Notes on Algebraic Geometry

Alessio Borzì

June 5, 2022

These are some informal notes on Algebraic Geometry, mostly on Scheme theory, that I wrote in the years 2020-2022 during my PhD at Warwick. They contain a lot of gaps and I am not certain that all proofs are correct. If you are able to fill in some of the gaps or have some corrections to suggest, do not hesitate to contact me at:
alessioborzi.math (at) gmail.com

Contents

Chapter 1

Category theory

1.1 Categories

Definition 1.1.1. A **category** $\mathscr{A}$ consists of

- a collection of **objects** $\mathrm{ob}(\mathscr{A})$;
- for every $A, B \in \mathrm{ob}(\mathscr{A})$, a collection of **morphisms** (or **maps**) $\mathrm{Hom}_{\mathscr{A}}(A, B)$ from A to B;
- for each $A, B, C \in \mathrm{ob}(\mathscr{A})$ a **composition** map

$$\mathrm{Hom}_{\mathscr{A}}(B, C) \times \mathrm{Hom}_{\mathscr{A}}(A, B) \to \mathrm{Hom}_{\mathscr{A}}(A, C), \quad (g, f) \mapsto g \circ f;$$

- for each $A \in \mathrm{ob}(\mathscr{A})$, an **identity** element $1_A \in \mathrm{Hom}_{\mathscr{A}}(A, A)$;

satisfying the following axioms: for each $f \in \mathrm{Hom}_{\mathscr{A}}(A, B)$, $g \in \mathrm{Hom}_{\mathscr{A}}(B, C)$ and $h \in \mathrm{Hom}_{\mathscr{A}}(C, D)$

1. $h \circ (g \circ f) = (h \circ g) \circ f$ (associativity)
2. $1_B \circ f = f = f \circ 1_A$ (identity)

We will simply write $\mathrm{Hom}(A, B)$ when the category is clear. Also, we will write $f : A \to B$ instead of $f \in \mathrm{Hom}(A, B)$.

Definition 1.1.2. A **subcategory** $\mathscr{A}$ of a category $\mathscr{B}$ has as its objects some of the objects of $\mathscr{B}$, and as morphisms some of the morphisms of $\mathscr{B}$, such that they are closed under composition and include the identity morphisms of the objects in $\mathscr{A}$. A subcategory $\mathscr{A}$ of $\mathscr{B}$ is **full** if $\mathrm{Hom}_{\mathscr{A}}(A, B) = \mathrm{Hom}_{\mathscr{B}}(A, B)$ for every $A, B \in \mathrm{ob}(\mathscr{A})$.

Example 1.1.3 (Categories of mathematical structures)**.**

1. The category **Set** whose objects are sets and the morphisms are the functions between them. The composition is the composition of functions and the identity is the usual identity map.

2. The category **Grp** whose objects are groups and the morphisms are the homomorphisms between them. We will denote by **Ab** the subcategory of **Grp** of abelian groups.

3. The category **Ring** whose objects are rings and morphisms are homomorphisms of rings.

4. The category **Top** whose objects are topological spaces and the morphisms are the continuous maps.

Example 1.1.4 (Categories as mathematical structures)**.**

1. The category **0** with no objects or maps at all.

2. The category **1** with one object and one identity map.

3. The category **2** with two objects and one non-identity map: $A \xrightarrow{f} B$.

4. The cateogry **3** with three objects and three non-identity maps:

$$\begin{tikzcd} & B \arrow[dr, "g"] & \\ A \arrow[ur, "f"] \arrow[rr, "g\circ f"] & & C \end{tikzcd}$$

5. A group G can be seen as a category with one object A, in which every morphism is an isomorphism, so that $G = \mathrm{Hom}(A, A)$.

6. A **discrete** category, that is a category in which every morphism is an identity.

7. Let P be a set. A **preorder** on P is a reflexive and transitive binary relation $\leq$. A preordered set $(P, \leq)$ can be seen as a category in which the objects are the elemnts of P and the morphisms are the relations $A \leq B$. Note that a partially ordered set (poset) is a particular instance of a preordered set.

Definition 1.1.5. A morphism $f : A \to B$ in a category $\mathscr{A}$ is

1. an **isomorphism** if there exists a map $g : B \to A$ such that $g \circ f = 1_A$ and $f \circ g = 1_B$ (in this case we say that A and B are **isomorphic**, written $A \simeq B$ and g is said to be an **inverse** of f, sometimes denoted by f^{-1});

2. an **epimorphism** if $g \circ f = h \circ f$ implies $g = h$, for every $g, h : B \to C$ (in this case we say that f is **epic**);

3. a **monomorphism** if $f \circ g = f \circ h$ implies $g = h$ for every $g, h : C \to A$ (in this case we say that f is **monic**).

Remark 1.1.6. It is easy to see that an isomorphism is both a monomorphism and an epimorphism (it is enough to compose with the inverse function). However, the converse is not true in general. It is enough to consider the category **2** in which the non-identity map is both monic and epic, but it is not an isomorphism.

Nevertheless, note that in the category **Set**, the epimorphisms are the surjective functions, the monomirphisms are the injective functions, and the isomorphisms are the bijective functions. Therefore, in this particular case, a morphism is an isomorphism if and only if it is both monic and epic.

1.2 Functors

Definition 1.2.1. Let $\mathscr{A}$ and $\mathscr{B}$ be two categories. A (covariant) **functor** $F : \mathscr{A} \to \mathscr{B}$ consists of two functions (that, with abuse of notation, will be both denoted with F): one from the objects of $\mathscr{A}$ to the objects of $\mathscr{B}$, and the other from the morphisms of $\mathscr{A}$ to the morphisms of $\mathscr{B}$, such that if $f \in \mathrm{Hom}_{\mathscr{A}}(A, B)$ then $F(f) \in \mathrm{Hom}_{\mathscr{B}}(F(A), F(B))$, and

1. $F(g \circ f) = F(g) \circ F(f)$,
2. $F(1_A) = 1_{F(A)}$,

for every $f \in \mathrm{Hom}_{\mathscr{A}}(A, B)$, $g \in \mathrm{Hom}_{\mathscr{A}}(B, C)$ and $A, B, C \in \mathrm{ob}(\mathscr{A})$.

Definition 1.2.2. Given a category $\mathscr{A}$, the **opposite** category of $\mathscr{A}$, denoted by $\mathscr{A}^{op}$, is a category with objects the same objects of $\mathscr{A}$, and morphisms reversed, that is if $f \in \mathrm{Hom}_{\mathscr{A}}(A, B)$, then $f \in \mathrm{Hom}_{\mathscr{A}^{op}}(B, A)$. A **contravariant functor** from $\mathscr{A}$ to $\mathscr{B}$ is a functor $F : \mathscr{A}^{op} \to \mathscr{B}$.

Example 1.2.3.

1. A first trivial example is the **identity functor** $1_{\mathscr{A}} : \mathscr{A} \to \mathscr{A}$ defined by the identity functions on the objects and on the morphisms.

2. The **forgetful functor** goes from a category of mathematical structures to the category of sets, assigning to each object its underlying set. More precisely, for instance consider the functor $F : \mathbf{Top} \to \mathbf{Set}$ that assign to each topological space X its support and to each continuous function, the same function seen as a map of sets.

3. The functor $\mathcal{P} : \mathbf{Set} \to \mathbf{Set}$ that assign to each set X its power set $\mathcal{P}(X)$, and to each $f : X \to Y$ the function $\mathcal{P}(f) : \mathcal{P}(X) \to \mathcal{P}(Y)$ defined by $\mathcal{P}(f)(X') = f(X') \subseteq Y$, for every $X' \subseteq X$.

4. The i-th homology functor $H_i(\cdot, \mathbb{Z}) : \mathbf{Top} \to \mathbf{Ab}$ that assign to each topological space its i-th homology group $H_i(X, \mathbb{Z})$ and to every continuous function $f : X \to Y$ its induced homomoprhism on the homologies $H_i(f, \mathbb{Z}) : H_i(X, \mathbb{Z}) \to H_i(Y, \mathbb{Z})$.

Functors can be composed. In fact, let $F : \mathscr{A} \to \mathscr{B}$ and $G : \mathscr{B} \to \mathscr{C}$ be two functors of categories, its composition $G \circ F$ consists of the compositions of the functions of F and G on the objects and the compositions of the functions of F and G on the morphisms. Also, this composition of functors is associative, and obviusly $1_{\mathscr{B}} \circ F = F = F \circ 1_{\mathscr{A}}$. Therefore we can consider the "category of categories", that is a category **Cat** whose objects are the categories and the morphisms are functors between categories.

Definition 1.2.4. A functor $F : \mathscr{A} \to \mathscr{B}$ is **faithful** (respectively, **full**) if for every $A, B \in \mathrm{ob}(\mathscr{A})$ the function $F : \mathrm{Hom}_{\mathscr{A}}(A, B) \to \mathrm{Hom}_{\mathscr{B}}(F(A), F(B))$ is injective (respectively, surjective). A functor that is full and faithful is said to be **fully faithful**. The functor F is an **isomorphism** if it is so in the category **Cat**, that is, if there exists a functor $G : \mathscr{B} \to \mathscr{A}$ such that $F \circ G = 1_{\mathscr{B}}$, $G \circ F = 1_{\mathscr{A}}$. Equivalently, F is an isomorphism if it is a bijection on the objects and on the morphisms. In this case the categories $\mathscr{A}$ and $\mathscr{B}$ are said to be **isomorphic**.

If $\mathscr{A}$ is a subcategory of $\mathscr{B}$, then there is an inclusion functor $i : \mathscr{A} \to \mathscr{B}$, and such a functor is full if and only if $\mathscr{A}$ is a full subcategory of $\mathscr{B}$ (note that inclusions are always faithful, so there is no need of a notion of "faithful subcategory").

1.3 Natural transformations

As a functor is a "map" between two categories, intuitively a natural transformation can be thought of a "map" between two functors (with the same domain and codomain).

Definition 1.3.1. Let $F, G : \mathscr{A} \to \mathscr{B}$ be two functors. A **natural transformation** $\alpha : F \to G$ consists of, for each object $A \in \text{ob}(\mathscr{A})$, a morphism $\alpha_A : F(A) \to G(A)$ in $\mathscr{B}$ such that, for every $A, B \in \text{ob}(\mathscr{A})$ and $f \in \text{Hom}_{\mathscr{A}}(A, B)$, the following diagram

$$\begin{tikzcd} F(A) \arrow[r, "F(f)"] \arrow[d, "\alpha_A"] & F(B) \arrow[d, "\alpha_B"] \\ G(A) \arrow[r, "G(f)"] & G(B) \end{tikzcd}$$

commutes. The maps α_A are called **components** of α.

We can consider the identity natural transformation 1_F of a functor $F : \mathscr{A} \to \mathscr{B}$, defined by $(1_F)_A = 1_{F(A)}$. Also, we can define the composition of two natural transformations in the obvius way. Therefore we have defined another category, called the **functor category** and denoted by $[\mathscr{A}, \mathscr{B}]$, in which the objects are the functors from $\mathscr{A}$ to $\mathscr{B}$ and the morphisms are natural transformations.

Definition 1.3.2. Let $F, G : \mathscr{A} \to \mathscr{B}$ be two functors. A **natural isomorphism** is a natural transformation $\alpha : F \to G$ such that $\alpha_A : F(A) \to G(A)$ is an isomorphism for every $A \in \text{ob}(\mathscr{A})$.

Lemma 1.3.3. *Let $F, G : \mathscr{A} \to \mathscr{B}$ be two functors. A natural transformation $\alpha : F \to G$ is a natural isomorphism if and only if it is an isomorphism in the functor category*

Proof.

- $\Rightarrow$ For every $A \in \text{ob}(\mathscr{A})$, $\alpha_A : F(A) \to G(A)$ is an isomorphism. Therefore there exists a morphism $\beta_A : G(A) \to F(A)$ such that $\beta_A \circ \alpha_A = 1_{F(A)}$, $\alpha_A \circ \beta_A = 1_{G(A)}$. Hence, the natural transformation $\beta : G \to F$ given by the morphisms β_A is such that $\alpha \circ \beta = 1_G$, $\beta \circ \alpha = 1_F$, that is, α is an isomorphism in the functor category.

- $\Leftarrow$ By hypothesis, there exists a natural transformation $\beta : G \to F$ such that $\alpha \circ \beta = 1_G$, $\beta \circ \alpha = 1_F$. This implies that for every $A \in \text{ob}(\mathscr{A})$, $\alpha_A \circ \beta_A = 1_{G(A)}$, $\beta_A \circ \alpha_A = 1_{F(A)}$, that is α_A is an isomorphisms, as required. $\square$

Definition 1.3.4. An **equivalence of categories** $\mathscr{A}$ and $\mathscr{B}$ consists of two functors $F : \mathscr{A} \to \mathscr{B}$ and $G : \mathscr{B} \to \mathscr{A}$ such that $F \circ G$ is naturally isomorphic to the identity functor $1_{\mathscr{B}}$, and $G \circ F$ is naturally isomorphic to the identity functor $1_{\mathscr{A}}$. In this case, we say that the categories $\mathscr{A}$ and $\mathscr{B}$ are **equivalent**, written $\mathscr{A} \simeq \mathscr{B}$. We also say that the functors F and G are **equivalences**.

Definition 1.3.5. A functor $F : \mathscr{A} \to \mathscr{B}$ is **essentially surjective** (on objects) if for every $B \in \mathrm{ob}(\mathscr{B})$ there exists $A \in \mathrm{ob}(\mathscr{A})$ such that $F(A) \simeq B$.

Proposition 1.3.6. *A functor $F : \mathscr{A} \to \mathscr{B}$ is an equivalence if and only if it is an essentially surjective, fully faithful functor.*

Proof.

- $\Rightarrow$ Let $G : \mathscr{B} \to \mathscr{A}$ be the equivalence associated to F, and let $\alpha : 1_{\mathscr{A}} \to G \circ F$ be a natural isomorphism. First, we prove that F is faithful. Suppose that $f, g \in \mathrm{Hom}_{\mathscr{A}}(A, B)$ are such that $F(f) = F(g)$. By hypothesis the following diagram commutes

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\alpha_A"] & B \arrow[d, "\alpha_B"] \\ G(F(A)) \arrow[r, "G(F(f))"] & G(F(B)) \end{tikzcd}$$

therefore $\alpha_B \circ f \circ \alpha_A^{-1} = G(F(f))$. Similarly we have $\alpha_B \circ g \circ \alpha_A^{-1} = G(F(g))$. Since $F(f) = F(g)$ we have

$$\alpha_B \circ f \circ \alpha_A^{-1} = G(F(f)) = G(F(g)) = \alpha_B \circ g \circ \alpha_A^{-1} \Rightarrow f = g,$$

where we used the fact that α_A^{-1} and α_B are isomorphisms, and so they are both monic and epic. This proves that F is faithful. Now we prove that F is full. Let $g \in \mathrm{Hom}_{\mathscr{B}}(F(A), F(B))$ and consider a natural isomorphism $\beta : 1_{\mathscr{B}} \to F \circ G$. If there exists an $f \in \mathrm{Hom}_{\mathscr{A}}(A, B)$ such that $F(f) = g$, then, by the naturality of α applied to f, we will have $f = \alpha_B^{-1} \circ G(g) \circ \alpha_A$. Hence, our strategy will be to set $f = \alpha_B^{-1} \circ G(g) \circ \alpha_A$ and prove $F(f) = g$. In order to do this, first apply the naturality of α to the map α_A. We obtain $G(F(\alpha_A)) \circ \alpha_A = \alpha_{G(F(A))} \circ \alpha_A$ that implies $G(F(\alpha_A)) = \alpha_{G(F(A))}$. Now by applying the naturality of α to the map $G(g)$, we obtain

$$G(g) = \alpha_{G(F(B))} \circ G(F(G(g))) \circ \alpha_{G(F(A))}^{-1} = G(F(\alpha_B \circ G(g) \circ \alpha_A^{-1})) = G(F(f)).$$

By what we have proved above, since G is an equivalence, it is faithful, therefore $g = F(f)$. This proves that F is full. Finally, for the essential surjectivity it is enough to note that for every $B \in \mathrm{ob}(\mathscr{B})$, set $A = G(B)$, we have $B \simeq F(A)$ via the map β_B.

- $\Leftarrow$ First, we define a functor $G : \mathscr{B} \to \mathscr{A}$ as follows: for each $B \in \mathrm{ob}(\mathscr{B})$, using the axiom of choice, we choose an object $A \in \mathrm{ob}(\mathscr{A})$ such that there is an isomorphism $\beta_B : B \to F(A)$, and define $G(B) = A$. For every $B, B' \in \mathrm{ob}(\mathscr{B})$ and every map $g : B \to B'$, let $\beta_B : B \to F(A)$ and $\beta_{B'} : B' \to F(A')$ isomorphisms as above, we have a map $\beta_{B'} \circ g \circ \beta_B^{-1} \in \mathrm{Hom}(F(A), F(A'))$. Since F is full, there exists $f \in \mathrm{Hom}(A, A')$ such that $F(f) = \beta_{B'} \circ g \circ \beta_B^{-1}$. Define $G(g) = f$. It is not hard to verify that G is a well defined functor, and that the isomorphisms β_B constitutes a natural isomorphism between $F \circ G$ and $1_{\mathscr{B}}$. Now let $A \in \mathrm{ob}(\mathscr{A})$, $F(A)$ is isomorphic to some $F(B)$ such that $G(F(A)) = B$. Since F is full, the previous isomorphism is of the form $F(\alpha_A)$. It is not hard to verify that the isomorphisms α_A constitutes a natural isomorphism between $G \circ F$ and $1_{\mathscr{A}}$. $\square$

Remark 1.3.7. It is possible to compose natural transformations with functors, in the following way. If $F : \mathscr{A} \to \mathscr{A}'$, $F', G' : \mathscr{A}' \to \mathscr{A}''$ are functors, and $\eta' : F' \to G'$ is a natural transformation, then the maps $(\eta' F)_A = \eta'_{F(A)}$ define a natural transformation $\eta' F : F' \circ F \to G' \circ F$. In a similar way, if $F, G : \mathscr{A} \to \mathscr{A}'$ and $F' : \mathscr{A}' \to \mathscr{A}''$ are functors and $\eta : F \to G$ is a natural transformation, then the maps $(F'\eta)_A = F'(\eta_A)$ define a natural transformation $F'\eta : F' \circ F \to F' \circ G$.

1.4 Limits and colimits

Definition 1.4.1. Let $\mathscr{A}$ be a category. An object $I \in \mathrm{ob}(\mathscr{A})$ is **initial** if for every object $A \in \mathrm{ob}(\mathscr{A})$ there exists an unique map $f : I \to A$. An object $T \in \mathrm{ob}(\mathscr{A})$ is **terminal** if for every $A \in \mathrm{ob}(\mathscr{A})$ there exists a unique map $f : A \to T$. A **zero object** is an object that is both initial and terminal.

Lemma 1.4.2. *If I and I' are two initial (respectively terminal) objects in a category, then there exists a unique isomorphism $f : I \to I'$. In particular, $I \simeq I'$.*

Proof. Since I and I' are initial, then there exists a unique morphism $f : I \to I'$ and a unique morphism $f' : I \to I'$. Since the only morphism from I to I is the identity, and similarly for I', we must have $f' \circ f = 1_I$ and $f \circ f' = 1_{I'}$. The proof in the case when I and I' are terminal is analogous. $\square$

Definition 1.4.3. A **small category** is a category in which the objects are some sets and the morphisms are some functions between sets.

Definition 1.4.4. Let $\mathscr{A}$ be a category and $\mathbf{I}$ be a small category. A functor $D : \mathbf{I} \to \mathscr{A}$ is called a **diagram**. A **cone** on the diagram D is an object $A \in \mathrm{ob}(\mathscr{A})$ together with a family of morphisms $\left(A \xrightarrow{f_i} D(i)\right)_{i \in \mathbf{I}}$ in $\mathscr{A}$ such that for all maps $g : i \to j$ in $\mathbf{I}$, the following diagram commutes.

$$\begin{tikzcd} & A \arrow[dl, "f_i"'] \arrow[dr, "f_j"] & \\ D(i) \arrow[rr, "D(g)"] & & D(j) \end{tikzcd}$$

A morphism from a cone $\left(A \xrightarrow{f_i} D(i)\right)_{i \in \mathbf{I}}$ to a cone $\left(B \xrightarrow{g_i} D(i)\right)_{i \in \mathbf{I}}$ consist of a morphism $\varphi : A \to B$ such that $g_i \circ \varphi = f_i$ for every $i \in \mathbf{I}$.

We have defined the category of cones on a diagram $D : \mathbf{I} \to \mathscr{A}$.

1.4.1 (Inverse) limit

Definition 1.4.5. Let $D : \mathbf{I} \to \mathscr{A}$ be a diagram. The **limit** (or **projective limit**, or **inverse limit**) of the diagram D is a terminal object in the category of cones on D. Explicitly, the limit consists of:

$$\text{an object } \varprojlim_{i \in \mathbf{I}} D(i) \in \mathrm{ob}(\mathscr{A}), \text{ together with maps } f_i : \varprojlim_{i \in \mathbf{I}} D(i) \to D(i)$$

for every $i \in \mathbf{I}$, such that for every cone $\left(B \xrightarrow{g_i} D(i)\right)_{i\in\mathbf{I}}$ there exists a unique map $\varphi : B \to \varprojlim_{i\in\mathbf{I}} D(i)$ such that $f_i \circ \varphi = g_i$ for all $i \in \mathbf{I}$.

We note that limits do not always exist. When a category has a limit of a given diagram, from Lemma 1.4.2 the limit is unique up to isomorphisms.

Example 1.4.6. Let $\mathbf{I}$ be a small category. In the category **Set**, the inverse limit of a diagram $D : \mathbf{I} \to \mathbf{Set}$ is the set

$$\varprojlim_{i\in\mathbf{I}} D(i) = \left\{ (a_i) \in \prod_{i\in\mathbf{I}} D(i) : D(f)(a_i) = a_j \text{ for every } f \in \mathrm{Hom}_{\mathbf{I}}(i,j) \right\}$$

together with the canonical projections $\pi_j : \varprojlim_{i\in\mathbf{I}} D(i) \to D(j)$ for every $j \in I$. In fact, if $(B \xrightarrow{g_i} D(i))_{i\in\mathbf{I}}$ is a cone on D, then there is a unique morphism of cones $\varphi : B \to \varprojlim_{i\in\mathbf{I}} D(i)$ defined by $\varphi(b) = (g_i(b))_{i\in\mathbf{I}}$ for every $b \in B$.

1.4.2 Product

Definition 1.4.7 (Product)**.** If $\mathbf{I}$ is a discrete small category, then the limit of a diagram $D : \mathbf{I} \to \mathscr{A}$ is the **product** of the objects $D(i)$ for $i \in \mathbf{I}$, denoted $\prod_{i\in I} D(i)$.

Example 1.4.8.

1. In the category **Set** the product of two sets A_1 and A_2 exists and it is the cartesian product $A_1 \times A_2$ (together with the canonical projections $\pi_i : A_1 \times A_2 \to A_i$, $i \in \{1, 2\}$).

2. In the category **Top** the product of two topological spaces X_1 and X_2 exists, and it is the product topology $X_1 \times X_2$ together with the canonical projections.

3. In the category **Ab** the product of two groups A_1 and A_2 exists and it is the direct sum $A_1 \oplus A_2$ together with canonical projections.

4. In a poset $(P, \leq)$ (seen as a category), the product of two objects a and b, if it exists, is their meet (i.e. greatest lower bound) $a \wedge b$ together with the inequalities $a \wedge b \leq a$, $a \wedge b \leq b$.

1.4.3 Pullback (fibered product)

Definition 1.4.9 (Fibered product)**.** If $\mathbf{I}$ is the small category depicted below

$$\begin{tikzcd} & j \arrow[d] \\ i \arrow[r] & k \end{tikzcd}$$

then a limit of a diagram $D : \mathbf{I} \to \mathscr{A}$ is the **fibered product** (or **pullback**), denoted by $D(i) \times_{D(k)} D(j)$.

When a category $\mathscr{A}$ has a terminal object, then the fibered product with respect to the terminal object coincides with the product in the category $\mathscr{A}$.

Example 1.4.10. In the category **Set**, if we have a diagram

$$\begin{tikzcd} & C \arrow[d, "g"] \\ B \arrow[r, "f"] & A \end{tikzcd}$$

its fibered product is

$$B \times_A C = \{(b, c) \in B \times C : f(b) = g(c)\}.$$

1.4.4 Equalizer

Definition 1.4.11. If **I** is the small category depicted below

$$i \underset{g}{\overset{f}{\rightrightarrows}} j$$

then a limit of a diagram $D : \mathbf{I} \to \mathscr{A}$ is the **equalizer** of $D(f)$ and $D(g)$.

Example 1.4.12.

1. In the category **Set**, if we have a diagram

$$X \underset{g}{\overset{f}{\rightrightarrows}} Y,$$

   its equalizer is the set $E = \{x \in X : f(x) = g(x)\}$, together with the inclusion map $i : E \to X$ and its compositions with f and g.

2. (Kernels) In the category **Grp**, if we have a diagram

$$G \underset{\omega}{\overset{f}{\rightrightarrows}} H,$$

   where f is a homomorphisms of groups, and ω is the trivial homomorphism, then the equalizer of f and ω is $\ker f$ with the inclusion $\ker f \subseteq G$ and its compositions.

1.4.5 Filtered category

Definition 1.4.13. A cateogry $\mathscr{I}$ is **filtered** if

1. it is not the empty category;

2. for every $A, B \in \mathrm{ob}(\mathscr{I})$ there exists $C \in \mathrm{ob}(\mathscr{I})$ and two morphisms $A \to C$, $B \to C$;

3. for every pair of morphisms $f, g : A \to B$ there exists a morphism $h : B \to C$ such that $h \circ f = h \circ g$.

An example of filtered category is a **directed** poset $(I, \leq)$, that is, a nonempty poset in which for every $a, b \in I$ there exists $c \in I$ such that $a \leq c$ and $b \leq c$. In other words, every finite subset of I has an upper bound.

Example 1.4.14.

1. Let $\mathscr{A}$ be a category in which the objects are some sets and the morphisms are inclusions between them. Consider the directed poset $(\mathbb{N}, \leq)$, a diagram $D : \mathbb{N} \to \mathscr{A}$ corresponds to a chain of sets
$$D(0) \subseteq D(1) \subseteq D(2) \subseteq \dots$$
In this case, $\varprojlim_{i \in \mathbb{N}} D(i) = D(0)$.

2. (p-adic integers) Let $p \in \mathbb{Z}$ be a prime integer. Consider the filtered category $(\mathbb{N}, \leq)^{op}$, and let $D : \mathbb{N}^{op} \to \mathbf{Ring}$ be the following diagram
$$\dots \to \mathbb{Z}/p^3 \to \mathbb{Z}/p^2 \to \mathbb{Z}/p,$$
in which an element in $\mathbb{Z}/p^{i+1}$ is sent in $\mathbb{Z}/p^i$ to its class modulo p^i. The (inverse) limit of D is the ring of **p-adic integers**
$$\mathbb{Z}_p = \left\{(a_i)_{i \geq 1} : a_{i+1} \equiv a_i \pmod{p^i} \text{ for every } i \geq 1\right\},$$
where the sum is componentwise, and the product is the Cauchy product (by seeing the elements of $\mathbb{Z}_p$ as series). CHECK

1.4.6 (Direct) limit

If $D : \mathbf{I} \to \mathscr{A}$ is a diagram, then denote by $D^{op} : \mathbf{I}^{op} \to \mathscr{A}^{op}$ the corresponding functor in the opposite categories.

Definition 1.4.15. A **cocone** on a diagram $D : \mathbf{I} \to \mathscr{A}$ is a cone on the diagram D^{op}. Explicitly, it is an object $A \in \mathrm{ob}(\mathscr{A})$ together with a family of morphisms $(D(i) \xrightarrow{f_i} A)_{i \in \mathbf{I}}$ in $\mathscr{A}$ such that for all maps $g : i \to j$ in $\mathbf{I}$, the triangle

$$\begin{tikzcd} & A & \\ D(i) \arrow[ur, "f_i"] \arrow[rr, "D(g)"] & & D(j) \arrow[ul, "f_j"'] \end{tikzcd}$$

commutes. The morphisms between cocones on D are the morphisms between cones on D^{op}. Explicitly, a morphism from a cocone $\left(D(i) \xrightarrow{f_i} A\right)_{i \in \mathbf{I}}$ to a cocone $\left(D(i) \xrightarrow{g_i} B\right)_{i \in \mathbf{I}}$ consists of a morphism $\varphi : A \to B$ such that $g_i = \varphi \circ f_i$ for every $i \in \mathbf{I}$.

Definition 1.4.16. Let $D : \mathbf{I} \to \mathscr{A}$ be a diagram. The **colimit** (or **injective limit**, or **direct limit**) of the diagram D is a limit of the diagram D^{op}, or equivalently, an initial object in the category of cocones. Explicitly, the colimit consists of:

$$\text{an object } \varinjlim_{i \in \mathbf{I}} D(i) \in \mathrm{ob}(\mathscr{A}) \text{ together with maps } f_i : D(i) \to \varinjlim D(i)$$

for every $i \in \mathbf{I}$, such that for every cocone $\left(D(i) \xrightarrow{g_i} B\right)_{i\in\mathbf{I}}$ there exists a unique map $\varphi : \varinjlim_{i\in\mathbf{I}} D(i) \to B$ such that $\varphi \circ f_i = g_i$ for all $i \in \mathbf{I}$.

Example 1.4.17. Let $\mathbf{I}$ be a small filtered category. In the category **Set**, the direct limit of a diagram $D : \mathbf{I} \to \mathbf{Set}$ is the set

$$\varinjlim_{i\in\mathbf{I}} D(i) = \coprod_{i\in\mathbf{I}} D(i) \Big/ \sim$$

where $\coprod_{i\in\mathbf{I}} D(i) = \bigcup_{i\in\mathbf{I}} D(i) \times \{i\}$ is the disjoint union (see Example 1.4.20), and we have $(a_i, i) \sim (a_j, j)$ if there exists $k \in \mathrm{ob}(\mathbf{I})$ and two morphisms $f : i \to k$, $g : j \to k$ such that $D(f)(a_i) = D(g)(a_j)$. The maps $f_i : D(i) \to \varinjlim_{i\in\mathbf{I}} D(i)$ are defined as the composition of the inclusion $D(i) \subseteq \coprod D(i)$ and the canonical projection $\coprod D(i) \to \coprod D(i)\big/\sim$ (the filtered hypothesis ensures that $\sim$ is an equivalence relation).

In fact, if $(D(i) \xrightarrow{g_i} B)_{i\in\mathbf{I}}$ is a cocone on D, then there is a unique morphism of cocones $\varphi : \varinjlim D(i) \to B$ defined by $\varphi([(a_i, i)]) = g_i(a_i)$ (it is not hard to verify that φ is well defined).

Example 1.4.18.

1. Consider the diagram $D : \mathbb{N} \to \mathscr{A}$ of Example 1.4.14 (1)

$$D(0) \subseteq D(1) \subseteq D(2) \subseteq \dots$$

   In this case, $\varinjlim_{i\in\mathbb{N}} D(i) = \bigcup_{i\in\mathbb{N}} D(i)$.

2. (Localization) Let A be a (commutative, unitary) ring, and $S \subseteq A$ a multiplicative set. The poset $(S, |)$ (S with divisibility) is directed, in fact if $s, t \in S$, then $st \in S$ with $s \mid st$ and $t \mid st$. Consider the functor $D : S \to \mathbf{Ring}$ defined by $D(s) = A_s = A[1/s]$ and for every morphism $f : s \to t$, that is if $s \mid t$, then $D(f) : A_s \to A_t$ is the inclusion $A_s \subseteq A_t$. The colimit of the diagram D is the localization $S^{-1}A$. CHECK

1.4.7 Coproduct

Definition 1.4.19 (Coproduct)**.** If $\mathbf{I}$ is a discrete small category, then the colimit of a diagram $D : \mathbf{I} \to \mathscr{A}$ is the **coproduct** of the objects $D(i)$ for $i \in \mathbf{I}$, denoted $\coprod_{i\in I} D(i)$.

Example 1.4.20.

1. In the category **Set** the coproduct of a family of sets $\{A_i\}_{i\in I}$ exists and it is the disjoint union $\coprod_{i\in I} A_i = \bigcup_{i\in I} A_i \times \{i\}$ (together with the maps $A_j \hookrightarrow \bigcup_{i\in I} A_i \times \{i\}$ defined by $a_j \mapsto (a_j, j)$).

2. In the category **Ab** the coproduct of a finite number of groups exists and it coincides with the product, that is their direct sum together with canonical projections.

1.4.8 Pushout (fibered coproduct)

Definition 1.4.21 (Pushout)**.** If $\mathbf{I}$ is the small category depicted below

$$\begin{tikzcd} & j \\ i & k \arrow[l] \arrow[u] \end{tikzcd}$$

then a colimit of a diagram $D : \mathbf{I} \to \mathscr{A}$ is the **pushout**.

Example 1.4.22.

1. In the category **Set**, if we have a diagram

$$\begin{tikzcd} & C \\ B & A \arrow[l, "f"] \arrow[u, "g"'] \end{tikzcd}$$

   the pushout is $B \amalg C\big/ \sim$, where $f(a) \sim g(a)$ for every $a \in A$.

2. (Tensor product of A-algebras) Let A be a (commutative unitary) ring, the pushout (or fibered coproduct) in the category of commutative unitary rings of the follwing diagram

$$\begin{tikzcd} & C \\ B & A \arrow[l, "f"] \arrow[u, "g"'] \end{tikzcd}$$

   is the same as the tensor product $B \otimes_A C$ of A-algebras.

1.4.9 Coequalizer

Definition 1.4.23. If $\mathbf{I}$ is the small category depicted below

$$i \overset{f}{\underset{g}{\rightrightarrows}} j$$

then a colimit of a diagram $D : \mathbf{I} \to \mathscr{A}$ is the **coequalizer** of $D(f)$ and $D(g)$.

Example 1.4.24.

1. In the category **Set**, if we have a diagram

$$X \overset{f}{\underset{g}{\rightrightarrows}} Y,$$

   its coequalizer is the set $C = Y/\sim$, where $\sim$ is the equivalence relation generated by $\{(f(x), g(x)) : x \in X\}$, together with the canonical projection $\pi : Y \to Y/\sim$.

2. (Cokernels) In the category **Grp**, if we have a diagram

$$G \overset{f}{\underset{\omega}{\rightrightarrows}} H,$$

   where f is a homomorphisms of groups, and ω is the trivial homomorphism, then the coequalizer of f and ω is $\operatorname{coker} f$ with the projection $H \to H/\operatorname{im} f$.

1.5 Abelian categories

Definition 1.5.1. A category $\mathscr{C}$ is **additive** if

1. For every $A, B \in \mathrm{ob}(\mathscr{C})$, $\mathrm{Hom}(A, B)$ is an abelian group over an operation $+$, and the composition of morphisms is both left and right distributive with respect to $+$;

2. $\mathscr{C}$ has a zero object, denoted 0;

3. every pair of objects in $\mathscr{C}$ has a product.

If $\mathscr{C}$ is a category with a 0-object, the 0-morphism in $\mathrm{Hom}(A, B)$ is the composition $A \to 0 \to B$.

We have seen that the kernel of a homomorphism of groups can be seen as a limit (precisely, as an equalizer). Now we generalize this notion to a category with a zero object, and we proceed similarly for cokernels.

Definition 1.5.2. Let $\mathscr{C}$ be a category with a zero object 0. The **kernel** (resp. **cokernel**) of a morphism $f : A \to B$ is the limit (resp. colimit) of the diagram

$$\begin{tikzcd} & 0 \arrow[dr] & \\ A \arrow[ur] \arrow[rr] & & B \end{tikzcd}$$

Definition 1.5.3. An **abelian category** is an additive category satisfying the following additional properties:

1. every map has a kernel and cokernel;

2. every monomorphism is the kernel of its cokernel;

3. every epimorphism is the cokernel of its kernel.

1.6 Adjoints

Definition 1.6.1. Let $\mathscr{A}$ and $\mathscr{B}$ be categories, and let $F : \mathscr{A} \to \mathscr{B}$ and $G : \mathscr{B} \to \mathscr{A}$ be functors. We say that F is **left adjoint** to G, and G is **right adjoint** to F, if for every $A \in \mathrm{ob}(\mathscr{A})$ and $B \in \mathrm{ob}(\mathscr{B})$, there is a bijection

$$\varphi_{A,B} : \mathrm{Hom}_{\mathscr{B}}(F(A), B) \to \mathrm{Hom}_{\mathscr{A}}(A, G(B))$$

(denoted by φ when the objects A and B are clear) such that for every pair of diagrams

$$A' \xrightarrow{p} A \xrightarrow{f} G(B) \ , \qquad F(A) \xrightarrow{g} B \xrightarrow{q} B'$$

we have

$$\varphi^{-1}(f \circ p) = \varphi^{-1}(f) \circ F(p), \quad \text{and} \quad \varphi(q \circ g) = G(q) \circ \varphi(g). \tag{1.1}$$

An **adjunction** between F and G is a choice of a family of bijections $\{\varphi_{A,B}\}$ as above.

Proposition-Definition 1.6.2. *Let $F : \mathscr{A} \to \mathscr{B}$ and $G : \mathscr{B} \to \mathscr{A}$ be adjoint functors with adjunction $\{\varphi_{A,B}\}$. Then the maps $\eta_A = \varphi(1_{F(A)})$ and $\epsilon_B = \varphi^{-1}(1_{G(B)})$ define two natrual transformations*

$$\eta : 1_{\mathscr{A}} \to G \circ F, \qquad \epsilon : F \circ G \to 1_{\mathscr{B}}$$

*called the **unit** and **counit** of the adjunction.*

Proof. Let $f : A \to A'$ be a map in $\mathscr{A}$, then the diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\eta_A"] & A' \arrow[d, "\eta_{A'}"] \\ G(F(A)) \arrow[r, "G(F(f))"] & G(F(A')) \end{tikzcd}$$

commutes. In fact, we have

$$\begin{aligned} G(F(f)) \circ \eta_A = G(F(f)) \circ \varphi(1_{F(A)}) = \varphi(F(f) \circ 1_{F(A)}) = \varphi(1_{F(A')} \circ F(f)) = \\ = \varphi(1_{F(A')}) \circ f = \eta_{A'} \circ f. \end{aligned}$$

The proof for ϵ is analogous. □

Lemma 1.6.3. *Let $F : \mathscr{A} \to \mathscr{B}$ and $G : \mathscr{B} \to \mathscr{A}$ be adjoint functors with adjunction $\{\varphi_{A,B}\}$, unit η and counit ϵ. Then*

$$\varphi(g) = G(g) \circ \eta_A, \quad \varphi^{-1}(f) = \epsilon_B \circ F(f)$$

for every $f : A \to G(B)$ and $g : F(A) \to B$, with $A \in \mathrm{ob}(\mathscr{A})$ and $B \in \mathrm{ob}(\mathscr{B})$.

Proof. We have $G(g) \circ \eta_A = G(g) \circ \varphi(1_{F(A)}) = \varphi(g \circ 1_{F(A)}) = \varphi(g)$, the other identity is analogous. □

Recall that it is possible to compose natural transformations with functors (Remark 1.3.7).

Proposition 1.6.4 (Triangle identities)**.** *Let $F : \mathscr{A} \to \mathscr{B}$ and $G : \mathscr{B} \to \mathscr{A}$ be adjoint functors with unit η and counit ϵ. Then, the following triangles commutes*

$$\begin{tikzcd} F \arrow[r, "F\eta"] \arrow[dr, "1_F"'] & FGF \arrow[d, "\epsilon F"] \\ & F \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, "\eta G"] \arrow[dr, "1_G"'] & GFG \arrow[d, "G\epsilon"] \\ & G \end{tikzcd} \tag{1.2}$$

*These are also called **triangle identities**.*

Proof. We have to verify that, for any $A \in \mathrm{ob}(\mathscr{A})$, the diagram

$$\begin{tikzcd} F(A) \arrow[r, "F(\eta_A)"] \arrow[dr, "1_{F(A)}"'] & FGF(A) \arrow[d, "\epsilon_{F(A)}"] \\ & F(A) \end{tikzcd}$$

commutes (the proof for the other diagram is analogous). In order to verify that, it is enough to write

$$\epsilon_{F(A)} \circ F(\eta_A) = \varphi^{-1}(1_{GF(A)}) \circ F(\eta_A) = \varphi^{-1}(1_{GF(A)} \circ \eta_A) = \varphi^{-1}(\eta_A) = \varphi^{-1}(\varphi(1_{F(A)})) = 1_{F(A)}.$$

□

Theorem 1.6.5. *Let $F : \mathscr{A} \to \mathscr{B}$ and $G : \mathscr{B} \to \mathscr{A}$ be functors. There is a one to one correspondence between*

1. *adjunctions between F and G (with F on the left and G on the right);*
2. *pairs of natural transformations $\eta : 1_{\mathscr{A}} \to G \circ F$, $\epsilon : F \circ G \to 1_{\mathscr{B}}$ satisfying the triangle identities* (1.2).

Proof. We have already seen in Proposition 1.6.4 that an adjunction gives rise to a pair (η, ϵ) satisfying the triangle identities. Conversely, let (η, ϵ) be a pair of natural trasnformations that satisfies the triangle identities. We must show that there exists a unique adjunction between F and G. For existence, for every $A \in \mathrm{ob}(\mathscr{A})$ and $B \in \mathrm{ob}(\mathscr{B})$ define

$$\varphi_{A,B}(g) = G(g) \circ \eta_A, \quad \varphi^{-1}_{A,B}(f) = \epsilon_B \circ F(f).$$

From the following commutative diagram

$$\begin{tikzcd} F(A) \arrow[r, "F(\eta_A)"] \arrow[dr, "1_{F(A)}"'] & FGF(A) \arrow[r, "FG(g)"] \arrow[d, "\epsilon_{F(A)}"] & FG(B) \arrow[d, "\epsilon_B"] \\ & F(A) \arrow[r, "g"] & B \end{tikzcd}$$

it easy to see that $\varphi_{A,B}$ and $\varphi^{-1}_{A,B}$ are inverse of each other. Further, to show that the family $\{\varphi_{A,B}\}$ is an adjunction between F and G, we just need to prove they verify (1.1):

$$\varphi^{-1}(f \circ p) = \epsilon_B \circ F(f \circ p) = \epsilon_B \circ F(f) \circ F(p) = \varphi^{-1}(f) \circ F(p),$$
$$\varphi(q \circ g) = G(q \circ g) \circ \eta_A = G(q) \circ G(g) \circ \eta_A = G(q) \circ \varphi(g).$$

Finally, uniqueness follows from Lemma 1.6.3. □

Let $G : \mathscr{B} \to \mathscr{A}$ be a functor and $A \in \mathrm{ob}(\mathscr{A})$, let $(A \Rightarrow G)$ be the category in which objects are pair (B, f) where $B \in \mathrm{ob}(\mathscr{B})$ and $f \in \mathrm{Hom}(A, G(B))$, and the morphisms are morphisms $q : B \to B'$ in $\mathscr{B}$ such that the diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "f'"'] & G(B) \arrow[d, "G(q)"] \\ & G(B') \end{tikzcd}$$

commutes.

Lemma 1.6.6. *Let $F : \mathscr{A} \to \mathscr{B}$ and $G : \mathscr{B} \to \mathscr{A}$ be adjoint functors. Then for every $A \in \mathrm{ob}(\mathscr{A})$, the unit map $\eta_A : A \to G(F(A))$ is an initial object of $(A \Rightarrow G)$.*

Proof. Let $(B, f) \in \mathrm{ob}(A \Rightarrow G)$, explicitly, $f : A \to G(B)$. A morphism $(F(A), \eta_A) \to (B, f)$ in $(A \Rightarrow G)$ is a map $q : F(A) \to B$ such that

$$f = G(q) \circ \eta_A = G(q) \circ \varphi(1_{F(A)}) = \varphi(q \circ 1_{F(A)}) = \varphi(q).$$

It follows that $q = \varphi^{-1}(f)$ and thus it is uniquely determined. □

Theorem 1.6.7. *Let $F : \mathscr{A} \to \mathscr{B}$ and $G : \mathscr{B} \to \mathscr{A}$ be functors. There is a one to one correspondence between*

1. *adjunctions between F and G (with F on the left and G on the right);*
2. *natural transformations $\eta : 1_{\mathscr{A}} \to G \circ F$ such that $\eta_A : A \to GF(A)$ is an initial object in $(A \Rightarrow G)$ for every $A \in \mathrm{ob}(\mathscr{A})$.*

Proof. TO DO □

Chapter 2

Sheaves and presheaves

2.1 Definition of (pre)sheaf

Let X be a topological space, let $\mathcal{U}$ be the family of open sets of X, and consider the poset $(\mathcal{U}, \subseteq)$. Note that its associated category and its opposite are filtered.

Definition 2.1.1. Let $\mathscr{A}$ be a category. A **presheaf** on a topological space X is a contravariant functor $\mathscr{F} : \mathcal{U}^{op} \to \mathscr{A}$.

Explicitly, if for instance $\mathscr{A}$ is the category **Set**, a presheaf $\mathscr{F}$ of sets on X consists of:

1. for every open set $U \subseteq X$ a set $\mathscr{F}(U)$,
2. for every open sets $U \subseteq V$ a map

$$\rho_{V,U} : \mathscr{F}(V) \to \mathscr{F}(U)$$

   called **restriction map**, such that

   (a) $\rho_{U,U} = 1_{\mathscr{F}(U)}$,
   (b) if $U \subseteq V \subseteq W$ are open sets, then $\rho_{V,U} \circ \rho_{W,V} = \rho_{W,U}$.

Note that $\mathscr{A}$ can be any category, in fact if $\mathscr{A}$ is the category **Ab**, then we have a presheaf of abelian groups on X, if $\mathscr{A}$ is the category **Ring**, then we have a presheaf of rings on X, etc.

The elements of $\mathscr{F}(U)$ are called **sections** of $\mathscr{F}$ over U and the elements of $\mathscr{F}(X)$ are called **global sections**. The object $\mathscr{F}(U)$ will be sometimes denoted by $\Gamma(U, \mathscr{F})$ or $H^0(U, \mathscr{F})$. We will also denote by $s_{|U}$ the restriction $\rho_{V,U}(s)$ of $s \in \mathscr{F}(V)$ to U.

Example 2.1.2. Let X be an algebraic variety endowed with the Zariski topology. The functor $\mathcal{O}$ that assign to each open set U the ring of regular functions $\mathcal{O}(U)$, and to each inclusion of open sets $U \subseteq V$ the (usual) restriction map $\mathcal{O}(V) \to \mathcal{O}(U)$, is a presheaf of rings on X.

Example 2.1.3. Let X and Y be a topological spaces and let $\mathscr{C}$ be the functor that assign to each open subset U of X the ring of continuous functions from U to Y

$$\mathscr{C}(U, Y) = \{f : U \to Y : f \text{ is continuous}\},$$

and to each inclusion of open sets $U \subseteq V$ the (usual) restriction function $\mathscr{C}(V,Y) \to \mathscr{C}(U,Y)$ defined by $f \mapsto f_{|U}$. Then $\mathscr{C}$ is a presheaf of sets on X. If $Y = \mathbb{R}$ with the usual topology, $\mathscr{C}$ is a presheaf of rings on X.

Example 2.1.4. Let X and Y be a manifolds and let $\mathscr{F}$ be the functor that assign to every open subset U of X the set of differentiable functions from U to Y, and to every inclusion of open sets $U \subseteq V$ the (usual) restriction map $\mathscr{F}(V) \to \mathscr{F}(U)$. Then $\mathscr{F}$ is a presheaf of sets on X.

Example 2.1.5 (Constant presheaf)**.** Let X be a topological space and let A be an object of a category $\mathscr{A}$. Define the functor $\mathscr{F}$ that assign to each open set U the object $\mathscr{F}(U) = A$, and to every inclusion of open sets $U \subseteq V$ the identity map on A. Then $\mathscr{F}$ is a presheaf on X, called the **constant presheaf** (not to be confused with the constant sheaf, see Example 2.1.11).

Definition 2.1.6. A presheaf $\mathscr{F}$ of sets or abelian groups on a topological space X is a **sheaf** if for every open set $U \subseteq X$ and every open cover $\{U_i\}_{i \in I}$ of U, the following properties are satisfied:

1. if $s_1, s_2 \in \mathscr{F}(U)$ with $s_{1|U_i} = s_{2|U_i}$ for all $i \in I$, then $s_1 = s_2$;
2. for every family of sections $s_i \in \mathscr{F}(U_i)$ such that $s_{i|U_i \cap U_j} = s_{j|U_i \cap U_j}$ for all $i, j \in I$, there exists $s \in \mathscr{F}(U)$ such that $s_{|U_i} = s_i$ for all $i \in I$.

Note that condition (1) implies that the section s in condition (2) is unique.

Remark 2.1.7. If $\mathscr{F} : \mathcal{U}^{op} \to \mathscr{A}$ is a sheaf on X, and $U = \bigcup_{i \in I} U_i$ is an open cover that is also a base for the induced topology on U, then from the properties of the sheaf we have

$$\mathscr{F}(U) = \varprojlim_{i \in I} \mathscr{F}(U_i).$$

(alternatively, if $\{U_i\}_{i \in I}$ is an arbitrary open cover, it is enough to enlarge the family $\{U_i\}_{i \in I}$ by adding the intersections $U_i \cap U_j$).

Note that if $U = \emptyset$ and then $\{U_i\}_{i \in \emptyset}$ is the empty cover, this means that $\mathscr{F}(\emptyset)$ is the inverse limit of a diagram from the empty category to $\mathscr{A}$, that is, $\mathscr{F}(\emptyset)$ is a terminal object in $\mathscr{A}$. This observation, also implies that, if U and V are disjoint open subsets of X, then

$$\mathscr{F}(U \cup V) = \mathscr{F}(U) \times \mathscr{F}(V),$$

since the fibered product on a terminal object is the same as the product.

Example 2.1.8 (A presheaf which is not a sheaf)**.** Consider the presheaf $\mathscr{B}$ of bounded holomorphic functions on $\mathbb{C}$. For every $i \in \mathbb{N}$, let $U_i = \{z \in \mathbb{C} : |z| < i\}$ and $f_i \in \mathscr{B}(U_i)$ defined by $f_i(z) = z$. Then $\{U_i\}_{i \in \mathbb{N}}$ is an open cover of $\mathbb{C}$, but there is no $f \in \mathscr{B}(\mathbb{C})$ such that $f_{|U_i} = f_i$ for every $i \in \mathbb{N}$ (in fact, by Liouville's theorem we have $\mathscr{B}(\mathbb{C}) = \mathbb{C}$).

Example 2.1.9 (Restriction of a sheaf)**.** Let X be a topological space, $\mathcal{U}$ be the family of open sets of X and let $\mathscr{F} : \mathcal{U}^{op} \to \mathscr{A}$ be a sheaf on X. Fix an open set U of X and define a contravariant functor $\mathscr{F}_{|U}$ from the category of open subsets of U with respect to inclusion to $\mathscr{A}$ by $\mathscr{F}_{|U}(V) = \mathscr{F}(V)$ for every open set $V \subseteq U$ with the same restriction maps. Then $\mathscr{F}_{|U}$ is a sheaf on U, called the **restriction** of $\mathscr{F}$ to U.

Example 2.1.10 (Skyscraper sheaf)**.** Let X be a topological space and $\mathcal{U}$ be the family of open sets of X. Fix a point $p \in X$ and a group $G \in \mathbf{Ab}$, and define $\mathscr{S} : \mathcal{U}^{op} \to \mathbf{Ab}$ by

$$\mathscr{S}(U) = \begin{cases} G & p \in U, \\ 0 & p \notin U, \end{cases}$$

where the restriction map associated to the inclusion $U \subseteq V$ is

$$\rho_{V,U} = \begin{cases} 1_G & p \in U \subseteq V, \\ G \to 0 & p \in V, p \notin U, \\ 1_0 & p \notin V. \end{cases}$$

Then $\mathscr{S}$ is a sheaf on X, called the **skyscraper sheaf** defined by G at p.

Example 2.1.11 (Constant sheaf)**.** Let G be an Abelian group and X a topological space. Endow G with the discrete topology, and for every open set U of X define

$$\mathscr{F}(U) = \{f : U \to G : f \text{ continuous}\},$$

with the usual restriction maps. Since G has the discrete topology, if U is a connected open set, every function $f \in \mathscr{F}(U)$ is constant, in other words $\mathscr{F}(U) \simeq G$. We have that $\mathscr{F}$ is a sheaf of Abelian groups, called the **constant sheaf** $\mathscr{F}$ on X determined by G. We will sometimes denote it by $\underline{G}$.

Remark 2.1.12 (Constant sheaf vs constant preshaef)**.** In general, the constant presheaf is not a sheaf. In fact, let $\mathscr{A}$ be the category **Set**, $A = \{a, b\}$ and let $X = \{x, y\}$ with the discrete topology. Let $\mathscr{F}$ be the constant presheaf on X defined by A. The family of sections $a \in \mathscr{F}(\{x\})$ and $b \in \mathscr{F}(\{y\})$ cannot glue to any global section. Alternatively, if X is any topological space and A is a set with at least two elements, then pick $a, a' \in A = \mathscr{F}(\emptyset)$ with $a \neq a'$, a and a' coincide on the empty cover, but they are not equal, so $\mathscr{F}$ is not a sheaf (in other words, $\mathscr{F}(\emptyset)$ is not a terminal object). On the other hand, for the constant sheaf we have

$$\mathscr{F}(\emptyset) = \{f : \emptyset \to G : f \text{ continuous}\} = \{\emptyset\},$$

that is, $\mathscr{F}(\emptyset)$ is a group with one element, so it is a terminal object in the category **Ab**.

Definition 2.1.13. Let $f : X \to Y$ be a continuous function of topological spaces, and let $\mathscr{F}$ be a presheaf on X. The **pushoforward** (or **direct image**) of $\mathscr{F}$ is a presheaf $f_*\mathscr{F}$ defined by

$$f_*\mathscr{F}(U) = \mathscr{F}(f^{-1}(U)),$$

for every open set U of Y. If $U \subseteq V$, the restriction map on $f_*\mathscr{F}$ is the restriction map of $\mathscr{F}$ associated to the inclusion $f^{-1}(U) \subseteq f^{-1}(V)$. If $\mathscr{F}$ is a sheaf, then so is $f_*\mathscr{F}$.

If $f : X \to Y$ is continuous, it induces a functor $F : \mathcal{U}_Y^{op} \to \mathcal{U}_X^{op}$ from the category of open sets of Y to the category of open sets of X ordered both by reverse inclusion, defined by $F(U) = f^{-1}(U)$ (in fact, if $U \subseteq V$ then $f^{-1}(U) \subseteq f^{-1}(V)$). The pushforward presheaf can be seen as the composition $\mathscr{F} \circ F$.

Remark 2.1.14 (Skyscraper sheaf as pushforward)**.** Let X be a topological space and fix $p \in X$. Let $i_p : \{p\} \to X$ be the inclusion map $\{p\} \subseteq X$. The skyscraper sheaf $\mathscr{S}$ at p defined by an abelian group G can be seen as the pushforward of the constant sheaf $\underline{G}$ by i_p, that is $\mathscr{S} = i_{p*}(\underline{G})$. In fact, we have

$$i_{p*}(\underline{G})(U) = \underline{G}(i_p^{-1}(U)) = \begin{cases} G & p \in U, \\ 0 & p \notin U. \end{cases}$$

2.2 Morphisms of (pre)sheaves

Definition 2.2.1. Let $\mathscr{F}$ and $\mathscr{G}$ be two presheaves on X. A **morphism of presheaves** is a natural transformation $\varphi : \mathscr{F} \to \mathscr{G}$ (seeing $\mathscr{F}$ and $\mathscr{G}$ as functors). If $\mathscr{F}$ and $\mathscr{G}$ are sheaves, a **morphisms of sheaves** $\varphi : \mathscr{F} \to \mathscr{G}$ is a morphism of preasheaves.

Explicitly, a morphism of presheaves $\varphi : \mathscr{F} \to \mathscr{G}$ consist of, for every open set U of X, a morphism $\varphi(U) : \mathscr{F}(U) \to \mathscr{G}(U)$ such that, for every inclusion of open sets $U \subseteq V$, the following diagram commutes

$$\begin{tikzcd} \mathscr{F}(V) \arrow[r, "\rho_{V,U}"] \arrow[d, "\varphi(V)"] & \mathscr{F}(U) \arrow[d, "\varphi(U)"] \\ \mathscr{G}(V) \arrow[r, "\rho'_{V,U}"] & \mathscr{G}(U) \end{tikzcd}$$

With this definition, we have defined the category of presheaves, and the category of sheaves. In particular, from the definitions, the second is a full subcategory of the first. We will denote the category of sheaves (resp. presheaves) of objects in a category $\mathscr{A}$ on X by $\mathscr{A}_X$ (resp. $\mathscr{A}_X^{\text{pre}}$). For instance, a sheaf of sets on X is an object in $\mathbf{Set}_X$.

Two (pre)sheaves are **isomorphic** if they are so as objects of their category.

Example 2.2.2. Let X, Y and Z be topological spaces, and let $\mathscr{C}_Y$ (resp. $\mathscr{C}_Z$) be the sheaf of continuous functions from open sets of X to Y (resp. Z) as in Example 2.1.3. A continuous function $g : Y \to Z$ induces a morphism of sheaves $\varphi : \mathscr{C}_Y \to \mathscr{C}_Z$ defined by $\varphi(U)(f) = g \circ f$ for every $f \in \mathscr{C}_Y$.

If $\varphi : \mathscr{F} \to \mathscr{G}$ is a morphism of presheaves on X, and $f : X \to Y$ is a continuous function, then the pushforward induces a morphism $f_*\varphi : f_*\mathscr{F} \to f_*\mathscr{G}$ of presheaves on Y defined by

$$f_*\varphi(U) = \varphi(f^{-1}(U)).$$

With this definition, if $\mathscr{A}$ is a category, and $f : X \to Y$ is a continuous function of topological spaces, the pushforward induces a functor $f_* : \mathscr{A}_X \to \mathscr{A}_Y$.

Definition 2.2.3 (Restriction of morphism of sheaves)**.** Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves on X. The **restriction** of φ to an open set U of X is the morphism of sheaves $\varphi_{|U} : \mathscr{F}_{|U} \to \mathscr{G}_{|U}$ defined by $\varphi_{|U}(V) = \varphi(V)$ for every open set $V \subseteq U$.

Proposition-Definition 2.2.4 (Sheaf Hom)**.** *Let $\mathscr{A}$ be a category, let $\mathscr{F}$ be a presheaf and $\mathscr{G}$ be a sheaf of objects of $\mathscr{A}$ on a topological space X. Define the presheaf of sets $\mathscr{H}om_{\mathscr{A}_X}(\mathscr{F}, \mathscr{G})$ on X by*

$$\mathscr{H}om_{\mathscr{A}_X}(\mathscr{F}, \mathscr{G})(U) = \mathrm{Hom}_{\mathscr{A}_X}(\mathscr{F}_{|U}, \mathscr{G}_{|U}),$$

and if $U \subseteq V$, the restriction map is $\rho_{V,U} : \mathrm{Hom}_{\mathscr{A}_X}(\mathscr{F}_{|V}, \mathscr{G}_{|V}) \to \mathrm{Hom}_{\mathscr{A}_X}(\mathscr{F}_{|U}, \mathscr{G}_{|U})$ defined by $\rho_{V,U}(\varphi) = \varphi_{|U}$. The presheaf $\mathscr{H}om_{\mathscr{A}_X}(\mathscr{F}, \mathscr{G})$ is a sheaf of sets, called the ***sheaf Hom****.*

Proof. Let $\{U_i\}_{i\in I}$ be an open cover of an open set U, and let $s_i \in \mathscr{H}om_{\mathscr{A}_X}(\mathscr{F}, \mathscr{G})(U_i)$ for each $i \in I$ such that $s_{i|U_i\cap U_j} = s_{j|U_i\cap U_j}$ for every $i, j \in I$. Enlarge the family $\{U_i\}_{i\in I}$ with the intersections $U_i \cap U_j$. Since $\mathscr{G}$ is a sheaf, for every open set $V \subseteq U$ we have

$$\mathscr{G}(V) = \varprojlim_{i\in I} \mathscr{G}(V \cap U_i).$$

Therefore, if we consider the composition maps $s_i(V \cap U_i) \circ \rho_{V,V\cap U_i} : \mathscr{F}(V) \to \mathscr{G}(V \cap U_i)$ for every $i \in I$, then there exists a unique map $s(V) : \mathscr{F}(V) \to \mathscr{G}(V)$. Thus, we can check that we have defined a unique element $s \in \mathscr{H}om_{\mathscr{A}_X}(\mathscr{F}, \mathscr{G})(U)$ such that $s_{|U_i} = s_i$. □

Remark 2.2.5. If $\mathscr{F}$ and $\mathscr{G}$ are sheaves of abelian groups, then $\mathscr{H}om_{\mathbf{Ab}_X}(\mathscr{F}, \mathscr{G})$ has the structure of sheaf of abelian groups in a natural way. Later, we will see that, if $\mathscr{F}$ and $\mathscr{G}$ are $\mathcal{O}_X$-modules, then the sheaf Hom $\mathscr{H}om_{\mathrm{Mod}_{\mathcal{O}_X}}(\mathscr{F}, \mathscr{G})$ is also an $\mathcal{O}_X$-module.

Example 2.2.6. Let $\mathscr{F}$ be a sheaf of sets on X, and let $\underline{\{p\}}$ be the constant sheaf on X determined by the one element set $\{p\}$. Then we have $\mathscr{H}om_{\mathbf{Set}_X}(\underline{\{p\}}, \mathscr{F}) \simeq \mathscr{F}$. The isomorphism $\varphi : \mathscr{H}om_{\mathbf{Set}_X}(\underline{\{p\}}, \mathscr{F}) \to \mathscr{F}$ is defined as follows. For every open set $U \subseteq X$, $\varphi(U) : \mathscr{H}om_{\mathbf{Set}_X}(\underline{\{p\}}, \mathscr{F})(U) \to \mathscr{F}(U)$ is defined by $\varphi(U)(f) = f(U)(p) \in \mathscr{F}(U)$, where $f \in \mathscr{H}om_{\mathbf{Set}_X}(\underline{\{p\}}, \mathscr{F})(U)$, so $f(U) : \{p\} \to \mathscr{F}(U)$.

Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism of presheaves of abelian groups on X. We want to define a presheaf $\ker_{\mathrm{pre}} \varphi$ by assigning for each open set $U \subseteq X$, the kernel $\ker \varphi(U)$, and for each inclusion $U \subseteq V$, the restricion map obtained as follows. Consider the diagram

$$\begin{tikzcd}
& 0 \arrow[dr] & \\
\mathscr{F}(V) \arrow[ur] \arrow[rr, "\varphi(V)"] \arrow[dd, "\rho_{V,U}"] & & \mathscr{G}(V) \arrow[dd, "\rho'_{V,U}"] \\
& 0 \arrow[dr] & \\
\mathscr{F}(U) \arrow[ur] \arrow[rr, "\varphi(U)"] & & \mathscr{G}(U).
\end{tikzcd}$$

Since the kernel $\ker \varphi(V)$ is the limit of the upper triangle in the diagram, we have some morphisms $f_V : \ker \varphi(V) \to \mathscr{F}(V)$ and $g_V : \ker \varphi(V) \to \mathscr{G}(V)$. Now, by considering the compositions with the respective restrictions maps $\rho_{V,U} \circ f_V$ and $\rho'_{V,U} \circ g_V$, since the previous diagram commutes and $\ker \varphi(U)$ is the limit of the lower triangle, there exists a unique map $\tau_{V,U} : \ker \varphi(V) \to \ker \varphi(U)$ such that $f_U \circ \tau_{V,U} = \rho_{V,U} \circ f_V$ and

$g_U \circ \tau_{V,U} = \rho'_{V,U} \circ g_V$. The map $\tau_{V,U}$ is the restriction map of $\ker_{\mathrm{pre}} \varphi$ associated to the inclusion $U \subseteq V$. Clearly $\tau_{U,U} = 1_{\ker \varphi(U)}$. If $U \subseteq V \subseteq W$ are open sets, from the uniqueness of the map $\tau_{W,U}$, it follows that $\tau_{W,U} = \tau_{V,U} \circ \tau_{W,V}$. Therefore $\ker_{\mathrm{pre}} \varphi$ is a presheaf.

Similarly, by using the same argument we can define dually the presheaf $\operatorname{coker}_{\mathrm{pre}} \varphi$.

Definition 2.2.7. Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism of presheaves of abelian groups on X. The presheaf $\ker_{\mathrm{pre}} \varphi$ defined above is the **presheaf kernel** and the presheaf $\operatorname{coker}_{\mathrm{pre}} \varphi$ is the **presheaf cokernel** of φ.

Proposition-Definition 2.2.8. *If $\varphi : \mathscr{F} \to \mathscr{G}$ is a morphism of sheaves of abelian groups on X, then the presheaf $\ker_{\mathrm{pre}} \varphi$ is a sheaf, called the **kernel sheaf**, denoted by $\ker \varphi$.*

Proof. Let $\{U_i\}_{i \in I}$ be an open cover of an open set U of X. Let $s_i \in (\ker_{\mathrm{pre}} \varphi)(U_i)$ such that $s_{i|U_i \cap U_j} = s_{j|U_i \cap U_j}$ for every $i, j \in I$. Since $s_i \in \ker \varphi(U_i) \subseteq \mathscr{F}(U_i)$ and $\mathscr{F}$ is a sheaf, there exists a unique section $s \in \mathscr{F}(U)$ such that $s_{|U_i} = s_i$. From the fact that φ is a morphism of sheaves, we have $\varphi(U)(s)_{|U_i} = \varphi(U_i)(s_i) = 0$ since $s_i \in \ker \varphi(U_i)$ for every $i \in I$. This implies that $\varphi(U)(s) = 0$ given that $\mathscr{G}$ is a sheaf. $\square$

The previous result does not hold for the presheaf cokernel.

Example 2.2.9 (A presheaf cokernel that is not a sheaf)**.** Let $X = \mathbb{C}$ and consider the two sheaf of abelian groups: $\mathcal{O}_X$ the sheaf of holomorphic functions with respect to addition and $\mathcal{O}_X^*$ the sheaf of nowhere zero holomorphic functions with respect to multiplication. Define the morphism of sheaves $\varphi : \mathcal{O}_X \to \mathcal{O}_X^*$ by, for each open set $U \subseteq X$, $\varphi(U)(f(z)) = e^{f(z)}$.

Now let $U = \mathbb{C} \setminus \{0\}$, $U_1 = \mathbb{C} \setminus (]-\infty, 0] \times \{0\})$ and $U_2 = \mathbb{C} \setminus ([0, +\infty[\times \{0\})$, note that $U = U_1 \cup U_2$. Define $f_1 : U_1 \to \mathbb{C}$ by $f_1(z) = \log|z| + i\theta$ such that $\theta \in]-\pi, \pi[$ and $z = |z|e^{i\theta}$; and similarly $f_2 : U_2 \to \mathbb{C}$ by $f_2(z) = \log|z| + i\theta$ such that $\theta \in]0, 2\pi[$ and $z = |z|e^{i\theta}$. Now, we have $f_j \in \mathcal{O}_X(U_j)$ and $\varphi(U_j)(f_j) = z_{|U_j} \in \mathcal{O}_X^*(U_j)$ for $j \in \{1, 2\}$. Therefore, the class of $z_{|U_j}$ in $(\operatorname{coker}_{\mathrm{pre}} \varphi)(U_j)$ is zero $[0]$ for $j \in \{1, 2\}$. Now consider the function $z_{|U} \in \mathcal{O}_X^*(U)$. Its class in the cokernel $[z_{|U}] \in (\operatorname{coker}_{\mathrm{pre}})(U)$ restricts to the zero class in U_1 and U_2, but it is not the zero class, since there is no holomorphic function $f(z) \in \mathcal{O}_X(U)$ such that $\varphi(f(z)) = e^{f(z)} = z$ in U. This proves that $\operatorname{coker}_{\mathrm{pre}} \varphi$ is not a sheaf.

Definition 2.2.10. Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves of abelian groups on X. The **presheaf image** is the presheaf $\operatorname{im}_{\mathrm{pre}} \varphi$ defined by, for each open set $U \subseteq X$, $(\operatorname{im}_{\mathrm{pre}} \varphi)(U) = \operatorname{im} \varphi(U)$, and for each inclusion of open sets $U \subseteq V$, the restriction map $(\operatorname{im}_{\mathrm{pre}} \varphi)(U) \to (\operatorname{im}_{\mathrm{pre}} \varphi)(V)$ is given by the restricion of the restriction map $\rho'_{V,U}$ of $\mathscr{G}$ to $\operatorname{im} \varphi(V)$.

Example 2.2.11 (A presheaf image that is not a sheaf)**.** By using the same notation of Example 2.2.9, the presheaf image of φ is not a sheaf. In fact, $\varphi(U_i)(f_i) = z_{|U_i} \in (\operatorname{im}_{\mathrm{pre}} \varphi)(U_i)$ for $i \in \{1, 2\}$, but there is no $f \in (\operatorname{im}_{\mathrm{pre}} \varphi)(U)$ such that $f_{|U_i} = f_i$ for $i \in \{1, 2\}$.

2.3 Stalks and sheafification

2.3.1 Stalks

Let X be a topological space and denote by $\mathcal{U}$ the family of its open sets. Fix $p \in X$ and let

$$\mathcal{U}_p = \{U \in \mathcal{U} : p \in U\},$$

the opposite category of $(\mathcal{U}_p, \subseteq)$ is filtered, since for every $U, V \in \mathcal{U}_p$ we have $U \cap V \in \mathcal{U}_p$.

Definition 2.3.1. Let $\mathscr{F}$ be a sheaf on X and let $p \in X$. The **stalk** $\mathscr{F}_p$ of $\mathscr{F}$ at p is the direct limit

$$\mathscr{F}_p = \varinjlim_{U \ni p} \mathscr{F}(U).$$

Explicitly, we have

$$\mathscr{F}_p = \coprod_{U \ni p} \mathscr{F}(U) / \sim$$

where $(f, U) \sim (g, V)$ if there exists $W \subseteq U, V$ such that $f_{|W} = g_{|W}$. If $p \in U$, then the image of a section $f \in \mathscr{F}(U)$ in $\mathscr{F}_p$ is the **germ** of f at p, sometimes denoted by $f_p = [(f, U)] \in \mathscr{F}_p$.

Example 2.3.2.

1. (Constant sheaf) Let $\mathscr{F}$ be the constant sheaf on X determined by G, and fix $p \in X$. We have $\mathscr{F}_p \simeq G$. In fact, the homomorphism of groups $\mathscr{F}_p \to G$ defined by $[(f, U)] \mapsto f(p)$ is an isomorphism.

2. (Skyscraper sheaf) Suppose that X is Hausdorff, fix $p \in X$ and let $\mathscr{S}$ be the skyscraper sheaf on X at p. Then, we have

$$\mathscr{S}_q = \begin{cases} G & q = p, \\ 0 & q \neq p. \end{cases}$$

   In fact, if $q = p$, then the map $\mathscr{S}_p \to G$ defined by $[(g, U)] \mapsto g$ is an isomorphism, on the other hand if $q \neq p$, since X is Hausdorff there exists an open set U such that $p \notin U$ and $q \in U$, this means that $\mathscr{S}(U) = 0$, and therefore $\mathscr{S}_q = 0$.

Definition 2.3.3. Let $\mathscr{F}$ be a sheaf of abelian groups on X. The **support** of a section $s \in \mathscr{F}(U)$ is

$$\mathrm{Supp}(s) = \{p \in U : s_p \neq 0 \text{ in } \mathscr{F}_p\}.$$

Definition 2.3.4. Let $\mathscr{F}$ be a sheaf on X, and let U be an open set of X. We say that $\big([(s(p), U_p)]\big)_{p \in U} \in \prod_{p \in U} \mathscr{F}_p$ consists of **compatible germs** if for all $p \in U$ there exists an open neighbourhood V_p of p contained in U and an element $t(p) \in \mathscr{F}(V_p)$ such that $t(p)_q = s(q)_q$ for all $q \in V_p$.

Proposition 2.3.5 (Sections are determined by germs)**.** *Let $\mathscr{F}$ be a sheaf of sets on X, and let U be an open set of X. The map*

$$\psi : \mathscr{F}(U) \to \prod_{p \in U} \mathscr{F}_p, \quad \psi(s) = (s_p)_{p \in U} = \big([(s, U)] \in \mathscr{F}_p\big)_{p \in U}$$

is injective. Further, its image is the set of compatible germs.

Proof. Let $s, t \in \mathscr{F}(U)$ such that $\psi(s) = \psi(t)$. This means that for every $p \in U$, we have $[(s, U)] = [(t, U)]$ in $\mathscr{F}_p$, that is, for every $p \in U$ there exists an open set $V_p \subseteq U$ containing p such that $s_{|V_p} = t_{|V_p}$. The family $\{V_p\}_{p\in U}$ is an oper cover of U, therefore, since $\mathscr{F}$ is a sheaf, we have $s = t$.

For the second statement, it is clear that the image is contained in the set of compatible germs. For the reverse inclusion, suppose that $\big([(s(p), U_p)]\big)_{p\in U} \in \prod_{p\in U} \mathscr{F}_p$ consists of compatible germs. For every $p \in U$, let V_p and $t(p) \in \mathscr{F}(V_p)$ as in Definition 2.3.4. Then, for all $p, p' \in U$, we have $t(p)_q = t(p')_q$ for every $q \in V_p \cap V_{p'}$. From the previous paragraph, it follows that $t(p)_{|V_p\cap V_{p'}} = t(p')_{|V_p\cap V_{p'}}$. Since $\mathscr{F}$ is a sheaf, there exists $t \in \mathscr{F}(U)$ such that $t_{|V_p} = t(p)$, that is $\psi(t) = \big([(s(p), U_p)]\big)_{p\in U}$. □

Fix $p \in X$, a morphism of presheaves $\varphi : \mathscr{F} \to \mathscr{G}$ induces a morphisms on the stalks $\varphi_p : \mathscr{F}_p \to \mathscr{G}_p$ that can be described as follows. Since the stalks are colimits, they come with morphisms $f_U : \mathscr{F}(U) \to \mathscr{F}_p$ and $g_U : \mathscr{G}(U) \to \mathscr{G}_p$. If we consider the composition maps $g_U \circ \varphi(U) : \mathscr{F}(U) \to \mathscr{G}_p$, by the definition of colimit, there exists a unique map $\varphi_p : \mathscr{F}_p \to \mathscr{G}_p$ such that $\varphi_p \circ f_U = g_U \circ \varphi(U)$. Explicitly, if $[(s, U)] \in \mathscr{F}_p$, then

$$\varphi_p\big([(s, U)]\big) = \varphi_p\big(f_U(s)\big) = g_U\big(\varphi(U)(s)\big) = [(\varphi(U)(s), U)] \in \mathscr{G}_p. \tag{2.1}$$

In other words, for every open set $U \subseteq X$ and every section $s \in \mathscr{F}(U)$, the germ $\varphi(U)(s)_p$ of $\varphi(U)(s)$ at $p \in U$ is equal to the image under φ_p of the germ s_p of s at p:

$$\big(\varphi(U)(s)\big)_p = \varphi_p(s_p).$$

Proposition 2.3.6. *Let $\mathscr{F}$ be a presheaf of sets and $\mathscr{G}$ be a sheaf of sets on X. If $\varphi_1, \varphi_2 : \mathscr{F} \to \mathscr{G}$ are two morphisms such that ${\varphi_1}_p = {\varphi_2}_p$ for every $p \in X$, then $\varphi_1 = \varphi_2$.*

Proof. Let U be an open set of X. Fix $s \in \mathscr{F}(U)$, for every $p \in U$ by hypothesis we have

$$\big(\varphi_1(U)(s)\big)_p = {\varphi_1}_p(s_p) = {\varphi_2}_p(s_p) = \big(\varphi_2(U)(s)\big)_p.$$

Hence, the image of $\varphi_1(U)(s)$ and $\varphi_2(U)(s)$ under the map $\psi : \mathscr{G}(U) \to \prod_{p\in U} \mathscr{G}_p$ are the same. Since $\mathscr{G}$ is a sheaf, from Proposition 2.3.5 ψ is injective, therefore $\varphi_1(U)(s) = \varphi_2(U)(s)$. By the arbitrary choice of $s \in \mathscr{F}(U)$ it follows $\varphi_1(U) = \varphi_2(U)$ for every open set U, that is $\varphi_1 = \varphi_2$. □

Proposition 2.3.7. *Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves of sets on X. Then φ is an isomorphism if and only if the induced map on the stalks $\varphi_p : \mathscr{F}_p \to \mathscr{G}_p$ is an isomorphism for every $p \in X$.*

Proof. Suppose that φ is an isomorphism. The, there exists an isomorphism $\varphi^{-1} : \mathscr{G} \to \mathscr{F}$ such that $\varphi \circ \varphi^{-1} = 1_{\mathscr{G}}$ and $\varphi^{-1} \circ \varphi = 1_{\mathscr{F}}$. For every $p \in X$, this implies that $\varphi_p \circ \varphi_p^{-1} = 1_{\mathscr{G}_p}$ and $\varphi_p^{-1} \circ \varphi_p = 1_{\mathscr{F}_p}$ (it is enough to consider the composition maps $\mathscr{G}(U) \xrightarrow{\varphi^{-1}(U)} \mathscr{F}(U) \to \mathscr{F}_p$ and use the universal properties of the stalk).

Conversely, suppose that φ_p is an isomorphism for every $p \in X$. Let U be an open set of X, from Lemma 1.3.3 it is enough to show that $\varphi(U) : \mathscr{F}(U) \to \mathscr{G}(U)$ is an isomorphism. First we prove injectivity. Let $s, t \in \mathscr{F}(U)$ such that $\varphi(U)(s) = \varphi(U)(t)$. For every $p \in U$ we have $\varphi_p(s_p) = \big(\varphi(U)(s)\big)_p = \big(\varphi(U)(t)\big)_p = \varphi_p(t_p)$. By hypothesis, φ_p

is an isomorphism, in particular it is injective, therefore $s_p = t_p$ for every $p \in U$. That is, if $\psi : \mathscr{F}(U) \to \prod_{p\in U} \mathscr{F}_p$, we have $\psi(s) = \psi(t)$. Since $\mathscr{F}$ is a sheaf, from Proposition 2.3.5 ψ is injective, hence $s = t$. Now we prove surjectivity. Let $t \in \mathscr{G}(U)$, then its image $(t_p)_{p\in U}$ in $\prod_{p\in U} \mathscr{G}_p$ consists of compatible germs. Since φ_p is surjective, for each $p \in U$, we have $t_p = \varphi_p([(s(p), U_p)])$ for some $[(s(p), U_p)] \in \mathscr{F}_p$. This menas:

$$[(t, U)] = t_p = \varphi_p([(s(p), U_p)]) = \Big[\Big(\varphi(U_p)\big(s(p)\big), U_p\Big)\Big],$$

that is, there exists an open neighbourhood V_p of p such that

$$t_{|V_p} = \varphi(U_p)\big(s(p)\big)_{|V_p} = \varphi(V_p)\big(s(p)_{|V_p}\big).$$

From now on, simplicity of notation, we set $s(p) = s(p)_{|V_p}$. The element $\big([(s(p), V_p)]\big)_{p\in U}$ in $\prod_{p\in U} \mathscr{F}_p$ consists of compatible germs. In fact, for all $p \in U$ and $q \in V_p$, we have

$$\varphi_q(s(p)_q) = \Big(\varphi(V_p)\big(s(p)\big)\Big)_q = t_q = \Big(\varphi(V_q)\big(s(q)\big)\Big)_q = \varphi_q(s(q)_q).$$

Since φ_q is injective, it follows $s(p)_q = s(q)_q$ for all $q \in V_p$, so $\big([(s(p), V_p)]\big)_{p\in U}$ consists of compatible germs. From Proposition 2.3.5, $\big([(s(p), V_p)]\big)_{p\in U}$ is the image under ψ of some section $s \in \mathscr{F}(U)$. Now we verify that $t = \varphi(U)(s)$. In fact, for all $p \in U$

$$t_p = \varphi_p(s_p) = \big(\varphi(U)(s)\big)_p,$$

in other words, the germs of t and $\varphi(U)(s)$ are the same on all $p \in U$. Finally, apply Proposition 2.3.5. $\square$

2.3.2 Sheafification

Definition 2.3.8. Let $\mathscr{F}$ be a presheaf on X. A morphism of presheaves $\mathrm{sh} : \mathscr{F} \to \mathscr{F}^{\mathrm{sh}}$ is a **sheafification** of $\mathscr{F}$ if $\mathscr{F}^{\mathrm{sh}}$ is a sheaf, and for any sheaf $\mathscr{G}$, and any presheaf morphism $g : \mathscr{F} \to \mathscr{G}$, there exists a unique morphism of sheaves $f : \mathscr{F}^{\mathrm{sh}} \to \mathscr{G}$ such that $g = f \circ \mathrm{sh}$.

$$\begin{tikzcd}
\mathscr{F} \arrow[r, "\mathrm{sh}"] \arrow[dr, "g"'] & \mathscr{F}^{\mathrm{sh}} \arrow[d, dashed, "\exists! f"] \\
& \mathscr{G}
\end{tikzcd}$$

It is clear that if $\mathscr{F}$ is already a sheaf, then the identity map $1_{\mathscr{F}}$ is a sheafification. Further, by using the universal property of sheafification, if a sheafification exists, it is unique up to isomorphism.

Remark 2.3.9. The sheafification can be seen as a functor $\mathrm{sh} : \mathscr{A}_X^{\mathrm{pre}} \to \mathscr{A}_X$ that assign to each presheaf $\mathscr{F}$ the sheaf $\mathscr{F}^{\mathrm{sh}}$, and to every presheaf morphism $\varphi : \mathscr{F} \to \mathscr{G}$ the sheaf morphism $\varphi^{\mathrm{sh}} : \mathscr{F}^{\mathrm{sh}} \to \mathscr{G}^{\mathrm{sh}}$ defined by using the universal property of the sheafification with respect to the composition $\mathrm{sh} \circ \varphi$.

$$\begin{tikzcd}
\mathscr{F} \arrow[r, "\mathrm{sh}"] \arrow[d, "\varphi"'] & \mathscr{F}^{\mathrm{sh}} \arrow[d, dashed, "\varphi^{\mathrm{sh}}"] \\
\mathscr{G} \arrow[r, "\mathrm{sh}"'] & \mathscr{G}^{\mathrm{sh}}
\end{tikzcd}$$

Now we prove that every presheaf of sets (or abelian groups) admits a sheafification with an explicit construction.

Proposition 2.3.10 (Construction of the sheafification)**.** *Let $\mathscr{F}$ be a presheaf of sets on X. Define a presheaf $\mathscr{F}^{\mathrm{sh}}$ on X as follows. For any open set U of X set*

$$\mathscr{F}^{\mathrm{sh}}(U) = \left\{ \big([(s(p), U_p)]\big)_{p\in U} \in \prod_{p\in U} \mathscr{F}_p \text{ that consists of compatible germs} \right\}.$$

For any inclusion of open sets $U \subseteq V$, the restriction map $\rho_{V,U} : \mathscr{F}^{\mathrm{sh}}(V) \to \mathscr{F}^{\mathrm{sh}}(U)$ is induced by the projection $\prod_{p\in V} \mathscr{F}_p \to \prod_{p\in U} \mathscr{F}_p$.

Now define the presheaf morphism $\mathrm{sh} : \mathscr{F} \to \mathscr{F}^{\mathrm{sh}}$ *as follows. For every open set U of X, the map* $\mathrm{sh}(U) : \mathscr{F}(U) \to \mathscr{F}^{\mathrm{sh}}(U) \subseteq \prod_{p\in U} \mathscr{F}_p$ *is given by* $\mathrm{sh}(U)(s) = (s_p)_{p\in U}$*. From Proposition 2.3.5, the map* $\mathrm{sh}(U)$ *is well defined. Then, the presheaf morphism* $\mathrm{sh} : \mathscr{F} \to \mathscr{F}^{\mathrm{sh}}$ *is a sheafification of $\mathscr{F}$.*

Proof. That $\mathscr{F}^{\mathrm{sh}}$ is a sheaf is immediate. Now let $\mathscr{G}$ be a sheaf and $g : \mathscr{F} \to \mathscr{G}$ be presheaf morphism. It is clear from the definition that that for every $p \in X$, $\mathscr{F}_p = \mathscr{F}^{\mathrm{sh}}_p$ and $\mathrm{sh}_p = 1_{\mathscr{F}_p}$. Therefore, any map $f : \mathscr{F} \to \mathscr{G}$ such that $g = f \circ \mathrm{sh}$ must satisfy $f_p = g_p$. From Proposition 2.3.6 it follows that such a map must be unique. Now we prove existence. For every $p \in X$, the morphism g induces a morphism on stalks $g_p : \mathscr{F}_p \to \mathscr{G}_p$. These maps induce a morphism $f : \mathscr{F}^{\mathrm{sh}} \to \mathscr{G}$ defined as follows. For every open set U of X, $f(U) : \mathscr{F}^{\mathrm{sh}}(U) \to \mathscr{G}(U)$ is the map that assign to an element $(s(p)_p)_{p\in U} \in \mathscr{F}^{\mathrm{sh}}(U)$ the unique section of $\mathscr{G}(U)$ that corresponds to the compatible germs $(g_p(s(p)_p))_{p\in U} \in \prod_{p\in U} \mathscr{G}_p$ (the existence and uniqueness of this section follows from Proposition 2.3.5, since $\mathscr{G}$ is a sheaf). Now it is clear from the definition that $f_p = g_p$, therefore by the argument used above we have $g = f \circ sh$. □

Definition 2.3.11. Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves on X. Define the **sheaf cokernel** and the **sheaf image** of φ as the sheafification of the presheaf cokernel and of the presheaf image resepctively. In symbols, we will denote the sheaf cokernel by $\operatorname{coker}\varphi = (\operatorname{coker}_{\mathrm{pre}}\varphi)^{\mathrm{sh}}$ and the sheaf image by $\operatorname{im}\varphi = (\operatorname{im}_{\mathrm{pre}}\varphi)^{\mathrm{sh}}$

From Remark 2.3.9, we can view the sheafification as a functor $\mathrm{sh} : \mathscr{A}_X^{\mathrm{pre}} \to \mathscr{A}_X$. Now Consider the inclusion functor $\mathcal{F} : \mathscr{A}_X \to \mathscr{A}_X^{\mathrm{pre}}$ by which a sheaf is viewed just as a presheaf. The following result shows that sh and $\mathcal{F}$ are adjoints.

Proposition 2.3.12. *The sheafification functor* $\mathrm{sh} : \mathscr{A}_X^{\mathrm{pre}} \to \mathscr{A}_X$ *is left adjoint to the inclusion functor $\mathcal{F} : \mathscr{A}_X \to \mathscr{A}_X^{\mathrm{pre}}$. In other words, for every presheaf $\mathscr{F}$ and sheaf $\mathscr{G}$ on X, there exists a bijection*

$$\mathrm{Hom}_{\mathscr{A}_X}(\mathrm{sh}(\mathscr{F}), \mathscr{G}) \simeq \mathrm{Hom}_{\mathscr{A}_X^{\mathrm{pre}}}(\mathscr{F}, \mathcal{F}(\mathscr{G}))$$

satisfying (1.1).

Proof. From the definition of the sheafification functor, for every pesheaf $\mathscr{F}$ on X we have that the map $\eta_{\mathscr{F}} : \mathscr{F} \to \mathcal{F}(\mathscr{F}^{\mathrm{sh}})$ is an initial object in the category $(\mathscr{F} \Rightarrow \mathcal{F})$. Now apply Theorem 1.6.7. □

2.4 Subsheaves and quotient sheaves

Proposition-Definition 2.4.1. *Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves of sets on X. The following are equivalent:*

1. *φ is a monomorphism in the category of sheaves.*
2. *φ_p is injective for all $p \in X$.*
3. *$\varphi(U)$ is injective for all open $U \subseteq X$.*

*If these conditions hold, we say that φ is **injective**, $\mathscr{F}$ is a **subsheaf** of $\mathscr{G}$, and φ is their **inclusion**.*

Proof.

(1) $\Rightarrow$ (3) Let $s, t \in \mathscr{F}(U)$ such that $\varphi(U)(s) = \varphi(U)(t)$. Now let $\mathscr{H}$ be a sheaf on X defined as follows. For every open set V of X

$$\mathscr{H}(V) = \begin{cases} \{h\} & V \subseteq U, \\ \emptyset & V \nsubseteq U. \end{cases}$$

The restriction maps are the identities on $\{h\}$ or the inclusions $\emptyset \subseteq \{h\}$ (i.e. the unique map from $\emptyset$ to $\{h\}$)(the sheaf $\mathscr{H}$ is called **indicator sheaf**). Now define two morphisms $\psi, \psi' : \mathscr{H} \to \mathscr{F}$ as follows. For every open set $V \subseteq X$, $\psi(V)$ and $\psi'(V)$ are the inclusions $\emptyset \subseteq \mathscr{F}(V)$ if $V \nsubseteq U$, whereas $\psi(V)(h) = \rho_{U,V}(s)$ and $\psi'(V)(h) = \rho_{U,V}(t)$ if $V \subseteq U$. Since $\varphi(U)(s) = \varphi(U)(t)$, we have $\varphi \circ \psi = \varphi \circ \psi'$, by hypothesis this implies $\psi = \psi'$, in particular $s = \psi(U)(h) = \psi'(U)(t) = t$. This proves the injectivity of $\varphi(U)$.

(3) $\Rightarrow$ (2) Let $p \in X$, and suppose that $\varphi_p(s_p) = \varphi_p(t_p)$ for some $s_p = [(s, U_s)] \in \mathscr{F}_p$ and $t_p = [(t, U_t)] \in \mathscr{F}_p$. This means

$$[(\varphi(U_s)(s), U_s)] = (\varphi(U_s)(s))_p = \varphi_p(s_p) = \varphi_p(t_p) = (\varphi(U_t)(t))_p = [(\varphi(U_t)(t), U_t)].$$

That is, there exists an open set $V \subseteq U_s \cap U_t$ such that $s_{|V} = t_{|V}$. This implies, $s_p = t_p$.

(2) $\Rightarrow$ (1) Let $\psi, \psi' : \mathscr{H} \to \mathscr{F}$ be two morphisms of sheaves of sets on X such that $\varphi \circ \psi = \varphi \circ \psi'$. Since the stalks of φ_p are injective, the equality $\varphi_p \circ \psi_p = \varphi_p \circ \psi'_p$ implies $\psi_p = \psi'_p$ for all $p \in X$. From Proposition 2.3.6 it follows $\psi = \psi'$. □

Proposition-Definition 2.4.2. *Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves of sets on X. The following are equivalent:*

1. *φ is an epimorphism in the category of sheaves.*
2. *φ_p is surjective for all $p \in X$.*

*If these conditions hold, we say that φ is **surjective** and $\mathscr{G}$ is a **quotient sheaf** of $\mathscr{F}$.*

Proof.

(1) $\Rightarrow$ (2) Fix $p \in X$ and let $i_p : \{p\} \to X$ be the inclusion map. Consider the skyscraper sheaf $i_{p*}\underline{\{0,1\}}$ (see Remark 2.1.14). Now let $\psi : \mathscr{G} \to i_{p*}\underline{\{0,1\}}$ be a morphism of sheaves defined by

$$\psi(U)(g) = \begin{cases} 0 & [(g,U)] \in \operatorname{im} \varphi_p, \\ 1 & \text{otherwise,} \end{cases}$$

if the open set U of X contains p, and the trivial map if $p \notin U$. Now define another morphism $\psi' : \mathscr{G} \to i_{p*}\underline{\{0,1\}}$ by $\psi'(U)(g) = 0$ if $p \in U$ and the trivial map otherwise. Now we have $\psi \circ \varphi = \psi' \circ \varphi$, since φ is an epimorphism, it follows $\psi = \psi'$, that is φ_p is surjective.

(2) $\Rightarrow$ (1) Let $\psi, \psi' : \mathscr{G} \to \mathscr{H}$ be two morphisms of sheaves such that $\psi \circ \varphi = \psi' \circ \varphi$. Then for every $p \in X$ we have $\psi_p \circ \varphi_p = \psi'_p \circ \varphi_p$, by hypothesis φ_p is surjective, therefore $\psi_p = \psi'_p$ for every $p \in X$. Finally, from Proposition 2.3.6 we have $\psi = \psi'$. □

The previous propositions can be generalized for sheaves with values in any category with surjective and injective substituted by monic and epic.

The equivalent conditions in the definition of quotient sheaf are not equivalent to the surjectivity of the maps $\varphi(U)$ for every open set U of X, as the following example shows.

Example 2.4.3. Let $\varphi : \mathcal{O}_X \to \mathcal{O}_X^*$ as in Example 2.2.9. We note that for every $p \in X$ the map φ_p is surjective. In fact, every nowhere zero holomorphic function $f \in \mathcal{O}_X^*(U)$ admits a logarithm locally, meaning that for every point $p \in U$ and every sufficiently small open neighbourhood V of $f(p) \in \mathbb{C} \setminus \{0\}$ we can define a holomorphic function $\log f \in \mathcal{O}_X(W)$, where $W = f^{-1}(V)$, such that $\varphi(W)(\log f) = f_{|W}$. This implies that $f_p = \varphi_p\big((\log f)_p\big) \in \operatorname{im} \varphi_p$. However, it is not true that $\varphi(U)$ is surjective for every open set U of X. For instance, set $U = \mathbb{C} \setminus \{0\}$, the restriction on U of the identity map $z_{|U} \in \mathcal{O}_X^*(U)$ is not in $\operatorname{im} \varphi(U)$.

2.5 Sheaf on a base

Let X be a topological space. Recall that a base $\mathcal{B}$ on X is a collection of open sets such that every open set of X is a union of elements of $\mathcal{B}$. As before, we can consider the poset $(\mathcal{B}, \subseteq)$ and its associated category.

Definition 2.5.1. Let $\mathscr{A}$ be a category, X be a topological space, and let $\mathcal{B}$ be a base on X. A **presheaf on the base $\mathcal{B}$** (or **$\mathcal{B}$-presheaf**) is a contravariant functor $\mathscr{F} : \mathcal{B}^{op} \to \mathscr{A}$.

Explicitly, if $\mathscr{A}$ is the category **Set**, a $\mathcal{B}$-presheaf $\mathscr{F}$ of sets on X consists of:

1. for every $B \in \mathcal{B}$ a set $\mathscr{F}(B)$,
2. for every inclusion $B \subseteq B'$ of elements in $\mathcal{B}$, a map $\rho_{B',B} : \mathscr{F}(B') \to \mathscr{F}(B)$, such that
   (a) $\rho_{B,B} = 1_{\mathscr{F}(B)}$,

(b) if $B \subseteq B' \subseteq B''$ are open sets, then $\rho_{B',B} \circ \rho_{B'',B'} = \rho_{B'',B}$.

Definition 2.5.2. Let X be a topological space, and let $\mathcal{B}$ be a base on X. A $\mathcal{B}$-presheaf $\mathscr{F}$ on X is a **sheaf on the base** $\mathcal{B}$ (or **$\mathcal{B}$-sheaf**) if for all $B \in \mathcal{B}$ and all open cover $\{B_i\}_{i\in I} \subseteq \mathcal{B}$ of B, the following properties are satisfied:

1. if $s_1, s_2 \in \mathscr{F}(B)$ with $s_{1|B_i} = s_{2|B_i}$ for all $i \in I$, then $s_1 = s_2$;
2. for every family of sections $s_i \in \mathscr{F}(B_i)$ such that $s_{i|B_i\cap B_j} = s_{j|B_i\cap B_j}$ for all $i, j \in I$, there exists $s \in \mathscr{F}(B)$ such that $s_{|B_i} = s_i$ for all $i \in I$.

Definition 2.5.3. If $\mathscr{F}$ is a $\mathcal{B}$-sheaf on X, define the **stalk** of $\mathscr{F}$ at $p \in X$ by

$$\mathscr{F}_p = \varinjlim_{p\in B\in\mathcal{B}} \mathscr{F}(B).$$

For every open set U of X, an element $\big([(s(p), U_q)]\big)_{p\in U} \in \prod_{p\in U} \mathscr{F}_p$ consists of **compatible germs** if for every $p \in U$ there exist an $B_p \in \mathcal{B}$ such that $p \in B_p \subseteq U$, and an element $t(p) \in \mathscr{F}(B_p)$ such that $t(p)_q = s(q)_q$ for every $q \in B_p$.

Theorem 2.5.4 (Sheaf on a base)**.** *For every $\mathcal{B}$-sheaf $\mathscr{F}'$ of sets on X, there exists a sheaf of sets $\mathscr{F}$ on X such that $\mathscr{F}'(B) \simeq \mathscr{F}(B)$ for every $B \in \mathcal{B}$ and $\mathscr{F}$ is unique up to isomorphism.*

Proof. For every open set U of X, define $\mathscr{F}$ by

$$\mathscr{F}(U) = \left\{ \big([(s(p), U_p)]\big)_{p\in U} \in \prod_{p\in U} \mathscr{F}_p \text{ that consists of compatible germs} \right\},$$

with the restriction maps induced by the natural projections. As in Proposition 2.3.10, it is immediate to verify that $\mathscr{F}$ is a sheaf. The isomorphism $\mathscr{F}'(B) \simeq \mathscr{F}(B)$ is the same as the map given in Proposition 2.3.5. From Remark 2.1.7 we have

$$\mathscr{F}(U) = \varprojlim_{U\supseteq B\in\mathcal{B}} \mathscr{F}(B) \simeq \varprojlim_{U\supseteq B\in\mathcal{B}} \mathscr{F}'(B),$$

from which it follows that $\mathscr{F}$ is unique up to isomorphism (and this last observation gives also an alternative definition of $\mathscr{F}$ using inverse limits). $\square$

Definition 2.5.5. If $\mathscr{F}$ and $\mathscr{G}$ are two $\mathcal{B}$-sheaves on X, a **morphism of $\mathcal{B}$-sheaves** $\varphi : \mathscr{F} \to \mathscr{G}$ is a collection of maps $\varphi(B) : \mathscr{F}(B) \to \mathscr{G}(B)$ for each $B \in \mathcal{B}$ that commutes with restriction maps: $\varphi(B) \circ \rho_{B',B} = \rho_{B',B} \circ \varphi(B')$ for each inclusion $B \subseteq B'$ of elements of $\mathcal{B}$.

Proposition 2.5.6 (Morphism on a base)**.** *Let $\mathscr{F}'$ and $\mathscr{G}'$ be two $\mathcal{B}$-sheaves on X, and let $\mathscr{F}$ and $\mathscr{G}$ the corresponding sheaves on X. For every morphism of $\mathcal{B}$-sheaves $\varphi' : \mathscr{F}' \to \mathscr{G}'$ there exists a unique morphism of sheaves $\varphi : \mathscr{F} \to \mathscr{G}$ such that $\varphi'(B) = \varphi(B)$.*

Sketch of proof. The morphism φ' induces morphisms on the stalks $\varphi'_p : \mathscr{F}'_p \to \mathscr{G}'_p$ (note that $\mathscr{F}'_p \simeq \mathscr{F}_p$ and similarly for $\mathscr{G}$). From Proposition 2.3.6 these maps correspond to a unique morphism of sheaves $\varphi : \mathscr{F} \to \mathscr{G}$. $\square$

Proposition 2.5.7 (Glueing sheaves)**.** *Let X be a topological space and let $\{U_i\}_{i\in I}$ be an open cover of X. Suppose that for each $i \in I$ we have a sheaf of sets $\mathscr{F}_i$ on U_i and for every $i, j \in I$ isomorphisms of sheaves $\phi_{ij} : \mathscr{F}_{i|U_i\cap U_j} \to \mathscr{F}_{j|U_i\cap U_j}$ such that for every $i, j, k \in I$*

1. $\phi_{ii} = 1_{\mathscr{F}_i}$,
2. $\phi_{ji} = (\phi_{ij})^{-1}$,
3. $\phi_{jk} \circ \phi_{ij} = \phi_{ik}$ *on* $U_i \cap U_j \cap U_k$ *(this property is called* ***cocycle condition****).*

Then, there exists a sheaf of sets $\mathscr{F}$ on X together with isomorphisms $\phi_i : \mathscr{F}_{|U_i} \to \mathscr{F}_i$ such that, for every $i, j \in I$ the following diagram commutes:

$$\begin{tikzcd} & \mathscr{F}_{|U_i\cap U_j} \arrow[dl, "\phi_{i|U_i\cap U_j}"'] \arrow[dr, "\phi_{j|U_i\cap U_j}"] & \\ \mathscr{F}_{i|U_i\cap U_j} \arrow[rr, "\phi_{ij}"] & & \mathscr{F}_{j|U_i\cap U_j} \end{tikzcd}$$

Further, $\mathscr{F}$ is unique up to isomorphism.

Proof. Let $\mathcal{B}$ be the base of X that consists of all the open sets U of X for which there exists $i \in I$ such that $U \subseteq U_i$. Now we define a $\mathcal{B}$-sheaf $\mathscr{F}$ on X as follows. For each $U \in \mathcal{B}$ we set $\mathscr{F}(U) = \mathscr{F}_i(U)$ for some $i \in I$ such that $U \subseteq U_i$. For every inclusion $U \subseteq V$ of elements of $\mathcal{B}$, there exist $i, j \in I$ such that $V \subseteq U_j$ and $U \subseteq U_i$ (this implies that $U \subseteq U_i \cap U_j$, $\mathscr{F}(U) = \mathscr{F}_i(U)$ and $\mathscr{F}(V) = \mathscr{F}_j(V)$), the restriction map $\rho_{V,U} : \mathscr{F}(V) \to \mathscr{F}(U)$ is given by the composition of the restriction map $\rho^{\mathscr{F}_j}_{V,U} : \mathscr{F}_j(V) \to \mathscr{F}_j(U)$ and the isomorphism $\phi_{ji}(U) : \mathscr{F}_j(U) \to \mathscr{F}_i(U)$. Now we verify that $\mathscr{F}$ is a $\mathcal{B}$-sheaf. It is clear that $\rho_{U,U} = 1_{\mathscr{F}(U)}$ (since $\varphi_{ii} = 1_{\mathscr{F}_i}$). Now suppose that $U \subseteq V \subseteq W$ are elements of $\mathcal{B}$ such that $W \subseteq U_k$, $V \subseteq U_j$, $U \subseteq U_i$ for $i, j, k \in I$. Now, from the cocycle condition, we have

$$\begin{aligned} \rho_{V,U} \circ \rho_{W,V} &= \phi_{ji}(U) \circ \rho^{\mathscr{F}_j}_{V,U} \circ \phi_{kj}(V) \circ \rho^{\mathscr{F}_k}_{W,V} = \\ &= \phi_{ji}(U) \circ \phi_{kj}(U) \circ \rho^{\mathscr{F}_k}_{V,U} \circ \rho^{\mathscr{F}_k}_{W,V} = \\ &= \phi_{ki}(U) \circ \rho^{\mathscr{F}_k}_{W,U} = \rho_{W,U}. \end{aligned}$$

From Theorem 2.5.4, $\mathscr{F}$ defines a sheaf of sets on X (that we continue to denote by $\mathscr{F}$). Finally, for every $i \in I$ define the isomorphism $\phi_i : \mathscr{F}_{|U_i} \to \mathscr{F}_i$ as follows. For every open sets $U \subseteq U_i$, suppose that $\mathscr{F}(U) = \mathscr{F}_j(U)$ for some $j \in I$ such that $U \subseteq U_j$ (therefore $U \subseteq U_i \cap U_j$). Then, set $\phi_i(U) = \phi_{ji}(U)$. The uniqueness of $\mathscr{F}$ follows from the fact that the isomorphisms ϕ_i define an isomorphism on the base $\mathcal{B}$, that from Proposition 2.5.6 extends to an isomorphism on the sheaves. $\square$

2.6 The inverse image sheaf

Let $\pi : X \to Y$ be a continuous map of topological spaces. Let $\mathscr{G}$ be a sheaf on Y, define the presheaf

$$\pi^{-1}_{\text{pre}}\mathscr{G}(U) = \varinjlim_{V \supseteq \pi(U)} \mathscr{G}(V),$$

where the restriction maps is defined as follows. Let $U \subseteq V$ be an inclusion of open sets. We have $\pi(U) \subseteq \pi(V)$, so every open set W containining $\pi(V)$, contains also $\pi(U)$. Therefore, we have maps $\mathscr{G}(W) \to \pi_{\text{pre}}^{-1}\mathscr{G}(U)$. Finally, since $\pi_{\text{pre}}^{-1}\mathscr{G}(V)$ is a colimit, the restriction map $\rho_{V,U}$ will be the unique map $\rho_{V,U} : \pi_{\text{pre}}^{-1}\mathscr{G}(V) \to \pi_{\text{pre}}^{-1}\mathscr{G}(U)$.

In general, π_{pre}^{-1} is not a sheaf, as the following example shows.

Example 2.6.1. Let $X = \{x, y\}$ with the discrete topology, and $Y = \{y\}$. Let $\pi : X \to Y$ be the unique map from X to Y, and let $\mathscr{F}$ be a sheaf of sets on Y defined by $\mathscr{F}(\{y\}) = \{a, b\}$ (whereas $\mathscr{F}(\emptyset)$ must be a terminal object in the category of sets, that is, a singleton, say $\{a\}$; the restriction map $\rho_{\{y\},\emptyset}$ is the unique map from $\mathscr{F}(\{y\})$ to $\mathscr{F}(\emptyset)$). We obtain a presheaf $\pi_{\text{pre}}^{-1}\mathscr{F}$ where

$$\pi_{\text{pre}}^{-1}\mathscr{F}(\{x,y\}) = \{a,b\}, \qquad \pi_{\text{pre}}^{-1}\mathscr{F}(\{x\}) = \{a,b\},$$
$$\pi_{\text{pre}}^{-1}\mathscr{F}(\emptyset) = \{a\}, \qquad \pi_{\text{pre}}^{-1}\mathscr{F}(\{y\}) = \{a,b\},$$

and the restriction maps, where not obvious, are the identities. Now the family of sections $a \in \pi_{\text{pre}}^{-1}(\{x\})$ and $b \in \pi_{\text{pre}}^{-1}(\{y\})$ cannot glue to any global section. Hence, $\pi_{\text{pre}}^{-1}\mathscr{F}$ is not a sheaf.

Definition 2.6.2. Let $\pi : X \to Y$ be a continuous map of topological spaces. Let $\mathscr{G}$ be a sheaf on Y, the **inverse image sheaf** is defined as $\pi^{-1}\mathscr{G} = (\pi_{\text{pre}}^{-1}\mathscr{G})^{\text{sh}}$.

Example 2.6.3.

1. Let $\mathscr{G}$ be a sheaf on a topological space X, and fix $p \in X$. Consider the inclusion map $i_p : \{p\} \to X$. The inverse image sheaf is the stalk of $\mathscr{G}$ at p: $i_p^{-1}\mathscr{G}(\{p\}) = \mathscr{G}_p$. In fact, $i_{p_{\text{pre}}}^{-1}\mathscr{G}$ is already a sheaf, therefore we have $i_p^{-1}\mathscr{G} = i_{p_{\text{pre}}}^{-1}\mathscr{G}$, and in addition

$$i_p^{-1}\mathscr{G}(\{p\}) = \varinjlim_{U \ni p} \mathscr{G}(U) = \mathscr{G}_p.$$

2. Let $\pi : X \to Y$ be a continuous map of topological spaces and let $\mathscr{G}$ be a sheaf on Y. If π is an open map, meaning that for every open set U of X, $\pi(U)$ is open on Y, we have

$$\varinjlim_{V \supseteq \pi(U)} \mathscr{G}(V) = \mathscr{G}(\pi(U)).$$

Therefore, $\pi_{\text{pre}}^{-1}\mathscr{G}$ is a sheaf, so $\pi_{\text{pre}}^{-1}\mathscr{G} = \pi^{-1}\mathscr{G}$. Further, for every open set U of X

$$\pi^{-1}\mathscr{G}(U) = \mathscr{G}(\pi(U)).$$

3. Let $\mathscr{F}$ be a sheaf on a topological space X, and fix an open subset U of X. Let $i : U \to X$ be the inclusion map. The map i is open, thus

$$i^{-1}\mathscr{F}(V) = \mathscr{F}(i(V)) = \mathscr{F}(V),$$

for every open set $V \subseteq U$. In other words, the inverse image sheaf of the open inclusion $U \subseteq X$ is the restriction: $i^{-1}\mathscr{F} = \mathscr{F}_{|U}$.

If $\pi : X \to Y$ is a continuous map, and $f : \mathscr{F} \to \mathscr{G}$ is a morphism of sheaves on Y, then the composition maps $\mathscr{F}(V) \xrightarrow{f(V)} \mathscr{G}(V) \to \pi_{\text{pre}}^{-1}\mathscr{G}(U)$, for every open subsets $U \subseteq X$ and $V \supseteq \pi(U)$, gives us a map $\pi_{\text{pre}}^{-1}\mathscr{F}(U) \to \pi_{\text{pre}}^{-1}\mathscr{G}(U)$. This defines a morphism of presheaves $\pi_{\text{pre}}^{-1}f : \pi_{\text{pre}}^{-1}\mathscr{F} \to \pi_{\text{pre}}^{-1}\mathscr{G}$. Thus π^{-1} can be seen as a functor, given by the composition of functors $\pi^{-1} = \text{sh} \circ \pi_{\text{pre}}^{-1}$.

Proposition 2.6.4 (π_{pre}^{-1} and π_* are adjoints in the category of presheaves)**.** *Let $\pi : X \to Y$ be a continuous map. The functors $\pi_* : \mathscr{A}_X^{\text{pre}} \to \mathscr{A}_Y^{\text{pre}}$ and $\pi_{\text{pre}}^{-1} : \mathscr{A}_Y^{\text{pre}} \to \mathscr{A}_X^{\text{pre}}$ are adjoints (with π_{pre}^{-1} on the left and π_* on the right), that is, for every presheaf $\mathscr{F}$ on X and every presheaf $\mathscr{G}$ on Y there are bijections*

$$\text{Hom}_{\mathscr{A}_Y^{\text{pre}}}(\pi_{\text{pre}}^{-1}\mathscr{G}, \mathscr{F}) \simeq \text{Hom}_{\mathscr{A}_X^{\text{pre}}}(\mathscr{G}, \pi_*\mathscr{F})$$

satisfying (1.1).

Proof. From Theorem 1.6.7, we just need to show that for every $\mathscr{G} \in \mathscr{A}_Y^{\text{pre}}$ there exists a morphism $\eta_{\mathscr{G}} : \mathscr{G} \to \pi_*\pi_{\text{pre}}^{-1}\mathscr{G}$ with the property that for every presheaf $\mathscr{F} \in \mathscr{A}_X^{\text{pre}}$ and every morphism $f : \mathscr{G} \to \pi_*\mathscr{F}$, there exists a unique morphism $g : \pi_*\pi_{\text{pre}}^{-1}\mathscr{G} \to \pi_*\mathscr{F}$ such that $f = g \circ \eta_{\mathscr{G}}$. Let V be an open subset of Y, we have $V \supseteq \pi(\pi^{-1}(V))$, therefore there is a canonical map

$$\mathscr{G}(V) \to \varinjlim_{W \supseteq \pi(\pi^{-1}(V))} \mathscr{G}(W) = \pi_{\text{pre}}^{-1}\mathscr{G}(\pi^{-1}(V)) = \pi_*\pi_{\text{pre}}^{-1}\mathscr{G}(V).$$

By varying V, this is compatible with the restriction maps, therefore we obtain a morphism of presheaves $\eta_{\mathscr{G}} : \mathscr{G} \to \pi_*\pi_{\text{pre}}^{-1}\mathscr{G}$. Now let $f : \mathscr{G} \to \pi_*\mathscr{F}$ be a morphism of presheaves. Fix an open subset V of Y. For every open subset $W \supseteq \pi(\pi^{-1}(V))$, we have $\pi^{-1}(W) \supseteq \pi^{-1}(V)$, therefore we can condiser the composition maps

$$\mathscr{G}(W) \xrightarrow{f} \mathscr{F}(\pi^{-1}(W)) \xrightarrow{\rho} \mathscr{F}(\pi^{-1}(V))$$

these maps give rise to a unique morphism

$$\pi_{\text{pre}}^{-1}\pi_*\mathscr{G}(V) = \pi_{\text{pre}}^{-1}\mathscr{G}(\pi^{-1}(V)) \to \mathscr{F}(\pi^{-1}(V)) = \pi_*\mathscr{F}(V)$$

and from the properties of direct limit, we have that the following diagram

$$\begin{tikzcd} \mathscr{G}(V) \arrow[r, "f"] \arrow[dr, "\eta_{\mathscr{G}}"'] & \mathscr{F}(\pi^{-1}(V)) \\ & \pi_*\pi_{\text{pre}}^{-1}\mathscr{G}(V) \arrow[u, "g"'] \end{tikzcd}$$

is commutative. Therefore $f = g \circ \eta_{\mathscr{G}}$ as required. $\square$

Corollary 2.6.5 (π^{-1} and π_* are adjoints in the category of sheaves)**.** *Let $\pi : X \to Y$ be a continuous map. The functors $\pi_* : \mathscr{A}_X \to \mathscr{A}_Y$ and $\pi^{-1} : \mathscr{A}_Y \to \mathscr{A}_X$ are adjoints (with π^{-1} on the left and π_* on the right), that is, for every sheaf $\mathscr{F}$ on X and every sheaf $\mathscr{G}$ on Y there are bijections*

$$\text{Hom}_{\mathscr{A}_X}(\pi^{-1}\mathscr{G}, \mathscr{F}) \simeq \text{Hom}_{\mathscr{A}_Y}(\mathscr{G}, \pi_*\mathscr{F})$$

satisfying (1.1).

Proof. Recall that $\pi^{-1} = \mathrm{sh} \circ \pi_{\mathrm{pre}}^{-1}$, that sh is left adjoint to the inclusion functor $\mathcal{F} : \mathscr{A}_X \to \mathscr{A}_X^{\mathrm{pre}}$ (Proposition 2.3.12) and that from the previous proposition, π_{pre}^{-1} is left adjoint to π_* in the category of presheaves. Now for every sheaf $\mathscr{F}$ on X and every sheaf $\mathscr{G}$ on Y, we have

$$\begin{aligned}
\mathrm{Hom}_{\mathscr{A}_X}(\pi^{-1}\mathscr{G}, \mathscr{F}) &\simeq \mathrm{Hom}_{\mathscr{A}_X}(\mathrm{sh}\, \pi_{\mathrm{pre}}^{-1}\mathscr{G}, \mathscr{F}) \\
&\simeq \mathrm{Hom}_{\mathscr{A}_X^{\mathrm{pre}}}(\pi_{\mathrm{pre}}^{-1}\mathscr{G}, \mathcal{F}(\mathscr{F})) \\
&\simeq \mathrm{Hom}_{\mathscr{A}_Y^{\mathrm{pre}}}(\mathscr{G}, \pi_*\mathcal{F}(\mathscr{F})) \\
&\simeq \mathrm{Hom}_{\mathscr{A}_X}(\mathscr{G}, \pi_*\mathscr{F})
\end{aligned}$$

where in the last bijection we used the fact that the pushforward of a sheaf is a sheaf. All the bijections above come from an adjoint pair, therefore their composition do form an adjoint pair. $\square$

Lemma 2.6.6. *Let $\pi : X \to Y$ be a continuous map, $\mathscr{G}$ be a presheaf on Y and let $x \in X$. There is a canonical isomorphism*

$$(\pi_{\mathrm{pre}}^{-1}\mathscr{G})_x \simeq \mathscr{G}_{f(x)}$$

Proof.

$$\begin{aligned}
(\pi_{\mathrm{pre}}^{-1}\mathscr{G})_x &\simeq \varinjlim_{x\in U} \pi_{\mathrm{pre}}^{-1}\mathscr{G}(U) \\
&\simeq \varinjlim_{x\in U} \varinjlim_{V\supseteq\pi(U)} \mathscr{G}(V) \\
&\simeq \varinjlim_{V\ni\pi(x)} \mathscr{G}(V) = \mathscr{G}_{f(x)}.
\end{aligned}$$

$\square$

Corollary 2.6.7. *Let $\pi : X \to Y$ be a continuous map, $\mathscr{G}$ be a sheaf on Y and let $x \in X$. There is a canonical isomorphism*

$$(\pi^{-1}\mathscr{G})_x \simeq \mathscr{G}_{f(x)}$$

Proof. Follow from Lemma 2.6.6 and from the fact that sheafification does not change the stalks. $\square$

Chapter 3

Schemes and affine schemes

3.1 Zariski topology on $\operatorname{Spec} R$

Every ring will be always commutative with unity. In this section R will be a ring.

Definition 3.1.1. The **spectrum** $\operatorname{Spec} R$ of R is the set of all prime ideals.

Definition 3.1.2. For every $S \subseteq R$, the **vanishing set** of S is

$$V(S) = \{p \in \operatorname{Spec}(R) : \forall f \in S \; f(p) = 0\} = \{P \in \operatorname{Spec}(R) : P \supseteq S\}.$$

For every $E \subseteq \operatorname{Spec}(R)$ define

$$\mathscr{I}(E) = \{f \in R : \forall p \in E \; f(p) = 0\} = \bigcap_{P \in E} P$$

(where the second equality holds if $E \neq \emptyset$).

Note that $V(S) = V(\langle S \rangle)$, where $\langle S \rangle$ is the ideal generated by the set $S \subseteq R$. Therefore, we can restrict the function V to the ideals of R. Further, $\mathscr{I}(E)$ is an ideal.

Proposition 3.1.3. *Let $I, J \subseteq R$ be two ideals and let $X, Y \subseteq \operatorname{Spec}(R)$, and let $\mathfrak{N}(R)$ denote the nilradical of R.*

1a. $I \subseteq \mathscr{I}(V(I))$	*1b.* $X \subseteq V(\mathscr{I}(X))$
2a. $I \subseteq J \Rightarrow V(I) \supseteq V(J)$	*2b.* $X \subseteq Y \Rightarrow \mathscr{I}(X) \supseteq \mathscr{I}(Y)$
3a. $V(\mathscr{I}(V(I))) = V(I)$	*3b.* $\mathscr{I}(V(\mathscr{I}(X))) = \mathscr{I}(X)$
4a. $V(1) = \emptyset$, $V(\mathfrak{N}(R)) = \operatorname{Spec} R$	*4b.* $\mathscr{I}(\operatorname{Spec} R) = \mathfrak{N}(R), \mathscr{I}(\emptyset) = R$
5a. $V\left(\sum_{\lambda \in \Lambda} I_\lambda\right) = \bigcap_{\lambda \in \Lambda} V(I_\lambda)$	*5b.* $\mathscr{I}(\bigcup_{\lambda \in \Lambda} X_\lambda) = \bigcap_{\lambda \in \Lambda} \mathscr{I}(X_\lambda)$
6a. $V(IJ) = V(I \cap J) = V(I) \cup V(J)$	*6b.* $\mathscr{I}(X \cap Y) \supseteq \mathscr{I}(X) + \mathscr{I}(Y)$

Proof.

1a. If $f \in I$, then $f(p) = 0$ for every $p \in V(I)$, that is $f \in \mathscr{I}(V(I))$.

2a. If $p \in V(J)$ then $f(p) = 0$ for every $f \in I \subseteq J$, therefore $p \in V(I)$.

3a. From (1a) and (2a) we have $I \subseteq \mathscr{I}(V(I)) \Rightarrow V(I) \supseteq V(\mathscr{I}(V(I)))$, the inclusion $V(I) \subseteq V(\mathscr{I}(V(I)))$ follows from (1b).

4a. Every prime ideal is proper by definition, that is, if $p \in \operatorname{Spec} R$ then $p \not\supseteq R = \langle 1 \rangle$, that is $V(1) = \emptyset$. On the other hand, the nilradial $\mathfrak{N}(R)$ is contained in every prime ideal $p \in \operatorname{Spec} R$.

5a. It is enough to observe that $p \supseteq \sum_{\lambda \in \Lambda} I_\lambda$ if and only if $p \supseteq I_\lambda$ for every $\lambda \in \Lambda$.

6a. Let $p \in V(IJ)$, that is $IJ \subseteq p$. Since p is prime, this means $I \subseteq p$ or $J \subseteq p$, that is $p \in V(I) \cup V(J)$. Therefore, $V(IJ) \subseteq V(I) \cup V(J)$.
Now $I \cap J \subseteq I \Rightarrow V(I \cap J) \supseteq V(I)$, and similarly $V(I \cap J) \supseteq V(J)$ thus we have $V(I \cap J) \supseteq V(I) \cup V(J)$. Further $IJ \subseteq I \cap J$, therefore $V(IJ) \supseteq V(I \cap J)$. Finally we obtain
$$V(IJ) \supseteq V(I \cap J) \supseteq V(I) \cup V(J) \supseteq V(IJ).$$

1b. If $p \in X$, then $f(p) = 0$ for every $f \in \mathscr{I}(X)$, that is $p \in V(\mathscr{I}(X))$.

2b. If $f \in \mathscr{I}(Y)$ then $f(p) = 0$ for every $p \in X \subseteq Y$, therefore $f \in \mathscr{I}(X)$.

3b. From (1b) and (2b) we have $X \subseteq V(\mathscr{I}(X)) \Rightarrow \mathscr{I}(X) \supseteq \mathscr{I}(V(\mathscr{I}(X)))$, the inclusion $\mathscr{I}(X) \subseteq \mathscr{I}(V(\mathscr{I}(X)))$ follows from (1a).

4b. We have $\mathscr{I}(R) = \bigcap_{p \in \operatorname{Spec} R} p = \mathfrak{N}(R)$. The second statement follows from the definition.

5b. We have
$$\mathscr{I}\left(\bigcup_{\lambda \in \Lambda} X_\lambda\right) = \bigcap_{p \in \bigcup_{\lambda \in \Lambda} X_\lambda} p = \bigcap_{\lambda \in \Lambda} \bigcap_{p \in X_\lambda} p = \bigcap_{\lambda \in \Lambda} \mathscr{I}(X_\lambda).$$

6b. Since $X \cap Y \subseteq X, Y$, from (2b) we have $\mathscr{I}(X \cap Y)$ contains both $\mathscr{I}(X)$ and $\mathscr{I}(Y)$, therefore $\mathscr{I}(X \cap Y) \supseteq \mathscr{I}(X) + \mathscr{I}(Y)$. $\square$

In the last proposition, we have seen that the family of sets $V(S)$ for $S \subseteq R$ satisfies the properties of closed sets in a topological space.

Definition 3.1.4. The **Zariski topology** on $\operatorname{Spec}(R)$ is the topology in which the family of closed sets are of the form $V(S)$ for $S \subseteq R$.

Proposition 3.1.5. *For every ideal $I \subseteq R$ and subset $X \subseteq \operatorname{Spec}(R)$ we have*

1a. $\mathscr{I}(V(I)) = \sqrt{I}$

1b. $V(\mathscr{I}(X)) = \overline{X}$

2a. $V(I) = V(\sqrt{I})$

2b. $\mathscr{I}(X) = \mathscr{I}(\overline{X})$

Therefore $\mathscr{I}$ and V are inclusion reversing bijections between radical ideals and closed sets of $\operatorname{Spec}(R)$.

Proof.

1a. $\mathscr{I}(V(I)) = \bigcap_{P \in V(I)} P = \bigcap_{P \supseteq I} P = \sqrt{I}$.

2a. Follows from applying (1a) and (3a) of Proposition 3.1.3.

1b. $\subseteq$ $X \subseteq \overline{X} \Rightarrow \mathscr{I}(X) \supseteq \mathscr{I}(\overline{X}) \Rightarrow V(\mathscr{I}(X)) \subseteq V(\mathscr{I}(\overline{X})) = \overline{X}$. For the last equality, set $\overline{X} = V(S)$, then $V(\mathscr{I}(\overline{X})) = V(\mathscr{I}(V(S))) = V(S) = \overline{X}$.

$\supseteq$ $V(\mathscr{I}(X))$ is a closed set containing X.

2b. Follows from applying (2a) and (3b) of Proposition 3.1.3. □

3.2 Distinguished open sets

In this section, we set $X = \operatorname{Spec} R$, where R is a ring.

Definition 3.2.1. For every $f \in R$ define $X_f = X \setminus V(f)$. The open sets of the form X_f are called **distinguished**.

More explictely, we have

$$\begin{aligned} X_f = \operatorname{Spec}(R)_f &= \{x \in \operatorname{Spec}(R) : f(x) \neq 0\} = \\ &= \{P \in \operatorname{Spec}(R) : f \notin P\} = \\ &= \operatorname{Spec}(R) \setminus V(f) = X \setminus V(f). \end{aligned}$$

Note that X itself is a distinguished open set: $X = X_1$.

Proposition 3.2.2. *For every open set $U = X \setminus V(S)$, we have $U = \bigcup_{f \in S} X_f$. In particular, the distinguished open sets of* $\operatorname{Spec} R$ *form a basis for the Zariski topology.*

Proof. We have

$$U = X \setminus V\left(\bigcup_{f \in S} f\right) = X \setminus \bigcap_{f \in S} V(f) = \bigcup_{f \in S} (X \setminus V(f)) = \bigcup_{f \in S} X_f$$

(where in the second equality we used (3a) of Proposition 3.1.3). □

Corollary 3.2.3. *Let $S \subseteq R$. Then*

$$1 \in \langle S \rangle \Leftrightarrow V(S) = \emptyset \Leftrightarrow X = \bigcup_{f \in S} X_f.$$

Proposition 3.2.4. *For every $f, g \in R$ and $g_i \in R$ with $i \in \{1, \dots, n\}$, we have*

1. $X_f \cap X_g = X_{fg}$,
2. $X_f \subseteq \bigcup_{i=1}^n X_{g_i} \iff f \in \sqrt{(g_1, \dots, g_n)}$,
3. $X = X_1$, $\emptyset = X_0$.

Proof.

1. $X_f \cap X_g = \big(X \setminus V(f)\big) \cap \big(X \setminus V(g)\big) = X \setminus \big(V(f) \cup V(g)\big) = X \setminus V(fg) = X_{fg}$.

2. $$\begin{aligned} X_f \subseteq \textstyle\bigcup_{i=1}^n X_{g_i} &\iff X \setminus V(f) \subseteq X \setminus V(g_1, \dots, g_n) \\ &\iff V(f) \supseteq V(g_1, \dots, g_n) \\ &\iff (f) \subseteq \mathscr{I}(V(f)) \subseteq \mathscr{I}(V(g_1, \dots, g_n)) = \sqrt{(g_1, \dots, g_n)} \\ &\iff f \in \sqrt{(g_1, \dots, g_n)}. \end{aligned}$$

3. Clear from the definitions. □

Lemma 3.2.5. *Every homomorphism of rings $\varphi : A \to B$ induce a continuous map $\psi : \operatorname{Spec} B \to \operatorname{Spec} A$ defined by $\psi(p) = \varphi^{-1}(p)$.*

Proof. The map ψ is well defined since the inverse image of a prime ideal is a prime ideal. The fact that is continuous follows from

$$\psi^{-1}(V(J)) = \{p \in \operatorname{Spec} B : \psi(p) \supseteq J\} = \{p \in \operatorname{Spec} B : \varphi^{-1}(p) \supseteq J\} = V(\varphi(J)),$$

(where we used $p \supseteq \varphi(J) \Leftrightarrow \varphi^{-1}(p) \supseteq J$). □

From the previous lemma, we can see Spec as a (contravariant) functor from **Ring** to **Top**. We will see later that Spec is actually a (contravariant) functor from **Ring** to the category of affine schemes.

Proposition 3.2.6. *Let R be a ring, $f \in R$, I an ideal of R and set $X = \operatorname{Spec} R$.*

1. *$\operatorname{Spec}(R_f)$ is homeomorphic to the open set $X_f \subseteq X$ with the induced topology.*
2. *$\operatorname{Spec}(R/I)$ is homeomorphic to the closed set $V(I) \subseteq X$ with the induced topology.*

Proof.

1. From Lemma 3.2.5, the map $\varphi : R \to R_f$ induce a map continuous map on the spectra $\psi : \operatorname{Spec} R_f \to \operatorname{Spec} R$ given by $\psi(p) = \varphi^{-1}(p)$. Now, $p \mapsto \psi(p)$ is a one to one correspondence between the prime ideals of R_f and the prime ideals of R that do not intersect $\{1, f, f^2, \dots, \}$. Finally, we prove that ψ is a closed map. Let I be an ideal of R_f. Since all the ideals in R_f are extended, I is generated by $\varphi(J)$ for some ideal J of R. We have

$$\psi(V(I)) = \{\varphi^{-1}(p) : p \supseteq \varphi(J)\} = \{q \in X_f : q \supseteq J\} = X_f \cap V(J).$$

2. Similarly as above, the quotient map $\varphi : R \to R/I$ induce a well defined continuous map on the spectra $\psi : \operatorname{Spec} R/I \to \operatorname{Spec} R$ given by $\psi(p) = \varphi^{-1}(p)$. Now $\psi(\operatorname{Spec} R/I) \subseteq V(I)$, in fact $\psi(p) = \varphi^{-1}(p) \supseteq I$. Further, $p \mapsto \psi(p)$ is a bijection between prime ideals of R/I and prime ideals of R that contain I. Finally, we prove that ψ is a closed map. Let J be an ideal of R/I, then J is the image under φ of an ideal J' of R containing I. We have

$$\psi(V(J)) = \{\varphi^{-1}(p) : p \supseteq J\} = \{q \in V(I) : q \supseteq J'\} = V(J').$$ □

3.3 Topological properties of $\operatorname{Spec} R$

3.3.1 Connectedness

Definition 3.3.1. A topological space is **connected** if it cannot be written as a disjoint union of two nonempty open sets (or, equivalently, closed sets).

Proposition 3.3.2. *The following statements are equivalent.*

1. $X = \operatorname{Spec} R$ *is not connected.*
2. *There exists two proper ideals* $I, J \subseteq R$ *such that* $IJ = 0$, $I + J = R$.
3. $R \simeq R_1 \times R_2$.
4. R *has an idempotent element* $e^2 = e \in R \setminus \{0, 1\}$.

Proof.

(1) $\Rightarrow$ (2) By hypothesis, X is the disojoint union of two nonempty closed sets. That is, $X = V(I) \cup V(J) = V(IJ)$ with $\emptyset = V(I) \cap V(J) = V(I + J)$. This implies $\sqrt{I + J} = R$, so $I + J = R$, and $IJ \subseteq \sqrt{IJ} = \mathfrak{N}(R)$. Therefore, there exists $i \in I$ and $j \in J$ such that $i + j = 1$ and $ij \in \mathfrak{N}(R)$, so $(ij)^n = 0$ for some $n \in \mathbb{N}$. Now let $I' = (i^n)$ and $J' = (j^n)$, by construction we have $I'J' = 0$, further

$$\sqrt{I' + J'} = \sqrt{\sqrt{I'} + \sqrt{J'}} = \sqrt{(i) + (j)} = R \Rightarrow I' + J' = R.$$

(2) $\Rightarrow$ (3) From the Chinese remainder theorem the map $R \to R/I \times R/J$ defined by $r \mapsto (r + I, r + J)$ is an isomorphism.

(3) $\Rightarrow$ (1) We have $X = X_{(1,1)} = X_{(1,0)} \cup X_{(0,1)}$, with $X_{(1,0)} \cap X_{(0,1)} = X_{(0,0)} = \emptyset$.

(2) $\Rightarrow$ (4) Let $e \in I$ and $e' \in J$ such that $e + e' = 1$ and $ee' = 0$. Then $e' = 1 - e$ and $0 = ee' = e(1 - e) = e - e^2 \Rightarrow e^2 = e$.

(4) $\Rightarrow$ (2) Set $I = (e)$ and $J = (1 - e)$. $\square$

An alternative proof of (1) $\Rightarrow$ (2) of the previous proposition can be given in terms of the *structure sheaf* (that we will define later). In fact, if $X = U \cup V$ is the disjoint union of two open sets, then $\mathcal{O}_X(U \cup V) = \mathcal{O}_X(U) \times \mathcal{O}_X(V)$.

3.3.2 Irreducibility

Definition 3.3.3. A topological space X is **irreducible** if whenever $X = F \cup G$ with F and G closed subsets of X, we have $X = F$ or $X = G$. The space X is **reducible** if it is not irreducible.

Note that irreducibility implies connectedness. Further, every singleton is irreducible.

Proposition 3.3.4. *Let* $f : X \to Y$ *be a continuous function of topological spaces. If* X *is irreducible, then* $f(X)$ *is irreducible.*

Proof. Suppose that $f(X) = F \cup G$, for F and G closed in $f(X)$. By taking the preimages we have $X = f^{-1}(F) \cup f^{-1}(G)$. Since f is continuous, $f^{-1}(F)$ and $f^{-1}(G)$ are closed, therefore, from the irreducibility of X, we have $X = f^{-1}(F)$ or $X = f^{-1}(G)$, that is $f(X) = F$ or $f(X) = G$. □

Proposition 3.3.5. *For a topological space X the following are equivalent:*

1. *X is irreducible.*
2. *If U, V are nonempty open sets of X, then $U \cap V \neq \emptyset$.*
3. *Every open set of X is dense.*

Proof.

(1) $\Leftrightarrow$ (2) Follows from the definition by taking the complements.

(2) $\Leftrightarrow$ (3) Follows from the fact that a set is dense if it intersects every open set. □

Corollary 3.3.6. *Let X be a topological space, for every $Y \subseteq X$ the following are equivalent:*

1. *Y is irreducible.*
2. *If U, V are two open sets that intersect Y, then $U \cap V$ intesects Y.*
3. *$\overline{Y}$ is irriducible.*

Proof.

(1) $\Leftrightarrow$ (2) Follows from (2) of Proposition 3.3.5.

(2) $\Leftrightarrow$ (3) Follows from the fact that an open U intersects Y if and only if it intersects $\overline{Y}$. In fact, if U intersects Y then $U \cap \overline{Y} \supseteq U \cap Y \neq \emptyset$. Conversely, if U intersects $\overline{Y}$, let $x \in U \cap \overline{Y}$. The open set U is a neighbourhood of $x \in \overline{Y}$, therefore $U \cap Y \neq \emptyset$. □

Corollary 3.3.7. *In an irreducible topological space X the nonempty open sets are irreducible.*

Proof. From (2) of Proposition 3.3.5 the open sets of X have nonempty intersections. Now the result follows from (1) $\Leftrightarrow$ (2) of Corollary 3.3.6. □

Lemma 3.3.8. *Let $Y \subseteq \operatorname{Spec}(R)$. Then Y is irreducible if and only if $\mathscr{I}(Y)$ is prime.*

Proof.

$\Rightarrow$ Suppose $IJ \subseteq \mathscr{I}(Y)$, then $V(I) \cup V(J) = V(IJ) \supseteq \overline{Y} \supseteq Y$. Since Y is irreducible, we have $Y \subseteq V(I)$ or $Y \subseteq V(J)$, that is $\mathscr{I}(Y) \supseteq \sqrt{I} \supseteq I$ or $\mathscr{I}(Y) \supseteq \sqrt{J} \supseteq J$.

$\Leftarrow$ Suppose $Y \subseteq V(I) \cup V(J) = V(IJ)$, then $\mathscr{I}(Y) \supseteq \sqrt{IJ} \supseteq IJ$, Since $\mathscr{I}(Y)$ is prime, it follows $\mathscr{I}(Y) \supseteq I$ or $\mathscr{I}(Y) \supseteq J$, that is $Y \supseteq \overline{Y} \subseteq V(I)$ or $Y \subseteq \overline{Y} \subseteq V(J)$. □

3.3.3 Irreducible componentes and Noetherianity

Definition 3.3.9. An **irreducible component** of a topological space X is a maxiaml (with respect to inclusion) irreducible subset of X.

From Corollary 3.3.6 it follows that every irreducible component is closed.

Proposition 3.3.10. *Let X be a topological space.*

1. *Every irreducible subset of X is contained in an irreducible component.*

2. *X is the union of its irreducible components.*

Proof.

1. Let $Y \subseteq X$ be irreducible, and set $\Sigma = \{Z \subseteq X : Z \text{ is irreducible, } Z \supseteq Y\}$. Clearly, $Y \in \Sigma \neq \emptyset$. Let $\{X_i\}_{i \in I} \subseteq \Sigma$ be a chain and set $Z = \bigcup_{i \in I} X_i$. If U and V are two open sets that intersect Z, then they intersect X_i and X_j respectively, for some $i, j \in I$. If for instance $X_i \subseteq X_j$, then $U \cap V$ intersect X_j, thus also Z. From Corollary 3.3.6 it follows $Z \in \Sigma$. By Zorn's lemma Σ has a maxiaml element, hence Y is contained in an irreducible component.

2. Follows from (1) since every singleton is irreducible. □

Definition 3.3.11. A topological space X is **Noetherian** if every descending chain of closed sets $X_0 \supseteq X_1 \supseteq \ldots$ eventually stabilizes, that is there exists $n \in \mathbb{N}$ such that $X_n = X_{n+1} = \ldots$

Remark 3.3.12. From Proposition 3.1.5 we have that if R is a Noetherian ring, then $\operatorname{Spec} R$ is a Noetherian topological space. The converse is not true in general. For example, the ring $R = k[x_i : i \in \mathbb{N}]/(x_i^i : i \in \mathbb{N})$ is not Noetherian, but $\operatorname{Spec} R$ has one point, so it is Noetherian.

The preceding condition on descending chains can be given in general for a partially ordered set $(\Sigma, \leq)$. It is called **Descending Chain Condition** (D.C.C.). It is easy to see that it is equivalent to the **Minimal condition**, that is the fact that every nonempty subset of Σ has a minimal element.

Proposition 3.3.13. *In a Noetherian topological space X every closed set is a finite union of irreducible sets.*

Proof. Let Σ be the family of closed subsets of X that are not a finite union of irreducible sets. Suppose by contardiction that $\Sigma \neq \emptyset$. Since X is Noetherian, let $Z \in \Sigma$ be a minimal element. By construction Z is not irreducible, hence $Z = F \cup G$ for some closed sets F and G contained in Z. By the minimality of Z it follows that $F, G \notin \Sigma$, that is F and G are a finite union of irreducible sets. This implies that also Z is a finite union of irreducible sets, contradiction. □

The preceding result is the topological translation of the fact that in a Noetherian ring, every ideal is a finite intersection of irreducible ideals.

Corollary 3.3.14. *A Noetherian topological space X has a finite number of irreducible components. Every irreducible component is not contained in the union of the others.*

Proof. Since X is closed, from Proposition 3.3.13, X is a finite union of irreducible sets. From (1) of Proposition 3.3.10 we can write $X = X_1 \cup \ldots \cup X_n$ with X_i distinct irreducible components of X. Now suppose that $Y \subseteq X$ is an irreducible component of X, then $Y \subseteq \bigcup_{i=1}^{n} X_i$. Since Y is irreducible and the X_i are closed, it follows $Y = X_i$ for some $i \in \{1, \ldots, n\}$. Therefore the irreducible components of X are all and only of the form X_i with $i \in \{1, \ldots, n\}$, in particular they are finitely many. Assume by contradiction that $X_i \subseteq \bigcup_{j \neq i} X_j$, similarly as before we have $X_i = X_j$ for some $j \neq i$, contradiction. $\square$

Remark 3.3.15. Since the irreducible components of $\operatorname{Spec} R$ are the maximal irreducible closed subsets, from Lemma 3.3.8 and Proposition 3.1.5 they corresponds to the minimal prime ideals of R.

3.3.4 Connected components

Definition 3.3.16. A **connected component** of a topological space X is a maxiaml (with respect to inclusion) connected subset of X.

Since the closure of a connected set is connected, the connected components are closed. It is not always the case that the connected components are open. For example, consider $\mathbb{Q}$ with the induced Euclidean topology. It is a totally disconnected space, that is, the connected components are the singletons, and they are not open.

However, it is the case if the topological space is Noetherian, as the following results show.

Proposition 3.3.17. *If a topological space X has a finite number of connected components, then those are open in X.*

Proof. Since X is the union of its connected components $X = X_1 \cup \cdots \cup X_n$, then $X_i = X \setminus \bigcup_{j \neq i} X_j$, therefore X_i is open. $\square$

Proposition 3.3.18. *Every connected component of a topological space X is the union of irreducible components.*

Proof. Let C be a connected component of X. For every $x \in C$, from Proposition 3.3.10 the singleton $\{x\}$ is contained in an irreducible F component of X. Since every irreducible set is connected, we have $x \in F \subseteq C$. It follows that C is a union of irreducible components of X. $\square$

Corollary 3.3.19. *A Noetherian topological space X has a finite number of connected components. In particular, they are open.*

Proof. From Corollary 3.3.14 X has a finite number of irreducible components. From Proposition 3.3.18 it follows that X has also a finite number of connected components. The last claim follows from Proposition 3.3.17. $\square$

3.3.5 (Quasi)compactness

Definition 3.3.20. A topological space X si **quasicompact** (or **compact**) if every open cover $X = \bigcup_{i \in I} U_i$ has a finite subcover $X = U_{i_1} \cup \cdots \cup U_{i_n}$.

Note that a space X is compact if and only if every family of closed sets that satisfy the finite intersection property (FIP) has nonempty intersection (it is enough to consider the complements in the defintion of quasicompactness).

Proposition 3.3.21. *Every Noetherian topological space is quasicompact.*

Proof. Let X be a Noetherian topological space. Let $\mathscr{F} = \{F_i\}_{i \in I}$ be a family of closed sets that statisy the FIP (every finite intersection of elements of $\mathscr{F}$ is nonempety). Suppose by contradiction that $\bigcap_{F \in \mathscr{F}} F = \emptyset$. Then, for every finite set $\{F_{i_1}, \ldots, F_{i_n}\}$ of elements of $\mathscr{F}$, and every $x \in F_{i_1} \cap \cdots \cap F_{i_n} \neq \emptyset$, there exists $F_{i_{n+1}} \in \mathscr{F}$ such that $x \notin F_{i_1} \cap \cdots \cap F_{i_{n+1}}$. In particular $F_{i_1} \cap \cdots \cap F_{i_n} \subsetneq F_{i_1} \cap \cdots \cap F_{i_{n+1}}$. Therefore, we can inductively construct a not stationary descending chain of closed sets

$$F_{i_1} \supsetneq F_{i_1} \cap F_{i_2} \supsetneq F_{i_1} \cap F_{i_2} \cap F_{i_3} \supsetneq \ldots$$

contradicting the fact that X is Noetherian. □

Proposition 3.3.22. *The topological space $X = \operatorname{Spec}(R)$ is quasicompact.*

Proof. Because distinguished open sets form a basis, it suffices to check that every covering of distinguished open sets $\{X_f : f \in S\}$ for some $S \subseteq R$ has a finite subcover. From Corollary 3.2.3 we have $1 \in \langle S \rangle$, so there exist $f_1, \ldots, f_n \in S$ such that $1 = \sum_{i=1}^{n} a_i f_i$ for some $a_i \in R$, this means $1 \in \langle f_1, \ldots, f_n \rangle$, again from Corollary 3.2.3 $\big\{X_{f_i} : i \in \{1, \ldots, n\}\big\}$ is a finite subcover. □

3.3.6 Generic and closed points

Definition 3.3.23. A point p in a topological space X is a **closed point** if $\{p\}$ is closed in X.

In $X = \operatorname{Spec} R$ the closed points correspond to maximal ideals. In fact, for every $p \in X$, we have

$$\overline{\{p\}} = V(\mathscr{I}(\{p\})) = V(p) = \{q \in X : q \supseteq p\}.$$

Therefore, $\{p\}$ is closed if and only if p is maximal.

Definition 3.3.24. A point p in a topological space X is a **generic point** for a closed subset $Y \subseteq X$ if $\overline{\{p\}} = Y$.

Proposition 3.3.25. *Let $X = \operatorname{Spec}(R)$. Any irreducible closed subset $Y \subseteq X$ has a unique generic point. In other words, there is a bijection between points of $\operatorname{Spec} R$ and irreducible closed subsets, given by $p \mapsto \overline{\{p\}}$.*

Proof. Let $p = \mathscr{I}(Y) \subseteq R$. Since Y is irreducible, from Lemma 3.3.8 p is prime, so it can be seen as a point of X. From Proposition 3.1.5 we have

$$\overline{\{p\}} = V(\mathscr{I}(\{p\})) = V(p) = V(\mathscr{I}(Y)) = \overline{Y} = Y.$$

Finally, if p_1, p_2 are two generic points of Y, then applying the previous equality we have

$$p_1 \in Y = \overline{\{p_2\}} = V(p_2) \Rightarrow p_1 \supseteq p_2$$

and similarly we obtain $p_1 \subseteq p_2$. □

Remark 3.3.26. If R is a domain, then the zero ideal $(0) = \mathscr{I}(\operatorname{Spec} R)$ is prime and, from Lemma 3.3.8, $\operatorname{Spec} R$ is irreducible. Therefore, it has a unique generic point, and it is the zero ideal, since

$$\overline{\{(0)\}} = V(\mathscr{I}(\{(0)\})) = V(0) = \operatorname{Spec} R.$$

3.4 Structure sheaf $\mathcal{O}_X$

In this section we will define a structure sheaf of rings on $\operatorname{Spec} R$. First, we will need some results about localization.

Definition 3.4.1. Let R be a ring, a multiplicative set $S \subseteq R$ is **satured** if from $xy \in S$ we have $x \in S$ and $y \in S$. The **saturation** $\overline{S}$ of a multiplicative set $S \subseteq R$ is the intersection of all satured multiplicative sets containing S.

Since satured multiplicative sets are closed under intersection, the saturation is again a satured multiplicative set. It is easy to see that

$$\overline{S} = \{x \in R : xy \in S \text{ for some } y \in S\}.$$

Lemma 3.4.2. *Let I be an ideal of R and $S \subseteq R$ a multiplicative set such that $I \cap S = \emptyset$. The set*

$$\Sigma = \{P \textit{ ideal of } R : I \subseteq P,\ P \cap S = \emptyset\}$$

has a maximal element (with respect to inclusion). Further, every maximal element of Σ is a prime ideal.

Proof. To show that Σ has a maximal element is an easy application of Zorn's lemma. Now let P be a maximal element of Σ, and let $a, b \notin P$. Since P is maximal we have

$$I \subseteq P \subsetneq P + (a) \Rightarrow \exists s \in (P + (a)) \cap S,$$
$$I \subseteq P \subsetneq P + (b) \Rightarrow \exists t \in (P + (b)) \cap S,$$

it follows $st \in (P + (ab)) \cap S$, therefore $ab \notin P$, otherwise we will have $st \in P \cap S = \emptyset$. This poves that P is prime. □

Proposition 3.4.3. *A multiplicative set $S \subseteq R$ is satured if and only if $R \setminus S$ is a union of prime ideals.*

Proof.

$\Rightarrow$ Let $x \in R \setminus S$, since S is satured, we have $(x) \cap S = \emptyset$, from Lemma 3.4.2 there exists a prime ideal containing (x) such that $P \cap S = \emptyset$, that is $x \in P \subseteq R \setminus S$. By the arbitrary choice of $x \in R \setminus S$, it follows that $R \setminus S$ is a union of prime ideals.

$\Leftarrow$ Let $R \setminus S = \bigcap_{i \in I} P_i$, with P_i prime ideals. Suppose $xy \in S = R \setminus \bigcup_{i \in I} P_i$, that is $xy \notin P_i \Rightarrow x, y \notin P_i$ for every $i \in I$, thus $x, y \in S = R \setminus \bigcup_{i \in I} P_i$, so S is satured. $\square$

Corollary 3.4.4. *The saturation of a multiplicative set $S \subseteq R$ is*

$$\overline{S} = R \setminus \bigcup_{\substack{P \cap S = \emptyset \\ P \text{ prime}}} P.$$

Proof. If S is satured, the result follows from the proof of the previous proposition. If S is not satured, it is enough to observe that, for a prime ideal P, we have $P \cap \overline{S} = \emptyset$ if and only if $P \cap S = \emptyset$. $\square$

Remark 3.4.5. Set $X = \operatorname{Spec} R$. In the particular case when $S = \{1, f, f^2, \dots\}$ for some $f \in R$, the saturation of S is

$$\overline{S} = R \setminus \bigcup_{p \in X_f} p.$$

In fact, for a prime ideal $p \in X$ we have $p \cap S = \emptyset \Leftrightarrow f \notin p \Leftrightarrow p \in X_f$.

Proposition 3.4.6. *Let $S \subseteq R$ be a multiplicative set, then $S^{-1}R \simeq \overline{S}^{-1}R$.*

Proof. Since $S \subseteq \overline{S}$, we can define $\varphi : S^{-1}R \to \overline{S}^{-1}R$ by $\varphi(r/s) = r/s$. Now, φ is clearly an homomorphism of rings. For injectivity, suppose that $\varphi(r/s) = \varphi(r'/s')$, this means $r/s = r'/s'$ in $\overline{S}^{-1}R$, that is, there exists $x \in \overline{S}$ such that $x(rs' - r's) = 0$. Since $x \in \overline{S}$, there exists $y \in S$ such that $xy \in S$. Therefore $xy(rs' - r's) = 0$, that is $r/s = r'/s'$ in $S^{-1}R$. For surjectivity, let $r/s \in \overline{S}^{-1}R$. There exists $s' \in S$ such that $ss' \in S$, this means $\varphi(rs'/ss') = rs'/ss' = r/s$. $\square$

From Proposition 3.4.6 and Remark 3.4.5 we have the following result.

Corollary 3.4.7. *Let $f \in R$, set $X = \operatorname{Spec} R$ and $S_{X_f} = R \setminus \bigcup_{p \in X_f} p$. Then*

$$R_f \simeq S_{X_f}^{-1} R.$$

Lemma 3.4.8 (Double localization)**.** *Let R be a ring, $S \subseteq T$ be two multiplicative sets of R, and $\widetilde{T}$ be the image of T in $S^{-1}R$. The map $\varphi : T^{-1}R \to \widetilde{T}^{-1}(S^{-1}R)$ defined by $\varphi(r/t) = (r/1)/(t/1)$ is an isomorphism. Further, the following diagram commutes*

$$\begin{tikzcd} R \arrow[r, "S^{-1}"] \arrow[d, "T^{-1}"] & S^{-1}R \arrow[d, "\widetilde{T}^{-1}"] \\ T^{-1}R \arrow[r, "\varphi"] & \widetilde{T}^{-1}(S^{-1}R) \end{tikzcd}$$

In particular, the homomorphism $\varphi^{-1} \circ \widetilde{T}^{-1} : S^{-1}R \to T^{-1}R$ commutes with the localization maps (alternatively, it is a homomorphism of R-algebras).

Proof. It is clear that φ is an isomorphism of rings. For the injectivity, suppose that $\varphi(r/t) = \varphi(r'/t')$, this means that there exists $u \in \widetilde{T}$ such that $u(rt' - r't) = 0$ (as elements of $S^{-1}R$), that is, there exists $s \in S$ such that $us(rt' - r't) = 0$ (as elements of R), since $us \in T$, this means that $r/t = r'/t'$ (as elements of $T^{-1}R$). For surjectivity, it is enough to note that $(r/s)/(t/1) = \varphi(r/st)$. The commutativity of the diagram is clear. $\square$

Remark 3.4.9. Let R be a ring. If $f_1, \dots, f_n \in R$ are such that $\langle f_1, \dots, f_n \rangle = R$, then we have also $\langle f_1^{m_1}, \dots, f_n^{m_n} \rangle = R$ for every $m_1, \dots, m_n \in \mathbb{N}$, since

$$\sqrt{(f_1^{m_1}) + \cdots + (f_n^{m_n})} = \sqrt{\sqrt{(f_1^{m_1})} + \cdots + \sqrt{(f_n^{m_n})}} = \sqrt{(f_1) + \cdots + (f_n)} = R.$$

Now we are going to define a sheaf of rings on the topological space $X = \operatorname{Spec} R$. In order to do this, from Theorem 2.5.4 it is enough to define a $\mathcal{B}$-sheaf of rings on $\operatorname{Spec} R$. The base we will consider is the base of distinguished open sets $\mathcal{B} = \{X_f : f \in R\}$

Proposition-Definition 3.4.10. *Let $X = \operatorname{Spec} R$ and $\mathcal{B}$ be the family of distinguished open sets of X ordered by inclusion. Define a $\mathcal{B}$-presheaf of rings $\mathcal{O}_X : \mathcal{B}^{op} \to$ **Ring** as follows. For every $B \in \mathcal{B}$ let $S_B = R \setminus \bigcup_{p \in B} p$ and set*

$$\mathcal{O}_X(B) = S_B^{-1} R.$$

For every $B \subseteq B'$ we have $S_B \supseteq S_{B'}$, the restriction map $\rho_{B',B} : \mathcal{O}_X(B') \to \mathcal{O}_X(B)$ is the map described in Lemma 3.4.8 (therefore $\mathcal{O}_X$ is a $\mathcal{B}$-presheaf). $\mathcal{O}_X$ is a $\mathcal{B}$-sheaf of rings on X. The induced sheaf of rings by Theorem 2.5.4, that we continue to denote by $\mathcal{O}_X$, is called the ***structure sheaf*** *of X.*

Proof. First, we prove the two properties of a sheaf on a base for a finite open cover $\{X_{f_i} : i \in \{1, \dots, n\}\} \subseteq \mathcal{B}$ of X.

(1) Let $s \in \mathcal{O}_X(X) = R$ such that $s_{|X_{f_i}} = 0$ for every $i \in \{1, \dots, n\}$. Since $S_{X_{f_i}}$ is the saturation of $\{1, f_i, f_i^2, \dots\}$ (Remark 3.4.5), this means that $f_i^{m_i} s = 0$ for some $m_i \in \mathbb{N}$ and every $i \in \{1, \dots, n\}$. Since $\{X_{f_i} : i \in \{1, \dots, n\}\}$ is a cover of X, from Corollary 3.2.3 we have $\langle f_1, \dots, f_n \rangle = R$, therefore also $\langle f_1^{m_1}, \dots, f_n^{m_n} \rangle = R$ (Remark 3.4.9). Thus $\sum_{i=1}^n a_i f_i^{m_i} = 1$ for some $a_i \in R$. Finally

$$s = s\left(\sum_{i=1}^n a_i f_i^{m_i}\right) = \sum_{i=1}^n a_i f_i^{m_i} s = 0.$$

(2) Let $s_i \in \mathcal{O}_X(X_{f_i}) = S_{X_{f_i}}^{-1} R$ such that $s_{i|X_{f_i} \cap X_{f_j}} = s_{j|X_{f_i} \cap X_{f_j}}$. Since $S_{X_{f_i}}$ is the saturation of $\{1, f_i, f_i^2, \dots\}$ (Remark 3.4.5), the elements s_i can be written in the form $s_i = r_i / f_i^{\alpha_i}$ for some $\alpha_i \in \mathbb{N}$ and every $i \in \{1, \dots, n\}$. From Proposition 3.2.4, we have $X_{f_i} \cap X_{f_j} = X_{f_i f_j}$, thus by a similar argument, the equality on overlaps implies that

$$(f_i f_j)^\beta (r_i f_j^{\alpha_j} - r_j f_i^{\alpha_i}) = 0, \tag{3.1}$$

for some $\beta \in \mathbb{N}$ independent of the choice of $i, j \in \{1, \dots, n\}$. Now, from Remark 3.4.9 we have $\langle f_i^{\alpha_i+\beta} : i \in \{1, \dots, n\} \rangle = R$, therefore $\sum_{i=1}^n a_i f_i^{\alpha_i+\beta} = 1$ for some in $a_i \in R$. Now set $r = \sum_{i=1}^n a_i f_i^\beta r_i \in R$. For each $i \in \{1, \dots, n\}$ we have

$$r f_i^{\alpha_i+\beta} = \sum_{j=1}^n a_j \big(r_j f_i^{\alpha_i}\big)\big(f_i f_j\big)^\beta = \sum_{j=1}^n a_j \big(r_i f_j^{\alpha_j}\big)\big(f_i f_j\big)^\beta = r_i f_i^\beta \sum_{j=1}^n a_j f_j^{\alpha_j+\beta} = r_i f_i^\beta,$$

where in the second equality we used (3.1). The previous equality can be rewritten as $f_i^\beta(r f_i^{\alpha_i} - r_i) = 0$, that is $r_{|X_{f_i}} = r/1 = r_i/f_i^{\alpha_i} = s_i$.

Now, since X is quasicompact (Proposition 3.3.22), it is clear that what we proved above suffice for property (1) also in the case of infinite open covers. For property (2), let $\{X_{f_i} : i \in I\} \subseteq \mathcal{B}$ be an infinite open cover, and let $s_i \in \mathcal{O}_X(X_{f_i})$ for every $i \in I$ that agree on overlaps. Since X is quasicompact, there exists a finite subcover $\{X_{f_j} : j \in J\}$ with $J \subseteq I$. From what we have proved above, there exists some $s \in R$ such that $s_{|X_{f_j}} = s_j$ for every $j \in J$. Now fix $\bar{i} \in I \setminus J$, to verify that also $s_{|X_{f_{\bar{i}}}} = s_{\bar{i}}$, repeat the process with the finite open cover $\{X_{f_j} : j \in J \cup \{\bar{i}\}\}$, thus there exists a $s' \in R$ such that $s'_{|X_{f_j}} = s_j$ for every $j \in J \cup \{\bar{i}\}$, in particular s and s' agree on X_{f_j} for $j \in J$. From property (1) this implies $s = s'$, therefore $s_{|X_{f_{\bar{i}}}} = s_{\bar{i}}$.

Finally, from Proposition 3.2.6, the proof for open covers of a distinguished open set X_f is obtained by the same steps above, substituting the ring R with $S_{X_f}^{-1} R \simeq R_f$. $\square$

Proposition 3.4.11. *Let R be a ring. The stalk of the structure sheaf of $X = \operatorname{Spec} R$ at a point $p \in X$ is isomorphic to the localization of R at p:*

$$\mathcal{O}_{X,p} \simeq R_p.$$

Proof. Let $\mathcal{B}$ be the base of distinguished open sets of X. We have

$$\mathcal{O}_{X,p} = \varinjlim_{p \in U} \mathcal{O}_X(U) \simeq \varinjlim_{p \in X_f \in \mathcal{B}} \mathcal{O}_X(X_f) \simeq \varinjlim_{f \in R \setminus p} R_f \simeq R_p.$$

$\square$

3.5 Definition of scheme

Definition 3.5.1. A **ringed space** is a pair $(X, \mathcal{O}_X)$, where X is a topological space, and $\mathcal{O}_X$ is a sheaf of rings on X.

In this section, $\mathcal{O}_X$ will be always the sheaf of rings of a ringed space $(X, \mathcal{O}_X)$. We will denote the restriction of $\mathcal{O}_X$ to an open set U of X by simply $\mathcal{O}_U = \mathcal{O}_{X|U}$. The stalk of $\mathcal{O}_X$ at a point $p \in X$ will be denoted by $\mathcal{O}_{X,p} = (\mathcal{O}_X)_p$.

We now give the definition of morphisms between ringed spaces, thus we define their category. Further, we will define the subactegory of *locally ringed space*. This notions will be needed to define schemes.

Definition 3.5.2. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be two ringed space. A **morphism of ringed spaces** from $(X, \mathcal{O}_X)$ to $(Y, \mathcal{O}_Y)$ is a pair $(\pi, \pi^\#)$ of a continuous map $\pi : X \to Y$ and a morphism of sheaves $\pi^\# : \mathcal{O}_Y \to \pi_*\mathcal{O}_X$ (where $\pi_*\mathcal{O}_X$ is the direct image of $\mathcal{O}_X$ under π, Definition 2.1.13).

In analogy with smooth manifolds, the geometric motivation for the previous definition is the following. Let $\pi : X \to Y$ be a map of smooth manifolds, and let $\mathcal{O}_X$ and $\mathcal{O}_Y$ be the sheaves of smooth function on X and Y respectively. A smooth function f on Y pulls back to a smooth function f^* on X. More precisely, given an open subset $U \subseteq Y$, there is a natural map $\mathcal{O}_Y(U) \to \mathcal{O}_X(\pi^{-1}(U))$. The pullback commutes with restrictions, thus we have a map of sheaves $\pi^\# : \mathcal{O}_Y \to \pi_*\mathcal{O}_X$. In the case of smooth manifolds, the pullback map is induced by the smooth map π. However, this is not the case for continuous functions between ringed spaces. This is the reason why we have to add the map $\pi^\# : \mathcal{O}_Y \to \pi_*\mathcal{O}_X$ in the definition.

Remark 3.5.3. Let $(\pi, \pi^\#) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. For every $p \in X$, the map $\pi^\# : \mathcal{O}_Y \to \pi_*\mathcal{O}_X$ induces a map on the stalks at $\pi(p)$: $\pi^\#_{\pi(p)} : \mathcal{O}_{Y,\pi(p)} \to (\pi_*\mathcal{O}_X)_{\pi(p)}$, see (2.1). Now we have $(\pi_*\mathcal{O}_X)_{\pi(p)} \simeq \mathcal{O}_{X,p}$, in fact

$$(\pi_*\mathcal{O}_X)_{\pi(p)} = \varinjlim_{\pi(p)\in U} \pi_*\mathcal{O}_X(U) = \varinjlim_{p\in\pi^{-1}(U)} \mathcal{O}_X(\pi^{-1}(U)) \simeq \mathcal{O}_{X,p}.$$

Definition 3.5.4. Let A and B be two local rings with maximal ideals $\mathfrak{m}_A$ and $\mathfrak{m}_B$ respectively. A homomorphism $\varphi : A \to B$ is a **local homomorphism** if $\varphi(\mathfrak{m}_A) \subseteq \mathfrak{m}_B$.

Definition 3.5.5. A ringed space $(X, \mathcal{O}_X)$ is a **locally ringed space** if for each point $p \in X$, the stalk $\mathcal{O}_{X,p}$ is a local ring. A **morphism of locally ringed spaces** from $(X, \mathcal{O}_X)$ to $(Y, \mathcal{O}_Y)$ is a morphism of ringed spaces $(\pi, \pi^\#)$ such that, for every $p \in X$, the induced map on the stalks $\pi^\#_{\pi(p)} : \mathcal{O}_{Y,\pi(p)} \to \mathcal{O}_{X,p}$ is a local homomorphism (note that we used Remark 3.5.3).

It is a routine check that ringed spaces with their morphisms and locally ringed spaces with their morphisms do form two categories. From Proposition 3.4.11 we have the following corollary.

Corollary 3.5.6. *The spectrum of a ring R with its structure sheaf* $(\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R})$ *is a locally ringed space.*

Definition 3.5.7 (Affine scheme)**.** An **affine scheme** is a locally ringed space $(X, \mathcal{O}_X)$ which is isomorphic (as a locally ringed space) to $(\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R})$ for some ring R. The **morphisms of affine schemes** are the morphisms as locally ringed spaces.

Definition 3.5.8 (Scheme)**.** A **scheme** is a locally ringed space $(X, \mathcal{O}_X)$ in which every point $p \in X$ has an open neighbourhood U such that $(U, \mathcal{O}_{X|U})$ is an affine scheme. The **morphisms of schemes** are the morphisms as locally ringed spaces.

3.6 Affine schemes

The aim of this section is, informally, to show that an affine scheme retain no more information than its associated ring.

Lemma 3.6.1. *A homomorphism of rings $\varphi : A \to B$ induces a morphism of affine schemes $(\pi, \pi^\#)$ from $X = \operatorname{Spec} B$ to $Y = \operatorname{Spec} A$.*

Proof. The continuous map π is given by Lemma 3.2.5. Now we define $\pi^\#$. From Proposition 2.5.6 it is enough to describe $\pi^\#$ on a distinguished open set Y_f with $f \in A$. First, note that

$$\begin{aligned}\pi^{-1}(Y_f) &= \{p \in X : \pi(p) \in Y_f\} \\ &= \{p \in X : f \notin \varphi^{-1}(p)\} \\ &= \{p \in X : \varphi(f) \notin p\} = X_{\varphi(f)}.\end{aligned}$$

Further, we have

$$\begin{aligned}\mathcal{O}_Y(Y_f) &= S_{Y_f}^{-1}A \\ \pi_*\mathcal{O}_X(Y_f) &= \mathcal{O}_X(\pi^{-1}(Y_f)) = \mathcal{O}_X(X_{\varphi(f)}) = S_{X_{\varphi(f)}}^{-1}B = S_{Y_f}^{-1}B,\end{aligned}$$

where in the last equality, we are using the fact that $\varphi(S_{Y_f}) = S_{X_{\varphi(f)}}$, and we are viewing B as an A-algebra. Now, define the map

$$\pi^\#(Y_f) : \mathcal{O}_Y(Y_f) \to \pi_*\mathcal{O}_X(Y_f),$$

to be the localization of φ at S_{Y_f}, that is $S_{Y_f}^{-1}\varphi : S_{Y_f}^{-1}A \to S_{Y_f}^{-1}B$. Note that, as in Proposition-Definition 3.4.10, this definition does not depend on the choice of $f \in A$. From the fact that diagrams of the form

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d, "S^{-1}"'] & B \arrow[d] \\ S^{-1}A \arrow[r, "S^{-1}\otimes\varphi"] & S^{-1}B \end{tikzcd}$$

commutes, it follows that $\pi^\#$ is a morphism of sheaves. Finally, for every $p \in X$, the map on stalks $\pi^\#_{\pi(p)}$ is $\varphi_{\pi(p)} : A_{\pi(p)} \to B_{\pi(p)}$, which is a local homomorphism (note that $B_{\pi(p)} = B_p$). $\square$

Theorem 3.6.2. *The category of affine schemes is equivalent to the category of rings.*

Proof. Let $\mathscr{A}$ denote the the category of affine schemes. Let $F : \mathbf{Ring} \to \mathscr{A}^{op}$ be a functor defined by $F(R) = (\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R})$ for every ring R, and for every morphism of rings $\varphi : R \to R'$, $F(\varphi)$ is defined as in Lemma 3.6.1. We want to show that F is an equivalence. From Proposition 1.3.6 it is enough to show that F is essentially surjective and fully faithful. The fact that F is essentially surjective follows from the definition of affine scheme. Let $\varphi_1, \varphi_2 : R \to R'$ be two homomorphisms of rings such that

$$(\pi_1, \pi_1^\#) = F(\varphi_1) = F(\varphi_2) = (\pi_2, \pi_2^\#),$$

then $\varphi_1 = \pi_1^\#(\operatorname{Spec} R) = \pi_2^\#(\operatorname{Spec} R) = \varphi_2$, hence F is faithful. Now suppose that $(\pi, \pi^\#) : F(R') \to F(R)$ is a morphism of affine schemes, since for every distinguished open set U of $\operatorname{Spec} R$ the maps $\pi^\#(\operatorname{Spec} R)$ and $\pi^\#(U)$ commutes with the restriction maps, we must have $\pi^\#(U) = S_U^{-1}\pi^\#(\operatorname{Spec} R)$, in other words, $(\pi, \pi^\#) = F(\pi^\#(\operatorname{Spec} R))$, and this proves that F is full. $\square$

The previous theorem in particular tells us that every morphism of affine schemes comes from a morphism of rings. This is not the case for morphism of ringed spaces, as the next example shows.

Example 3.6.3 (Not local morphism of ringed spaces)**.** Let R be a discrete valuation ring (DVR) and set $X = \operatorname{Spec} R$. The topological space X has two points: the zero ideal $x_0 = (0)$ which is open and dense, and the maximal ideal $x_1 = \mathfrak{m}$ which is closed. Let $Q(R)$ be the field of fractions of R, and set $Y = \operatorname{Spec} Q(R)$. The inclusion map $R \subseteq Q(R)$ induces a morphism of affine schemes $(Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ that sends the unique point $y \in Y$ into $x_0 \in X$. There exists another morphism of ringed spaces $(\pi, \pi^\#) : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ defined by $\pi(y) = x_1$, and $\pi^\#$ is such that $\pi^\#(X) : R \to K$ is the inclusion $R \subseteq K$, $\pi^\#(\{x_0\}) : K \to 0$ and $\pi^\#(\emptyset) : 0 \to 0$ are the obvious maps. This morphism is not a morphism of locally ringed spaces, because the map on stalks $\pi^\#_{\pi(y)} : \mathcal{O}_{X,\pi(y)} \to \mathcal{O}_{Y,y}$, that is, $\pi^\#_{x_1} : R_{x_1} \to Q(R)_y$ is the inclusion map $R \subseteq Q(R)$, which is not a local homomorphism (the maximal ideal $\mathfrak{m}$ of R is not contained in the maximal ideal (0) of $Q(R)$).

Proposition 3.6.4. *Let $\varphi : A \to B$ be a morphism of rings, set $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} A$, and let $(\pi, \pi^\#) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be the induced morphism of affine schemes as in Lemma 3.6.1.*

1. *φ is injective if and only if $\pi^\#$ is injective. In this case $\pi(X)$ is dense in Y (that is, π is **dominant**).*

2. *φ is surjective if and only if $\pi^\#$ is surjective. In this case π is a homeomorphism of X and the closed subset $V(\ker \varphi)$ of Y.*

Proof.

1. It follows from the fact that $\varphi = \pi^\#(Y)$ and that the localization of an injective map is injective. For the second statement, since φ is injective, an element $a \in A$ is nilpotent if and only if $\varphi(a) \in B$ is nilpotent, in other words $\varphi^{-1}(\mathfrak{N}_B) = \mathfrak{N}_A$. Now write
$$\overline{\pi(Y)} = V\Big(\mathscr{I}\big(\pi(Y)\big)\Big) = V\Big(\bigcap_{p \subseteq B \text{ prime}} \varphi^{-1}(p)\Big) = V\Big(\varphi^{-1}(\mathfrak{N}_B)\Big) = V(\mathfrak{N}_A) = X.$$

2. It follows from the fact that φ is surjective if and only if so are all the localizations of φ at a prime ideal of A. In fact, this last condition is the same as requiring that all the induced maps on stalks by $\pi^\#$ are surjective. For the second statement, we have $B \simeq A/\ker\varphi$, so from Proposition 3.2.6 π is an homeomorphism of X and the closed subset $V(\ker \varphi)$ of Y. □

Proposition 3.6.5. *Let R be a ring and set $X = \operatorname{Spec} R$. For every $f \in R$ we have $(X_f, \mathcal{O}_{X|X_f}) \simeq (\operatorname{Spec} R_f, \mathcal{O}_{\operatorname{Spec} R_f})$ as locally ringed spaces.*

Proof. Let $(\pi, \pi^\#)$ be the morphism of affine schemes induced by the homomorphism $R \to R_f$. From Proposition 3.2.6 the map π is a homeomorphism from $\operatorname{Spec} R_f$ to X_f. Further, the restriction of $\pi^\#$ to X_f is an isomorphism of sheaves, since for every open set $U \subseteq X_f$ the map $\pi^\#(U) : \mathcal{O}_X(U) \to \mathcal{O}_{\operatorname{Spec} R_f}(U)$ is an isomorphism. □

Corollary 3.6.6. *Let $(X, \mathcal{O}_X)$ be a scheme. For every open set $U \subseteq X$, $(U, \mathcal{O}_{X|U})$ is also a scheme.*

Proof. Let $x \in U$, since X is a scheme, there exists an open neighbourhood $V \subseteq X$ of x such that $(V, \mathcal{O}_{X|V}) \simeq (\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R})$, for some ring R. Now $U \cap V$ is an open set of $\operatorname{Spec} R$. Since the distinguished open sets form a basis of $\operatorname{Spec} R$, there exists $f \in R$ such that $x \in V_f \subseteq U \cap V$. Thus, from Proposition 3.6.5, V_f is an open neighbourhood of x in U such that $(V_f, \mathcal{O}_{X|V_f}) \simeq (\operatorname{Spec} R_f, \mathcal{O}_{\operatorname{Spec} R_f})$. □

Definition 3.6.7. Let $(X, \mathcal{O}_X)$ be a scheme. For every open set $U \subseteq X$, we say that $(U, \mathcal{O}_{X|U})$ is an **open suscheme** of X. If $(U, \mathcal{O}_{X|U})$ is also an affine scheme, we say that U is an **affine open subscheme**.

In particular, from Corollary 3.6.6, the affine open subschemes form a basis for the topology of X.

3.7 Glueing schemes

Proposition 3.7.1 (Glueing topological spaces)**.** *Let $\{X_i\}_{i \in I}$ be a family of topological spaces such that for each $i, j \in I$ there exists an open subset $X_{ij} \subseteq X_i$ and a homeomorphism $\phi_{ij} : X_{ij} \to X_{ji}$ such that*

1. $X_{ii} = X_i$ *and* $\phi_{ii} = 1_{X_i}$,
2. $\phi_{ij}^{-1} = \phi_{ji}$,
3. $\phi_{ik} = \phi_{jk} \circ \phi_{ij}$ *in* $X_{ij} \cap X_{ik}$ *(and necessarily* $\phi_{ij}(X_{ij} \cap X_{ik}) \subseteq X_{ji} \cap X_{jk}$*).*

Then, there exists a topological space X with a family of open subsets $\{U_i\}_{i \in I}$ and homeomorphism $\phi_i : X_i \to U_i$ such that

- $\phi_i(X_{ij}) = U_i \cap U_j$,
- $\phi_{ij} = \phi^{-1}_{j|U_i \cap U_j} \circ \phi_{i|U_{ij}}$.

Proof. As a set, $X = \coprod_{i \in I} X_i / \sim$, where $\sim$ is the relation defined by, for every $x_i \in X_{ij}$ and $x_j \in X_{ji}$, $x_i \sim x_j$ if $\phi_{ij}(x_i) = x_j$. From the properties $1, 2$ and 3 of the maps ϕ_{ij}, we have that $\sim$ is an equivalence relation. Now, for every $i \in I$, let $\phi_i : X_i \to X$ be the composition of the inclusion map $X_i \to \coprod_{i \in I} X_i$ and the quotient $\coprod_{i \in I} X_i \to X$, and set $U_i = \phi_i(X_i)$. Note that ϕ_i is bijective. Define a topology on X by declaring $U \subseteq X$ open if and only if $\phi_i^{-1}(U) \subseteq X_i$ is open for every $i \in I$. Note that for every $i \in I$ the subsets $U_i \subseteq X$ are open and the maps $\phi_i : X_i \to U_i$ are homemomorphisms. The last two properties follows by construction. □

In Proposition 3.7.1 we glued different topological spaces along some patches, in Proposition 2.5.7 we glued together different patches of a sheaf in the same topological space. By combining the two constructions, we can glue a family of schemes along some patches. The same result holds more generally for (locally) ringed spaces.

Proposition 3.7.2 (Glueing schemes)**.** *Let $\{(X_i, \mathcal{O}_{X_i})\}_{i\in I}$ be a family of schemes such that for each $i, j \in I$ there exists an open subscheme $X_{ij} \subseteq X_i$ and an isomorphism of schemes $(\phi_{ij}, \phi_{ij}^{\#}) : X_{ij} \to X_{ji}$ such that*

1. *$X_{ii} = X_i$ and $(\phi_{ii}, \phi_{ii}^{\#}) = (1_{X_i}, 1_{\mathcal{O}_{X_i}})$,*

2. *$(\phi_{ij}, \phi_{ij}^{\#})^{-1} = (\phi_{ji}, \phi_{ji}^{\#})$,*

3. *$(\phi_{ik}, \phi_{ik}^{\#}) = (\phi_{jk}, \phi_{jk}^{\#}) \circ (\phi_{ij}, \phi_{ij}^{\#})$ in $X_{ij} \cap X_{ik}$ (and necessarily $\phi_{ij}(X_{ij} \cap X_{ik}) \subseteq X_{ji} \cap X_{jk}$).*

Then, there exists a scheme $(X, \mathcal{O}_X)$ with a family of open subschemes $\{U_i\}_{i\in I}$ and isomorphisms of schemes $(\phi_i, \phi_i^{\#}) : X_i \to U_i$ such that

- *$\phi_i(X_{ij}) = U_i \cap U_j$,*

- *$\phi_{ij} = \phi_{j|U_i\cap U_j}^{-1} \circ \phi_{i|U_{ij}}$.*

Proof. From Proposition 3.7.1, there exists a topological space X with a family of open subsets $\{U_i\}_{i\in I}$ and homeomorphisms $\phi_i : X_i \to U_i$ that satisfies the required properties. Now we can give the structure of scheme to each open subset $U_i \subseteq X$ by considering the pushforward sheaf $\phi_{i*}\mathcal{O}_{X_i}$. Futher, we have the following isomorphism of schemes $(\phi_i, 1_{\phi_{i*}\mathcal{O}_{X_i}}) : (X_i, \mathcal{O}_{X_i}) \to (U_i, \phi_{i*}\mathcal{O}_{X_i})$. Now we are in the hypothesis of Proposition 2.5.7, so we can glue the sheaves $\phi_{i*}\mathcal{O}_{X_i}$ together to obtain a sheaf of rings $\mathcal{O}_X$ on X such that, the schemes $(U_i, \phi_{i*}\mathcal{O}_{X_i})$ are open subschemes of X. □

The proof of the following result is just a routine check.

Proposition 3.7.3 (Glueing of morphisms)**.** *Let X and Y be schemes. For every open subscheme $U \subseteq X$ let $\mathscr{F}(U)$ be the set of morphisms of schemes from U to Y. Then $\mathscr{F}$, with the obvious restriction maps, is a sheaf of sets on X.*

In other words, if $\{U_i\}_{i\in I}$ is an open covering of X, then a family of morphisms $U_i \to Y$ glues to a morphism $X \to Y$ if and only if the morphisms coincide on the intersections $U_i \cap U_j$, and the resulting morphism $X \to Y$ is uniquely determined.

3.8 Examples of schemes

Definition 3.8.1. Let k be a field. Then the affine scheme $\mathbb{A}_k^n = \operatorname{Spec} k[x_1, \dots, x_n]$ is the n-dimensional **affine space** of k.

Example 3.8.2 (Affine plane minus the origin)**.** Let $\mathbb{A}_k^2 = \operatorname{Spec} k[x, y]$ be the affine plane over a field k. Consider the open subscheme $X = \mathbb{A}_k^2 \setminus \{(x, y)\}$ of $\mathbb{A}_k^2$. From Corollary 3.6.6, $(X, \mathcal{O}_X)$ is a scheme (where $\mathcal{O}_X$ is the restriction to X of the structure sheaf of $\mathbb{A}_k^2$), but it is not an affine scheme. In fact, suppose by contradiction that $(X, \mathcal{O}_X)$ is affine. Consider the inclusion map $\pi : X \to \mathbb{A}_k^2$, and define the morphism of sheaves $\pi^{\#} : \mathcal{O}_{\mathbb{A}_k^2} \to \pi_*\mathcal{O}_X$ where $\pi^{\#}(U) : \mathcal{O}_{\mathbb{A}_k^2}(U) \to \mathcal{O}_{\mathbb{A}_k^2}(U \cap X)$ are the restriction maps of the structure sheaf $\mathcal{O}_{\mathbb{A}_k^2}$ for every open set $U \subseteq \mathbb{A}_k^2$ (recall that $\mathcal{O}_X = \mathcal{O}_{\mathbb{A}_k^2|X}$). Thus, $(\pi, \pi^{\#})$ is a morphism of affine schemes, from X to $\mathbb{A}_k^2$. From Theorem 3.6.2 this morphism must come from the morphism of rings $\pi^{\#}(\mathbb{A}^2_k) : k[x,y] \to \mathcal{O}_X(X)$. This is precisely the localization map $k[x,y] \to S^{-1}k[x,y]$ at the multiplicative set

$$S = k[x,y] \setminus \bigcup_{p \in X} p.$$

Now, since $k[x,y]$ is a UFD, every non invertible element of $k[x,y]$ is contained in a principal prime ideal, that is different from (x,y) since the last is not principal. This means that $\bigcup_{p\in X} p = \bigcup_{p \in \mathbb{A}^2_k} p$. In other words, S is the set of invertible elements, and $\pi^{\#}(\mathbb{A}^2_k) : k[x,y] \to S^{-1}k[x,y]$ is an isomorphism. From Theorem 3.6.2 this means that $(\pi, \pi^{\#})$ is an isomorphism of affine schemes, in particular that the inclusion $\pi : X \to \mathbb{A}^2_k$ is an homeomorphism, but π is not surjective, contradiction.

Example 3.8.3 (Affine line with doubled origin)**.** Consider two copies of the affine line: $X = \operatorname{Spec} k[x]$, $Y = \operatorname{Spec} k[y]$. Consider the open subschemes $X_x = \operatorname{Spec} k[x, 1/x] \subseteq X$, $Y_y = \operatorname{Spec} k[y, 1/y] \subseteq Y$, and consider the isomorphism of (affine) schemes $(\phi, \phi^{\#}) : Y_y \to X_x$ induced by the isomorphism of rings $\varphi : k[x, 1/x] \to k[y, 1/y]$ defined by $\varphi(x) = y$. From Proposition 3.7.2, we can glue X and Y along the open subschemes X_x and Y_y using the isomorphism $(\phi, \phi^{\#})$. We obtain a scheme $(W, \mathcal{O}_W)$ that has X and Y as open subschemes, and it is called the **affine line with doubled origin**. Explicitly, as a set

$$W = \{[p] : p \in \operatorname{Spec} k[x]\} \cup \{[(y)]\} = (X \amalg Y)/\sim,$$

and a subset $U \subseteq W$ is open if and only if $U \cap X$ and $U \cap Y$ are both open in X and Y respectively. Now the ring of global sections is

$$\mathcal{O}_W(W) = \varprojlim_{U \subseteq X \text{ or } Y} \mathcal{O}_W(U) \simeq \mathcal{O}_W(X) \times_{\mathcal{O}_W(X \cap Y)} \mathcal{O}_W(Y) \simeq k[x] \times_{k[x,1/x]} k[x] \simeq k[x],$$

where $k[x] \times_{k[x,1/x]} k[x]$ is the fibered product of rings with respect to two copies of the inclusion map $k[x] \subseteq k[x, 1/x]$. Therefore, the restriction map $\mathcal{O}_W(W) \to \mathcal{O}_W(X)$ is the isomorphism $k[x] \to k[x]$. Hence, by using the same argument of Example 3.8.2 we obtain that W is not affine.

Example 3.8.4 (Projective line)**.** With the same notation and proceeding similarly as in Example 3.8.3, but changing the definition of the ring map $\varphi : k[x, 1/x] \to k[y, 1/y]$ by $\varphi(x) = 1/y$, we obtain another scheme W called the **projective line**. In this case, the ring of global sections is given by the fibered product

$$k[x] \times_{k[x,1/x]} k[1/x] \simeq k,$$

that is $\mathcal{O}_W(W) = k$, so also in this case W is not affine because otherwise we must have $W = \operatorname{Spec} \mathcal{O}_W(W) = \operatorname{Spec} k$, but the last consists just of a point, contradiction.

The construction of the previous example can be generalized by gluing $n+1$ copies of the n-dimensional affine space $X_i = \operatorname{Spec} k[x_0, , \dots, \hat{x_i}, \dots, x_n] \simeq \mathbb{A}^n_k$ (for $i \in \{0, \dots, n\}$) along the open subschemes $U_{ij} = \operatorname{Spec} k[x_0, \dots, \hat{x_i}, \dots, x_n]_{x_j} \subseteq X_i$, to obtain what is called the n-dimensional **projective space** $\mathbb{P}^n_k$. The gluing maps $\phi_{ij} : U_{ij} \to U_{ji}$ are induced by the ring maps

$$\varphi_{ij} : k[x_0, \dots, \hat{x_j}, \dots, x_n]_{x_i} \to k[x_0, \dots, \hat{x_i}, \dots, x_n]_{x_j}$$

defined by $\varphi_{ij}(x_k) = x_k/x_j$ for $k \neq i$, and $\varphi(x_i) = 1/x_j$. Then, the ring of global sections will be the following fibered product of rings over $k[x_0^{\pm 1}, \dots, x_n^{\pm 1}]$

$$\prod_{i=0}^{n} k[x_0, \dots, \hat{x}_i, \dots, x_n],$$

which is isomorphic to k. By the same argument used above, this implies that $\mathbb{P}_k^n$ is not affine.

3.9 Projective schemes

In this section, with *graded ring* we will always mean a positively $\mathbb{Z}$-graded ring. Let $S = \bigoplus_{i \in \mathbb{N}} S_i$ be a graded ring and let $S_+ = \bigoplus_{i>0} S_i$ be the **irrelevant ideal** of S. We want to define a scheme, $\operatorname{Proj} S$. In order to do this, we will follow a similar path to what we have done for the spectrum of a ring. First, as a set

$$\operatorname{Proj} S = \{p \subseteq S : p \text{ is a homogeneous prime ideal, } p \not\supseteq S_+\}.$$

Note that the zero ideal (0) is considered a homogeneous ideal, so if S is a domain, $(0) \in \operatorname{Proj} S$. In particular, since the radical of a homogeneous ideal is homogeneous, the nilradical $\mathfrak{N}(S) = \sqrt{(0)}$ is also a homogeneous ideal. Now we define the topology of $\operatorname{Proj} S$.

Definition 3.9.1. For every homogeneous ideal $I \subseteq S$, the **projecitve vanishing set** of T is

$$V^{\mathbb{P}}(I) = \{p \in \operatorname{Proj} S : p \supseteq I\}.$$

For every $Z \subseteq \operatorname{Proj} S$ define

$$\mathscr{I}^{\mathbb{P}}(Z) = \bigcap_{p \in Z} p.$$

Note that the ideal $\mathscr{I}^{\mathbb{P}}(Z)$ is homogeneous since the arbitrary intersection of homogeneous ideals is homogeneous.

Proposition 3.9.2. *Let $I, J \subseteq S$ be two homogeneous ideals and let $X, Y \subseteq \operatorname{Proj} S$.*

1a. $I \subseteq \mathscr{I}^{\mathbb{P}}(V^{\mathbb{P}}(I))$

1b. $X \subseteq V^{\mathbb{P}}(\mathscr{I}^{\mathbb{P}}(X))$

2a. $I \subseteq J \Rightarrow V^{\mathbb{P}}(I) \supseteq V^{\mathbb{P}}(J)$

2b. $X \subseteq Y \Rightarrow \mathscr{I}^{\mathbb{P}}(X) \supseteq \mathscr{I}(Y)$

3a. $V^{\mathbb{P}}(\mathscr{I}^{\mathbb{P}}(V^{\mathbb{P}}(I))) = V^{\mathbb{P}}(I)$

3b. $\mathscr{I}^{\mathbb{P}}(V^{\mathbb{P}}(\mathscr{I}^{\mathbb{P}}(X))) = \mathscr{I}^{\mathbb{P}}(X)$

4a. $V^{\mathbb{P}}(S_+) = \emptyset$, $V^{\mathbb{P}}(\mathfrak{N}(S)) = \operatorname{Proj} S$

4b. $\mathscr{I}^{\mathbb{P}}(\operatorname{Proj} S) = \mathfrak{N}(S), \mathscr{I}^{\mathbb{P}}(\emptyset) = S$

5a. $V^{\mathbb{P}}\left(\sum_{\lambda \in \Lambda} I_\lambda\right) = \bigcap_{\lambda \in \Lambda} V^{\mathbb{P}}(I_\lambda)$

5b. $\mathscr{I}^{\mathbb{P}}(\bigcup_{\lambda \in \Lambda} X_\lambda) = \bigcap_{\lambda \in \Lambda} \mathscr{I}^{\mathbb{P}}(X_\lambda)$

6a. $V^{\mathbb{P}}(IJ) = V^{\mathbb{P}}(I \cap J) = V^{\mathbb{P}}(I) \cup V^{\mathbb{P}}(J)$

6b. $\mathscr{I}^{\mathbb{P}}(X \cap Y) \supseteq \mathscr{I}^{\mathbb{P}}(X) + \mathscr{I}^{\mathbb{P}}(Y)$

Proof.

1a. Clear from the definitions.

2a. $p \in V^{\mathbb{P}}(J) \Rightarrow p \supseteq J \supseteq I \Rightarrow p \in V^{\mathbb{P}}(I)$.

3a. From (1a) and (2a) we have $I \subseteq \mathscr{I}^{\mathbb{P}}(V^{\mathbb{P}}(I)) \Rightarrow V^{\mathbb{P}}(I) \supseteq V^{\mathbb{P}}(\mathscr{I}^{\mathbb{P}}(V^{\mathbb{P}}(I)))$, the inclusion $V^{\mathbb{P}}(I) \subseteq V^{\mathbb{P}}(\mathscr{I}^{\mathbb{P}}(V^{\mathbb{P}}(I)))$ follows from (1b).

4a. The first statement follows from definition. For the second, the nilradial $\mathfrak{N}(S)$ is contained in every homogeneous prime ideal $p \in \operatorname{Proj} S$.

5a. It is enough to observe that $p \supseteq \sum_{\lambda\in\Lambda} I_\lambda$ if and only if $p \supseteq I_\lambda$ for every $\lambda \in \Lambda$.

6a. Let $p \in V^{\mathbb{P}}(IJ)$, that is $IJ \subseteq p$. Since p is prime, this means $I \subseteq p$ or $J \subseteq p$, that is $p \in V^{\mathbb{P}}(I) \cup V^{\mathbb{P}}(J)$. Therefore, $V^{\mathbb{P}}(IJ) \subseteq V^{\mathbb{P}}(I) \cup V^{\mathbb{P}}(J)$.
Now $I \cap J \subseteq I \Rightarrow V^{\mathbb{P}}(I \cap J) \supseteq V^{\mathbb{P}}(I)$, and similarly $V^{\mathbb{P}}(I \cap J) \supseteq V^{\mathbb{P}}(J)$ thus we have $V^{\mathbb{P}}(I \cap J) \supseteq V^{\mathbb{P}}(I) \cup V^{\mathbb{P}}(J)$. Further $IJ \subseteq I \cap J$, therefore $V^{\mathbb{P}}(IJ) \supseteq V^{\mathbb{P}}(I \cap J)$. Finally we obtain

$$V^{\mathbb{P}}(IJ) \supseteq V^{\mathbb{P}}(I \cap J) \supseteq V^{\mathbb{P}}(I) \cup V^{\mathbb{P}}(J) \supseteq V^{\mathbb{P}}(IJ).$$

1b. Clear from the definitions.

2b. If $f \in \mathscr{I}(Y)$ then $f \in p$ for every $p \in X \subseteq Y$, therefore $f \in \mathscr{I}(X)$.

3b. From (1b) and (2b) we have $X \subseteq V^{\mathbb{P}}(\mathscr{I}^{\mathbb{P}}(X)) \Rightarrow \mathscr{I}^{\mathbb{P}}(X) \supseteq \mathscr{I}^{\mathbb{P}}(V^{\mathbb{P}}(\mathscr{I}^{\mathbb{P}}(X)))$, the inclusion $\mathscr{I}^{\mathbb{P}}(X) \subseteq \mathscr{I}^{\mathbb{P}}(V^{\mathbb{P}}(\mathscr{I}^{\mathbb{P}}(X)))$ follows from (1a).

4b. In a graded ring, the intersection of all the homogeneous prime ideals is equal to the nilradical $\mathfrak{N}(S)$. The second statement follows from the definition.

5b. We have

$$\mathscr{I}^{\mathbb{P}}\left(\bigcup_{\lambda\in\Lambda} X_\lambda\right) = \bigcap_{p\in\bigcup_{\lambda\in\Lambda} X_\lambda} p = \bigcap_{\lambda\in\Lambda}\bigcap_{p\in X_\lambda} p = \bigcap_{\lambda\in\Lambda} \mathscr{I}^{\mathbb{P}}(X_\lambda).$$

6b. Since $X \cap Y \subseteq X, Y$, from (2b) we have $\mathscr{I}^{\mathbb{P}}(X \cap Y)$ contains both $\mathscr{I}^{\mathbb{P}}(X)$ and $\mathscr{I}^{\mathbb{P}}(Y)$, therefore $\mathscr{I}^{\mathbb{P}}(X \cap Y) \supseteq \mathscr{I}^{\mathbb{P}}(X) + \mathscr{I}^{\mathbb{P}}(Y)$. $\square$

In the last proposition, we have seen that the family of sets $V^{\mathbb{P}}(I)$ for homogeneous ideals $I \subseteq S$ satisfies the properties of closed sets in a topological space.

Definition 3.9.3. The **(projective) Zariski topology** on $\operatorname{Proj} S$ is the topology in which the family of closed sets are of the form $V^{\mathbb{P}}(I)$, for a homogeneous ideal $I \subseteq S$.

Proposition 3.9.4. *For every homogeneous ideal $I \subseteq S$ and subset $X \subseteq \operatorname{Proj} S$ we have*

1a. $\mathscr{I}^{\mathbb{P}}(V^{\mathbb{P}}(I)) = \sqrt{I}$

1b. $V^{\mathbb{P}}(\mathscr{I}^{\mathbb{P}}(X)) = \overline{X}$

2a. $V^{\mathbb{P}}(I) = V^{\mathbb{P}}(\sqrt{I})$

2b. $\mathscr{I}^{\mathbb{P}}(X) = \mathscr{I}^{\mathbb{P}}(\overline{X})$

Therefore $\mathscr{I}^{\mathbb{P}}$ and $V^{\mathbb{P}}$ are inclusion reversing bijections between homogeneous radical ideals and closed sets of $\operatorname{Proj} S$.

Proof.

1a. $\mathscr{I}^{\mathbb{P}}(V^{\mathbb{P}}(I)) = \bigcap_{p \in V^{\mathbb{P}}(I)} p = \sqrt{I}$.

2a. Follows from applying (1a) and (3a) of Proposition 3.9.2.

1b. $\subseteq$ $X \subseteq \overline{X} \Rightarrow \mathscr{I}^{\mathbb{P}}(X) \supseteq \mathscr{I}^{\mathbb{P}}(\overline{X}) \Rightarrow V^{\mathbb{P}}(\mathscr{I}^{\mathbb{P}}(X)) \subseteq V^{\mathbb{P}}(\mathscr{I}^{\mathbb{P}}(\overline{X})) = \overline{X}$. For the last equality, set $\overline{X} = V(T)$, then $V^{\mathbb{P}}(\mathscr{I}^{\mathbb{P}}(\overline{X})) = V^{\mathbb{P}}(\mathscr{I}^{\mathbb{P}}(V^{\mathbb{P}}(T))) = V^{\mathbb{P}}(T) = \overline{X}$.

$\supseteq$ $V^{\mathbb{P}}(\mathscr{I}^{\mathbb{P}}(X))$ is a closed set containing X.

2b. Follows from applying (2a) and (3b) of Proposition 3.9.2. □

Proposition 3.9.5. *Let $I \subseteq S$ be a homogeneous ideal. The following are equivalent:*

1. $V^{\mathbb{P}}(I) = \emptyset$
2. $\sqrt{I} \supseteq S_+$

Proof.

(1) $\Rightarrow$ (2) $V^{\mathbb{P}}(I) = \emptyset \Rightarrow \sqrt{I} = \mathscr{I}^{\mathbb{P}}(V^{\mathbb{P}}(I)) = \mathscr{I}^{\mathbb{P}}(\emptyset) = S \supseteq S_+$

(2) $\Rightarrow$ (1) $\sqrt{I} \supseteq S_+ \Rightarrow V^{\mathbb{P}}(I) = V^{\mathbb{P}}(\sqrt{I}) \subseteq V^{\mathbb{P}}(S_+) = \emptyset$. □

The previous proposition motivates why S_+ is called "irrelevant" ideal.

Definition 3.9.6. Set $X = \operatorname{Proj} S$, for any homogeneous element $f \in S$ of positive degree, define $X_f = X \setminus V^{\mathbb{P}}(f)$, called **projective distinguished** open set.

Similarly for the spectrum of a ring, the projective distinguished open subsets form a basis for $\operatorname{Proj} S$ and satisfies similar properties.

Proposition 3.9.7. *Set $X = \operatorname{Proj} S$, then we have:*

1. *the projective distinguished open sets form a basis for* $\operatorname{Proj} S$*;*
2. $X_f \cap X_g = X_{fg}$*;*
3. $X_f \subseteq \bigcup_{i=1}^{n} X_{g_i} \iff f \in \sqrt{(g_1, \dots, g_n)}$.

Proof.

1. Let $U = X \setminus V^{\mathbb{P}}(I)$ be an open set, with $I \subseteq S$ an homogeneous ideal, we have

$$U = X \setminus V^{\mathbb{P}}(I) = X \setminus \bigcap_{\substack{f \in I \cap S_+ \\ \text{homogeneous}}} V(f) = \bigcup_{\substack{f \in I \cap S_+ \\ \text{homogeneous}}} X_f.$$

2. $X_f \cap X_g = \big(X \setminus V^{\mathbb{P}}(f)\big) \cap \big(X \setminus V^{\mathbb{P}}(g)\big) = X \setminus \big(V^{\mathbb{P}}(f) \cup V^{\mathbb{P}}(g)\big) = X \setminus V^{\mathbb{P}}(fg) = X_{fg}$.

3.

$$\begin{aligned}
X_f \subseteq \textstyle\bigcup_{i=1}^n X_{g_i} &\iff X \setminus V^{\mathbb{P}}(f) \subseteq X \setminus V^{\mathbb{P}}(g_1, \dots, g_n) \\
&\iff V^{\mathbb{P}}(f) \supseteq V^{\mathbb{P}}(g_1, \dots, g_n) \\
&\iff (f) \subseteq \mathscr{I}^{\mathbb{P}}(V^{\mathbb{P}}(f)) \subseteq \mathscr{I}^{\mathbb{P}}(V^{\mathbb{P}}(g_1, \dots, g_n)) = \sqrt{(g_1, \dots, g_n)} \\
&\iff f \in \sqrt{(g_1, \dots, g_n)}. \qquad \square
\end{aligned}$$

In order to define the scheme structure on $X = \operatorname{Proj} S$, we will give each projective dinstiguished open set X_f the structure of affine scheme and then glue these sheaves together.

If S is a graded ring and $f \in S_+$ is homogeneous, then also the localization S_f has a grading, given by $\deg(s/f^n) = \deg(s) - \deg(f^n)$. The next lemma, shows that the homogeneous prime ideals of S_f are in bijection with the prime ideals of $(S_f)_0$.

Lemma 3.9.8. *Let S be a graded ring and suppose there exists a homogeneous invertible element $f \in S_+$. Then, the map $P \mapsto P \cap S_0$ is a bijection between homogeneous prime ideals of S and prime ideals of S_0.*

Proof. Let $P_0 \subseteq S_0$ be a prime ideal. For every $i \in \mathbb{N}$ set

$$P_i = \{s \in S : s^{\deg f}/f^i \in P_0\} \subseteq S_i,$$

note that this definition is consistent for $i = 0$. Finally set $P = \bigoplus_{i \in \mathbb{N}} P_i$. Now if $s \in S_i$ and $t \in P_j$ then $st \in P_{i+j}$. Further, if $s, t \in P_i$, then $(s+t)^2 = s^2 + 2st + t^2 \in P_{2i}$, that is $(s+t)^{2\deg f}/f^{2i} = \left((s+t)^{\deg f}/f^i\right)^2 \in P_0$ which implies $s + t \in P_i$. Furthermore, if $st \in P_i$ then $s \in P_i$ or $t \in P_i$. This proves that P is a homogeneous prime ideal of S, and $P \cap S_0 = P_0$, therefore $P \mapsto P \cap S_0$ is a bijection. $\square$

Lemma 3.9.9. *Let S be a graded ring, $f, g \in S_+$ homogeneous elements and set $X = \operatorname{Proj} S$ and $Y = \operatorname{Spec}(S_f)_0$.*

1. *The composition $\operatorname{Spec}(S_f)_0 \to \operatorname{Proj} S_f \to X_f$ where the first map is described in Lemma 3.9.8, is a homeomorphism.*

2. *The affine schemes $\operatorname{Spec}(S_{fg})_0$ and $Y_{g^{\deg f}/f^{\deg g}}$ are isomorphic.*

Proof.

1. First, we prove that the second map $\psi : \operatorname{Proj} S_f \to X_f$ is an homeomorphism. The map ψ is induced by the map of graded rings $\varphi : S \to S_f$, in fact, for every homogeneous prime ideal $p \in \operatorname{Proj} S_f$, $\psi(p) = \varphi^{-1}(p)$. In particular, ψ is bijective. Now the rest of the proof is similar to the proof of Proposition 3.2.6, in fact, for every homogeneous ideal $I \subseteq S$ such that $I \cap \{1, f, f^2, \dots\} = \emptyset$, if J is the extension of I in S_f, that is, the homogeneous ideal generated by $\varphi(I)$, then

$$\psi^{-1}(V^{\mathbb{P}}(I) \cap X_f) = V^{\mathbb{P}}(J), \quad \psi(V^{\mathbb{P}}(J)) = V^{\mathbb{P}}(I) \cap X_f,$$

therefore ψ is continuous and closed.

Now set $\psi : \operatorname{Spec}(S_f)_0 \to \operatorname{Proj} S_f$ as in Lemma 3.9.8. The map ψ is bijective. Furthermore, if I is a homogeneous ideal of S_f, then $\psi^{-1}(V^{\mathbb{P}}(I)) = V(I \cap (S_f)_0) \subseteq \operatorname{Spec}(S_f)_0$. On the other hand, if I_0 is a homogeneous ideal of $(S_f)_0$, then $\psi(V(I_0)) = V^{\mathbb{P}}(I)$, where I is the homogeneous ideal generated by $I_0 \subseteq (S_f)_0 \subseteq S_f$. Therefore, ψ is a homeomorphism.

2. The isomorphism of affine schemes is given by the following isomorphism of rings $(S_{fg})_0 \simeq \left((S_f)_0\right)_{g^{\deg f}/f^{\deg g}}$. □

Lemma 3.9.9 allow us to give each projective distinguished open subsets X_f of $X = \operatorname{Proj} S$ the structure of affine scheme. Further, we have isomorphisms between the overlaps $X_f \cap X_g = X_{fg}$ and they satisfy the cocycle condition. Hence, we can glue this structure together by applying Proposition 2.5.7, obtaining a sheaf $\mathcal{O}_{\operatorname{Proj} S}$ such that $(\operatorname{Proj} S, \mathcal{O}_{\operatorname{Proj} S})$ is a scheme, since the base of distinguished open subsets form a base and $(X_f, \mathcal{O}_{\operatorname{Proj} S|X_f})$ is isomorphic to the affine scheme $\operatorname{Spec}(S_f)_0$.

Definition 3.9.10. The n-dimensional **projective space** over a ring R is defined by $\mathbb{P}^n_R = \operatorname{Proj} R[x_0, \ldots, x_n]$.

For example, the projective n-space over a field k is $\mathbb{P}^n_k = \operatorname{Proj} k[x_0, \ldots, x_n]$.

3.10 Values of sections

Definition 3.10.1. Let $(X, \mathcal{O}_X)$ be a locally ringed space. For each $p \in X$, denote by $\mathfrak{m}_p$ the maximal ideal of $\mathcal{O}_{X,p}$. Let $f \in \mathcal{O}_X(U)$ be a section, for some open subset $U \subseteq X$. The **value** of f at $p \in U$, denoted by $f(p)$, is the image of the stalk f_p in the **residue field** $k(p) = \mathcal{O}_{X,p}/\mathfrak{m}_p$. We say that f **vanishes** at p if its value is zero (or equivalently if $f_p \in \mathfrak{m}_p$).

If $X = \operatorname{Spec} R$ is an affine scheme, then the residue field $k(p)$ at a point $p \in X$ is isomorphic to the quotient field of R/p. In fact

$$k(p) = \mathcal{O}_{X,p}/\mathfrak{m}_p \simeq R_p/pR_p \simeq Q(R/p).$$

Further, if $f \in \mathcal{O}_X(X)$ is a global section, we have

$$V(f) = \{p \in X : f_p \in \mathfrak{m}_p\},$$

since $p \in V(f) \Leftrightarrow f \in p \Leftrightarrow f_p \in pR_p = \mathfrak{m}_p$. This observation allows us to give the following definition.

Definition 3.10.2. Let $(X, \mathcal{O}_X)$ be a locally ringed space and let $f \in \mathcal{O}_X(U)$ be a section, for some open subset $U \subseteq X$. The **vanishing set** of f is

$$V(f) = \{p \in U : f_p \in \mathfrak{m}_p\}.$$

Note that this notation is consistent with the case when X is an affine scheme and f is a global section.

Corollary 3.10.3. *Let $(X, \mathcal{O}_X)$ be a locally ringed space and let $f \in \mathcal{O}_X(X)$ be a global section. The vanishing set $V(f)$ of f is closed.*

Proof. We show that the complement $X \setminus V(f)$ is open. Let $x \in X \setminus V(f)$, that is $f_x \in \mathcal{O}_{X,x} \setminus \mathfrak{m}_x$, this means that $f_x = [(f, U)]$ is invertible. Thus there exists some $g_x = [(g, V)] \in \mathcal{O}_{X,x}$ such that $f_x g_x = 1$. This means that $fg = 1$ on some open subset $W \subseteq U \cap V$, that is f_x is invertible (or f is nonzero) for every $x \in W \subseteq X \setminus V(f)$. □

The vanishing set of a section of a scheme should not be confused with (the complement of) its support (Definition 2.3.3).

3.11 Reducedness and integrality

Definition 3.11.1. A scheme $(X, \mathcal{O}_X)$ is **irreducible**, **connected** or **quasicompact**, if so is its underlying topological space X.

Recall that a ring R is **reduced** if it has no nilpotent elements.

Proposition-Definition 3.11.2. *A scheme $(X, \mathcal{O}_X)$ is* ***reduced*** *if it satisfies one of the following equivalent conditions:*

1. *$\mathcal{O}_X(U)$ is reduced for every open subset $U \subseteq X$,*
2. *$\mathcal{O}_{X,p}$ is reduced for every $p \in X$.*

Proof.

(1) ⇒ (2) Let $p \in X$, then there exists an affine open neighbourhood $U \subseteq X$ of p. In particular, $\mathcal{O}_{X,p}$ is isomorphic to the localization of $\mathcal{O}_X(U)$ at some prime ideal. Since $\mathcal{O}_X(U)$ is reduced, then also $\mathcal{O}_{X,p}$ is reduced.

(2) ⇒ (1) Suppose that $f \in \mathcal{O}_X(U)$ is nilpotent, for some open subset $U \subseteq X$. By hypothesis, the germs are zero $f_p = 0$ for all $p \in U$ since $\mathcal{O}_{X,p}$ is reduced. From Proposition 2.3.5 we obtain $f = 0$, hence $\mathcal{O}_X(U)$ is reduced. □

Definition 3.11.3. A scheme $(X, \mathcal{O}_X)$ is **integral** if it is nonempty and for every nonempty open set $U \subseteq X$, $\mathcal{O}_X(U)$ is a domain.

Proposition 3.11.4. *Let* $\operatorname{Spec} R$ *be an affine scheme. The following are equivalent*

1. $\operatorname{Spec} R$ *is integral,*
2. $\operatorname{Spec} R$ *is reduced and irreducible,*
3. *R is a domain.*

Proof.

(1) ⇒ (3) Trivial.

(3) ⇒ (1) Follows from the fact that the localization of a domain is a domain.

(2) $\Leftrightarrow$ (3) $\operatorname{Spec} R$ is reduced if and only if R is reduced (i.e. $\mathfrak{N}(R) = (0)$), since the localization of a reduced ring is reduced. In addition, from Propoisition 3.3.8 $\operatorname{Spec} R$ is irreducible if and only if $\mathscr{I}(\operatorname{Spec} R) = \mathfrak{N}(R)$ is prime. Therefore $\operatorname{Spec} R$ is reduced and irreducible if and only if the zero ideal (0) is prime, that is, if and only if R is a domain. $\square$

Proposition 3.11.5. *A scheme $(X, \mathcal{O}_X)$ is integral if and only if it is reduced and irreducible.*

Proof.

- $\Rightarrow$ Suppose that X is integral. Since every domain is reduced, X is reduced. If X is reducible, then there exists two nonempety disjoint open subsets $U, V \subseteq X$ and $\mathcal{O}_X(U \cup V) = \mathcal{O}_X(U) \times \mathcal{O}_X(V)$ which is not a domain, contradiction.

- $\Leftarrow$ Conversely, suppose that X is reduced and irreducible. Since every open subscheme of X is also irreducible and reduced, it is enough to show that the ring of global sections $\mathcal{O}_X(X)$ is a domain. Suppose that $fg = 0$ for some $f, g \in \mathcal{O}_X(X)$. For every $x \in X$ we have $f_x g_x = 0 \in \mathfrak{m}_x \Rightarrow f_x \in \mathfrak{m}_x$ or $g_x \in \mathfrak{m}_x$, that is $X = V(f) \cup V(g)$. From Corollary 3.10.3 the sets $V(f)$ and $V(g)$ are closed. Since X is irreducible, we have $X = V(f)$ or $X = V(g)$. Suppose $X = V(f)$, then the restriction $f_{|U}$ of f to every affine open subscheme $U \subseteq X$ is nilpotent. In fact, from Proposition 3.2.4, $U_{f_{|U}} = \emptyset = U_0$ implies $f_{|U} \in \sqrt{(0)}$. Finally, since X is reduced, we have that $f = 0$. $\square$

We have seen in Proposition 3.3.25 that, for an affine scheme X, every irreducible closed subset has a unique generic point. In otehr words, there is a correspondence between points in X and irreducible closed subsets of X. The next result shows that this is true more generally for schemes.

Proposition 3.11.6. *Let $(X, \mathcal{O}_X)$ be a scheme. Any irreducible closed subset $Z \subseteq X$ contains a unique generic point. In other words, the map*

$$\begin{aligned} X &\to \{Z \subseteq X : Z \text{ closed and irreducible}\} \\ x &\to \overline{\{x\}} \end{aligned}$$

is a bijection.

Proof. From Proposition 3.3.25 we know that the statement is true for affine schemes. Let $U \subseteq X$ be an affine open subscheme such that $Z \cap U \neq \emptyset$. Then $Z \cap U$ is closed in U, and irreducible, since it is open in Z, that is irreducible. Since U is affine, $Z \cap U$ has a unique generic point, that is also a generic point for Z, since $Z \cap U$ is dense in Z. For uniqueness, if $z \in Z$ is a generic point, it is contained in every open set that meets Z, hence also in every open set U as above, therefore the uniqueness follows from the uniqueness in the affine case. $\square$

Let X be an integral scheme. From Proposition 3.11.5 we know that X is irreducible. Hence, from Proposition 3.11.6 X has a unique generic point $\eta \in X$.

Proposition-Definition 3.11.7. *Let $(X, \mathcal{O}_X)$ be an integral scheme.*

1. *The local ring $\mathcal{O}_{X,\eta}$ is a field, callled the* **function field** *of X, denoted by $K(X)$.*

2. *For every affine open subscheme $U \subseteq X$, the fraction field of $\mathcal{O}_X(U)$ is isomorphic to the function field $K(X)$.*

3. *For every nonempty open subset $U \subseteq X$ the map $\mathcal{O}_X(U) \to \mathcal{O}_{X,\eta}$ is injective.*

4. *For every open subsetes $U \subseteq V \neq \emptyset$ the restriction map $\rho_{V,U} : \mathcal{O}_X(V) \to \mathcal{O}_X(U)$ is injective.*

Proof.

1. 2. Let $U = \operatorname{Spec} R \subseteq X$ be an affine open subscheme. Since η is the genereic point of X, $\eta \in U$ and η is also the generic point of U. From Remark 3.3.26, the generic point $\eta \in U$ corresponds to the zero ideal of R. Therefore $\mathcal{O}_{X,\eta} \simeq \mathcal{O}_{U,\eta} \simeq R_{(0)} \simeq Q(R)$.

3. If U is affine, for what we have just proved, the map $\mathcal{O}_X(U) \to \mathcal{O}_{X,\eta}$ is the inclusion of $\mathcal{O}_X(U)$ in its field of fractions. For the general case, let $f, g \in \mathcal{O}_X(U)$ such that $f_\eta = g_\eta$. Let $\{U_i\}_{i\in I}$ be an affine open cover of U, since the maps $\mathcal{O}_X(U_i) \to \mathcal{O}_{X,\eta}$ are injective, from $f_\eta = g_\eta$ we have that $f_{|U_i} = g_{|U_i}$ for all $i \in I$, that is, $f = g$.

4. From the previous point, the map $\mathcal{O}_X(V) \to \mathcal{O}_{X,\eta}$ is injective. Therefore the composition $\mathcal{O}_X(V) \xrightarrow{\rho_{V,U}} \mathcal{O}_X(U) \to \mathcal{O}_{X,\eta}$ is injective, hence the map $\rho_{V,U}$ must be injective. □

3.12 Noetherian schemes

Definition 3.12.1. A scheme $(X, \mathcal{O}_X)$ is **locally Noetherian** if it can be covered by affine schemes $\operatorname{Spec} R$ where R is a Noetherian ring. X is a **Noetherian** scheme if it is locally Noetherian and quasicompact.

Equivalently, a scheme X is Noetherian if it can be covered by a finite number of affine schemes $\operatorname{Spec} R$ where R is a Noetherian ring.

Proposition 3.12.2. *If $(X, \mathcal{O}_X)$ is a Noetherian scheme, then X is a Noetherian topological space.*

Proof. Suppose that X is covered by affine open subschemes $U_i = \operatorname{Spec} R_i$ with R_i Noetherian for every $i \in \{1, \dots, n\}$. From Remark 3.3.12 we know that each topological space U_i is Noetherian. Let $X_0 \supseteq X_1 \supseteq \dots$ be a descending chain of closed subsets of X. For every $i \in \{1, \dots, n\}$ we know that the descending chain $X_0 \cap U_i \supseteq X_1 \cap U_i \supseteq \dots$ stabilizes, that is, there exists $n_i \in \mathbb{N}$ such that $X_{n_i} \cap U_i = X_k \cap U_i$ for every $k \geq n_i$. Now let $N = \max\big\{n_i : i \in \{1, \dots, n\}\big\}$, we have that $X_N \cap U_i = X_k \cap U_i$ for every $i \in \{1, \dots, n\}$, hence $X_N = X_N \cap \bigcup_{i=1}^n U_i = X_k \cap \bigcup_{i=1}^n U_i = X_k$ for every $k \geq N$. □

Lemma 3.12.3. *Let R be a ring and let $f_1, \dots, f_n \in R$ such that $(f_1, \dots, f_n) = R$. Then R is Noetherian if and only if R_{f_i} is Noetherian for every $i \in \{1, \dots, n\}$.*

Proof. Necessity follows from the fact that the localization of a Noetherian ring is Noetheria. For sufficiency, let $\varphi_i : R \to R_{f_i}$ the localization maps. We first prove that for every ideal $I \subseteq R$ we have

$$I = \bigcap_{i=1}^{n} \varphi_i^{-1}(\varphi_i(I)A_{f_i}).$$

The inclusion $\subseteq$ is obvious. Conversely, let $b \in R$ contained in this intersection, we have $\varphi_i(b) = a_i/f_i^{n_i}$ in R_{f_i} for each i, where $a_i \in I$. This means that $f_i^{m_i}(f_i^{n_i}b - a_i) = 0$ for every $i \in \{1, \dots, n\}$. Therefore, there exists $N \in \mathbb{N}$ such that $f_i^N b \in I$ for every $i \in \{1, \dots, n\}$. From Remark 3.4.9 we have $\sum_{i=1}^n c_i f_i^N = 1$ for some $c_i \in R$. Hence

$$b = \sum_{i=1}^{n} c_i f_i^N b \in I.$$

Now let $I_1 \subseteq I_2 \subseteq \dots$ be an ascending chain of ideals in R. Then for each $i \in \{1, \dots, n\}$, $\varphi_i(I_1)R_{f_i} \subseteq \varphi_i(I_2)R_{f_i} \subseteq \dots$ is and ascending chain of ideals in R_{f_i}, which must become stationary because R_{f_i} is Noetherian. By intersecting all these (finitely many) chains for every $i \in \{1, \dots, n\}$, from what we have proved above we obtain that the chain $I_1 \subseteq I_2 \subseteq \dots$ must be stationary. $\square$

Corollary 3.12.4. *An affine scheme* $\operatorname{Spec} R$ *is Noetherian if and only if* R *is Noetherian.*

Proof. First, note that $\operatorname{Spec} R$ is quasicompact (Proposition 3.3.22), so it is locally Noetherian if and only if it is Noetherian. Now, if R is Noetherian, then $\operatorname{Spec} R$ is Noetherian by definition. Conversely, suppose that $X = \operatorname{Spec} R$ is Noetherian. This means that it can be covered by a finite number of affine schemes $U = \operatorname{Spec} R'$ with R' Noetherian. Recall that $R' \simeq S_U^{-1}R$, where $S_U = R \setminus \bigcup_{p \in U}$. Since the distinguished open subsets form a base, there exists $f \in R$ such that $X_f \subseteq U = \operatorname{Spec} R'$. This implies that $S_{X_f} \supseteq S_U$, from Lemma 3.4.8 we have that $R_f \simeq R'_{\overline{f}}$, where $\overline{f}$ is the image of f in R'. In particular, this implies that R_f is Noetherian, since R' is Noetherian. Hence, $X = \operatorname{Spec} R$ can be covered by distinguished open subsets $\operatorname{Spec} R_f$ with R_f Noetherian. Since $\operatorname{Spec} R$ is quasicompact, we can choose such a cover to be finite. Therefore, there exists $f_1, \dots, f_n \in R$ with R_{f_i} Noetherian and such that $\bigcup_{i=1}^n X_{f_i} = X$. From Corollary 3.2.3 this is equivalent to $(f_1, \dots, f_n) = R$. Finally, from Lemma 3.12.3 R is Noetherian. $\square$

Proposition 3.12.5. *Let* $(X, \mathcal{O}_X)$ *be a locally Noetherian scheme. Every open subscheme* $(U, \mathcal{O}_{X|U})$ *of* X *is locally Noetherian.*

Proof. Let $\{\operatorname{Spec} R_i\}_{i \in I}$ be an affine open cover of X with R_i Noetherian. Recall that if R is a Noetherian ring, so is every localization R_f for every $f \in R$. Thus, the distinguished open subsets $\operatorname{Spec} R_{i f_i}$, for $f_i \in R_i$, form a basis for the topology of X consisting of the spectra of Noetherian rings. In particular, every open subscheme $U \subseteq X$ can be covered by the spectra of Noetherian rings. $\square$

Proposition 3.12.6. *A scheme* $(X, \mathcal{O}_X)$ *is locally Noetherian if and only if for every affine open subscheme* $\operatorname{Spec} R$ *of* X*,* R *is a Noetherian ring.*

Proof. Follows from Proposition 3.12.5 and Corollary 3.12.4. $\square$

3.13 Normality and factoriality

Definition 3.13.1. A scheme $(X, \mathcal{O}_X)$ is **normal** if $\mathcal{O}_{X,p}$ is an integrally closed domain (in its field of fractions) for every $p \in X$.

From Proposition-Definition 3.11.2 we have that every normal scheme is reduced. We also note that the localization of an integrally closed domain is integrally closed. Therefore, if R is an integrally closed domain, $\operatorname{Spec} R$ is normal.

Definition 3.13.2. A scheme $(X, \mathcal{O}_X)$ is **factorial** if $\mathcal{O}_{X,p}$ is a unique factorization domain (UFD) for every $p \in X$.

The nonzero localizations of a UFD are again UFDs. Thus, if R is a UFD, then $\operatorname{Spec} R$ is factorial. The converse need not hold. Further, every UFD is an integrally closed domain, therefore factoriality implies normality.

Corollary 3.13.3. *Every factorial scheme is normal.*

Chapter 4

Morphisms of schemes

4.1 Functor of points and S-schemes

4.1.1 S-schemes

Definition 4.1.1. Let S be a scheme. An **S-scheme** is a scheme X together with a morphism $\varphi_X : X \to S$, called **structure morphism**. If $\varphi_Y : Y \to S$ is another S-scheme, a **morphism of S-schemes** is a morphism of schemes $\pi : X \to Y$ such that $\varphi_Y \circ \pi = \varphi_X$, in other words, if the following diagram commmutes:

$$\begin{tikzcd} X \arrow[rr, "\pi"] \arrow[dr, "\varphi_X"'] & & Y \arrow[dl, "\varphi_Y"] \\ & S & \end{tikzcd}$$

It is easy to see that S-schemes forms a category. If R is a ring, and R-scheme is a $(\operatorname{Spec} R)$-scheme. Note that we can give to the ring of sections $\mathcal{O}_X(U)$ of an R-scheme $(X, \mathcal{O}_X)$ the structure of an R-algebra via the composition $R \to \mathcal{O}_X(X) \xrightarrow{\rho_{X,U}} \mathcal{O}_X(U)$.

4.1.2 Morphisms to affine schemes

Proposition 4.1.2. *Let $(X, \mathcal{O}_X)$ be a scheme and $Y = \operatorname{Spec} R$ be an affine scheme. The map*

$$\varphi : \operatorname{Hom}(X, Y) \to \operatorname{Hom}(R, \mathcal{O}_X(X)), \quad \varphi(\pi, \pi^\#) = \pi^\#(Y)$$

is a bijection.

Proof. If X is affine, the result follows from Theorem 3.6.2. For the general case, it is enough to consider an affine open cover $\{U_i\}_{i \in I}$ of X and apply Proposition 3.7.3. $\square$

Corollary 4.1.3. *The scheme* $\operatorname{Spec} \mathbb{Z}$ *is a final object in the category of schemes.*

Proof. It follows from the fact that $\mathbb{Z}$ is an initial object in the category **Ring** and from Proposition 4.1.2. $\square$

Corollary 4.1.4. *Let $S = \bigoplus_{i \geq 0} S_i$ be a graded ring. There is a canonical morphism* $\operatorname{Proj} S \to \operatorname{Spec} S_0$.

Proof. We have $\mathcal{O}_{\operatorname{Proj} S}(\operatorname{Proj} S) = S_0$ (see Example 6.3.4 and 6.3.5 and the discussion after them). Now from Proposition 4.1.2, the inclusion $S_0 \subseteq S$ corresponds to a morphism of schemes $\operatorname{Proj} S \to \operatorname{Spec} S_0$. $\square$

4.1.3 Morphisms from affine schemes

Let $(X, \mathcal{O}_X)$ be a scheme and consider a point $p \in X$. Recall from Definition 3.10.1 that the residue field of p is $k(p) = \mathcal{O}_{X,p}/\mathfrak{m}_p$. We would like to understand the inclusion $p \to X$ in a scheme-theoretic fashion. Usually, the point $p \in X$ is identified with the affine scheme $\operatorname{Spec} k(p)$. The inclusion map $(i, i^{\#}) : \operatorname{Spec} k(p) \to X$ is defined as follows. The continuous map $i : \operatorname{Spec} k(p) \to X$ maps the zero ideal of $k(p)$ to $p \in X$. The map on sheaves $i^{\#} : \mathcal{O}_X \to i_* \mathcal{O}_{\operatorname{Spec} k(p)}$ is defined as follows. Let $U \subseteq X$ be an open subset, if $p \notin U$, then $i^{\#}(U) : \mathcal{O}_X(U) \to 0$ is the obvious map, on the other hand if $p \in U$ then $i^{\#}$ is given by the composition $\mathcal{O}_X(U) \to \mathcal{O}_{X,p} \to \mathcal{O}_{X,p}/\mathfrak{m}_p = k(p)$.

The maps $\mathcal{O}_X(U) \to k(p)$ can be thought of calculating the values of the sections of $\mathcal{O}_X(U)$ at p. For instance, if $X = \operatorname{Spec} \mathbb{Z}[x, y]/(x^2 + y^2 - 16)$ and $p = (x - 4, y) \in X$ that represents the point $(4, 0)$ in the affine plane, then the map $\mathcal{O}_X(X) \to k(p)$ is the map $\mathbb{Z}[x, y]/(x^2 + y^2 - 16) \to \mathbb{Q}$ that sends $x \mapsto 4$ and $y \mapsto 0$. Note that in this case $k(p) = \mathbb{Q}$, so we expect the values of the sections to be rational numbers. Now if we consider instead a morphism of schemes $\operatorname{Spec} \mathbb{C} \to X$, this gives us a homomorphism on the global sections $\mathbb{Z}[x, y]/(x^2 + y^2 - 16) \to \mathbb{C}$ that represent a different "point" of the scheme, or better another solution of the equation $x^2 + y^2 = 16$ in the field $\mathbb{C}$.

More in general, if $X = \operatorname{Spec} \mathbb{Z}[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$, morphisms of schemes of the type $\operatorname{Spec} R \to X$ are precisely the solutions to the equations

$$f_1(x_1, \ldots, x_n) = \ldots f_m(x_1, \ldots, x_n) = 0$$

in the ring R. This discussion motivates the following definition.

Definition 4.1.5. Let Z be a scheme. The Z**-valued points** of a scheme X, denoted $X(Z)$, are defined to be maps $Z \to X$. Therefore, any scheme X can be interpreted as a (contravariant) functor h_X from schemes to sets, called **functor of points** of X, that maps a scheme Z to $X(Z)$, and a morphism of schemes $Y \xrightarrow{\pi} Z$ to the map of sets $X(Z) \to X(Y)$ given by $f \mapsto f \circ \pi$.

If R is a ring, R-valued points are $\operatorname{Spec}(R)$-valued points.

4.1.4 Morphisms of projective schemes

Definition 4.1.6. Let $S = \bigoplus_i S_i$ and $R = \bigoplus_i R_i$ be graded rings. A map of graded rings $\varphi : S \to R$ is a homomorphism of rings such that $\varphi(S_i) \subseteq R_{id}$ for some $d > 0$.

Proposition 4.1.7. *A map of graded rings $\phi : S \to R$ induce a morphism of schemes $\varphi : \operatorname{Proj} R \setminus V^{\mathbb{P}}(\phi(S_+)) \to \operatorname{Proj} S$.*

Proof. Set $X = \operatorname{Proj} R$ and $Y = \operatorname{Proj} S$. By localizing ϕ at an element $f \in S_+$, we get a map of rings $\phi_f : S_f \to R_{\phi(f)}$, its degree zero part $(\phi_f)_0 : (S_f)_0 \to (R_{\phi(f)})_0$ corresponds to a map of affine schemes $\varphi_f : \operatorname{Spec}(R_{\phi(f)})_0 \to \operatorname{Spec}(S_f)_0$. We can rewrite it as $\varphi_f : X_{\phi(f)} \to Y_f$. By ranging $f \in S_+$, the maps φ_f agree on the overlaps, in fact for

$f, g \in S_+$, the maps φ_f and φ_g restricted to $X_{\phi(f)} \cap X_{\phi(g)} = X_{\phi(fg)}$ both coincide with φ_{fg}. Therefore we can glue these maps together (Proposition 3.7.3) to obtain a morphism $\varphi : X \setminus V^{\mathbb{P}}(\phi(S_+)) \to Y$ (note that $\bigcup_{f \in S_+} X_{\phi(f)} = X \setminus V^{\mathbb{P}}(\phi(S_+))$). □

4.2 Properties of morphisms

Definition 4.2.1. A morphism of schemes $\varphi : X \to Y$ is an **open embedding** (or **open immersion**) if φ factors as

$$(X, \mathcal{O}_X) \xrightarrow{\rho} (U, \mathcal{O}_{Y|U}) \xrightarrow{i} (Y, \mathcal{O}_Y)$$

where ρ is an isomorphism, and i is an inclusion of the open set U in Y.

Definition 4.2.2. A morphism of schemes $\varphi : X \to Y$ is **quasicompact** if for every quasicompact open subset $U \subseteq Y$, $\varphi^{-1}(U)$ is quasicompact.

Definition 4.2.3. A morphism of schemes $\varphi : X \to Y$ is **affine** if for every affine open subscheme $U \subseteq Y$, $\varphi^{-1}(U)$ is an affine scheme.

Definition 4.2.4. A morphism of schemes $\varphi : X \to Y$ is **finite** if φ is affine, and for every affine open subscheme $U = \operatorname{Spec} A$ of Y, with $\varphi^{-1}(U) = \operatorname{Spec} B$, B is a finite A-algebra (that is, a finitely generated A-module).

We recall the definition of an integral ring homomorphism.

Definition 4.2.5. A ring homomorphism $\varphi : A \to B$ is **integral** if B is integral over $\varphi(A)$. In other words, if $\varphi(A) \subseteq B$ is an integral extension.

Definition 4.2.6. A morphism of schemes $\varphi : X \to Y$ is **integral** if φ is affine, and for every affine open subscheme $U = \operatorname{Spec} A$ of Y, with $\varphi^{-1}(U) = \operatorname{Spec} B$, the induced map $A \to B$ is an integral ring homomorphism.

From the fact that every finite A-algebra is integral over A, we have that every finite morphism is integral.

Definition 4.2.7. A morphism of schemes $\varphi : X \to Y$ is **locally of finite type** if for every affine open subset $U = \operatorname{Spec} A$ of Y, and every affine open subset $V = \operatorname{Spec} B$ of $\pi^{-1}(U)$, the induced homomorphism of rings $A \to B$ expresses B as a finitely generated A-algebra. The morphism π is of **finite type** if it is locally of finite type and quasicompact. An S-scheme X is (locally) of finite type if the structure morphism $X \to S$ is (locally) of finite type.

Since an A-algebra is finite if and only if it is finitely genrated and integral over A, we have that a morphism of schemes π is finite if and only if it is of finite type and integral.

4.3 Associated points and rational maps

4.3.1 Associated points

Definition 4.3.1. Let R be a ring and M be an R-module. A prime ideal $P \subseteq R$ is a **associated** to M if there exists $x \in M$ such that

$$P = \{a \in R : ax = 0\} = (0 : x).$$

The set of associated primes of M is dentoed $\operatorname{Ass}(M)$.

The relation between associated primes of an R-module and primary decomposition of ideals of R is the following. If an ideal $I \subseteq R$ has a primary decomposition, it means that I is the intersection of primary ideals $Q_i \subseteq R$ for $i \in \{1, \dots, n\}$. If this primary decomposition is minimal, the radicals of these primary ideals $P_i = \sqrt{Q_i}$ are prime ideals of R, called the associated primes of I (as an ideal). These are actually the associated primes of A/I (as an R-module).

Note that a prime ideal P is associated to R (as an R-module) if it is one of the associated primes of the zero ideal (0) (as an ideal). In other words, if P appears among the radicals of the primary ideals in a minimal primary decomposition of (0).

Definition 4.3.2. Let $(X, \mathcal{O}_X)$ be a locally Noetherian scheme. A point $x \in X$ is an **associated point** of X if the maximal ideal $\mathfrak{m}_x$ is an associated prime of $\mathcal{O}_{X,x}$. The set of the associated points is denoted by $\operatorname{Ass}(X)$.

Example 4.3.3.

1. Consider the affine plane $X = \mathbb{A}^2_k = \operatorname{Spec} k[x, y]$. The origin $p = (0, 0)$, that corresponds to the ideal (x, y), is not an associated point, since in the ring $\mathcal{O}_{X,p} \simeq k[x, y]_{(x,y)}$ the maximal ideal (x, y) is not an associated prime. In fact, $k[x, y]_{(x,y)}$ is a domain, hence the zero ideal is prime (0), therefore its unique associated prime is itself. On the other hand, the generic point η of X, that corresponds to the zero ideal (Remark 3.3.26) is an associated point of X. In fact, the ring $\mathcal{O}_{X,\eta} \simeq k[x, y]_{(0)} \simeq k(x, y)$ is a field, the maximal ideal is the zero ideal (0) and it is an associated prime.

2. Let $X = \operatorname{Spec} k[x, y]/(y^2, xy)$. The ideal (y^2, xy) in $k[x, y]$ has the following minimal primary decomposition
$$(y^2, xy) = (y) \cap (x, y^2),$$
since the localization of a minimal primary decomposition is again a minimal primary decomposition, the associated primes of $k[x, y]/(y^2, xy)$ are (y) and (x, y), and these are also the associated points of X.

4.3.2 Rational functions

Definition 4.3.4. Let X be a locally Noetherian scheme. An open subset $U \subseteq X$ is **schematically dense** if it contains the associated points of X.

Definition 4.3.5. Let $(X, \mathcal{O}_X)$ be a locally Noetherian scheme. A **rational function** is an element of the direct limit

$$R(X) = \varinjlim_{U \supseteq \mathrm{Ass}(X)} \mathcal{O}_X(U).$$

Explicitly, $R(X)$ consists of classes of pairs $[(f, U)]$ with U a schematically dense open subset and $f \in \mathcal{O}_X(U)$, where two pairs (f, U), (g, V) are equivalent if there exists a schematically dense open set $W \subseteq U \cap V$ such that $f_{|W} = g_{|W}$. The ring $R(X)$ is called the **total fraction ring** of X.

Remark 4.3.6. If X is a locally Noetherian integral scheme, then the generic point $\eta \in X$ is the unique associated point and every nonempty open set is schematically dense. Therefore the set of rational functions coincides with the stalk at η, that is, the total fraction ring coincide with the function field:

$$R(X) = \mathcal{O}_{X,\eta} = K(X).$$

Remark 4.3.7. If $X = \operatorname{Spec} R$ is an affine Noetherian scheme, then, from the fact that the localization of a minimal primary decomposition is again a minimal prime decomposition, the associated points of X are the associated primes $\{p_1, \dots, p_n\}$ of the zero ideal (0) of R (as an ideal). The union of these prime ideals $D = \bigcup_i p_i$ is the set of zerodivisors of R. It follows that the total fraction ring of X coincide with the total fraction ring of R, that is the localization of R at $S = R \setminus D$, in fact

$$R(X) = \varinjlim_{U \supseteq \mathrm{Ass}(X)} \mathcal{O}_X(U) = \varinjlim_{U \supseteq \{p_i\}_i} S_U^{-1} R = S^{-1} R,$$

where $S_U = R \setminus \bigcup_{p \in U} p$.

4.3.3 Rational maps

Definition 4.3.8. Let X and Y be locally Noetherian schemes. Consider the sheaf $\mathscr{F}$ of sets on X consisting of morphisms from an open subscheme U of X to Y

$$\mathscr{F}(U) = \{f : U \to Y \text{ as schemes}\},$$

with the obvious restriction maps (this is a sheaf by Proposition 3.7.3). A **rational map** from X to Y, denoted $\varphi : X \dashrightarrow Y$, is an element of the direct limit

$$R(X, Y) = \varinjlim_{U \supseteq \mathrm{Ass}(X)} \mathscr{F}(U).$$

By using the isomorphism $\mathrm{Hom}_{\mathbf{Ring}}(\mathbb{Z}[x], \mathcal{O}_X(U)) \simeq \mathcal{O}_X(U)$ given by $\varphi \mapsto \varphi(x)$, and by Proposition 4.1.2 we get the idenfitication

$$R(X) = R(X, \mathbb{A}^1_{\mathbb{Z}}),$$

that is we can identify the rational functions as particular rational maps.

Definition 4.3.9. Let $\varphi : X \to Y$ a rational function. The **domain of definition** $\mathrm{dom}(\varphi)$ of φ is the set of points $x \in X$ such that there exists a representative $[(f, U)]$ of φ with $x \in U$.

In some cases, a rational map $\varphi : X \dashrightarrow Y$ can be thought as a morphism of schemes $\varphi : \mathrm{dom}(\varphi) \to Y$. This is the case for rational functions, when viewed as rational maps of the type $\varphi : X \dashrightarrow \mathbb{A}^1_{\mathbb{Z}}$. We say that a rational function φ is **regular** at a point $p \in X$ if $p \in \mathrm{dom}(\varphi)$.

4.4 Subschemes and embeddings

We use the terms *immersion* and *embedding* as synonyms, although in topology they use to have different meanings.

4.4.1 Open immersions

Definition 4.4.1. A morphism of schemes $(\pi, \pi^{\#}) : X \to Y$ is an **open immersion** (or **open embedding**) if π is an homeomorphism between X and an open subset U of Y, and for every open subset $V \subseteq U$, $\pi^{\#}(V) : \mathcal{O}_Y(V) \to \pi_*\mathcal{O}_X(V)$ is an isomorphism.

For every open subscheme $U \subseteq Y$, there exists a canonical morphism of schemes $(i, i^{\#}) : U \to Y$, where $i : U \to Y$ is the inclusion map, and $i^{\#}$ is given by the restriction maps $i^{\#}(V) = \rho_{V,U\cap V}$ of the sheaf $\mathcal{O}_Y$, for every open subset $V \subseteq Y$.

A morphism of schemes $\pi : X \to Y$ is an open immersion if it factors as $X \xrightarrow{\varphi} U \hookrightarrow Y$ where φ is an isomorphism of schemes, and $U \hookrightarrow Y$ is the canonical inclusion of the open subscheme U into Y.

Definition 4.4.2. Let $\pi : X \to Y$ be a morphism of schemes, and let $U \subseteq X$ and $V \subseteq Y$ be open subsets. The **(domain) restriction** of π to U is the composition $\pi_{|U} = \pi \circ i$, where $i : U \to X$ is the canonical inclusion. The **(codomain) restriction** of π to V is the morphism of schemes given by the continuous map $\pi_{|\pi^{-1}(V)} : \pi^{-1}(V) \to V$ and the sheaf morphism $\pi^{\#}_{|V} : \mathcal{O}_{Y|V} \to \pi_*\mathcal{O}_{X|V}$ (cf. Definition 2.2.3).

4.4.2 Closed embeddings

Definition 4.4.3. A morphism of schemes $(\pi, \pi^{\#}) : X \to Y$ is a **closed embedding** (or a **closed immersion**) if π is an homeomorphism of X onto a closed subset of Y and $\pi^{\#}$ is surjective. If the map π is an inclusion of sets, then X is a **closed subscheme** of Y.

From Prospotions 3.2.6 and 3.6.4 it is clear that the morphism $\mathrm{Spec}\, R/I \to \mathrm{Spec}\, R$, induced by the quotient map $R \to R/I$, is a closed embedding.

Theorem 4.4.4. *Let $X = \mathrm{Spec}\, R$ be an affine scheme. Every closed subscheme of X is of the form $\mathrm{Spec}\, R/I$, identifying $\mathrm{Spec}\, R/I$ with $V(I) \subseteq X$ (cf. Proposition 3.2.6), for some ideal I of R.*

Proof. TO DO. □

Proposition 4.4.5. *Let $(\pi, \pi^\#) : X \to Y$ be a morphism of schemes. The following are equivalent:*

1. *$(\pi, \pi^\#)$ is a closed embedding.*

2. *$(\pi, \pi^\#)$ is an affine morphism, and for every affine open subset $U = \operatorname{Spec} A \subseteq Y$ with $\pi^{-1}(U) \simeq \operatorname{Spec} B$, the map $A \to B$ is surjective (that is, it is of the type $A \to A/I$ for some ideal I of A).*

Proof.

(1) $\Rightarrow$ (2) Let $U \subseteq Y$ be an affine open subset, then the restriction $(\pi_{|\pi^{-1}(U)}, \pi^\#_{|U})$ (cf. Defintiion 2.2.3) is a closed embedding of $\pi^{-1}(U)$ into the affine open subscheme U. From Theorem 4.4.4 $\pi^{-1}(U)$ is affine. This proves that π is affine. The surjectivity of the ring homomorphisms given by the affine restrictions follows from the surjectivity of $\pi^\#$ and Proposition 3.6.4

(2) $\Rightarrow$ (1) If X and Y are affine, the result is true by Proposition 3.6.4. For the general case, let $\{U_i\}_{i\in I}$ be an affine open cover of Y, by hypothesis $\{\pi^{-1}(U_i)\}_{i\in I}$ is an affine open cover of X. Since surjectivity can be cheked on stalks (Proposition 2.4.2) and each restriction $\pi^\#_{|U_i}$ is surjective, $\pi^\#$ is surjective. Similarly, π is injective because it is so each restriction $\pi_{|\pi^{-1}(U_i)}$. Finally, π is closed, in fact for every closed subset $F \subseteq X$, $\pi(F) \cap U_i = \pi_{|\pi^{-1}(U_i)}(F \cap \pi^{-1}(U_i))$ is closed for all $i \in I$ because each $\pi_{|\pi^{-1}(U_i)}$ is closed. It follows that π is a homeomorphism from X to a closed subset of Y. $\square$

4.4.3 (Locally closed) subschemes

Definition 4.4.6. Let X be a topological space. A **locally closed** subset is the intersection of an open subset and a closed subset of X. Equivalently, it is a closed subset of an open subset of X.

Definition 4.4.7. A (locally closed) **embedding** (or **immersion**) is a morphism of schemes $(\pi, \pi^\#) : X \to Y$ such that π is an homeomorphism between X and a locally closed subset of Y, and $\pi^\#$ is surjective. If X is a subset of Y, then X is a (locally closed) **subscheme** of Y.

An immersion $X \to Y$ can be factored into a closed embedding $X \to U$ followed by an open immersion $U \to Y$.

Lemma 4.4.8. *Let $(\pi, \pi^\#) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of schemes. If π injective and $\pi^\#$ surjective, then $(\pi, \pi^\#)$ is a monomorphism in the category of schemes.*

Proof. Let $(f_i, f_i^\#) : (T, \mathcal{O}_T) \to (X, \mathcal{O}_X)$ for $i \in \{1, 2\}$ be two morphisms of schemes such that thei compositions with $(\pi, \pi^\#)$ are the same. This means that $f_1 \circ \pi = f_2 \circ \pi$, and it follows $f_1 = f_2$ since π is injective. Further from $\pi^\# \circ f_1^\# = \pi^\# \circ f_2^\#$ we have $f_1^\# = f_2^\#$ since $\pi^\#$ is surjective, that is, an epimorphism in the category of sheaves (Proposition-Definition 2.4.2). $\square$

Corollary 4.4.9. *The locally closed embeddings of schemes are monomorphisms in the category of schemes.*

4.4.4 The reduced subscheme

Let $(X, \mathcal{O}_X)$ be a scheme. The sheaf $\mathcal{N}_X$ associated to the presheaf $U \mapsto \mathfrak{N}(\mathcal{O}_X(U))$ is an ideal sheaf such that $(X, \mathcal{O}_X/\mathcal{N}_X)$ is a reduced closed subscheme of X, called the **reduced subscheme** of X, denoted by X_{red}. This construction is functorial, in the sense that if $f : X \to Y$ is a morphism of schemes, there exists a morphism $f_{red} : X_{red} \to Y_{red}$ that commutes with the corresponding closed embeddings, and such that if $g : Y \to Z$ is another morphism of schemes, $(g \circ f)_{red} = g_{red} \circ f_{red}$.

4.4.5 Regular embeddings

Definition 4.4.10. A locally closed embedding $\pi : X \to Y$ is a **regular embedding** (of codimension r) at a point $p \in X$ if the ideal of X in the local ring $\mathcal{O}_{Y,p}$ is generated by a regular sequence (of length r). We say that π is a **regular embedding** if it is a regular embedding at all $p \in X$.

The condition for an embedding to be regular is open, in the sense that if π is a regular embedding at p, then it is a regular embedding in an open neighbourhood of p.

4.5 Fibered products of schemes

In this section, we show that the fibered product $X \times_S Y$ of two S-schemes X and Y exists. The proof will be divided in several steps. First, we discuss two particular cases.

Lemma 4.5.1. *In the category of schemes, the fibered product of two affine schemes* $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$ *over an affine scheme* $Z = \operatorname{Spec} C$ *exists, and we have*

$$\operatorname{Spec} A \times_{\operatorname{Spec} C} \operatorname{Spec} B = \operatorname{Spec}(A \otimes_C B).$$

Proof. We need to show that $\operatorname{Spec}(A \otimes_C B)$ is the fibered product $X \times_Z Y$. Let W be a scheme and let $p : W \to X$ and $q : W \to Y$ be two morphism of schemes that commutes with the structure morphisms of X and Y as Z-schemes. Thus W is a Z-scheme, in particular $\mathcal{O}_W(W)$ is a C-algebra. From Proposition 4.1.2 the morphisms p and q correspond to C-algebra homomorphisms $A \to \mathcal{O}_W(W)$, $B \to \mathcal{O}_W(W)$. Since $A \otimes_C B$ is the fibered coproduct in the category of rings (Example 1.4.22), there exists a unique C-algebra homomorphism $A \otimes_C B \to \mathcal{O}_W(W)$ such that the following diagram commutes

$$\begin{tikzcd}
 & & \mathcal{O}_W(W) \\
A \arrow[r] \arrow[urr, bend left] & A \otimes_C B \arrow[ur, dashed, "\exists!"] & \\
C \arrow[u] \arrow[r] & B \arrow[u] \arrow[uur, bend right] &
\end{tikzcd}$$

The C-algebra homomorphism $A \otimes_C B \to \mathcal{O}_W(W)$ corresponds to a unique Z-schemes morphism $W \to \operatorname{Spec}(A \otimes_C B)$ such that its compositions with the structure morphisms $X \to Z$, $Y \to Z$ commute with the morphisms p, q. $\square$

Lemma 4.5.2. *Let $\rho : Y \to Z$ be a morphism of schemes and let $i : U \to Z$ be an open embedding. The open subscheme $\rho^{-1}(U)$ of Y together with the canonical open embedding $\rho^{-1}(U) \to Y$ and the codomain restriction $\rho^{-1}(U) \to U$ of ρ to U (Definition 4.4.2), is the fibered product $U \times_Z Y$.*

Proof. Let W be a scheme and let $p : W \to Y$ and $q : W \to U$ be morphisms such that $\rho \circ p = i \circ q$. Hence, as sets, we have

$$\rho(p(W)) = i(q(W)) = q(W) \subseteq U \Rightarrow p(W) \subseteq \rho^{-1}(U).$$

Therefore, we can restrict the codomain of p to $\rho^{-1}(U)$ (Definition 4.4.2), obtaining a morphism of schemes $\varphi : W \to \rho^{-1}(U)$ that commutes with p and q, and is uniquely determined by p. $\square$

In particular, if U and V are open subschemes of X, the fibered product $U \times_X V$ is the open subscheme given by the intersection $U \cap V$.

Theorem 4.5.3. *The fibered product exists in the category of schemes.*

Proof. Let X,Y and Z be schemes and let $\alpha : X \to Z$, $\beta : Y \to Z$ be morphisms of schemes. We will divide the proof in several steps.

Step 1 Assume that X and Z are affine and suppose that the morphism β factors as $Y \xrightarrow{i} Y' \to Z$, with Y' affine and i is an open embedding. From Lemma 4.5.1 we know that the fibered product $W' = X \times_Z Y'$ exists. Now from Lemma 4.5.2 the fibered product $W = W' \times_Y' Y$ exists, and the projection $W \to W'$ is an open embedding. It is easy to verify that W, together with the two compositions $W \to W' \to X$ and $W \to Y' \to Y$, is the fibered product $X \times_Z Y$.

$$\begin{array}{ccc} W & \longrightarrow & Y \\ \downarrow & & \downarrow \\ W' & \longrightarrow & Y' \\ \downarrow & & \downarrow \\ X & \longrightarrow & Z \end{array}$$

Step 2 Assume that X and Z are affine. Let $\{U_i\}_{i \in I}$ be an affine open cover of Y. From Lemma 4.5.1, for each $i \in I$, the fibered product $W_i = X \times_Z U_i$ exists, let $\alpha_i : W_i \to X$, $\beta_i : W_i \to U_i$ be the corresponding canonical projections. For every $i, j \in I$, let $U_{ij} = U_i \cap U_j$ and first, consider it as an open subscheme of the affine scheme U_i. From Step 1, the fibered product $W_{ij} = U_{ij} \times_Z X$ exists, with projections $\alpha_{ij} : W_{ij} \to U_{ij}$, $\beta_{ij} : W_{ij} \to X$. By changing the role of i and j, we obtain another fibered product W_{ji}, therefore there exists a unique scheme isomorphism $\phi_{ij} : W_{ij} \to W_{ji}$. The uniqueness implies that the morphisms ϕ_{ij} satify the cocyle condition, therefore, from Proposition 3.7.2, we can glue the schemes W_i together and obtain a scheme W. From Proposition 3.7.3, we can also glue the morphisms α_i and β_i, obtaining two morphisms $\alpha' : W \to X$ and $\beta' : W \to Y$.

Now we prove that the scheme W with the morphisms α' and β' is the fibered product $X \times_Z Y$. Let T be a scheme and let $\alpha'' : T \to X$, $\beta'' : T \to Y$ be two morphisms of schemes. For every $i \in I$ let $T_i = \beta'^{-1}(U_i)$, then the codomain restriction of β'' on U_i induces a morphism $\alpha''_i : T_i \to U_i$. Then, there exists a unique morphism $\gamma_i : T_i \to W_i$. We can repeat the same argument on the intersections $U_i \cap U_j$ and, similarly as before, we obtain that the morphisms γ_i glue together to a unique morphism $\gamma : T \to W$.

Step 3 Assume that Z is affine. To show the existence of $X \times_Z Y$, it is enough to repeat the procedure in Step 2 on X, considering an affine open cover $\{U_i\}_{i\in I}$ of X, using the fact that, from Step 2 we konw that the fibered product of an affine scheme and an arbitrary scheme exists.

Step 4 Finally, for the general case, we use the same technique on the scheme Z. □

Since the category of schemes has a final object, $\operatorname{Spec}\mathbb{Z}$, the previous theorem proves the existence of product of schemes $X \times Y$, interpreted as the fibered product $X \times_{\operatorname{Spec}\mathbb{Z}} Y$.

Definition 4.5.4 (Fibre of morphism)**.** Let $f : X \to Y$ be a morphism of schemes, let $p \in Y$ be a point and let $i : \operatorname{Spec} k(p) \to Y$ be the canonical inclusion (cf. Section 4.1.3). The (scheme-theoretic) **fibre** of the morphism f at p (or above p) is defined as the fibered product $f^{-1}(p) = \operatorname{Spec} k(p) \times_Y X$. If Y is irreducible, the fibre at the generic point of Y is the **generic fibre**.

$$\begin{tikzcd} \operatorname{Spec} k(p) \times_Y X \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \operatorname{Spec} k(p) \arrow[r, "i"] & Y \end{tikzcd}$$

We have to distinguish *generic fibre* with the expression *general fibre*. As we have seen above, the first means the fibre over the generic point (of an irreducible scheme). Instead, if we say that the general fibre has some property, we mean that the fibre over any point of some open dense subset has that property.

With the above definition, a morphism of schemes $\pi : X \to Y$ can be interpreted as a *family* of schemes $X_y = f^{-1}(y)$ for every $y \in Y$. For example, the family of curves $y^2 = x^3 + tx$ parametrized by t, can be interpreted as the morphism of schemes $\operatorname{Spec} k[x, y, t]/(y^2 - x^3 - tx) \to \operatorname{Spec} k[t]$.

The fibered product can be also interpreted in the following way: if $\pi : Y \to Z$ is a morphism of schemes, and $f : X \to Z$ is anoterh morphism of schemes, the projection $X \times_Z Y \to X$ is seen as a *pullback* of π by f, or a **base change**. Informally, we say that a property is **preserved by base change** if whenever a morphism $\pi : Y \to Z$ has the property, its pullback $X \times_Z Y \to X$ by any morphism of schemes $X \to Z$ has the property.

4.6 Separated morphisms

A topological space X is Hausdorff is and only if the diagonal $\Delta \subseteq X \times X$ is closed in the product topology. The notion of separated schemes tries to mimic the Hausdorff property for topological space. Their definition is given in light of the above characterization.

Definition 4.6.1. Let $\pi : X \to Y$ be a morphism of schemes. From the properties of the fibered product, by considering two copies of the identity morphism $X \to X$ there exists a unique morphism $\delta_\pi : X \to X \times_Y X$, called the **diagonal morphism**.

If π is the morphism $X \to \operatorname{Spec} \mathbb{Z}$, we denote the diagonal morphism simply by δ. The diagonal morphism is a locally closed embedding.

Definition 4.6.2. A morphism of schemes $\pi : X \to Y$ is **separated** if the diagonal morphism δ_π is a closed embedding. An S-scheme is **separated over** S if the structure morphism $X \to S$ is separated. A scheme X is **separated** if it is separated over $\operatorname{Spec} \mathbb{Z}$.

The next fact is true in any category with fibered product, the proof is immediate.

Lemma 4.6.3. *Let $\mathscr{A}$ be a category with fibered product, and let $f : X \to Y$ be a morphism. The following are equivalent:*

1. *f is a monomorphism,*
2. *X with two copies of the identity is a fibered product $X \times_Y X$,*
3. *the diagonal morphism $\delta_f : X \to X \times_Y X$ is an isomorphism.*

Since an isomorphism of schemes is in particular a closed embedding, we obtain the following result.

Corollary 4.6.4. *Every monomorphism of schemes $f : X \to Y$ is separated.*

In particular, from Corollary 4.4.9 we have the following result.

Corollary 4.6.5. *Every locally closed embedding is separated.*

In particular, open and closed embeddings are separated since they are locally closed embeddings.

Proposition 4.6.6. *Every morphism of affine schemes $f : \operatorname{Spec} B \to \operatorname{Spec} A$ is separated.*

Proof. From Lemma 4.5.1 we know that $\operatorname{Spec} B \times_{\operatorname{Spec} A} \operatorname{Spec} B = \operatorname{Spec}(B \otimes_A B)$. The diagonal morphism $\delta_f : \operatorname{Spec} B \to \operatorname{Spec}(B \otimes_A B)$ is given by the ring homomorphism $B \otimes_A B \to B$ defined by $b \otimes b' \mapsto bb'$, which is surjective. Therefore, from Theorem 4.4.4, δ_f is a closed embedding. $\square$

Theorem 4.6.7 (Valuative criterion for separatedness)**.** *Let $f : X \to Y$ be a morphism of schemes, and suppose that X is Noetherian. Then f is separated if and only if the following condition holds. For any discrete valuation ring R with quotient field K, for every pair of morphisms $\operatorname{Spec} K \to X$ and $\operatorname{Spec} R \to Y$ that makes the following diagram commutative*

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d, "i"] & X \arrow[d, "f"] \\ \operatorname{Spec} R \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

where $i : \operatorname{Spec} K \to \operatorname{Spec} R$ is induced by the inclusion $R \subseteq K$, there exists at most one morphism from $\operatorname{Spec} R$ to X that makes the whole diagram commutative.

4.7 Proper morphisms

A topological space X is compact if and only if for every topological space Y the projection map $X \times Y \to Y$ is closed. In other words, if the (closed continous) map $X \to \{p\}$ remains closed under base change. This observation motivates the following definition.

Definition 4.7.1. A morphism of schemes $f : X \to Y$ is **universally closed** if for every other morphism of schemes $Z \to Y$, the induced morphism $X \times_Y Z \to Z$ is closed, that is, the continuous map on the topological spaces is closed.

In other words, a morphism of schemes is universally closed if it reamins closed under base change.

Definition 4.7.2. A morphism of schemes $\pi : X \to Y$ is **proper** if it is separated, finite type and universally closed. An S-scheme is **proper over** S if the structure morphism $X \to S$ is proper. A scheme X is **proper** if it is proper over $\operatorname{Spec} \mathbb{Z}$.

Theorem 4.7.3. *Let A be a ring. The morphism of schemes $\mathbb{P}^n_A \to \operatorname{Spec} A$ is proper.*

Proof. TO DO $\square$

Theorem 4.7.4 (Valuative criterion of properness)**.** *Let $f : X \to Y$ be a morphism of schemes, and suppose that X is Noetherian. Then f is proper if and only if the following condition holds. For any valuation ring R with quotient field K, for every pair of morphisms $\operatorname{Spec} K \to X$ and $\operatorname{Spec} R \to Y$ that makes the following diagram commutative*

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d, "i"] & X \arrow[d, "f"] \\ \operatorname{Spec} R \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

where $i : \operatorname{Spec} K \to \operatorname{Spec} R$ is induced by the inclusion $R \subseteq K$, there exists a unique morphism from $\operatorname{Spec} R$ to X that makes the whole diagram commutative.

Definition 4.7.5. An **abstract variety** is a reduced, separated scheme of finite type over a field k. If it is proper over k we will also say it is **complete**.

Definition 4.7.6. The **projective n-space** over a scheme Y, denoted $\mathbb{P}^n_Y$ is defined as the product $\mathbb{P}^n_{\mathbb{Z}} \times Y$. A morphism of schemes $f : X \to Y$ is **projective** if it factors into a closed immersion $X \to \mathbb{P}^n_Y$ and the projection $\mathbb{P}^n_Y \to Y$. A morphism of schemes $f : X \to Y$ is **quasi-projective** if it factors into an open immersion $X \to X'$ and a projective morphism $X' \to Y$. An S-scheme X is **(quasi)projective over** S if so is the structure morphism $X \to Y$. A scheme X is **(quasi)projective** if it so over $\operatorname{Spec} \mathbb{Z}$.

Chapter 5

Geometric properties

5.1 Dimension and codimension

Definition 5.1.1. Let X be a topological space. The **dimension** of X, denoted by $\dim X$, is the supremum of the lengths of chains of nonempty distinct irreducible closed subsets. The space X is **pure dimensional** (or **equidimensional**, or just **pure**) if all its irreducible components have the same dimension.

Note that the above definition is compatible with the notion of Krull dimension for a ring, in the sense that, if R is a ring, then $\dim R = \dim \operatorname{Spec} R$.

It is easy to show that if $Y \subseteq X$ then $\dim Y \leq \dim X$.

Proposition 5.1.2. *Every Noetherian scheme X of dimension 0 is finite.*

Proof. Since X is a Noetherian scheme, it has a finite open cover of affine open susbsets $U_i = \operatorname{Spec} R_i$ with R_i Noetherian. Each open subset $U_i = \operatorname{Spec} R_i$ has dimension 0, therefore R_i is a Noetherian ring of dimension 0, that is, an Artinian ring. This implies that each R_i has a finite number of prime ideals, hence X is finite. □

Definition 5.1.3. Let X be a topological space. The **codimension** of an irreducible subset $Y \subseteq X$ is the supremum $\operatorname{codim}_X Y$ of the lengths of chains of nonempty distinct irreducible closed subsets starting with $\overline{Y}$.

If $X = \operatorname{Spec} R$, the codimension of $V(I)$ coincides with the heigth of the ideal I.

Proposition 5.1.4. *If Y is an irreducible closed subset of a scheme X, and η is the generic point of Y, then $\operatorname{codim}_X Y = \dim \mathcal{O}_{X,\eta}$.*

Proof. First suppose that $X = \operatorname{Spec} R$ is affine. Then, $Y = V(p)$ for some prime ideal p, and $\eta = p$ (Proposition 3.3.25). We have

$$\dim \mathcal{O}_{X,\eta} = \dim R_p = h(p) = \operatorname{codim}_X V(p) = \operatorname{codim}_X Y.$$

For the general case, let $U \subseteq X$ be an affine open subset such that $U \cap Y \neq \emptyset$. Since $U \cap \overline{\{\eta\}} \neq \emptyset$, it follows $\eta \in U$. In particular $\mathcal{O}_{X,\eta} = \mathcal{O}_{U,\eta}$. Now let

$$Y \subsetneq X_0 \subsetneq \cdots \subsetneq X_n \subseteq X$$

be a chain of distinct irreducible closed subsets of lenght $n = \operatorname{codim}_X Y$. For every index i, we have $\eta \in U \cap X_i \neq \emptyset$. Further, $U \cap X_i$ is irreducible, since it is an open subset of an irreducible set, and closed in U. Since $U \cap X_i$ is dense in X_i, its closure in X is X_i, hence the subsets $U \cap X_i$ are distinct. It follows that $\operatorname{codim}_X Y \leq \operatorname{codim}_U Y$. On the other hand, if we consider a chain of distinct irreducible closed subsets of U:

$$Y \cap U \subsetneq U_0 \subsetneq \cdots \subsetneq U_m \subseteq X$$

then taking the closures gives rise to a chain of distinct irreducible closed subsets of X from Y to X. Therefore $\operatorname{codim}_X Y \geq \operatorname{codim}_U Y$ and we have the equality. Finally

$$\operatorname{codim}_X Y = \operatorname{codim}_U Y = \dim \mathcal{O}_{U,\eta} = \dim \mathcal{O}_{X,\eta}. \qquad \square$$

5.2 Regularity and smoothness

Definition 5.2.1. The **Zariski tangent space** of a local ring $(A, \mathfrak{m})$ is $\mathfrak{m}/\mathfrak{m}^2$, it is a vector space over the residue field $A/\mathfrak{m}$. The **Zariski tangent space** is the dual vector space $(\mathfrak{m}/\mathfrak{m}^2)^\vee$. If X is a scheme, the **Zariski (co)tangent space** $T_{X,p}$ (resp. $T^\vee_{X,p}$) at a point p is defined to be the Zariski (co)tangent space of the local ring $\mathcal{O}_{X,p}$.

Let $(A, \mathfrak{m}, k)$ be a Noetherian local ring. From the Nakayama's Lemma we have that

$$\dim A \leq \dim_k \mathfrak{m}/\mathfrak{m}^2.$$

Definition 5.2.2. A Noetherian local ring $(A, \mathfrak{m})$ is a **regular local ring** if we have $\dim A = \dim_k \mathfrak{m}/\mathfrak{m}^2$. A Noetherian ring A is **regular** if so is the localization A_p at any prime ideal p of A.

A locally Noetherian scheme X is **regular** (or **nonsingular**) at a point $p \in X$ if the (Noetherian local) ring $\mathcal{O}_{X,p}$ is regular. A point $p \in X$ is **singular** (or **nonregular**) if it is not regular. The scheme X is **regular** if it is regular at all points. It is **singular** otherwise.

Definition 5.2.3. Let k be a field. A k-scheme X locally of finite type of dimension d is **smooth** if there exists a cover by affine open sets $U = \operatorname{Spec} k[x_1, \ldots, x_n]/(f_1, \ldots, f_r)$ where the Jacobian matrix $J(p)$ at any point $p \in U$

$$J(p) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(p) & \cdots & \frac{\partial f_r}{\partial x_1}(p) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_1}{\partial x_n}(p) & \cdots & \frac{\partial f_r}{\partial x_n}(p) \end{bmatrix}$$

has corank d.

Theorem 5.2.4 (Smoothness vs. Regularity)**.** *Let X be a k-scheme. Then the following implications hold:*

$$X \textit{ smooth} \Longrightarrow X \textit{ regular},$$
$$X \textit{ regular} + k \textit{ perfect} \Longrightarrow X \textit{ smooth}.$$

Chapter 6

Quasicoherent sheaves

6.1 $\mathcal{O}_X$-modules

In this section, $\mathcal{O}_X$ will be always the sheaf of rings of a ringed space $(X, \mathcal{O}_X)$.

Definition 6.1.1. An **$\mathcal{O}_X$-module** is a sheaf of abelian groups $\mathscr{F}$ on X with the following additional structure. For each open set U of X, $\mathscr{F}(U)$ is an $\mathcal{O}_X(U)$-module. Further, for every inclusion of open sets $U \subseteq V$ the following diagram commutes

$$\begin{array}{ccc} \mathcal{O}_X(V) \times \mathscr{F}(V) & \longrightarrow & \mathscr{F}(V) \\ {\scriptstyle \rho_{V,U} \times \rho_{V,U}} \big\downarrow & & \big\downarrow {\scriptstyle \rho_{V,U}} \\ \mathcal{O}_X(U) \times \mathscr{F}(U) & \longrightarrow & \mathscr{F}(U) \end{array}$$

where the vertical left arrow is the product of the restriction maps of $\mathcal{O}_X$ and $\mathscr{F}$, the vertical right arrow is the restriction map of $\mathscr{F}$, and the horizontal arrows are the scalar multiplications of the modules.

A **morphism of $\mathcal{O}_X$-modules** is a morphism of sheaves $\varphi : \mathscr{F} \to \mathscr{G}$, where $\mathscr{F}, \mathscr{G}$ are $\mathcal{O}_X$-modules, such that for each open set $U \subseteq X$, $\varphi(U) : \mathscr{F}(U) \to \mathscr{G}(U)$ is a homomorphism of $\mathcal{O}_X(U)$-modules.

The category of $\mathcal{O}_X$-modules will be denoted by $\mathrm{Mod}_{\mathcal{O}_X}$. We also note that each stalk $\mathscr{F}_p$ has the structure of an $\mathcal{O}_{X,p}$-module by defining the scalar multiplication as follows

$$[(s, U)] \cdot [(f, V)] = [(s \cdot f, U \cap V)],$$

where $[(s, U)] \in \mathcal{O}_{X,p}$ and $[(f, V)] \in \mathscr{F}_p$. With this structure, if $\varphi : \mathscr{F} \to \mathscr{G}$ is a morphism of $\mathcal{O}_X$-modules, the induced map on stalks φ_p is a morphism of $\mathcal{O}_{X,p}$-modules.

Example 6.1.2.

1. The sheaf $\mathcal{O}_X$, seen as a sheaf of abelian groups, is an $\mathcal{O}_X$-module.

2. The $\mathcal{O}_X$-module $\mathscr{F}$ such that $\mathscr{F}(U) = \{0\}$ for all open subsets $U \subseteq X$ is called the **zero $\mathcal{O}_X$-module**, simply denoted by 0.

3. Let X be a topological space and let $\underline{\mathbb{Z}}$ be the constant sheaf of rings on X determined by $\mathbb{Z}$ (Example 2.1.11). Then a $\underline{\mathbb{Z}}$-module is simply a sheaf of abelian groups on X. (So everything we will prove for $\mathcal{O}_X$-modules, will be true also for sheaves of abelian groups.)

4. Let R be a ring and let $X = \{x\}$ be a topological space that consists of a single point. Let $\mathcal{O}_X$ be the sheaf of rings given by $\mathcal{O}_X(X) = R$ (and as always $\mathcal{O}_X(\emptyset) = 0$). Then an $\mathcal{O}_X$-module is just an R-module, in the sense that there is a bijection between R-modules and $\mathcal{O}_X$-modules defined by $M \mapsto \mathscr{F}$, where $\mathscr{F}(X) = M$.

5. Let $\pi : E \to X$ be a vector bundle of rank n on a smooth manifold X. Let $\mathcal{O}_X$ be the sheaf of rings of real-valued smooth functions on X, i.e. $\mathcal{O}_X(U) = C^\infty(U, \mathbb{R})$. The presheaf of local smooth sections on E

$$\mathscr{F}(U) = \{s : U \to E \text{ smooth}, \pi_{|U} \circ s = 1_U\}$$

with the obvious restriction maps, is a sheaf of abelian groups that is an $\mathcal{O}_X$-module. The operations are given as follows. For every $s, t \in \mathscr{F}(U)$, $f \in \mathcal{O}_X(U)$ and $x \in U$ with $s(x) = (x, v) \in \pi^{-1}(U) \simeq U \times \mathbb{R}^n$, $t(x) = (x, w) \in \pi^{-1}(U) \simeq U \times \mathbb{R}^n$ we have

- $(s + t)(x) = (x, v + w)$
- $(f \cdot s)(x) = (x, f(x)v)$

Sections of a vector bundle on a smooth manifold are also a motivating example for the notion of *locally free sheaf* of $\mathcal{O}_X$-modules (see Definition 6.1.10).

Definition 6.1.3 (Submodules)**.** An **$\mathcal{O}_X$-submodule** $\mathscr{F}$ of an $\mathcal{O}_X$-module $\mathscr{G}$ is a subsheaf of $\mathscr{G}$, that is, there is an injective morphism $\varphi : \mathscr{F} \to \mathscr{G}$ (Proposition-Definition 2.4.1). An **ideal sheaf** $\mathscr{I}$ is a $\mathcal{O}_X$-submodule of $\mathcal{O}_X$.

Definition 6.1.4 (Quotients)**.** Let $\mathscr{F}$ be an $\mathcal{O}_X$-submodule of an $\mathcal{O}_X$-module $\mathscr{G}$, and let $\varphi : \mathscr{F} \to \mathscr{G}$ be the corresponding injective morphism. The cokernel sheaf of φ with the scalar multiplication induced by $\mathscr{G}$ is again an $\mathcal{O}_X$-module, denoted by $\mathscr{G}/\mathscr{F} = \operatorname{coker} \varphi$, called the **quotient** of $\mathscr{G}$ by $\mathscr{F}$. The canonical morphism of sheaves $\mathscr{G} \to \mathscr{G}/\mathscr{F}$ is a morphism of $\mathcal{O}_X$-modules, called the **canonical projection**.

Remark 6.1.5 (The sheaf $\mathscr{H}om$ is an $\mathcal{O}_X$-module)**.** Let $\mathscr{F}$ and $\mathscr{G}$ be two $\mathcal{O}_X$-modules. We can endow the sheaf $\mathscr{H}om_{\mathrm{Mod}_{\mathcal{O}_X}}(\mathscr{F}, \mathscr{G})$ the structure of $\mathcal{O}_X$-module as follows. For every open subset $U \subseteq X$, let $h \in \mathscr{H}om_{\mathrm{Mod}_{\mathcal{O}_X}}(\mathscr{F}, \mathscr{G})(U) = \mathrm{Hom}_{\mathrm{Mod}_{\mathcal{O}_X}}(\mathscr{F}_{|U}, \mathscr{G}_{|U})$ and $s \in \mathcal{O}_X(U)$, for every open subset $V \subseteq U$ define

$$(s \cdot h)(V)(f) = \rho_{U,V}(s)h(V)(f).$$

Definition 6.1.6. If $\mathscr{F}$ is an $\mathcal{O}_X$-module, its **dual** is defined as $\mathscr{F}^\vee = \mathscr{H}om_{\mathrm{Mod}_{\mathcal{O}_X}}(\mathscr{F}, \mathcal{O}_X)$.

Example 6.1.7. Let $\mathscr{F}$ be an $\mathcal{O}_X$-module. We have $\mathscr{H}om_{\mathrm{Mod}_{\mathcal{O}_X}}(\mathcal{O}_X, \mathscr{F}) \simeq \mathscr{F}$, and the isomorphism $\varphi : \mathscr{H}om_{\mathrm{Mod}_{\mathcal{O}_X}}(\mathcal{O}_X, \mathscr{F}) \to \mathscr{F}$ is given by $\varphi(U)(f) = f(U)(1_{\mathcal{O}_X})$.

Proposition-Definition 6.1.8 (Direct sum)**.** *Let $\mathscr{F}$ and $\mathscr{G}$ be two $\mathcal{O}_X$-modules. The the presheaf $\mathscr{F} \oplus \mathscr{G}$ defined by*

$$(\mathscr{F} \oplus \mathscr{G})(U) = \mathscr{F}(U) \oplus \mathscr{G}(U),$$

*for every open set U, and $\rho_{V,U} = \rho'_{V,U} \oplus \rho''_{V,U}$ for every inclusion of open sets $U \subseteq V$, is a sheaf, called the **direct sum** of $\mathscr{F}$ and $\mathscr{G}$. The direct sum is both the product and the coproduct of $\mathscr{F}$ and $\mathscr{G}$ in the category of $\mathcal{O}_X$-modules.*

Proof. Let $\{U_i\}_{i \in I}$ be an open cover of an open set U of X, and for every $i \in I$, let $s_i = (s'_i, s''_i) \in (\mathscr{F} \oplus \mathscr{G})(U_i)$ such that $s_{i|U_i \cap U_j} = s_{j|U_i \cap U_j}$ for every $i, j \in I$. It follows that $s'_{i|U_i \cap U_j} = s'_{j|U_i \cap U_j}$ and $s''_{i\,|U_i \cap U_j} = s''_{j\,|U_i \cap U_j}$ for every $i, j \in I$. Since $\mathscr{F}$ and $\mathscr{G}$ are sheaves of abelian groups, there exist a unique $s = (s', s'') \in \mathscr{F}(U) \oplus \mathscr{G}(U) = (\mathscr{F} \oplus \mathscr{G})(U)$ such that $s_{|U_i} = s_i$.

Now let $\mathscr{H}$ be an $\mathcal{O}_X$-modules and let $\varphi' : \mathscr{H} \to \mathscr{F}$ and $\varphi'' : \mathscr{H} \to \mathscr{G}$ be morphisms of $\mathcal{O}_X$-modules. For each open subset U of X, we have maps $\varphi'(U) : \mathscr{H}(U) \to \mathscr{F}(U)$ and $\varphi''(U) : \mathscr{H}(U) \to \mathscr{G}(U)$. Since the direct sum is the product in the category of modules over a ring, there exists a unique map $\varphi(U) : \mathscr{H}(U) \to \mathscr{F}(U) \oplus \mathscr{G}(U)$ such that $\varphi = (\varphi', \varphi'')$. By the arbitrary choice of U, this gives us a unique map of $\mathcal{O}_X$-modules $\varphi : \mathscr{H} \to \mathscr{F} \oplus \mathscr{G}$ such that $\varphi = (\varphi', \varphi'')$. This proves that $\mathscr{F} \oplus \mathscr{G}$ is the product in the category of $\mathcal{O}_X$-modules of $\mathscr{F}$ and $\mathscr{G}$. The proof of the analogous statement for coproducts is similar since the direct sum is also the coproduct of two modules over a ring. $\square$

$\mathscr{F}^{\oplus n}$ will be a shorthand for $\underbrace{\mathscr{F} \oplus \cdots \oplus \mathscr{F}}_{n \text{ times}}$.

Definition 6.1.9 (Tensor product)**.** Let $\mathscr{F}$ and $\mathscr{G}$ be two $\mathcal{O}_X$-modules. The sheafification of the presheaf $\mathscr{F} \otimes^{\text{pre}}_{\mathcal{O}_X} \mathscr{G}$ defined by

$$(\mathscr{F} \otimes^{\text{pre}}_{\mathcal{O}_X} \mathscr{G})(U) = \mathscr{F}(U) \otimes_{\mathcal{O}_X(U)} \mathscr{G}(U),$$

for every open set U, and $\rho_{V,U} = \rho'_{V,U} \otimes \rho''_{V,U}$ for every inclusion of open sets $U \subseteq V$, is called the **tensor product** of $\mathscr{F}$ and $\mathscr{G}$, denoted by $\mathscr{F} \otimes_{\mathcal{O}_X} \mathscr{G}$.

$\mathscr{F}^{\otimes n}$ will be a shorthand for $\underbrace{\mathscr{F} \otimes_{\mathcal{O}_X} \cdots \otimes_{\mathcal{O}_X} \mathscr{F}}_{n \text{ times}}$.

6.1.1 Locally free sheaves

Definition 6.1.10. An $\mathcal{O}_X$-module $\mathscr{F}$ is **free** if it is isomorphic to $\mathcal{O}_X^{\oplus I}$, for some index set I. It is **locally free** if X can be covered by open sets U for which $\mathscr{F}_{|U}$ is a free $\mathcal{O}_{X|U}$-module. If for every open set U in the cover we have $\mathscr{F}_{|U} \simeq \mathcal{O}_{X|U}^{\oplus n}$ for some $n \in \mathbb{N}$, then $\mathscr{F}$ is a **locally free sheaf of rank** n. A locally free sheaf of rank 1 is also called an **invertible sheaf** or a **line bundle**.

Proposition 6.1.11. *Let $\mathscr{F}$ and $\mathscr{G}$ be two locally free sheaves on X of rank n and m respectively.*

1. *$\mathscr{H}om(\mathscr{F}, \mathscr{G})$ is locally free of rank nm.*

2. $\mathscr{F}^\vee = \mathscr{H}om(\mathscr{F}, \mathcal{O}_X)$ *is locally free of rank* n.

3. $\mathscr{F} \oplus \mathscr{G}$ *is locally free of rank* $n+m$.

4. $\mathscr{F} \otimes \mathscr{G}$ *is locally free of rank* nm.

Proof. Suppose $\{U_i\}_{i\in I}$ and $\{V_i\}_{i\in I}$ are open covers of X such that $\mathscr{F}_{|U_i} \simeq \mathcal{O}_{X|U_i}^{\oplus n}$ and $\mathscr{G}_{|V_i} \simeq \mathcal{O}_{X|V_i}^{\oplus m}$. To prove the above statements it is enough to consider the open cover $\{U_i \cap V_i\}_{i\in I}$ of X. $\square$

6.1.2 The $\widetilde{M}$ construction

The construction used in Proposition-Definition 3.4.10 can be generalized to every R-module M, as we now explain, and the proof is essentially the same, *mutatis mutandis*.

Proposition-Definition 6.1.12. *Let* M *be an* R*-module, set* $X = \operatorname{Spec} R$ *and let* $\mathcal{B} = \{X_f : f \in R\}$*. Let* $\widetilde{M}$ *be the* $\mathcal{B}$*-presheaf of abelian groups defined as follows. For every* $B \in \mathcal{B}$

$$\widetilde{M}(B) = S_B^{-1}M \simeq S_B^{-1}R \otimes_R M,$$

for every $B \subseteq B'$*, the restiction map* $\rho_{B',B}$ *is induced by the map of Lemma 3.4.8. Then* $\widetilde{M}$ *is a* $\mathcal{B}$*-sheaf of abelian groups. The induced sheaf of abelian groups by Theorem 2.5.4 that we continue to denote by* $\widetilde{M}$*, has the structure of* $\mathcal{O}_X$*-module.*

Note that with this definition, we have $\widetilde{R} = \mathcal{O}_X$.

6.2 Quasicoherent sheaves

Definition 6.2.1. Let X be a scheme. An $\mathcal{O}_X$-module $\mathscr{F}$ is **quasicoherent** if for every affine open subset $U \subseteq X$, $\mathscr{F}_{|U} \simeq \widetilde{M}$ for some $\mathcal{O}_X(U)$-module M.

Example 6.2.2 (Not every $\mathcal{O}_X$-module is quasicoherent)**.** Let $X = \operatorname{Spec} k[t]$ and let $\mathscr{F}$ be the skyscraper sheaf defined by the $k[t]$-module $k(t)$ at the origin (t) (Example 2.1.10). Then $\mathscr{F}$ is an $\mathcal{O}_X$-module that is not quasicoherent. In fact, X is an affine open set, and $\mathscr{F}_{|X} = \mathscr{F} \not\simeq \widetilde{k(t)}$, since $\mathscr{F}(X_t) = 0$, whereas $\widetilde{k(t)}(X_t) = k(t)_t = k(t)$.

From the definition, every locally free sheaf is quasicoherent, in fact $\mathcal{O}_X^{\oplus I} \simeq \widetilde{R^I}$.
If M is an R-module, then there exists an exact sequence of R-modules

$$R^J \to R^I \to M \to 0$$

for some index sets I and J.

Proposition 6.2.3. *Let* X *be a scheme and let* $\mathscr{F}$ *be an* $\mathcal{O}_X$*-module. The following are equivalent.*

1. $\mathscr{F}$ *is quasicoherent.*

2. *X can be covered by open subsets U in which there exists an exact sequence of $\mathcal{O}_{X|U}$-modules of the form*

$$\mathcal{O}_{X|U}^{\oplus J} \to \mathcal{O}_{X|U}^{\oplus I} \to \mathscr{F}_{|U} \to 0$$

where I and J are arbitrary index sets.

Proof. TO DO $\square$

Definition 6.2.4. Let M be an R-module. M is

- **finitely generated** if it has a finite number of generators, that is, there exists an exact sequence
$$R^n \to M \to 0$$

- **finitely presented** if it has a finite number of generators with finite relations, that is, there exists an exact sequence
$$R^m \to R^n \to M \to 0$$

- **coherent** if it is finitely generated and any map $R^n \to M$ (not necessarily surjective) has a finitely generated kernel.

It is clear from the definitions that

$$\text{coherent} \implies \text{finitely presented} \implies \text{finitely generated}$$

Proposition 6.2.5. *Let M be an R-module. If R is Noetherian then the following are equivalent*

1. *M is finitely generated,*

2. *M is finitely presented,*

3. *M is coeherent.*

Proof. Suppose that M is finitely generated and let $\varphi : R^n \to M$. Then $\ker\varphi \subseteq R^n$, since R is Noetherian, $\ker\varphi$ is finitely generated. $\square$

Definition 6.2.6. A quasicoherent sheaf $\mathscr{F}$ is **finite type** (resp. **finitely presented**, **coherent**) if for every affine open $U \subseteq X$, $\mathscr{F}(U)$ is a finitely generated (resp. finitely presented, coherent) $\mathcal{O}_X(U)$-module.

From the previous proposition, on a locally Noetherian scheme the three notions defined above are the same.

6.3 Quasicoherent sheaves on projective schemes

Let S be a graded ring, and let M be a graded S-module. We define a quasicoherent sheaf $\widetilde{M}$ on $X = \operatorname{Proj} S$ as follows. The restriction on each projective distinguished open set X_f is $\widetilde{(M_f)_0}$. Similarly as what we did for the Proj construction, we can define isomorphisms between overlaps $X_f \cap X_g = X_{fg}$ that satisfy the cocyle condition. By gluing everything together, we obtain a quasicoherent sheaf $\widetilde{M}$.

Proposition 6.3.1. *Let S be a graded ring and let M and N be two graded S-modules such that $M_d \simeq N_d$ for every $d \geq d_0$, for some $d_0 \in \mathbb{N}$. Then $\widetilde{M} \simeq \widetilde{N}$.*

Proof. Let $f \in S_+$ homogeneous with $\deg(f) = e$, we have

$$\begin{aligned} (M_f)_0 \simeq (M_{f^{d_0}})_0 = & \left\{ \frac{m}{f^{d_0 t}} : m \in M_{ed_0 t} \right\} \\ \simeq & \left\{ \frac{n}{f^{d_0 t}} : n \in N_{ed_0 t} \right\} \simeq (N_{f^{d_0}})_0 \simeq (N_f)_0. \end{aligned}$$

By varying $f \in S_+$, the above isomorphisms induce isomorphisms between $\widetilde{M}$ and $\widetilde{N}$ on the distinguished open sets, that glue to an isomorphism $\widetilde{M} \simeq \widetilde{N}$. □

Recall that if S is a graded ring and M is a graded S-module, the S-module $M(n)$, where

$$M(n)_m = M_{n+m}$$

for every $n, m \in \mathbb{Z}$, is again a graded module. In particular $S(n)$ for every $n \in \mathbb{Z}$ is a graded S-module.

Definition 6.3.2. Let S be a graded ring and $X = \operatorname{Proj} S$. For any $n \in \mathbb{Z}$ define

$$\mathcal{O}_X(n) = \widetilde{S(n)}.$$

The **twisting sheaf** is $\mathcal{O}_X(1)$, the **tautological line bundle** is $\mathcal{O}_X(-1)$. For any sheaf of $\mathcal{O}_X$-modules $\mathscr{F}$, the **twisted sheaf** is $\mathscr{F}(n) = \mathscr{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n)$.

Proposition 6.3.3. *Let S be a graded ring and let $X = \operatorname{Proj} S$. Assume that S is generated by S_1 as an S_0-algebra.*

1. *$\mathcal{O}_X(n)$ is an invertible sheaf on X.*
2. *$\widetilde{M}(n) \simeq \widetilde{M(n)}$, in particular $\mathcal{O}_X(n) \otimes \mathcal{O}_X(m) \simeq \mathcal{O}_X(n+m)$.*

Proof.

1. We want to show that the sheaf $\mathcal{O}_X(n)$ is locally free of rank 1, that is, X can be covered by open subsets U such that $\mathcal{O}_X(n)_{|U} \simeq \mathcal{O}_{X|U}$.

   The fact that S is generated in degree 1 is equivalent of saying that S_+ is generated by degree one elements. This implies that $\{X_f : f \in S_1\}$ is an open cover of X. In fact, if $p \in X$, then there exists some homogeneous element $f \in S_1$ such that $f \notin p$ (otherwise we would have $S_+ \subseteq p$), that is $p \in X_f \subseteq X$.

Let $f \in S_1$, recall that by definition we have

$$\mathcal{O}_X(n)_{|X_f} = \widetilde{S(n)}_{|X_f} = (\widetilde{S(n)_f})_0$$

where the last $\widetilde{\ }$ is interpreted in $\operatorname{Spec}(S_f)_0$. On the other hand, we have

$$\mathcal{O}_{X|X_f} = \operatorname{Spec}(S_f)_0 = \widetilde{(S_f)_0}.$$

Therefore, it suffices to prove that $(S(n)_f)_0 \simeq (S_f)_0$ as $(S_f)_0$-modules. We have

$$\begin{aligned}(S(n)_f)_0 &= \left\{ \frac{g}{f^m} : g \in S(n)_m = S_{n+m} \right\} \\ &= \left\{ f^n \frac{g}{f^{n+m}} : g \in S_{n+m} \right\} = f^n \cdot (S_f)_0 \simeq (S_f)_0.\end{aligned}$$

2. This follows from the fact that $\widetilde{M \otimes_S N} \simeq \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N}$ for any two graded S-modules M and N, when S is generated by S_1. Indeed for every $f \in S_1$ we have $\big((M \otimes_S N)_f\big)_0 = (M_f)_0 \otimes_{(S_f)_0} (N_f)_0$. $\square$

Example 6.3.4. Let k be a field and let $X = \mathbb{P}^1_k = \operatorname{Proj} S$ with $S = k[x_0, x_1]$. Consider the line bundle $\mathcal{O}_X(1)$. In order to describe it, consider the open cover of X given by the two distinguished open sets X_{x_0} and X_{x_1}. By definition, we have $\mathcal{O}_X(1)_{|X_{x_0}} \simeq (\widetilde{S(1)_{x_0}})_0$, in particular

$$\mathcal{O}_X(1)(X_{x_0}) = (S(1)_{x_0})_0 = \left\{ \frac{f}{x_0^t} : f \in S(1)_t = S_{t+1} \right\} = x_0 \cdot (S_{x_0})_0$$

(note that $x_0 \cdot (S_{x_0})_0 \simeq (S_{x_0})_0$ as $(S_{x_0})_0$-modules, see the proof of Proposition 6.3.3). In a similar way, we have

$$\mathcal{O}_X(1)(X_{x_1}) = (S(1)_{x_1})_0 = x_1 \cdot (S_{x_1})_0.$$

Now consider the distinguished open subset $X_{x_0x_1}$. We have two inclusion maps: $X_{x_0x_1} \to X_{x_0}$ and $X_{x_0x_1} \to X_{x_1}$ that correspond to the restriction maps

$$\begin{aligned}x_0 \cdot (S_{x_0})_0 &\longrightarrow x_0 \cdot \big((S_{x_0})_0\big)_{\frac{x_1}{x_0}} \simeq x_0 \cdot (S_{x_0x_1})_0 \\ x_0 \cdot (S_{x_1})_0 &\longrightarrow x_1 \cdot \big((S_{x_1})_0\big)_{\frac{x_0}{x_1}} \simeq x_1 \cdot (S_{x_0x_1})_0 \simeq x_0 \cdot (S_{x_0x_1})_0\end{aligned}$$

(here we used the ring isomorphism $(S_{fg})_0 \simeq \big((S_f)_0\big)_{g^{\deg f}/f^{\deg g}}$, see the proof of Lemma 3.9.9. Also, it might be useful to recall that $(S_{x_0})_0 \simeq k[\frac{x_1}{x_0}]$ and $(S_{x_0x_1})_0 \simeq k[\frac{x_0}{x_1}, \frac{x_1}{x_0}] = k[(\frac{x_1}{x_0})^{\pm 1}]$). Now, since $\{X_{x_0}, X_{x_1}, X_{x_0x_1}\}$ consistutes an open cover of X closed by intersection (Remark 2.1.7), the global sections $\mathcal{O}_X(1)(X)$ are the (inverse) limit of the following diagram of abelian groups

$$\begin{array}{ccc} & & x_1 \cdot (S_{x_1})_0 \\ & & \downarrow \\ x_0 \cdot (S_{x_0})_0 & \longrightarrow & x_0 \cdot (S_{x_0x_1})_0 \simeq x_1 \cdot (S_{x_0x_1})_0 \end{array}$$

(recall that $\mathcal{O}_X(X) \simeq k$, see Example 3.8.4, therefore $\mathcal{O}_X(1)(X)$ is a k-vector space). In other words, if $f \in \mathcal{O}_X(1)(X)$ is a global section, let

$$ax_0 + bx_1 + x_0 p, \text{ with } p \in k[x_1/x_0]_{\geq 2}$$

be its image in $x_0 \cdot (S_{x_0})_0$, and let

$$a'x_0 + b'x_1 + x_1 q, \text{ with } q \in k[x_0/x_1]_{\geq 2}$$

be its image in $x_1 \cdot (S_{x_1})_0$. This images must coincide in $x_0 \cdot (S_{x_0 x_1})_0 \simeq x_1 \cdot (S_{x_0 x_1})_0$, this means that $a = a'$, $b = b'$ and $p = q = 0$, therefore $f = ax_0 + bx_1 \in S_1$. This proves that $\mathcal{O}_X(1)(X) = S_1$.

Example 6.3.5. Let k be a field and let $X = \mathbb{P}^2_k = \operatorname{Proj} S$ with $S = k[x_0, x_1, x_2]$. Consider the line bundle $\mathcal{O}_X(1)$. In order to compute its global sections, we can proceed similarly as in the previous example. Consider the open cover

$$\{X_{x_0}, X_{x_1}, X_{x_2}, X_{x_0x_1}, X_{x_0x_2}, X_{x_1x_2}, X_{x_0x_1x_2}\}$$

which is closed by intersection. Then $\mathcal{O}_X(1)(X)$ is the inverse limit of the diagram

$$\begin{tikzcd}
x_0 \cdot (S_{x_0})_0 \arrow[d] \arrow[dr] & x_1 \cdot (S_{x_1})_0 \arrow[dl] \arrow[dr] & x_2 \cdot (S_{x_2})_0 \arrow[d] \arrow[dl] \\
x_0 \cdot (S_{x_0x_1})_0 \arrow[dr] & x_2 \cdot (S_{x_0x_2})_0 \arrow[d] & x_1 \cdot (S_{x_1x_2})_0 \arrow[dl] \\
& x_0 \cdot (S_{x_0x_1x_2})_0 & .
\end{tikzcd}$$

It might be useful to recall that

$$\begin{aligned}
(S_{x_0})_0 &\simeq k\left[\frac{x_1}{x_0}, \frac{x_2}{x_0}\right] \\
(S_{x_0x_1})_0 &\simeq k\left[\left(\frac{x_1}{x_0}\right)^{\pm 1}, \frac{x_2}{x_0}\right] \\
(S_{x_0x_1x_2})_0 &\simeq k\left[\left(\frac{x_1}{x_0}\right)^{\pm 1}, \left(\frac{x_2}{x_0}\right)^{\pm 1}\right].
\end{aligned}$$

Therefore, if $f \in \mathcal{O}_X(1)(X)$ is a global section, its images in $x_i \cdot (S_{x_i})_0$ for $i \in \{0, 1, 2\}$ are of the form

$$\begin{aligned}
ax_0 + bx_1 + cx_2 + p &\text{ with } p \in k[x_1/x_0, x_2/x_0]_{\geq 2} \\
a'x_0 + b'x_1 + c'x_2 + q &\text{ with } q \in k[x_0/x_1, x_2/x_1]_{\geq 2} \\
a''x_0 + b''x_1 + c''x_2 + r &\text{ with } r \in k[x_0/x_2, x_1/x_2]_{\geq 2}
\end{aligned}$$

these elements must coincide inside $x_0 \cdot (S_{x_0x_1x_2})_0$, therefore we get $a = a' = a''$, $b = b' = b''$, $c = c' = c''$ and $p = q = r = 0$, therefore $f = ax_0 + bx_1 + cx_2 \in S_1$. Therefore $\mathcal{O}_X(1)(X) = S_1$.

More in general, for $\mathbb{P}^n_k = \operatorname{Proj} S$ with $S = k[x_0, \ldots, x_n]$, for every $d \in \mathbb{Z}$ we have that

$$\mathcal{O}_{\mathbb{P}^n_k}(d)(\mathbb{P}^n_k) = S_d,$$

(note that if $d < 0$, then $S_d = 0$) where the previous formula can be deducted by applying the same methods used in the previous examples, noting that $(S(d)_{x_0})_0 = x_0^d \cdot (S_{x_0})_0$.

6.4 Divisors

6.4.1 Preliminaries on discrete valuation rings

Definition 6.4.1. Let K be a field. A **discrete valuation** on K is a surjective map $v : K^* \to \mathbb{Z}$ (where $K^* = K \setminus \{0\}$ is the multiplicative group of K) such that

1. $v(xy) = v(x) + v(y)$ (i.e. v is a group homomorphism),
2. $v(x + y) \geq \min(v(x), v(y))$.

The set

$$R = \{0\} \cup \{x \in K^* : v(x) \geq 0\}$$

is a ring, called the **valuation ring** of v.

An integral domain R is a **discrete valuation ring** (DVR) if there is a discrete valuation of its field of fractions K such that R is the valuation ring of v.

DVRs can be defined in many equivalent ways, as the following result, which we will not prove, shows.

Proposition 6.4.2. *Let R be a domain. The following conditions are equivalent:*

1. *R is a discrete valuation ring.*
2. *R is a Noetherian valuation ring.*
3. *R is a local principal ideal domain.*
4. *R is local, Noetherian and its maximal ideal is principal.*
5. *R is local, Noetherian, one-dimensional and integrally closed.*

Remark 6.4.3 (1-dim regular local rings $\Leftrightarrow$ DVRs)**.** From the Auslander-Buchsbaum theorem we know that every regular local ring is a UFD. This fact, together with the above proposition, imply that one dimensional regular local rings are discrete valuations rings. Note that the converse is also easily seen to be true: discrete valuation rings are regular local rings.

Definition 6.4.4. A ring R is **regular in codimension one** if for every prime ideal $p \in \operatorname{Spec} R$ of codimension (or height) one, the localization R_p is a (one-dimensional) regular local ring.

A scheme X is **regular in codimension one** if for every point $p \in X$ such that $\dim \mathcal{O}_{X,p} = 1$, the local ring $\mathcal{O}_{X,p}$ is regular.

The previous definition can be interpreted in the following way. We know that points on a scheme X correspond to irreducible closed subsets (Proposition 3.11.6). Therefore we can identify a point $p \in X$ with its corresponding irreducible closed subset $\overline{\{p\}} \subseteq X$. From Proposition 5.1.4, this subset has codimension one if and only if the dimension of the local ring $\mathcal{O}_{X,p}$ is one. In this case, we informally say that $p \in X$ is a "codimension one point", in the sense that its corresponding irreducible subset $\overline{\{p\}}$ has codimension one in X.

Remark 6.4.5. From Remark 6.4.3 we have that in a scheme X regular in codimension one, the one dimensional local rings $\mathcal{O}_{X,p}$ are DVRs, therefore they have a valuation $v : \mathcal{O}_{X,p} \to \mathbb{Z}$.

Remark 6.4.6. For a Noetherian domain the following implications hold:

$$\text{UFD} \implies \text{integrally closed} \implies \text{regular in codimension one.}$$

Hence for a locally Noetherian scheme, we have similar implications:

$$\text{factorial} \implies \text{normal} \implies \text{regular in codimension one.}$$

The implications of the previous remark are strict. Some examples are $\mathbb{Z}[\sqrt{-5}]$ for the first and $k[x^2, x^3, xy, y]$ for the second.

6.4.2 Weil divisors

In this section, X will be a Noetherian, integral, separated scheme which is regular in codimension one.

Definition 6.4.7. A **prime** (or **irreducible**) **divisor** on X is an integral closed subscheme of X of codimension one. A **Weil divisor** on X is an element of the free abelian group $\mathrm{WDiv}(X)$ generated by prime divisors. A Weil divisor $D \in \mathrm{WDiv}(X)$ will be written as a sum

$$D = \sum n_i Y_i \in \mathrm{WDiv}(X)$$

where $n_i \in \mathbb{Z}$, all but a finite number of which are zero. If $n_i \geq 0$ for all i, then D is **effective**, written $D \geq 0$, and for $D_1, D_2 \in \mathrm{WDiv}(X)$, by $D_1 \geq D_2$ we will mean $D_1 - D_2 \geq 0$.

Remark 6.4.8 (Valuation of rational functions)**.** Let Y be a prime divisor, and let $\eta_Y \in X$ be its generic point. Since X is regular in codimension one, from Remark 6.4.5 the local ring $\mathcal{O}_{X,\eta_Y}$ has a valuation $v_Y : \mathcal{O}_{X,\eta_Y} \to \mathbb{Z}$. Note also that, since X is integral, a rational function f of X is an element of the function field $K(X)$ (Remark 4.3.6). Now let U be an affine open subset that contains η_Y. From Proposition 3.11.7 we have that $K(X)$ is the fraction field of $\mathcal{O}_X(U)$, and from Proposition 3.4.11 we have that $\mathcal{O}_{X,\eta_Y}$ is the localization of $\mathcal{O}_X(U)$ at the prime ideal of $Y \cap U$. Therefore $K(X)$ is also the fraction field of $\mathcal{O}_{X,\eta_Y}$, hence it makes sense to consider $v_Y(f)$ (considering the valuation to be defined on the valuation field).

Proposition-Definition 6.4.9. *The function* $\operatorname{div} : K(X)^* \to \operatorname{WDiv}(X)$*, where*

$$\operatorname{div}(f) = \sum v_Y(f) Y$$

is well defined. Divisors of the form $\operatorname{div}(f)$ *are called* ***principal****.*

Proof. We have to show that if $f \in K(X)^*$ is a nonzero rational function of X, then $v_Y(f) = 0$ except finitely many prime divisors. Recall that $K(X) = \mathcal{O}_{X,\eta}$, where η is the generic point of X. Let $[(f', U_1)]$ and $[(1/f', U_2)]$ be representatives for f and $1/f$ respectively. The intersection $U = U_1 \cap U_2$ is nonempty, since X is irreducible. Therefore, $f'_{|U} \in \mathcal{O}_X(U)$ is invertible, in particular for any prime divisor Y contained in U we have $v_Y(f) = 0$. Now, the complement $Z = X \setminus U$ is a closed subset of X, thus it is a Noetherian topological space, since X is Noetherian. This implies that Z has finitely many irreducible components, that is, Z contains only a finite number of codimension one closed subschemes of X. □

Note that the function div is a homomorphism because of the properties of valuations. Therefore the group of principal divisors, denoted $\operatorname{Princ}(X)$, is a subgroup of $\operatorname{WDiv}(X)$.

Definition 6.4.10. Two Weil divisors $D_1, D_2 \in \operatorname{WDiv}(X)$ are **linearly equivalent** if $D_1 - D_2$ is principal. The quotient of the group of Weil divisors by linear equivalence is the **class group** of X, denoted $\operatorname{Cl}(X) = \operatorname{WDiv}(X)/\operatorname{Princ}(X)$.

Lemma 6.4.11 (Algebraic Hartogs's Lemma)**.** *Let R be an integrally closed Noetherian domain, then*

$$R = \bigcap_{\substack{p \in \operatorname{Spec} R \\ h(p)=1}} R_p.$$

Proposition 6.4.12. *Let R be an integrally closed, Noetherian, domain. Then R is a UFD if and only if* $\operatorname{Cl}(\operatorname{Spec} R) = 0$.

Proof. Recall that a Noetherian domain is a UFD if and only if every prime ideal of height one is principal. Thus, we will prove that $\operatorname{Cl}(\operatorname{Spec} R) = 0$ if and only if every prime ideal of R of height one is principal.

Necessity. Let p a prime ideal of height one, and let Y be the corresponding prime divisor on $\operatorname{Spec} R$. Then $Y = \operatorname{div}(f)$ for some $f \in Q(R)$. This means that $v_Y(f) = 1$, so $f \in R_p$ and f generates pR_p, and $v_{Y'}(f) = 0$ for any other prime divisor Y' that corresponds to a prime ideal p', so $f \in R_{p'}$. By the algebraic Hartogs's lemma 6.4.11, we have that $f \in R$, in particular $f \in R \cap pR_p = p$. Now if $g \in p$, then $v_Y(g) \geq 1$ and $v_{Y'}(g) \geq 0$. Hence $v_Z(g/f) \geq 0$ for any prime divisor Z of $\operatorname{Spec} R$. Similarly as before, we have $g/f \in R$, that is $g \in (f)$. This proves that $p = (f)$.

Sufficiency. Let Y be a prime divisor of $\operatorname{Spec} R$, then Y corresponds to a prime ideal p of height one, which by hypothesis is principal. If p is generated by $f \in R$, then $\operatorname{div}(f) = Y$. In fact, it is easy to see that $v_Y(f) = 1$. On the other hand, if Z is a prime divisor different from Y, then Z corresponds to a prime ideal $q \neq p$ of R of height 1, so $f \notin q$ and therefore f is invertible in R_q, that is, $v_Z(f) = 0$. This proves that any prime divisor is principal, that is $\operatorname{Cl}(\operatorname{Spec} R) = 0$. □

From the previuos proposition, it follows that $\mathrm{Cl}(\mathbb{A}^n_k) = \mathrm{Cl}(\operatorname{Spec} k[x_1, \dots, x_n]) = 0$ since $k[x_1, \dots, x_n]$ is a UFD.

Now consider the projective space $\mathbb{P}^n_k = \operatorname{Proj} S$ over a field k, where $S = k[x_0, \dots, x_n]$. $\mathbb{P}^n_k$ satisfies the standing hypothesis of this section. Recall that the ring $K(\mathbb{P}^n_k)$ of rational functions of $\mathbb{P}^n_k$ is, by definition, $Q(S)_0$, that is the degree zero part of the fraction field of S (recall that the grading of $Q(S)$ is given by $\deg(f/g) = \deg(f) - \deg(g)$). More explicitly, this is $k(x_0, \dots, x_n)_0$. Notice that $K(\mathbb{P}^n_k)$ is also isomorphic to $k(x_1, \dots, x_n)$, the isomorphism is given by the dehomogeneization map $x_i \mapsto x_i/x_0$. The prime divisors of $\mathbb{P}^n_k$ are the irreducible hypersurfaces, that corresponds to principal homogeneous prime ideals of S, i.e. irreducible homogeneous polynomials in S. We define the **degree** $\deg(Y)$ of a prime divisor Y of $\mathbb{P}^n_k$ to be the degree of its corresponding homogeneous polynomial in S.

Proposition 6.4.13. *Let $\mathbb{P}^n_k$ be the projective space over a field k. The function*

$$\deg : \mathrm{Cl}(\mathbb{P}^n_k) \to \mathbb{Z} \quad \textit{defined by} \quad \deg\left(\sum_i n_i[Y_i]\right) = \sum_i n_i \deg(Y_i)$$

is an isomorphism. In particular $\mathrm{Cl}(\mathbb{P}^n_k) \simeq \mathbb{Z}$.

Proof. Any divisor D of degree d (that is, $\deg([D]) = d$) can be writte as the difference of two effective divisors $D = D_1 - D_2$ of degree d_1 and d_2 respectively, with $d = d_1 - d_2$. Now let $D_i = \operatorname{div}(g_i)$ for $i \in \{1, 2\}$, and let $f = g_1/(g_2 x_0^d)$, then $\operatorname{div}(f) = D - d \cdot H$ where H is the hyperplane corresponding to the polynomial x_0. Hence $D \sim d \cdot H$. This proves all at once that deg is a well-defined isomorphism of groups. □

Let U be a nonempty open subscheme of X. It is not hard to see, just by using the definitions, that if Y is a prime divisor of X, then $Y \cap U$ is either empty or a prime divisor of U. Conversely, if Y' is a prime divisor of U, then its closure on X is a prime divisor of X (it is yet irreducible, of codimension one and a closed subscheme of X (TO DO: Vakil Exercise 8.1.M)). This means that the homomorphism $\mathrm{WDiv}(X) \to \mathrm{WDiv}(U)$ defined by $Y \mapsto Y \cap U$ for each prime divisor Y, is well-defined and surjective. The image of a divisor D via this map will be denoted by $D_{|U}$.

Definition 6.4.14. A Weil divisor $D \in \mathrm{WDiv}(X)$ is **locally principal** if X can be covered by open subsets U such that $D_{|U}$ is principal.

Since the restriction $D \mapsto D_{|U}$ is a homomorphism, the locally principal divisors are a subgoup of $\mathrm{WDiv}(X)$.

Proposition 6.4.15. *If X is factorial, then every Weil divisor is locally principal.*

Proof. Since the sum of two locally principal divisors is easily seen to be locally principal, it is enough to prove the statement for prime divisors. Let Y be a prime divisor of X. We have to find a cover of open subsets U of X in which $Y \cap U$ is principal. If $U = X \setminus Y$, then $Y \cap U = \emptyset = \operatorname{div}(1)$. Now let $y \in Y$ and let U be an open affine neighbourhood of y. Let p be the corresponding prime ideal of height one in $\mathcal{O}_X(U)$. Since $y \in Y$, the prime ideal p is contained in the prime ideal of y in $\mathcal{O}_X(U)$. This menas that the ideal p is nonzero in the local ring $\mathcal{O}_{X,y}$, which by hypothesis is a UFD. Therefore, p is principal in $\mathcal{O}_{X,y}$ generated by an element $f \in \mathcal{O}_{X,y} \subseteq K(X)$. Now the prime ideal p, seen as a point of $\operatorname{Spec} \mathcal{O}_X(U) = U \subseteq X$ is the generic point of Y. From the fact that the image of f in the local ring $\mathcal{O}_{X,p}$ generates the maximal ideal, we have $v_Y(f) = 1$. Thus, the coefficient of Y of the divisor $\operatorname{div}(f)$ is one. Further, the number of zeros and poles of $\operatorname{div}(f)$ is finite, so we can consider the open set V given by U minus all the zeros and poles of $\operatorname{div}(f)$ different from Y. By construction we have $Y_{|V} = \operatorname{div}(f)_{|V}$. By the arbitrary choice of $y \in Y$, this proves that Y is locally principal. $\square$

It is not hard to show that if $f \in K(X)^*$ with $\operatorname{div}(f) = \sum_i n_i Y_i$, then consider $f_{|U}$ (that is f seen as an element of $K(U) \simeq K(X)$), we have $\operatorname{div}(f_{|U}) = \sum_i n_i (Y_i \cap U)$. Hence, we have a well defined surjective homomorphism $\operatorname{Cl}(X) \to \operatorname{Cl}(U)$ given by $[Y] \mapsto [Y \cap U]$ for classes of prime divisors.

Proposition 6.4.16. *Let Z be a prime divisor of X. Then, there is an exact sequence*

$$\mathbb{Z} \to \operatorname{Cl}(X) \to \operatorname{Cl}(X \setminus Z) \to 0$$

where the first map is defined by $n \mapsto n \cdot [Z]$, and the second is the map described above.

Proof. It is enough to observe that the kernel of the map in the middle is the subgroup generated by $[Z]$. $\square$

Example 6.4.17. Let k be a field, let $R = k[x, y, z]/(xy - z^2)$ and let $X = \operatorname{Spec} R$. Note that X satisfies the standing hypothesis of this section. Consider the prime ideal (y, z) and let Y be the corresponding prime divisor of X. From proposition 6.4.16 we have the exact sequence

$$\mathbb{Z} \to \operatorname{Cl}(X) \to \operatorname{Cl}(X \setminus Y) \to 0.$$

Notice that $X \setminus Y = X \setminus V(y)$, in fact Y is (set-theoretically) equal to $V(y)$, even though the closed subscheme given by the ideal (y) is not reduced, so it is not (scheme-theoretically) equal to Y. It follows that $X \setminus Y = \operatorname{Spec} R_y = \operatorname{Spec} A[y^{\pm 1}, z]$ as $x = z^2/y$ in R_y. Thus $\operatorname{Cl}(X \setminus Y) = 0$ since R_y is a UFD. From the above exact sequence we have that $\operatorname{Cl}(X)$ is generated by $[Y]$. Now $2 \cdot [Y] = 0$ as the divisor of the rational function $y \in Q(R)$ is $2 \cdot Y$. Further, the class $[Y]$ is nonzero as the ideal (y, z) is not principal in R. It follows that

$$\operatorname{Cl}(X) \simeq \mathbb{Z}/2\mathbb{Z}.$$

6.4.3 Cartier divisors

In this section, X will be a scheme with no further hypothesis. Sometimes we will use the notation $\Gamma(U, \mathscr{F})$ to denote the sections over U of a sheaf $\mathscr{F}$ on X. Recall that, if R is a ring, we denote by $Q(R)$ its *total ring of fractions*, that is, its localization on the mulitplicative set consisting of all nonzero divisors. If R is a domain, it coincides with the field of fractions.

Proposition-Definition 6.4.18. *Let X be a scheme. The functor*

$$U \mapsto Q(\mathcal{O}_X(U))$$

is a presheaf on the base of affine open subsets of X. Its sheafification $\mathcal{K}_X$ is called the ***sheaf of total ring of fractions*** *of X.*

Proof. Let $V \subseteq U$ be two affine open subsets of X, and set $A = \mathcal{O}_X(U)$ and $B = \mathcal{O}_X(V)$. The restriction map $\rho_{U,V} : Q(A) \to Q(B)$ is the composition of the maps

$$Q(A) = S_A^{-1}A \to S_A^{-1}B \to S_B^{-1}B = Q(B),$$

where S_A (resp. S_B) is the set of nonzero divisors of A (resp. B), the first map is the localization at S_A of the restriction map $A \to B$, and the second map is the localization of $S_A^{-1}B$ at the (image of) S_B. Now let W be another affine subset of X that is contained in V, and set $C = \mathcal{O}_X(W)$. Then from the commutativity of the following diagram

$$\begin{tikzcd}
S_A^{-1}A \arrow[r] & S_A^{-1}B \arrow[r] \arrow[d] & S_A^{-1}C \arrow[d] \\
 & S_B^{-1}B \arrow[r] & S_B^{-1}C \arrow[d] \\
 & & S_C^{-1}C
\end{tikzcd}$$

it follows that $\rho_{U,W} = \rho_{V,W} \circ \rho_{U,V}$. $\square$

Remark 6.4.19 (Misconceptions about $\mathcal{K}_X$)**.** It is important to define the above presheaf just on the base of affine open subsets, as in general a presheaf defined by $U \mapsto Q(\mathcal{O}_X(U))$ for every open subset U of X may fail to exists! Further, it is not true that $\mathcal{K}_X(U) = Q(\mathcal{O}_X(U))$, even if U is affine. Finally, it is not true that $\mathcal{K}_{X,p} = Q(\mathcal{O}_{X,p})$ for every $p \in X$. For more information, see Kleiman's paper "Misconceptions about $\mathcal{K}_X$".

On an arbitrary scheme, the sheaf $\mathcal{K}_X$ plays the role of the function field $K(X)$ of an integral scheme. We denote by $\mathcal{K}_X^*$ (resp. $\mathcal{O}_X^*$) the sheaf (of multiplicative groups) of invertible elements in the sheaf (of rings) $\mathcal{K}_X$ (resp. $\mathcal{O}_X$). Note that $\mathcal{O}_X^*$ is a subsheaf of $\mathcal{K}_X^*$, since $\mathcal{O}_X^*(U)$ is a subgroup of $\mathcal{K}_X^*(U)$, so there exists an injective morphism of sheaves $\mathcal{O}_X^* \to \mathcal{K}_X^*$. The cokernel sheaf (or quotient sheaf) $\mathcal{K}_X^*/\mathcal{O}_X^*$ is the sheafification of the cokernel presheaf $U \mapsto \mathcal{K}_X^*(U)/\mathcal{O}_X^*(U)$, and we have the following exact sequence of sheaves

$$0 \to \mathcal{O}_X^* \to \mathcal{K}_X^* \to \mathcal{K}_X^*/\mathcal{O}_X^* \to 0.$$

Definition 6.4.20. A **Cartier divisor** on a scheme X is an element of $\Gamma(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$, that is, a global section of the quotient sheaf $\mathcal{K}_X^*/\mathcal{O}_X^*$. A Cartier divisor is **principal** if it is in the image of the map $\Gamma(X, \mathcal{K}_X^*) \to \Gamma(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$. Two Cartier divisors are **linearly equivalent** if their difference is principal. (Although the group operation on $\mathcal{K}_X^*/\mathcal{O}_X^*$ is multiplication, we will use the language of additive groups when speaking of Cartier divisors, so as to preserve the analogy with Weil divisors.)

We want to express in a more explicit way the data of a Cartier divisor. Recall that $\mathcal{K}_X^*/\mathcal{O}_X^*$ is the sheafification of the presheaf defined by $U \mapsto \mathcal{K}_X^*(U)/\mathcal{O}_X^*(U)$. In general, for a presheaf $\mathscr{F}$, a section of the sheafification of $\mathscr{F}$ is a compatible germ (Proposition 2.3.10). The data of a compatible germ on U can be expressed as a family of pairs (f_i, U_i) where $\{U_i\}_i$ is an open cover of U and $f_i \in \mathscr{F}(U_i)$ such that $f_i = f_j$ on $U_i \cap U_j$ (see Definition 2.3.4 or the proof of Porposition 2.3.5).

In our case, this means that a Cartier divisor consists of the data $\{([f_i], U_i)\}_i$, where $\{U_i\}_i$ is an open cover of X, and $[f_i] \in \mathcal{K}_X^*(U_i)/\mathcal{O}_X^*(U_i)$ such that $[f_i] = [f_j]$ on $U_i \cap U_j$, that is $(f_i/f_j)_{|U_i \cap U_j} \in \mathcal{O}_X^*(U_i \cap U_j)$.

Equivalently, a Cartier divisor on X consists of the data $\{(f_i, U_i)\}_i$, where $\{U_i\}_i$ is an open cover of X, and $f_i \in \mathcal{K}_X^*(U_i)$ are such that $(f_i/f_j)_{|U_i \cap U_j} \in \mathcal{O}_X^*(U_i \cap U_j)$.

Example 6.4.21. Let k be a field and let $X = \mathbb{P}_k^2 = \operatorname{Proj} S$ with $S = k[x_0, x_1, x_2]$. In this case we have $\mathcal{K}_X = K(X) \simeq k(x_1/x_0, x_2/x_0)$. The three distinguished open subsets $X_{x_0}, X_{x_1}, X_{x_2}$ constitute an open cover of X. Then the data

$$\left\{(X_{x_0}, 1), \left(X_{x_1}, \frac{x_0}{x_1}\right), \left(X_{x_2}, \frac{x_0}{x_2}\right)\right\}$$

represents a Cartier divisor H. In fact, the restrictions of x_0/x_1 on $X_{x_0} \cap X_{x_1} = X_{x_0 x_1}$ and of x_0/x_2 on $X_{x_0} \cap X_{x_2} = X_{x_0 x_2}$ are both invertible, and the restrictions of x_0/x_1 and x_0/x_2 on $X_{x_1 x_2}$ are obtained one from another by multiplying an invertible element of $\mathcal{O}_X^*(X_{x_1 x_2})$. Notice that H is effective (Definition 6.4.32), since

$$\begin{aligned}
&1 \in \mathcal{O}_X^*(X_{x_0}),\\
&\frac{x_0}{x_1} \in \mathcal{O}_X^*(X_{x_1}) \simeq k\left[\frac{x_0}{x_1}, \frac{x_2}{x_1}\right],\\
&\frac{x_0}{x_2} \in \mathcal{O}_X^*(X_{x_2}) \simeq k\left[\frac{x_0}{x_2}, \frac{x_1}{x_2}\right].
\end{aligned}$$

Further, H corresponds to the coordinate hyperplane of equation $x_0 = 0$.

Proposition 6.4.22 (Cartier divisors are locally principal Weil divisors)**.** *Let X be a Noetherian, integral, separated, normal scheme. Then, there exists an injective homomorphism $\Gamma(X, \mathcal{K}_X^*/\mathcal{O}_X^*) \to \operatorname{WDiv}(X)$ such that its image is the subgroup of locally principal divisors.*

Proof. First, note that X is regular in codimension one (Remark 6.4.6), therefore it makes sense to talk about Weil divisors. Now let $\{(f_i, U_i)\}_i$ be a Cartier divisor. Note that X is quasicompact since it is Noetherian, therefore we can assume that the open cover $\{U_i\}_i$ is finite. Since X is integral, the sheaf $\mathcal{K}_X^*$ is the constant sheaf on X determined by the function field $K(X)$. We have $\mathcal{K}_X^*(U) = K(X)$ for every open subset U, in particular $f_i \in K(X)$ for every i. We define the associated Weil divisor D as follows. For each prime divisor Y, take the coefficient of Y to be $v_Y(f_i)$, where i is any index for which $Y \cap U_i \neq \emptyset$. This is choice produce a well defined Weil divisor, in fact if j is another such index, then both f_i/f_j and f_j/f_i, restricted to $U_i \cap U_j$, are in $\mathcal{O}_X^*(U_i \cap U_j)$. This means that $f_i/f_{j|U_i \cap U_j}$ is invertible on $\mathcal{O}_X(U_i \cap U_j)$, that is $v_Y(f_i/f_j) = 0$, which implies $v_Y(f_i) = v_Y(f_j)$. The divisor $D = \sum v_Y(f_i)Y$ is clearly locally principal, since $D_{|U_i} = \operatorname{div}(f_i)$. Further, from the properties of valuations, it is also clear that the map just defined is a homomorphism.

For injectivity, let $\{(U_i, f_i)\}_i$ and $\{(V_i, g_i)\}_i$ be two Cartier divisors. Up to refining the open covers, we can assume that $U_i = V_i$ for all i and that they are affine. Now suppose that these Cartier divisors are mapped to the same locally principal Weil divisor. We have to show that f_i and g_i represents the same class in $K(X)/\mathcal{O}_X(U_i) = K(U_i)/\mathcal{O}_X(U_i)$, that is $f_i/g_i \in \mathcal{O}_X(U_i)$. By hypothesis we have $v_Y(f_i) = v_Y(g_i)$, that is $v_Y(f_i/g_i) = 0$, for some prime divisor Y such that $Y \cap U_i \neq \emptyset$. This implies that $f_i/g_i \in \mathcal{O}_{X,p_Y}$, where p_Y is the height one prime ideal in $\mathcal{O}_X(U_i)$ corresponding to $Y \cap U_i$. By the arbitrary choice of Y, we have

$$f_i/g_i \in \bigcap_{\substack{p \in \operatorname{Spec} \mathcal{O}_X(U_i) \\ h(p)=1}} \mathcal{O}_X(U_i)_p = \mathcal{O}_X(U_i),$$

where we used the fact that U_i is affine, and that $\mathcal{O}_X(U_i)$ is an integrally closed Noetherian domain, so the algebraic Hartogs's lemma applies (Lemma 6.4.11). □

Corollary 6.4.23. *Let X be an Noetherian, integral, separated, factorial scheme. Then the group of Cartier divisors is isomorphic to the group of Weil divisors.*

Proof. Follows from Proposition 6.4.15 and 6.4.22. □

6.4.4 Line bundles associated to a divisor

Recall that a line bundle is an $\mathcal{O}_X$-module locally free of rank 1.

Proposition-Definition 6.4.24. *Let X be a ringed space. The isomorphism classes of invertible sheaves (line bundles) on X form a group with respect to the tensor product, called the* ***Picard group*** $\operatorname{Pic}(X)$ *of X.*

Proof. The fact that the tensor product of two line bundles is a line bundle follows from Proposition 6.1.11. Further, from the definition, the tensor product is clearly associative. The identity element is $\mathcal{O}_X$. The inverse of a line bundle $\mathscr{L}$ is its dual $\mathscr{L}^\vee = \mathscr{H}om(\mathscr{L}, \mathcal{O}_X)$. In fact, TO DO □

Definition 6.4.25. Let D be a Cartier divisor on a scheme X, represented by $\{(U_i, f_i)\}_i$. We define the **sheaf associated** to D as the subsheaf $\mathcal{O}_X(D)$ of $\mathcal{K}_X$ defined to be: on U_i, the sub-$\mathcal{O}_X$-module of $\mathcal{K}_X$ generated by f_i^{-1}. This is well-defined, since f_i/f_j is invertible on $U_i \cap U_j$, so f_i^{-1} and f_j^{-1} generate the same $\mathcal{O}_X$-module on $U_i \cap U_j$.

Proposition 6.4.26. *Let X be a scheme. The map $D \mapsto \mathcal{O}_X(D)$ that assign to each Cartier divisor its associated sheaf, satisfies the following properties:*

1. *$\mathcal{O}_X(D)$ is an invertible subsheaf of $\mathcal{K}_X$;*
2. *it is bijective (on the invertible subsheaves of $\mathcal{K}_X$);*
3. *$\mathcal{O}_X(D_1 + D_2) = \mathcal{O}_X(D_1) \otimes \mathcal{O}_X(D_2)$;*
4. *$D_1 \sim D_2$ if and only if $\mathcal{O}_X(D_1) \simeq \mathcal{O}_X(D_2)$ as abstract invertible sheaves (disregarding the embedding in $\mathcal{K}_X$)*

Proof.

1. 2. Suppose that D is represented by $\{(U_i, f_i)\}_i$. By definition, $\mathcal{O}_X(D)_{|U_i}$ is the principal sub-$\mathcal{O}_X$-module of $\mathcal{K}_{X|U_i}$ generated by ${f_i^{-1}}_{|U_i}$. Therefore, it is isomorphic to $\mathcal{O}_{X|U_i}$. Hence, $\mathcal{O}_X(D)$ is an invertible subsheaf of $\mathcal{K}_X$. Now let $\mathscr{L}$ be an invertible subsheaf of $\mathcal{K}_X$. By definition, there exists an open cover $\{U_i\}_i$ such that $\mathscr{L}_{|U_i} \simeq \mathcal{O}_{X|U_i}$. Since $\mathscr{L}$ is a subsheaf of $\mathcal{K}_X$, this means that $\mathscr{L}_{|U_i} \simeq \widetilde{(f_i^{-1})}$, for some $f_i \in \mathcal{K}_X(U_i)$ (where the $\sim$ construction is thought in $\mathcal{O}_{X|U_i} = \mathcal{O}_{U_i}$). By construction, for any other index j, f_i/f_j is invertible in $\mathcal{O}_X(U_i \cap U_j)$. Therefore, we have constructed a unique Cartier divisor, thus the map $D \mapsto \mathcal{O}_X(D)$ is bijective.

3. It is enough to observe that $\widetilde{(f_i^{-1})} \otimes \widetilde{(g_i^{-1})} \simeq \widetilde{(f_i^{-1} g_i^{-1})}$.

4. From the previous point, we have $D_1 \sim D_2$ if and only if $D = D_1 - D_2$ is principal, if and only if

$$\mathcal{O}_X \simeq \widetilde{(f^{-1})} \simeq \mathcal{O}_X(D) = \mathcal{O}_X(D_1 - D_2) = \mathcal{O}_X(D_1) \otimes \mathcal{O}_X(D_2)^{-1}.$$ □

Corollary 6.4.27. *On a scheme X, the map $D \mapsto \mathcal{O}_X(D)$ gives an injective homomorphism from the group of Cartier divisors modulo linear equivalence* $\operatorname{CaCl}(X)$ *to* $\operatorname{Pic}(X)$.

Example 6.4.28. Let X and H be as in Example 6.4.21. By comparing with Example 6.3.5 it is not hard to show that $\mathcal{O}_X(H) \simeq \mathcal{O}_X(1)$. More generally, if $X = \mathbb{P}^n$ then $\mathcal{O}_X(d) \simeq \mathcal{O}_X(dH)$ where H is the divisor of an hyperplane.

Proposition 6.4.29. *On an integral scheme X, every invertible sheaf is a subsheaf of $\mathcal{K}_X = \underline{K(X)}$.*

Proof. Let $\mathscr{L}$ be an invertible sheaf, and consider the sheaf $\mathscr{L} \otimes \mathcal{K}_X$. On any open set U where $\mathscr{L} \simeq \mathcal{O}_X$, we have $\mathscr{L} \otimes \mathcal{K}_X \simeq \mathcal{K}_X$, so it is a constant sheaf on U. Now because X is irreducible, this means that the sheaf $\mathscr{L} \otimes \mathcal{K}_X$ is constant. Thus $\mathscr{L} \otimes \mathcal{K}_X$ is isomorphic to the constant sheaf $\mathcal{K}_X$, and the natural map $\mathscr{L} \to \mathscr{L} \otimes \mathcal{K}_X \simeq \mathcal{K}_X$ expresses $\mathscr{L}$ as a subsheaf of $\mathcal{K}_X$. □

Corollary 6.4.30. *If X is an integral scheme, the map* $\operatorname{CaCl}(X) \to \operatorname{Pic}(X)$ *is an isomorphism.*

Corollary 6.4.31. *If X is a Noetherian, integral, separated, factorial scheme, then* $\operatorname{Cl}(X) \simeq \operatorname{Pic}(X)$.

Proof. This follows from Corollary 6.4.30 and 6.4.23. □

Definition 6.4.32. A Cartier divisor on a scheme X is **effective** if it can be represented by $\{(U_i, f_i)\}_i$ where $f_i \in \mathcal{O}_X(U_i)$ for all i. In that case, we define the **associated subscheme** of codimension one Y, to be the closed subscheme defined by the sheaf of ideals $\mathscr{I}_Y$ which is locally generated by f_i.

In the hypothesis of Corollary 6.4.23, effective Cartier divisors correspond exactly to effective Weil divisors. Note also that, if Y is the associated closed subscheme of an effective Cartier divisor D, then by definition $\mathscr{I}_Y \simeq \mathcal{O}_X(-D)$.

The definition of associated sheaf of a divisor can be extended also to Weil divisors, in the following way. Let X be a Noetherian, integral, separated normal scheme. Let D be a Weil divisor on X. Define the **sheaf associated** to D by

$$\Gamma(U, \mathcal{O}_X(D)) = \{f \in K(X)^* : \mathrm{div}(f)_{|U} + D_{|U} \geq 0\} \cup \{0\}.$$

It is not hard to see that if D is locally principal (i.e. it is a Cartier divisor), this definition coincides with the one given previously.

6.5 Pullback and pushforward of $\mathcal{O}_X$-modules

Let $f : A \to B$ be a morphism of rings, and let M be a B-module. Clearly, M has a structure of an A-module as well. In other words, ther exists a functor $\cdot_A : \mathrm{Mod}_B \to \mathrm{Mod}_A$ that sends a B-module M into itself with its natural A-module structure, using the notation M_A. On the other hand, if N is an A-module, then $N \otimes_A B$ is a B-module, in other words we can consider the functor $\cdot \otimes_A B : \mathrm{Mod}_A \to \mathrm{Mod}_B$.

Lemma 6.5.1. *Let $f : A \to B$ be a morphism of rings. Then the functor $\cdot \otimes_A B$ is left adjoint to the functor $\cdot_A$ as described above.*

Proof. Let N be an A-module and M be a B-module. Notice that there is a canonical map of A-modules $\eta_N : N \to (N \otimes_A B)_A$ defined by $n \mapsto n \otimes 1_B$. From Theorem 1.6.7 we just need to show that for every morphism of A-modules $p : N \to M_A$ there exists a unique morphism of A-modules $q : (N \otimes_A B)_A \to M_A$ such that $p = q \circ \eta_N$. From this last equality it follows that the map q is the unique A-module morphism defined by $n \otimes b \mapsto b \cdot p(n)$. □

Proposition 6.5.2. *Let $\pi : X \to Y$ be a morphism of ringed spaces, let $\mathscr{F}$ be a (pre)sheaf of $\mathcal{O}_X$-modules and $\mathscr{G}$ be a (pre)sheaf of $\mathcal{O}_Y$-modules. Then, the pushforward $\pi_*\mathscr{F}$ is a (pre)sheaf of $\pi_*\mathcal{O}_X$-modules, and the inverse image (pre)sheaf $\pi^{-1}\mathscr{G}$ (resp. $\pi^{-1}_{\mathrm{pre}}\mathscr{G}$) is a (pre)sheaf of $\pi^{-1}\mathcal{O}_Y$-modules (resp. $\pi^{-1}_{\mathrm{pre}}\mathcal{O}_Y$-modules).*

Proof. TO DO □

Similarly as in the case for rings, if $\mathcal{O}_X \to \mathcal{O}_Y$ is a morphism of sheaves of rings, then every (pre)sheaf $\mathscr{G}$ of $\mathcal{O}_Y$-modules has the structure of (pre)sheaf of $\mathcal{O}_X$-modules. In other words, there exist two restriction functors $\cdot_{\mathcal{O}_X} : \mathrm{PMod}_{\mathcal{O}_Y} \to \mathrm{PMod}_{\mathcal{O}_Y}$ and $\cdot_{\mathcal{O}_X} : \mathrm{Mod}_{\mathcal{O}_Y} \to \mathrm{Mod}_{\mathcal{O}_Y}$. On the other hand, we have already defined the two tensor product functors $\cdot \otimes^{\mathrm{pre}}_{\mathcal{O}_X} \mathcal{O}_Y : \mathrm{PMod}_{\mathcal{O}_X} \to \mathrm{PMod}_{\mathcal{O}_Y}$ and $\cdot \otimes_{\mathcal{O}_Y} \mathcal{O}_X : \mathrm{Mod}_{\mathcal{O}_X} \to \mathrm{Mod}_{\mathcal{O}_Y}$.

Corollary 6.5.3. *Let $\mathcal{O}_X \to \mathcal{O}_Y$ be a morphism of sheaves of rings.*

1. *The functor $\cdot \otimes^{\mathrm{pre}}_{\mathcal{O}_X} \mathcal{O}_Y$ is left adjoint to the restriction functor $\cdot_{\mathcal{O}_X}$.*

2. *The functor $\cdot \otimes_{\mathcal{O}_X} \mathcal{O}_Y$ is left adjoint to the restriction functor $\cdot_{\mathcal{O}_X}$.*

Proof.

1. It follows from Lemma 6.5.1.

2. It follows from point 1, that $\cdot \otimes_{\mathcal{O}_X} \mathcal{O}_Y = \mathrm{sh} \circ (\cdot \otimes^{\mathrm{pre}}_{\mathcal{O}_X} \mathcal{O}_Y)$ and the fact that the sheafification functor is left adjoint to the inclusion functor (Proposition 2.3.12).

□

Definition 6.5.4. Let $\pi : X \to Y$ be a morphism of ringed spaces and let $\mathscr{G}$ be an $\mathcal{O}_Y$-module. The **pullback** of the $\mathcal{O}_Y$-module $\mathscr{G}$ is defined as the following $\mathcal{O}_X$-module

$$\pi^*\mathscr{G} = \pi^{-1}\mathscr{G} \otimes_{\pi^{-1}\mathcal{O}_Y} \mathcal{O}_X.$$

Proposition 6.5.5 (π^* and π_* are adjoints)**.** *Let $\pi : X \to Y$ be a morphism of ringed spaces. The functor $\pi^* : \mathrm{Mod}(\mathcal{O}_Y) \to \mathrm{Mod}(\mathcal{O}_X)$ is left adjoint to $\pi_* : \mathrm{Mod}(\mathcal{O}_X) \to \mathrm{Mod}(\mathcal{O}_Y)$. In other words, for every $\mathcal{O}_X$-module $\mathscr{F}$ and every $\mathcal{O}_Y$-module $\mathscr{G}$ there is a bijection*

$$\mathrm{Hom}_{\mathrm{Mod}(\mathcal{O}_X)}(\pi^*\mathscr{G}, \mathscr{F}) \simeq \mathrm{Hom}_{\mathrm{Mod}(\mathcal{O}_Y)}(\mathscr{G}, \pi_*\mathscr{F})$$

satisfying (1.1).

Proof. We have

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{Mod}(\mathcal{O}_X)}(\pi^*\mathscr{G}, \mathscr{F}) &= \mathrm{Hom}_{\mathrm{Mod}(\mathcal{O}_X)}(\pi^{-1}\mathscr{G} \otimes_{\pi^{-1}\mathcal{O}_Y} \mathcal{O}_X, \mathscr{F}) \\
&\simeq \mathrm{Hom}_{\mathrm{Mod}(\pi^{-1}\mathcal{O}_Y)}(\pi^{-1}\mathscr{G}, \mathscr{F}_{\pi^{-1}\mathcal{O}_Y}) \\
&\simeq \mathrm{Hom}_{\mathrm{Mod}(\pi^{-1}\mathcal{O}_Y)}(\mathrm{sh}\, \pi^{-1}_{\mathrm{pre}}\mathscr{G}, \mathscr{F}_{\pi^{-1}\mathcal{O}_Y}) \\
&\simeq \mathrm{Hom}_{\mathrm{PMod}(\pi^{-1}_{\mathrm{pre}}\mathcal{O}_Y)}(\pi^{-1}_{\mathrm{pre}}\mathscr{G}, \mathcal{F}(\mathscr{F}_{\pi^{-1}\mathcal{O}_Y})) \\
&\simeq \mathrm{Hom}_{\mathrm{PMod}(\mathcal{O}_Y)}(\mathscr{G}, \pi_*\mathcal{F}(\mathscr{F}_{\pi^{-1}\mathcal{O}_Y})) \\
&\simeq \mathrm{Hom}_{\mathrm{Mod}(\mathcal{O}_Y)}(\mathscr{G}, \pi_*\mathscr{F}).
\end{aligned}$$

□

Remark 6.5.6 (Global sections as morphisms $\mathcal{O}_X \to \mathscr{F}$)**.** Suppose that $\mathscr{F}$ is a $\mathcal{O}_X$-module. Then, a global section of $\mathscr{F}$ is the data of a morphism $\mathcal{O}_X \to \mathscr{F}$. In fact, note that if R is a ring and M is an R-module, then maps of the form $f : R \to M$ are in bijection with elements of M, the bijection is given by $f \mapsto f(1)$. Now a morphism $\mathcal{O}_X \to \mathscr{F}$ consists of, for each open subset U of X, a map $\mathcal{O}_X(U) \to \mathscr{F}(U)$, that is the data of an section $s(U) \in \mathscr{F}(U)$. Since the maps $\mathcal{O}_X(U) \to \mathscr{F}(U)$ commute with the restriction maps, we have that $s(U) = \rho_{X,U}(s(X))$, therefore the family of sections $\{s(U)\}_U$ consists of the data of a single global section $s \in \mathscr{F}(X)$.

Theorem 6.5.7. *Let $\pi : X \to Y$ be a morphism of ringed spaces and let $\mathscr{G}$ be an $\mathcal{O}_Y$-module. Then*

1. *$\pi^*\mathcal{O}_Y \simeq \mathcal{O}_X$.*

2. *If $\mathscr{G}$ is a locally free sheaf of rank r, then so is $\pi^*\mathscr{G}$.*

3. *For every global section $s : \mathcal{O}_Y \to \mathscr{G}$ (see Remark 6.5.6), there is a natural global section $\pi^*s : \mathcal{O}_X \to \pi^*\mathscr{G}$ called the **pullback** of the section s.*

4. *The pullback π^* is a right-exact functor.*

Proof.

1. $\pi^*\mathcal{O}_Y = \pi^{-1}\mathcal{O}_Y \otimes_{\pi^{-1}\mathcal{O}_Y} \mathcal{O}_X \simeq \mathcal{O}_X$.

2. Let U be an open subset of Y such that $\mathscr{G}_{|U} \simeq \mathcal{O}_{Y|U}^{\oplus r}$. Then, we have

$$(\pi^*\mathscr{G})_{|\pi^{-1}(U)} \simeq \pi^*_{|\pi^{-1}(U)}(\mathscr{G}_{|U}) \simeq \pi^*_{|\pi^{-1}(U)}(\mathcal{O}_{Y|U}^{\oplus r}) \simeq \mathcal{O}_{X|\pi^{-1}(U)}^{\oplus r}$$

where in the last isomorphism, we used point 1 (with respect to the morphism of ringed spaces $\pi_{|\pi^{-1}(U)} : \pi^{-1}(U) \to U$). Finally, note that if $\{U_i\}_i$ is an open cover of Y, then $\{\pi^{-1}(U_i)\}_i$ is an open cover of X.

3. The fact that $\pi^*s : \mathcal{O}_X \to \pi^*\mathscr{G}$ is a global section of $\pi^*\mathscr{G}$ follows from point 1 and the discussion above.

4. It follows from the fact that π^{-1} is exact (since exactness can be checked on the stalks, and π^{-1} preserves the stalks), and the fact that $\cdot_{\pi^{-1}\mathcal{O}_Y}\mathcal{O}_X$ is right-exact. $\square$

Remark 6.5.8 (Alternative definition of pullback of sections)**.** Let $\pi : X \to Y$ be a morphism of ringed spaces, let $\mathscr{G}$ be an $\mathcal{O}_Y$-module and let $s \in \mathscr{G}(Y)$ be a global section. We can define the pullback π^*s of the section s in the following alternative way. Since π^* and π_* are adjoints, there is a canonical map $\eta_{\mathscr{G}} : \mathscr{G} \to \pi_*\pi^*\mathscr{G}$ given by the unit η of the adjunction. By taking the global sections, we have a map

$$\eta_{\mathscr{G}}(Y) : \mathscr{G}(Y) \to \pi_*\pi^*\mathscr{G}(Y) = \pi^*\mathscr{G}(\pi^{-1}(Y)) = \pi^*\mathscr{G}(X),$$

and we define $\pi^*s = (\eta_{\mathscr{G}}(Y))(s)$.

Definition 6.5.9. A morphism of ringed spaces $\pi : X \to Y$ is **flat** if the functor π^* from the category of quasicoherent sheaves on Y to the category of quasicoherent sheaves on X is exact.

6.6 Globally generated line bundles

Definition 6.6.1. Let X be a scheme, and let $\mathscr{F}$ be an $\mathcal{O}_X$-module. We say that $\mathscr{F}$ is **globally generated** if there is a surjective morphism

$$\phi : \bigoplus_{i\in I} \mathcal{O}_X \to \mathscr{F}.$$

If the index set I is finite, we say that $\mathscr{F}$ is **finitely globally generated**.

Note that, from Remark 6.5.6 we can view global sections as morphisms $s_i : \mathcal{O}_X \to \mathscr{F}$, therefore a family of global sections $\{s_i\}_{i\in I}$ give rise to a morphism $\phi : \bigoplus_{i\in I} \mathcal{O}_X \to \mathscr{F}$ defined, for every open subset U of X, by

$$\phi(U) : \bigoplus_{i\in I} \mathcal{O}_X(U) \to \mathscr{F}(U), \quad \phi(U)((r_i)_{i\in I}) = \sum_{i\in I} s_i(U)(r_i).$$

Conversely, any morphism $\phi : \oplus_{i\in I}\mathcal{O}_X \to \mathscr{F}$ give rise to a family of sections $\{s_i\}_{i\in I}$ by splitting ϕ component by component. Surjectivity implies that for every $p \in X$, the images of the global sections $\{s_i\}_{i\in I}$ generate the stalk $\mathscr{F}_p$ (as an $\mathcal{O}_{X,p}$-module), see Proposition-Definition 2.4.2.

Definition 6.6.2. Let $\mathscr{F}$ be an $\mathcal{O}_X$-module. A section $f \in \mathscr{F}(U)$ **vanish** at a point $p \in X$ if $f_p \in \mathfrak{m}_p \mathscr{F}_p$.

Definition 6.6.3. Let $\mathscr{L}$ be an invertible sheaf on a scheme X. The points $p \in X$ in which all the global sections of X vanish are called **base points** of $\mathscr{L}$, and the set of base points is called **base locus** of $\mathscr{L}$. If $\mathscr{L}$ has no base points, it is **base point free**.

Proposition 6.6.4. *An invertible sheaf $\mathscr{L}$ on a scheme X is globally generated if and only if it is base point free.*

Proof. Since $\mathscr{L}$ is locally free of rank 1, for every $p \in X$ we have $\mathscr{L}_p \simeq \mathcal{O}_{X,p}$. Now $\mathscr{L}$ is globally generated if and only if $\mathscr{L}_p \simeq \mathcal{O}_{X,p}$ is generated by stalks of global sections, that is, if there exists $f \in \mathscr{L}(X)$ such that $f_p \notin \mathfrak{m}_p \mathscr{L}_p$. $\square$

Example 6.6.5. Let k be a field and $X = \mathbb{P}^n_k = \operatorname{Proj} S$ with $S = k[x_0, \ldots, x_n]$, then $\mathcal{O}_X(1)$ is globally generated by the global sections $x_0, \ldots, x_n$.

Theorem 6.6.6 (Serre). *Let S be a graded ring generated in degree 1, finitely generated over S_0. Let $\mathscr{F}$ be any finite type quasicoherent sheaf on $\operatorname{Proj} S$. Then there exists some n_0 such that for all $n \geq n_0$, $\mathscr{F}(n) = \mathscr{F} \otimes \mathcal{O}(n)$ is finitely globally generated.*

Proof. TO DO. $\square$

6.7 (Very) ample line bundles

Let R be a ring, and consider the projective space $\mathbb{P}^n_R = \operatorname{Proj} R[x_0, \ldots, x_n]$. On $\mathbb{P}^n_R$ we have the invertible sheaf $\mathcal{O}_{\mathbb{P}^n_R}(1)$ which is globally generated by $x_0, \ldots, x_n \in \Gamma(\mathbb{P}^n_R, \mathcal{O}_{\mathbb{P}^n_R}(1))$. The next theorem, shows that the data of a morphism of R-schemes $\varphi : X \to \mathbb{P}^n_R$ is the same as the data of an invertible sheaf $\mathscr{L}$ on X and global sections $s_0, \ldots, s_n \in \Gamma(X, \mathscr{L})$ that generate the sheaf $\mathscr{L}$.

Theorem 6.7.1. *Let R be a ring and let X be a scheme over R.*

1. *If $\varphi : X \to \mathbb{P}^n_R$ is an R-morphism, then $\varphi^* \mathcal{O}_{\mathbb{P}^n_R}(1)$ is an invertible sheaf on X, which is generated by the global sections $s_i = \varphi^*(x_i)$ for $i \in \{0, 1, \ldots, n\}$.*

2. *Conversely, if $\mathscr{L}$ is an invertible sheaf on X, and $s_0, \ldots, s_n \in \Gamma(X, \mathscr{L})$ generate $\mathscr{L}$, then there exists a unique R-morphism $\varphi : X \to \mathbb{P}^n_R$ such that $\mathscr{L} \simeq \varphi^* \mathcal{O}_{\mathbb{P}^n_R}(1)$ and $s_i = \varphi^*(x_i)$ under this isomorphism.*

Proof.

1. The sections $\{x_i\}_i$ generates $\mathcal{O}_{\mathbb{P}^n_R}(1)$, so the morphism $\oplus_{i=0}^n x_i : \oplus_{i=0}^n \mathcal{O}_{\mathbb{P}^n_R} \to \mathcal{O}_{\mathbb{P}^n_R}(1)$ is surjective. Now, the morphism $\bigoplus_{i=0}^n \varphi^*(x_i) : \oplus_{i=0}^n \mathcal{O}_X \to \varphi^*(\mathcal{O}_{\mathbb{P}^n_R}(1))$ is surjective since φ^* is right-exact, that is, $\{\varphi^*(x_i)\}_i$ generates $\varphi^*(\mathcal{O}_{\mathbb{P}^n_R}(1))$. The fact that $\varphi^*(\mathcal{O}_{\mathbb{P}^n_R}(1))$ is an invertible sheaf follows from Theorem 6.5.7.

2. For each $i \in \{0, \dots, n\}$ let $X_i = \{p \in X : (s_i)_p \notin \mathfrak{m}_p \mathscr{L}_p\}$. Then X_i is an open subset of X (TO DO) and from Proposition 6.6.4, $\{X_i\}_i$ is an open cover of X. Let U_i be the distinguished open set of $\mathbb{P}^n_R$ associated to x_i. Recall that

$$U_i \simeq \operatorname{Spec}(R_{x_i})_0 = \operatorname{Spec} R\left[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}\right].$$

Note that, by the definition of X_i, the global section s_i viewed as a morphism and restricted to X_i, $s_{i|X_i} : \mathcal{O}_{X|X_i} \to \mathscr{L}_{|X_i}$ is an isomorphism, in fact, for every $p \in X_i$, $(s_i)_p$ generates $\mathscr{L}_p$, since $(s_i)_p \notin \mathfrak{m}_p \mathscr{L}_p$. In other words, $\mathscr{L}_{|X_i} \simeq s_i \cdot \mathcal{O}_{X|X_i}$, so the transition functions on $X_i \cap X_j$ are (the image of) s_j/s_i. It makes sense to consider elements of the form s_j/s_i since (the image of) s_i on X_i is invertible. Now define a ring homomorphism $R[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}] \to \mathcal{O}_X(X_i)$ by sending $x_j/x_i \mapsto s_j/s_i$. This makes sense since $\mathscr{L}$ is locally free of rank 1, so the quotient s_i/s_j is a well-defined element of $\mathcal{O}_X(X_i)$. From Proposition 4.1.2 this ring homomorphism determines a morphism of schemes $\varphi_i : X_i \to U_i$. Finally, these morphisms glue to a morphism $\varphi : X \to \mathbb{P}^n_R$. It is clear from the construction (TO DO) that φ is an R-morphism, $\varphi^* \mathcal{O}_{\mathbb{P}^n_R}(1) \simeq \mathscr{L}$, and that $\varphi^*(x_i) = s_i$. □

Definition 6.7.2. Let X be a scheme over Y. An invertible sheaf $\mathscr{L}$ on X is **very ample** realtive to Y, if there is a closed embedding $i : X \to \mathbb{P}^n_Y$ such that $i^* \mathcal{O}_{\mathbb{P}^n_Y}(1) \simeq \mathscr{L}$. The sheaf $\mathscr{L}$ is **ample** if $\mathscr{L}^{\otimes n}$ is very ample, for some $n > 0$. A Cartier divisor D on X is **ample** or **very ample** if so is $\mathcal{O}_X(D)$.

Corollary 6.7.3 (very ample $\Rightarrow$ globally generated)**.** *If $\mathscr{L}$ is a very ample line bundle on an R-scheme X, with R a ring, then $\mathscr{L}$ is globally generated.*

Proof. By hypothesis, there exists a closed embedding $i : X \to \mathbb{P}^n_R = \operatorname{Proj} R[x_0, \dots, x_n]$ such that $i^* \mathcal{O}(1) \simeq \mathscr{L}$, therefore $\mathscr{L}$ is globally generated by $\{i^*(x_j)\}_j$. □

Lemma 6.7.4. *Let $\mathscr{L}$ and $\mathscr{L}'$ be two line bundles on X. If $\mathscr{L}$ is very ample and $\mathscr{L}'$ is globally generated then $\mathscr{L} \otimes \mathscr{L}'$ is very ample.*

Proof. TO DO □

Theorem 6.7.5. *Let X be a Noetherian scheme over* $\operatorname{Spec} A$*, with structure morphism* $X \to \operatorname{Spec} A$ *proper. For a line bundle $\mathscr{L}$ on X the following are equivalent:*

1. *$\mathscr{L}$ is ample.*
2. *$\mathscr{L}^{\otimes n}$ is very amply for $n \gg 0$.*
3. *For all finite type quasicoherent sheaves $\mathscr{F}$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $\mathscr{F} \otimes \mathscr{L}^{\otimes n}$ is globally generated.*
4. *The open subsets $X_f = \{p \in X : f(p) \neq 0\}$ for all $f \in \mathscr{L}^{\otimes n}(X)$ and all $n > 0$, form a base for the topology of X.*
5. *The affine open subsets of the form $X_f = \{p \in X : f(p) \neq 0\}$ for all $f \in \mathscr{L}^{\otimes n}(X)$ and all $n > 0$, form a base for the topology of X.*

Proof. TO DO □

6.8 Linear systems

In this section, X is a smooth projective variety over an algebraically closed field k. In this hypothesis the notion of Weil and Cartier divisors coincide. Furthermore, the class group is isomorphic to the Picard group. Recall that, since X is a k-scheme, the ring of sections $\mathcal{O}_X(U)$, for some open subset $U \subseteq X$, is a k-algebra. Therefore, if $\mathscr{F}$ is an $\mathcal{O}_X$-module, $\mathscr{F}(U)$ is a module over a k-algebra, in particular it is a k-vector space. If $\mathscr{F}$ is in addition an invertible sheaf, then $\Gamma(X, \mathscr{F})$ is a finite dimensional k-vector space (TO DO).

Definition 6.8.1. Let $\mathscr{L}$ e an invertible sheaf on X, and let $s \in \Gamma(X, \mathscr{L})$ be a nonzero section of $\mathscr{L}$. We define the **divisor of zeros** $D = (s)_0$ of s as follows. Over any open subset $U \subseteq X$ where $\mathscr{L}$ is trivial, let $\varphi : \mathscr{L}_{|U} \to \mathcal{O}_{X|U}$ be an isomorphism of sheaves. Then $\varphi(s_{|U}) \in \Gamma(U, \mathcal{O}_{X|U})$. As U ranges over a covering of X, the collection $\{(U, \varphi(s_{|U}))\}$ determines an effective Cartier divisor D on X. Indeed, φ is determined up to multiplication by an element of $\Gamma(U, \mathcal{O}^*_{X|U})$, so we get a well-defined Cartier divisor.

Proposition 6.8.2. *Let D be a divisor on X represented by $\{(U_i, f_i)\}_i$, and let $\mathscr{L} = \mathcal{O}_X(D)$ be its associated invertible sheaf. Then:*

1. *for each nonzero $s \in \Gamma(X, \mathscr{L})$, the divisor of zeros $(s)_0$ is an effective divisor linearly equivalent to D;*
2. *every effective divisor linearly equivalent to D is of the form $(s)_0$ for some section $s \in \Gamma(X, \mathscr{L})$;*
3. *two sections $s, s' \in \Gamma(X, \mathscr{L})$ have the same divisor of zeros if and only if there is a $\lambda \in k^*$ such that $s' = \lambda s$.*

Proof.

1. We have to show that the Cartier divisor $(s)_0 - D$ is principal. This divisor is represented by $\{(U_i, \varphi_i(s_{|U_i})/f_i)\}$, where $\varphi_i : \mathscr{L}_{|U_i} \to \mathcal{O}_{X|U_i}$ is the isomorphism defined by multiplying by f_i. Thus, on $U_i \cap U_j = U_{ij}$ we have $\varphi_i(s_{|U_{ij}})/f_i = \varphi_j(s_{|U_{ij}})/f_j$, therefore these sections glue together, that is, $(s)_0 - D$ is principal.
2. Let E be an effective divisor linearly equivalent to D, which, up to refining the open coverings, we can assume it is represented by $\{(U_i, g_i)\}$. This means that there exists $f \in K(X)^*$ such that $E = D + \operatorname{div}(f) \geq 0$, in particular $\operatorname{div}(f) \geq -D$ TO DO

$\square$

Definition 6.8.3. A **complete linear system** on X is defined as the set of all effective divisors linearly equivalent to some given divisor D. It is denoted by $|D|$.

From the previous proposition we see that there is a one to one correspondence between the sets $|D|$ and $(\Gamma(X, \mathscr{L}) \setminus \{0\})/k^*$, giving $|D|$ the structure of (the set of closed points of a) projective space over k.

Definition 6.8.4. A **linear system** $\mathfrak{d}$ on X is a linear subspace of the projective space $|D|$ for some divisor D on X. The dimension of $\mathfrak{d}$ is its dimension as a linear projective space.

In other words, a linear system is a vector subspace $V \subseteq \Gamma(X, \mathscr{L})$. The dimension of a linear system is finite since $\Gamma(X, \mathscr{L})$ is a finite dimensional vector space.

Example 6.8.5. Let $X = \mathbb{P}^n$, and let H be the divisor of an hyperplane. Then $|dH|$ is a complete linear system of dimension $\binom{n+d}{n} - 1$. It corresponds to the invertible sheaf $\mathcal{O}_X(d)$ (Example 6.4.28), whose global sections consists of the space of all homogenoeus polynomials of degree d.

6.9 Relative Proj

Definition 6.9.1. Let X be a scheme. A sheaf of $\mathcal{O}_X$-algebras is a sheaf of rings on X that has the structure of $\mathcal{O}_X$-modules. A sheaf of graded $\mathcal{O}_X$-algebras is a sheaf (in the category) of graded rings that has the structure of $\mathcal{O}_X$-module.

In this section, X is a Noetherian scheme, $\mathscr{S}$ is a quasi-coherent sheaf of $\mathcal{O}_X$-modules, which has the structure of a sheaf of graded $\mathcal{O}_X$-algebras, thus $\mathscr{S} = \bigoplus_{d \geq 0} \mathscr{S}_d$, where $\mathscr{S}_d$ is the homogeneous part of degree d. Furthermore, we assume that $\mathscr{S}_0 = \mathcal{O}_X$, that $\mathscr{S}_1$ is a coherent $\mathcal{O}_X$-module, and that $\mathscr{S}$ is locally generated by $\mathscr{S}_1$ as an $\mathcal{O}_X$-algebra (it follows that $\mathscr{S}_d$ is coherent for all $d \geq 0$).

Lemma 6.9.2. *Let $U \subsetneq V$ be affine open subsets of X. The morphism of graded rings $\phi : \mathscr{S}(V) \to \mathscr{S}(U)$ induces an open embedding $\varphi : \operatorname{Proj} \mathscr{S}(U) \to \operatorname{Proj} \mathscr{S}(V)$ such that $\operatorname{Proj} \mathscr{S}(U)$ is the fibre product $\operatorname{Proj} \mathscr{S}(V) \times_V U$ in the diagram*

$$\begin{tikzcd}
\operatorname{Proj} \mathscr{S}(U) \arrow[r, "\varphi"] \arrow[d, "\pi_U"] & \operatorname{Proj} \mathscr{S}(V) \arrow[d, "\pi_V"] \\
U \arrow[r, "\subseteq"] & V
\end{tikzcd}$$

Proof. First of all, since $\mathscr{S}$ is quasicoherent and V is affine, we have $\mathscr{S}_{|V} \simeq \widetilde{S}$, for some graded ring $S = \bigoplus_{i \geq 0} S_i$ generated in degree one with $S_0 = \mathcal{O}_X(V)$. The map π_V is the canonical map

$$\pi_V : \operatorname{Proj} \mathscr{S}(V) = \operatorname{Proj} S \longrightarrow \operatorname{Spec} S_0 = \operatorname{Spec} \mathcal{O}_X(V) = V$$

from Corollary 4.1.4.The definition of π_U is analogous. The map $\phi : \mathscr{S}(V) \to \mathscr{S}(U)$ is such that $\emptyset = V^{\mathbb{P}}(\phi(S_+)) \subseteq \operatorname{Proj} \mathscr{S}(U)$ (WHY?). Therefore, from Proposition 4.1.7, it induces a morphism of schemes $\varphi : \operatorname{Proj} \mathscr{S}(U) \to \operatorname{Proj} \mathscr{S}(V)$. Now if we restrict to distinguished open subsets V' and $U' = U \cap V'$ such that the corresponding open subset of $\operatorname{Proj} \mathscr{S}(V)$ is affine, say $\operatorname{Spec} S'$ for some ring S', and the corresponding open subset of $\operatorname{Proj} \mathscr{S}(U)$ is affine, say $\operatorname{Spec} S_U^{-1} S'$ for some multiplicative set $S_U \subseteq \mathcal{O}_X(V')$ (note that S' is a $\mathcal{O}_X(V')$-algebra). We have tje diagram

$$\begin{tikzcd}
\operatorname{Spec} S_U^{-1} S' \arrow[r] \arrow[d] & \operatorname{Spec} S' \arrow[d] \\
U' \arrow[r] & V'
\end{tikzcd}$$

that corresponds to the diagram in the category of rings

$$\begin{tikzcd} S_U^{-1}S' & S' \arrow[l] \\ S_U^{-1}\mathcal{O}_X(V') \arrow[u] & \mathcal{O}_X(V') \arrow[l] \arrow[u] \end{tikzcd}$$

note that $\mathcal{O}_X(U') = S_U^{-1}\mathcal{O}_X(V')$. The previous diagram is a fibre product diagram, since $S_U^{-1}S \simeq S_U^{-1}\mathcal{O}_X(V') \otimes_{\mathcal{O}_X(V')} S'$. By the arbitrary choice of the distinguished open subset V', it follows that $\operatorname{Proj} \mathscr{S}(U)$ is the fibre product $\operatorname{Proj} \mathscr{S}(V) \times_V U$. Further, since open embeddings are preserved under base change (Lemma 4.5.2) and $U \subseteq V$ is clearly an open embedding, φ is an open embedding as well.
(TO DO: of course I am cheating here with these distinguished open substes V' and U', even though the idea should be correct, or at least it works with relative Spec)

□

Proposition-Definition 6.9.3. *The morphisms $\pi_U : \operatorname{Proj} \mathscr{S}(U) \to U$, where U ranges over the affine subsets of X, can be glued together to form a scheme $\mathbf{Proj}\, \mathscr{S}$, that we call **relative Proj** of $\mathscr{S}$, and a **structure morphism** $\pi : \mathbf{Proj}\, \mathscr{S} \to X$.*

Proof. From Lemma 6.9.2, for U and V affine open subsets of X, $\operatorname{Proj} \mathscr{S}(U \cap V)$ is an open subscheme of both $\operatorname{Proj} \mathscr{S}(U)$ and $\operatorname{Proj} \mathscr{S}(V)$, and these open embeddings satisfy the cocyle conditions (WHY?), therefore, from Proposition 3.7.2 these glue together into a scheme $\mathbf{Proj}\, \mathscr{S}$. The family of morphisms $\pi_U : \operatorname{Proj} \mathscr{S}(U) \to U$ coincide in the overlaps (inside $\mathbf{Proj}\, \mathscr{S}$), therefore from Proposition 3.7.3 they glue together into a morphism $\pi : \mathbf{Proj}\, \mathscr{S} \to X$. □

6.10 Blowing up

Definition 6.10.1. Let R be a ring, let $I \subseteq R$ be an ideal and let t be an indeterminate. The **Rees algebra** (or **blow-up algebra**) of I is the graded subalgebra of $R[t]$ defined by

$$R[It] = \bigoplus_{n \geq 0} I^n t^n = \left\{ \sum_i a_i t^i : a_i \in I^i \right\}$$

where $I^0 = R$.

The previous blow-up algebra construction can be done also relatively to a quasicoherent sheaf of ideals $\mathcal{I}$ of a scheme X.

Proposition-Definition 6.10.2. *Let X be a Noetherian scheme, and let $\mathcal{I}$ be a coherent sheaf of ideals of X. The sheaf of graded algebras*

$$\mathcal{O}_X[\mathcal{I}t] = \bigoplus_{n \geq 0} \mathcal{I}^n t^n$$

is quasicoherent, and such that $\mathcal{O}_X[\mathcal{I}t]_1$ is coherent and locally generates $\mathcal{O}_X[\mathcal{I}t]$ as an $\mathcal{O}_X$-algebra. It makes sense to consider

$$\operatorname{Bl}_Y X = \mathbf{Proj}\, \mathcal{O}_X[\mathcal{I}t]$$

*that we define to be the **blow-up** of X along Y, or with respect to $\mathcal{I}$, where Y is the closed subscheme of X corresponding to $\mathcal{I}$.*

Proof. TO DO $\square$

Example 6.10.3. Consider $\mathbb{A}^n_k = \operatorname{Spec} R$ with $R = k[x_1, \dots, x_n]$. Let $I = (x_1, \dots, x_n)$ be the ideal associated to the origin $P = (0, \dots, 0) \in \mathbb{A}^n_k$. Then, $\mathcal{I} = \widetilde{I}$ and we have

$$\mathrm{Bl}_P \mathbb{A}^n_k = \operatorname{Proj} R[It].$$

The map $\mathrm{Bl}_P \mathbb{A}^n_k \to \mathbb{A}^n_k$ is given by the inclusion map of rings $R \to R[It]$. Further, we can define a surjective map of graded rings $\varphi : R[y_1, \dots, y_n] \to R[It]$ defined by $y_i \mapsto x_i t$. Thus $\mathrm{Bl}_Y X$ is isomorphic to a closed subscheme of $\mathbb{P}^{n-1}_R = \mathbb{P}^{2n-1}_k$, defined by the homogeneous polynomials which generates the kernel of φ, it is easy to see that they are $\{x_i y_j - x_j y_i : 1 \leq i \leq j \leq n\}$.

Definition 6.10.4. Let $f : X \to Y$ be a morphism of schemes, and let $\mathcal{I}$ be and sheaf of ideals on Y. We define the **inverse image ideal sheaf** $f^{-1}\mathcal{I} \cdot \mathcal{O}_X$ to be the ideal sheaf generated by the image of the map $f^*\mathcal{I} \to \mathcal{O}_X$ coming from the pullback of the inclusion $\mathcal{I} \subseteq \mathcal{O}_Y$. Alternatively, it is the ideal sheaf generated by the image of $f^{-1}\mathcal{I}$ under the map $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$.

The inverse image ideal sheaf $f^{-1}\mathcal{I} \cdot \mathcal{O}_X$ does not always coincide with the inverse image sheaf $f^*\mathcal{I}$, since the pullback functor is not always left exact, so $f^*\mathcal{I}$ is not always a subsheaf of $\mathcal{O}_X$, as the following example shows.

Example 6.10.5. Let k be a field and let $\phi : k[x] \to k \simeq k[x]/(x)$ be the canonical projection. Let $f : \operatorname{Spec} k \to \operatorname{Spec} k[x]$ be the corresponding morphism of affine schemes. Consider the ideal $I = (x) \subseteq k[x]$ and let $\mathcal{I} = \widetilde{I}$. Note that $I \simeq k[x]$ as a $k[x]$-module, therefore $\mathcal{I} \simeq \mathcal{O}_{\operatorname{Spec} k[x]}$, and so $f^*\mathcal{I} \simeq f^*\mathcal{O}_{\operatorname{Spec} k[x]} \simeq \mathcal{O}_{\operatorname{Spec} k}$. On the other hand, it is easy to verify that $f^{-1}\mathcal{O}_{\operatorname{Spec} k[x]} = \widetilde{k[x]_{(x)}}$ and $f^{-1}\mathcal{I} = \widetilde{(x)k[x]_{(x)}}$, where bot of the $\sim$ constructions are taken over $\mathcal{O}_{\operatorname{Spec} k}$. The image of $f^{-1}\mathcal{I}$ under the map $f^{-1}\mathcal{O}_{\operatorname{Spec} k[x]} \to \mathcal{O}_{\operatorname{Spec} k}$ is zero, thus

$$f^*\mathcal{I} \neq f^{-1}\mathcal{I} \cdot \mathcal{O}_{\operatorname{Spec} k}.$$

TO DO: inverse image ideal sheaf corresponds (in the affine case) to the ideal extension (for $\operatorname{Spec} B \to \operatorname{Spec} A$ corresponding to $A \to B$, the inverse image ideal sheaf of an ideal I of A is the ideal in B generated by the image of I in B) (is this true if we look at affine open subsets U: $(\pi^{-1}\mathcal{I} \cdot \mathcal{O}_X)(U) = \mathcal{I}(U)^e$?)

Proposition-Definition 6.10.6. *Let X be a Noetherian scheme, $\mathcal{I}$ a coherent sheaf of ideals, Y its associated closed subscheme, and let $\pi : \mathrm{Bl}_Y X \to X$ be the blow-up of X along Y. The inverse image ideal sheaf $\pi^{-1}\mathcal{I} \cdot \mathcal{O}_{\mathrm{Bl}_Y X}$ is an invertible sheaf on $\mathrm{Bl}_Y X$. We denote by $E_Y X$ (or simply by E) its associated closed subscheme, which is an effective Cartier divisor called the **exceptional divisor** of $\mathrm{Bl}_Y X$.*

Proof. The blow-up $\mathrm{Bl}_Y X$ has an invertible sheaf $\mathcal{O}_{\mathrm{Bl}_Y X}(1)$ defined as follows: for every open affine subset $U \subseteq X$, set $R = \mathcal{O}_X(U)$ and $I = \mathcal{I}(U)$, define

$$\mathcal{O}_{\mathrm{Bl}_Y X}(1)_{|U} = \widetilde{R[It]}(1) = \mathcal{O}_{\operatorname{Proj} R[It]}(1).$$

From Proposition 6.3.1, we have that $\widetilde{R[It]}(1) \simeq I \cdot \widetilde{R[It]}$, where the last coincide with the inverse image ideal sheaf $\pi^{-1}\mathcal{I} \cdot \mathcal{O}_{\mathrm{Bl}_Y X}$ (WHY?). $\square$