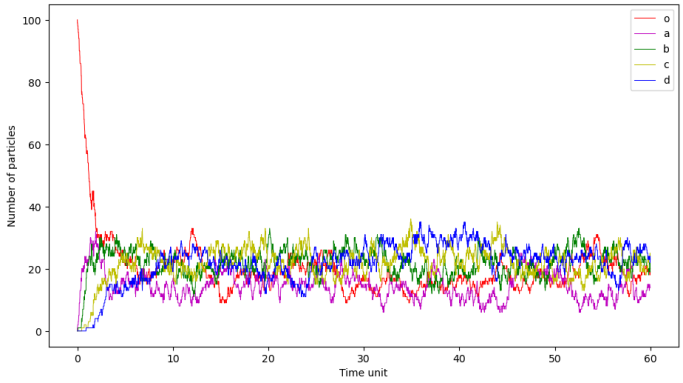

Fig. 2. Node distribution Exercise 2.C

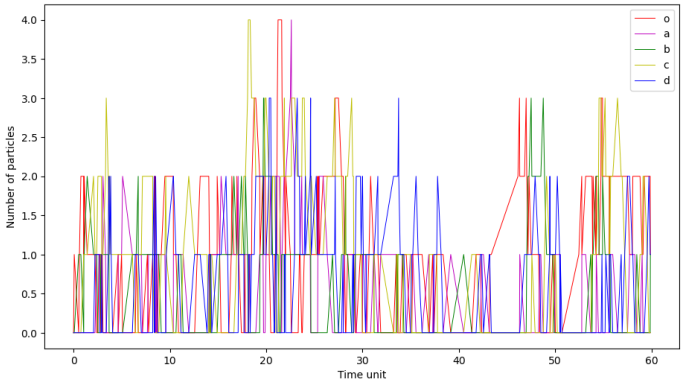

Fig. 4. Particles distribution in the nodes for Exercise 3.A

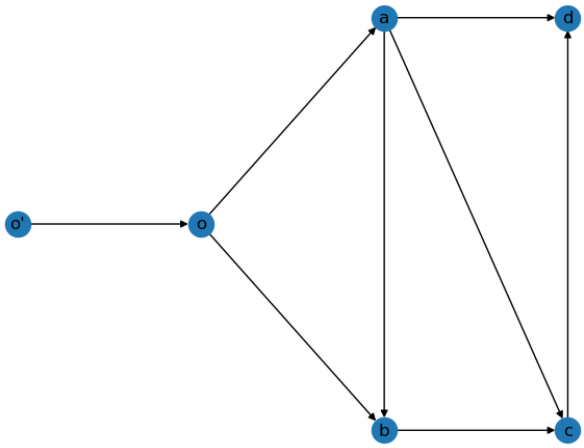

Fig. 3. Graph for Exercise 3

the number of particles in each node during the simulation time.
The number of particles in each node over time is shown in Figure 2.

E) Compare the simulation result in the first point above with the stationary distribution of the continuous-time random walk followed by the single particles.
The stationary distribution of the continuous random walk is $\bar{\pi}$ = [0.18518519, 0.14814815, 0.22222222, 0.22222222, 0.22222222]. The average number of particles in the nodes is $\bar{\pi} * 100$ = [18.51851852, 14.81481481, 22.22222222, 22.22222222, 22.22222222], with a global network average of 20. Close results are obtained in point C).

III. EXERCISE 3

In this exercise the aim is to see how different particles affect each other when moving around in a continuous time network, and the particles are seen from the system perspective. With respect to the original graph in the exercise, I added a node o' having an edge connected to node o, and that I consider to be always full (there is always one particle in o' ready to enter the network). Such a new node has a clock with constant rate λ=1 and, when its clock ticks, the number of particles in node o, as well as in the network, increases by one. Node d, instead, sends the particles outside the network and has a clock with ω_d=2, hence if its clock ticks the number of particles in the network only decreases by one. All other nodes have a clock that depends on the considered scenario, and the "fastest" clock determines which node to pass along a particle, and the selected particle moves to a neighbor node according to $P = \frac{1}{\omega}\Lambda_{open}$, $\omega = \mathbf{1}\Lambda_{open}$.
Also, Λ_{open} has been modified introducing a new row for node o' and modifying the node d row in order to avoid divisions by 0:

$$\Lambda_{open} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3/4 & 3/8 & 0 & 0 \\ 0 & 0 & 0 & 1/4 & 1/4 & 2/4 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

A. Proportional Rate

In the proportional rate scenario, each node passes along particles according to a Poisson process with rate equal to the number of particles in the node ($n_i(t)$) times the rate of the local Poisson clock of node i (ω_i):

$$t_{o'} = -\frac{ln(u)}{\lambda} \quad t_d = -\frac{ln(u)}{\omega_d * n_d(t)} \quad t_i = -\frac{ln(u)}{\omega_i * n_i(t)} \quad i \neq d,\ o' \tag{9}$$

A) Simulate the system for 60 time units and plot the evolution of the number of particles in each node over time.
After the system is simulated for 60 time units, the number of particles over time follows:

B) What is the largest input rate that the system can handle without blowing up?
Any input size λ is valid and the system does not blow up. This is due to the proportional clock rates, therefore, the node with the highest number of particles is the more likely

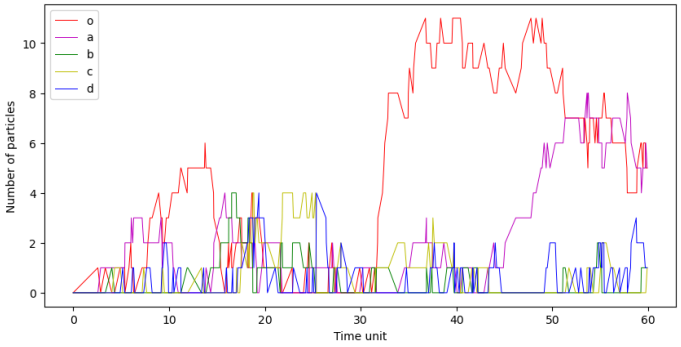

Fig. 5. Particles distribution in the nodes for Exercise 3.C

to be selected to move along particles and no bottleneck is is present in the network.

B. Fixed Rate

In the fixed rate scenario, each node passes along particles according to a Poisson process with rate equal to the local Poisson clock of node i (ω_i):

$$t_{o'} = -\frac{ln(u)}{\lambda} \quad t_d = -\frac{ln(u)}{\omega_d} \quad t_i = -\frac{ln(u)}{\omega_i} \quad i \neq d,\ o' \quad (10)$$

C) Simulate the system for 60 time units and plot the evolution of the number of particles in each node over time.
After the system is simulated for 60 time units, the number of particles over time is plotted in Figure 5.

D) What is the largest input rate that the system can handle without blowing up? Why is this different from the other case?
The largest input rate that the system can manage is λ=1. This is due to the fact that the input rates are fixed and the node with the fastest clock is the one selected to move along the particle. Therefore, node o is a bottleneck for the network, since its rate is $\omega_o = 1.125$. Therefore, if we increase the input rate to a value greater than 1, node o is not able to manage the amount of particles entering the system. In Figure 6 the particle distribution is shown when the input rate is set to $\lambda = 5$. As expected, node o is not able to manage the number of incoming particles and its size linearly increases in time.

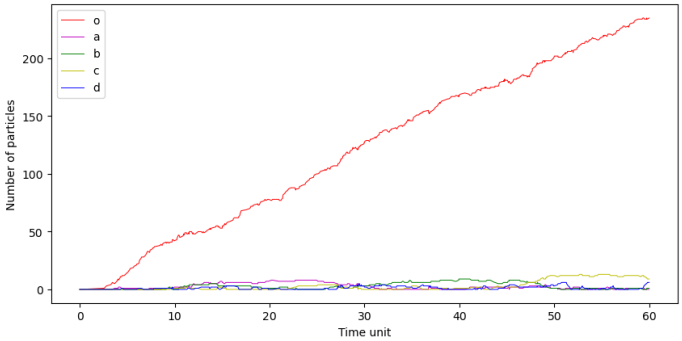

Fig. 6. Particles distribution when input rate changes to λ=5