End lect. Oct 22

X

Expectation values

- Let Y be some function of x, and $x \sim p(x)$

Then the expectation value of $Y(x)$ is

$$\boxed{E[Y] \equiv \int Y(x)\, p(x)\, dx}$$

some arbitrary function

A pdf

Alt. notation:

$$\langle Y \rangle \equiv \int Y(x)\, p(x)\, dx$$

We will use this notation in Project 4 because it is common in the physics literature

Discrete case:

$$E[Y] = \sum_{x \in \mathbb{D}} Y(x)\, p(x)$$

sum over possible outcomes

- So it's a weighted sum of all possible outcomes for $Y(x)$ (by running through all possible inputs x), where each term is weighted by the corresponding prob. $p(x)$ for the input x.

- Example: Y: money earned in a single coin flip bet:

$$Y(x) = \begin{cases} -1 & \text{for } x = \text{heads} \\ +1 & \text{for } x = \text{tails} \end{cases}$$

$$p(x) = \begin{cases} 0.5 & x = \text{heads} \\ 0.5 & x = \text{tails} \end{cases}$$

Note: Here there is no possible outcome that corresp. to the expected value

$$\Rightarrow E[Y] = Y(\text{heads})\, p(\text{heads}) + Y(\text{tails})\, p(\text{tails})$$

$$= -1 \times 0.5 + 1 \times 0.5 = \underline{\underline{0}}$$

Moments: particularly useful expectation values

- Analogous concept in physics: $r^n \times \begin{bmatrix}\text{physical} \\ \text{quantity} \\ \text{at } r\end{bmatrix}$

$$\int r^n \times \begin{bmatrix}\text{physical quantity} \\ \text{density} \\ \text{at } r\end{bmatrix} dr$$

- Example: Given a mass density $\rho(\vec{r})$ such that

$$M = \int \rho(\vec{r})\, d^3\vec{r}$$

  - 1st moment ($n=1$): Center of mass $\vec{R} = \frac{1}{M}\int \vec{r}\, \rho(\vec{r})\, d\vec{r}$

    where's the center?

  - 2nd moment: Moment of inertia $I = \int \vec{r}^{\,2} \rho(\vec{r})\, d\vec{r}$

    How spread out is the mass?

- Moments of prob. distributions:

$$E[x^n] \equiv \int x^n p(x)\, dx$$

- Zeroth moment is just norm. cond.:

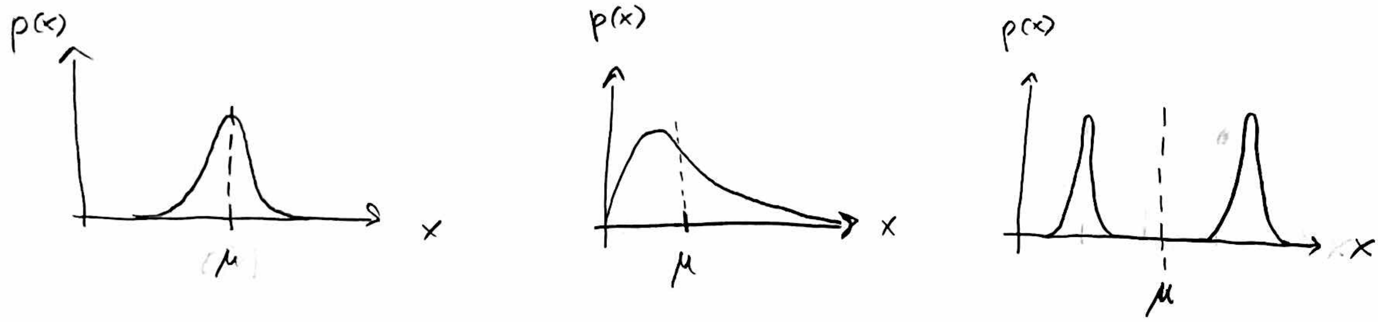
$$E[x^0] = E[1] = 1 = \int x^0 p(x)\, dx = \int p(x)\, dx$$

- First moment $E[x]$ (or $\langle x \rangle$) is called the mean of $p(x)$

$$\boxed{E[x] = \mu \equiv \int x\, p(x)\, dx}$$

The average x value. (Sums x-values weighted by their prob.)

- Examples of means



- Moments are relative to some point

$$\int (x-c)^n \, p(x)\, dx$$

with c some constant.

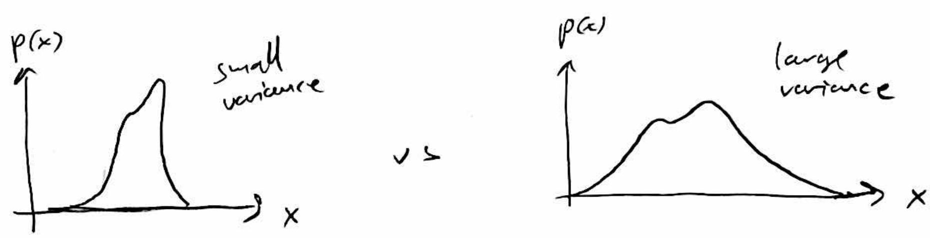
- Special case: take $c = \mu$, "central moments"

$$\int (x-\mu)^n \, p(x)\, dx$$

  - $n=0 \Rightarrow E[(x-\mu)^0] = 1$
  - $n=1 \Rightarrow E[(x-\mu)] = E[x] - \mu = \mu - \mu = 0$
  - <u>$n=2$</u> : <u>Variance</u> (σ^2)

$$\mathrm{Var}(X) = \sigma^2 = E[(x-\mu)^2] = \int (x-\mu)^2 \, p(x)\, dx$$

A measure of how spread-out $p(x)$ is. I.e. if large values of $(x-\mu)^2$ take up much of the prob. in $p(x)$, the distribution must be <u>wide</u> $\Longleftrightarrow$ large variance.



- Useful relation

$$\mathrm{Var}(x) = E\left[(x-\mu)^2\right]$$

$$= \int (x-\mu)^2 p(x)\, dx$$

$$= \int (x^2 - 2x\mu + \mu^2)\, p(x)\, dx$$

$$= E[x^2] - 2\mu E[x] + \mu^2$$

$$= E[x^2] - 2\mu^2 + \mu^2$$

$$= E[x^2] - \mu^2$$

$$= E[x^2] - (E[x])^2$$

$$\boxed{\mathrm{Var}(x) = E[x^2] - E[x]^2}$$

- Note that variance has units of $[x^2]$
- Standard deviation: $\boxed{\sigma = \sqrt{\mathrm{Var}(x)}}$

[Computation of different means and variances will be important in Proj. 4.]

Summarizing a prob. distribution with a single number

- In project 4 we will look at a lot of mean values, e.g $\langle E \rangle$

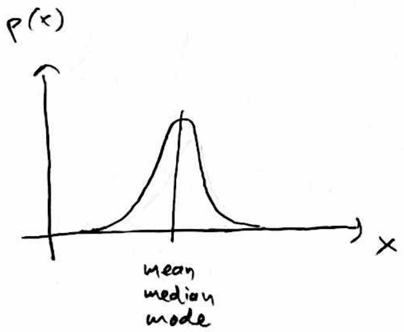
- But keep in mind that there are other point estimates that may be useful to summarize a pdf, e.g. the median and the mode

  - median: The point x such that $\text{Prob}(X \leq x) = \frac{1}{2}$, i.e.

$$\int_{-\infty}^{x} p(x')\,dx' = \frac{1}{2}$$

  - mode: The point x where $p(x)$ has its maximum

- For a normal distribution, the mean, median and mode are the same point:



Moral:
Keep in mind that $\langle x \rangle$ is not the full story. $p(x)$ is the full story!

But it's not true in general: