Face poset and Grothendieck-Tutte polynomial of matroids over a domain

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(joint work with Ivan Martino)

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Matroids

A (classical) matroid M is a pair $(E, \mathcal{I})$ where E is a finite set,called the ground set, and $\mathcal{I}$ is a family of subsets of E, called independent sets, such that

1. $\emptyset \in \mathcal{I}$,
2. $A \subseteq B \in \mathcal{I} \Rightarrow A \in \mathcal{I}$,
3. $A, B \in \mathcal{I}$, $|A| < |B| \Rightarrow \exists b \in B \setminus A : A \cup \{b\} \in \mathcal{I}$.

Motivating example

Let V be a vector space over a field k and set $E = \{v_1, \ldots, v_n\} \subseteq V$. If

$$\mathcal{I} = \{\text{linear independent subsets of } E\}$$

then $(E, \mathcal{I})$ is a matroid.

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then $(E, \mathcal{I})$ is a matroid.

If a matroid M can be realized from a set of vectors in a vector space, then M is said realizable (or representable).

Rank and corank

If $M = (E, \mathcal{I})$ is a matroid, for every $A \subseteq E$, the rank of A, denoted by $\mathrm{rk}(A)$, is the cardinality of a maximal independent set contained in A.

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The corank of $A \subseteq E$ is $\mathrm{cork}(A) = \mathrm{rk}(E) - \mathrm{rk}(A)$.

Since their introduction, new definitions of matroids have appeared. For instance *arithmetic matroids*, *valuated matroids*, *oriented matroids* ...

Fink-Moci matroids over a ring

A matroid over a commutative ring R on the ground set E is a function

$$\mathcal{M} : 2^E \to \{\text{f.g. } R\text{-modules}\}$$

such that for every $A \subseteq E$, $a, b \in E \setminus A$ there exist $x, y \in \mathcal{M}(A)$ such that

$$\mathcal{M}(A \cup \{b\}) \simeq \mathcal{M}(A)/(x)$$
$$\mathcal{M}(A \cup \{c\}) \simeq \mathcal{M}(A)/(y)$$
$$\mathcal{M}(A \cup \{b, c\}) \simeq \mathcal{M}(A)/(x, y)$$

$$
\begin{tikzcd}
\mathcal{M}(A) \arrow[r, "/(x)"] \arrow[d, "/(y)"] & \mathcal{M}(A \cup \{a\}) \arrow[d, "/(\overline{y})"] \\
\mathcal{M}(A \cup \{b\}) \arrow[r, "/(\overline{x})"] & \mathcal{M}(A \cup \{a, b\})
\end{tikzcd}
$$

Example of matroid over $\mathbb{Z}$

Set $R = \mathbb{Z}$, $E = \{1, 2\}$

$$\begin{tikzcd} \mathcal{M}(\emptyset) \arrow[r] \arrow[d] & \mathcal{M}(1) \arrow[d] \\ \mathcal{M}(2) \arrow[r] & \mathcal{M}(12) \end{tikzcd} \simeq \begin{tikzcd} \mathbb{Z}^2 \arrow[r] \arrow[d] & \mathbb{Z} \oplus \mathbb{Z}/(2) \arrow[d] \\ \mathbb{Z} \arrow[r] & \mathbb{Z}/(2) \end{tikzcd}$$

in this case $x = (2, 0), y = (0, 1) \in \mathcal{M}(\emptyset) = \mathbb{Z}^2$.

Realizable matroids over R

A matroid $\mathcal{M}$ over R on the gruond set E is realizable if there exists

$$\varphi : E \to \mathcal{M}(\emptyset) \quad \text{such that} \quad \mathcal{M}(A) = \mathcal{M}(\emptyset)/(\varphi(i) : i \in A).$$

The map φ is a realization of $\mathcal{M}$.

Direct sum and tensor product

Let $\mathcal{M}, \mathcal{M}'$ be two matroids over R on respective ground sets E and E', their direct sum is a matroid on the ground set $E \amalg E'$ defined by

$$(\mathcal{M} \oplus \mathcal{M}')(A \amalg A') = \mathcal{M}(A) \oplus \mathcal{M}'(A').$$

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If $R \to S$ is a ring homomorphism, then the function

$$(\mathcal{M} \otimes_R S)(A) = \mathcal{M}(A) \otimes_R S$$

is a matroid over S.

Essential and empty matroids

The empty (projective) matroid for a (projective) R-module N is a matroid on the ground set $\emptyset$ which maps $\emptyset$ to N.

A matroid $\mathcal{M}$ over R is essential if no nontrivial projective module is a direct summand of $\mathcal{M}(E)$.

Every matroid $\mathcal{M}$ is a direct sum of an essential matroid and an empty projective matroid

$$\mathcal{M} = \mathcal{M}_E \oplus \mathcal{M}_P$$

If R is a domain, $\mathcal{M}_E \otimes Q(R)$ is the generic matroid of $\mathcal{M}$.

Proposition (Fink-Moci, 2018)

If $\mathcal{M}$ is an essential matroid over a field k, then the function

$$\operatorname{cork} : 2^E \to \mathbb{N} \quad \textit{defined by} \quad \operatorname{cork}(A) = \dim_k \mathcal{M}(A)$$

is the corank of a (classical) matroid.

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If R is a domain, the generic matroid $\mathcal{M}_E \otimes Q(R)$ is a matroid over the field $Q(R)$, we denote by $\Delta\mathcal{M}$ the associated family of independent sets.

More generally, let $\mathcal{M}$ be an essential matroid over R.

If R is a field	then $\mathcal{M}$ is associated to a *classical* matroid
If R is $\mathbb{Z}$	then $\mathcal{M}$ is associated to a *quasi-arithmetic* matroid
If R is a DVR	then $\mathcal{M}$ is associated to a *valuated* matroid

Tutte polynomial

Let M be a classical matroid, the Tutte polynomial of M is

$$T_M(x,y) = \sum_{A \subseteq E} (x-1)^{\mathrm{rk}(E)-\mathrm{rk}(A)} (y-1)^{|A|-\mathrm{rk}(A)}$$

Dual matroid, deletion and contraction

Let M be a (classical matroid). The maximal independent sets are the bases of M. The dual of M is the matroid M^* with bases equal to the complements of the bases of M.

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A loop in M is an element $i \in E$ such that $\{i\}$ is a dependent set. A coloop is an element $i \in E$ that is a loop in M^*.

Deletion-Contraction property

Theorem (Deletion-Contraction property)

$$T_M(x,y) = \begin{cases} yT_{M\setminus i}(x,y) & \textit{if } i \textit{ is a loop} \\ xT_{M/i}(x,y) & \textit{if } i \textit{ is a coloop} \\ T_{M\setminus i}(x,y) + T_{M/i}/(x,y) & \textit{otherwise} \end{cases}$$

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In addition, every function defined on matroids that satisfy the previous property can be obtained as a specialization of the Tutte polynomial.

Hilbert series of the face ring

The independent sets of a matroid M constitute a simplicial complex. If $k[M]$ denote the *Stanley-Reisner* ring of M, $H(k[M],t)$ is the Hilbert series of $k[M]$ and $r = \mathrm{rk}(E)$, then

Theorem (Björner's Appendix of De Concini-Procesi, 2008)

$$H(k[M],t) = \frac{t^r}{(1-t)^r} T_M(1/t, 1)$$

Grothendieck style ring

Let R be a domain. If N is a finitely generated R-module, denote with $[N]$ the isomorphism class of N. Let $L_0(R\text{-mod})$ be the commutative ring with a $\mathbb{Z}$-linear basis $\{[N] : N \text{ f.g. } R\text{-modules}\}$, and product given by $[N] \cdot [N'] = [N \oplus N']$.

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Let $\mathcal{M}$ be a matroid over R, we indicate with $\operatorname{tor}(A)$ the torsion submodule of $\mathcal{M}(A)$. Denote with $\vee$ the controvariant functor $\operatorname{Hom}(-, Q(R)/R)$.

Grothendieck-Tutte polynomial for a matroid over a domain

Let $\mathcal{M}$ be a matroid a domain R. The Grothendieck-Tutte polynomial of $\mathcal{M}$ is

$$T_{\mathcal{M}}(x,y) = \sum_{A \subseteq E} [\mathrm{tor}(A)^{\vee}](x-1)^{\mathrm{rk}(E)-\mathrm{rk}(A)}(y-1)^{|A|-\mathrm{rk}(A)}$$

here the rank function rk is of the generic matroid $\mathcal{M}_E \otimes Q(R)$.

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This polynomial is a bit more explicit and general than the polynomial defined by Fink-Moci and generalizes as well the multiplicity polynomial defined by Moci for arithmetic matroids.

Deletion Contraction for matroids over a ring

If $\mathcal{M}$ is a matroid over R and $i \in E$, then

$$(\mathcal{M} \setminus i)(A) = \mathcal{M}(A)$$
$$(\mathcal{M}/i)(A) = \mathcal{M}(A \cup \{i\})$$

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Theorem (B.-Martino, 2019)

If $\mathcal{M}$ is a matroid over a domain R such that $\mathcal{M}(\emptyset)$ is torsion-free and $\mathcal{M}(E) = 0$, then

$$T_{\mathcal{M}}(x,y) = \begin{cases} yT_{\mathcal{M}\setminus i}(x,y) & \textit{if } i \textit{ is a loop,} \\ xT_{\mathcal{M}/i}(x,y) & \textit{if } i \textit{ is a coloop,} \\ T_{\mathcal{M}\setminus i}(x,y) + T_{\mathcal{M}/i}(x,y) & \textit{otherwise.} \end{cases}$$

Poset of torsions

Let $\mathcal{M}$ be a *realizable* matroid ovar a domain R with a fixed realization $\psi : E \to \mathcal{M}(\emptyset)$. The poset of torsions (or face poset) of $\mathcal{M}$ is

$$\mathrm{Gr}\,\mathcal{M} = \{(A, l) : A \in \Delta\mathcal{M}, l \in \mathrm{tor}(A)^\vee\}.$$

(recall that $\Delta\mathcal{M}$ is the family of independent sets of the generic matroid of $\mathcal{M}$)

Let $A \cup \{b\} \in \Delta\mathcal{M}$, with $b \notin A$, consider the quotient map

$$\overline{\pi_{A,b}} : \mathcal{M}(A) \twoheadrightarrow \mathcal{M}(A \cup \{b\}) \simeq \mathcal{M}(A)/(\psi(b))$$

consider its restriction to $\mathrm{tor}(A)$

$$\pi_{A,b} : \mathrm{tor}(A) \twoheadrightarrow \mathrm{tor}(A \cup \{b\})$$

and finally apply the controvariant functor $\vee$

$$\pi_{A,b}^{\vee} : \mathrm{tor}(A \cup \{b\})^{\vee} \to \mathrm{tor}(A)^{\vee}$$

Recall that if P is a poset, b covers a, denoted with $a \lhd b$, if $a < b$ and $\nexists c : \; a < c < b$.

If $(A, l), (A \cup \{b\}, h) \in \mathrm{Gr}\,\mathcal{M}$ then we set

$$(A, l) \lhd (A \cup \{b\}, h) \overset{\text{def}}{\iff} \pi^{\vee}_{A,b}(h) = l.$$

Example of poset of torsions

Let $R = \mathbb{Z}$, $E = \{1, 2\}$, consider the (realizable) matroid

$$\begin{tikzcd} \mathcal{M}(\emptyset) \arrow[r] \arrow[d] & \mathcal{M}(1) \arrow[d] \\ \mathcal{M}(2) \arrow[r] & \mathcal{M}(12) \end{tikzcd} \simeq \begin{tikzcd} \mathbb{Z}^2 \arrow[r] \arrow[d] & \mathbb{Z} \oplus \mathbb{Z}/(2) \arrow[d] \\ \mathbb{Z} \arrow[r] & \mathbb{Z}/(2) \end{tikzcd}$$

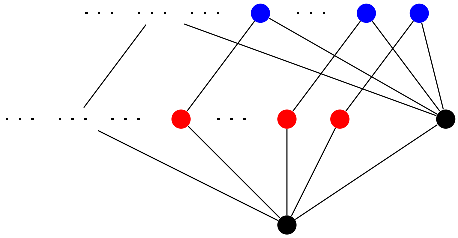
fix the realization $\psi : 2^E \to \mathbb{Z}^2$ defined by
$\psi(1) = (2, 0), \psi(2) = (0, 1)$. The poset of torsions of $\mathcal{M}$ is

$$\begin{tikzcd} & (\{1,2\}, \overline{0}) \arrow[dl, dash] \arrow[drr, dash] & & (\{1,2\}, \overline{1}) \arrow[dl, dash] \arrow[d, dash] \\ (\{1\}, \overline{0}) & & (\{1\}, \overline{1}) & (\{2\}, e) \\ & & (\emptyset, e) \arrow[ull, dash] \arrow[u, dash] \arrow[ur, dash] & \end{tikzcd}$$

Example of infinite poset of torsions

$$
\begin{array}{ccc}
\mathcal{M}(\emptyset) \longrightarrow \mathcal{M}(1) & & \mathbb{Z}[x]^2 \longrightarrow \mathbb{Z}[x] \oplus {\color{red}\mathbb{Z}} \\
\downarrow \qquad\qquad \downarrow & \simeq & \downarrow \qquad\qquad \downarrow \\
\mathcal{M}(2) \longrightarrow \mathcal{M}(12) & & \mathbb{Z}[x] \longrightarrow {\color{blue}\mathbb{Z}}
\end{array}
$$

Realized by $\psi(1) = (x, 0)$, $\psi(2) = (0, 1)$.



Simplicial posets

Recall that a Boolean lattice P is a distributive lattice, that has a maximum $\hat{1}$ and a minimum $\hat{0}$, in which for every element $a \in P$ there exists a complement $a' \in P$ such that $a \vee a' = \hat{1}$ and $a \wedge a' = \hat{0}$. The rank of P is the length of any maximal chain in P.

A poset P is a simplicial poset if it has a minimum $\hat{0}$ and for every $a \in P$ the segment $[\hat{0}, a] = \{b \in P : \hat{0} \leq b \leq a\}$ is a Boolean lattice. The rank of an element $a \in P$ is the rank of $[\hat{0}, a]$ as a Boolean lattice.

Theorem (B.-Martino, 2019)

If $\mathcal{M}$ is a realizable matroid over a domain R such that $\mathcal{M}(\emptyset)$ is torsion-free, then $\operatorname{Gr}\mathcal{M}$ is a simplicial poset.

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Recall that, if P is a poset and $a \in P$, then
$\operatorname{link} a = \{b \in P : a \leq b\}$.

Theorem (B.-Martino, 2019)

If $\mathcal{M}$ is a realizable matroid over a domain R, then $\operatorname{Gr}\mathcal{M}$ is a disjoint union of simplicial posets isomorphic to $\operatorname{link}(\emptyset, e)$.

Face ring of a simplicial poset

Let P be a simplicial poset. If $a, b \in P$, denote with $M(a,b)$ the set of *minimal* upper bounds of $\{a, b\}$. The face ideal of P in the polynomial ring $k[x_a : a \in P]$ is

$$I_P = \left(x_{\hat{0}} - 1,\ x_a x_b - x_{a \wedge b} \left(\sum_{c \in M(a,b)} x_c \right) : a, b \in P \right)$$

where $x_{a \wedge b} = 0$ if $a \wedge b$ does not exists, and $\sum_{c \in M(a,b)} x_c = 0$ if $M(a,b) = \emptyset$. The face ring of P is the quotient

$$k[P] = \frac{k[x_a : a \in P]}{I_P}$$

we set on $k[P]$ the grading $\deg(x_a) = \mathrm{rk}(a)$.

Face *module* for a matroid over a domain

Let $\mathcal{M}$ be a matroid over a domain R. If L is the link of $(\emptyset, e)$ in $\mathrm{Gr}\,\mathcal{M}$, then we define the face module of $\mathcal{M}$ to be

$$N_{\mathcal{M}} = k[L]^{|\operatorname{tor}(\emptyset)|}$$

Application to rings of integers

Recall that an number field $\mathbb{F}$ is a finite extension of $\mathbb{Q}$. A ring of integers is the integral closure of $\mathbb{Z}$ in a number field $\mathbb{F}$.

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Assume that R is a ring of integers that is also a PID. Then every f.g. torsion R-module N is finite and $N \simeq N^\vee$. Further, any f.g. R-module N is the direct sum of a free module and its torsion submodule:

$$N = N/\operatorname{tor}(N) \oplus \operatorname{tor}(N).$$

Now consider the homomorphism $\varphi : L_0(R\text{-mod}) \to \mathbb{Z}$ defined by

$$\begin{aligned} \varphi([F]) &= 1 && \text{for every free module } F, \\ \varphi([N]) &= |N| && \text{for every torsion module } N. \end{aligned}$$

It induces an homomorphism of polynomial rings

$$\tilde{\varphi} : L_0(R\text{-mod})[x, y] \to \mathbb{Z}[x, y].$$

Let $\mathcal{M}$ be a matroid over R, recall that the Grothendieck-Tutte polynomial of $\mathcal{M}$ is

$$T_{\mathcal{M}}(x, y) = \sum_{A \subseteq E} [\mathrm{tor}(A)^\vee](x-1)^{\mathrm{rk}(E)-\mathrm{rk}(A)}(y-1)^{|A|-\mathrm{rk}(A)}.$$

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Set $\tilde{T}_{\mathcal{M}}(x,y) = \tilde{\varphi}(T_{\mathcal{M}}(x,y))$, it results

$$\tilde{T}_{\mathcal{M}}(x,y) = \sum_{A\subseteq E} |\,\mathrm{tor}(A)|(x-1)^{\mathrm{rk}(E)-\mathrm{rk}(A)}(y-1)^{|A|-\mathrm{rk}(A)}$$

Recall that $H(N_{\mathcal{M}}, t)$ is the Hilbert series of the face module $N_{\mathcal{M}}$ of $\mathcal{M}$.

Theorem (B.-Martino, 2019)

If $\mathcal{M}$ is a relizable matroid over R with $r = \mathrm{rk}(E)$, then

$$H(N_{\mathcal{M}}, t) = \frac{t^r}{(1-t)^r} \tilde{T}_{\mathcal{M}}(1/t, 1).$$

A worked example

Set $R = \mathbb{Z}[i]$, $E = \{1,2\}$, consider the (realizable) matroid

$$\begin{array}{ccc} \mathcal{M}(\emptyset) \longrightarrow \mathcal{M}(1) & & \mathbb{Z}[i]^2 \longrightarrow \mathbb{Z}[i] \\ \downarrow \qquad\qquad \downarrow & \simeq & \downarrow \qquad\qquad \downarrow \\ \mathcal{M}(2) \longrightarrow \mathcal{M}(12) & & \mathbb{Z}[i] \oplus \mathbb{Z}[i]/(1+i)\mathbb{Z}[i] \longrightarrow \mathbb{Z}[i]/2\mathbb{Z}[i] \end{array}$$

and fix the realization $\psi : 2^E \to \mathbb{Z}[i]^2$ defined by
$\psi(1) = (1, 1+i)$, $\psi(2) = (1+i, 0)$.

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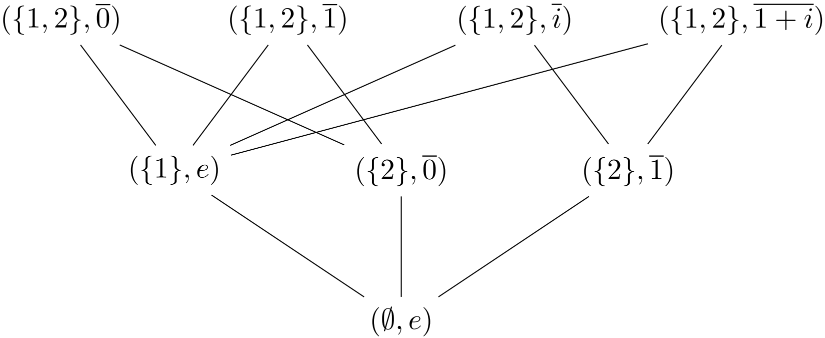
The poset of torsions $\mathrm{Gr}\,\mathcal{M}$ of $\mathcal{M}$ is



A worked example

Since $\mathcal{M}(\emptyset) = \mathbb{Z}[i]^2$ is torsion-free, $\mathrm{Gr}\,\mathcal{M} = \mathrm{link}(\emptyset, e)$ is a simplicial poset, and its face ring $k[\mathrm{Gr}\,\mathcal{M}]$ conincide with the face module $N_{\mathcal{M}}$ of $\mathcal{M}$. We can calculate the Hilbert series of $N_{\mathcal{M}}$ and the Grothendieck-Tutte polynomial of $\mathcal{M}$:

$$H(N_{\mathcal{M}}, t) = \frac{1 + t + 2t^2}{(1-t)^2}$$

$$\tilde{T}_{\mathcal{M}}(x, y) = x^2 + x + 1$$

$$\Rightarrow N_{\mathcal{M}}(t) = \frac{1 + t + 2t^2}{(1-t)^2} = \frac{t^2}{(1-t)^2}\tilde{T}_{\mathcal{M}}(1/t, 1).$$

Thank you for your attention!