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ARF PROPERTY FOR QUADRATIC QUOTIENTS OF THE REES ALGEBRA

TESI DI LAUREA

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Introduction

The value semigroup of an analytically irreducible ring is a numerical semigroup. Thus, the study of this class of rings is intimately related to the study of numerical semigroups. One of the main applications of these numerical and algebraic tools is the classification of curve singularities. The usual way to study a curve singularity is to blow-up the curve in its singular point. The blow-up change the space in which the curve lies. In this new space the singular point is substituted with the so called *exceptional divisor*. The information is derived from the multiplicity of the points in the intersection of the transformed curve with the exceptional divisor. If one of these points is again singular, we can reiterate the process on it. For this purpose, we restrict our attention to such pieces of information at each singular point on the curve. This corresponds algebraically to assume that (the completion of) the coordinate ring of the curve, localized at the singular point, is analytically irreducible. In this manner, we carry out a sequence of blow-ups, and consequently a sequence of multiplicity of the point under study. This sequence of integers corresponds to the so called *multiplicity sequence* of the coordinate ring and its valuesemigroup. Roughly speaking, the classification is obtained by comparing this sequences, so two singularities are considered to be equivalent if they share the same multiplicity sequence. This is the main idea behind the solution of the classification problem for curve singularities given by Arf [1] (for a better heuristic exposition see [31]). Later, Lipman [24] defined the notion of *Arf ring* in a quite more general context, and the numerical counterpart is the notion of *Arf numerical semigroup* (see for instance [4], [30]). In addition, the definition of Arf numerical semigroup can be further generalized to the more general context of patterns on numerical semigroups [9].

The aim of this thesis is to present some original results about characterizing the Arf property in certain family of rings [6], and some related developments of the theory of patterns on numerical semigroups [7].

More precisely, the structure of this thesis is the following. In Chapter 1 and 2 we give some preliminary notions about Cohen-Macaulay and Gorenstein rings and some basic properties about valuations, discrete valuations rings and completion. In Chapter 3 we study analytically irreducible rings, we show their connection with numerical semigroups and we report some of the basic properties that connect these two kind of objects. Finally, as a classical example of the spirit of this connection, we report Kunz's Theorem 3.3.13. In Chapter 4 we expose the theory of Arf rings and Arf numerical semigroups, showing why one is the numerical counterpart of the other, and giving all the basic notions and properties. In Chapter 5 we present two constructions. For the first one, let R be a ring, I be an ideal of R and t be an indeterminate; if $R[It] = \bigoplus_{n\geq 0} I^n t^n$ is the Rees algebra, we review some results on the following family of quadratic quotients of the Rees algebra

$$R(I)_{a,b} = \frac{R[It]}{(t^2 + at + b) \cap R[It]} \quad a, b \in R.$$

For the second one, let S be a numerical semigroup, E a semigroup ideal of S and $d \in S$ be an odd integer, the *numerical duplication* of S with respect to E and d is defined as

$$S \bowtie^d E = 2 \cdot S \cup (2 \cdot E + d) = \{2s : s \in S\} \cup \{2e + d : e \in E\}.$$

In Theorem 5.2.3 we prove that the value semigroup of the quotient $R(I)_{0,-b}$ is the numerical duplication $v(R) \bowtie^{v(b)} v(I)$. Finally, in Theorem 5.3.4 we give a (numerical) characterization of the Arf property on the numerical duplication and from that, in Theorem 5.3.11, we give an (algebraic) characterization of the Arf property on the quotients $R(I)_{0,-b}$. In Chapter 6 we present some developments on the theory of patterns on numerical semigroups; in particular we determine the family of patterns equivalent to the Arf pattern (Theorem 6.2.3) and we characterize when a monic pattern is admitted by the numerical duplication $S \bowtie^d E$ for $d \gg 0$ (Theorem 6.5.8).

Contents

Chapter 1

Cohen-Macaulay and Gorenstein rings

In this chapter we will briefly discuss about Cohen-Macaulay and Gorenstein rings, and their main properties using homological methods. Our main references for this chapter are [10], [25], [23] and [27].

1.1 Injective, projective and global dimension

Definition 1.1.1. Let R be a ring and M be an R-module. A **projective resolution** of M is an exact complex

$$\cdots \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0 \tag{1.1}$$

where all the modules P_i are projective. The length of 1.1 is the smallest integer n such that $P_n \neq 0$. If no such integer exists, then the length is said to be infinite. An **injective resolution** of M is an exact complex

$$0 \to M \to I_0 \to I_1 \to \cdots \to I_{n-1} \to I_n \to \dots \tag{1.2}$$

where all the modules I_i are injective. The length of 1.2 is the smallest integer n such that $I_n \neq 0$. If no such integer exists, then the length is said to be infinite.

Since every free module is projective and every module is an epimorphic image of a free module, it is clear that every module M has a projective resolution that can be constructed as follows. Let $F_0 \to M$ be a surjective map from the free module F_0 to M. This map has a kernel K_0. Consider a surjective map $F_1 \to K_0$ from the free module F_1 to K_0, this map has a kernel K_1. Compose $F_1 \to K_0$ with the inclusion $K_0 \hookrightarrow F_0$ and repeat the construction on K_1. Iterating this process, we obtain an exact complex

$$\cdots \to F_n \to F_{n-1} \to \cdots \to F_0 \to M \to 0$$

which is a free (hence projective) resolution of M.

On the other hand, it is not so easy to prove that every module M has an injective resolution. The existence of an injective resolution relies on the following theorem, that we leave without proof.

Theorem 1.1.2. *[23, Theorem 3.20] Every R-module can be embedded in an injective module.*

Thus, given a module M, we can consider an injection $M \to I_0$ where I_0 is injective. This map has a cokernel K_0 that can be embedded in another injective module I_1, and this inclusion has a cokernel K_1. Now compose $I_0 \to K_0$ with the injection $K_0 \to I_1$ and repeat the construction on K_1. Iterating this process we obtain an exact complex

$$0 \to M \to I_0 \to I_1 \to \dots \to I_{n-1} \to I_n \to \dots$$

which is an injective resolution of M.

Definition 1.1.3. Given an R-module M, the **projective dimension** (resp. **injective dimension**) of M, denoted by $\operatorname{proj\,dim} M$ ($\operatorname{inj\,dim} M$), is the smallest integer n for which there exists a projective (injective) resolution of M of length n. If no such integer exits, then we will write $\operatorname{proj\,dim} M = \infty$ ($\operatorname{inj\,dim} M = \infty$).

It is clear from the definition that M is projective (resp. injective) if and only if $\operatorname{proj\,dim} M = 0$ ($\operatorname{inj\,dim} M = 0$).

Theorem 1.1.4. *Let M be an R-module and $n \geq 0$. The following are equivalent*

1. $\operatorname{proj\,dim} M \leq n$.
2. *There exists a projective resolution of M of length n.*
3. $\operatorname{Ext}^i(M, N) = 0$ *for every $i > n$ and every R-module N.*
4. $\operatorname{Ext}^{n+1}(M, N) = 0$ *for every R-module N.*

Proof. (1) $\Leftrightarrow$ (2) and (2) $\Rightarrow$ (3) follows from the definition of proj dim and Ext functor. The implication (3) $\Rightarrow$ (4) is trivial. Thus it is enough to prove (4) $\Rightarrow$ (2).

Using the construction described above, we can construct a projective resolution of M. If we stop at step n we obtain

$$0 \to X \to P_{n-1} \to P_{n-2} \to \dots \to P_0 \to M \to 0$$

where all the P_i are projective. Note that $\operatorname{Ext}^1(X, N) = \operatorname{Ext}^{n+1}(M, N) = 0$, hence X is projective. □

The following theorem is the injective version of the preceding one, and the proof is analogous.

Theorem 1.1.5. *Let M be an R-module and $n \geq 0$. The following are equivalent*

1. $\operatorname{inj\,dim} M \leq n$.
2. *There exists an injective resolution of M of length n.*
3. $\operatorname{Ext}^i(N, M) = 0$ *for every $i > n$ and every R-module N.*
4. $\operatorname{Ext}^{n+1}(N, M) = 0$ *for every R-module N.*

The previous theorems allow us to give the following definition.

Definition 1.1.6. Let R be a ring, the **global dimension** of R is the integer

$$\begin{aligned} \operatorname{gl\,dim} R &= \sup\{n : \operatorname{Ext}^n(M, N) \neq 0 \text{ for some } R\text{-modules } M, N\} \\ &= \sup\{\operatorname{proj\,dim} M : M\ R\text{-module}\} \\ &= \sup\{\operatorname{inj\,dim} M : M\ R\text{-module}\}. \end{aligned}$$

1.2 Cohen Macaulay rings

Definition 1.2.1. Let M be a module over a ring R. We say that $x \in R$ is an M**-regular element** (or **regular in** M) if $xm = 0 \Rightarrow m = 0$ for every $m \in M$.
A sequence $x_1, \dots, x_n$ of elements of R of length n, is called an M**-regular sequence** (or simply an M**-sequence**) if the following conditions are satisfied

1. $\overline{x_i}$ is regular in $M/(x_1, \dots, x_{i-1})M$, for every $i \in \{1, \dots, n\}$,
2. $M/(x_1, \dots, x_n)M \neq 0$.

An M-sequence $x_1, \dots, x_n$ (contained in an ideal I) is **maximal** (in I), if $x_1, \dots, x_{n+1}$ is not an M-sequence for any $x_{n+1} \in R$ ($x_{n+1} \in I$). A **regular sequence** is an R-sequence.

From the definition, it follows that if $x_1, \dots, x_r, \dots, x_n$ is a sequence of elements in R and $x_1, \dots, x_r$ is an M-sequence, then $x_1, \dots, x_n$ is an M-sequence if and only if $\overline{x_{r+1}}, \dots, \overline{x_n}$ is an $M/(x_1, \dots, x_r)M$-sequence.

If M is a module over a ring R, then for every M-sequence $x_1, \dots, x_n$ contained in an ideal I, we can consider the ascending chain

$$(x_1) \subset (x_1, x_2) \subset (x_1, x_2, x_3) \subset \dots$$

If R is Noetherian, then the preceding chain eventually terminates. Therefore every M-sequence can be extended to a maximal sequence.

Lemma 1.2.2. *Let R be a ring and M, N be R-modules. If there exists an element $x \in \operatorname{Ann}(N)$ which is M-regular, then* $\operatorname{Hom}(N, M) = 0$.

Proof. Let $f \in \operatorname{Hom}(N, M)$ and $n \in N$, then

$$xf(n) = f(nx) = f(0) = 0 \Rightarrow f(n) = 0.$$

By the arbitrary choice of $n \in N$ we get $f = \underline{0}$. □

Proposition 1.2.3. *Let R be a ring, M, N be R-modules and $x_1, \dots, x_n$ an M-sequence in* $\operatorname{Ann}(N)$. *Then*

$$\operatorname{Hom}(N, M/(x_1, \dots, x_n)M) \simeq \operatorname{Ext}^n(N, M).$$

Proof. We use induction on n. If $n = 0$ there is nothing to prove. Let $n > 0$, then the inductive hypothesis implies that $\operatorname{Ext}^{n-1}(N, M) \simeq \operatorname{Hom}(N, M/(x_1, \dots, x_{n-1})M)$. As x_n is $M/(x_1, \dots, x_{n-1})M$-regular, from the previous lemma we have $\operatorname{Ext}^{n-1}(N, M) = 0$. Therefore the exact sequence

$$0 \longrightarrow M \xrightarrow{x_1} M \longrightarrow M/x_1M \longrightarrow 0$$

yields an exact sequence

$$\dots 0 \longrightarrow \operatorname{Ext}^{n-1}(N, M/x_1M) \xrightarrow{\psi} \operatorname{Ext}^n(N, M) \xrightarrow{x_1} \operatorname{Ext}^n(N, M) \dots$$

Since $x_1 \in \operatorname{Ann}(N)$, we have $x_1 \operatorname{Ext}^n(N, M) = 0$, so ψ is an isomorphism. Now apply the inductive hypothesis on N and M/x_1M. □

Theorem 1.2.4 (Rees)**.** *Let R be a Noetherian ring, M a finite R-module, and I an ideal such that $IM \neq M$. Then all maximal M-sequences in I have the same length n given by*

$$n = \min\{i : \operatorname{Ext}^i(R/I, M) \neq 0\}.$$

Proof. Let $x_1, \ldots, x_n$ be a maximal M-sequence in I. Since for every $i \in \{1, \ldots, n\}$, I contains an $M/(x_1, \ldots, x_{i-1})M$-regular element, from the previous lemma and proposition it results

$$\operatorname{Ext}^{i-1}(R/I, M) \simeq \operatorname{Hom}(R/I, M/(x_1, \ldots, x_{i-1})M) = 0.$$

On the other hand, since $IM \neq M$ and $x_1, \ldots, x_n$ is a maximal M-sequence, there are no $M/(x_1, \ldots, x_n)M$-regular elements in I. Hence, since R is Noetherian there exists $\overline{m} \in M/(x_1, \ldots, x_n)M$ such that $I\overline{m} = 0$. Now the homomorphism $1 + I \mapsto \overline{m}$ in $\operatorname{Hom}(R/I, M/(x_1, \ldots, x_n)M)$ is non zero, therefore

$$\operatorname{Ext}^n(R/I, M) \simeq \operatorname{Hom}(R/I, M/(x_1, \ldots, x_n)M) \neq 0. \qquad \square$$

The previous theorem allows us to give the following definition.

Definition 1.2.5. Let R be a Noetherian ring, M a finite R-module, and I an ideal such that $IM \neq M$. Then the length of a maximal M-sequence in I is called the **grade** of I in M, denoted by

$$\operatorname{grade}(I, M).$$

Definition 1.2.6. Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring, and M a finite R-module. The **depth** of M is defined as

$$\operatorname{depth}(M) := \operatorname{grade}(\mathfrak{m}, M).$$

Now we rephrase Theorem 1.2.4 in the local case as follows

$$\operatorname{depth}(M) = \min\{i : \operatorname{Ext}^i(k, M) \neq 0\}.$$

Definition 1.2.7. Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring, and M a finite R-module with $r = \operatorname{depth}(M)$. The number $t(M) = \dim_k \operatorname{Ext}^r_R(k, M)$ is called the **type** (or **Cohen-Macaulay type**) of M.

Lemma 1.2.8. *Let R be a Noetherian ring. If an ideal I of R is generated by a regular sequence $x_1, \ldots, x_n$, then* $\operatorname{height}(I) = n$.

Proof. We proceed by induction on n. If $n = 0$ there is nothing to prove. For the inductive step, assume that $x_1, \ldots, x_{n+1}$ is a regular sequence, $I = (x_1, \ldots, x_{n+1})$, and $\operatorname{height}(x_1, \ldots, x_n) = n$. By hypothesis, $\overline{x_{n+1}}$ is regular in $R/(x_1, \ldots, x_n)$, therefore it is not contained in any minimal prime of $R/(x_1, \ldots, x_n)$. Since the minimal primes in $R/(x_1, \ldots, x_n)$ corresponds to minimal primes of $(x_1, \ldots, x_n)$ in R, it follows that $\operatorname{height}(I) \geq n + 1$. The other inequality follows from Krull's generalized principal ideal theorem. $\square$

Proposition 1.2.9. *Let R be a Noetherian ring. For every ideal I of R we have*

$$\operatorname{grade}(I, R) \leq \operatorname{height}(I).$$

Proof. Set $n = \text{grade}(I,R)$ and let $x_1,\dots,x_n$ be a maximal regular sequence in I. By the previous Lemma we have

$$\text{grade}(I,R) = n = \text{height}(x_1,\dots,x_n) \le \text{height}(I). \qquad \square$$

Corollary 1.2.10. *Let $(R,\mathfrak{m})$ be a Noetherian local ring. Then*

$$\text{depth}(R) \le \dim R.$$

Proof. From the previous proposition we have

$$\text{depth}(R) = \text{grade}(\mathfrak{m},R) \le \text{height}(\mathfrak{m}) = \dim R. \qquad \square$$

Definition 1.2.11. A Noetherian local ring R is **Cohen-Macaulay** if $\text{depth}\, R = \dim R$. A Noetherian (not necessarily local) ring R is **Cohen-Macaulay** if $R_\mathfrak{m}$ is Cohen-Macaulay, for every maximal ideal $\mathfrak{m}$ of R.

Lemma 1.2.12. *If $(R,\mathfrak{m})$ is a local Noetherian ring and $x_1,\dots,x_n$ is a regular sequence, then*

$$\dim R/(x_1,\dots,x_n) = \dim R - n.$$

Proof. For $n = 1$ the statement follows from Krull's principal ideal theorem, in fact if $x \in R$ is regular, then

$$\dim R/(x) = \text{height}(\mathfrak{m}/(x)) = \text{height}(\mathfrak{m}) - 1 = \dim R - 1.$$

For the general case, it is suffice to iteratively apply the preceding formula. $\square$

Corollary 1.2.13. *If R is Noetherian, local, Cohen-Macaulay, then the quotient of R by a regular sequence is again Cohen-Macaulay.*

Proposition 1.2.14. *If R is a Noetherian, local, Cohen-Macaulay ring, then for every ideal I of R we have* $\text{grade}(I,R) = \text{height}(I)$.

Proof. Proceed by induction on $n = \dim R = \text{depth}\, R$. If $n = 0$ there is nothing to prove. For the inductive step, let $x \in R$ be a regular element, then

$$\text{grade}(I,R) = \text{grade}(I/(x), R/(x)) - 1 = \text{height}(I/(x)) - 1 = \text{height}(I). \qquad \square$$

Proposition 1.2.15. *Let R be a Noetherian Cohen-Macaulay ring. For every ideal I of R we have*

$$\text{height}(I) + \dim(R/I) = \dim(R).$$

Proof. We proceed by induction on $n = \dim R = \text{depth}\, R$. For $n = 0$ there is nothing to prove. For the inductive step let $x \in R$ be a regular element, then

$$\text{height}(I) + \dim(R/I) = \text{height}(I/(x)) - 1 + \dim\left(\frac{R/(x)}{I/(x)}\right) = \dim(R/(x)) - 1 = \dim R. \ \square$$

1.3 Regular rings

Definition 1.3.1. Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring. The **embedding dimension** of R is

$$\operatorname{emdim}(R) := \dim_k \mathfrak{m}/\mathfrak{m}^2.$$

The local ring R is **regular** if $\operatorname{emdim}(R) = \dim R$.

By the Nakayama's lemma, the embedding dimension of R is also the number of any minimal system of generators of $\mathfrak{m}$. Therefore, from Krull's generalized principal ideal theorem, in general we have

$$\operatorname{depth}(R) \leq \dim R \leq \operatorname{emdim}(R).$$

When the first inequality is an equality, R is Cohen-Macaulay. When this happen for the second inequality, R is regular. We will prove that if R is regular, it is also Cohen-Macaulay. Geometrically, a regular ring is the ring of a non-singular point of a variety.

Lemma 1.3.2. *If $(R, \mathfrak{m})$ is a regular local ring and $x \in \mathfrak{m} \setminus \mathfrak{m}^2$, then $R/(x)$ is regular.*

Proof. By Krull's principal ideal theorem we have $\dim R - 1 \leq \dim R/(x)$. For what we observed above, $\operatorname{emdim}(R/(x)) = \operatorname{emdim}(R) - 1$. Therefore

$$\operatorname{emdim}(R/(x)) = \operatorname{emdim}(R) - 1 \leq \dim R - 1 \leq \dim R/(x) \leq \operatorname{emdim} R/(x). \qquad \square$$

Theorem 1.3.3. *If $(R, \mathfrak{m})$ is a regular local ring, then R is a domain.*

Proof. We proceed by induction on $d = \dim R$. If $d = 0$, then $\mathfrak{m} = (0)$ and R is a field. For the inductive step, let $P_1, \ldots, P_m$ be the minimal prime ideals of R. The ideal $\mathfrak{m}$ is not contained in any of the ideals $P_1, \ldots, P_m, \mathfrak{m}^2$, therefore, by the prime avoidance, there exists $x \in \mathfrak{m} \setminus \mathfrak{m}^2$ which is not contained in $\bigcup_{i=1}^m P_i$. Therefore x is regular and, by the previous lemma, $R/(x)$ is regular. By the induction hypothesis $R/(x)$ is a domain. Hence (x) is a prime ideal of R, so it contains a minimal prime ideal P_i. Now if $y \in P_i \subseteq (x)$, then $y = zx$ for some $z \in R$, but since $x \notin P_i$, we have $z \in P_i$. It follows that $P_i = xP_i$. Finally by the Nakayama's lemma $P_i = (0)$, so R is a domain. $\square$

Theorem 1.3.4. *If $(R, \mathfrak{m})$ is a regular local ring, then $\mathfrak{m}$ can be generated by a regular sequence of length $d = \dim R$.*

Proof. We proceed by induction on d. If $d = 0$, R is a domain of dimension 0, i.e. R is a field and $\mathfrak{m}$. For the inductive step, let $x_1 \in \mathfrak{m} \setminus \mathfrak{m}^2$. Since R is a domain, x_1 is regular, so $\dim R/(x_1) = d-1$, and by the previous lemma $R/(x_1)$ is regular, hence for the inductive hypothesis there exists a regular sequence $\overline{x_2}, \ldots, \overline{x_d}$ in $R/(x_1)$ that generates $\overline{\mathfrak{m}}$. It follows that $x_1, \ldots, x_d$ is a regular sequence in R that generates $\mathfrak{m}$. $\square$

Corollary 1.3.5. *Every regular local ring R is Cohen-Macaulay.*

Proof. From the previous theorem we have $\operatorname{depth} R \leq \dim R = \operatorname{emdim} R \leq \operatorname{depth} R$. $\square$

Regular local rings have been characterized in terms of the global dimension. We report this characterization without proof (see [10, Theorem 2.2.7]).

Theorem 1.3.6 (Auslander, Buchsbaum, Serre)**.** *Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring. Then R is regular if and only if $\operatorname{gl\,dim} R < \infty$.*

1.4 Gorenstein rings

Definition 1.4.1. A Noetherian local ring R is **Gorenstein** if $\operatorname{inj}\dim R < \infty$. A Noetherian ring is **Gorenstein** if every localization at every maximal ideal is a Gorenstein local ring.

We report, without proof, a characterization of Gorenstein rings (see [10, Theorem 3.2.10] or [25, Theorem 18.1]).

Theorem 1.4.2. *Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring of Krull dimension n. The following are equivalent*

1. *R is Gorenstein.*
2. *R is Cohen-Macaulay of type* 1.
3. $\operatorname{Ext}^i(k, R) = 0$ *for some $i > n$.*
4. $\operatorname{Ext}^i(k, R) = \begin{cases} 0 & i \neq n \\ k & i = n. \end{cases}$
5. $\operatorname{inj}\dim R = n$.

Proposition 1.4.3. *Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring. Then we have the following implications:*

$$R \text{ is regular} \Longrightarrow R \text{ is Gorenstein} \Longrightarrow R \text{ is Cohen-Macaulay.}$$

Proof. For the first implication, if R is regular then $\operatorname{gl}\dim R < \infty$ (Theorem 1.3.6), in particular $\operatorname{Ext}^i(k, R) = 0$ for $i \gg 0$, therefore from Theorem 1.4.2 R is Gorenstein. The second implication follows directly from Theorem 1.4.2. $\square$

Chapter 2

Valuation and completion

In this chapter we will briefly discuss about two important topics in commutative algebra: valuations and completions. In the first and the second sections of this chapter, we will mainly follow Swanson-Huneke [19].

2.1 Valuations and valuation rings

Let K be a field and G a totally ordered abelian group (written additively). We indicate with $K^* = K \setminus \{0\}$ the multiplicative group of K.

Definition 2.1.1. A **valuation** on K is a map $v : K^* \to G$ such that for all $x, y \in K$

1. $v(xy) = v(x) + v(y)$, (i.e. v is a homomorphism of groups)
2. $v(x + y) \geq \min\{v(x), v(y)\}$.

The image $v(K^*)$ is the **value group** of v.

It easily follows from the definition that $v(\pm 1) = 0$, $v(x^{-1}) = -v(x)$, $v(x) = v(-x)$. In addition, if $v(x) \neq v(y)$ then $v(x + y) = \min\{v(x), v(y)\}$. To show this, suppose that $v(x) > v(y)$, then $v(y) = v((x + y) - x) \geq \min\{v(x + y), v(x)\} \geq v(y)$, forcing $v(x + y) = v(y) = \min\{v(x), v(y)\}$.

Definition 2.1.2. Two valuations $v : K^* \to G$, $w : K^* \to G'$ on K are **equivalent** if there exists an order preserving isomorphism of groups $\varphi : v(G) \to w(G')$ such that $w = \varphi \circ v$.

Definition 2.1.3. Let R be a domain and K its field of fractions. Then R is a **valuation ring** if for every $x \in K^*$ either $x \in R$ or $x^{-1} \in R$.

Given a domain R with field of fractions K, we denote with R^* the multiplicative group of invertible elements of R. Let $\Gamma_R = K^*/R^*$ and $v_R : K^* \to \Gamma_R$ the natural group homomorphism. Γ_R is abelian since so is K^*. We can endow Γ_R with an order as follows. Let $v_R(x), v_R(y) \in \Gamma_R$ (v_R surjective) and define $v_R(x) \geq v_R(y)$ if $xy^{-1} \in R$. It is not difficult to check that this is an order and it is compatible with the group structure of Γ_R.

Proposition 2.1.4. *Let R be a domain and K its field of fractions. Then, endowing Γ_R with the order as above, the following conditions are equivalent*

1. *R is a valuation ring.*

2. *Γ_R is totally ordered and v_R is a valuation on K.*

Proof.

(1) $\Rightarrow$ (2) Since R is a valuation ring, the order of Γ_R is total. Now v_R is a homomorphism by definition. Let $x, y \in K^*$, suppose that $xy^{-1} \in R$, then $xy^{-1}+1 = (x+y)y^{-1} \in R$, therefore $v(x+y) \geq v(y) \geq \min\{v(x), v(y)\}$.

(2) $\Rightarrow$ (1) If $x \in K^*$, then, since the order of Γ_R is total, either $v_R(x) \geq 0$ or $v_R(x) \leq 0$, in other words either $v_R(x) \geq 0$ or $v_R(x^{-1}) \geq 0$, this means $x \in R$ or $x^{-1} \in R$. $\square$

Remark 2.1.5. From the preceding proof, it is not difficult to show that, if v is a valuation on a field K, then the set $R_v = \{x \in K : v(x) \geq 0\}$ is a valuation ring and its field of fractions is K. In this case, v_R and v are equivalent. Therefore, from the previous proposition, there is a one to one correspondence between equivalence class of valuations on K and valuation rings with field of fractions K.

Proposition 2.1.6. *Let R be a valuation ring with field of fractions K.*

1. *The ideals of R are totally ordered by inclusion.*

2. *R is a local ring.*

3. *R is integrally closed.*

4. *Every finitely generated ideal of R is principal.*

Proof.

1. Let I, J be ideals of R. Suppose that $I \not\subseteq J$ and fix $x \in I \backslash J$. Let $y \in J$, if $xy^{-1} \in R$, then $x = (xy^{-1})y \in J$, a contradiction. Hence $x^{-1}y \in R$, so that $y = x(x^{-1}y) \in I$. By the arbitrary choice of $y \in J$ we obtain $J \subseteq I$.

2. Follows from 1.

3. Let $x \in K$ integral over R, then

$$x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0, \quad a_i \in R.$$

If $x \in R$ there is nothing to prove. On the other hand, if $x^{-1} \in R$, then from the previous equality we have $x = -(a_{n-1} + a_{n-2}x^{-1} + \cdots + a_0 x^{1-n}) \in R$.

4. We can reduce by induction to the case of an ideal with two generators (x, y). Suppose that $xy^{-1} \in R$, then $x = (xy^{-1})y \in (y)$, so $(x, y) = (y)$. The other case is similar. $\square$

2.2 Discrete valuation rings

Definition 2.2.1. Let v be a valuation on the field K. The **rank** of v is the Krull dimension of $R_v = \{x \in K : v(x) \geq 0\}$. The valuation v is **discrete** if its value group is isomorphic ot $\mathbb{Z}^n$.

Before proving a characterization for the discrete valuation rings, we need the Krull intersection theorem. The short proof reported can be found in Clark's notes [11].

Theorem 2.2.2 (Krull intersection theorem)**.** *Let R be a Noetherian ring and I an ideal of R. If $x \in \bigcap_{n\in\mathbb{N}} I^n$, then $x \in xI$.*

Proof. Since R is Noetherian, every ideal is finitely generated, so $I = (a_1, \dots, a_m)$. For all $n \in \mathbb{N}$, since $x \in I^n$ there exists a homogeneous polynomial $P_n(x_1, \dots, x_m) \in R[x_1, \dots, x_m]$ of degree m such that $x = P_n(a_1, \dots, a_m)$. Now set $J_n = (P_1, \dots, P_n) \subseteq R[x_1, \dots, x_m]$. By the Hilbert basis theorem, $R[x_1, \dots, x_m]$ is Noetherian, therefore there exists $k \in \mathbb{N}$ such that $J_{k+1} = J_k$. Hence

$$P_{k+1} = Q_k P_1 + \cdots + Q_1 P_k,$$

where $Q_i \in R[x_1, \dots, x_m]$ are homogeneous of (positive) degree i. If we substitute in the preceding equality $x_i = a_i$ we obtain

$$x = x\Big(Q_k(a_1, \dots, a_m) + \cdots + Q_1(a_1, \dots, a_m)\Big) \in xI. \qquad \square$$

Corollary 2.2.3. *If R is a local Noetherian ring and I is an ideal of R, then we have*

$$\bigcap_{n\in\mathbb{N}} I^n = (0).$$

Proof. Set $J = \bigcap_{n\in\mathbb{N}} I^n$. From the Krull intersection theorem 2.2.2 we have $J \subseteq IJ$. Since R is local Noetherian, from Nakayama's lemma we obtain $J = 0$. $\square$

Lemma 2.2.4. *[2, Proposition 5.1] Let $R \subseteq S$ be a ring extension. The following conditions are equivalent*

1. *$x \in S$ is integral over R.*
2. *There exists a faithful $R[x]$-module M which is finitely generated as an R-module.*

Proof.

(1) $\Rightarrow$ (2) It is enough to prove that $R[x]$ is a finitely generated R-module. By hypothesis we have

$$x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 = 0, \quad a_i \in R$$
$$\Rightarrow x^n = -(a_{n-1}x^{n-1} + \ldots + a_1 x + a_0).$$

It easily follows by induction that $x^m \in R + Rx + \ldots + Rx^{n-1}$ for all $m \geq n$. Hence $R[x] = R + Rx + \ldots + Rx^{n-1}$.

(2) $\Rightarrow$ (1) By hypothesis $M = Rx_1 + \ldots + Rx_n$, therefore we have

$$\begin{aligned} xx_1 &= a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \\ xx_2 &= a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \\ &\vdots \\ xx_n &= a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n \end{aligned}$$

for some $a_{ij} \in R$. So for all $i \in \{1, \ldots, n\}$

$$\sum_{j=1}^{n} (\delta_{ij}x - a_{ij})x_j = 0.$$

Multiplying by the transpose of the adjoint of the matrix $A = (\delta_{ij}x - a_{ij})$ we obtain $\det(A)x_i = 0$ for all $i \in \{1, \ldots, n\}$. Since M is faithful we must have $\det(A) = 0$ which is a relation of integral dependence of x over R. □

Proposition-Definition 2.2.5. *Let R be a domain and K its field of fractions, suppose $R \neq K$. The following conditions are equivalent*

1. *R is a Noetherian valuation ring.*
2. *R is a local principal ideal domain.*
3. *R is local, Noetherian, and its maximal ideal $\mathfrak{m}$ is principal.*
4. *R is local Noetherian, one-dimensional, and integrally closed.*
5. *R is a valuation ring with value group isomorphic to $\mathbb{Z}$.*

If R satisfies any of the preceding conditions, then R is a ***discrete valuation ring*** *(DVR).*

Proof.

(1) $\Rightarrow$ (2) It follows from Proposition 2.1.6.

(2) $\Rightarrow$ (3) Trivial.

(3) $\Rightarrow$ (5) Set $\mathfrak{m} = (x)$. Let I be a non zero ideal of R. By the Corollary of Krull intersection theorem 2.2.3 we have $\bigcap_{n \in \mathbb{N}} \mathfrak{m}^n = (0)$. Therefore there exists an integer $r \in \mathbb{N}$ such that $I \subseteq \mathfrak{m}^r$, $I \nsubseteq \mathfrak{m}^{r+1}$. Hence there exists $y \in I$ such that $y \in \mathfrak{m}^r \setminus \mathfrak{m}^{r+1}$. So $y = \lambda x^r$ with $\lambda \notin \mathfrak{m}$, that is λ is invertible, thus $x^r = \lambda^{-1}y \in I$. This means that $\mathfrak{m}^r = (x^r) \subseteq I \subseteq \mathfrak{m}^r \Rightarrow I = \mathfrak{m}^r = (x^r)$. Now let $a \in R$, for what we have proved so far $(a) = (x^r)$ for some $r \in \mathbb{N}$. Set $v(a) = r$ and extend v to K by defining $v(a/b) = v(a) - v(b)$. Now it is easy to see that $v : K \to \mathbb{Z}$ is a valuation on K.

(5) $\Rightarrow$ (1) Let I be a non zero ideal of R, and let $k = \min v(I)$. It is not difficult to check that $I = \{y \in R : v(y) \geq k\}$. So if we let $x \in R$ such that $v(x) = 1$, then $I = (x^k)$, so I is principal, in particular finitely generated.

(1) $\Rightarrow$ (4) It follows from Proposition 2.1.6.

(4) $\Rightarrow$ (3) Let $\Sigma = \{I \text{ ideal of } R : R \neq (R :_K I)\}$. Note that if $x \in R$ is not invertible, $(x) \in \Sigma \neq \emptyset$. Since R is Noetherian, there exists a maximal element $J \in \Sigma$. Now let $a, b \notin J$ and $z \in (R : J) \setminus R$. From the maximality of J we have $z(J + (a)) \nsubseteq R$, otherwise $J + (a) \in \Sigma$. Therefore $za \in (R : J) \setminus R$. Repeting the preceding argument, we obtain $z(ab) = (za)b \in (R : J) \setminus R$. It follows $ab \notin J$, hence J is a prime ideal of R. Since R is a local, one-dimensional, domain, $J = \mathfrak{m}$ and it results $R \subsetneq (R :_K \mathfrak{m})$. Now let $x \in (R :_K \mathfrak{m}) \setminus R$, by definition $x\mathfrak{m} \subseteq R$. If we assume $x\mathfrak{m} \subseteq \mathfrak{m}$, then $\mathfrak{m}$ is a faithful $R[x]$-module, finitely generated as an R-module, so from Lemma 2.2.4 x is integral over R, hence $x \in R$ since R is integrally closed, a contradiction. Therefore the ideal $x\mathfrak{m}$ of R contains some invertible element, it follows $x\mathfrak{m} = R \Rightarrow \mathfrak{m} = (1/x)$. □

From the previous characterization it follows that a ring R is DVR if and only if it is a rank one discrete valuation domain.

Remark 2.2.6. For what we have proved so far, if R is a discrete valuation ring, then

$$R = \{x \in Q(R) : v(x) \geq 0\}.$$

Further, if $z \in R$ is an element of valuation 1, then the ideals of R are all of the form

$$(z^n) = \{x \in Q(R) : v(x) \geq n\} \quad n \in \mathbb{N}.$$

2.3 Fractional ideals

Definition 2.3.1. Let R be a domain and $Q(R)$ its field of fractions. An R-submodule M of $Q(R)$ is a **fractional ideal** if there exists $x \in R$, $x \neq 0$ such that $xM \subseteq R$.

(The previous definition can be given more generally for a ring R, where $Q(R)$ is its total ring of fractions). Clearly, every ideal of R is fractional ideal (take $x = 1$).

Proposition 2.3.2. *If R is a Noetherian domain, then an R-submodule M of $Q(R)$ is a fractional ideal if and only if it is finitely generated.*

Proof. If M is a fractional ideal, note that $M \subseteq x^{-1}R$. Conversely, if M is generated by $x_1, \dots, x_n \in M \subseteq Q(R)$, we can write $x_i = y_i/z_i$ with $y_i, z_i \in R$, then, setting $z = z_1 z_2 \dots z_n \neq 0$, we have $zM \subseteq R$. □

2.4 Completion

In this section R is a Noetherian local ring and $\mathfrak{m}$ is the maximal ideal of R. The concept of completion can be defined in a more general setting, but , for simplicity, we reduce to the Noetherian local case.

Definition 2.4.1. A sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq R$ is a **Cauchy sequence** if

$$\forall N \in \mathbb{N}, \exists \nu \in \mathbb{N} : n, m \geq \nu \Rightarrow x_n - x_m \in \mathfrak{m}^N.$$

Definition 2.4.2. Two sequences $\{x_n\}_{n\in\mathbb{N}}, \{y_n\}_{n\in\mathbb{N}} \subseteq R$ are **equivalent** if the sequence $\{y_n - x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. The relation is denoted by $(x_n) \sim (y_n)$.

Proposition-Definition 2.4.3. *The the set*

$$\hat{R}_{\mathfrak{m}} = \big\{\{x_n\}_{n\in\mathbb{N}} \subseteq R \text{ Cauchy sequence}\big\}/\sim$$

(sometimes denoted by $\hat{R}$) with the operations

- $(x_n) + (y_n) = (x_n + y_n)$,
- $(x_n) \cdot (y_n) = (x_n y_n)$,

is a ring, called the ***completion*** *of R with respect to the maximal ideal $\mathfrak{m}$. Further $R \subseteq \hat{R}$.*

Proof. The facts that the operations are well defined and $\hat{R}$ is a ring are easy to check and we omit the proofs. For the inclusion $R \subseteq \hat{R}$, consider the homomorphism of rings $\phi : R \to \hat{R}$ defined by $\phi(x) = (x)$ (the constant sequence equal to x). From the Corollary of Krull intersection theorem 2.2.3 we have

$$\ker\phi = \bigcap_{n\in\mathbb{N}} \mathfrak{m}^n = (0),$$

therefore ϕ is injective. $\square$

Remark 2.4.4. On R we can define a topology assigning a metric in the following manner. If $x \in R$, set

$$v(x) = \begin{cases} \max\{n \in \mathbb{N} : x \in \mathfrak{m}^n\} & x \neq 0, \\ \infty & x = 0. \end{cases}$$

Now for every $x, y \in R$ set $d(x,y) = 2^{-v(x-y)}$ (with the convention $2^{-\infty} = 0$). With this definitions (R,d) is a metric space. It can be proved that the completion of R as a metric space, coincide with the completion of R as a ring.

Chapter 3

Analytically irreducible rings

In this chapter we will discuss about the connection between numerical semigroups and analytically irreducible rings. We will mainly follow [21], [29] and [4].

3.1 Analytically irreducible and analytically unramified rings

In this section, $(R, \mathfrak{m})$ is a one-dimensional local Cohen-Macaulay ring. We denote by $\mathrm{Reg}(R)$ the set of regular elements of R. Note that $\mathrm{Reg}(R)$ is a multiplicative set, the localization of R at $\mathrm{Reg}(R)$ is the **total ring of fractions** $Q(R)$. The integral closure of R in $Q(R)$ will be denoted by $\overline{R}$.

Definition 3.1.1. A local ring $(R, \mathfrak{m})$ is **analytically irreducible** if its completion $\hat{R}$ (with respect to the maximal ideal $\mathfrak{m}$) is a domain. We say that R is **analytically unramified** if $\hat{R}$ is reduced (i.e. it does not contain nilpotent elements).

The following theorems gives us some useful characterizations of analytically irreducible and analytically unramified rings. The proofs are omitted, but they can be found in [21].

Theorem 3.1.2. *[21, Theorem 3.17] The following statements are equivalent.*

1. *R is analytically irreducible.*
2. *$Q(R)$ is a field, and $\overline{R}$ is a DVR and a finitely generated R-module.*

Theorem 3.1.3. *[21, Theorem 3.22] The following statements are equivalent.*

1. *R is analytically unramified.*
2. *$\overline{R}$ is a finitely generated R-module.*

3.2 Numerical semigroups

Definition 3.2.1. A numerical semigroup S is a submonoid of $(\mathbb{N}, +)$ with finite complement in $\mathbb{N}$. In other words, $S \subseteq \mathbb{N}$ is such that

1. $0 \in S$.
2. $a, b \in S \Rightarrow a + b \in S$.
3. $\mathbb{N} \setminus S$ is finite.

Throughout, S will be always a numerical semigroup.

Definition 3.2.2. A **relative ideal** of S is a subset $E \subseteq \mathbb{Z}$ such that

1. $E + S \subseteq E$.
2. There exists $s \in S$ such that $s + E \subseteq S$.

A relative ideal E contained in S is simply an **ideal** of S. The set $M = S \setminus \{0\}$ is easily seen that is an ideal of S, called the **maximal ideal** of S.

If I, J are two relative ideals of S, then also the following sets

- $I \cap J$,
- $I + J = \{i + j : i \in I, j \in J\}$
- $I - J = \{z \in \mathbb{Z} : z + J \subseteq I\}$

are relative ideals of S.

Remark 3.2.3. It is easy to see (see for instance [29, Theorem 2.7]) that every numerical semigroup has a unique finite minimal system of generators given by $M \setminus (M + M)$.

Definition 3.2.4. We define the following constants associated to S:

multiplicity	$\mu(S) = \min(S \setminus \{0\})$
embedding dimension	$\mathrm{e}(S) = \lvert M \setminus (M + M) \rvert$
Frobenius number	$F(S) = \max(\mathbb{N} \setminus S)$
conductor	$c(S) = F(S) + 1$
genus	$g(S) = \lvert \mathbb{N} \setminus S \rvert$.

So $\mathrm{e}(S) = |M \setminus (M + M)|$ is the number of the (minimal) generators of S; it is easy to see that $\mathrm{e}(S) \leq \mu(S)$ and that $\mu(S)$ is the least generator of S.

The Frobenius number is associated to a problem of Frobenius, that is to find a formula for $F(S)$ in terms of the generators of the numerical semigroup S. The problem in the case with two generators was first posed in 1884 by Sylvester [32]. In this case a formula exists, and it is easily seen to be

$$F(\langle a, b \rangle) = ab - b - a$$

(for a short proof, see [29, Proposition 2.13]). In the case with three or more generators, Curtis [12] proved that a polynomial formula involving the generators and the Frobenius number does not exist. For a fixed number n of generators, Kannan [20] gave a polynomial time algorithm that finds the Frobenius number. If the number n of the generators is variable, Ramírez-Alfonsín [28] proved that the Frobenius problem is NP-hard.

The genus is associated to a conjecture of Bras-Amorós [8]. If n_g is the number of of numerical semigroups with genus g, the sequence n_g seems to have a Fibonacci-like behavior. It was conjectured that

1. $n_g \geq n_{g-1} + n_{g-2} \quad g \geq 2$,
2. $\lim\limits_{g\to\infty} \frac{n_{g-1}+n_{g-2}}{n_g} = 1$,
3. $\lim\limits_{g\to\infty} \frac{n_g}{n_{g-1}} = \phi = \frac{1+\sqrt{5}}{2}$.

Zhai [33] proved that the limit

$$\lim_{g\to\infty} \frac{n_g}{\phi^g}$$

is constant. This proves part 2 and 3 of the conjecture, and implies that part 1 can fail to hold for only finitely many values of g. Nevertheless, the first part of the conjecture is still open. Even the weaker version $n_g \geq n_{g-1}$ is still unsolved. Some recent works on this problem include [5], [15], [18], [17].

The Frobenius number has the following property: $F(S) + s \in S$ for all $s \in S$, $s \neq 0$ and of course $F(S) \notin S$.

Definition 3.2.5. An integer $x \in \mathbb{Z}$ is a **pseudo-Frobenius number** if $x \notin S$ and $x+s \in S$ for all $s \in S$, $s \neq 0$. The set of pseudo-Frobenius numbers is denoted by $\mathrm{PF}(S)$. The cardinality of $\mathrm{PF}(S)$ is the **type** $t(S)$ of S.

Rephrasing the previous definition, we have

$$PF(S) = \{x \in \mathbb{Z} : x \notin S, x + m \in M \text{ for all } m \in M\} = (M - M) \setminus S,$$

and $t(S) = |(M - M) \setminus S|$. Of course we have $F(S) \in \mathrm{PF}(S)$.

Proposition-Definition 3.2.6. *[16, Lemma 1, Propostion 2] The following conditions are equivalent*

1. $x \notin S \iff F(S) - x \in S \quad \forall x \in \mathbb{Z}$.
2. *Among the numbers* $0, 1, \ldots, F(S)$ *there are just as many elements in* S *as in* $\mathbb{N}\setminus S$.
3. $\mathrm{PF}(S) = \{F(S)\}$.
4. $t(S) = 1$.

*A numerical semigroup that satisfy one of the preceding conditions is said to be **symmetric**.*

Proof.

(1) $\Leftrightarrow$ (2) Set $T = \{0, 1, \ldots, F(S)\}$, the map $\varphi : T \to T$, defined by $\varphi(x) = F(S) - x$, is bijective. Restricting φ to $T\cap S$, we obtain a bijection between $T\cap S$ and $T\cap(\mathbb{N}\setminus S)$.

(1) $\Rightarrow$ (3) Let $f \in \mathrm{PF}(S)$, then $f \notin S$, so $s = F(S) - f \in S$. If $s \neq 0$, then $F(S) = f + s \in S$, a contradiction, so $F(S) - f = s = 0 \Rightarrow f = F(S)$.

(3) $\Rightarrow$ (1) Trivial.

(4) $\Rightarrow$ (1) Assume by contradiction that

$$H(S) = \{x \in \mathbb{Z} : x \notin S, F(S) - x \notin S\} \neq \emptyset,$$

and set $h(S) = \max H(S)$. Suppose that exists $s \in S$, $s \neq 0$ such that $h(S)+s \notin S$, then the maximality of $h(S)$ will imply that $F(S) - (h(S) + s) = s' \in S$, thus $F(S) - h(S) = s + s' \in S$, a contradiction. This proves that $h(S) \in PF(S)$ and clearly $h(S) \neq F(S)$, therefore $t(S) \geq 2$, contradicting the hypothesis. □

3.3 The value semigroup

Throughout this section (unless otherwise specified), $(R, \mathfrak{m}, k)$ will be a one-dimensional, Noetherian, local, analytically irreducible, ring. Further, we will assume also that R is **residually rational**, i.e. R and its integral closure $\overline{R}$ in its total ring of fractions $Q(R)$ have the same residue field: $R/\mathfrak{m} \simeq \overline{R}/\tilde{\mathfrak{m}}$ ($\tilde{\mathfrak{m}}$ being the maximal ideal of $\overline{R}$). With this assumptions, $\overline{R}$ is a discrete valuation ring, and we denote with $v : Q(R) \setminus \{0\} \to \mathbb{Z}$ the valuation associated with $\overline{R}$.

Theorem 3.3.1. *If R is a one-dimensional, Noetherian, local, analytically irreducible, ring, then the value semigroup $v(R)$ of R is a numerical semigroup.*

Proof. Since v is a homomorphism of groups, if we restrict v to the (multiplicative) semigroup $R\setminus\{0\}$, then the image $v(R) \subseteq \mathbb{N}$ is a subsemigroup of $\mathbb{N}$, and $v(1) = 0 \in v(R)$. Finally, since $\overline{R}$ is a finitely generated R-submodule of $Q(R)$, it is a fractional ideal (Proposition 2.3.2), therefore there exists $z \in R$, $z \neq 0$ such that $z\overline{R} \subseteq R$. Thus

$$v(z) + \mathbb{N} = v(z) + v(\overline{R}) = v(z\overline{R}) \subseteq v(R),$$

and $\mathbb{N} \setminus v(R) \subseteq \{0, 1, \ldots, v(z)\}$ is finite. □

The preceding theorem shows the connection between analytically irreducible rings and numerical semigroups.

Remark 3.3.2. Note that, if R is as above, and I is a (fractional) ideal of R, then $v(I)$ is a (relative) ideal of $v(R)$.

Proposition 3.3.3. *Let I, J be two fractional ideals of R.*

1. $v(IJ) \supseteq v(I) + v(J)$.

2. $v(I : J) \subseteq v(I) - v(J)$.

3. $v(I \cap J) \subseteq v(I) \cap v(J)$.

Proof.

1. Let $n \in v(I) + v(J)$, then $n = v(i) + v(j) = v(ij) \in v(IJ)$, for some $i \in I$, $j \in J$.
2. Let $n \in v(I : J)$, then $n = v(x)$ for some $x \in I : J$. Let $j \in J$, we have $n + v(j) = v(z) + v(j) = v(zj) \in v(I)$, this proves $n \in v(I) - v(J)$.
3. Trivial. $\square$

Definition 3.3.4. The **conductor** of R is the fractional ideal $C(R) = (R : \overline{R})$.

Proposition 3.3.5. *The conductor $C(R)$ is the biggest ideal shared by both R and $\overline{R}$.*

Proof. $C(R)$ is a fractional ideal of R. Further, from

$$R \cdot C(R) \subseteq \overline{R} \cdot C(R) \subseteq R \subseteq \overline{R},$$

it is clear that it is an ideal of both R and $\overline{R}$. Now let $I \subseteq R$ be an ideal of both R and $\overline{R}$. If $i \in I$, then

$$i\overline{R} \subseteq I \subseteq R \Rightarrow i \in C(R).$$ $\square$

Set $c(R) = \min v(C(R))$. From Remark 2.2.6 we have

$$C(R) = \{x \in Q(R) : v(x) \geq c(R)\}.$$

Lemma 3.3.6. *If $x, y \in Q(R)$ are such that $v(x) = v(y)$, then there exists an invertible $u \in R$ such that*

$$v(y - ux) > v(x) = v(y).$$

Proof. Since x/y is a unit in $\overline{R}$ and since the residue fields of R and $\overline{R}$ are isomorphic, there exists a unit $u \in R$ such that $x/y - u$ is in the maximal ideal of $\overline{R}$, that is

$$v(y - ux) - v(x) = v(y/x - u) > 0.$$ $\square$

Lemma 3.3.7. *Every fractional ideal I of R contains an ideal of $\overline{R}$. In other words, there exists $n \in \mathbb{N}$ such that*

$$\{x \in Q(R) : v(x) \geq n\} \subseteq I.$$

Proof. Since I is a fractional ideal, there exists $x \in I$, $x \neq 0$, such that $xI \subseteq R$. Then

$$x^2 C(R) = x(xC(R)) \subseteq xI \subseteq R \subseteq \overline{R},$$

therefore $x^2 C(R)$ is an ideal of $\overline{R}$ contained in I. The second statement follows from Remark 2.3.2. $\square$

Lemma 3.3.8. *If $J \subseteq I$ are fractional ideals of R such that $v(I) = v(J)$, then $I = J$.*

Proof. From Lemma 3.3.7, there exists $n \in \mathbb{N}$ such that

$$\{x \in Q(R) : v(x) \geq n\} \subseteq J.$$

Let $i \in I$, by hypothesis $v(i) = v(j)$ for some $j \in J$. By the previous lemma there exists a unit $u \in R$ such that $v(i - uj) > v(i)$. Note that $i - uj \in I$, so by iterating this process there exists $j' \in J$ such that $v(i - j') \geq n$. This proves that $i - j' \in J \Rightarrow i \in J$. $\square$

Let I be a fractional ideal of R, then $xI \subseteq R$, for some $x \in I$, $x \neq 0$. Then $v(x) + v(I) = v(xI) \subseteq v(R) \subseteq \mathbb{N}$. It follows that $v(I)$ has a minimum.

Now if I, J are two fractional ideals of R, then both $v(I)$ and $v(J)$ are contained in $m + \mathbb{N}$, for some $m \in \mathbb{Z}$ (apply the previous observation to I and J and take the minimum). Further, there exists $n \in \mathbb{N}$ such that $n + \mathbb{N}$ is contained in both $v(I)$ and $v(J)$ (take the maximum of the two integers obtained applying Lemma 3.3.7 to I and J). It follows that

$$|v(I) \setminus v(J)| \subseteq \{m, m+1, \ldots, n-1, n\}$$

is finite.

Proposition 3.3.9. *[4, Proposition II.1.4] If $J \subseteq I$ are nonzero fractional ideals of R, then*

$$\lambda(I/J) = |v(I) \setminus v(J)|.$$

Proof. We proceed by induction on $n = |v(I) \setminus v(J)|$. If $n = 0$, the assertion follows from Lemma 3.3.8. If $n = 1$, let M be a fractional ideal of R such that

$$J \subsetneq M \subseteq I.$$

From Lemma 3.3.8 $v(J) \subsetneq v(M) \subseteq v(I)$, and since $|v(I) \setminus v(J)| = 1$, we have $v(M) = v(I)$, and again from Lemma 3.3.8 $M = I$. This proves $\lambda(I/J) = 1$. Now if $n > 1$, set $v(I) \setminus v(J) = \{m_1, \ldots, m_n\}$ with $m_1 > m_2 > \cdots > m_n$ and

$$J_t = J + \{x \in I : v(x) \geq m_t\} \quad t \in \{1, \ldots, n\}.$$

We obtain a chain

$$J \subsetneq J_1 \subsetneq J_2 \subsetneq \cdots \subsetneq J_n = I$$

that is maximal by the induction hypothesis. □

Lemma 3.3.10. *Let R be a commutative ring and $J \subseteq I$ be two ideals of R. Then*

$$\mathrm{Hom}(R/I, R/J) \simeq (J : I)/J.$$

Sketch of proof. For every $x \in (J : I)$, define the map $\varphi_x(r + I) = xr + J$, then φ_x is well defined and $\varphi_x \in \mathrm{Hom}(R/I, R/J)$. Now define $\varphi : (J : I) \to \mathrm{Hom}(R/I, R/J)$ by $\varphi(x) = \varphi_x$, then φ is well defined, surjective and $\ker \varphi = J$. □

In the following theorem, for points 3 and 4 see [26, Theorem 1], [4, Proposition II.1.16].

Theorem 3.3.11. *Let $(R, \mathfrak{m}, k)$ be a one-dimensional, Noetherian, local, analytically irreducible, residually rational ring. Then*

1. $\mathrm{emdim}(R) \leq \mathrm{e}(v(R))$.
2. $c(R) = c(v(R))$.
3. *R is a Cohen-Macaulay ring of type $t(R) = |v(\mathfrak{m} : \mathfrak{m}) \setminus v(R)|$.*
4. $t(R) \leq t(v(R))$.

Proof.

1. From Proposition 3.3.3 we have

$$v(\mathfrak{m}^2) \supseteq v(\mathfrak{m}) + v(\mathfrak{m}) \Rightarrow v(\mathfrak{m}) \setminus v(\mathfrak{m}^2) \subseteq v(\mathfrak{m}) \setminus (v(\mathfrak{m}) + v(\mathfrak{m})),$$

   therefore

$$\begin{aligned} \operatorname{emdim}(R) = \lambda(\mathfrak{m}/\mathfrak{m}^2) = |v(\mathfrak{m}) \setminus v(\mathfrak{m}^2)| \leq \\ \leq |v(\mathfrak{m}) \setminus (v(\mathfrak{m}) + v(\mathfrak{m}))| = e(v(R)). \end{aligned}$$

2. Let $z \in \overline{R}$ be an element of value 1. Then, from Remark 2.2.6 and Proposition 3.3.5 we have

$$m = c(R) = \min\{i : (z^i) \subseteq R\}.$$

   On the other hand, by definition we have

$$n = c(v(R)) = \min\{i : i + \mathbb{N} \subseteq v(R)\}.$$

   Clearly $m + \mathbb{N} = v((z^m)) \subseteq v(R)$, therefore $n \leq m$. Conversely, we have

$$\{x \in R : v(x) \geq n\} \subseteq (z^n)$$

   and $v(\{x \in R : v(x) \geq n\}) = n + \mathbb{N} = v((z^n))$, so from Lemma 3.3.8 equality hold, hence $n \geq m$.

3. Since R is analytically irreducible, from Theorem 3.1.2 it R is a (one-dimensional) domain, therefore it is Cohen-Macaulay. Let $x \in \mathfrak{m}$, $x \neq 0$, from Proposition 1.2.3, Lemma 3.3.10 and by definition of type (respectively of rings and numerical semigroups), we have

$$\begin{aligned} \operatorname{Ext}^1(R/\mathfrak{m}, R) \simeq \operatorname{Hom}(R/\mathfrak{m}, R/xR) \simeq \\ \simeq (xR : \mathfrak{m})/xR = x(\mathfrak{m} : \mathfrak{m})/xR \simeq (\mathfrak{m} : \mathfrak{m})/R. \end{aligned}$$

   Therefore

$$t(R) = \lambda\big(\operatorname{Ext}^1(k, R)\big) = \lambda\big((\mathfrak{m} : \mathfrak{m})/R\big) = |v(\mathfrak{m} : \mathfrak{m}) \setminus v(R)|.$$

4. From Proposition 3.3.3 we have $v(\mathfrak{m} : \mathfrak{m}) \subseteq v(\mathfrak{m}) - v(\mathfrak{m})$, therefore

$$t(R) = |v(\mathfrak{m} : \mathfrak{m}) \setminus v(R)| \leq |(v(\mathfrak{m}) - v(\mathfrak{m})) \setminus v(R)| = t(v(R)). \qquad \square$$

The preceding inequalities may be strict, as the following example shows.

Example 3.3.12. *[4, Example II.1.19, Remark II.2.1] Let k be a field with* $\operatorname{char}(k) \neq 2$, *and $R = k[[t^4, t^6 + t^7, t^{10}]]$. Since*

$$\begin{aligned} t^{11} &= t^4(t^6 + t^7) - t^{10} \in R \\ t^{13} &= 1/2\big((t^6 + t^7)^2 - (t^4)^3 - (t^4)^2 t^6\big) \in R, \end{aligned}$$

we have $v(R) = \langle 4, 6, 11, 13\rangle$. *In addition* $\mathfrak{m} = (t^4, t^6 + t^7, t^{10})$, *thus*

$$\operatorname{emdim}(R) = 3 < 4 = e(v(R)).$$

It is easy to see that $v(\mathfrak{m}) - v(\mathfrak{m}) = \{2, 4, 6, \rightarrow\}$. *Now we prove that* $2 \notin v(\mathfrak{m} : \mathfrak{m})$. *Let* $y = a_2 t^2 + a_3 t^3 + \cdots \in k[[t]]$ *with* $a_2 \neq 0$ *(*$v(y) = 2$*). If* $yt^4 \in R$*, then* $a_2 = a_3$*, so that* $y(t^6 + t^7) = a_2 t^8 + 2a_2 t^9 + \ldots \notin R$ *since* $2a_2 \neq 0$*. This proves* $v(\mathfrak{m} : \mathfrak{m}) \subsetneq v(\mathfrak{m}) - v(\mathfrak{m})$*, in particular*

$$t(R) = |v(\mathfrak{m} : \mathfrak{m}) \setminus v(R)| < |(v(\mathfrak{m}) - v(\mathfrak{m})) \setminus v(R)| = t(v(R)).$$

Theorem 3.3.13. *(Kunz,[22]) R is Gorenstein* $\iff$ $v(R)$ *is symmetric.*

Proof. Set $n = F(v(R))$, the Frobenius number of $v(R)$.

$\Rightarrow$ Omitted.

$\Leftarrow$ Let $x \in (\mathfrak{m} : \mathfrak{m}) \setminus R$. Since $v(R)$ is symmetric, from Theorem 3.2.6 $n - v(x) \in v(R)$. If there exists $r' \in \mathfrak{m}$ such that $v(r') = n - v(x)$, then $v(xr') = v(x) + v(r') = n \notin v(R)$, contradicting $x \in \mathfrak{m} : \mathfrak{m}$. Therefore $v(x) = n$. This prove that $v(\mathfrak{m} : \mathfrak{m}) \setminus v(R) = \{n\}$, therefore

$$t(R) = |v(\mathfrak{m} : \mathfrak{m}) \setminus v(R)| = 1,$$

so by Theorem 1.4.2 R is Gorenstein. □

3.4 Semigroup rings

In this section we study semigroup rings, a subfamily of the rings studied in the previous section.

Definition 3.4.1. Let k be a field and t be a variable. If $S = \langle n_1, n_2, \ldots, n_e\rangle$ is a numerical semigroup, the **semigroup ring** associated to S is

$$k[[S]] = k[[t^s : s \in S]] = \left\{\sum a_i t^s : a_i \in k, s \in S\right\} \subseteq k[[t]].$$

The semigroup ring $k[[S]]$ is Noetherian, one-dimensional, local, analytically irreducible, residually rational, the maximal ideal of $k[[S]]$ is $\mathfrak{m} = (t^{n_1}, \ldots, t^{n_e})$. Further, $k[[S]]$ is the completion of the subring $k[S] = k[t^s : s \in S]$ of $k[t]$, with respect to the maximal ideal generated by the monomials $t^{n_1}, \ldots, t^{n_e}$. Consider the map $\varphi : k[x_1, \ldots, x_e] \to k[t]$ defined by $x_i \mapsto t^{n_i}$, then

$$k[S] \simeq \frac{k[x_1, \ldots, x_e]}{\ker \varphi}.$$

Therefore, $k[S]$ is the coordinate ring of the monomial curve parametrized by $x_i = t^{n_i}$.

A semigroup ring $k[[S]]$ satisfy additional properties that relates the ring to its associated numerical semigroup. For example, some of the inequalities of Theorem 3.3.11 are equalities.

Definition 3.4.2. An ideal I of $k[x_1, \dots, x_n]$ (or $k[[x_1, \dots, x_n]]$) is a **monomial ideal** if it is generated by monomials.

Note that the maximal ideal $\mathfrak{m}$ of $k[[S]]$ is monomial.

Proposition 3.4.3. *Let S be a numerical semigroup and $R = k[[S]]$, if I and J are monomial ideals, then*

1. $v(IJ) = v(I) + v(J)$,
2. $v(I : J) = v(I) - v(J)$.

Proof.

1. $\supseteq$ Proposition 3.3.3.

   $\subseteq$ Let $n \in v(IJ)$, there exist $i \in I$ and $j \in J$ such that $n = v(ij)$. Since I and J are monomial, we can assume that

   $$ij = \sum_{k=1}^{t} a_k t^{i_k} t^{j_k}$$

   with a_k invertible and $t^{i_k} \in I$, $t^{j_k} \in J$ for all k. Therefore, if $t^{i_1}t^{j_1}$ is the monomial of minimum degree, we have

   $$n = v(ij) = v(t^{i_1}t^{j_1}) = v(t^{i_1}) + v(t^{j_1}) = i_1 + j_1 \in v(I) + v(J).$$

2. $\subseteq$ Proposition 3.3.3.

   $\supseteq$ Let $n \in v(I) - v(J)$, it is enough to prove that $t^n \in (I : J)$. If $j \in J$ then $v(t^n j) = n + v(j) \in v(I)$, so, since I and J are monomial, we have $t^n j \in I$.

□

Corollary 3.4.4. *Let S be a numerical semigroup and $R = k[[S]]$, then*

1. $\mathrm{emdim}(R) = \mathrm{e}(v(R))$.
2. $t(R) = t(v(R))$.

Proof. Since the maximal ideal $\mathfrak{m}$ of R is monomial, from the previous Proposition we have

- $v(\mathfrak{m}^2) = v(\mathfrak{m}) + v(\mathfrak{m})$,
- $v(\mathfrak{m} : \mathfrak{m}) = v(\mathfrak{m}) - v(\mathfrak{m})$.

Now the equalities follows applying the proofs of Theorem 3.3.11. □

Chapter 4

Stable ideals and Arf rings

In this chapter we will discuss the theory of Arf rings and Arf numerical semigroups. Our main references for this chapter are [19], [24], [30] and [4].

4.1 Integral closure of an ideal and the Rees algebra

Definition 4.1.1. Let I be an ideal of a ring R. An element $r \in R$ is **integral over** I if there exist $n \in \mathbb{N}$ and elements $a_i \in I^i$ for $i \in \{1, \ldots, n\}$ such that

$$r^n + a_1 r^{n-1} + \cdots + a_{n-1} r + a_n = 0.$$

The set of all elements in R integral over I is called the **integral closure** of I denoted by $\overline{I}$. If $I = \overline{I}$, then I is **integrally closed**.

We want to show that the integral closure of an ideal of a ring R is again an ideal of R. To do this, we introduce the so-called Rees algebra

Definition 4.1.2. Let R be a ring, I an ideal of R and t an indeterminate. The **Rees algebra** of I is the subring of $R[t]$ defined as

$$R[It] = \bigoplus_{n \in \mathbb{N}} I^n t^n = \left\{ \sum_{i=1}^{n} a_i t^i : a_i \in I^i, n \in \mathbb{N} \right\}.$$

We now prove a weaker version of [19, Proposition 5.2.1] (see also [24, Section 2]).

Proposition 4.1.3. *An element of a ring $z \in R$ is integral over an ideal I if and only if $zt \in R[t]$ is integral over the Rees algebra $R[It]$.*

Proof. Necessity. Suppose that

$$z^n + a_1 z^{n-1} + \cdots + a_n = 0$$

for some $a_i \in I^i$. Then by multiplying by t^n we obtain

$$(zt)^n + (a_1 t)(zt)^{n-1} + \cdots + a_n t^n = 0$$

which is a relation of integral dependence of $zt \in R[t]$ in $R[It]$.
Sufficiency. Suppose that $zt \in R[t]$ is integral over $R[It]$, so

$$(zt)^n + a_1(zt)^{n-1} + \cdots + a_n = 0$$

for some $a_i \in (It)^i$. Expanding each $a_i = \sum_{j=1}^{k_i} a_{i,j}t^j$, the homogeneous part of degree n with respect to t of the preceding equation is

$$t^n(z^n + a_{1,1}z^{n-1} + \cdots + a_{n,n}) = 0$$

where $a_{i,i} \in I^i$, hence we get a relation of integral dependence of z on the ideal I. □

Corollary 4.1.4. *The integral closure of an ideal is and ideal.*

Proof. Let I be an ideal of the ring R. If $x.y \in \overline{I}$, then $xt, yt \in R[t]$ are integral over $R[It]$, therefore, since the integral closure of $R[It]$ in $R[t]$ is a ring ([2, Corollary 5.3]), $(x+y)t$ is also integral over $R[It]$, so $x + y \in \overline{I}$. Similarly, for every $r \in R$ we have $rx \in \overline{I}$. □

Now we report, without proof, two results about the integral closure of an ideal.

Proposition 4.1.5. *[19, Proposition 1.6.1] Let $R \subseteq S$ be an integral extension of rings, and I be an ideal of R. Then $\overline{IS} \cap R = \overline{I}$.*

Proposition 4.1.6. *[19, Proposition 6.8.1] Let R be an integral domain with field of fractions $Q(R)$, I be and ideal of R and V be a valuation ring between R and $Q(R)$. Then $IV = \overline{I}V = \overline{IV}$.*

If R is a Noetherian, one-dimensional, local, analytically irreducible, domain then $\overline{R}$ is a (discrete) valuation ring between R and $Q(R)$ and clearly $R \subseteq \overline{R}$ is integral. Now set $v : Q(R) \to \mathbb{Z}$ the valuation associated to R, and let I be an ideal of R. From the previous two propositions we have

$$\overline{I} = \overline{I\overline{R}} \cap R = I\overline{R} \cap R = \{x \in R : v(x) \geq \min v(I)\}.$$

Recall that $v(R)$ is a numerical semigroup, and from Remark 3.3.2, $v(I)$ is a (semigroup) ideal of S. The previous equality leads to the following definition, which is the numerical counterpart of integral closure and integrally closed ideals.

Definition 4.1.7. Let E be an ideal of a numerical semigroup S. The **integral closure** of E is

$$\overline{E} = \{x \in \mathbb{N} : x \geq \min E\}.$$

If $E = \overline{E}$, then E is **integrally closed**.

Lemma 4.1.8. *Let R be as above. If R is in addition residually rational, then every ideal I of R is integrally closed if and only if the ideal $v(I)$ of $v(R)$ is integrally closed.*

Proof. If I is integrally closed, then clearly also $v(I)$ is integrally closed. Conversely, if $v(I)$ is integrally closed, then $v(I) = v(\overline{I})$, so from Lemma 3.3.8 we have $I = \overline{I}$. □

4.2 Stable ideals

In this and the following sections, $(R, \mathfrak{m})$ will be a one-dimensional Noetherian local Cohen-Macaulay ring. Any fractions of elements of R will be implicitly intended in the total ring of fractions of R.

Definition 4.2.1. An ideal I of R is **open** if it contains a regular element. An element x of and open ideal I is I**-transversal** if $xI^n = I^{n+1}$ for some $n > 0$.

From [21, Proposition 1.18, page 74] if $R/\mathfrak{m}$ is infinite, then every open ideal has an I-transversal element.

Proposition-Definition 4.2.2. *An ideal I of R is* ***stable*** *if there exists $x \in I$ that satisfy one of the following equivalent conditions*

1. *$I^2 = xI$.*

2. *x is regular and Ix^{-1} is a ring.*

Proof.

(1) $\Rightarrow$ (2) If $I^2 = xI$, then x is regular and $(Ix^{-1})^2 = Ix^{-1}$, so Ix^{-1} is closed under multiplication, therefore it is a ring.

(2) $\Rightarrow$ (1) $(Ix^{-1})^2 = Ix^{-1} \Rightarrow I^2 = xI$. □

If we assume in addition that R is analytically irreducible, and $v : Q(R) \to \mathbb{Z}$ is its associated valuation, then if I is a stable ideal of R we have

$$2v(I) = v(I) + v(I) = v(I^2) = v(xI) = v(x) + v(I),$$

where necessarily $v(x) = \min v(I)$. This observation leads to the following definition, which is the numerical counterpart of stable ideals.

Definition 4.2.3. If E is an ideal of a numerical semigroup S, then E is **stable** if $2E = E + \min E$.

4.3 Arf rings

Definition 4.3.1. The ring R is an **Arf ring** if

1. every integrally closed open ideal I has an I-transversal element;

2. for every $x, y, z \in R$ such that x is regular and $y/x, z/x \in \overline{R}$, we have $\dfrac{yz}{x} \in R$.

Lemma 4.3.2. *Let I, J be two ideal of R with I open, and let $x, z \in R$ with x regular.*

1. *z is integral over (x) if and only if $z/x \in \overline{R}$.*

2. *If $zI \subseteq JI$ then z is integral over J.*

Proof.

1. It is enough to observe that

$$z^n + a_1 x z^{n-1} + \cdots + x^n a_n = 0 \iff \left(\frac{z}{x}\right)^n + a_1 \left(\frac{z}{x}\right)^{n-1} + \cdots + a_0 = 0,$$

where the first equation is a relation of integral dependence of z over the ideal (x), whereas the second is a relation of integral dependence of z/x over R.

2. Suppose that $I = (x_1, \ldots, x_n)$, then from $zI \subseteq JI$, for every $i \in \{1, \ldots, n\}$, we have

$$zx_i = \sum_{j=1}^{n} a_{i,j} x_j \Rightarrow \sum_{j=1}^{n} (a_{i,j} - \delta_{i,j} z) x_j = 0$$

for some $a_{i,j} \in J$, where $\delta_{i,j}$ is the Kronecker delta. By multiplying by the transpose of the adjoint matrix of $A = (a_{i,j} - \delta_{i,j} z)$ we obtain $\det(A) x_i = 0$. Since I is open, it contains some regular element y, therefore $\det(A) y = 0$, so $\det(A) = 0$, and this is an equation of integral dependence of z over J. □

Corollary 4.3.3. *Let I be an open ideal of R. If x is an I-transversal element of I, then $Ix^{-1} \subseteq \overline{R}$.*

Proof. If $z \in I$, then $zI^n \subseteq I^{n+1} = xI^n$ for some $n \in \mathbb{N}$. From the previous lemma, z is integral over (x), equivalently $z/x \in \overline{R}$. □

Theorem 4.3.4. *The following conditions on R are equivalent*

1. *R is an Arf ring.*
2. *Every integrally closed open ideal in R is stable.*

Proof.

(1) ⇒ (2) Let I be an integrally closed open ideal of R. If x is an I-transversal element, from 4.2.2 it is enough to show that Ix^{-1} is a ring, i.e. it is multiplicatively closed. Let $y/x, z/x \in Ix^{-1}$, from Corollary 4.3.3 $y/x, z/x \in \overline{R}$, then by hypothesis we have $w = yz/x \in R$. Now

$$\frac{w}{x} = \frac{y}{x} \cdot \frac{z}{x} \in \overline{R}$$

so from Lemma 4.3.2 $w \in \overline{I} = I$, so $y/x \cdot z/x = w/x \in Ix^{-1}$.

(2) ⇒ (1) If an integrally closed open ideal I is stable, then I has an I-transversal element. Now if $x, y, z \in R$ such that x is regular and $y/x, z/x \in \overline{R}$, denote by I the integral closure of (x). From Lemma 4.3.2 we have that Ix^{-1} is a ring, so from Proposition 4.2.2 $I^2 = xI$. Now, again from Lemma 4.3.2 $y, z \in I$, so $yz \in I^2 = xI \subseteq xR$. □

We now report, without proof, the existence of the smallest Arf ring that contains a fixed ring R.

Proposition-Definition 4.3.5. *[24, Proposition-Definition 3.1] Among the Arf rings between R and $\overline{R}$ there exists one of them,* $\mathrm{Arf}(R)$, *which is contained in all the others; it is called the* ***Arf closure*** *of R.*

4.4 Arf numerical semigroups

In this section we want to introduce the notion of Arf numerical semigroups. For further reading see [30].

Definition 4.4.1. A numerical semigroup S is an **Arf numerical semigroup** if for every $x, y, z \in S$, $x \geq y \geq z$ we have $x + y - z \in S$.

The preceding definition codify numerically, via the valuation, the second condition of Definition 4.3.1. In fact, if R is an Arf ring, then its value semigroup $v(R)$ is an Arf numerical semigroup. However, the converse is not true. In fact, while as we saw in Theorem 3.3.13, the symmetry of the value semigroup determines whether the ring R is Gorestein or not, this does not happen for Arf rings, as the following example shows.

Example 4.4.2. *[4, Remark II.2.14] Let k be a field of characteristic not 2 and set $R = k[[t^4, t^6 + t^7, t^{10}]]$. The value semigroup of R is $v(R) = \langle 4, 6, 11, 13\rangle = \{0, 4, 6, 8, 10, \rightarrow\}$ which is an Arf numerical semigroup (it is enough to verify the relation $x + y - z$ for all $z \leq y \leq x < c(v(R)) = 10$). However, the maximal ideal $\mathfrak{m} = (t^4, t^6 + t^7, t^10)$ (that is trivially integrally closed) is not stable, in fact $\mathfrak{m}^2 = (t^8, t^{10} + t^{11}, t^{12} + 2t^{13}, t^{14}) \neq x\mathfrak{m}$ for every $x \in \mathfrak{m}$.*

The following result easily follows from the definition.

Proposition 4.4.3. *[30, Proposition 1] A finite intersection of Arf numerical semigroups is again an Arf numerical semigroup.*

Given a numerical semigroup S, from definition, the family of numerical semigroups that contains S is finite, so a fortiori also the family $\mathcal{A}$ of Arf numerical semigroups that contains S is finite. By the previous proposition, the intersection of the semigroups of $\mathcal{A}$ is again an Arf numerical semigroup, which is the minimum of $\mathcal{A}$, with respect to inclusion.

Definition 4.4.4. Let S be a numerical semigroup and let $\mathcal{A}$ be the family of Arf numerical semigroups that contains S. The **Arf closure** of S is the Arf numerical semigroup

$$\mathrm{Arf}(S) = \bigcap_{T \in \mathcal{A}} T.$$

From the previous observations, the Arf closure of a numerical semigroup S is the smallest, with respect to inclusion, Arf numerical semigroup that contains S. For any $X \subseteq \mathbb{N}$, we set

$$\mathrm{Arf}(X) = \mathrm{Arf}(\langle X \rangle).$$

Lemma 4.4.5. *[30, Lemma 11] Let S be an Arf numerical semigroup. If $x, x + 1 \in S$, then $x + \mathbb{N} \subseteq S$.*

Proof. We prove that $x + n \in S$ by induction on $n \in \mathbb{N}$. The base case follows from the hypothesis. Now for the inductive step $x + (n + 1) = 2(x + n) - (x + n - 1) \in S$. $\square$

Definition 4.4.6. Let $S = \{0 = s_0 < s_1 < s_2 < \dots\}$ be an Arf numerical semigroup, set $e_i = s_{i+1} - s_i$. The sequence $(e_0, e_1, e_2, \dots)$ is the **multiplicity sequence** of S.

Lemma 4.4.7. *If S is an Arf numerical semigroup and $x \in S$, then $\{0\} \cup (x+S)$ is an Arf numerical semigroup.*

Proof. It is easy to see that $\{0\} \cup (x+S)$ is a numerical semigroup. Now let $s_1, s_2, s_3 \in S$ such that $s_1 \geq s_2 \geq s_3$. Since S is Arf, we have $s_1 + s_2 - s_3 \in S$, therefore

$$(x+s_1) + (x+s_2) - (x+s_3) = x + (s_1 + s_2 - s_3) \in x + S. \qquad \square$$

Proposition-Definition 4.4.8. *A sequence of non negative integers $(e_0, e_1, e_2, \dots)$ is the multiplicity sequence of some Arf numerical semigroup if and only if it satisfies the following two conditions.*

1. *Exists $n \in \mathbb{N}$ such that for every $k \geq n$ we have $e_k = 1$.*
2. *$e_i = \sum_{j=1}^{h} e_{i+j}$ for some $h \in \mathbb{N}$.*

*Every sequence that satisfies the previous conditions is said to be an **Arf sequence**.*

Proof. Necessity. Suppose that $\{e_i\}_{i \in \mathbb{N}}$ is the multiplicity sequence of some Arf numerical semigroup $S = \{s_0 < s_1 < \dots\}$, so that $s_i = \sum_{j=0}^{i-1} e_j$. Then, by definition of numerical semigroup, for $n \gg 0$ we have $e_n = s_{n+1} - s_n = 1$. Now, for every $i \in \mathbb{N}$ since S is Arf, $2s_{i+1} - s_i = s_{i+h+1} \in S$ for some $h \in \mathbb{N}$. Therefore

$$e_i = s_{i+1} - s_i = s_{i+h+1} - s_{i+1} = \sum_{j=0}^{i+h} e_j - \sum_{j=0}^{i} e_j = \sum_{j=i+1}^{i+h} e_j = \sum_{j=1}^{h} e_{i+j}.$$

Sufficiency. It is enough to show that $S = \{0, e_0, e_0 + e_1, e_0 + e_1 + e_2, \dots\}$ is an Arf numerical semigroup. In order to do that, set

$$\begin{aligned} &S_0 = \mathbb{N}, \\ &S_i = \{0\} \cup (e_{n-1} + S_{i-1}) \quad i \geq 1. \end{aligned}$$

From Lemma 4.4.7 every S_i is an Arf numerical semigroup, further $S = S_n$ for $n \gg 0$. $\square$

Finally, we can give a numerical version of Theorem 4.3.4. The proof is the same *mutatis mutandis*.

Theorem 4.4.9. *For a a numerical semigroup S, the following conditions are equivalent.*

1. *S is Arf.*
2. *Every integrally closed ideal of S is stable.*

Now let $(R, \mathfrak{m})$ be a one-dimensional, Noetherian, local, analytically irreducible, residually rational, domain. Under this hypothesis, the **multiplicity** of R is

$$\mu(R) = \mu(v(R))$$

(see for instance [4, II.2.4]).

Now, for any two R-submodules E, F of $\overline{R}$, set

$$(E : F) = \{x \in \overline{R} : xF \subseteq E\}.$$

Definition 4.4.10. The **Lipman ring** of R is

$$L(R) = \bigcup_{n \in \mathbb{N}} (\mathfrak{m}^n : \mathfrak{m}^n).$$

Now, set $R_0 = R$ and $R_{i+1} = L(R_i)$ for every $i \in \mathbb{N}$. The sequence $(\mu(R_0), \mu(R_1), \mu(R_2), \dots)$ is **the multiplicity sequence** of R.

We saw in Example 4.4.2 that the Arf property of the value semigroup is not enough to imply that the corresponding ring is Arf. Nonetheless, the Arf property can be characterized if we consider in addition a condition on the corresponding multiplicity sequences. We report the characterization without proof.

Theorem 4.4.11. *[4, Theorem II.2.13] The ring R is Arf if and only if $v(R)$ is Arf and the multiplicity sequences of R and $v(R)$ coincides.*

Chapter 5

Quadratic quotients of the Rees algebra

In this chapter we will present some results about a family of quadratic quotients of the Rees algebra. We will follow [3], [14], [13] and [6].

5.1 Basic properties

In this section, R is a ring, I is an ideal of R and t is an indeterminate. If $f \in R[t]$, we denote by $(f(t))$ the ideal generated by $f(t)$ in $R[t]$. For every $f \in R[t]$, from [3, Lemma 1.1, Lemma 1.2] we have the following inclusions

$$R \subseteq \frac{R[It]}{(f(t)) \cap R[It]} \subseteq \frac{R[t]}{(f(t))} \tag{5.1}$$

Proposition 5.1.1. *[3, Proposition 1.3] The ring extensions (5.1) are integral and the three rings have the same Krull dimension.*

Proof. It is enough to note that the class of t in $R[t]/(f(t))$ is integral over R. Now the statement follows from the transitivity of integral extensions and from well known theorems on the Krull dimension. □

We now recall two ring constructions that can be obtained as quotients of the Rees algebra. The first is the **Nagata's idealization** of R with respect to an ideal I of R (that could be defined for any R-module M); it is defined as the R-module $R \oplus I$ endowed with the multiplication $(r, i)(s, j) = (rs, rj + si)$ and it is denoted by $R \ltimes I$. The second is the **duplication** of R with respect to I, defined as the R-module $R \oplus I$ endowed with the multiplication $(r, i)(s, j) = (rs, rj + si + ij)$.

Proposition 5.1.2. *[3, Proposition 1.4] We have the following isomorphisms of rings:*

1. $R \bowtie I \simeq \dfrac{R[It]}{(t^2) \cap R[It]}$,
2. $R \ltimes I \simeq \dfrac{R[It]}{(t^2 - t) \cap R[It]}$.

Proof. It is enough to verify that the map that assigns (r, i) to the class of $r + it$ is an isomorphisms of rings in both cases. □

More generally, it is now natural to consider the family of quadratic quotients

$$R(I)_{a,b} = \frac{R[It]}{(t^2+at+b)\cap R[It]} \quad a,b \in R.$$

The previous proposition can be restated as $R(I)_{0,0} \simeq R \bowtie I$, $R(I)_{-1,0} \simeq R \ltimes I$.

We now report, without proof, some properties of the family $R(I)_{a,b}$.

Proposition 5.1.3. *[3, Corollary 1.8] If I is a regular ideal, then the rings $R(I)_{a,b}$ and $R[t]/(t^2+at+b)$ have the same total ring of fractions and the same integral closure. In particular, $R(I)_{a,b}$ is integrally closed if and only if so is $R[t]/(t^2+at+b)$.*

Proposition 5.1.4. *[3, Proposition 1.11] The following conditions are equivalent:*

1. *R is a Noetherian ring;*
2. *$R(I)_{a,b}$ is a Noetherian ring for all $a,b \in R$;*
3. *$R(I)_{a,b}$ is a Noetherian ring for some $a,b \in R$.*

Proposition 5.1.5. *[3, Proposition 2.1] R is local with maximal ideal $\mathfrak{m}$ if and only if $R(I)_{a,b}$ is local with maximal ideal $\mathfrak{m} + It$.*

Definition 5.1.6. Let R be a one-dimensional, Noetherian, local, ring. A regular fractional ideal $\omega \subseteq Q(R)$ is a **canonical ideal** if

$$\omega : (\omega : F) = F$$

for every regular fractional ideal $F \subseteq Q(R)$.

Proposition 5.1.7. *[3, Corollary 3.3] Let R be a one-dimensional, Noetherian, local ring and I is a regular ideal. Then, $R(I)_{a,b}$ is Gorenstein if and only if I is a canonical ideal of R, for every $a,b \in R$.*

Proposition 5.1.8. *[14, Corollary 1.3] $R(I)_{a,b}$ is an integral domain if and only if R is an integral domain and t^2+at+b is irreducible in $Q(R)[t]$.*

5.2 The numerical duplication

For every $A \subseteq \mathbb{Z}$, we set $2 \cdot A = \{2a : a \in A\}$, note that $2 \cdot A \neq 2A = A + A$.

Definition 5.2.1. Let S be a numerical semigroup, E and ideal of S and $d \in S$ an odd integer. The **numerical duplication** of S with respect to E and d is the following subset of $\mathbb{N}$

$$S \bowtie^d E = 2 \cdot S \cup (2 \cdot E + d).$$

It is straightforward to check that $S \bowtie^d E$ is a numerical semigroup. We now report without proof some basic properties of the numerical duplication. If $E \subseteq S$ is a semigroup ideal of S, we denote by $F(E)$ and $g(E)$ respectively the Frobenius number and the genus of the numerical semigroup $E \cup \{0\}$).

Proposition 5.2.2. *[13, Proposition 2.1]*

1. $F(S \bowtie^d E) = 2F(E) + d$.
2. $g(S \bowtie^d E) = g(S) + g(E) + \frac{d-1}{2}$.
3. *$S \bowtie^d E$ is symmetric if and only if E is a canonical ideal of S.*

Now we want to show what is the connection between quadratic a family of quotients of the Rees algebra and the numerical duplication.

Let $(R, \mathfrak{m})$ be a Noetherian, analytically irreducible, residually rational, one-dimensional, local, domain with $\mathrm{char}(R) \neq 2$. Let $v : Q(R) \to \mathbb{Z}$ be the valuation on $Q(R)$ associated to $\overline{R}$. Let I be an ideal of R, t be an indeterminate and $b \in R$ such that $v(b) = d$ is odd. Denote by $\alpha = t+(t^2-b)$ the class of t in $R[t]/(t^2-b)$. Then, $R(I)_{0,-b}$ is a subring of $R[\alpha]$. Furthermore, $R[\alpha]$ and $R(I)_{0,-b}$ have the same ring of fractions $Q(R)[\alpha]$ (see [3, Corollary 1.8]), and the same integral closure $\overline{R(I)_{0,-b}}$, which is equal to the integral closure of R in $Q(R)[\alpha]$. The extension field $Q(R) \subseteq Q(R)[\alpha]$ is finite and since $\mathrm{char}(R) \neq 2$, it is also separable.

Theorem 5.2.3. *[6, Theorem 3.1] The ring $R(I)_{0,-b}$ is a Noetherian one-dimensional local domain analytically irreducible and residually rational. If $v' : Q(R)[\alpha] \to \mathbb{Z}$ is the extension on $Q(R)[\alpha]$ of the valuation of $\overline{R(I)_{0,-b}}$, then $v'_{|Q(R)} = 2v$, $v'(\alpha) = m$ and*

$$v'(R(I)_{0,-b}) = v(R) \bowtie^{v(b)} v(I).$$

Proof. Denote by $\mathcal{R} = R(I)_{0,-b}$. From [3] we know that if R is Noetherian, one-dimensional and local so is $\mathcal{R}$. Moreover, since $v(b)$ is odd, the polynomial $t^2 - b$ is irreducible in $Q(R)[t]$, then, from [14, Corollary 1.3], $\mathcal{R}$ is a domain.
Now we prove that $\mathcal{R}$ is analytically irreducible. It is enough to prove that $\overline{\mathcal{R}}$ is local and a finitely generated $\mathcal{R}$-module. Let $x \in \overline{R}$ be an element of valuation 1, $k = \frac{m-1}{2}$ and $\beta = \frac{\alpha}{x^k} \in Q(R)[\alpha]$; then $\beta^2 = \frac{b}{x^{m-1}} \in \overline{R}$ since $v(\frac{b}{x^{m-1}}) = 1 > 0$, so β is integral over $\overline{R}$. We prove that $\overline{\mathcal{R}} = \overline{R} + \overline{R}\beta$. The inclusion $\overline{R} + \overline{R}\beta \subseteq \overline{\mathcal{R}}$ follows from the fact that $\overline{R} \subseteq \overline{\mathcal{R}}$ and that β is integral over $\overline{R}$. Conversely, let $p + q\alpha \in \overline{\mathcal{R}} = \overline{R}^{Q(R)[\alpha]}$, where $p, q \in Q(R)$, $q \neq 0$. Then, since $Q(R) \subseteq Q(R)[\alpha]$ is algebraic, from [19, Theorem 2.1.17] the coefficients $2p$ and $p^2 - q^2 b$ of the minimal polynomial of $p + q\alpha$ over $Q(R)$ are in $\overline{R}$. In addition, $v(p^2) = 2v(p)$ is even and $v(q^2 b) = 2v(q) + m$ is odd, then $v(p^2) \neq v(q^2 b)$, therefore

$$0 \leq v(p^2 - q^2 b) = \min\{2v(p), 2v(q) + m\}$$
$$\Rightarrow \begin{cases} v(p) \geq 0 \Rightarrow p \in \overline{R} \\ 2v(q) + m \geq 0 \Rightarrow v(q) \geq -k \Rightarrow qx^k \in \overline{R}. \end{cases}$$

Hence $p+q\alpha = p+qx^k\beta \in \overline{R}+\overline{R}\beta$. Now if we denote by $\overline{\mathfrak{m}}$ the maximal ideal of $\overline{R}$, the ring $\overline{R}+\overline{R}\beta$ is local with maximal ideal $\overline{M} = \overline{\mathfrak{m}}+\overline{R}\beta$; indeed the inverse of $p+q\beta \in \overline{R}+\overline{R}\beta$ with $p \in \overline{R} \setminus \overline{\mathfrak{m}}$, is $\frac{p-q\beta}{p^2-q^2\beta^2}$, in fact $p^2 - q^2\beta^2$ is invertible since $0 = v(p^2) \neq v(q^2\beta^2) = 2v(q) + 1$ and

$$v(p^2 - q^2\beta^2) = \min\{v(p^2), v\left(q^2\beta^2\right)\} = v(p^2) = 0.$$

It follows that $\overline{\mathcal{R}} = \overline{R} + \overline{R}\beta$ is local.

Now we prove that $\overline{\mathcal{R}}$ is a finitely generated $\mathcal{R}$-module. The field extension $Q(R) \subseteq Q(R)[\alpha]$ is finite and separable, then, by [19, Theorem 3.1.3], the integral closure of $\overline{R}$ in $Q(R)[\alpha]$, which is equal to $\overline{R}^{Q(R)[\alpha]} = \overline{\mathcal{R}}$, is a finite module over $\overline{R}$. Now $\overline{R}$ and $\mathcal{R} \simeq R + I\alpha$ are finite modules over R, so $\overline{\mathcal{R}}$ is a finite module over $\mathcal{R}$.

Now let $v'(Q(R)) = d\mathbb{Z}$ for some $d \in \mathbb{N}$. It results $v'(x) = d$ and $b \in (x^m) \setminus (x^{m+1})$, namely $b = ux^m$ with $u \in \overline{R}$ invertible. It follows that $v'(b) = mv'(x)$; in addition from $\alpha^2 = b$ we obtain $2v'(\alpha) = v'(b) = mv'(x) = md$. Since

$$\begin{aligned} v'(Q(R)[\alpha]) = v'(Q(R) + Q(R)\alpha) = v'(Q(R)) \cup [v'(Q(R)) + v'(\alpha)] = \\ = d\mathbb{Z} \cup (d\mathbb{Z} + v(\alpha)) = \mathbb{Z}, \end{aligned}$$

it must be $d = 2$, so $v'_{|Q(R)} = 2v$ and also $v'(\alpha) = m$. It easily follows that

$$v'(\mathcal{R}) = v(R) \bowtie^{v(b)} v(I) = S \bowtie^m v(I).$$

Finally we show that $\mathcal{R}$ is residually rational. Recall that $\overline{\mathfrak{m}}$ is the maximal ideal of $\overline{R}$ and $\overline{M} = \overline{\mathfrak{m}} + \overline{R}\beta$ is the maximal ideal of $\overline{\mathcal{R}}$. From [3, Proposition 2.1] the maximal ideal of $\mathcal{R}$ is $M = \mathfrak{m} + I\alpha$. Thus

$$\overline{\mathcal{R}}/\overline{M} = \frac{\overline{R} + \overline{R}\beta}{\overline{\mathfrak{m}} + \overline{R}\beta} \simeq \overline{R}/\overline{\mathfrak{m}} \simeq R/\mathfrak{m} \simeq \frac{R + I\alpha}{\mathfrak{m} + I\alpha} = \mathcal{R}/M. \qquad \square$$

In the light of the Theorem of Kunz 3.3.13 and Theorem 5.2.3, point (3) of Proposition 5.2.2 is the numerical version of a special case of Proposition 5.1.7.

5.3 A characterization of the Arf property

In this section we want to show how the Arf property can be characterized, first numerically on the numerical duplication, this characterization algebraically on the family of quadratic quotients of the Rees algebra of Theorem 5.2.3.

5.3.1 Numerical characterization

In this subsection, S will be a numerical semigroup, E a semigroup ideal of S and $d \in S$ an odd integer. Recall that the **quotient** of S by a positive integer k is

$$\frac{S}{k} = \{x \in \mathbb{N} : kx \in S\}.$$

Proposition 5.3.1. *For every $k > 0$, if S is Arf so is $\frac{S}{k}$.*

Proof. Let $x, y, z \in \frac{S}{k}$ with $x \geq y \geq z$, then we have $kx, ky, kz \in S$ with $kx \geq ky \geq kz$ and since S is Arf it follows that

$$k(x + y - z) = kx + ky - kz \in S,$$

hence $x + y - z \in \frac{S}{k}$. $\square$

By definition, the numerical duplication is the inverse of the quotient by two. In other words $(S \bowtie^d E)/2 = S$, hence we immediately get the following

Corollary 5.3.2. *If $S \bowtie^d E$ is Arf so is S.*

The next lemma gives us another necessary condition for the numerical duplication to be Arf.

Lemma 5.3.3. *If $S \bowtie^d E$ is Arf then E is integrally closed.*

Proof. Suppose by contradiction that E is not integrally closed. Then there exists $i \in \mathbb{N}$ such that $s_i \in E$ and $s_{i+1} \notin E$. Consider $2s_{i+1}, 2s_i + m, 2s_i \in S \bowtie^m E$, since $2s_{i+1} \geq 2s_i$, $2s_i + m \geq 2s_i$ and $S \bowtie^m E$ is Arf, we have

$$2s_{i+1} + 2s_i + m - 2s_i = 2s_{i+1} + m \in S \bowtie^m E,$$

which means $s_{i+1} \in E$, contradiction. □

As the following theorem shows, the previous two necessary conditions, for the numerical duplication to be Arf, are, in some cases, also sufficient. In the other cases, an additional condition on the multiplicity sequence is required, and this allows us to characterize the Arf property.

First, recall that if $S = \{0 = s_0 < s_1 < s_2 < \dots\}$ is an Arf numerical semigroup, its **multiplicity sequence** is $(e_0, e_1, e_2, \dots)$, where $e_i = s_{i+1} - s_i$. We fix $n \in \mathbb{N}$ to be the smallest integer such that $e_m = 1$ for every $m \geq n$. In particular $s_{n+1} = s_n + 1$ and $s_n = c(S)$ is the conductor of S.

Theorem 5.3.4. *[6, Theorem 2.4] The numerical duplication $D = S \bowtie^d E$ is Arf if and only if S is Arf, E is integrally closed and, if $\min(E) < c(S)$, the multiplicity sequence of S is $(d, d, \dots, d, 1, 1, \dots,)$.*

Proof. Necessity. From Corollary 5.3.2 and Lemma 5.3.3, S is Arf and E is integrally closed. Now if $\min(E) < c(S) = s_n$ then $s_{n-1} \in E$. Suppose that $d \geq 2e_{n-1} = 2(s_n - s_{n-1})$, then

$$2s_{n-1} + d \geq 2s_n.$$

Since $2s_{n-1} + d + 1$ is even and s_n is the conductor of S we obtain $2s_{n-1} + d + 1 = 2s_k$ for some $k \in \mathbb{N}$. Setting $x = 2s_{n-1} + d$ we have $x, x+1 \in D$ which is Arf, hence from Lemma 4.4.5 we have $x + \mathbb{N} \subseteq D$, and so $x + 2 \in D$; this means

$$\begin{gathered} 2s_{n-1} + d + 2 = 2(s_{n-1} + 1) + d \in D \Rightarrow \\ \Rightarrow s_{n-1} + 1 \in S \Rightarrow s_n = s_{n-1} + 1 \Rightarrow \\ \Rightarrow e_{n-1} = s_n - s_{n-1} = 1, \end{gathered}$$

which is a contradiction. Therefore $d < 2e_{n-1}$, hence $2s_{n-1} + d < 2s_n$, since D is Arf, this implies

$$2s_n + 2s_n - (2s_{n-1} + d) = 2s_n + 2e_{n-1} - d \in D.$$

Furthermore $2s_n + 2e_{n-1} - d$ is odd and

$$2s_n + 2e_{n-1} - d \geq 2s_n > 2s_{n-1} + d.$$

It follows that

$$\begin{gathered} 2s_n + 2e_{n-1} - d \geq 2s_n + d \\ \Rightarrow d \leq e_{n-1} \leq e_0. \end{gathered}$$

Since $d \in S$ and it is odd, we must have $d = e_0 = e_1 = \ldots = e_{n-1}$.
Sufficiency. If $\min(E) \geq s_n$, then $E = x + \mathbb{N}$ with $x \geq s_n$, so it results

$$D = 2S \cup ((2x + d) + \mathbb{N})$$

and it is easy to check that D is Arf.
Otherwise if $\min(E) < s_n$ and $d = e_0 = e_1 = \ldots = e_{n-1}$, then $S = e_0\mathbb{N} \cup (ne_0 + \mathbb{N})$ and $E = \{ie_0, (i+1)e_0, \ldots, (n-1)e_0\} \cup (ne_0 + \mathbb{N})$ for some $i \leq n$. Hence, if $D = \{0 = d_0 < d_1 < \ldots < d_k < \ldots\}$ then after some easy calculations it results

$$(d_{k+1} - d_k : k \in \mathbb{N}) = (\underbrace{2e_0, 2e_0, \ldots, 2e_0}_{i \text{ times}}, \underbrace{e_0, e_0, \ldots, e_0}_{2(n-i) \text{ times}}, \underbrace{2, 2, \ldots, 2}_{\frac{e_0-1}{2} \text{ times}}, 1, \ldots),$$

which is an Arf sequence, so D is Arf. □

Example 5.3.5. *Let $S = \langle 3, 7, 8 \rangle = \{0, 3, 6, \rightarrow\}$, S is Arf and its multiplicity sequence is $(3, 3, 1, \ldots)$, so $n = 2$. Let $E = S \setminus \{0\}$ and $d = 3$, E is integrally closed, $\min(E) = 3 < 6 = s_n$ and $d = e_0 = e_1$. The numerical duplication is*

$$S \bowtie^d E = \langle 6, 9, 14, 16, 17, 19 \rangle = \{0, 6, 9, 12, 14, \rightarrow\},$$

and it is an Arf numerical semigroup.

Let $\tilde{E}$ be the integral closure in $\mathrm{Arf}(S)$ of the ideal generated by E in $\mathrm{Arf}(S)$. More explicitly, if $\tilde{e} = \min E$, then $\tilde{E} = \{s \in \mathrm{Arf}(S) : s \geq \tilde{e}\}$. Since if the numerical duplication $S \bowtie^d E$ is Arf then S is Arf and E is integrally closed, it is natural to ask what is the relation between $\mathrm{Arf}(S \bowtie^d E)$ and $\mathrm{Arf}(S) \bowtie^d \tilde{E}$.

Proposition 5.3.6. *With the notation introduced above we have*

$$\mathrm{Arf}(S) \bowtie^d \tilde{E} \subseteq \mathrm{Arf}(S \bowtie^d E).$$

Proof. Since $S = (S \bowtie^d E)/2 \subseteq \mathrm{Arf}(S \bowtie^d E)/2$ and, by Proposition 5.3.1, $\mathrm{Arf}(S \bowtie^d E)/2$ is Arf, we got

$$\mathrm{Arf}(S) \subseteq \frac{\mathrm{Arf}(S \bowtie^d E)}{2};$$

it follows that $2 \cdot \mathrm{Arf}(S) \subseteq \mathrm{Arf}(S \bowtie^d E)$. Now if $e \in \tilde{E}$, then $2e \geq 2\tilde{e}$ and $2\tilde{e} + d \geq 2\tilde{e}$, and, since $\mathrm{Arf}(S \bowtie^d E)$ is Arf and $2e, 2\tilde{e}, 2\tilde{e} + d \in \mathrm{Arf}(S \bowtie^d E)$, we have

$$2e + 2\tilde{e} + d - 2\tilde{e} = 2e + d \in \mathrm{Arf}(S \bowtie^d E).$$

So, $2\tilde{E} + d \subseteq \mathrm{Arf}(S \bowtie^d E)$ and therefore $\mathrm{Arf}(S) \bowtie^d \tilde{E} \subseteq \mathrm{Arf}(S \bowtie^d E)$. □

Theorem 5.3.4 gives us sufficient conditions so that the inclusion of Proposition 5.3.6 is an equality. Recall that $\overline{E}$ is the integral closure in S of the semigroup ideal E.

Corollary 5.3.7. *If one of the following conditions holds*

1. $\min(E) \geq c(S)$,

2. *S is Arf and $d = e_0 = e_1 = \ldots = e_{n-1}$,*

then $\overline{E} = \tilde{E}$ *and* $\mathrm{Arf}(S) \bowtie^d \overline{E} = \mathrm{Arf}(S \bowtie^d E)$.

It is easy to see that the previous equality is not true in the general case. In particular, the following example shows that, in general, the inclusion

$$\mathrm{Arf}(S) \subseteq \frac{\mathrm{Arf}(S \bowtie^d E)}{2},$$

may be strict.

Example 5.3.8. *Let* $S = \langle 5, 8, 11, 12, 14 \rangle = \{0, 5, 8, 10, \rightarrow\}$, $E = S \setminus \{0\}$ *and* $d = 5$. *Note that S is Arf, so* $S = \mathrm{Arf}(S)$. *The numerical duplication of S with respect to E and d is*

$$S \bowtie^d E = \{0, 10, 15, 16, 20, 21, 22, 24, \rightarrow\}.$$

Since $15, 16 \in S \bowtie^d E$, *from Lemma 4.4.5 its Arf closure is* $\mathrm{Arf}(S \bowtie^d E) = \{0, 10, 15 \rightarrow\}$; *moreover* $9 \in \mathrm{Arf}(S \bowtie^d E)/2$, *but* $9 \notin \mathrm{Arf}(S) = S$.

5.3.2 Algebraic characterization

In this subsection we will follow the same notation of Section 5.2, and for convenience we will sometimes set $\mathcal{R} = R(I)_{0,-b}$. In the light of Theorem 5.2.3, we want to extend the previous numerical characterization (Theorem 5.3.4) to the family of quotients $R(I)_{0,-b}$.

Proposition 5.3.9. *If $R(I)_{0,-b}$ is Arf, so is R.*

Proof. Let J be an integrally closed ideal of R and let $x \in J$ such that $v(x) = \min v(J)$. Fix $\tilde{J} = \{y \in \mathcal{R} : v'(y) \geq v'(x)\}$, $\tilde{J}$ is an integrally closed ideal of $\mathcal{R}$, so it is stable, namely $(\tilde{J} : \tilde{J}) = x^{-1}\tilde{J}$. Furthermore, since $\overline{R} \cap \mathcal{R} = R$, we have $J = \tilde{J} \cap R = \tilde{J} \cap \overline{R}$. It suffices to prove that $(J : J) = x^{-1}J$. It is clear that $(J : J) \subseteq x^{-1}J$, in fact, if $j \in (J : J)$, then by definition $xj \in J$, so $j \in x^{-1}J$. Conversely let $j \in x^{-1}J$ and $j' \in J$, we have

$$\left.\begin{array}{r} j \in x^{-1}J \subseteq x^{-1}\tilde{J} = (\tilde{J} : \tilde{J}) \\ j' \in J \subseteq \tilde{J} \end{array}\right\} \Rightarrow jj' \in \tilde{J}.$$

Further

$$\left.\begin{array}{r} j \in x^{-1}J \subseteq \overline{R} \\ j' \in J \subseteq R \subseteq \overline{R} \end{array}\right\} \Rightarrow jj' \in \overline{R},$$

it follows that $jj' \in \tilde{J} \cap \overline{R} = J$, so $j \in (J : J)$. $\square$

Lemma 5.3.10. *If $R(I)_{0,-b}$ is Arf, then I is integrally closed.*

Proof. If $\mathcal{R}$ is Arf, then $v'(\mathcal{R}) = v(R) \bowtie^{v(b)} v(I)$ is an Arf numerical semigroup. From Lemma 5.3.3 we have that $v(I)$ is integrally closed. Finally, from Lemma 4.1.8 I is integrally closed. $\square$

The previous necessary condition are, in some cases, also sufficient. Similarly as in the numerical case, we need an additional condition on the multiplicity sequence of the ring. The proof of sufficiency is very technical and we omit it. however, it can be found in [6, Theorem 3.9].

Theorem 5.3.11. *[6, Theorem 3.9] $R(I)_{0,-b}$ is Arf if and only if R is Arf, I is integrally closed and if $C(R) \subsetneq I$ then the multiplicity sequence of R is $(v(b), v(b), \dots, v(b), 1, 1, \dots)$.*

Proof. Necessity. From Proposition 5.3.9 and Lemma 5.3.10 we have that R is Arf and I is integrally closed. Furthermore, if $C(R) \subsetneq I$, then $v(C(R)) \subsetneq v(I)$. Since I (and then also $v(I)$) is integrally closed, this is equivalent to $c(v(R)) = c(R) = \min v(C(R)) > \min v(I)$ (see Theorem 3.3.11). Since $\mathcal{R}$ is Arf, then also $v'(\mathcal{R}) = v(R) \bowtie^{v(b)} v(I)$ is Arf, so from the numerical characterization (Theorem 5.3.4) we have that the multiplicity sequence of $v(R)$ is $(v(b), v(b), \dots, v(b), 1, 1, \dots)$. Finally, since R is Arf, from Theorem 4.4.11, the multiplicity sequence of $v(R)$ coincide with the multiplicity sequence of R.
Sufficiency. Omitted. $\square$

Chapter 6

Patterns on numerical semigroups

We have seen that an Arf numerical semigroup S is a numerical semigroup that satisfies the linear relation $x + y - z$ for every $x, y, z \in S$ with $x \geq y \geq z$. This definition can be generalized to any linear polynomial $p(x_1, \ldots, x_n)$, that we will call pattern.

In this chapter we will show the basic properties and some developments of the theory of patterns on numerical semigroups. We will follow [9] and [7].

6.1 Basic properties and definitions

Definition 6.1.1. A **pattern** of length n is a linear homogeneous polynomial in n variables $x_1, \ldots, x_n$ with non-zero integer coefficients (for $n = 0$ the unique pattern is $p = 0$). A numerical semigroup S **admits** a pattern $p(x_1, \ldots, x_n)$ if for every $s_1, \ldots, s_n \in S$ with $s_1 \geq \cdots \geq s_n$ we have $p(s_1, \ldots, s_n) \in S$.

We denote by $\mathscr{S}(p)$ the family of numerical semigroups that admits the pattern p.

Definition 6.1.2. Given two patterns p_1 and p_2, we say that p_1 **induces** p_2 if every numerical semigroup that admits p_1, admits also p_2, that is $\mathscr{S}(p_1) \subseteq \mathscr{S}(p_2)$. The patterns p_1 and p_2 are **equivalent** if they induce each other, that is $\mathscr{S}(p_1) = \mathscr{S}(p_2)$.

Let p be a pattern of length n, set

$$p(x_1, \ldots, x_n) = \sum_{i=1}^{n} a_i x_i,$$

and $b_i = \sum_{j \leq i} a_j$, we will keep this notation throughout. Note that we can write

$$\begin{aligned} p(x_1, \ldots, x_n) &= a_1 x_1 + \cdots + a_n x_n = \\ &= b_1(x_1 - x_2) + \cdots + b_{n-1}(x_{n-1} - x_n) + b_n x_n, \end{aligned} \tag{6.1}$$

we will use frequently this decomposition in the sequel. Further, set

$$p' = \begin{cases} p - x_1 & \text{if } a_1 > 1 \\ p(0, x_1, \ldots, x_{n-1}) & \text{if } a_1 = 1, \end{cases}$$

and define recursively

$$p^{(0)} = p$$
$$p^{(i)} = (p^{(i-1)})' \quad \text{for } i \in \mathbb{N} \setminus \{0\}.$$

Definition 6.1.3. The **admissibility degree** of p, denoted by $\mathrm{ad}(p)$, is the least integer k such that $p^{(k)}$ is not admissible, if such integer exists, otherwise is infinite. The pattern p is **admissible** if it is admitted by some numerical semigroup, that is $\mathscr{S}(p) \neq \emptyset$. If p' is admissible, p is **strongly admissible**.

With this definitions, p is admissible if $\mathrm{ad}(p) \geq 1$, strongly admissible if $\mathrm{ad}(p) \geq 2$.

Proposition 6.1.4. *[9, Theorem 12] For a pattern p the following conditions are equivalent*

1. *p is admissible,*
2. *$\mathbb{N}$ admits p,*
3. *$b_i \geq 0$ for all $i \in \{1, \ldots, n\}$.*

Proof.

(2) $\Rightarrow$ (1) Trivial.

(1) $\Rightarrow$ (3) If S admits p, for every $s \in S$ and $i \in \{1, \ldots, n\}$ we have

$$p(\underbrace{s, \ldots, s}_{i}, 0, \ldots, 0) = sb_i \in S \subseteq \mathbb{N} \Rightarrow b_i \geq 0.$$

(3) $\Rightarrow$ (2) If $s_1, \ldots, s_n \in S$ with $s_1 \geq \cdots \geq s_n$, from 6.1 we have

$$p(s_1, \ldots, s_n) = b_1(s_1 - s_2) + \cdots + b_{n-1}(s_{n-1} - s_n) + b_n s_n \geq 0. \qquad \square$$

Corollary 6.1.5. *If p has admissibility degree 1, then there exists $i \in \{1, \ldots, n\}$ such that $b_i = 0$.*

Proof. By hypothesis p' is not admissible, then from Proposition 6.1.4 there exists i such that $(a_1 - 1) + \sum_{j=2}^{i} a_j = -1 \Rightarrow b_i = \sum_{j=1}^{i} a_j = 0$. $\square$

Of course if $b_i \geq k$ for all $i \in \{1, \ldots, n\}$ then $\mathrm{ad}(p) \geq k + 1$. The converse is not true, as Example 6.1.7 shows. More in general, we have the following proposition.

Proposition 6.1.6. *If a pattern p has admissibility degree at least $k+1$, then $b_i \geq \min\{i, k\}$ for all $i \in \{1, \ldots, n\}$.*

Proof. Let a'_i be the coefficients of p' and $b'_i = \sum_{j \leq i} a'_j$. We proceed by induction on k. The base case follows from Proposition 6.1.4. For the inductive step, firstly we assume that p is monic. For all $i \in \{1, \ldots, n-1\}$ we have $b_{i+1} = b'_i + 1$, then

$$\mathrm{ad}(p) \geq k + 1 \Rightarrow \mathrm{ad}(p') \geq k \Rightarrow b'_i \geq \min\{i, k-1\} \Rightarrow b_{i+1} \geq \min\{i+1, k\},$$

in addition $b_1 = 1 \geq \min\{1, k\}$. On the other hand, if p is not monic, for all $i \in \{1, \ldots, n\}$ we have $b_i = b'_i + 1$, then

$$\mathrm{ad}(p) \geq k + 1 \Rightarrow \mathrm{ad}(p') \geq k \Rightarrow$$
$$\Rightarrow b'_i \geq \min\{i, k-1\} \Rightarrow b_i \geq \min\{i+1, k\} \geq \min\{i, k\}. \qquad \square$$

Example 6.1.7. *Proposition 6.1.6 cannot be inverted. For instance consider the pattern $p = x_1 + 3x_2 - x_3$, then $b_i \geq \min\{i, k\}$ for all $k \in \mathbb{N}$, but p has admissibility degree 4.*

6.1.1 The standard decomposition

The next result gives us a canonical decomposition of the patterns.

Lemma 6.1.8. *[7, Lemma 2.3][9, Lemma 42] An admissible pattern p with finite admissibility degree can be written uniquely as*

$$p(x_1, \dots, x_n) = H_p(x_1, \dots, x_h) + C_p(x_h, \dots, x_t) + T_p(x_{t+1}, \dots, x_n),$$

where either $H_p = 0$ or all the coefficients of H_p are positive and their sum is equal to $\mathrm{ad}(p) - 1$, C_p is admissible and the sum of all its coefficients is zero, $\mathrm{ad}(T_p) > 1$.

Proof. Set $\mathrm{ad}(p) = k + 1$, then p can be written uniquely as the sum

$$p(x_1, \dots, x_n) = H_p(x_1, \dots, x_h) + p^{(k)}(x_h, \dots, x_n)$$

where H_p is a pattern with positive coefficients and their sum is equal to $k = \mathrm{ad}(p) - 1$, and $p^{(k)}$ is admissible with $\mathrm{ad}(p^{(k)}) = 1$. If a'_i are the coefficients of $p^{(k)}$, by Corollary 6.1.5 there exists an integer i such that $\sum_{j=h}^{i} a'_j = 0$, set t to be the largest of such integers. Set

$$C_p(x_h, \dots, x_t) = \sum_{i=h}^{t} a'_i x_i, \quad T_p(x_{t+1}, \dots, x_n) = \sum_{i=t+1}^{n} a'_i x_i.$$

By the choice of t it follows $\sum_{i=t+1}^{m} a'_i = \sum_{i=h}^{m} a'_i > 0$ for all $m \in \{t + 1, \dots, n\}$, hence $\mathrm{ad}(T_p) > 1$. □

If the pattern p has admissibility degree ∞, we set $H_p = p$ and $C_p = T_p = 0$. Therefore, we can write every pattern p as

$$p(x_1, \dots, x_n) = H_p(x_1, \dots, x_h) + C_p(x_h, \dots, x_t) + T_p(x_{t+1}, \dots, x_n) \tag{6.2}$$

we will keep this notation throughout.

Definition 6.1.9. Let p be a pattern. With the notation of Lemma 6.1.8 we call H_p the **head**, C_p the **center** and T_p the **tail** of p. The decomposition (6.2) is the **standard decomposition** of p.

Example 6.1.10. *Let $p = x_1 + 3x_2 + x_3 - 2x_4 + x_5 + x_6$, the admissibility degree of p is 4, the standard decomposition of p is*

$$\begin{aligned} H_p(x_1, x_2) &= x_1 + 2x_2, \\ C_p(x_2, x_3, x_4) &= x_2 + x_3 - 2x_4, \\ T_p(x_5, x_6) &= x_5 + x_6. \end{aligned}$$

Lemma 6.1.11. *[9, Proposition 34][7, Lemma 3.1] A numerical semigroup S admits every pattern of admissibility degree greater or equal than $\lceil \frac{\mathrm{c}(S)}{\mu(S)} \rceil + 1$.*

Proof. Write

$$p(x_1, \dots, x_n) = H_p(x_1, \dots, x_h) + C_p(x_{h+1}, \dots, x_t) + T_p(x_{t+1}, \dots, x_n),$$

and recall that the coefficients of H_p are positive and their sum is equal to $\mathrm{ad}(p) - 1 \geq \lceil \frac{\mathrm{c}(S)}{\mu(S)} \rceil$. Let $s_1, \dots, s_n \in S$ with $s_1 \geq \dots \geq s_n$. If $s_{h+1} < \mu(S)$, then $s_{h+1} = s_{h+2} = \dots = s_n = 0$ and

$$p(s_1, \dots, s_n) = \sum_{i=1}^{h} a_i s_i \in S.$$

On the other hand, if $s_{h+1} \geq \mu(S)$, then $s_1 \geq \dots s_h \geq \mu(S)$, therefore

$$\begin{aligned} p(s_1, \dots, s_n) \geq H_p(s_1, \dots, s_h) \geq (\mathrm{ad}(p) - 1)\mu(S) \geq \\ \geq \left\lceil \frac{\mathrm{c}(S)}{\mu(S)} \right\rceil \mu(S) \geq \mathrm{c}(S). \end{aligned}$$

□

Proposition 6.1.12. *If p has admissibility degree k, then there exists a numerical semigroup S that admits every pattern of admissibility degree $k+1$ but it does not admit p.*

Proof. If $k = 0$ take $S = \mathbb{N}$. Assume $k \geq 1$. The sum of the coefficients of C_p si zero, therefore we can write

$$p(x_1, \dots, x_n) = H_p(x_1, \dots, x_h) + \sum_{i=h+1}^{t} c_i(x_i - x_{i+1}) + T_p(x_{t+1}, \dots, x_n)$$

for some $c_i \in \mathbb{N}$. Note that there exists $r \in \{h+1, \dots, t\}$ such that $c_r > 0$. Now let $q \in \mathbb{N}$ such that $q > c_r + k - 1$. Set $S = \langle q, q+1 \rangle \cup (kq + \mathbb{N})$, then

$$p(\underbrace{q+1, \dots, q+1}_{r}, \underbrace{q, \dots, q}_{t-r}, 0, \dots, 0) = (k-1)(q+1) + c_r = \lambda,$$

with $(k-1)q + k - 1 < \lambda < kq$, therefore $\lambda \notin S$, so S does not admit p. Nonetheless, since $\mathrm{c}(S) = kq = k\mu(S)$, from the preceding lemma S admits every pattern of admissibility degree $k+1$. □

Corollary 6.1.13. *[7, Corollary 3.3] Let p and q be two patterns.*

1. *If p induces q, then* $\mathrm{ad}(p) \leq \mathrm{ad}(q)$.
2. *If p and q are equivalent, then* $\mathrm{ad}(p) = \mathrm{ad}(q)$.

6.1.2 Some criterions for the admissibility

Proposition 6.1.14. *Let p be an admissible pattern such that the sum of its coefficients is zero. A numerical semigroup S admits p if and only if the monoid generated by the integers $b_1, \dots, b_n$ (defined as in 6.1) is a subset of S.*

Proof. *Necessity.* Let $i \in \{1,\dots,n\}$ and $\lambda \in \mathbb{N}$ such that $\lambda, \lambda+1 \in S$. Then

$$p(\underbrace{\lambda+1,\dots,\lambda+1}_{i},\lambda,\dots,\lambda) = \sum_{j=1}^{n} a_j\lambda + \sum_{j=1}^{i} a_j = b_n\lambda + b_i = b_i \in S.$$

Sufficiency. It is enough to consider the decomposition 6.1. □

Proposition 6.1.15. *If p has admissibility degree* 1*, then a numerical semigroup S admits p if and only if it admits C_p and T_p.*

Proof. Sufficiency follows from $p = C_p + T_p$. For the necessity, it is enough to write

$$p(x_1,\dots,x_t,0,\dots,0) = C_p(x_1,\dots,x_t),$$
$$p(x_{t+1},\dots,x_{t+1},x_{t+2},\dots,x_n) = T_p(x_{t+1},\dots,x_n),$$

where t is the same index used in the proof of Lemma 6.1.8. □

From the previous two propositions, we obtain the following corollary.

Corollary 6.1.16. *[7, Corollary 2.10] If p has admissibility degree* 1*, then a numerical semigroup S admits p if and only if S admits T_p and contains the monoid generated by $b_1,\dots,b_t$.*

Lemma 6.1.17. *If p is a strongly admissible pattern, then for every $s_1 \geq \cdots \geq s_n$ we have that $p(s_1,\dots,s_n) \geq s_1$.*

Proof. Since p is strongly admissible, then p' is admissible, so from Proposition 6.1.4 we have

$$p(s_1,\dots,s_n) = s_1 + p'(s_1,\dots,s_n) \geq s_1.$$ □

Corollary 6.1.18. *[9, Corollary 16] Let p be a strongly admissible pattern. Every numerical semigroup S admits p if and only if $p(s_1,\dots,s_n) \in S$ for every $s_1,\dots,s_n \in S$ with $c(S) > s_1 \geq \cdots \geq s_n$*

By combining Corollary 6.1.16 and Corollary 6.1.18 we can check computationally when a numerical semigroup S admits a pattern p. Explicitly, if $\mathrm{ad}(p) = 0$ then S does not admit p, if $\mathrm{ad}(p) = 1$ apply Corollary 6.1.16 on p and Corollary 6.1.18 on T_p, finally if $\mathrm{ad}(p) > 1$ (so p is strongly admissible) apply directly Corollary 6.1.18 on p.

Proposition 6.1.19. *[7, Proposition 2.11] If p is monic and has admissibility degree* 2 *with*

$$p(x_1,\dots,x_n) = x_1 + C_p(x_2,\dots,x_t) + T_p(x_{t+1},\dots,x_n),$$

then S admits p if and only if it admits $p_i(x_1,x_2,x_3) = x_1 + (b_i-1)(x_2-x_3)$ for all $i \in \{2,\dots,n\}$, and $x_1 + T_p$.

Proof. First, write

$$p(x_1,\dots,x_n) = x_1 + \sum_{i=2}^{t}(b_i-1)(x_i-x_{i+1}) + T_p(x_{t+1},\dots,x_n).$$

Necessity. Let $i \in \{2, \ldots, n\}$, we have

$$p(x_1, \underbrace{x_2, \ldots, x_2}_{i-1}, \underbrace{x_3, \ldots, x_3}_{t-i}, 0, \ldots, 0) = x_1 + (b_i - 1)(x_2 - x_3),$$
$$p(\underbrace{x_1, \ldots, x_1}_{t}, x_{t+1}, \ldots, x_n) = x_1 + T_p(x_{t+1}, \ldots, x_n).$$

Sufficiency. Let $\lambda_1, \ldots, \lambda_n \in S$ with $\lambda_1 \geq \ldots \lambda_n$. We can write

$$p(\lambda_1, \ldots, \lambda_t, 0, \ldots, 0) = \lambda_1 + \sum_{i=2}^{t}(b_i - 1)(\lambda_i - \lambda_{i+1}).$$

By hypothesis $\lambda_1 + (b_2 - 1)(\lambda_2 - \lambda_3) \in S$ and it is greater than λ_1. Thus also $\Big(\lambda_1 + (b_2 - 1)(\lambda_2 - \lambda_3)\Big) + (b_3 - 1)(\lambda_3 - \lambda_4) \in S$. By iterating this process we obtain $p(\lambda_1, \ldots, \lambda_t, 0, \ldots, 0) \in S$ and it is greater than λ_1. Finally, since S admits $x_1 + T_p$, we have

$$p(\lambda_1, \ldots, \lambda_n) = p(\lambda_1, \ldots, \lambda_t, 0, \ldots, 0) + T_p(\lambda_{t+1}, \ldots, \lambda_n) \in S. \qquad \square$$

6.2 Patterns equivalent to the Arf pattern

In this section, we determine (Theorem 6.2.3) all the patterns equivalent to the Arf pattern.

Definition 6.2.1. The **Arf pattern** is the pattern $x_1 + x_2 - x_3$.

Note that $\mathscr{S}(x_1 + x_2 - x_3)$ is the family of Arf numerical semigroups.

Proposition 6.2.2. *[7, Proposition 3.5] The Arf pattern induces every strongly admissible pattern.*

Proof. Let p be a strongly admissible pattern, so $\mathrm{ad}(p) \geq 2$. We proceed by induction on the number of variables n of the pattern p. If $n = 1$ then p is equivalent to the zero pattern, so the Arf pattern induces p. Now, for the inductive step, suppose that the Arf pattern induces every pattern of admissibility degree at least 2 with at most $n - 1$ variables. Since p' induces p, it is enough to prove that the Arf pattern induces every pattern of admissibility degree 2 with n variables. So assume $\mathrm{ad}(p) = 2$. Suppose that S admits the Arf pattern. Let $s_1, \ldots, s_n \in S$ with $s_1 \geq \cdots \geq s_n$. From Lemma 6.1.8 we have

$$p(s_1, \ldots, s_n) = s_1 + \sum_{i=1}^{t-1}(b_i - 1)(s_i - s_{i+1}) + T_p(s_{t+1}, \ldots, s_n),$$

note that $b_t - 1 = 0$ and $t > 1$. From Lemma **??** the Arf pattern induces the pattern $x_1 + (b_1 - 1)(x_2 - x_3)$, so $s'_1 = s_1 + (b_1 - 1)(s_1 - s_2) \in S$ with $s'_1 \geq s_1$. Similarly, since the Arf pattern induces the pattern $x_1 + (b_2 - 1)(x_2 - x_3)$, then

$$s'_2 = s'_1 + (b_2 - 1)(s_2 - s_3) = s_1 + (b_1 - 1)(s_1 - s_2) + (b_2 - 1)(s_2 - s_3) \in S,$$

with $s_2' \geq s_1$. Iterating this process we obtain that

$$s = s_1 + \sum_{i=1}^{t-1} (b_i - 1)(s_i - s_{i+1}) \in S.$$

Since $t > 1$, the number of variables of the pattern $x_1 + T_p$ is less than n. By the inductive hypothesis, the Arf pattern induces $x_1 + T_p(x_{t+1}, \ldots, x_n)$, so

$$p(s_1, \ldots, s_n) = s + T_p(s_{t+1}, \ldots, s_n) \in S. \qquad \square$$

Theorem 6.2.3. *[7, Theorem 3.6] A pattern $p = \sum_{i=1}^{n} a_i x_i$ is equivalent to the Arf pattern if and only if it has admissibility degree 2 and there exists $i \in \{1, \ldots, n\}$ such that $b_i = \sum_{j=1}^{i} a_j = 2$.*

Proof. From Corollary 6.1.13, we can assume that $\mathrm{ad}(p) = 2$. Now from Proposition 6.2.2, the Arf pattern induces p. If there exists i such that $b_i = 2$, then

$$p(x_1, \ldots, x_n) = x_1 + \sum_{i=1}^{t} (b_i - 1)(x_i - x_{i+1}) + T_p(x_{t+1}, \ldots, x_n) \Rightarrow$$

$$\Rightarrow p(\underbrace{x_1, \ldots, x_1}_{i}, \underbrace{x_2, \ldots, x_2}_{t-i}, 0, \ldots, 0) = x_1 + (b_i - 1)(x_1 - x_2) = 2x_1 - x_2.$$

Therefore p induces the pattern $2x_1 - x_2$ which is equivalent to the Arf pattern. On the other hand, suppose that $b_i \neq 2$ for all $i \in \{1, \ldots, n\}$. Then, from Proposition 6.1.6, either $b_i = 1$ or $b_i \geq 3$. Let $q > 1$ and $S = \{q, q+1, q+3, \rightarrow\}$. From Lemma 6.1.11, S admits every pattern of admissibility degree greater or equal than 3. In particular, S admits $x_1 + T_p$. Now let $s_1, \ldots, s_n \in S$ with $s_1 \geq \cdots \geq s_n$. If for every $i \in \{1, \ldots, t\}$ either $b_i = 1$ or $s_i = s_{i+1}$, then

$$p(s_1, \ldots, s_n) = s_1 + T_p(s_{t+1}, \ldots, s_n) \in S.$$

Otherwise, there exists $i \in \{1, \ldots t\}$ such that $s_i > s_{i+1}$ and $b_i \geq 3$, then $s_1 \geq s_i > s_{i+1} \geq q \Rightarrow s_1 \geq q + 1$, and

$$p(s_1, \ldots, s_n) \geq s_1 + (b_i - 1)(s_{i+1} - s_i) \geq q + 3 = \mathrm{c}(S).$$

Clearly, S is not Arf since $2(q+1) - q = q + 2 \notin S$, therefore p is not equivalent to the Arf pattern. $\square$

6.3 Substraction and Boolean patterns

6.3.1 Substraction patterns

Definition 6.3.1. The pattern $x_1 + x_2 + \cdots + x_n - x_{n+1}$ is the **substraction pattern** of order n.

It is easy to see that the substraction pattern of order n has admissibility degree n. Therefore, from Corollary 6.1.13 we obtain the next result.

Proposition 6.3.2. *[9, Proposition 38] Two substraction patterns are equivalent if and only if they have the same order.*

The Arf pattern is the substraction pattern of order 2. The substraction pattern of order 1 is $x_1 - x_2$, also called the **trivializing pattern**, note that $\mathscr{S}(x_1 - x_2) = \{\mathbb{N}\}$, so from Proposition 6.1.4, the trivializing pattern induces every admissible pattern. Further, from Proposition 6.2.2, what we have so far is that for $k = 1, 2$, the subtraction pattern of order k induces all patterns of admissibility degree at least k. As the following example shows, this cannot be extended to $k \geq 3$.

Example 6.3.3. *[9, Example 50] The numerical semigroup $S = \langle 5, 6, 13\rangle$ admits the pattern*

$$p_1 = x_1 + x_2 + x_3 - x_4,$$

but it does not admit the pattern

$$p_2 = x_1 + x_2 + x_3 + x_4 - x_5 - x_6.$$

(p_1 and p_2 have both admissibility degree 3.)

If we denote by $\mathscr{S}_i = \mathscr{S}(x_1 + \cdots + x_i - x_{i+1})$, the chain

$$\mathscr{S}_0 \subsetneq \mathscr{S}_1 \subsetneq \mathscr{S}_2 \subsetneq \cdots \subsetneq \mathscr{S}_n \subsetneq \dots$$

contains all numerical semigroups and it is non-stabilizing. Note that $\mathscr{S}_0 = \emptyset$, $\mathscr{S}_1 = \{\mathbb{N}\}$ and $\mathscr{S}_2$ is the family of Arf numerical semigroups.

6.3.2 Boolean patterns

Substraction patterns belong to the more general class of Boolean patterns.

Definition 6.3.4. A pattern is **Boolean** if all its coefficients are either 1 or -1.

Clearly, every substraction pattern is Boolean.

Proposition 6.3.5. *[9, Proposition 43] A Boolean pattern with admissibility degree k induces the substraction pattern of degree k.*

Proof. Let p be a Boolean pattern of admissibility degree k. From Lemma 6.1.8, write

$$p(x_1, \dots, x_n) = H_p(x_1, \dots, x_{k-1}) + C_p(x_k, \dots, x_t) + T_p(x_{t+1}, \dots, x_n).$$

Assume that S admits p and let $s_1, \dots, s_n \in S$ with $s_1 \geq \cdots \geq s_n$. Since p is Boolean, also C_p is Boolean, so decomposing C_p as in 6.1 it is easy to deduce that $C_p(s_k, s_{k+1}, \dots, s_{k+1}) = s_k - s_{k+1}$. Hence

$$\begin{aligned} p(s_1, \dots, s_k, s_{k+1}, \dots, s_{k+1}, 0, \dots, 0) = &\\ = H_p(s_1, \dots, s_{k-1}) + C_p(s_k, s_{k+1}, \dots, s_{k+1}) + T_p(0, \dots, 0) = &\\ = s_1 + \cdots + s_{k-1} + s_k - s_{k+1} \in S. & \end{aligned}$$

□

If p is a Boolean pattern of admissibility degree k, then $b_k = k$. Therefore, from Theorem 6.2.3 and Corollary 6.1.16 we obtain the following result.

Proposition 6.3.6. *[9, Proposition 48] For a fixed $i \in \{0, 1, 2\}$, all Boolean patterns with admissibility degree i are equivalent to each other.*

As Example 6.3.3 shows, the previous proposition cannot be extended for patterns with admissibility degree greater or equal than 3.

6.4 Frobenius varieties

The notion of Arf closure can be generalized to every pattern p, defining the p-closure of a numerical semigroup (see for instance [9, Section 4]). Similarly, the notion of Arf systems of generators (see [30, Section 1]) can be generalized to every pattern p, defining the notion of p-system of generators (see [9, Section 5]). Finally, as for Arf numerical semigroups, the family $\mathscr{S}(p)$ can be arranged in a tree rooted in $\mathbb{N}$ and such that S is a son of T if $S = T \cup \{F(T)\}$.

All of this properties can be unified in the more general framework of Frobenius varieties (see [29, Chapter 6]).

Definition 6.4.1. A **Frobenius variety** is a nonempty family of numerical semigroups $\mathscr{F}$ such that

1. $S, T \in \mathscr{F} \Rightarrow S \cap T \in \mathscr{F}$,
2. $S \in \mathscr{F} \setminus \{\mathbb{N}\} \Rightarrow S \cup \{F(S)\} \in \mathscr{F}$.

Note that, if $\mathscr{F}$ is a Frobenius variety, then $\mathbb{N} \in \mathscr{F}$. Clearly, the family of all numerical semigroups is a Frobenius variety. This family can be arranged in a tree as above (see [29, Proposition 7.1]). Moreover, every numerical semigroup has a minimal system of generators (Remark 3.2.3). In this case the closure of a set $X \subseteq \mathbb{N}$ will be the numerical semigroup $\langle X \rangle$ generated by X.

Proposition 6.4.2. *[29, Proposition 7.17] Let p be an admissible pattern. Then $\mathscr{S}(p)$ is a Frobenius variety.*

Proof. It follows from Corollary 6.1.16 and Corollary 6.1.18. □

As a consequence of the preceding result, if we consider the Arf pattern $x_1 + x_2 - x_3$, the family of Arf numerical semigroups is a Frobenius variety.

Definition 6.4.3. If $\mathscr{F}$ is a Frobenius variety, the **$\mathscr{F}$-closure** of a set $X \subseteq \mathbb{N}$ is

$$\mathscr{F}(X) = \bigcap_{\substack{S \in \mathscr{F} \\ X \subseteq S}} S.$$

In this case, we say that X is a **$\mathscr{F}$-system of generators** of $\mathscr{F}(X)$.

Our aim is now to prove that minimal $\mathscr{F}$-systems of generators of a numerical semigroup are unique and have finitely many elements.

Proposition 6.4.4. *[29, Proposition 7.23] Let $\mathscr{F}$ be a Frobenius variety. Every numerical semigroup has a $\mathscr{F}$-system of generators with finitely mane elements.*

Proof. It follows from $\mathscr{F}(X) = \mathscr{F}(\langle X \rangle)$ and from the fact that every numerical semigroup have a finite minimal system of generators (Remark 3.2.3). □

Finally, we report without proof the following theorem.

Theorem 6.4.5. *[29, Theorem 7.26] Let $\mathscr{F}$ be a Frobenius variety. If S is a numerical semigroup and A, B are two minimal $\mathscr{F}$-systems of generators of S, then $A = B$.*

Therefore, every numerical semigroup has a unique minimal $\mathscr{F}$-system of generators and this set is finite.

The elements of a Frobenius variety can be arranged in a tree rooted in $\mathbb{N}$. More precisely, given a Frobenius variety $\mathscr{F}$, define $\mathcal{G}(\mathscr{F}) = (V, E)$ to be the graph with set of vertices $V = \mathscr{F}$ and edges $E = \big\{(S,T) \in \mathscr{F} \times \mathscr{F} : S = T \cup \{F(T)\}\big\}$. The graph $\mathcal{G}(\mathscr{F})$ is a tree rooted in $\mathbb{N}$ as the following theorem shows (we omit the proof).

Theorem 6.4.6. *[29, Theorem 7.30] Let $\mathscr{F}$ be a Frobenius variety. The graph $\mathcal{G}(\mathscr{F})$ is a tree with root equal to $\mathbb{N}$. Furthermore, the sons of $S \in \mathscr{F}$ are $S \setminus \{x_1\}, \dots, S \setminus \{x_r\}$, where $x_1, \dots, x_r$ are the elements of the minimal $\mathscr{F}$-system of generators of S which are greater than $F(S)$.*

6.5 Patterns on the numerical duplication

The aim of this section is to discuss some possible extensions of the characterization given in Theorem 5.3.4 about the numerical duplication $S \bowtie^d E$ to every pattern. As the following example shows, in this general setting such a characterization would be very complicated.

Example 6.5.1. *The following tables show for which values of d the numerical duplication $S \bowtie^d E$ admits p.*

$$S = \langle 3, 19, 20 \rangle \quad E = 3 + S \quad p(x_1, x_2) = 3x_1 - x_2.$$

d	*admits* p
3	✓
9	✓
15	✓
19	
21	✓
23	
25	
27	✓
29	✓

$$S = \langle 5, 8, 19, 22 \rangle \quad E = 5 + S \quad p(x_1, x_2, x_3) = 4x_1 - x_2 - x_3.$$

d	*admits* p
5	
13	✓
15	
19	✓
21	
23	
25	✓
27	✓
29	✓

Hence, we will concentrate our study to the cases when $d \gg 0$. Many proofs of the results presented are very technical, so we will omit most of them. The interested reader may consult [7, Section 4].

In what follows, S will be a numerical semigroup, E will be an ideal of S, $d \in S$ will be an odd integer and $p = \sum_{i=1}^{n} a_i x_i$. We say that the numerical duplication $S \bowtie^d E$ **admits p eventually with respect to d** if there exists $d' \in \mathbb{N}$ such that $S \bowtie^d E$ admits p for all $d \geq d'$.

Proposition 6.5.2. *If S admits p then also $\frac{S}{k}$ admits p for every $k \geq 1$.*

Proof. If $\lambda_1 \geq \cdots \geq \lambda_n$ are elements of $\frac{S}{k}$, then $k\lambda_1 \geq \cdots \geq k\lambda_n$ are in S. Therefore

$$p(k\lambda_1, \ldots, k\lambda_n) = kp(\lambda_1, \ldots, \lambda_n) \in S \Rightarrow p(\lambda_1, \ldots, \lambda_n) \in \frac{S}{k}. \qquad \square$$

For the next result, recall that $\dfrac{S \bowtie^d E}{2} = S$.

Corollary 6.5.3. *If $S \bowtie^d E$ admits p then S admits p.*

Throughout we will assume that S admits the pattern p. Note that if p has admissibility degree 2, then by applying Corollary 6.1.5 to the center of p, we obtain that the set $B = \{i : b_i - 1 = 0\}$ is nonempty.

Proposition 6.5.4. *Suppose that p has admissibility degree 2 and set*

$$B = \{i : b_i - 1 = 0\}, \quad r = \min B, \quad t = \max B.$$

If $S \bowtie^d E$ admits p eventually with respect to d, then

1. *for every $1 \leq i \leq t$, $\lfloor \frac{b_i}{2} \rfloor \in E - E$;*
2. *for every $r \leq i \leq t$, if b_i is even then $b_i/2 \geq \mathrm{c}(E) - \min(E)$.*

Proposition 6.5.5. *If p has admissibility degree 2 and is monic, then $S \bowtie^d E$ admits p eventually with respect to d if and only if*

1. *for every $i \in \{1, \ldots, t\}$*
   - *if b_i is odd then $(b_i - 1)/2 \in E - E$;*
   - *if b_i is even then $b_i/2 \geq \mathrm{c}(E) - \min(E)$.*
2. *$S \bowtie^d E$ admits $x_1 + T_p$.*

Proposition 6.5.6. *If p has admissibility degree at least 3 and it is not monic (i.e. $a_1 \geq 2$), then $S \bowtie^d E$ admits p eventually with respect to d.*

Proposition 6.5.7. *If p is monic with admissibility degree at least 3, then $p'(S) \subseteq E - E$ if and only if $S \bowtie^d E$ admits p eventually with respect to d.*

Assembling Corollary 6.1.16, Proposition 6.5.5 and Proposition 6.5.7 and iterating these results on $H_p + T_p$, we are able to characterize when the numerical duplication $S \bowtie^d E$ admits a monic pattern p for $d \gg 0$.

Theorem 6.5.8. *[7, Theorem 4.7] Let p be a monic pattern, written as*

$$p(x_1, \ldots, x_n) = H_p(x_1, \ldots, x_h) + C_p(x_{h+1}, \ldots, x_t) + T_p(x_{t+1}, \ldots, x_n).$$

Then $S \bowtie^d E$ admits p eventually with respect to d if and only if one of the following cases occurs:

1. $\mathrm{ad}(p) = 1$*,* $S \bowtie^d E = \mathbb{N}$*.*

2. $\mathrm{ad}(p) = 2$*, for every* $i \in \{1, \ldots, t\}$

   - *if* b_i *is odd then* $(b_i - 1)/2 \in E - E$*;*
   - *if* b_i *is even then* $b_i/2 \geq \mathrm{c}(E) - \min(E)$*;*

   and $S \bowtie^d E$ *admits* $x_1 + T_p$*.*

3. $\mathrm{ad}(p) \geq 3$ *and* $p'(S) \subseteq E - E$*.*

From Proposition 6.5.4, Proposition 6.5.6 and Corollary 6.1.16, in order to extend the previous theorem to not monic patterns, we would need just a sufficient condition in the case $\mathrm{ad}(p) = 2$.

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