**Alessio Borzì**

# Matroids over a domain

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**DIPLOMA DI LICENZA MAGISTRALE**

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# Introduction

The theory of matroids, introduced inedpendently by Hassler Whitney [Whi35] and Takeo Nakasawa [Nak09a, Nak09d, Nak09c, Nak09b] around 1935, is an abstraction of linear independence and, from a certain point of view, a generalization of graph theory. Since its introduction, many new notions of matroids were given, including oriented matroids [BLVS$^+$99], valuated matroids [DW92], arithmetic matroids [DM13] and orientable arithmetic matroids [Pag20]. These objects have several applications, among others, in hyperplane arrengements [DCP11], toric arregements [DM13], zonotopes [DM12] and tropical geometry [MR18]. This new notions of matroids usually consist of a matroid with an extra structure.

There are also some attempts to unify all these definitions under a common framework. One of the earliest is the notion of matroids with coefficients in a fuzzy ring [DW92]. Another important definition is that of matroids over hyperfields [BB19]. In this thesis we will focus on a different approach, with the definition of matroids over a ring given by Fink and Moci [FM16]. While a (classical) matroid can be seen as a rank function defined on the power set $2^E$ of the ground set $E$ of the matroid, a matroid $\mathcal{M}$ over a ring $R$ can be seen as a function that assigns to each subset $A$ of the ground set $E$ a finitely generated $R$-module $\mathcal{M}(A)$. When $R$ is a field, the dimension as vector spaces of these modules provide a rank function, so matroids over a field are classical matroids. When $R$ is the ring of integers $\mathbb{Z}$, a matroid over $\mathbb{Z}$ defines a quasi arithmetic matroid. Finally, when $R$ id a discrete valuation ring, then a matroid over $R$ defines a valuated matroid.

The aim of this thesis is to present some results proved in [BM19] that focus on matroids over a domain. We will generalize the notion of Tutte-Grothendieck polynomial given in [FM16], and the notion of independence complex, called *poset of torsions*, previously defined for matroids over $\mathbb{Z}$ in [Mar18]. As an application, we will show the interplay of these objects by generalizing some results for classical matroids, by showing for instance that the Hilbert series of the face module of a matroid over a ring of integers that is a PID, is a specialization of the Tutte-Grothendieck poylnomial.

# Contents

# Chapter 1

# Classical Matroids

In this chapter, we review the basics of the classical theory of matroids. A matroid can be seen as a generalization of: linear independence among vectors of a vector space, algebraic independence among elements in a field extension and subtrees of a graph, as we shall see in Section 1.3. Further, a matroid is one of few objects in mathematics that has many different equivalent definitions, called *criptomorphisms.* In Section 1.1 we will define a matroid in terms of its independent sets, circuits, bases and rank function. Each description has its advantages and disadvantages, and we will define a matroid using the one that suits better our needs.

## 1.1 Definition and criptomorphisms

### 1.1.1 Independet sets

The criptomorphism that we are going to review in this subsection uses the independent sets. We will use this criptomorphism as our formal definition of matroid, and show that the other criptomorphisms are equivalent to it.

**Definition 1.1.1.** *A matroid is a pair $M = (E, \mathcal{I})$ consisting of a finite set $E$, called the* ***ground set****, and a collection $\mathcal{I} \subseteq 2^E$ of subsets of $E$, called* ***independent sets****, such that*

(*I*1) $\emptyset \in \mathcal{I}$.

(*I*2) *If $A \in \mathcal{I}$ and $B \subseteq A$ then $B \in \mathcal{I}$.*

(*I*3) *If $A, B \in \mathcal{I}$ with $|A| < |B|$, then there exists $x \in B$ such that $A \cup \{x\} \in \mathcal{I}$.*

*The subsets of $E$ not independent are called* ***dependent****.*

### 1.1.2 Circuits

The minimal dependent sets are the **circuits**. Clearly, a subset $A \subseteq E$ is independent if and only if it does not contain a circuit. Therefore a matroid can be defined from its circuits. The properties that characterize the set of circuits of a matroid are given by the following result.

**Theorem 1.1.2.** *[Oxl11, Corollary 1.1.5] A collection of subsets $\mathcal{C} \subseteq 2^E$ is a set of circuits of a matroid $M$ if and only if it satisfies the following properties:*

(*C*1) $\emptyset \notin \mathcal{C}$.

(*C*2) *If $C_1, C_2 \in \mathcal{C}$ and $C_1 \subseteq C_2$ then $C_1 = C_2$.*

(*C*3) *If $C_1, C_2 \in \mathcal{C}$ are distinct and $x \in C_1 \cap C_2$ then there exists $C_3 \in \mathcal{C}$ such that $C_3 \subseteq C_1 \cup C_2 \setminus \{x\}$.*

*Proof. Necessity.* Property $(C_1)$ follows from $(I1)$, $(C2)$ follows from the maximality of the circuits. Now we prove $(C3)$. Let $C_1, C_2 \in \mathcal{C}$ distinct and let $x \in C_1 \cap C_2$. Assume that $C_1 \cup C_2 \setminus \{x\}$ does not contain a circuit. Then $(C_1 \cup C_2) \setminus \{x\} \in \mathcal{I}$. By $(C2)$, $C_1 \subsetneq C_2$ so there exists $y \in C_2 \setminus C_1$. As $C_2$ is a circuit, $C_2 \setminus \{y\} \in \mathcal{I}$. Now choose a subset $I \subseteq C_1 \cup C_2$ which is maximal with the properties that it contains $C_2 \setminus \{y\}$ and is independent. Clearly $y \notin I$, otherwise $C_2 \subseteq I$. Moreover $C_1 \subsetneq I$, so there exists $z \in C_1 \setminus I$. As $y \in C_2 \setminus C_1$, $y$ and $z$ are distinct. Hence

$$|I| \leq |(C_1 \cup C_2) \setminus \{y, z\}| = |C_1 \cup C_2| - 2 < |(C_1 \cup C_2) \setminus \{x\}|.$$

Now, applying $(I3)$ with $I$ and $C_1 \cup C_2 \setminus \{x\}$ we obtain an independent set that contradicts the maximality of $I$.

*Sufficiency.* It is enough to prove that the collection $\mathcal{I}$ of subsets of $E$ that do not contain any member of $\mathcal{C}$ satisfy properties $(I1) - (I3)$. Property $(I1)$ follows from $(C1)$. If $B \subseteq E$ does not contain any member of $\mathcal{C}$, so it is for $A \subseteq B$, hence $(I2)$ holds true. To prove $(I3)$, let $A, B \in \mathcal{I}$ with $|A| < |B|$ and suppose by contradiction that $(I3)$ fails. Let

$$\mathcal{G} = \{J \subseteq A \cup B : |J| > |A|, J \in \mathcal{I}\},$$

clearly $B \in \mathcal{G} \neq \emptyset$. Choose $I \in \mathcal{G}$ fow which $|A \setminus I|$ is minimal. As $(I3)$ fails, $A \setminus I \neq \emptyset$, so there exists $x \in A \setminus I$. Now for each $y \in I \setminus A$, let $T_y = (I \cup \{x\}) \setminus \{y\}$. Then $T_y \subseteq A \cup B$ and $|A \setminus T_y| < |A \setminus I|$. Therefore, by the choice of $I$, $T_y \notin \mathcal{I}$, so $T_y$ contains some set $C_y \in \mathcal{C}$. Clearly $y \notin C_y$. Moreover $x \in C_y$, otherwise $C_y \subseteq I$, contradicting $I \in \mathcal{I}$. Fix $z \in I \setminus A$. If $C_z \cap (I \setminus A) = \emptyset$, then $C_z \subseteq ((A \cap I) \cup \{x\}) \setminus \{z\} \subseteq A$, a contradiction. Therefore there exists $t \in C_z \cap (I \setminus A)$. Then $C_z$ and $C_t$ are distinct since $t \in C_z$ and $t \notin C_t$. Now $x \in C_z \cap C_t$, so by $(C3)$ there exists $C \in \mathcal{C}$ such that $C \subseteq (C_z \cup C_t) \setminus \{x\}$. But $C_z, C_t \subseteq I \cup \{x\}$, hence $C \subseteq I$, a contradiction. □

### 1.1.3 Bases

The maximal independent sets of a matroid are called **bases**.

**Lemma 1.1.3.** *All the bases of a matroid $M$ have the same cardinality.*

*Proof.* Let $B_1, B_2$ be two bases of $M$. If $|B_1| < |B_2|$, then by $(I3)$ there exists $x \in B_2 \setminus B_1$ such that $B_1 \cup \{x\} \in \mathcal{I}$, contradicting the maximality of $B_1$. □

If $M = (E, \mathcal{I})$ is a matroid, a subset $A \subseteq E$ is independent if and only if it is contained in some basis. Thereofore, the bases provide another equivalent definition of matroid.

**Theorem 1.1.4.** *[Oxl11, Corollary 1.2.5] A collection of subsets $\mathcal{B} \subseteq 2^E$ is a set of bases of a matroid $M$ if and only if it satisfies the following properties:*

$(B1)$ $\mathcal{B} \neq \emptyset$.

$(B2)$ *If $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \setminus B_2$ then there exists an element $y \in B_2 \setminus B_1$ such that $(B_1 \setminus \{x\}) \cup \{y\} \in \mathcal{B}$.*

*Proof. Necessity.* Property $(B1)$ follows from $(I1)$. To prove $(B2)$ let $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \setminus B_2$. Both $B_1 \setminus \{x\}$ and $B_2$ are independent sets. Moreover from the preceding lemma $|B_1 \setminus \{x\}| < |B_2|$. Therefore, by $(I3)$ there exists $y \in B_2 \setminus (B_1 \setminus \{x\})$ such that $(B_1 \setminus \{x\}) \cup \{y\} \in \mathcal{I}$. Furthermore, if $(B_1 \setminus \{x\}) \cup \{y\}$ is contained in a basis $B$ then by the previous lemma we have

$$|B| = |B_1| = |(B_1 \setminus \{x\}) \cup \{y\}|,$$

hence $(B_1 \setminus \{x\}) \cup \{y\} = B$, that is, $(B_1 \setminus \{x\}) \cup \{y\}$ is a basis.

*Sufficiency.* It is enough to prove that the collection $\mathcal{I}$ of subsets of $E$ that are contained in some member of $\mathcal{B}$ satisfy properties $(I1) - (I3)$. Property $(I1)$ follows from $(B1)$. If $I \in \mathcal{I}$ there exists $B \in \mathcal{B}$ such that

$I \subseteq B$. Thus if $I' \subseteq I$, then $I' \subseteq B$, so $I' \in \mathcal{I}$. Hence $(I2)$ holds true. To prove $(I3)$, we first prove that the members of $\mathcal{B}$ are equicardinal.

Suppose that $B_1$ and $B_2$ are distinct members of $\mathcal{B}$ for which $|B_1| > |B_2|$ so that, among all such pairs, $|B_1 \setminus B_2|$ is minimal. Clearly $B_1 \setminus B_2 \neq \emptyset$. Thus let $x \in B_1 \setminus B_2$, applying $(B2)$ we can find an element $y \in B_2 \setminus B_1$ so that $(B_1 \setminus \{x\}) \cup \{y\} \in \mathcal{B}$. Evidently $|(B_1 \setminus \{x\}) \cup \{y\}| = |B_1| > |B_2|$ and $|((B_1 \setminus \{x\}) \cup \{y\}) \setminus B_2| < |B_1 \setminus B_2|$, contradicting the choice of $B_1$ and $B_2$. Therefore the members of $\mathcal{B}$ are equicardinal.

Now suppose that $(I3)$ fails for $\mathcal{I}$. Then there exist $I_1, I_2 \in \mathcal{I}$ with $|I_1| < |I_2|$ such that, for all $z \in I_2 \setminus I_1$, the set $I_1 \cup \{z\} \notin \mathcal{I}$. By definition, there are $B_1, B_2 \in \mathcal{B}$ such that $I_i \subseteq B_i$ $(i = 1, 2)$. Assume that such a set $B_2$ is chosen so that $|B_2 \setminus (I_2 \cup B_1)|$ is minimal. By the choice of $I_1$ and $I_2$,

$$I_2 \setminus B_1 = I_2 \setminus I_1, \tag{1.1}$$

Now suppose that $B_2 \setminus (I_2 \cup B_1)$ is non-empty. Then we can choose an element $x$ from this set. By $(B2)$, there is an element $y \in B_1 \setminus B_2$ such that $(B_2 \setminus \{x\}) \cup \{y\} \in \mathcal{B}$. But then $|((B_2 \setminus \{x\}) \cup \{y\}) \setminus (I_2 \cup B_1)| < |B_2 \setminus (I_2 \cup B_1)|$ and the choice of $B_2$ is contradicted. Hence $B_2 \setminus (I_2 \cup B_1)$ is empty and so $B_2 \setminus B_1 = I_2 \setminus B_1$. Thus by 1.1,

$$B_2 \setminus B_1 = I_2 \setminus I_1. \tag{1.2}$$

Next we show that $B_1 \setminus (I_1 \cup B_2)$ is empty. If not, then there is an element $x$ in this set and $y \in B_2 \setminus B_1$ so that $(B_1 \setminus \{x\}) \cup \{y\} \in \mathcal{B}$. Now $I_1 \cup \{y\} \subseteq (B_1 \setminus \{x\}) \cup \{y\}$ so $I_1 \cup \{y\} \in \mathcal{I}$. Since $y \in B_2 \setminus B_1 = I_2 \setminus I_1$, we have a contradiction to our assumption that $(I3)$ fails. We conclude that $B_1 \setminus (I_1 \cup B_2)$ is empty. Hence $B_1 \setminus B_2 = I_1 \setminus B_2$. Since the last set is contained in $I_1 \setminus I_2$, it follows that

$$B_1 \setminus B_2 \subseteq I_1 \setminus I_2 \tag{1.3}$$

Since the members of $\mathcal{B}$ are equicardinal, $|B_1| = |B_2|$, so $|B_1 \setminus B_2| = |B_2 \setminus B_1|$. Therefore by 1.2 and 1.3, $|I_1 \setminus I_1| \geq |I_2 \setminus I_1|$, so $|I_1| \geq |I_2|$. This is a contradiction. $\square$

### 1.1.4 Rank

Let $M = (E, \mathcal{I})$ be a matroid. We have seen that all the bases of $M$ have the same cardinality, which is the **dimension** $\dim(M)$ of $M$. For every subsets $X \subseteq E$, set $\mathcal{I}|X = \{I \subseteq X : I \in \mathcal{I}\}$. The pair $(X, \mathcal{I}|X)$ is a matroid, called the **restriction** of $M$ to $X$, denoted by $M|X$. The **rank** of a subset $X \subseteq E$ is the dimension of the restriction:

$$\mathrm{rk}_M(X) = \dim(M|X).$$

Thus we defined a map $\mathrm{rk}_M : 2^E \to \mathbb{N}$ called the **rank function**. The dimension of $M$ is also called the rank of $M$ and it is denoted $\mathrm{rk}_M(M) = \mathrm{rk}_M(E)$. Note that $X \subseteq E$ is independent if and only if $\mathrm{rk}_M(X) = |X|$. Therefore, the rank function provides another equivalent definition of matroid.

**Theorem 1.1.5.** *[Oxl11, Corollary 1.3.4] A function* $\mathrm{rk} : 2^E \to \mathbb{N}$ *is the rank function of a matroid* $M = (E, \mathcal{I})$ *if and only if for all* $X, Y \in 2^E$ *we have*

$(R1)$ $0 \leq \mathrm{rk}(X) \leq |X|$.

$(R2)$ $X \subseteq Y \Rightarrow \mathrm{rk}(X) \leq \mathrm{rk}(Y)$.

$(R3)$ $\mathrm{rk}(X \cup Y) + \mathrm{rk}(X \cap Y) \leq \mathrm{rk}(X) + \mathrm{rk}(Y)$.

For the proof of the theorem we will use the following lemma.

**Lemma 1.1.6.** *If* $\mathrm{rk} : 2^E \to \mathbb{N}$ *is a function that satisfy properties* $(R2)$ *and* $(R3)$*, then for every pair of subsets* $X, Y \subseteq E$ *such that, for all* $y \in Y \setminus X$, $\mathrm{rk}(X \cup \{y\}) = \mathrm{rk}(X)$*, it results* $\mathrm{rk}(X \cup Y) = \mathrm{rk}(X)$.

*Proof.* Let $Y \setminus X = \{y_1, \ldots, y_k\}$. We procede by induction on $k$. If $k = 1$, the result is obvious. Assume it is true for $k = n$ and let $k = n + 1$: Then, by the induction assumption and $(R3)$,

$$\begin{aligned}
\mathrm{rk}(X) + \mathrm{rk}(X) =& \,\mathrm{rk}(X \cup \{y_1, \ldots, y_n\}) + \mathrm{rk}(X \cup \{y_{n+1}\}) \\
\geq& \,\mathrm{rk}((X \cup \{y_1, \ldots, y_n\}) \cup (X \cup \{y_{n+1}\})) \\
&+ \mathrm{rk}((X \cup \{y_1, \ldots, y_n\}) \cap (X \cup \{y_{n+1}\})) \\
=& \,\mathrm{rk}(X \cup \{y_1, \ldots, y_{n+1}\}) + \mathrm{rk}(X) \\
\geq& \,\mathrm{rk}(X) + \mathrm{rk}(X).
\end{aligned}$$

The last inequality follows from $(R2)$. The first and the last member of the preceding chain of inequalities are equal. Therefore equality hold throughout, so

$$\mathrm{rk}(X \cup \{y_1, \ldots, y_{n+1}\}) = \mathrm{rk}(X). \qquad \square$$

*Proof of Theorem 1.1.5. Necessity.* Property $(R1)$ and $(R2)$ are clear from the definition. To prove $(R3)$, let $B_{X\cap Y}$ be a basis for $M|(X \cap Y)$. Then $B_{X\cap Y}$ is an independent set of $M|(X \cup Y)$. Therefore it is contained in a basis $B_{X\cup Y}$ of this matroid. Now $B_{X\cup Y} \cap X$ and $B_{X\cup Y} \cap Y$ are independent in $M|X$ and $M|Y$, respectively. Therefore

$$|B_{X\cup Y} \cap X| \leq \mathrm{rk}(X) \quad \text{ and } \quad |B_{X\cup Y} \cap Y| \leq \mathrm{rk}(Y).$$

So

$$\begin{aligned}
\mathrm{rk}(X) + \mathrm{rk}(Y) \geq& |B_{X\cup Y} \cap X| + |B_{X\cup Y} \cap Y| \\
=& |(B_{X\cup Y} \cap X) \cup (B_{X\cup Y} \cap Y)| + |(B_{X\cup Y} \cap X) \cap (B_{X\cup Y} \cap Y)| \\
=& |B_{X\cup Y} \cap (X \cup Y)| + |B_{X\cup Y} \cap (X \cap Y)| \\
=& |B_{X\cup Y}| + |B_{X\cap Y}| \\
=& \,\mathrm{rk}(X \cup Y) + \mathrm{rk}(X \cap Y).
\end{aligned}$$

*Sufficiency.* First, we prove that the collection $\mathcal{I}$ of subsets $X$ of $E$ such that $\mathrm{rk}(X) = |X|$ satisfy property $(I1) - (I3)$. By $(R1)$, $0 \leq \mathrm{rk}(\emptyset) \leq |\emptyset| = 0$, then $\mathrm{rk}(\emptyset) = 0 = |\emptyset|$, therefore $\emptyset \in \mathcal{I}$ and $(I1)$ holds true. Now let $A \in \mathcal{I}$ and $B \subseteq A$. Then $\mathrm{rk}(A) = |A|$. By $(R3)$,

$$\mathrm{rk}(B \cup (A \setminus B)) + \mathrm{rk}(B \cap (A \setminus B)) \leq \mathrm{rk}(B) + \mathrm{rk}(A \setminus B)$$

that is

$$\mathrm{rk}(A) + \mathrm{rk}(\emptyset) \leq \mathrm{rk}(B) + \mathrm{rk}(A \setminus B).$$

Since $\mathrm{rk}(A) = |A|$, $\mathrm{rk}(B) \leq |B|$, $\mathrm{rk}(A \setminus B) \leq |A \setminus B|$; it follows that

$$|A| = \mathrm{rk}(A) \leq \mathrm{rk}(B) + \mathrm{rk}(A \setminus B) \leq |B| + |A \setminus B| = |A|.$$

So the quality must hold throughout. Therefore, subtracting $|A \setminus B|$

$$|B| = \mathrm{rk}(B) + \mathrm{rk}(A \setminus B) - |A \setminus B| \leq \mathrm{rk}(B) \leq |B|.$$

Hence $\mathrm{rk}(B) = |B|$, that is, $B \in \mathcal{I}$. Then $(I2)$ hold true.

Now we prove $(I3)$. Let $I_1, I_2 \in \mathcal{I}$ with $|I_1| < |I_2|$. Suppose by contradiction that $(I3)$ fails for $I_1$ and $I_2$, so that for all $x \in I_2 \setminus I_1$, $I_1 \cup \{x\} \notin \mathcal{I}$. This means that

$$|I_1| + 1 = |I_1 \cup \{x\}| > \mathrm{rk}(I_1 \cup \{x\}) \geq \mathrm{rk}(I_1) = |I_1|,$$

and so $\mathrm{rk}(I_1 \cup \{x\}) = \mathrm{rk}(I_1)$. Now applying the preceding lemma to $I_1$ and $I_2$ we immediately get that $\mathrm{rk}(I_1) = \mathrm{rk}(I_1 \cup I_2)$. Therefore

$$|I_1| = \mathrm{rk}(I_1) = \mathrm{rk}(I_1 \cup I_2) \geq \mathrm{rk}(I_2) = |I_2|,$$

hence $|I_1| \geq |I_2|$, a contradiction.

To complete the proof we need to show that $r$ is the rank function of $M$, i.e. for every $X \subseteq E$, $\mathrm{rk}(X) = \mathrm{rk}_M(X) = \dim(M|X)$. In fact, if $X \in \mathcal{I}$, then, by definition, $\mathrm{rk}(X) = |X| = \mathrm{rk}_M(X)$. Now suppose that $X \notin \mathcal{I}$ and let $B$ be a basis for $M|X$. Then $\mathrm{rk}_M(X) = |B|$. Moreover, $B \cup \{x\} \notin \mathcal{I}$ for all $x \in X \setminus B$. Hence $|B| = \mathrm{rk}(B) \leq \mathrm{rk}(B \cup \{x\}) < |B \cup \{x\}|$, so $\mathrm{rk}(B) = \mathrm{rk}(B \cup \{x\})$. By Lemma 1.1.6 it follows that $\mathrm{rk}(B \cup X) = \mathrm{rk}(B)$, that is, $\mathrm{rk}(X) = \mathrm{rk}(B) = |B| = \mathrm{rk}_M(X)$. This proves $\mathrm{rk} = \mathrm{rk}_M$. □

### 1.1.5 Closure

Let $M = (E, \mathcal{I})$ be a matroid. For every subset $X \subseteq E$ set

$$\mathrm{cl}(X) = \{x \in E : \mathrm{rk}_M(X \cup \{x\}) = \mathrm{rk}_M(X)\}.$$

The function $\mathrm{cl} : 2^E \to 2^E$ is called the **closure operator of** $M$, $\mathrm{cl}(X)$ is the **closure** (or the **span**) of $X$ in $M$. The closure operator can be used to define a matroid.

**Proposition 1.1.7.** *[Oxl11, Corollary 1.4.6] A function $\mathrm{cl} : 2^E \to 2^E$ is the closure operator of a matroid if and only if it satisfies the following properties*

$(CL1)$ $X \subseteq \mathrm{cl}(X)$

$(CL2)$ $X \subseteq Y \Rightarrow \mathrm{cl}(X) \subseteq \mathrm{cl}(Y)$

$(CL3)$ $\mathrm{cl}(\mathrm{cl}(X)) = \mathrm{cl}(X)$

$(CL4)$ $y \in \mathrm{cl}(X \cup \{x\}) \setminus \mathrm{cl}(X) \Rightarrow x \in \mathrm{cl}(X \cup \{y\})$.

We observe that $(CL_1) - (CL_3)$ are the defining properties of a closure operator [BS81, Definition 5.1]. The last condition, called the *MacLane-Steinitz exchange property*, characterize closure operators of matroids. If $X = \mathrm{cl}(X)$ then $X$ is called a **flat** (or a **closed set**) of $M$.

## 1.2 Lattice of Flats

We give two equivalent definition of a lattice.

**Definition 1.2.1.** *[BS81, Definition 1.4] We say that a partially ordered set $(L, \leq)$ is a **lattice** if for every $x, y \in L$ both $\sup\{x, y\}$ and $\inf\{x, y\}$ exist (in $L$).*

**Definition 1.2.2.** *[BS81, Definition 1.1] Let $L$ be a nonempty set and $\vee, \wedge$ two binary operations on $L$. We say that $(L, \vee, \wedge)$ is a **lattice** if the two operations are associative, commutative and for every $x, y \in L$ it statisfies the following **absorption laws**:*

1. $x \wedge (x \vee y) = x$,
2. $x \vee (x \wedge y) = x$.

The previous two definitions of a lattice are equivalent, in fact:

(a) If $(L, \leq)$ is a lattice in the sense of Definition 1.2.1 then define

$$x \wedge y = \inf\{x, y\} \quad x \vee y = \sup\{x, y\}.$$

It follows that $(L, \vee, \wedge)$ is a lattice in the sense of Definition 1.2.2.

(b) If $(L, \vee, \wedge)$ is a lattice in the sense of Definition 1.2.2 then define

$$x \leq y \Longleftrightarrow x = x \wedge y.$$

It follows that $(L, \leq)$ is a lattice in the sense of Definition 1.2.1.

Now, if $M$ is a matroid we denote with $\mathcal{L}(M)$ the set of flats of $M$.

**Proposition 1.2.3.** *[Oxl11, Lemma 1.7.3] [BS81, Theorem 5.2] The partially ordered set $(\mathcal{L}(M), \subseteq)$ is a lattice, called the **lattice of flats**, and for every $X, Y \in \mathcal{L}(M)$*

$$X \wedge Y = X \cap Y, \quad X \vee Y = \mathrm{cl}(X \cup Y).$$

*Proof.* Let $X, Y \in \mathcal{L}(M)$. We have

$$X \cap Y \subseteq X \Rightarrow \mathrm{cl}(X \cap Y) \subseteq \mathrm{cl}(X) = X,$$
$$X \cap Y \subseteq Y \Rightarrow \mathrm{cl}(X \cap Y) \subseteq \mathrm{cl}(Y) = Y,$$

from which it follows $\mathrm{cl}(X \cap Y) \subseteq X \cap Y$, hence $X \cap Y = \mathrm{cl}(X \cap Y) \in \mathcal{L}(M)$, so $\inf\{X, Y\} = X \cap Y$. Now let $Z \in \mathcal{L}(M)$ such that $X, Y \subseteq Z$. We have $X \cup Y \subseteq Z \Rightarrow \mathrm{cl}(X \cup Y) \subseteq \mathrm{cl}(Z) = Z$. Therefore $\sup\{X, Y\} = \mathrm{cl}(X \cup Y)$. □

**Definition 1.2.4.** *Let $(P, \leq)$ be a finite partially ordered set. Let $x, y \in P$, we say that $y$ **covers** $x$, written $x \lhd y$, if $x < y$ and there are are no elements $z \in P$ such that $x < z < y$. A **chain** of length $n$ in $P$ is a subset $\{x_0, \ldots, x_n\} \subseteq P$ such that $x_0 < x_1 < \cdots < x_n$. The chain is **maximal** if $x_{i+1}$ covers $x_i$ for all $i \in \{0, \ldots, n-1\}$. If all maximal chains between two fixed elements of $P$ have the same length, then $P$ satisfies the **Jordan-Dedekind chain condition**.*

**Definition 1.2.5.** *Let $(P, \leq)$ be a finite partially ordered set. If $P$ has a minimum, we will denote it be $\hat{0}$. Similarly, if it has a maximum, we will denote with $\hat{1}$. If $P$ has a minimum $\hat{0}$, then the **rank** $\mathrm{rk}(x)$ of an element $x \in P$ is the maximum length of a chain from $\hat{0}$ to $x$. The elements of rank one are the **atoms**, in other words, an element $x \in P$ is an atom if it covers $0$.*

**Definition 1.2.6.** *A finite lattice $L$ is **semimodular** if for every $x, y \in L$*

$$x \wedge y \lhd x \Rightarrow x \lhd x \vee y \quad (\textit{semimodularity}).$$

*A finite lattice $L$ is **atomic** if every $x \in L$ is a join of atoms. A finite lattice $L$ is **geometric** if it is semimodular and atomic.*

**Definition 1.2.7.** *A matroid $M$ is **simple** if it has no circuits of cardinality less than 3.*

In a certain sense, simple matroids are the matroid-equivalent of simple graphs. The next theorem shows a connection between the lattice of flats of simple matroids and geometric lattices.

**Theorem 1.2.8.** *[Oxl11, Theorem 1.7.5] A lattice $L$ is geometric if and only if it is the lattic of flats of a simple matroid.*

## 1.3 Representability

In this section we will focus on representability of matroids. In particular, we will present three motivating examples of the notion of matroid: linear (or representable) matroids, that arise from linear independence of a set of vectors in a $\mathbb{K}$-vector space; graphic matroids, that arise from (multi)graphs; algebraic matroids, that arise from algebraic independence of a set o elements in a field extension $\mathbb{F} \supseteq \mathbb{K}$.

### 1.3.1 Linear matroids

In this subsection, $\mathbb{K}$ is a field.

**Proposition 1.3.1.** *Let $A \in \mathbb{K}^{m,n}$ be a matrix with coefficients in $\mathbb{K}$ with columns $E = \{v_1, \ldots, v_n\} \subseteq \mathbb{K}^m$. Set*

$$\mathcal{I} = \left\{ I \subseteq E : \text{ the vectors in } I \text{ are linearly indepedent } \right\}.$$

*The pair $M[A] = (E, \mathcal{I})$ is a matroid.*

*Proof.* The properties $(I1)$ and $(I2)$ are clear, $(I3)$ follows from Steinitz exchange lemma. $\square$

The matroid $M$ arising from a matrix $A \in \mathbb{K}^{m,n}$ as in Proposition 1.3.1 is denoted by $M[A]$.

**Definition 1.3.2.** *Let $M_i = (E_i, \mathcal{I}_i)$ for $i \in \{1,2\}$ be two matroids. We say that $M_1$ and $M_2$ are **isomorphic**, denoted by $M_1 \simeq M_2$, if there exists a function $\psi : E_1 \to E_2$ such that $E_1 \in \mathcal{I}_1$ if and only if $\psi(E_1) \in \mathcal{I}_2$.*

**Definition 1.3.3.** *A matroid $M = (E, \mathcal{I})$ is **linear** (or **representable**, or **realizable**) over a field $\mathbb{K}$ if there exists a matrix $A \in \mathbb{K}^{m,n}$ such that $M \simeq M[A]$. Further, $M$ is **binary** (resp. **ternary**) if it is representable over $\mathbb{F}_2$ (resp. $\mathbb{F}_3$).*

We will say that a matroid is linear (or representable, or realizable), if it is so over some field $\mathbb{K}$.

A natural question now is wether all the matroids are linear over some field. The answer is no, in fact there exist matroids that are not representable over any field, as the following examples shows.

**Example 1.3.4** (Non-Pappus matroid)**.** *The non-Pappus matroid is the rank 3 matroid on the ground set $E = \{1, \dots, 9\}$ in which a subset of $E$ with three elements is independent if and only if its elements are not on a line in Figure 1.1. Pappus's Theorem [Oxl11, Theorem 6.1.11] implies that in such a configuration the points 4, 5, 6 must be aligned (in the plane). However, Figure 1.1 has no line passing through 4, 5, 6. This fact implies that the non-Pappus matroid is not representable (over any field), see [Oxl11, Proposition 6.1.10].*

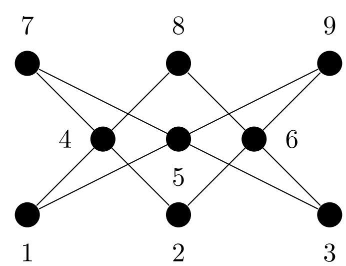



Figure 1.1: The non-Pappus matroid

Representability stongly depends on which field we work on, as the following examples shows.

**Example 1.3.5** (Fano and non-Fano matroid)**.** *The Fano matroid $F_7$ is the rank 3 matroid on the ground set $E = \{1, \dots, 7\}$ in which a subset of $E$ with three elements is independent if and only if its elements are not on a line in Figure 1.2 (where the circle passing through 3, 5, 6 is considered as a "line"). The Fano matroid is representable over a field $\mathbb{K}$ if and only if $\mathbb{K}$ has characteristic 2 (this can be shown by a matrix-completion argument, see [GM12, Section 6.4]). The non-Fano matroid $F_7^-$ is defined similarly starting from Figure 1.3. It is representable over $\mathbb{K}$ if and only if $\mathbb{K}$ has characteristic not equal to 2. Therefore, any matroid $M$ with two restrictions, one isomorphic to $F_7$ and the other to $F_7^-$ is another example of a matroid that is non representable over any field.*

**Definition 1.3.6.** *The **uniform matroid** $U_{m,n}$ is a matroid on an $n$-elment ground set $E = \{1, \dots, n\}$ where the independent sets are the all the subsets of $E$ with at most $m$ elements:*

$$\mathcal{I} = \{I \subseteq E : |I| \leq m\}.$$

**Example 1.3.7.** *Let $\mathbb{K}$ be a field with at least three elements. The uniform matroid $U_{2,4}$ is represented by the following matrix*

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & a \end{bmatrix}$$

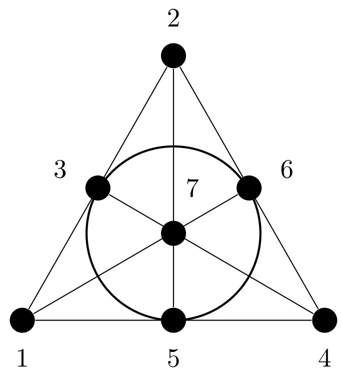



Figure 1.2: The Fano matroid

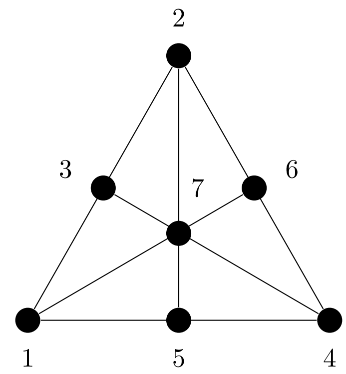



Figure 1.3: The non-Fano matroid

*where $a \in \mathbb{K} \setminus \{0, 1\}$. On the other hand, if the field $\mathbb{K}$ with two elements, than $U_{2,4}$ is not representable over $\mathbb{K}$, in other words, it is not binary. This is the smallest example of a matroid not representable over some field.*

### 1.3.2 Graphic and regular matroids

We allow a graph to have multiple edges and loops, that is, with *graph* we will always mean multigraph.

**Proposition 1.3.8.** *Let $G$ be a (multi)graph, denote by $E$ the set of its edges. Set*

$$\mathcal{I} = \big\{ I \subseteq E : \ I \text{ does not contain the edge set of a cycle } \big\}.$$

*Then the pair $M = (E, \mathcal{I})$ is a matroid, called the **cycle matroid** of $G$.*

*Proof.* Properties $(I1)$ and $(I2)$ are clear, $(I3)$ follows from the fact that every spanning tree of $G$ has the same number of edges. ☐

**Definition 1.3.9.** *A matroid $M$ is **graphic** if there exists some graph $G$ such that $M \simeq M[G]$.*

Now we explore the relations between graphic matroids and representability. In order to do this we introduce the notion of regular matroids.

**Definition 1.3.10.** *A matrix $A \in \mathbb{R}^{m,n}$ is **unimodular** if every of its square submatrix has determinant in $\{-1, 0, 1\}$. A matroid $M$ is **regular** if it has a unimodular representation, that is, it is represented by a unimodular matrix $A$: $M \simeq M[A]$.*

**Theorem 1.3.11.** *[Oxl11, Theorem 6.6.3] The following statements are equivalent for a matroid $M$:*

1. *$M$ is regular,*
2. *$M$ is representable over any field,*
3. *$M$ is binary and representable over a field of characteristic other than two.*

**Proposition 1.3.12.** *[Oxl11, Proposition 5.1.2] Every graphic matroid $M[G]$ is regular.*

*Proof.* From Theorem 1.3.11 it is enough to show that $M[G]$ is representable over any field. Let $D(G)$ be an arbitrary orientation of the edges of $G$. Let $A_{D(G)} = (a_{ij})$ be the incidence matrix of the directed graph $D(G)$. In other words, we have

$$a_{ij} = \begin{cases} \phantom{-}1 & \text{if the vertex } i \text{ is the tail tail of the non-loop arc } j, \\ -1 & \text{if the vertex } i \text{ is the head tail of the non-loop arc } j, \\ \phantom{-}0 & \text{otherwise.} \end{cases}$$

The matrix $A_{D(G)}$ can be interpreted has a matrix over any field $\mathbb{K}$ (if $\mathbb{K}$ has characteristic 2, then $1 = -1$). Now, if $C$ is a cycle in $M[G]$, this means that there is a path in $C$ where the first and last vertex coincide. This implies that, the sum of the associated columns in $A_{D(G)}$ of the path, each multipied by $\pm 1$ depending on whether the orientation of the path agrees or not with the orientation of $D(G)$, is zero. In particular, $C$ corresponds to a circuit in $M[A_{D(G)}]$. On the other hand, suppose that $C$ is a cycle in $M[A_{D(G)}]$. Then, there is a linear combination of the columns $\{v_1, \ldots, v_t\}$ of $C$ that is zero:

$$\sum_{i=1}^{t} a_i v_i = 0.$$

This implies that in the subgraph corresponding to the columns $v_i$ such that $a_i \neq 0$, every vertex has degree at least two. This implies that this subgraph contains a cycle. This proves that $M[A_{D(G)}]$ is a representation for $M[G]$. $\square$

### 1.3.3 Algebraic matroids

Let $\mathbb{K} \subseteq \mathbb{F}$ be an extension field. Recall that the elements $\alpha_1, \ldots, \alpha_n \in \mathbb{F}$ are algebraically independent over $\mathbb{K}$ if there is no non zero polynomial $f \in \mathbb{K}[x_1, \ldots, x_n]$ such that $f(\alpha_1, \ldots, \alpha_n) = 0$.

**Theorem 1.3.13.** *[Oxl11, Theorem 6.7.1] Let $\mathbb{K} \subseteq \mathbb{F}$ be a field extension, and let $E \subseteq \mathbb{F}$. Set*

$$\mathcal{I} = \big\{ I \subseteq E : \text{ the elements of } I \text{ are algebraically independent over } \mathbb{K} \big\}.$$

*Then $M = (E, \mathcal{I})$ is a matroid.*

**Definition 1.3.14.** *A matroid $M$ is* ***algebraic*** *over a field $\mathbb{K}$ if it is isomorphic to a matroid as in Theorem 1.3.13.*

We will say that a matroid is algebraic, if it is so over some field $\mathbb{K}$.

**Proposition 1.3.15.** *[Oxl11, Proposition 6.7.10] If a matroid $M$ is representable over $\mathbb{K}$, then it is algebraic over $\mathbb{K}$.*

*Proof.* Let $A \in \mathbb{K}^{m,n}$ be a matrix that represents $M$. Withouth loss of generality, we can assume that $A$ is of the form $[I_m|D]$, for some matrix $D \in \mathbb{K}^{m,n-m}$, and $M = M[A]$. Denote by $v_1, \ldots, v_{n-m}$ the columns of $D$, let $t_1, \ldots, t_n$ be be independent transcendentals over $\mathbb{K}$, and define a function $\phi : E \to \mathbb{K}(t_1, \ldots, t_n)$ by $\phi(e_i) = t_i$. The function $\phi$ is well-defined since $e_1, \ldots, e_m$ is a basis, so every vector $v_i$ can be expressed as a linear combination of the vectors $e_i$. Further, it is easy to see that $I \subseteq E$ is independent if and only if $\phi(I)$ is algebraically independent over $\mathbb{K}$. In other words, $\phi$ is an isomorphism of matroids. It follow that $M$ is algebraic over $\mathbb{K}$. $\square$

**Proposition 1.3.16.** *[Oxl11, Proposition 6.7.11] If a matroid $M$ is algebraic over a field $\mathbb{K}$ of characteristic zero, then $M$ is linear over $\mathbb{K}(T)$ for some finite set $T$ of trascendentals over $\mathbb{K}$.*

The previous proprosition does not hold for finite fields, as the following example show.

**Example 1.3.17** (Non linear algebraic matroids)**.** *The uniform matroid $U_{2,n}$ is realizable over a field $\mathbb{K}$ if and only if $|\mathbb{K}| \geq n-1$. In contrast, $U_{2,n}$ is algebraic over every field $\mathbb{K}$. To see this, let $E = \{xy, x^2y, \ldots, x^ny\} \subseteq \mathbb{K}(x, y)$ where $x$ and $y$ are trascendentals over $\mathbb{K}$. Then, $U_{2,n}$ is isomorphic to the algebraic matroid $(E, \mathcal{I})$, where $\mathcal{I}$ is the family of all subsets of $E$ algebraically indepdent over $\mathbb{K}$.*

**Example 1.3.18** (Vámos matroid)**.** *The Vámos matroid is a matroid that is not algebraic (over any field). Thus it is also an example of a non representable matroid over any field. It is defined as follows: $V_8 = (E, , \mathcal{I})$, where $E = 1, 2, \ldots, 8$ and the independent sets are all the cardinality 4 subsets, except for 5 of them (Figure 1.4):*

$$\mathcal{I} = \big\{ I \subseteq E : |I| = 4 \big\} \setminus \big\{ \{1,2,3,4\}, \{1,4,5,6\}, \{1,4,7,8\}, \{2,3,5,6\}, \{2,3,7,8\} \big\}.$$

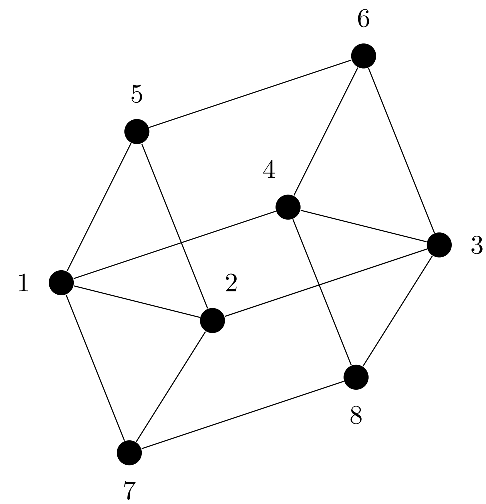



Figure 1.4: The Vámos matroid

## 1.4 Duality

**Proposition-Definition 1.4.1.** *[Oxl11, Theorem 2.1.1] Let $M = (E, \mathcal{I})$ be a matroid, and let $\mathcal{B}$ the set of its bases. The set*

$$\mathcal{B}^* = \{E \setminus B : B \in \mathcal{B}\}$$

*is the set of bases of a matroid $M^*$, called the* ***dual*** *of $M$.*

For example, the dual of a uniform matroid is again a uniform matroid. More precisely, it is easy to see that we have $U^*_{m,n} = U_{n-m,n}$. Further, it is clear from the definition that duality is an involution (that is $M^{**} = M$).

**Definition 1.4.2.** *Let $M = (E, \mathcal{I})$ be matroid. An element $e \in E$ is a* ***loop*** *of $M$ if $\{e\}$ is a circuit, and a* ***coloop*** *if it a loop of $M^*$.*

Let $M = (E, \mathcal{I})$ be a matroid and let $X \subseteq E$. If $X$ is an independent set, or a basis, or a circuit for the dual $M^*$, then $X$ is called **coindependent**, **cobasis** or **cocircuit** respectively. The rank function of $M^*$ is denoted $\mathrm{rk}^* = \mathrm{rk}_{M^*}$. The rank $\mathrm{rk}^*(X)$ of $X$ in the dual matroid $M^*$ is called the **dual rank** of $X$ (not to be confused with the *corank*, that is usually defined as $\mathrm{cork}(X) = \mathrm{rk}(M) - \mathrm{rk}(X)$).

**Proposition 1.4.3.** *[Oxl11, Proposition 2.1.9] Let $M$ be a matroid on the ground set $E$ with rank function $\mathrm{rk} : 2^E \to \mathbb{N}$. For every $X \subseteq E$ we have*

$$\mathrm{rk}^*(X) = |X| + \mathrm{rk}(E \setminus X) - \mathrm{rk}(E)$$

An immediate consequence of the previous proposition is that for every $A \subseteq E$ we have

$$|A| - \mathrm{rk}(A) = |A| - (|A| + \mathrm{rk}^*(E \setminus A) - \mathrm{rk}^*(E)) = \mathrm{rk}^*(E) - \mathrm{rk}^*(E \setminus A). \tag{1.4}$$

## 1.5 Tutte polynomial

**Definition 1.5.1.** *Let $M = (E, \mathcal{I})$ be a matroid. The* ***deletion*** *of a subset $X \subseteq E$ from $M$ is defined as the restriction of $M$ to $E \setminus X$, written*

$$M \setminus X = M|(E \setminus X).$$

*The* ***contraction*** *of $X$ from $M$ is defined by*

$$M/X = (M^* \setminus X)^*.$$

The deletion and contraction have some geometric interpretation for graphic matroids. Let $G$ be a graph and consider its cycle matroid $M[G]$. The deletion $M \setminus e$ of an element $e \in E$ is the cycle matroid of the graph $G$ with the edge $e$ removed. The contraction $M/e$ of $e \in E$ is the cycle matroid of the graph $G$ with the edge $e$ contracted (see Figure 1.5). In symbols

$$M[G] \setminus e = M[G \setminus e], \quad M[G]/e = M[G/e].$$

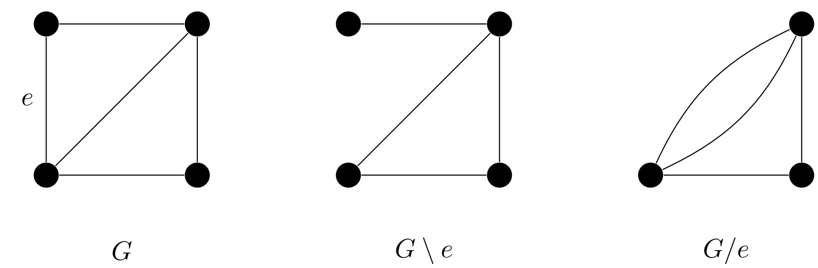



Figure 1.5

In this section, we will sometimes consider matroids defined by their rank function $\mathrm{rk} : 2^E \to \mathbb{N}$. Thus, to define a matroid we will consider the pair $M = (E, \mathrm{rk})$.

**Definition 1.5.2.** *Let $M = (E, \mathrm{rk})$ be a matroid. The* ***Tutte polynomial*** *of $M$ is the polynomial*

$$T_M(x, y) = \sum_{A \subseteq E} (x-1)^{\mathrm{rk}(M)-\mathrm{rk}(A)}(y-1)^{|A|-\mathrm{rk}(A)} \in \mathbb{Z}[x, y].$$

**Example 1.5.3.** *Let $M = U_{2,3}$, the Tutte polynomial of $M$ is*

$$T_M(x, y) = (x-1)^2 + 3(x-1) + 3 + (y-1) = x^2 + x + y \in \mathbb{Z}[x, y].$$

**Theorem 1.5.4** (Deletion-Contraction property)**.** *[GR01, Theorem 15.9.1] Let $M = (E, \mathrm{rk})$ be a matroid, and let $e \in E$. The Tutte polynomial satisfies the following deletion-contraction property*

$$T_M(x, y) = \begin{cases} y\, T_{M\setminus e}(x, y) & \text{if } e \text{ is a loop,} \\ x\, T_{M/e}(x, y) & \text{if } e \text{ is a coloop,} \\ T_{M\setminus e}(x, y) + T_{M/e}(x, y) & \text{otherwise.} \end{cases}$$

*Proof.* If $e$ is a loop or a coloop, the statement follows from the definition of Tutte polynomial. If $e$ is neither a loop ora coloop, then the subsets of $E$ that do not contain $e$ contribute $T_{M\setminus e}(x, y)$, while the subsets that contain $e$ contribute $T_{M/e}(x, y)$. $\square$

**Proposition 1.5.5.** *Let $M = (E, \mathrm{rk})$ be a matroid. We have*

$$T_M(x, y) = T_{M^*}(y, x).$$

*Proof.* From (1.4) we can write

$$\begin{aligned} T_M(x, y) &= \sum_{A \subseteq E} (x-1)^{\mathrm{rk}(M)-\mathrm{rk}(A)}(y-1)^{\mathrm{rk}^*(M)-\mathrm{rk}^*(E\setminus A)} = \\ &= \sum_{A \subseteq E} (y-1)^{\mathrm{rk}^*(M)-\mathrm{rk}^*(A)}(x-1)^{\mathrm{rk}(M)-\mathrm{rk}(E\setminus A)} = T_{M^*}(y, x). \end{aligned}$$

$\square$

From the definition of Tutte polynomial, we can easily deduce the following properties.

**Proposition 1.5.6.** *Let $M = (E, \mathrm{rk})$ be a matroid, then*

1. $T_M(2,2) = 2^{|E|}$,
2. $T_M(1,1)$ *is the number of bases of* $M$,
3. $T_M(2,1)$ *is the number of independent sets of* $M$.

The Tutte polynomial is the most general matroid parameter that can be computed by deletion-contraction. The proof of this powerful fact is a simple inductive argument.

**Theorem 1.5.7.** *[GR01, Theorem 15.9.3][Bol98, Theorem X.2.2] Let $F$ be a function defined on matroids, such that for the empty matroid we have $F(\emptyset) = 1$, and for all other matroids*

$$F(M) = \begin{cases} ayF(M \setminus e) & \text{if } e \text{ is a loop,} \\ bxF(M/e) & \text{if } e \text{ is a coloop,} \\ aF(M \setminus e) + bF(M/e) & \text{otherwise.} \end{cases}$$

*Then, for any matroid $M = (E, \mathrm{rk})$ we have $F(M) = a^{|E|-\mathrm{rk}(E)} b^{\mathrm{rk}(E)} T_M(x, y)$.*

As an application of the previous theorem, we express the chromatic polynomial of a simple graph $G$ in terms of the Tutte polynomial of its cycle matroid $M[G]$. Throughout the end of the section, we will assume that all graphs have no multiple edges.

**Definition 1.5.8.** *Let $G = (V(G), E(G))$ be an undirected simple graph. A **proper vertex $k$-coloring** is a function $\varphi : V(G) \to C$ such that $C$ is a set of cardinality $k$ and $\varphi$ is injective on the edges of $G$, that is, for each $\{x, y\} \in E(G)$ we have $\varphi(x) \neq \varphi(y)$.*

**Proposition-Definition 1.5.9.** *Let $G$ be a simple graph and let $\chi_G(\lambda)$ be the number of proper vertex $\lambda$-colorings on $G$. The function $\chi_G$ satisfies the following deletion-contraction property:*

$$\chi_G(\lambda) = \begin{cases} 0 & \text{if } e \text{ is a loop,} \\ \frac{(\lambda-1)}{\lambda}\chi_{G\setminus e}(\lambda) & \text{if } e \text{ is a coloop,} \\ \chi_{G\setminus e}(\lambda) - \chi_{G/e}(\lambda) & \text{otherwise.} \end{cases}$$

*It follows that $\chi_G(\lambda)$ is a polynomial, called the **chromatic polynomial** of $G$.*

*Proof.* Let $e = \{v, w\} \in E(G)$ that is not a loop or coloop. Then $\chi_{G\setminus e}(\lambda) = \chi_G(\lambda) + \chi_{G/e}(\lambda)$, since $\chi_G(\lambda)$ counts the number of $k$-colorings in which the vertices $v$ and $w$ are given different colors, and $\chi_{G/e}(\lambda)$ counts the number in which $v$ and $w$ are given the same color. If $e$ is a loop, then $\chi_G(\lambda) = 0$ by definition. If $e$ is a coloop, then $\chi_G(\lambda) = \frac{(\lambda-1)}{\lambda}\chi_{G\setminus e}(\lambda)$.

Now if the graph $G$ has no edges, then $\chi_G(\lambda) = \lambda^{|V(G)|}$, in particular it is a polynomial, therefore, recursively applying the preceding formula, it follows that $\chi_G(\lambda)$ is a polynomial. □

**Corollary 1.5.10.** *[Bol98, Theorem X.4.6] For any graph $G$ the chromatic polynomial is a specialization of the Tutte polynomial. More precisely:*

$$\chi_G(\lambda) = (-1)^{|V(G)|-k(G)} \lambda^{k(G)} T_{M[G]}(1-\lambda, 0),$$

*where $k(G)$ is the number of connected components of the graph $G$.*

*Proof.* Let $F$ be the following function defined on graphic matroids:

$$F(M[G]) = \frac{\chi_G(\lambda)}{\lambda^{k(G)}}.$$

From the previous proposition we have

$$F(M[G]) = \begin{cases} 0 & \text{if } e \text{ is a loop,} \\ -(1-\lambda)F(M[G] \setminus e) & \text{if } e \text{ is a coloop,} \\ F(M[G] \setminus e) - F(M[G]/e) & \text{otherwise.} \end{cases}$$

Therefore, we can apply Theorem 1.5.7 to $F$ with $a = 1$, $b = -1$, $x = 1 - \lambda$, $y = 0$, obtaining

$$\chi_G(\lambda) = (-1)^{|V(G)|-k(G)} \lambda^{k(G)} T_{M[G]}(1-\lambda, 0). \qquad \square$$

The Tutte polynomial also behaves well under direct sums of matroids.

**Definition 1.5.11.** *Let $M_i = (E_i, \mathcal{I}_i)$ for $i \in \{1, 2\}$ be two matroids on disjoint ground sets. Let $E = E_1 \cup E_2$ and $\mathcal{I} = \{I_1 \cup I_2 : I_1 \in \mathcal{I}_1, I_2 \in \mathcal{I}_2\}$. The matroid $M = (E, \mathcal{I})$ is the* ***direct sum*** *$M_1 \oplus M_2$ of the matroids $M_1$ and $M_2$.*

**Proposition 1.5.12.** *[Whi92, Proposition 6.2.5] Let $M_i = (E_i, \mathcal{I}_i)$ for $i \in \{1, 2\}$ be two matroids on disjoint ground sets. Then*

$$T_{M_1 \oplus M_2}(x, y) = T_{M_1}(x, y) T_{M_2}(x, y)$$

## 1.6 Matroids in tropical geometry

In this section, we do a little digression on some interactions between the theory of (valuated) matroids and tropical geometry. We will describe two important constructions of tropical geometry, that are related to matroids: the Bergman fan of a matroid and tropical ideals.

### 1.6.1 The Bergman Fan of a matroid

In this subsection, we will mainly follow [MS15, Chapter 4]. We denote by $\mathbb{R}^{n+1}/\mathbb{R}\mathbf{1}$ the quotient of $\mathbb{R}^{n+1}$ by the subvector space $\mathbb{R}\mathbf{1}$ spanned by the vector $\mathbf{1} = (1, \ldots, 1) \in \mathbb{R}^{n+1}$.

**Definition 1.6.1.** *[MS15, Definition 4.2.5] Let $M$ be a matroid on the ground set $\{0, 1, \ldots, n\}$. The* ***tropical linear space*** *$\operatorname{trop}(M)$ is the set of equivalence classes $[w] \in \mathbb{R}^{n+1}/\mathbb{R}\mathbf{1}$ where $w = (w_0, \ldots, w_n) \in \mathbb{R}^{n+1}$ is such that for any circuit $C$ of $M$, the minimum of the numbers $\{w_i : i \in C\}$ is attained at least twice.*

Note that if $w = (w_0, \ldots, w_n)$ is such that the minimum of $\{w_i : i \in C\}$ is attained at least twice, for any circuit $C$ of $M$, so is any vector $w + \lambda\mathbf{1}$ for every $\lambda \in \mathbb{R}$. Therefore, the previous definition is independent of the choice of the representative of each class.

It is possible to define several fan structures on a tropical linear space $\operatorname{trop}(M)$. We are going to describe the coarsest possible. Its corresponding fan is called the *Bergman fan* of $M$.

**Definition 1.6.2.** *[MS15, Definition 4.2.9] Let $M$ be a matroid on the ground set $\{0, 1, \ldots, n\}$. For a basis $B$ of $M$, denote by $e_B = \sum_{i \in B} e_i \in \mathbb{R}^{n+1}$. The* ***matroid polytope*** *is defined by*

$$P_M = \operatorname{conv}\{e_B : B \text{ basis of } M\}$$

The polytopes $P$ that arise from a matroid in the way described above, can be characterized by the property that each edge of $P$ is parallel to $e_i - e_j$ for some $i, j$ (see [MS15, Theorem 4.2.12]).

**Definition 1.6.3.** *[MS15, Definition 4.2.7] Let $M$ be a matroid on the ground set $\{0, 1, \ldots, n\}$. For any $w \in \mathbb{R}^{n+1}$ we define the* ***initial matroid*** *as the matroid $M_w$ on the same ground set of $M$, with the family of circuits $\mathcal{C}(M_w)$ given by*

$$\mathcal{C}(M_w) = \Big\{ \{j \in C : w_j = \min_{i \in C}(w_i)\} : C \text{ circuit of } M \Big\}.$$

**Proposition 1.6.4.** *[MS15, Proposition 4.2.10] For any $w \in \mathbb{R}^{n+1}$, the matroid polytope of $M_w$ is the face of the matroid polytope $P_M$ at which $w$ is maximized. Thus, $M_w$ is constant on the relative interior of the cones in the negative normal fan of $P_M$.*

**Corollary 1.6.5.** *[MS15, Corollary 4.2.11] We have*

$$\operatorname{trop}(M) = \{[w] \in \mathbb{R}^{n+1}/\mathbf{1}\mathbb{R} : M_w \text{ has no loops}\}.$$

*Proof.* The class $[w]$ lies in $\operatorname{trop}(M)$ if and only if the minimum $\min_{i \in C} w_i$ is achived at least twice for all circuits $C$ of $M$. This occurs if and only if all circuits of $M_w$ have size at least two, that is, $M_w$ has no loops. $\square$

**Definition 1.6.6.** *[AK06, FS05] The tropical linear space* $\operatorname{trop}(M)$ *with the fan structure given by Proposition 1.6.4 is the* ***Bergman fan*** *of $M$.*

We can define another fan structure on $\operatorname{trop}(M)$ in terms of chains of flats.

**Theorem 1.6.7.** *[MS15, Theorem 4.2.6] Let $M$ be a matroid on the ground set $E = \{0, 1, \ldots, n\}$. The collection of cones* $\operatorname{pos}(e_{F_1}, \ldots, e_{F_r}) + \mathbb{R}\mathbf{1}$*, where $e_F = \sum_{i \in F} e_i$ and $\emptyset \subsetneq F_1 \subsetneq \cdots \subsetneq F_r \subsetneq E$ runs over all chains of flats of $M$, forms a pure simplicial fan with support equal to the tropical linear space* $\operatorname{trop}(M)$.

The fan described in the previuos theorem is also sometimes referred as the Bergman fan of $M$ in the literature (see for instance [DR20]). For other possible fan structures of $\operatorname{trop}(M)$ defined by the nested sets of the lattice of flats of $M$, see [FY04, FS05].

What we have seen so far, is the definition of tropical linear space when the valuation is trivial. In fact, we could extend this definition to (valuated) matroids with non trivial valuation, but this goes beyond the scope of this thesis. For an account about tropical linear spaces with non trivial valuation, see [MS15, Section 4.4]

### 1.6.2 A glimpse on tropical ideals

From the definition of tropical scheme given in [GG16], Maclagan and Rincón [MR18] developed the notion of tropical ideal, proposing a solid algebraic foundation for tropical geometry. This definition is strictly related to the notion of valuated matroids, defined by Dress and Wenzel [DW92].

**Definition 1.6.8.** *A* ***valauted matroid*** *on the ground set $E$ of rank $r \in \mathbb{N}$, is a pair $M = (E, p)$ where $p : \binom{E}{r} \to \overline{\mathbb{R}}$ (with $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$) is a function defined on the family $\binom{E}{r}$ of subsets of $E$ of cardinality $r$, such that:*

1. *$p(B) \neq \infty$ for some $B \in \binom{E}{e}$,*
2. *For every $A, B \in \binom{E}{r}$ and every $a \in A \setminus B$, there exists $b \in B \setminus A$ such that*

$$p(A) + p(B) \geq p(A \cup \{b\} \setminus \{a\}) + p(B \cup \{a\} \setminus \{b\}).$$

Note that the relation in the second axiom of valuated matroids is equivalent to the *tropical Plücker relations* (see for instance [MS15, Section 4.4]).

If $M = (E, p)$ is a valuated matroid, then $\{B \in \binom{E}{e} : p(B) \neq \infty\}$ is a family of bases of a classical matroid of rank $r$ on the ground set $E$, called the *underlying matroid* of the valuated matroid $M$, denoted by

$\underline{M}$. From this point of view, a valaucted matroid can be seen as a classical matroid plus the extra structure of a valuation function $p$ on the bases. In fact, valuated matroids in which the image of $p$ is $\{0, \infty\}$ (that is, when we consider the *trivial valuation*) are precisely the classical matroids. In this particular case, the two axioms of valuated matroids translates into the axioms $(B1)$-$(B2)$ for the bases of a classical matroid.

**Definition 1.6.9.** *Let $M = (E, p)$ be a valuated matroid. Given a basis $B$ of the underlying matroid $\underline{M}$ and an element $e \in E \setminus B$, the (valuated)* ***fundamental circuit*** *$H(B, e)$ of $M$ is the vector of $\overline{\mathbb{R}}^E$ whose coordinates are given by*

$$H(B, e)_{e'} = p(B \cup e \setminus e') - p(B) \quad \textit{for any } e' \in E,$$

*where $p(B') = \infty$ if $|B'| > r$. A (valuated)* ***circuit*** *of $M$ is any vector in $\overline{\mathbb{R}}^E$ of the form $\lambda\mathbf{1} + H(B, e)$, where $B$ is a basis of $\underline{M}$, $e \in E \setminus B$ and $\lambda \in \mathbb{R}$.*

The collection of circuits of a valuated matroid $M$ is denoted by $\mathcal{C}(M)$, the **support** of a vector $H \in \overline{\mathbb{R}}^E$ is the set $\mathrm{supp}(H) = \{e \in E : H_e \neq \infty\}$. The supports of the valuated circuits of $M$ are the circuits of the underlying matroid $\underline{M}$. Furthermore, if two circuits $G$ and $H$ have the same support, then there exists $\lambda \in \mathbb{R}$ such that $G = \lambda\mathbf{1} + H$ [MT01, Theorem 3.1].

Now let $(\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}, \oplus, \odot)$ be the *tropical semiring*, where the tropical addition is the minimum $a \oplus b = \min(a, b)$, and the tropical multiplication is the usual addition $a \odot b = a + b$. Note that the additive identity of $\overline{\mathbb{R}}$ is $\infty$, and the multiplicative identity is 0. For a finite set $E$, the set $\overline{\mathbb{R}}^E$ can be seen as a semimodule over $\overline{\mathbb{R}}$, where the sum of two vectors is defined by taking the componentwise minimum, and the scalar multiplication of $\lambda \in \overline{\mathbb{R}}$ is the componentwise (classical) addition of $\lambda$. Thus, a linear combination of vectors $H_1, \dots, H_n \in \overline{\mathbb{R}}^E$ is of the form

$$(\lambda_1 \odot H_1) \oplus (\lambda_2 \odot H_2) \oplus \cdots \oplus (\lambda_n \odot H_n) = \min(\lambda_1\mathbf{1} + H_1, \dots, \lambda_n\mathbf{1} + H_n),$$

where the minimum is taken componentwise, and $\lambda_1, \dots, \lambda_n \in \overline{\mathbb{R}}$.

A **cycle** of a (classical) matroid is a union of circuits. The corresponding notion of cycle for a valuated matroid is *valuated vector.*

**Definition 1.6.10.** *Let $M = (E, p)$ be a valuated matroid. A* ***valuated vector*** *of $M$ is an element of the $\overline{\mathbb{R}}$-semimodule $\overline{\mathbb{R}}^E$ generated by valuated circuits. More explicitely, the set of valuated vectors of $M$ is*

$$\mathcal{V}(M) = \left\{ \bigoplus_{H \in \mathcal{C}(M)} \lambda_H \odot H : \lambda_H \in \overline{\mathbb{R}} \textit{ for all } H \right\}$$

Now consider the tropical semiring of polynomials $\overline{\mathbb{R}}[x_0, \dots, x_n]$. This is the semiring of polynomials in the variables $x_0, \dots, x_n$ with the two operations $\oplus$ and $\odot$, with coefficients in $\overline{\mathbb{R}}$. Note that here we are not regarding the polynomials as functions, for instance $x^2 \oplus 0 \neq x^2 \oplus 1 \odot x \oplus 0$, even though they identify the same function (note that with $x^2$ here we mean $x \odot x$, not $x \cdot x$).

Let $\mathrm{Mon}_d$ be the set of monomials of degree $d$ in $\overline{\mathbb{R}}[x_0, \dots, x_n]$. We will identify $\overline{\mathbb{R}}^{\mathrm{Mon}_d}$ with homogeneous polynomials of degree $d$ in $\overline{\mathbb{R}}[x_0, \dots, x_n]$. In this way, if $M$ is a valuated matroid on the ground set $\mathrm{Mon}_d$, the valauted vectors can be thought of as homogeneous polynomials of degree $d$ in $\overline{\mathbb{R}}[x_0, \dots, x_n]$.

**Definition 1.6.11** (Homogeneous tropical ideals)**.** *[MR18, Definition 2.1] A* ***homogeneous tropical ideal*** *is a homogeneous ideal $I \subseteq \overline{\mathbb{R}}[x_0, \dots, x_n]$ such that for each $d \geq 0$ the degree $d$ part $I_d$ is the collection of valuated vectors of some valuated matroid $M_d(I)$ on the ground set* $\mathrm{Mon}_d$*. In other words $I_d = \mathcal{V}(M_d(I))$.*

Another elegant way to rephrase the previous definition is by saying that each degree $d$ part $I_d$ is a tropical linear space (possibly, with non trivial valuation).

A more explicit definition of tropical ideals uses the axioms of valuated circuits of a valuated matroid. In fact, as for classical matroids, valuated matroids have many cryptomorphisms. We can define a valuated matroid starting from its valuated vectors. From [MT01, Theorem 3.4], a subset $\mathcal{V} \subseteq \overline{\mathbb{R}}^E$ is the collection of valuated vectors of a valuated matroid if and only if it is a subsemimodule of $\overline{\mathbb{R}}^E$ satisfying the following property:

- *Vector elimination axiom*: for any $G, H \in \mathcal{V}$ and any $e \in E$ such that $G_e = H_e \neq \infty$, there exists $F \in \mathcal{V}$ statisfying $F_e = \infty$, $F \geq G \oplus H$, and $F_{e'} = G_{e'} \oplus H_{e'}$ for all $e' \in E$ such that $G_{e'} \neq H_{e'}$.

The definition of tropical ideal, can be recast using the vector elimination axiom.

**Definition 1.6.12** (Tropical ideals)**.** *[MR18, Definition 1.1] An ideal $I \subseteq \overline{\mathbb{R}}[x_0, \ldots, x_n]$ is a **tropical ideal** if it satisfies the following property*

- monomial elimination axiom*: for any $f, g \in I_{\leq d}$ and any monomial $x^u$ for which $f_u = g_u \neq \infty$, there exists $h \in I_{\leq d}$ such that $h_u = \infty$ and $h_v \geq \min(f_v, g_v)$ for all $v \in \mathbb{N}^{n+1}$, with equality holding whenever $f_v \neq g_v$.*

*Where $I_{\leq d}$ is the set of polynomials in $I$ of degree at most $d$, $x^u$ is the monomial $x_0^{u_0} \cdot \ldots x_n^{u_n}$, for any $u = (u_0, \ldots, u_n) \in \mathbb{N}^{n+1}$, and $f_u$ is the coefficient of the monomial $x^u$ in $f$.*

**Definition 1.6.13.** *Let $K$ be a field with valuation $v : K \to \overline{\mathbb{R}}$ and let $f = \sum_{u \in \mathbb{N}^{n+1}} a_u x^u \in K[x_0, \ldots, x_n]$ be a (classical) polynomial. The **tropicalization** of $f$ is the polynomial*

$$\operatorname{trop}(f) = \bigoplus_{u \in \mathbb{N}^{n+1}} v(a_u) \odot x^u \in \overline{\mathbb{R}}[x_0, \ldots, x_n]$$

*(where $x^u$ is interpreted tropically, for instance $x^2 = x \odot x$). Now let $I \subseteq K[x_0, \ldots, x_n]$ be a (classical) ideal. The tropicalization of $I$ is the following ideal in $\overline{\mathbb{R}}[x_0, \ldots, x_n]$:*

$$\operatorname{trop}(I) = \langle \operatorname{trop}(f) : f \in I \rangle.$$

**Proposition-Definition 1.6.14.** *Let $I \subseteq K[x_0, \ldots, x_n]$, the tropicalization* $\operatorname{trop}(I)$ *of $I$ is a tropical ideal. The tropical ideals arising in this way are called **realizable tropical ideals**.*

For a proof of the previous result see [MR18, Example 2.2]. In addition, the class of tropical ideals strictly includes the tropicalization of classical ideals, see [MS15, Example 2.8] for an example of a *non-realizable* tropical ideal.

From the point of view of tropicalization, the monomial elimination axiom can be viewed as forcing the existence of the difference between two polynomials (something that in a semiring not necessarily exists). In fact, if we consider $\operatorname{trop}(f), \operatorname{trop}(g) \in \operatorname{trop}(I)$, a polynomial $h$ that satisfies the monomial elimination axiom is $\operatorname{trop}(f - g)$. From this interpretation, it is not surprising that the tropicalizations of classical ideals are tropical ideals.

# Chapter 2

# Matroids over a ring

In this chapter we introduce the main object of this thesis: matroid over a ring. In [FM16], Fink and Moci generalize the notion of matroid, by giving the definition of matroid over a (commutative and unitary) ring $R$, assigning to each subset of the ground set a finitely generated $R$-module according to some axioms. This construction generalizes the notion of matroid, arithmetic matroid and valuated matroid. In fact, the structure of a matroid over a ring $R$ defines a classical matroid, an arithmetic matroid or a valuated matroid in the cases when $R$ is a field, the ring of integers $\mathbb{Z}$ or a DVR respectively.

## 2.1 Definition and first properties

Let $R$ be a commutative ring and denote with $R$-mod the category of finitely generated $R$-modules.

**Definition 2.1.1.** *[FM16, Definition 2.1] A* ***matroid over a ring*** *$R$ on a ground set $E$ is a function*

$$\mathcal{M} : 2^E \to R\text{-}mod$$

*such that for every $A \subseteq E$ and $b, c \in E \setminus A$, there exist $x, y \in \mathcal{M}(A)$ such that*

$$\begin{aligned} \mathcal{M}(A \cup \{b\}) &\simeq \mathcal{M}(A)/\langle x \rangle \\ \mathcal{M}(A \cup \{c\}) &\simeq \mathcal{M}(A)/\langle y \rangle \\ \mathcal{M}(A \cup \{b, c\}) &\simeq \mathcal{M}(A)/\langle x, y \rangle \end{aligned}$$

*(note that the choice of $x$ and $y$ depends on both $b$ and $c$).*

Clearly, the choice of the modules $M(A)$ is only relevant up to isomorphism. A matroid over the ring $R$ will be more concisely referred also as an $R$-matroid. We now introduce some basic defintions and constructions of matroids over a ring.

**Definition 2.1.2.** *An $R$-matroid $\mathcal{M}$ on the ground set $E$ is* ***realizable*** *(or* ***representable****) if there exists a map $\psi : E \to \mathcal{M}(\emptyset)$ such that*

$$\mathcal{M}(A) \simeq \mathcal{M}(\emptyset)/\langle \psi(i) : i \in A \rangle \quad \text{for all } A \subseteq E.$$

*In this case, $\psi$ is a* realization *of $\mathcal{M}$.*

**Definition 2.1.3** (Direct sum)**.** *Let $\mathcal{M}$ and $\mathcal{M}'$ be two $R$-matroids on respective ground sets $E$ and $E'$. Their* ***direct sum*** *$\mathcal{M} \oplus \mathcal{M}'$ is an $R$-matroid on the ground set $E \amalg E'$ defined by*

$$(\mathcal{M} \oplus \mathcal{M}')(A \amalg A') = \mathcal{M}(A) \oplus \mathcal{M}'(A'), \quad \text{for all } A \subseteq E,\ A' \subseteq E'.$$

**Definition 2.1.4** (Deletion-contraction)**.** *Let $\mathcal{M}$ be an $R$-matroid on the ground set $E$ and let $i \in E$. The* ***deletion*** *$\mathcal{M} \setminus i$ and the* ***contraction*** *$\mathcal{M}/i$ of $i$ in $\mathcal{M}$ are $R$-matroids on the ground set $E \setminus i$ defined, for all $A \subseteq E \setminus i$, by*

$$(\mathcal{M} \setminus i)(A) = \mathcal{M}(A),$$
$$(\mathcal{M}/i)(A) = \mathcal{M}(A \cup \{i\}).$$

If $N$ is a finitely generated $R$-module, let the **empty matroid** for $N$ be the matroid over $R$ on the ground set $\emptyset$ which maps $\emptyset$ to $N$. By a **projective empty matroid** we mean an empty matroid for a projective module.

An $R$-matroid $\mathcal{M}$ is **essential** if no nontrivial projective module is a direct summand of $\mathcal{M}(E)$ (the term is adopted from the theory of hyperplane arrangements). The next lemma shows that very little is lost in restricting to essential matroids.

**Lemma 2.1.5.** *[FM16, Lemma 2.5] Every $R$-matroid $\mathcal{M}$ is the direct sum of an essential $R$-matroid $\mathcal{M}_E$ and a projective empty matroid $\mathcal{M}_P$:*

$$\mathcal{M} = \mathcal{M}_E \oplus \mathcal{M}_P. \tag{2.1}$$

*Proof.* Suppose that $\mathcal{M}$ is not essential. Then there is a projective $R$-module $P$ that is a direct summand of $\mathcal{M}(E)$. Then $P$ is a direct summand of every module $\mathcal{M}(A)$ since this property lifts back along the surjections $\mathcal{M}(A) \to \mathcal{M}(A \cup \{b\})$. Therefore $\mathcal{M}$ is the direct sum of another $R$-matroid $\mathcal{M}'$ and the empty matroid for $P$. Iterating this process, since $\mathcal{M}(E)$ is finitely generated, we will eventually express $\mathcal{M}$ as a direct sum of an essential matroid and a projective empty matroid. □

To avoid confusion, we will call matroids (in the sense of Chapter 1) *classical matroids.*

**Proposition 2.1.6.** *[FM16, Proposition 2.6] Let $\mathbb{K}$ be a field. Essential matroids over $\mathbb{K}$ corresponds to classical matroids, the correspondence is as follows. If $M = (E, \mathrm{rk})$ is a classical matroid then the function $\mathcal{M} : 2^E \to \mathbb{K}$-mod, given by $\mathcal{M}(A) = \mathbb{K}^{\mathrm{rk}(M)-\mathrm{rk}(A)}$ for every $A \subseteq E$, is a matroid over $\mathbb{K}$. Conversely, if $\mathcal{M}$ is a matroid over $\mathbb{K}$, the function* $\mathrm{rk} : 2^E \to \mathbb{N}$ *given by* $\mathrm{rk}(A) = \dim_{\mathbb{K}} \mathcal{M}(\emptyset) - \dim_{\mathbb{K}} \mathcal{M}(A)$ *is the rank function of a classical matroid.*

*Further, an essential matroid over $\mathbb{K}$ is realizable if and only if, as a classical matroid, is realizable over $\mathbb{K}$*

*Proof.* If $M = (E, \mathrm{rk})$ is a classical matroid, then $\mathcal{M}(A) = \mathbb{K}^{\mathrm{rk}(M)-\mathrm{rk}(A)}$ gives a matroid over $\mathbb{K}$, since for every $a, b \in E \setminus A$, the choice of elements $x, y \in \mathcal{M}(A) = \mathbb{K}^{\mathrm{rk}(M)-\mathrm{rk}(A)}$ in the definition is clear. Now suppose that $\mathcal{M}$ is a matroid over $\mathbb{K}$. Let $A \subseteq E$, and let $a \in A$, by the axiom of $\mathbb{K}$-matroid (with $a = b$), there exists $x \in \mathcal{M}(\emptyset)$ such that $\mathcal{M}(\{a\}) \simeq \mathcal{M}(\emptyset)/\langle a \rangle$. It follows that $\dim_{\mathbb{K}} \mathcal{M}(\emptyset) \le \dim_{\mathbb{K}} \mathcal{M}(\{a\}) + 1$. By iterating this argument with choices of elements in $A$, and since $A$ is finite, we will obtain the inequality

$$\dim_{\mathbb{K}} \mathcal{M}(\emptyset) \le \dim_{\mathbb{K}} \mathcal{M}(\{a\}) + 1 \le \cdots \le \dim_{\mathbb{K}} \mathcal{M}(A) + |A|,$$

from which it follows $\mathrm{rk}(A) = \dim_{\mathbb{K}} \mathcal{M}(\emptyset) - \dim_{\mathbb{K}} \mathcal{M}(A) \le |A|$. We also have a chain of surjective maps $\mathcal{M}(\emptyset) \to \mathcal{M}(\{a\}) \to \cdots \to \mathcal{M}(A)$, therefore $\dim_{\mathbb{K}} \mathcal{M}(\emptyset) \ge \dim_{\mathbb{K}} \mathcal{M}(A)$, that is $\mathrm{rk}(A) \ge 0$. This proves property $(R1)$ of Theorem 1.1.5. Property $(R2)$ follows from a similar argument. In fact, if $A \subseteq B$, as before, we have a chain of surjective maps $\mathcal{M}(A) \to \cdots \to \mathcal{M}(B)$, which implies that $\dim_{\mathbb{K}} \mathcal{M}(A) \ge \dim_{\mathbb{K}} \mathcal{M}(B)$, that is $\mathrm{rk}(A) \le \mathrm{rk}(B)$. To prove $(R3)$, let $a, b \in E \setminus A$, from the $\mathbb{K}$-matroid axiom we obtain

$$\dim_{\mathbb{K}} \mathcal{M}(A) + \dim_{\mathbb{K}} \mathcal{M}(A \cup \{a, b\}) \ge \dim_{\mathbb{K}} \mathcal{M}(A \cup \{a\}) + \dim_{\mathbb{K}} \mathcal{M}(A \cup \{b\}),$$

from which it follows $\mathrm{rk}(A) + \mathrm{rk}(A \cup \{a, b\}) \le \mathrm{rk}(A \cup \{a\}) + \mathrm{rk}(A \cup \{b\})$. An inductive use of the previous formula proves $(R3)$.

The statement about realizability follows from the definitions. □

Let $R \to S$ be a map of rings. Then every matroid over $S$ is naturally also a matroid over $R$. Furthermore, given such a map $R \to S$, the tensor product $- \otimes_R S$ is a functor $R$-mod $\to$ $S$-mod. One can use this to perform *base change* of matroids over $R$.

**Proposition-Definition 2.1.7** (Tensor product)**.** *[FM16, Proposition 2.7] Let $R \to S$ be an homomorphism of rings. If $\mathcal{M}$ is an $R$-matroid, define*

$$(\mathcal{M} \otimes_R S)(A) = \mathcal{M}(A) \otimes_R S \quad \text{for all } A \subseteq E.$$

*Then $\mathcal{M} \otimes_R S$ is a matroid over $S$.*

*Proof.* Since the tensor product is right exact (as a functor), it preserves cyclic kernels (quotient of cyclic modules are cyclic). Further, the tensor product is a left adjoin functor (of Hom), therefore it preserves pushouts, including the pushout from the definition of matrooids over a ring:

$$\begin{tikzcd}
\mathcal{M}(A) \arrow[r] \arrow[d] & \mathcal{M}(A \cup \{a\}) \arrow[d] \\
\mathcal{M}(A \cup \{b\}) \arrow[r] & \mathcal{M}(A \cup \{a, b\}).
\end{tikzcd}$$

□

Two particular cases of the tensor product are the following:

1. For every prime ideal $p$ of $R$, let $R_p$ be the localization of $R$ at $p$. We call

$$\mathcal{M}_p = \mathcal{M} \otimes_R R_p$$

the **localization** of $M$ at $p$.

2. Let $R$ be a domain, $Q(R)$ be its fraction field and let $\mathcal{M}$ be an $R$-matroid. From Lemma 2.1.5 we have that $\mathcal{M} = \mathcal{M}_E \oplus \mathcal{M}_P$, with $\mathcal{M}_E$ essential and $\mathcal{M}_P$ and empty projective matroid. The tensor product

$$\underline{\mathcal{M}} = \mathcal{M}_E \otimes_R Q(R)$$

is the **generic matroid** of $\mathcal{M}$.

Let $\mathcal{M}$ be a matroid over a domain $R$. Note that the generic matroid $\underline{\mathcal{M}}$ of $\mathcal{M}$ is an essential matroid over the field $Q(R)$, therefore, from Proposition 2.1.6, it corresponds to a classical matroid. This allows us to use the terminology of classical matroids (i.e. independent sets, rank, etc.) for a matroid $\mathcal{M}$ over a domain, by referring to its generic matroid $\underline{\mathcal{M}}$. The collection of independent sets of the generic matroid of $\mathcal{M}$ will be denoted by $\Delta\mathcal{M}$.

**Remark 2.1.8** (Duality)**.** *For a general ring $R$, it is hard to give a well posed definition of dual of a matroid over $R$. A duality theory for matroids over a Dedekind domain was developed in [FM16]. The details of the definition require some familiarity with homological algebra and are beyond the scope of this thesis. All we will need to know is that, for a Dedekind domain $R$, there is a well defined notion of* ***dual*** *$\mathcal{M}^*$ of a matroid $\mathcal{M}$ over $R$ [FM16, Definition 4.3], that it is an involution $\mathcal{M}^{**} = \mathcal{M}$ [FM16, Proposition 4.10], that behaves as expected under direct sums and minors [FM16, Proposition 4.9] and that in the realizable case it coincides with Gale duality [FM16, Proposition 4.8].*

## 2.2 Valuated matroids

In this section, we describe how a matroid over a discrete valuation ring (DVR) defines a valuated matroid. We have already seen valuated matroids in Section 1.6. Throughout this section, we will consider just *discrete* valuations $v : R \to \mathbb{Z} \cup \{\infty\}$, where $R$ is a DVR.

**Definition 2.2.1.** *A* ***valuated matroid*** *on the ground set $E$ of rank $r \in \mathbb{N}$, is a pair $M = (E, p)$ where $p : \binom{E}{r} \to \mathbb{Z} \cup \{\infty\}$ is a function defined on the family $\binom{E}{r}$ of subsets of $E$ of cardinality $r$, such that:*

1. $p(B) \neq \infty$ for some $B \in \binom{E}{e}$,

2. For every $A, B \in \binom{E}{r}$ and every $a \in A \setminus B$, there exists $b \in B \setminus A$ such that

$$p(A) + p(B) \geq p(A \cup \{b\} \setminus \{a\}) + p(B \cup \{a\} \setminus \{b\}).$$

**Corollary 2.2.2.** *[FM16, Corollary 5.9] Let $\mathcal{M}$ be a rank $r$ matroid on the ground set $E$, over a discrete valuation ring $R$ with maximal ideal $\mathfrak{m}$. The function $p : \binom{E}{r} \to \mathbb{Z} \cup \{\infty\}$ with $p(B) = \dim_{R/\mathfrak{m}} \mathcal{M}(B)$ defines a valuated matroid $M = (E, p)$.*

**Definition 2.2.3.** *A rank $r$ valuated matroid $M = (E, p)$ is **realizable** (or **representable**) over a discrete valuation ring $R$ with valuation $v : R \to \mathbb{Z} \cup \{\infty\}$, if there exists a matrix $A \in R^{r,n}$, where $n = |E|$, such that*

$$p(B) = v\big(\det(A[B])\big), \quad \textit{for all } B \in \binom{E}{r},$$

*where $A[B]$ is the square submatrix of $A$ with columns indexed by $B \subseteq E$.*

It is easy to see that, when a matroid over a DVR is realizable, so is its associated valuated matroid.

**Example 2.2.4.** *Let $R = k[[t]]$ the ring of formal power series with coefficients in a field $k$ over the variable $t$. The ring $R$ is a DVR with valuation $v : R \to \mathbb{Z} \cup \{\infty\}$ defined by*

$$v\left(\sum_{i \in \mathbb{N}} a_i t^i\right) = \min\{i \in \mathbb{N} : a_i \neq 0\},$$

*(where $v(0) = \infty$). Consider the following matrix*

$$A = \begin{bmatrix} t & 0 & 1 \\ 0 & t & t^2 \end{bmatrix} \in R^{2,3}.$$

*The matrix $A$ defines a rank 2 valuated matroid $M = (E, p)$ on the ground set $E = \{0, 1, 2\}$, where the function $p$ is defined by $p(B) = v(\det(A[B]))$ for every $B \subseteq E$ of cardinality 2. We have*

$$p(\{0,1\}) = v(t^2) = 2, \quad p(\{0,2\}) = v(t^3) = 3, \quad p(\{1,2\}) = v(-t) = 1,$$

*in particular the underlying matroid of $M$ is $\underline{M} = U_{2,3}$.*

*Now consider the matroid $\mathcal{M}$ over $R$ on the ground set $E$ defined by, for every $B \subseteq E$*

$$\mathcal{M}(B) = R^2 / \langle A[B] \rangle,$$

*where $\langle A[B] \rangle$ is the submodule of $R^2$ generated by the columns of $A$ indexed by $B$ (it is the zero module if $B = \emptyset$). The valuated matroid defined by $\mathcal{M}$ is exactly $M$, in particular the generic matroid of $\mathcal{M}$ is the same as the underlying matroid of $M$.*

Note that the structure of matroid over a DVR is richer than the structure of a valuated matroid. In fact, different matroids over a DVR can define the same valuated matroid. The same fact applies to (quasi)-arithmetic matroids, see Remark 2.3.3

## 2.3 Quasi-arithmetic matroids

**Definition 2.3.1.** *[DM13] A **quasi-arithmetic matroid** is a triple $\mathcal{A} = (E, \mathrm{rk}, m)$ where $M = (E, \mathrm{rk})$ is a matroid and $m : 2^E \to \mathbb{N}$ is a **multiplicity function** such that*

1. *For all $A \subseteq E$ and $e \in E$*

$$\begin{array}{ll} m(A \cup \{e\}) \textit{ divides } m(A) & \textit{if } \mathrm{rk}(A \cup \{e\}) = \mathrm{rk}(A), \\ m(A) \textit{ divides } m(A \cup \{e\}) & \textit{otherwise.} \end{array}$$

2. *If $A \subseteq B \subseteq E$ and $B$ is a disjoint union $B = A \cup F \cup T$ such that for all $A \subseteq C \subseteq B$ we have $\operatorname{rk}(C) = \operatorname{rk}(A) + |C \cap F|$, then*

$$m(A)\, m(B) = m(A \cup F)\, m(A \cup T).$$

Note that the condition on $A$ and $B$ is equivalent to saying that if we delete all the vectors in $E \setminus B$ and we contract all the vectors in $A$, we obtain a matroid consisting of loops $T$ and coloops $F$ only.

From the Fundamental theorem of finitely generated abelian groups, we know that every finitely generated $\mathbb{Z}$-module (abelian group) is the direct sum of free module and its torsion part (the submodule of torsion elements). This means that, if $\mathcal{M}$ is a matroid over $\mathbb{Z}$, we can decompose $\mathcal{M}(A)$ into the direct sum

$$\mathcal{M}(A) = \mathcal{M}(A)_{\text{proj}} \oplus \operatorname{tor}_{\mathcal{M}}(A),$$

where $\mathcal{M}(A)_{\text{proj}}$ is free (projective) and $\operatorname{tor}_{\mathcal{M}}(A)$ is the torsion part of $\mathcal{M}(A)$. Note that, if $\mathcal{M}$ is essential, the generic matroid is given by $\underline{\mathcal{M}}(A) = \mathcal{M}(A)_{\text{proj}} \otimes \mathbb{Q}$.

**Corollary 2.3.2.** *[FM16, Corollary 6.3] Let $\mathcal{M}$ be a matroid over $\mathbb{Z}$ on the ground set $E$. Consider the rank of the generic matroid*

$$\operatorname{rk}(A) = \dim_{\mathbb{Q}} \mathcal{M}(\emptyset) - \dim_{\mathbb{Q}} \mathcal{M}(A),$$

*and define the multiplicity function*

$$m(A) = |\operatorname{tor}_{\mathcal{M}}(A)|.$$

*The triple $(E, \operatorname{rk}, m)$ is a quasi-arithmetic matroid.*

**Remark 2.3.3.** *Note that, as for matroids over a DVR and valuated matroids, quasi-arithmetic matroids and matroids over $\mathbb{Z}$ are not equivalent. In fact, the information carried by a matroid over $\mathbb{Z}$ is richer: it retains isomorphism classes of torsion groups, not just their cardinalities.*

**Example 2.3.4.** *Consider the following matrix*

$$A = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 3 & 1 \end{pmatrix} \in \mathbb{Z}^{2,3}.$$

*The realizable matroid $\mathcal{M}$ on the ground set $E = \{0, 1, 2\}$ over $\mathbb{Z}$ associated to $A$ is*

$$\mathcal{M}(B) = \mathbb{Z}^2 / \langle A[B] \rangle \quad \text{for every } B \subseteq E.$$

*More explicitely, we have*

$$\begin{array}{ccc} \mathcal{M}(E) = 0 & & \\ \mathcal{M}(\{0,1\}) = \mathbb{Z}_2 \oplus \mathbb{Z}_3 & \mathcal{M}(\{0,2\}) = \mathbb{Z}_2 & \mathcal{M}(\{1,2\}) = \mathbb{Z}_3 \\ \mathcal{M}(\{0\}) = \mathbb{Z} \oplus \mathbb{Z}_2 & \mathcal{M}(\{1\}) = \mathbb{Z} \oplus \mathbb{Z}_3 & \mathcal{M}(\{2\}) = \mathbb{Z} \\ \mathcal{M}(\emptyset) = \mathbb{Z}^2 & & \end{array}$$

*The underlying generic matroid (seen as a classical matroid) is $\underline{\mathcal{M}} = U_{2,3}$. The multiplicity function $m$ defined by $\mathcal{M}$ results*

$$\begin{array}{ccc} m(E) = 1 & & \\ m(\{0,1\}) = 6 & m(\{0,2\}) = 2 & m(\{1,2\}) = 3 \\ m(\{0\}) = 2 & m(\{1\}) = 3 & m(\{2\}) = 1 \\ m(\emptyset) = 1 & & \end{array}$$

*the quasi-arithmetic matroid defined by $\mathcal{M}$ is $(U_{2,3}, m)$.*

### 2.3.1 Arithmetic Tutte polynomial

To every quasi-arithmetic matroid $\mathcal{A} = (E, \mathrm{rk}, m)$ we can associate a polynomial, called **arithmetic Tutte polynomial**

$$M_{\mathcal{A}}(x, y) = \sum_{A \subseteq E} m(A)(x-1)^{\mathrm{rk}(E)-\mathrm{rk}(A)}(y-1)^{|A|-\mathrm{rk}(A)}. \tag{2.2}$$

This polynomial has a deletion-contraction property [Moc12, Theorem 3.4] and has several applications to vector partition functions, toric arrangements and zonotopes [DM12].

## 2.4 Tutte-Grothendieck ring

In this section we will mainly follow [FM16, Section 7]. We will assume that $R$ is a Dedekind domain, so that we have a well defined notion of duality of $R$-matroids (Remark 2.1.8).

**Definition 2.4.1.** *We say that two matroids $\mathcal{M}_1$ and $\mathcal{M}_2$ over $R$ on the ground sets $E_1$ and $E_2$ respectively, are **isomorphic** if there exists a map $\sigma : E_1 \to E_2$ such that $\mathcal{M}_1(A) \simeq \mathcal{M}_2(\sigma(A))$ for every $A \subseteq E$.*

**Definition 2.4.2.** *The **Tutte-Grothendieck ring** $K(R\text{-}Mat)$ of matroids over $R$ is the ring generated (as a $\mathbb{Z}$-algebra) by the symbols $\boldsymbol{T}_{\mathcal{M}}$, one for each (isomorphism class of a) matroid $\mathcal{M}$ over $R$, modulo the relations*

$$\boldsymbol{T}_{\mathcal{M}} = \boldsymbol{T}_{\mathcal{M}\setminus e} + \boldsymbol{T}_{\mathcal{M}/e},$$
$$\boldsymbol{T}_{\mathcal{M}\oplus\mathcal{M}'} = \boldsymbol{T}_{\mathcal{M}}\boldsymbol{T}_{\mathcal{M}'},$$

*where $e$ is not a loop or coloop (of the generic matroid).*

The preceding relations mimic the deletion-contraction property (Theorem 1.5.4) and the property of the Tutte polynomial with respect to direct sums (Proposition 1.5.12).

Now let $\mathbb{Z}[R\text{-Mod}]$ be the ring generated (as a $\mathbb{Z}$-algebra) by the symbols $X^N$ (or alternatively $[N]$), one for each (isomoprhism class of a) finitely generated $R$-module $N$, modulo the relations $X^N X^{N'} = X^{N\oplus N'}$ (or, alternatively, $[N]\cdot[N'] = [N \oplus N']$).

**Theorem 2.4.3.** *[FM16, Theorem 7.1] The Tutte-Grothendieck ring $K(R\text{-}Mat)$ injects into*

$$\mathbb{Z}[R\text{-}Mod] \otimes_{\mathbb{Z}} \mathbb{Z}[R\text{-}Mod],$$

*in such a way that for every matroid $\mathcal{M}$ over $R$ on the ground set $E$, we have*

$$\boldsymbol{T}_{\mathcal{M}} \mapsto \sum_{A\subseteq E} X^{\mathcal{M}(A)} Y^{\mathcal{M}^*(E\setminus A)},$$

*where $\{X^N\}$ and $\{Y^N\}$ are the respective bases of the two tensor factors $\mathbb{Z}[R\text{-}Mod]$.*

If $R$ is a field, consider the isomorphism $\mathbb{Z}[R\text{-Mod}] \simeq \mathbb{Z}[x]$ given by $X^{R^d} \mapsto (x-1)^d$. For an essential matroid $\mathcal{M}$ over the field $R$, the image of $\mathbf{T}_{\mathcal{M}}$ into $\mathbb{Z}[R\text{-Mod}] \otimes \mathbb{Z}[R\text{-Mod}] \simeq \mathbb{Z}[x, y]$ coincides with the classical Tutte polynomial:

$$\mathbf{T}_{\mathcal{M}} \mapsto \sum_{A\subseteq E} X^{\mathcal{M}(A)} Y^{\mathcal{M}^*(E\setminus A)} \simeq \sum_{A\subseteq E} (x-1)^{\mathrm{rk}(E)-\mathrm{rk}(A)}(y-1)^{\mathrm{rk}^*(E)-\mathrm{rk}^*(E\setminus A)} = T_{\mathcal{M}}(x, y).$$

If $R = \mathbb{Z}$, consider the ring homomorphism $\mathbb{Z}[R\text{-Mod}] \to \mathbb{Z}[x]$ given by $X^{\mathbb{Z}^d} \mapsto (x-1)^d$, $X^G \mapsto |G|$ for every finite abelian group $G$. Then, for a matroid $\mathcal{M}$ over $\mathbb{Z}$, the image under this homomorphism of $\mathbf{T}_{\mathcal{M}}$ (as an element of $\mathbb{Z}[R\text{-Mod}] \otimes \mathbb{Z}[R\text{-Mod}]$) is the arithmetic Tutte polynomial (2.2) (see [FM16, Section 7.1]).

# Chapter 3

# Matroids over a domain

In this chapter we are going to study matroids over a domain and generalize some theorems of classical matroids. In order to do so, in the first section we first recall some notions about face rings of matroids and more generally simplicial complexes and simplicial posets.

## 3.1 Face rings

### 3.1.1 Simplicial complexes

**Definition 3.1.1.** *An (abstract)* ***simplicial complex*** $\Delta$ *on the vertex set* $[n] = \{1, \dots, n\}$ *is a collection of subsets of* $[n]$*, called* ***faces****, closed under taking subsets, that is, if* $A \subseteq B \in \Delta$*, then* $A \in \Delta$*. The dimension of a face* $\sigma \in \Delta$ *is the number* $|\sigma| - 1$*. The* ***dimension*** $\dim \Delta$ *of* $\Delta$ *is the maximum of the dimensions of its faces, or it is* $-\infty$ *if* $\Delta = \emptyset$ *has no faces.*

Note that, since the independent sets of a matroid are closed under taking subsets, matroids can be seen as simplicial complexes with extra properties.

Now let $\mathbb{K}$ be a field and consider the polynomial ring $S = \mathbb{K}[x_1, \dots, x_n]$. Denote by $\mathbf{x}^\sigma = \prod_{i \in \sigma} x_i$ for $\sigma \subseteq [n]$.

**Definition 3.1.2.** *Let* $\Delta$ *be a simplicial complex. The* ***face ring*** *(or* ***Stanley-Reisner ring****) of* $\Delta$ *is the quotient* $\mathbb{K}[\Delta] = S/I_\Delta$*, where*

$$I_\Delta = \langle x^\sigma : \sigma \notin \Delta \rangle$$

**Definition 3.1.3.** *Let* $\Delta$ *be a simplicial complex and let* $d = \dim \Delta + 1$*. The* ***f-vector*** *of* $\Delta$ *is* $(f_{-1}, f_0, \dots, f_{d-1})$ *where* $f_i$ *is the number of faces of* $\Delta$ *of dimension* $i$*. The* ***h-vector*** *of* $\Delta$ *is the vector* $(h_0, h_1, \dots, h_d)$ *where the integers* $h_i$ *are defined by the polynomial identity* $\sum_{i=0}^{d} h_i t^i = \sum_{i=0}^{d} f_{i-1} t^i (1-t)^{d-i}$.

**Definition 3.1.4.** *Let* $A$ *be a finitely generated* $\mathbb{N}$*-graded* $\mathbb{K}$*-algebra, and let* $N$ *be a finitely generated graded* $A$*-module. Denote by* $N_i$ *the homogeneous part of degree* $i$*. Since* $N$ *is finitely generated,* $N_i$ *is a finitely generated* $\mathbb{K}$*-vector space, and we denote its dimension with* $\dim_{\mathbb{K}} N_i$*. The* ***Hilbert series*** *of* $N$ *is*

$$H(N, t) = \sum_{i \in \mathbb{N}} \dim_{\mathbb{K}}(N_i) t^i.$$

There is a nice relation between the Hilbert series of the face ring of a simplicial complex and its $h$-vector.

**Theorem 3.1.5.** *[MS05, Corollary 1.15] Let* $\Delta$ *be a simplicial complex with face ring* $\mathbb{K}[\Delta]$ *and* $h$*-vector* $(h_0, \dots, h_d)$*. Then*

$$H(\mathbb{K}[\Delta], t) = \frac{h_0 + h_1 t + \cdots + h_d t^d}{(1-t)^d}.$$

Since the collection of independent sets of a matroid $M$ is a simplicial complex, we can talk about the face ring $\mathbb{K}[M]$ (later we will adopt the notation $A_M$) of a matroid $M$. The following result relates the Hilbert series of the face ring $A_M$ and the Tutte polynomial of $M$.

**Theorem 3.1.6** (Appendix by Björner in [DCP08])**.** *Let $M$ be a matroid of rank $r$ on the ground set $E$. Then*

$$H(A_M, t) = \frac{t^r}{(1-t)^r} T_M(1/t, 1).$$

### 3.1.2 Simplicial posets

The results and definitions of the previous subsection can be generalized from simplicial complexes to a classes of posets, called simplicial posets.

**Definition 3.1.7.** *A lattice $(P, \vee, \wedge)$ is **boolean** if it is **distributive** (i.e. $\vee$ and $\wedge$ satisfy the distributive law), it has a minimum $\hat{0}$ and a maximum $\hat{1}$, and every element $a \in P$ has a (necessarily unique) **complement**, that is, an element $a' \in P$ such that $a \vee a' = \hat{1}$ and $a \wedge a' = \hat{0}$. In a boolean lattice $P$, all maximal chains have the same length, and this number is the **rank** of $P$.*

A typical example of a boolean lattice is the power set $2^X$ of some set $X$, ordered by inclusion.

**Definition 3.1.8.** *A **simplicial poset** is a poset $(P, \leq)$ with a minimum element $\hat{0}$ and such that for every $a \in P$ the segment $[\hat{0}, a] = \{b \in P : \hat{0} \leq b \leq a\}$ is a boolean lattice. The **rank** $\operatorname{rk}(a)$ of an element $a$ is the rank of the boolean lattice $[\hat{0}, a]$. The maximum rank of all the elements of $P$ is the **rank** of the simplicial poset $P$, denoted by* $\operatorname{rk}(P)$.

Following [Sta91], we now define a face ring for a simplicial poset $P$. For every $a, b \in P$, let $M(a, b)$ denote the set of *minimal* upper bounds of $\{a, b\}$. In general, $M(a, b)$ may be infinite, but we will be concerned with the case when $M(a, b)$ is finite for all $a, b \in P$. Let $\mathbb{K}$ be a field and consider the $\mathbb{K}$-algebra $\mathbb{K}[x_a : a \in P]$ graded by $\deg(x_a) = \operatorname{rk}(a)$ for all $a \in P$.

**Definition 3.1.9.** *The **face ideal** of a simplicial poset $P$ is the ideal in $\mathbb{K}[x_a : a \in P]$ defined by*

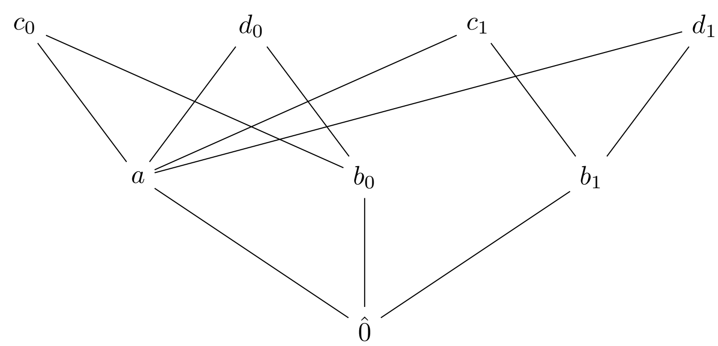
$$I_P = \left( x_{\hat{0}} - 1,\ x_a x_b - x_{a \wedge b} \left( \sum_{c \in M(a,b)} x_c \right) : a, b \in P \right),$$

*where $x_{a \wedge b} = 0$ if $a \wedge b$ does not exist, and $\sum_{c \in M(a,b)} x_c = 0$ if $M(a, b) = \emptyset$. The **face ring** (or **Stanley-Reisner ring**) of $P$ is the quotient*

$$A_P = \frac{\mathbb{K}[x_a : a \in P]}{I_P}.$$

The preceding definition generalizes the notion of Stanley-Reisner ring of a simplicial complex. In fact, if the (finite) simplicial poset $P$ is in addition a meet-semilattice, then $P$ is the poset of faces of some simplicial complex $\Delta$, and the face ring of $P$ coincides with the face ring of $\Delta$ (see [Sta91] or [Sta96, Chapter 3, Section 6]).

**Example 3.1.10.** *Consider the poset $P = \{\hat{0}, a, b_0, b_1, c_0, c_1, d_0, d_1\}$ where $\hat{0}$ is the minimum, $a \leq c_i, d_i$ and $b_i \leq c_i, d_i$ for every $i \in \{0, 1\}$.*




*The poset $P$ is a simplicial poset, and its face ring is*

$$A_P = \frac{\mathbb{K}[x_\alpha : \alpha \in P]}{I_P} \simeq \frac{\mathbb{K}[x_a, x_{b_0}, x_{b_1}, x_{c_0}, x_{c_1}, x_{d_0}, x_{d_1}]}{\left( \begin{array}{l} x_a x_{b_i} - (x_{c_i} + x_{d_i}), x_{b_0} x_{b_1}, \\ x_{c_i} x_{d_j}, x_{c_0} x_{c_1}, x_{d_0} x_{d_1}, \\ x_{b_i} x_{c_{\bar{i}}}, x_{b_i} x_{d_{\bar{i}}} \end{array} : \begin{array}{l} i, j \in \{0,1\} \\ \bar{i} = 1 - i \end{array} \right)}$$

**Definition 3.1.11.** *Let $P$ be a finite simplicial poset of rank $r$, and let $f_i$ be the number of elements of $P$ of rank $i+1$. The **$f$-vector** of $P$ is the vector $(f_{-1}, f_0, \dots, f_{r-1})$. The **$h$-vector** of $P$ is the vector $(h_0, \dots, h_r)$ defined by the polynomial identity $\sum_{i=0}^r f_{i-1}(t-1)^{r-i} = \sum_{i=0}^r h_i t^{r-i}$.*

**Theorem 3.1.12** (Stanley [Sta91, Proposition 3.8])**.** *Let $P$ be a finite simplicial poset of rank $r$ and $h$-vector $(h_0, \dots, h_r)$. With the grading of the face ring $A_P$ given by $\deg(x_a) = \operatorname{rk}(a)$ for all $a \in P$, we have*

$$H(A_P, t) = \frac{h_0 + h_1 t + \cdots + h_r t^r}{(1-t)^r}.$$

## 3.2 The Tutte-Grothendieck polynomial

We now give a new and more general definition of Tutte-Grothendieck polynomial for a matroid over a domain. We will prove that this polynomial satisfies the usual deletion-contraction property. In this section, $R$ is a domain with field of fractions $Q(R)$, and $\mathcal{M}$ is an $R$-matroid of rank $r$ on the ground set $E$.

**Definition 3.2.1.** *Let $M$ be an $R$-module, an element $x \in M$ is a **torsion element** if it has non trivial annihilator, that is $\operatorname{Ann}(x) = \{r \in R : rx = 0\} \neq 0$. The set of torsion elements is the **torsion part** of $M$:*

$$\operatorname{tor} M = \{x \in M : \exists r \in R, r \neq 0 \, rx = 0\}.$$

*Since $R$ is a domain, $\operatorname{tor} M$ is a submodule of $M$. If $\operatorname{tor} M = 0$, then $M$ is **torsion-free**.*

We recall some notations used on Chapter 2. For every subset $A \subseteq E$ the torsion part of $\mathcal{M}(A)$ is $\operatorname{tor}_{\mathcal{M}}(A)$, and the rank (associated to the generic matroid) is $\operatorname{rk}_{\mathcal{M}}(A)$. Further, we let $\vee$ denote the application of the variant functor $\operatorname{Hom}(-, Q(R)/R)$, so that $\operatorname{tor}(A)^\vee = \operatorname{Hom}(\operatorname{tor}(A), Q(R)/R)$. This functor mimics a generalization of the notion of *Pontryagin duality*, from $\mathbb{Z}$-modules (abelian groups) to $R$-modules, for any domain $R$, see also Remark 3.3.3.

**Definition 3.2.2.** *We define the **Tutte-Grothendieck polynomial** for the matroid $\mathcal{M}$ over a domain $R$ as the following polynomial with coefficients in $\mathbb{Z}[R\text{-mod}]$:*

$$T_{\mathcal{M}}(x, y) = \sum_{A \subseteq E} [\operatorname{tor}(A)^\vee](x-1)^{r - \operatorname{rk}(A)}(y-1)^{|A| - \operatorname{rk}(A)} \in \mathbb{Z}[R\text{-}Mod][x, y],$$

*(where $[\operatorname{tor}(A)^\vee]$ is the element in $\mathbb{Z}[R\text{-}Mod]$ corresponding to the isomorphism class of the $R$-module $\operatorname{tor}(A)^\vee$).*

The main result of this section is the following deletion-contraction property of the Tutte-Grothendieck polynomial.

**Theorem 3.2.3** (Deletion-contraction property)**.** *[BM19, Theorem 2.8] Let $R$ be a domain and $\mathcal{M}$ be an $R$-matroid of rank $r$ on the ground set $E$. If $\mathcal{M}(\emptyset)$ is torsion-free and if $\mathcal{M}(E) = 0$, then*

$$T_{\mathcal{M}}(x, y) = \begin{cases} yT_{\mathcal{M}\setminus i}(x, y) & \text{if } i \text{ is a loop,} \\ xT_{\mathcal{M}/i}(x, y) & \text{if } i \text{ is a coloop,} \\ T_{\mathcal{M}\setminus i}(x, y) + T_{\mathcal{M}/i}(x, y) & \text{otherwise.} \end{cases}$$

The proof of this result will be divided into three steps: the case when $i$ is not a loop or a coloop (Theorem 3.2.4), the case when $i$ is a loop (Corollary 3.2.7) and the case when $i$ is a coloop (Corollary 3.2.9).

First, we decompose the Tutte-Grothendieck polynomial in the following way:

$$\begin{aligned} T_{\mathcal{M}}(x, y) = &\sum_{A \subseteq E\setminus i} [\mathrm{tor}(A)^\vee](x-1)^{r-\mathrm{rk}(A)}(y-1)^{|A|-\mathrm{rk}(A)} + \\ &+ \sum_{A \subseteq E\setminus i} [\mathrm{tor}(A \cup \{i\})^\vee](x-1)^{r-\mathrm{rk}(A\cup\{i\})}(y-1)^{|A\cup\{i\}|-\mathrm{rk}(A\cup\{i\})} \end{aligned} \tag{3.1}$$

The sum runs over all subsets $A$ in $E \setminus i$, where the latter means $E \setminus \{i\}$. We will keep this abuse of notation all along this section.

**Theorem 3.2.4.** *Let $R$ be a domain and $\mathcal{M}$ be an $R$-matroid of rank $r$ on the ground set $E$. If $i \in E$ is not a loop or coloop, then*

$$T_{\mathcal{M}}(x, y) = T_{\mathcal{M}\setminus i}(x, y) + T_{\mathcal{M}/i}(x, y).$$

*Proof.* Since $i \in E$ is not a loop or coloop, as for classical matroids, for every $A \subseteq E \setminus i$ we have

$$\begin{aligned} \mathrm{rk}_{\mathcal{M}}(A) &= \mathrm{rk}_{\mathcal{M}\setminus i}(A), \\ \mathrm{rk}_{\mathcal{M}}(A \cup \{i\}) &= \mathrm{rk}_{\mathcal{M}/i}(A) + 1. \end{aligned}$$

Futher, by definition

$$\begin{aligned} \mathrm{tor}_{\mathcal{M}\setminus i}(A) &= \mathrm{tor}_{\mathcal{M}}(A), \\ \mathrm{tor}_{\mathcal{M}/i}(A) &= \mathrm{tor}_{\mathcal{M}}(A \cup \{i\}). \end{aligned}$$

Finally by substituting in (3.1) we obtain

$$T_{\mathcal{M}}(x, y) = T_{\mathcal{M}\setminus i}(x, y) + T_{\mathcal{M}/i}(x, y).$$ □

**Lemma 3.2.5.** *Let $M$ be a torsion-free $R$-module, then $M \otimes_R Q(R) = 0$ implies $M = 0$.*

*Proof.* Let $S = R \setminus \{0\}$ the multiplicative set of $R$. Since $M$ is torsion-free, $\varphi : M \to S^{-1}M$ defined by $\varphi(m) = m/1$ is injective. Hence

$$M \subseteq S^{-1}M \simeq M \otimes_R Q(R) = 0.$$ □

**Proposition 3.2.6.** *If $\mathcal{M}(\emptyset)$ is torsion-free and $i \in E$ is a loop, then for every $A \subseteq E \setminus i$ we have $\mathcal{M}(A) \simeq \mathcal{M}(A \cup \{i\})$, in particular* $\mathrm{tor}(A) \simeq \mathrm{tor}(A \cup \{i\})$.

*Proof.* First, write $\mathcal{M} = \mathcal{M}_E \oplus \mathcal{M}_P$, where $\mathcal{M}_E$ is an essential matroid and $\mathcal{M}_P$ is a projective empty matroid with $\mathcal{M}_P(\emptyset) = P$. We proceed by induction on the cardinality of $A$. For the base case, since $i \in E$ is a loop, we have

$$\mathcal{M}_E(\emptyset) \otimes Q(R) \simeq \mathcal{M}_E(i) \otimes Q(R).$$

By definition of matroid, $\mathcal{M}_E(i) \simeq \mathcal{M}_E(\emptyset)/(x)$ for some $x \in \mathcal{M}_E(\emptyset)$. Since $Q(R)$ is a flat $R$-module, it follows

$$\mathcal{M}_E(\emptyset) \otimes Q(R) \simeq \left(\frac{\mathcal{M}_E(\emptyset)}{(x)}\right) \otimes Q(R) \simeq \frac{\mathcal{M}_E(\emptyset) \otimes Q(R)}{(x) \otimes Q(R)},$$

from [Eis95, Corollary 4.4] the natural projection

$$\mathcal{M}_E(\emptyset) \otimes Q(R) \twoheadrightarrow \frac{\mathcal{M}_E(\emptyset) \otimes Q(R)}{(x) \otimes Q(R)}$$

is an isomorphism, so its kernel is zero $(x) \otimes Q(R) = 0$. Since $\mathcal{M}(\emptyset)$ is torsion-free, then also $(x)$ is torsion-free, so from Lemma 3.2.5 it follows that $(x) = 0$ implies $x = 0$. Hence $\mathcal{M}_E(\emptyset) \simeq \mathcal{M}_E(i)$, so

$$\mathcal{M}(\emptyset) = \mathcal{M}_E(\emptyset) \oplus P \simeq \mathcal{M}_E(i) \oplus P = \mathcal{M}(i).$$

Now for the inductive step, let $A \subseteq E \setminus i$ and $a \in A$. By definition of matroid there exist $y, z \in \mathcal{M}(A \setminus a)$ such that

$$\begin{aligned} \mathcal{M}(A) &\simeq \mathcal{M}(A \setminus a)/(y) \\ \mathcal{M}((A \cup \{i\}) \setminus a) &\simeq \mathcal{M}(A \setminus a)/(z) \\ \mathcal{M}(A \cup \{i\}) &\simeq \mathcal{M}(A \setminus a)/(y, z). \end{aligned}$$

By the inductive hypothesis, we have $\mathcal{M}(A \setminus a) \simeq \mathcal{M}(A \setminus a)/(z)$. Again from [Eis95, Corollary 4.4] the natural projection $\mathcal{M}(A \setminus a) \twoheadrightarrow \mathcal{M}(A \setminus a)/(z)$ is an isomorphism, so its kernel is zero, so $z = 0$. Now it easily follows $\mathcal{M}(A) \simeq \mathcal{M}(A \cup \{i\})$. □

**Corollary 3.2.7.** *If $\mathcal{M}(\emptyset)$ is torsion-free and $i \in E$ is a loop, then*

$$T_{\mathcal{M}}(x, y) = y T_{\mathcal{M} \setminus i}(x, y)$$

*Proof.* From Proposition 3.2.6, $[\mathrm{tor}(A)^\vee] = [\mathrm{tor}(A \cup \{i\})^\vee]$, further

$$\mathrm{rk}_{\mathcal{M}}(A \cup \{i\}) = \mathrm{rk}_{\mathcal{M}}(A) = \mathrm{rk}_{\mathcal{M} \setminus i}(A).$$

Hence, by substituting in (3.1) we obtain

$$T_{\mathcal{M}}(x, y) = T_{\mathcal{M} \setminus i}(x, y) + (y - 1) T_{\mathcal{M} \setminus i}(x, y) = y T_{\mathcal{M} \setminus i}(x, y).$$ □

**Proposition 3.2.8.** *If $\mathcal{M}(E) = 0$ and $i \in E$ is a coloop, then for every $A \subseteq E \setminus i$ we have $\mathcal{M}(A) \simeq \mathcal{M}(A \cup \{i\}) \oplus R$, in particular* $\mathrm{tor}(A) \simeq \mathrm{tor}(A \cup \{i\})$.

*Proof.* Note that $\mathcal{M}$ is an essential matroid. We want to prove the statement by induction on the cocardinality, $n - |A|$. For the base case, consider $E \setminus i$, by definition of matroid, we have $\mathcal{M}(E \setminus i)/(x) \simeq \mathcal{M}(E) = 0$ for some $x \in \mathcal{M}(E \setminus i)$. Since $i \in E$ is a coloop, we have

$$Q(R) \simeq \mathcal{M}(E \setminus i) \otimes Q(R) \simeq (x) \otimes Q(R),$$

so $x \in \mathcal{M}(E \setminus i)$ is not a torsion element, therefore $R \simeq (x) \simeq \mathcal{M}(E \setminus i)$. For the inductive step, let $A \subseteq E \setminus i$ and $a \in A$. By definition of matroid, there exist $y, x \in \mathcal{M}(A \setminus a)$ such that

$$\begin{aligned} \mathcal{M}(A) &\simeq \mathcal{M}(A \setminus a)/(y) \\ \mathcal{M}((A \cup \{i\}) \setminus a) &\simeq \mathcal{M}(A \setminus a)/(z) \\ \mathcal{M}(A \cup \{i\}) &\simeq \mathcal{M}(A \setminus a)/(y, z). \end{aligned}$$

By the inductive hypothesis, $\mathcal{M}(A) \simeq \mathcal{M}(A \cup \{i\}) \oplus R \simeq \mathcal{M}(A)/(\overline{z}) \oplus R$, in particular, $\overline{z}$ (and also $z$) is not a torsion element. Otherwise, tensoring by $Q(R)$, which is a flat $R$-module, we would have

$$\mathcal{M}(A) \otimes Q(R) \simeq \frac{\mathcal{M}(A) \otimes Q(R)}{(\overline{z}) \otimes Q(R)} \oplus Q(R) \simeq \big(\mathcal{M}(A) \otimes Q(R)\big) \oplus Q(R),$$

which is a contradiction. Hence $(z) \simeq (\overline{z}) \simeq R$ and $\mathcal{M}(A) \simeq \mathcal{M}(A)/(\overline{z}) \oplus (\overline{z})$, so the second row of the following diagram splits

$$\begin{tikzcd}
0 \arrow[r] & (z) \arrow[r] \arrow[d, leftrightarrow, "\simeq"] & \mathcal{M}(A \setminus a) \arrow[r] \arrow[d, two heads] & \mathcal{M}(A \setminus a)/(z) \arrow[r] \arrow[d, two heads] & 0 \\
0 \arrow[r] & (\overline{z}) \arrow[r, hook, shift right] & \mathcal{M}(A) \arrow[r] \arrow[l, dashed, shift right] & \mathcal{M}(A)/(\overline{z}) \arrow[r] & 0
\end{tikzcd}$$

further, since the composition $(z) \hookrightarrow \mathcal{M}(A \setminus a) \twoheadrightarrow \mathcal{M}(A) \dashrightarrow (\overline{z}) \simeq (z)$ is the identity on $(z)$, also the first row splits, so

$$\mathcal{M}(A \setminus a) \simeq \mathcal{M}(A \setminus a)/(z) \oplus (z) \simeq \mathcal{M}((A \cup \{i\}) \setminus a) \oplus R. \qquad \square$$

**Corollary 3.2.9.** *If $\mathcal{M}(E) = 0$ and $i \in E$ is a coloop, then*

$$T_{\mathcal{M}}(x, y) = xT_{\mathcal{M}/i}(x, y)$$

*Proof.* From Proposition 3.2.8, $[\mathrm{tor}(A)^\vee] = [\mathrm{tor}(A \cup \{i\})^\vee]$, further

$$\mathrm{rk}_{\mathcal{M}}(A \cup \{i\}) = \mathrm{rk}_{\mathcal{M}}(A) + 1 = \mathrm{rk}_{\mathcal{M}/i}(A) + 1.$$

Hence, by substituting in (3.1) we obtain

$$T_{\mathcal{M}}(x, y) = (x - 1)T_{\mathcal{M}/i}(x, y) + T_{\mathcal{M}/i}(x, y) = xT_{\mathcal{M}/i}(x, y). \qquad \square$$

### 3.2.1 The Grothendieck $f$-vector

We can give the corresponding notion of $f$-vector for matroids over a domain. Recall that $\Delta\mathcal{M}$ denotes the collection of independent sets of the generic matroid of $\mathcal{M}$.

**Definition 3.2.10.** *Let $\mathcal{M}$ be a matroid over a domain $R$. Define*

$$f_{i-1} = \sum_{\substack{A \in \Delta\mathcal{M}, \\ |A| = i}} [\mathrm{tor}(A)^\vee] \in \mathbb{Z}[R\text{-}Mod].$$

*The **Grothendieck $f$-vector** of the $R$-matroid $\mathcal{M}$, is the vector $(f_{-1}, f_0, \dots, f_{r-1}) \in \mathbb{Z}[R\text{-}Mod]^{r+1}$.*

The definition of Grothendieck $f$-vector of a matroid $\mathcal{M}$ over a domain, is inspired by a certain simplicial poset associated to $\mathcal{M}$, the *poset of torsions*, that we will discuss in Section 3.3. One can also define the prototype of the $h$-vector by using the usual polynomial identity

$$\sum_{i=0}^{r} h_i t^i = \sum_{i=0}^{r} f_{i-1} t^i (1 - t)^{r-i} \in \mathbb{Z}[R\text{-Mod}][t].$$

From the definitions, it is easy to see that the following relation holds

$$T_{\mathcal{M}}(t, 1) = \sum_{i=0}^{r} f_{i-1}(t - 1)^{r-i}.$$

## 3.3 Poset of torsions

We are going to define an independence poset, called the *poset of torsions*, of a matroid over a domain. In this section, $R$ is a domain with field of fractions $Q(R)$, and $\mathcal{M}$ is a realizable $R$-matroid on the ground set $E$, with a fixed realization $\psi : E \to \mathcal{M}(\emptyset)$. Recall that $\Delta\mathcal{M}$ is the collection of independent sets of the generic matroid of $\mathcal{M}$.

**Lemma 3.3.1.** *Let $M$ be an $R$-module. If $x \in M$ is not a torsion element and $\pi : M \to M/(x)$ is the natural surjection, then $\pi_{|\operatorname{tor} M}$ is injective.*

*Proof.* Let $y \in \ker \pi$ then either $y = 0$ or $y$ is not a torsion element. In fact, if $y \in \ker \pi = (x)$ and $y \neq 0$, then there exists $r \in R$, $r \neq 0$ such that $y = rx$, so

$$\operatorname{Ann}(y) \subseteq \operatorname{Ann}(x) = 0.$$

□

**Remark 3.3.2.** *Recall from Section 1.2 that, if $(P, \leq)$ is a partially ordered set and $a, b \in P$, we say that $b$* ***covers*** *$a$, written $a \lhd b$, if $a < b$ and there is no element $c \in P$ such that $a < c < b$. We can define a partial order just from the covering relation by declaring $a \leq b$ whenever there is an integer $n \in \mathbb{N}$ and elements $a_0, a_1, \ldots, a_n \in P$ such that there is a* chain *of covers*

$$a = a_0 \lhd a_1 \lhd \cdots \lhd a_{n-1} \lhd a_n = b.$$

Let $A$ be a subset of $E$ and $b \in E \setminus A$ such that $A \cup \{b\} \in \Delta\mathcal{M}$. Then, of course, $A \in \Delta\mathcal{M}$ and $\psi(b) \in \mathcal{M}(\emptyset)$ is not a torsion element. By Definition 2.1.1 there is the quotient map

$$\overline{\pi_{A,b}} : \mathcal{M}(A) \to \mathcal{M}(A \cup \{b\}) \simeq \mathcal{M}(A)/(\psi(b)).$$

We will denote the torsion part of $\mathcal{M}(A)$ by $\operatorname{tor}_{\mathcal{M}} A$, when there is no ambiguity, simply by $\operatorname{tor} A$. If we restrict $\overline{\pi_{A,b}}$ to $\operatorname{tor} A$, from Lemma 3.3.1, we obtain an injective map denoted by

$$\pi_{A,b} : \operatorname{tor}(A) \to \operatorname{tor}(A \cup \{b\}).$$

Recall that $\vee$ denotes the application of the contravariant functor $\operatorname{Hom}(-, {}^{Q(R)}/_{R})$, so that $\operatorname{tor}(A)^\vee = \operatorname{Hom}(\operatorname{tor}(A), {}^{Q(R)}/_{R})$. We obtain the sujective map

$$\pi_{A,b}^\vee : \operatorname{tor}(A \cup \{b\})^\vee \twoheadrightarrow \operatorname{tor}(A)^\vee.$$

**Remark 3.3.3.** *Since $R$ is a domain,* $\operatorname{tor}\mathcal{M}(A)$ *is* $\operatorname{Tor}_1^R(\mathcal{M}(A), {}^{Q(R)}/_{R})$*, see [Wei94, Exercise 3.1.2]. Moreover,* $\operatorname{tor}(A)^\vee = \operatorname{Hom}(\operatorname{tor}(A), {}^{Q(R)}/_{R})$ *is usually called the* full dual *of the $R$-module* $\operatorname{tor}(A)^\vee$ *and, by definition, this can also be written as* $\operatorname{Ext}_R^0(\operatorname{tor}\mathcal{M}(A), {}^{Q(R)}/_{R})$.

The next definition generalizes the poset of torsions introduced in [Mar18] for matroids over $\mathbb{Z}$.

**Definition 3.3.4.** *Let $\mathcal{M}$ be a realizable matroid over $R$, set*

$$\operatorname{Gr}\mathcal{M} = \{(A, l) : A \in \Delta\mathcal{M},\ l \in \operatorname{tor}(A)^\vee\}.$$

*Now, define a partial order on $\operatorname{Gr}\mathcal{M}$ by providing just the covering relations (Remark 3.3.2) as follows. If $(A \cup \{b\}, h), (A, l) \in \operatorname{Gr}\mathcal{M}$, then we set*

$$(A, l) \lhd (A \cup \{b\}, h) \overset{def}{\iff} \pi_{A,b}^\vee(h) = l.$$

*The poset $\operatorname{Gr}\mathcal{M}$ is the* ***poset of torsions*** *of $\mathcal{M}$.*

The following example shows that the poset of torsions $\operatorname{Gr}\mathcal{M}$ may be infinite.

**Example 3.3.5.** *Set $R = \mathbb{Q}[x]$ and let $\mathcal{M}$ be the $\mathbb{Q}[x]$-matroid on the ground set $\{1\}$ defined by*

$$\mathcal{M}(\emptyset) = \mathbb{Q}[x], \quad \mathcal{M}(\{1\}) = {}^{\mathbb{Q}[x]}/_{(x)} \simeq \mathbb{Q}.$$

*Then the poset of torsions $\operatorname{Gr}\mathcal{M} = \{(\emptyset, e), (\{1\}, q) : q \in \mathbb{Q}\}$ is infinite and as a poset every element $(\{1\}, q)$ covers $(\emptyset, e)$, while $(\{1\}, q)$ and $(\{1\}, q')$ are incomparable if $q \neq q'$.*

**Proposition 3.3.6.** *[BM19, Proposition 4.4] Let $\mathcal{M}$ be a realizable matroid over $R$ and let $\psi : E \to \mathcal{M}(\emptyset)$ be a realization of $\mathcal{M}$. For every $(A, h) \in \operatorname{Gr}\mathcal{M}$ and $B \subseteq A$ there exists a unique $l \in \operatorname{tor}(B)^\vee$ such that $(B, l) \leq (A, h)$ in $\operatorname{Gr}\mathcal{M}$.*

*Proof.* We start by showing the existence. Set $A \setminus B = \{b_1, \dots, b_m\}$, $B_0 = B$ and $B_{i+1} = B_i \cup \{b_{i+1}\}$ for every $i \in \{1, \dots, m\}$. Further, set $l_0 = h$, $l_{i+1} = \pi^\vee_{B_i, b_{i+1}}(l_i)$ for every $i \in \{1, \dots, m\}$ and $l = l_m$. Then, by definition we have a chain

$$(B, l) \triangleleft (B_1, l_{m-1}) \triangleleft \cdots \triangleleft (B_{m-1}, l_1) \triangleleft (A, h)$$

therefore $(B, l) \leq (A, h)$.

About the uniqueness, it is enough to note that the choice of $l$ does not depend on the order of the elements $b_i$. In fact, for any order of the elements $b_i$, the composition of the corresponding maps $\overline{\pi_{B_i, b_{i+1}}}$ will give always the same quotient map

$$\pi : \mathcal{M}(B) \to \mathcal{M}(A) \simeq \mathcal{M}(B)/(\psi(j) : j \in A \setminus B),$$

and $l = \pi^\vee_{|\operatorname{tor}(B)}(h)$. $\square$

We denote with $e \in \operatorname{tor}(\emptyset)^\vee$ the identity element of the torsion module $\operatorname{tor}(\emptyset)^\vee$.

**Theorem 3.3.7.** *[BM19, Theorem 4.5] For every realizable matroid $\mathcal{M}$ over $R$ such that $\mathcal{M}(\emptyset)$ is torsion-free,* $\operatorname{Gr}\mathcal{M}$ *is a simplicial poset.*

*Proof.* The element $(\emptyset, e)$ is the minimum of $\operatorname{Gr}\mathcal{M}$. Now let $(A, h) \in \operatorname{Gr}\mathcal{M}$, and set $I = [(\emptyset, e), (A, h)]$. From Proposition 3.3.6, for every subset $B \subseteq A$, there exists a unique $l \in \operatorname{tor}(B)^\vee$ such that $(B, l) \in I$. Thus, $I$ is isomorphic, as a poset, to the Boolean lattice $2^A$. $\square$

**Definition 3.3.8.** *Let $P$ be a poset, the **link** of $a \in P$ is*

$$\operatorname{link}_P a = \{b \in P : a \leq b\}.$$

When there is no ambiguity about the poset, we denote the link simply by $\operatorname{link} a$.

**Proposition 3.3.9.** *[BM19, Proposition 4.6] Let $\mathcal{M}$ be a realizable matroid over $R$ and let $A \in \Delta\mathcal{M}$. For every $t \in \operatorname{tor}(A)^\vee$,* $\operatorname{link}(A, e)$ *is isomorphic to* $\operatorname{link}(A, t)$ *as a poset.*

*Proof.* From Proposition 3.3.6, the two links do not intersect. We are going to define the isomorphism from $\operatorname{link}(A, e)$ to $\operatorname{link}(A, t)$, by providing the image of the low rank element first.

The atoms of $\operatorname{link}(A, e)$ are in bijection with the elements of the kernels of the surjective maps $\pi^\vee_{A,b} : \operatorname{tor}(A \cup \{b\})^\vee \to \operatorname{tor}(A)^\vee$, for all $b$ such that $A \cup \{b\}$ is an independent set of the generic matroid of $\mathcal{M}$.

Similarly, the atoms of $\operatorname{link}(A, t)$ are in bijections with elements of the cosets $m_b + \ker \pi^\vee_{A,b}$ for the same $b$ as above and for some $m_b$ in ${\pi^\vee_{A,b}}^{-1}(t)$. The correspondence among the atoms of $\operatorname{link}(A, e)$ and $\operatorname{link}(A, t)$ is given by sending an element $x$ of $\ker \pi^\vee_{A,b}$ to $m_b + x$ in $m_b + \ker \pi^\vee_{A,b}$.

We repeat the same construction to the atoms of $\operatorname{link}(A \cup \{b\}, h)$, extending the map to rank two elements and so on. We only need to take care that the choices of $m_b$ are coherent all along the construction. This comes from the fact that $\mathcal{M}$ is realizable and, as shown in the proof of Proposition 3.3.6, each map $\pi^\vee_{\_,\_}$ is given by a specific quotient:

$$\begin{tikzcd}
\operatorname{tor}(A)^\vee & \operatorname{tor}(A \cup \{b\})^\vee \arrow[l, "\pi^\vee_{A,b}"'] \\
\operatorname{tor}(A \cup \{c\})^\vee \arrow[u, "\pi^\vee_{A,c}"'] & \operatorname{tor}(A \cup \{b, c\})^\vee. \arrow[l, "\pi^\vee_{A \cup \{c\}, b}"] \arrow[u, "\pi^\vee_{A \cup \{b\}, c}"]
\end{tikzcd}$$

Therefore there exist $m_{bc}$ in $\operatorname{tor}(A \cup \{b, c\})^\vee$ that maps to $m_b$ in $\operatorname{tor}(A \cup \{b\})^\vee$, to $m_c$ in $\operatorname{tor}(A \cup \{c\})^\vee$, and that extends the bijection among the atoms to the a bijection among the rank two elements of $\operatorname{link}(A, e)$ and $\operatorname{link}(A, t)$ $\square$

In particular, from the previous proposition we have that $\operatorname{link}(\emptyset, e)$ is isomorphic to $\operatorname{link}(\emptyset, t)$, for every $t \in \operatorname{tor}(\emptyset)^\vee$.

**Proposition 3.3.10.** *[BM19, Proposition 4.7] Let $\mathcal{M}$ be a realizable matroid over $R$ with a fixed realization $\psi : E \to \mathcal{M}(\emptyset)$, then $\mathrm{link}_{\mathrm{Gr}\,\mathcal{M}}(\emptyset, e)$ is a simplicial poset.*

*Proof.* For every $A$ subset of $E$ we denote by $\psi[A] = (\psi(i) : i \in A)$ and one defines

$$\mathcal{M}'(A) = \frac{\mathcal{M}(\emptyset)}{\mathrm{tor}(\emptyset) + \psi[A]} \simeq \frac{\mathcal{M}(A)}{(\mathrm{tor}(\emptyset) + \psi[A])/\psi[A]}.$$

It is clear that $\mathcal{M}'$ is a realizable matroid over $R$ and a realization is given by the composition of $\psi$ with the quotient map $\mathcal{M}(\emptyset) \to {}^{\mathcal{M}(\emptyset)}/_{\mathrm{tor}(\emptyset)}$. We want to show that $\mathrm{Gr}\,\mathcal{M}'$ is isomorphic to $\mathrm{link}_{\mathrm{Gr}\,\mathcal{M}}(\emptyset, e)$ as posets. Since $\mathcal{M}'(\emptyset)$ is torsion-free, from Theorem 3.3.7 it will follow that $\mathrm{link}_{\mathrm{Gr}\,\mathcal{M}}(\emptyset, e)$ is a simplicial poset. Since

$$\mathcal{M}'(A) \otimes Q(R) \simeq \frac{\mathcal{M}(\emptyset) \otimes Q(R)}{\big(\mathrm{tor}(\emptyset) + \psi[A]\big) \otimes Q(R)} \simeq \frac{\mathcal{M}(\emptyset) \otimes Q(R)}{\psi[A] \otimes Q(R)} \simeq \mathcal{M}(A) \otimes Q(R),$$

then $\mathcal{M}' \otimes Q(R) \simeq \mathcal{M} \otimes Q(R)$.

For every subset $A$, let $\phi_A : \mathrm{tor}(A) \to \mathrm{tor}'(A)$ be the restriction to $\mathrm{tor}(A)$ of the quotient map from $\mathcal{M}(A)$ to $\mathcal{M}'(A)$, where $\mathrm{tor}'(A)$ denote the torsion part of $\mathcal{M}'(A)$. Consider its dual $\phi_A^\vee$. We want to show that

$$\varphi : \mathrm{Gr}\,\mathcal{M}' \to \mathrm{link}_{\mathrm{Gr}\,\mathcal{M}}(\emptyset, e), \quad \varphi(A, l) = (A, \phi_A^\vee(l))$$

is the desidered isomorphism. First, note that if $A \cup \{b\} \in \Delta\mathcal{M}$, the diagram

$$\begin{tikzcd} \dfrac{\mathcal{M}(\emptyset)}{\psi[A]} \arrow[r] \arrow[d] & \dfrac{\mathcal{M}(\emptyset)}{\psi[A] + \psi[b]} \arrow[d] \\ \dfrac{\mathcal{M}(\emptyset)}{\mathrm{tor}(\emptyset) + \psi[A]} \arrow[r] & \dfrac{\mathcal{M}(\emptyset)}{\mathrm{tor}(\emptyset) + \psi[A] + \psi[b]} \end{tikzcd}$$

clearly commutes, and if we restrict to the torsion parts and dualize we obtain the following commutative diagram

$$\begin{tikzcd} \mathrm{tor}(A)^\vee & \mathrm{tor}(A \cup \{b\})^\vee \arrow[l, "\pi_{A,b}^\vee"'] \\ \mathrm{tor}'(A)^\vee \arrow[u, "\phi_A^\vee"'] & \mathrm{tor}'(A \cup \{b\})^\vee \arrow[l, "\pi'^\vee_{A,b}"'] \arrow[u, "\phi_{A\cup\{b\}}^\vee"'] \end{tikzcd}$$

and then

$$\phi_A^\vee \circ \pi'^\vee_{A,b} = \pi_{A,b}^\vee \circ \phi_{A\cup\{b\}}^\vee.$$

The map $\phi_A$ is surjective, therefore its dual $\phi_A^\vee$ is injective. Now $(A, l) \lessdot (A \cup \{b\}, h)$ in $\mathrm{Gr}\,\mathcal{M}'$ if and only if

$$l = \pi_{A,b}^\vee(h) \iff \phi_A^\vee(l) = \phi_A^\vee(\pi_{A,b}^\vee(h)) = \pi'^\vee_{A,b}(\phi_{A\cup\{b\}}^\vee(h))$$

if and only if $(A, \phi_A(l)) \lessdot (A \cup \{b\}, \phi_{A\cup\{b\}}(h))$. Hence $\varphi$ is an order-embedding, and so well defined. The injectivity of $\varphi$ follows from the injectivity of $\phi_A^\vee$. For surjectivity, first we note that whenever $\varphi(A, l') \lessdot (A \cup \{b\}, h)$, we have $(A \cup \{b\}, h) \in \varphi(\mathrm{Gr}\,\mathcal{M}')$. In fact, by definition

$$\pi_{A,b}^\vee(h) = \phi_A^\vee(l') \Rightarrow h \in {\pi_{A,b}^\vee}^{-1}\big(\phi_A^\vee(l')\big) = \phi_{A\cup\{b\}}^\vee\big({\pi'^\vee_{A,b}}^{-1}(l')\big).$$

Now if $(A, l) \in \mathrm{link}_{\mathrm{Gr}\,\mathcal{M}}(\emptyset, e)$, then there is a chain of cover relations from $(\emptyset, e)$ to $(A, l)$, and since $(\emptyset, e) = \varphi(\emptyset, e')$, by iteratively apply the preceding remark, we obtain $(A, l) \in \varphi(\mathrm{Gr}\,\mathcal{M}')$. $\square$

The next result extends Theorem A of [Mar18] from matroids over $\mathbb{Z}$ to matroids over a domain.

**Theorem 3.3.11.** *[BM19, Theorem 4.8] If $\mathcal{M}$ is a realizable matroid over $R$, then $\operatorname{Gr}\mathcal{M}$ is a disjoint union of simplicial posets isomorphic to $\operatorname{link}(\emptyset, e)$.*

*Proof.* For each $t \in \operatorname{tor}(\emptyset)^\vee$, the pair $(\emptyset, t)$ is minimal in $\operatorname{Gr}\mathcal{M}$, therefore, from Proposition 3.3.6 we have

$$\operatorname{Gr}\mathcal{M} = \bigsqcup_{t\in\operatorname{tor}(\emptyset)^\vee} \operatorname{link}(\emptyset, t).$$

Finaly, from Proposition 3.3.9, for every $t \in \operatorname{tor}(\emptyset)$, $\operatorname{link}(\emptyset, t)$ is isomorphic to $\operatorname{link}(\emptyset, e)$ as posets, therefore, from Proposition 3.3.10, $\operatorname{link}(\emptyset, t)$ is a simplicial poset. $\square$

The ***f*-vector** of the poset of torsions $\operatorname{Gr}\mathcal{M}$ is, by definition, the $f$-vector of the $R$-matroid $\mathcal{M}$.

**Example 3.3.12.** *Let $R = \mathbb{Z}[i]$ and consider the matrix*

$$(v_1, v_2) = \begin{bmatrix} 1 & 1+i \\ 1+i & 0 \end{bmatrix} \in R^{2,2},$$

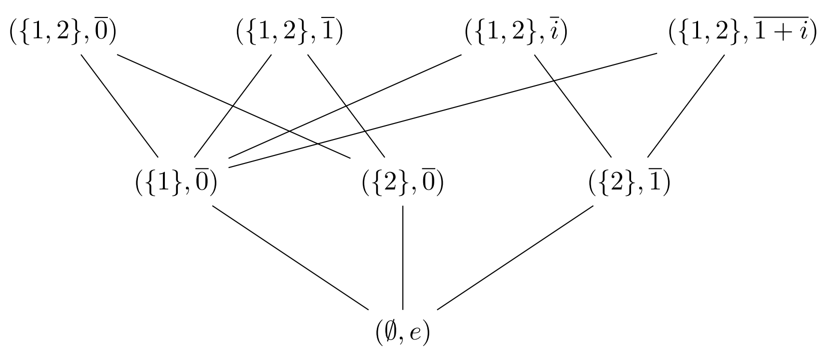
*where $v_1$ and $v_2$ are its columns. Let $[2] = \{1, 2\}$ and $\psi : [2] \to R^2$ with $\psi(i) = v_i$ and, for every $A \subseteq [2]$, set $\psi[A] = (\psi(i) : i \in A) \subseteq R^2$. Now define $\mathcal{M} : 2^{[2]} \to R$-mod with*

$$\mathcal{M}(A) = \frac{R^2}{\psi[A]} \quad A \subseteq [2].$$

*Thus $\mathcal{M}$ is a realizable $R$-matroid and $\psi$ is one of its realizations. More explicitely*

$$\begin{tikzcd} \mathcal{M}(\emptyset) \arrow[r] \arrow[d] & \mathcal{M}(1) \arrow[d] \\ \mathcal{M}(2) \arrow[r] & \mathcal{M}(12) \end{tikzcd} \simeq \begin{tikzcd} \mathbb{Z}[i]^2 \arrow[r] \arrow[d] & \mathbb{Z}[i] \arrow[d] \\ \mathbb{Z}[i] \oplus \mathbb{Z}[i]/(1+i)\mathbb{Z}[i] \arrow[r] & \mathbb{Z}[i]/2\mathbb{Z}[i] \end{tikzcd}$$

*The generic matroid of $\mathcal{M}$ is the uniform matroid $U_{2,2}$. We will see in Lemma 3.4.1 that, since $R = \mathbb{Z}[i]$, in this case we have $\operatorname{tor}(A) \simeq \operatorname{tor}(A)^\vee$. Therefore, the poset of torsions $\operatorname{Gr}\mathcal{M}$ can be represented by the following diagram*




*(it is the same poset of Example 3.1.10). The $f$-vector of $\operatorname{Gr}\mathcal{M}$ is*

$$(f_{-1}, f_0, f_1) = ([0], [0] + [\mathbb{Z}[i]/(1+i)\mathbb{Z}[i]], [\mathbb{Z}[i]/2\mathbb{Z}[i]]).$$

### 3.3.1 Face module for a matroid over a domain

To every classical matroid, we can associate the face ring of the simplicial complex given by its independent sets. In this subsection, we are going to define a similar notion for matroids over a domain.

Let $\mathcal{M}$ be a realizable matroid over a domain $R$ with a fixed realization $\psi : E \to \mathcal{M}(\emptyset)$, such that $\mathcal{M}(\emptyset)$ is torsion-free. From Theorem 3.3.7, $\operatorname{Gr}\mathcal{M}$ is a simplicial poset and we define $A_{\mathcal{M}}$ to be the face ring of $\operatorname{Gr}\mathcal{M}$ (Definition 3.1.9). Whenever $\mathcal{M}(\emptyset)$ has torsions, in general $\operatorname{Gr}\mathcal{M}$ is not a simplicial poset so in this case we cannot define a face ring. However, by Theorem 3.3.11 we know that $\operatorname{Gr}\mathcal{M}$ is a disjoint union of simplicial posets, therefore we can define a *face module* that is the direct sum of the face rings of each disjoint simplicial poset in $\operatorname{Gr}\mathcal{M}$.

**Definition 3.3.13.** *Let $\mathcal{M}$ be a realizable matroid over a domain $R$ with a fixed realization $\psi : E \to \mathcal{M}(\emptyset)$. Let $L$ be the link of $(\emptyset, e)$ in $\operatorname{Gr}\mathcal{M}$ and denote by $A_L$ the face ring of $L$. The **face module** $A_{\mathcal{M}}$ of $\mathcal{M}$ is the $A_L$-module*

$$A_{\mathcal{M}} = A_L^{|tor(\emptyset)|}$$

*(when $\operatorname{tor}(\emptyset)$ is not finite, we interpret $|\operatorname{tor}(\emptyset)|$ as a cardinal number).*

An alternative way to define the face module of $\mathcal{M}$ is to consider the matroid $\mathcal{M}'$ defined as in the proof of Proposition 3.3.10. For every subset $A \subseteq E$ set $\psi[A] = (\psi(i) : i \in A)$ and

$$\mathcal{M}'(A) = \frac{\mathcal{M}(\emptyset)}{\operatorname{tor}(\emptyset) + \psi[A]}.$$

Now $A_{\mathcal{M}} = A_{\mathcal{M}'}^{|\operatorname{tor}(\emptyset)|}$, since we have $\operatorname{Gr}\mathcal{M}' \simeq \operatorname{link}_{\operatorname{Gr}\mathcal{M}}(\emptyset, e)$ as posets, see the proof of Proposition 3.3.10.

**Example 3.3.14.** *If we consider the $\mathbb{Z}[x]$-matroid $\mathcal{M}$ of Example 3.3.12, then $\mathcal{M}(\emptyset)$ is torsion-free, so the face module of $\mathcal{M}$ is the face* ring *of $P = \operatorname{Gr}\mathcal{M}$ which coincides with the ring $A_P$ of Example 3.1.10.*

When $R$ is a field, then $A_{\mathcal{M}}$ is the classical Stanley-Reiner ring of a matroid; when $R$ is the ring of integers $\mathbb{Z}$, then $A_{\mathcal{M}}$ is the face module for a $\mathbb{Z}$-matroid defined in [Mar18]. In both cases, the poset of torsion is finite and $A_{\mathcal{M}}$ is Noetherian. However, in general we do not always have Noetherianity, as the following example shows.

**Example 3.3.15.** *Set $R = \mathbb{Z}[x]$ and consider the matrix*

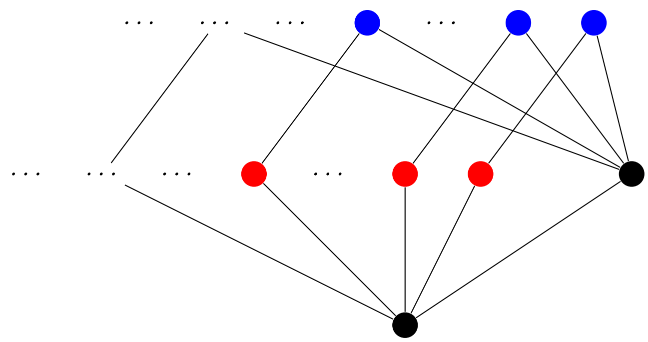
$$(v_1, v_2) = \begin{bmatrix} x & 0 \\ 0 & 1 \end{bmatrix} \in R^{2,2},$$

*where $v_1$ and $v_2$ are its columns. Similarly as what we have done in Example 3.3.12, we define $\psi : [2] \to R^2$ with $\psi(i) = v_i$ and and from $\psi$ we obtain a realizable $R$-matroid $\mathcal{M} : 2^{[2]} \to R$-mod*

$$\mathcal{M}(A) = \frac{R^2}{\psi[A]} \quad A \subseteq [2].$$

$$\begin{array}{ccccccc} \mathcal{M}(\emptyset) & \longrightarrow & \mathcal{M}(1) & & \mathbb{Z}[x]^2 & \longrightarrow & \mathbb{Z}[x] \oplus \mathbb{Z} \\ \downarrow & & \downarrow & \simeq & \downarrow & & \downarrow \\ \mathcal{M}(2) & \longrightarrow & \mathcal{M}(12) & & \mathbb{Z}[x] & \longrightarrow & \mathbb{Z} \end{array}$$

*(where $\mathbb{Z} \simeq \mathbb{Z}[x]/(x)$). The generic matroid of $\mathcal{M}$ is the uniform matroid $U_{2,2}$. In this case $\operatorname{tor}(\mathcal{M}(1)) = \operatorname{tor}(\mathcal{M}(12)) = \mathbb{Z}$ is not finite, and so is the poset of torsions $\operatorname{Gr}\mathcal{M}$:*



*in the preceding diagram the infinite red and blue dots corresponds respectively to the torsion part of $\mathcal{M}(1)$ and $\mathcal{M}(12)$. Note that in this case the face ring of* $\operatorname{Gr}\mathcal{M}$ *has infinitely many variables, therefore it is not Noetherian.*

## 3.4 Tutte polynomial for ring of integers of a number field

In this section, as an application of the notions developed in the previous two sections, we show that, when $R$ is the ring of integers of a number field, the Hilbert series of the face module of an $R$-matroid is the specialization of the Tutte-Grothendieck polynomial, generalizing Theorem 3.1.6.

Throughout this section, $\mathbb{F}$ is an *algebraic number field*, that is a finite field extension of the rational numbers $\mathbb{Q}$. The ring $R$ is the *ring of integers* of $\mathbb{F}$, i.e. the integral closure of $\mathbb{Z}$ in $\mathbb{F}$, so $Q(R) = \mathbb{F}$. Under this hypothesis, $R$ is a Dedekind domain [AM16, Theorem 9.5]. We further assume that $R$ is a PID.

**Lemma 3.4.1.** *For every ideal $I$ of $R$ we have*

$$\operatorname{Hom}_R({}^{R}\!/\!_{I}, {}^{Q(R)}\!/\!_{R}) \simeq {}^{R}\!/\!_{I}.$$

*Proof.* Since $R$ is a PID, we can write $I = (d)$. For every $f \in \operatorname{Hom}_R(R/I, {}^{Q(R)}\!/\!_{R})$ we have

$$d \cdot f(\overline{1}) = \overline{0} \Rightarrow f(\overline{1}) = \frac{r}{d} + R.$$

Now it is easy to check that the map $\varphi : \operatorname{Hom}_R(R/I, {}^{Q(R)}\!/\!_{R}) \to R/I$ defined by $\varphi(f) = r + I$ is an isomorphism. ☐

**Corollary 3.4.2.** *For every finitely generated $R$-module $N$ we have $\operatorname{tor} N \simeq \operatorname{tor} N^{\vee}$, in particular the cardinality $|\operatorname{tor} N^{\vee}| = |\operatorname{tor} N|$ is finite.*

*Proof.* Since $R$ is a PID, by the Smith normal form (see [DF04, Chapter 12] or [LQ15, Section IV.7]), we can write uniquely

$$\operatorname{tor}(N) = \frac{R}{I_1} \oplus \frac{R}{I_2} \oplus \cdots \oplus \frac{R}{I_m}$$

for a chain $I_1 \subseteq \cdots \subseteq I_m$ of nonzero ideals of $R$. Further, from [AW04, Theorem 9.1.3], for every nonzero ideal $I$ of $R$, the cardinality of $R/I$ is finite, hence $|\operatorname{tor}(N)| = \prod_{i=1}^{m} |R/I_m|$ is also finite. ☐

Now define the homomorphism of rings $\tilde{\varphi} : \mathbb{Z}[R\text{-Mod}] \to \mathbb{Z}$ by

$$\begin{aligned} \varphi([F]) &= 1 && \text{for every free module } F, \\ \varphi([N]) &= |N| && \text{for every torsion module } N. \end{aligned}$$

(the previous information are enough to define $\tilde{\varphi}$ since every finitely generated $R$-module is the direct sum of a free module and its torsion part).

The homomorphism $\tilde{\varphi}$ induces the homomorphism of polynomial rings

$$\varphi : \mathbb{Z}[R\text{-Mod}][x,y] \to \mathbb{Z}[x,y].$$

Thus, if $\mathcal{M}$ is a matroid over $R$, we can consider the image under $\varphi$ of the Tutte-Grothendieck polynomial

$$\varphi\big(T_{\mathcal{M}}(x,y)\big) = \sum_{A \subseteq E} |\operatorname{tor}(A)|(x-1)^{r-\operatorname{rk}(A)}(y-1)^{|A|-\operatorname{rk}(A)} \tag{3.2}$$

(note that from Corollary 3.4.2 we have $|\operatorname{tor}(A)| = |\operatorname{tor}(A)^\vee|$). In the case when $R = \mathbb{Z}$, as we have seen in Section 2.3, $\mathcal{M}$ defines a quasi-arithmetic matroid and $\varphi\big(T_{\mathcal{M}}(x,y)\big)$ is the arithmetic Tutte polynomial.

Now suppose that $\mathcal{M}$ is realizable. Fix a realization $\psi : E \to \mathcal{M}(\emptyset)$. For every subset $A \subseteq E$ denote by $\psi[A] = (\psi(i) : i \in A)$. As in the proof of Proposition 3.3.10, define the $R$-matroid

$$\mathcal{M}'(A) = \frac{\mathcal{M}(\emptyset)}{\operatorname{tor}(\emptyset) + \psi[A]}.$$

**Lemma 3.4.3.** *Let $\mathcal{M}$ be a realizable $R$-matroid and $\mathcal{M}'$ as above, then*

$$\varphi\big(T_{\mathcal{M}}(t,1)\big) = |\operatorname{tor}(\emptyset)|\varphi\big(T_{\mathcal{M}'}(t,1)\big).$$

*Proof.* Since we are evaluating the Tutte polynomial at $y = 1$, we are considering the term of the sum where $A$ is an independent set of $\Delta\mathcal{M}$. As we have seen in Section 3.3, the restriction of the quotient map $\operatorname{tor}(\emptyset) \to \operatorname{tor}(A)$ is injective, so by identifying $\operatorname{tor}(\emptyset)$ with its homomorphic image

$$\operatorname{tor}(\emptyset) \simeq \frac{\operatorname{tor}(\emptyset)}{\operatorname{tor}(\emptyset) \cap \psi[A]} \simeq \frac{\operatorname{tor}(\emptyset) + \psi[A]}{\psi[A]}.$$

We also know that

$$\mathcal{M}'(A) \simeq \frac{\mathcal{M}(A)}{(\operatorname{tor}(\emptyset)+\psi[A])/\psi[A]} \simeq \frac{\mathcal{M}(A)}{\operatorname{tor}(\emptyset)}.$$

This means that the torsion part of $\mathcal{M}'(A)$ is isomorphic to $\operatorname{tor}(A)/\operatorname{tor}(\emptyset)$, in particular, its cardinality is $|\operatorname{tor}(A)|/|\operatorname{tor}(\emptyset)|$ and now the statement follows easily. $\square$

From Theorem **??**, we know that the Hilbert series of the face ring of a classical matroid is a specialization of its Tutte polynomial. In [Mar18], it was proved that for realizable $\mathbb{Z}$-matroids, the Hilbert series of the face ring is a specialization of the arithmetic Tutte polynomial. The next result generalize this fact to matroids over the PID $R$.

**Theorem 3.4.4.** *[BM19, Theorem 5.4] If $\mathcal{M}$ is a realizable $R$-matroid of rank $r$, then*

$$H(A_{\mathcal{M}}, t) = \frac{t^r}{(1-t)^r}\varphi\big(T_{\mathcal{M}}(1/t, 1)\big).$$

*Proof.* From Lemma 3.4.3 and by the additivity property of the Hilbert series, it is enough to show that the theorem is true when $\mathcal{M}(\emptyset)$ is torsion-free, i.e. when $\operatorname{tor}(\emptyset) = 0$. In this case $\operatorname{Gr}\mathcal{M} = \operatorname{link}_{\operatorname{Gr}\mathcal{M}}(\emptyset, e)$ is a simplicial poset and $A_{\mathcal{M}}$ is the face ring of $\operatorname{Gr}\mathcal{M}$. The image under $\tilde{\varphi}$ of the components of the $f$-vector of the matroid $\mathcal{M}$ (Definition 3.2.10) coincides with the components $f_i$ of the $f$-vector of $\operatorname{Gr}\mathcal{M}$:

$$f_{i-1} = \sum_{\substack{A \in \Delta\mathcal{M} \\ |A| = i}} |\operatorname{tor}(A)^\vee|.$$

Thus

$$\varphi\big(T_{\mathcal{M}}(t,1)\big) = \sum_{A \in \Delta\mathcal{M}} |\operatorname{tor}(A)^\vee|(t-1)^{r-|A|} = \sum_{i=0}^{r} \left( \sum_{\substack{A \in \Delta\mathcal{M} \\ |A| = i}} |\operatorname{tor}(A)^\vee| \right) (t-1)^{r-i} =$$

$$= \sum_{i=0}^{r} f_{i-1}(t-1)^{r-i} = \sum_{i=0}^{r} h_i t^{r-i}$$

(where $h_i$ are the components of the $h$-vector of $\operatorname{Gr}\mathcal{M}$). Finally, from Theorem 3.1.12

$$H(A_{\mathcal{M}},t) = \frac{1}{(1-t)^r}\sum_{i=0}^{r} h_i t^i = \frac{t^r}{(1-t)^r}\varphi\big(T_{\mathcal{M}}(1/t,1)\big). \qquad \square$$

**Remark 3.4.5.** *The homomorphism $\varphi$, Lemma 3.4.3 and Theorem 3.4.4 can be defined and proved in the more general class of rings in which every module $M$ can be decomposed in the direct sum of its torsion part* $\operatorname{tor}(M)$ *and* $M/\operatorname{tor}(M)$, *and every torsion module is finite.*

**Example 3.4.6.** *Set $R=\mathbb{Z}[i]$ (note that $\mathbb{Z}[i]$ is the ring of integers of $\mathbb{Q}[i]$ and it is a PID), and consider the $R$-matroid $\mathcal{M}$ of Example 3.3.12. Set $P=\operatorname{Gr}\mathcal{M}$, the face module of $\mathcal{M}$ is the face ring of $P$, and it was computed in Example 3.1.10:*

$$A_{\mathcal{M}} = A_P \simeq \frac{\mathbb{K}[x_a, x_{b_0}, x_{b_1}, x_{c_0}, x_{c_1}, x_{d_0}, x_{d_1}]}{\left(\begin{array}{lcl} x_a x_{b_i} - (x_{c_i}+x_{d_i}), x_{b_0}x_{b_1}, & & i,j\in\{0,1\} \\ x_{c_i}x_{d_j}, x_{c_0}x_{c_1}, x_{d_0}x_{d_1}, & : & \bar{i}=1-i \\ x_{b_i}x_{c_{\bar{i}}}, x_{b_i}x_{d_{\bar{i}}} & & \end{array}\right)}$$

*We compute the Hilbert series of $A_{\mathcal{M}}$ using Macaulay2 [GS]:*

$$\begin{aligned} H(A_{\mathcal{M}},t) &= \frac{1-3t^2-2t^3+2t^4+8t^5+2t^6-12t^7-3t^8+8t^9+t^{10}-2t^{11}}{(1-t^2)^4(1-t)^3} = \\ &= \frac{1+t+2t^2}{(1-t)^2}. \end{aligned}$$

*The image under $\varphi$ of the Tutte polynomial is*

$$\begin{aligned} \varphi\big(T_{\mathcal{M}}(x,y)\big) &= \sum_{A\subseteq[2]} |\operatorname{tor}(A)|(x-1)^{2-\operatorname{rk}(A)}(y-1)^{|A|-\operatorname{rk}(A)} = \\ &= (x-1)^2+3(x-1)+4 = x^2+x+2. \end{aligned}$$

*(Note that $(1,3,4)$ and $(1,1,2)$ are the $f$-vector and the $h$-vector of $\operatorname{Gr}\mathcal{M}$.) Finally we have*

$$H(A_{\mathcal{M}},t) = \frac{1+t+2t^2}{(1-t)^2} = \frac{t^2}{(1-t)^2}\,\varphi\big(T_{\mathcal{M}}(1/t,1)\big).$$

# Bibliography


[AK06] Federico Ardila and Caroline J. Klivans. The Bergman complex of a matroid and phylogenetic trees. *J. Combin. Theory Ser. B*, 96(1):38–49, 2006.

[AM16] M. F. Atiyah and I. G. Macdonald. *Introduction to commutative algebra*. Addison-Wesley Series in Mathematics. Westview Press, Boulder, CO, economy edition, 2016.

[AW04] Şaban Alaca and Kenneth S. Williams. *Introductory algebraic number theory*. Cambridge University Press, Cambridge, 2004.

[BB19] Matthew Baker and Nathan Bowler. Matroids over partial hyperstructures. *Adv. Math.*, 343:821–863, 2019.

[BLVS+99] Anders Björner, Michel Las Vergnas, Bernd Sturmfels, Neil White, and Günter M. Ziegler. *Oriented matroids*, volume 46 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, second edition, 1999.

[BM19] Alessio Borzì and Ivan Martino. Set of independencies and tutte polynomial of matroids over a domain. *arXiv preprint arXiv:1909.00332*, 2019.

[Bol98] Béla Bollobás. *Modern graph theory*, volume 184 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1998.

[BS81] Stanley Burris and H. P. Sankappanavar. *A course in universal algebra*, volume 78 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1981.

[DCP08] C. De Concini and C. Procesi. Hyperplane arrangements and box splines. volume 57, pages 201–225. 2008. With an appendix by A. Björner, Special volume in honor of Melvin Hochster.

[DCP11] Corrado De Concini and Claudio Procesi. *Topics in hyperplane arrangements, polytopes and box-splines*. Universitext. Springer, New York, 2011.

[DF04] David S. Dummit and Richard M. Foote. *Abstract algebra*. John Wiley & Sons, Inc., Hoboken, NJ, third edition, 2004.

[DM12] Michele D'Adderio and Luca Moci. Ehrhart polynomial and arithmetic Tutte polynomial. *European J. Combin.*, 33(7):1479–1483, 2012.

[DM13] Michele D'Adderio and Luca Moci. Arithmetic matroids, the Tutte polynomial and toric arrangements. *Adv. Math.*, 232:335–367, 2013.

[DR20] Jan Draisma and Felipe Rincón. Tropical ideals do not realise all Bergman fans. *Sém. Lothar. Combin.*, 82B:Art. 69, 11, 2020.

[DW92] Andreas W. M. Dress and Walter Wenzel. Valuated matroids. *Adv. Math.*, 93(2):214–250, 1992.

[Eis95] David Eisenbud. *Commutative algebra*, volume 150 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995. With a view toward algebraic geometry.

[FM16] Alex Fink and Luca Moci. Matroids over a ring. *J. Eur. Math. Soc. (JEMS)*, 18(4):681–731, 2016.

[FS05] Eva Maria Feichtner and Bernd Sturmfels. Matroid polytopes, nested sets and Bergman fans. *Port. Math. (N.S.)*, 62(4):437–468, 2005.

[FY04] Eva Maria Feichtner and Sergey Yuzvinsky. Chow rings of toric varieties defined by atomic lattices. *Invent. Math.*, 155(3):515–536, 2004.

[GG16] Jeffrey Giansiracusa and Noah Giansiracusa. Equations of tropical varieties. *Duke Math. J.*, 165(18):3379–3433, 2016.

[GM12] Gary Gordon and Jennifer McNulty. *Matroids: a geometric introduction*. Cambridge University Press, Cambridge, 2012.

[GR01] Chris Godsil and Gordon Royle. *Algebraic graph theory*, volume 207 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2001.

[GS] Daniel R. Grayson and Michael E. Stillman. Macaulay2, a software system for research in algebraic geometry. Available at `http://www.math.uiuc.edu/Macaulay2/`.

[LQ15] Henri Lombardi and Claude Quitté. *Commutative algebra: constructive methods*, volume 20 of *Algebra and Applications*. Springer, Dordrecht, revised edition, 2015. Finite projective modules, Translated from the French by Tania K. Roblot.

[Mar18] Ivan Martino. Face module for realizable $\mathbb{Z}$-matroids. *Contrib. Discrete Math.*, 13(2):74–87, 2018.

[Moc12] Luca Moci. A Tutte polynomial for toric arrangements. *Trans. Amer. Math. Soc.*, 364(2):1067–1088, 2012.

[MR18] Diane Maclagan and Felipe Rincón. Tropical ideals. *Compos. Math.*, 154(3):640–670, 2018.

[MS05] Ezra Miller and Bernd Sturmfels. *Combinatorial commutative algebra*, volume 227 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2005.

[MS15] Diane Maclagan and Bernd Sturmfels. *Introduction to tropical geometry*, volume 161 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2015.

[MT01] Kazuo Murota and Akihisa Tamura. On circuit valuation of matroids. *Adv. in Appl. Math.*, 26(3):192–225, 2001.

[Nak09a] Takeo Nakasawa. Über die Abbildungskette vom Projektionsspektrum [Sci. Rep. Tokyo Bunrika Daigaku Sect. A **3** (1938), no. 64, 125–136; Zbl 0019.23804]. In *A lost mathematician, Takeo Nakasawa*, pages 132–143. Birkhäuser, Basel, 2009.

[Nak09b] Takeo Nakasawa. Zur Axiomatik der linearen Abhängigkeit. I [Sci. Rep. Tokyo Bunrika Daigaku Sect. A **2** (1935), no. 43, 129–149; Zbl 0012.22001]. In *A lost mathematician, Takeo Nakasawa*, pages 68–88. Birkhäuser, Basel, 2009.

[Nak09c] Takeo Nakasawa. Zur Axiomatik der linearen Abhängigkeit. II [Sci. Rep. Tokyo Bunrika Daigaku Sect. A **3** (1936), no. 51, 17–41; Zbl 0013.31406]. In *A lost mathematician, Takeo Nakasawa*, pages 90–114. Birkhäuser, Basel, 2009.

[Nak09d] Takeo Nakasawa. Zur Axiomatik der linearen Abhängigkeit. III. Schluss [Sci. Rep. Tokyo Bunrika Daigaku Sect. A **3** (1936), no. 55, 77–90; Zbl 0016.03704]. In *A lost mathematician, Takeo Nakasawa*, pages 116–129. Birkhäuser, Basel, 2009.

[Oxl11] James Oxley. *Matroid theory*, volume 21 of *Oxford Graduate Texts in Mathematics*. Oxford University Press, Oxford, second edition, 2011.

[Pag20] Roberto Pagaria. Orientable arithmetic matroids. *Discrete Math.*, 343(6):111872, 8, 2020.

[Sta91] Richard P. Stanley. $f$-vectors and $h$-vectors of simplicial posets. *J. Pure Appl. Algebra*, 71(2-3):319–331, 1991.

[Sta96] Richard P. Stanley. *Combinatorics and commutative algebra*, volume 41 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, second edition, 1996.

[Wei94] Charles A. Weibel. *An introduction to homological algebra*, volume 38 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1994.

[Whi35] Hassler Whitney. On the Abstract Properties of Linear Dependence. *Amer. J. Math.*, 57(3):509–533, 1935.

[Whi92] Neil White, editor. *Matroid applications*, volume 40 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1992.

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