Arf property and Patterns on the numerical duplication, with an application to quadratic quotients of the Rees algebra

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Numerical semigroups

A numerical semigroup is a subset $S \subseteq \mathbb{N}$ such that

1. $0 \in S$;
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3. $\mathbb{N} \setminus S$ is finite.

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Every numerical semigroup is finitely generated $S = \langle n_1, \ldots, n_e \rangle$ and we can always assume that $gcd(n_1, \ldots, n_e) = 1$.
The multiplicity of S is $\mu(S) = \min S \setminus 0$.

Connection with commutative algebra

Let $(R, \mathfrak{m})$ be a one-dimensional, Noetherian, local, domain. We also assume that R is analytically irreducible, that is the integral closure $\overline{R}$ of R is a DVR, finite as an R-module; and residually rational, that is R and $\overline{R}$ have the same residue field.

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Thus, there exists a valuation $v : \overline{R} \to \mathbb{N}$. Since $R \subseteq \overline{R}$, every element of R has a value in $\mathbb{N}$. The set of values $v(R)$ form a numerical semigroup.

Symmetric numerical semigroups

Let S be a numerical semigroup. The Frobenius number of S is $F(S) = \max(\mathbb{N} \setminus S)$. The conductor of S is $c(S) = F(S) + 1$

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S is symmetric if $\quad x \in S \iff F(S) - x \notin S \quad \forall x \in \mathbb{Z}$

A theorem of Kunz

With the above assumptions on R we have

Theorem (Kunz)

R is Gorenstein $\Longleftrightarrow v(R)$ is symmetric.

Semigroup ideals

$E \subseteq \mathbb{Z}$ is a relative ideal of S if

1. $E + S \subseteq E$;
2. $\exists s \in S$ such that $s + E \subseteq S$.

A relative ideal E contained in S is an ideal of S.

Quotient of a numerical semigroup

Let $a \in \mathbb{N}$ be a positive integer. The set

$$\frac{S}{a} = \{x \in \mathbb{N} : ax \in S\}$$

is again a numerical semigroup, called the quotient of S by a.

Numerical duplication

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Note that $\frac{S \bowtie^d E}{2} = S$.

Symmetry in the numerical duplication

The standard canonical ideal of S is the relative ideal

$$K(S) = \{x \in \mathbb{Z} : F(S) - x \notin S\}.$$

An ideal E of S is a canonical ideal if exists $x \in \mathbb{Z}$ such that $E = x + K(S)$.

Proposition (D'Anna-Strazzanti)

$S \bowtie^d E$ is symmetric $\Longleftrightarrow$ E is a canonical ideal of S.

Arf numerical semigroups

A numerical semigroup S is Arf if

$$\forall x, y, z \in S \text{ with } x \geq y \geq z \quad x + y - z \in S.$$

Multiplicity sequence

Suppose that $S = \{\ 0 = s_0 < s_1 < s_2 < \ldots\}$ is Arf, the multiplicity sequence of S is the sequence of differences

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Note that since $\mathbb{N} \setminus S$ is finite, $e_n = e_{n+1} = \cdots = 1$ for some n.

Integral closure

The integral closure of an ideal $E \subseteq S$ is

$$\overline{E} = \{s \in S : s \geq \min E\}.$$

If $\overline{E} = E$, then E is integrally closed.

Arf property in the numerical duplication

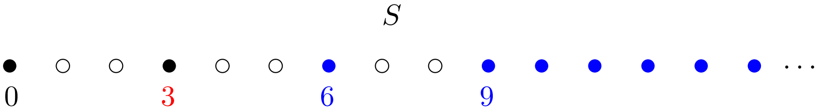
Theorem (B. 2018)

$S \bowtie^d E$ *is Arf* $\iff$ S *is Arf,* E *is integrally closed and if* $\min(E) < c(S)$*,* S *has multiplicity sequence* $(d, d, \ldots, d, 1, 1, \ldots)$.

Example

$S = \langle 3, 10, 11 \rangle = \{0, 3, 6, 9, \rightarrow\}$, $\quad d = 3 \in S$, $\quad E = \{6, 9, \rightarrow\}$

multiplicity sequence of S $(3, 3, 3, 1, \ldots)$



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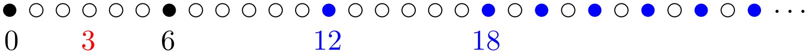

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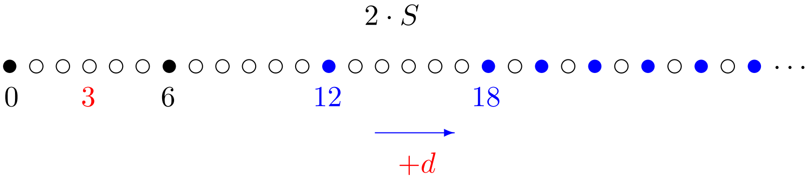

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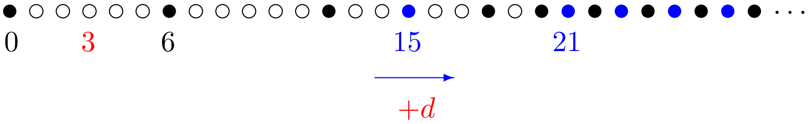
multiplicity sequence of $S \bowtie^d E$ $(6, 6, 3, 3, 2, 1, \dots)$

$$2 \cdot S \cup (2 \cdot E + d)$$



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0 3 6 15 21

Rees algebra

Let I be an ideal of R and t be an indeterminate. The Rees algebra associated with R and I is the graduated ring

$$R[It] = R \oplus It \oplus I^2t^2 \oplus \ldots = \bigoplus_{n \in \mathbb{N}} I^n t^n.$$

Quadratic quotients of the Rees algebra

In a work of Barucci-D'Anna-Strazzanti it was studied the family of quadratic quotients

$$R(I)_{a,b} = \frac{R[It]}{(t^2 + at + b) \cap R[It]} \quad a, b \in R.$$

This family generalise two constructions. The Nagata idealization

$$R \ltimes I = R \oplus I \qquad (r,i)\cdot(s,j) = (rs, rj + si),$$

and the amalgamated duplication

$$R \bowtie I = R \oplus I \qquad (r,i)\cdot(s,j) = (rs, rj + si + ij).$$

In fact they can be obtained as follows

$$R \ltimes I \simeq R(I)_{0,0} = \frac{R[It]}{(t^2) \cap R[It]} \qquad a = 0,\ b = 0$$

$$R \bowtie I \simeq R(I)_{-1,0} = \frac{R[It]}{(t^2 - t) \cap R[It]} \qquad a = -1,\ b = 0$$

Fix $b \in R$ with $v(b)$ odd. Consider the ring

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Theorem (B. 2018)

If $\mathrm{char}(R) \neq 2$ *then* $R(I)_{0,-b}$ *has the same properties of* R *and if* v' *is the valuation of* $\overline{R(I)_{0,-b}}$ *then* $v'_{|\overline{R}} = 2v$ *and*

$$v'\left(R(I)_{0,-b}\right) = v(R) \bowtie^{v(b)} v(I).$$

$$\begin{tikzcd}
R \arrow[r] \arrow[d] & R(I)_{0,-b} \arrow[d] & & 2\cdot v(R) \arrow[r] \arrow[d] & v(R) \bowtie^{v(b)} v(I) \arrow[d] \\
\overline{R} \arrow[r] \arrow[d] & \overline{R(I)_{0,-b}} \arrow[d] \arrow[r, phantom, "\xrightarrow{v'}"] & & 2\cdot \mathbb{N} \arrow[r] \arrow[d] & \mathbb{N} \arrow[d] \\
Q(R) \arrow[r] & Q(R(I)_{0,-b}) & & 2\cdot \mathbb{Z} \arrow[r] & \mathbb{Z}
\end{tikzcd}$$

Integral closure of an ideal (of a ring)

Let I be an ideal of R. An element $x \in R$ is integral over I if

$$x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_n = 0 \quad \text{for some } a_j \in I^j.$$

The set $\overline{I}$ of elements in R integral over I is an ideal called the integral closure of I. If $I = \overline{I}$, then I is integrally closed.

Stable ideals and Arf rings

An ideal I of R is stable if exists $x \in I$ such that $I^2 = xI$.

R is an Arf ring if every integrally closed ideal is stable.

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The multiplicity $\mu(R)$ of R is equal to the multiplicity of $v(R)$.

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The blow up of R is $L(R) = \bigcup_{n\in\mathbb{N}}(\mathfrak{m}^n : \mathfrak{m}^n)$. Fix

$$R_0 = R,$$
$$R_{i+1} = L(R_i).$$

The sequence $(\mu(R_0), \mu(R_1), \mu(R_2), \ldots)$ is the multiplicity sequence of R.

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The value semigroup of an Arf ring

R is Arf $\iff$ $v(R)$ is an Arf numerical semigroup and the multiplicity sequences of R and $v(R)$ coincide.

The conductor

Let $C = (R : \overline{R})$ be the conductor of R.

Note that $\min v(C)$ is the conductor of $v(R)$.

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If I is integrally closed, $C \subsetneq I$ is equivalent to $\min(v(I)) < \min(v(C))$.

Theorem (B. 2018)

$R(I)_{0,-b}$ *is Arf* $\iff$ R *is Arf,* I *is integrally closed and if* $C \subsetneq I$ *the multiplicity sequence of* R *is* $(v(b), v(b), \dots, v(b), 1, 1, \dots)$.

Patterns

A numerical semigroup S is Arf if

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A pattern $p(x_1, \ldots, x_n)$ is a lineare homogeneuos polynomial with non zero integer coefficients.

A numerical semigroup S admits the pattern p if

$$\forall \lambda_1 \geq \cdots \geq \lambda_n \in S \quad p(\lambda_1, \ldots, \lambda_n) \in S.$$

A pattern p is admissible if there exists a numerical semigroup S that admits it.

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Given a pattern $p(x_1, \ldots, x_n) = \sum_i a_i x_i$ set

$$p' = \begin{cases} p - x_1 & a_1 > 1 \\ p(0, x_1, \ldots, x_{n-1}) & a_1 \leq 1 \end{cases}$$

and define recursively

$$\begin{aligned} p^{(0)} &= p \\ p^{(i+1)} &= \left(p^{(i)}\right)' \end{aligned}$$

Admissibility degree

The admissibility degree of p, denoted $\mathrm{ad}(p)$, is the least integer k such that $p^{(k)}$ is not admissible, if k exists; $\mathrm{ad}(p) = \infty$ otherwise.

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$\mathrm{ad}(p) = 0$	not admissible patterns
$\mathrm{ad}(p) \geq 1$	admissible patterns
$\mathrm{ad}(p) \geq 2$	strongly admissible patterns (p' is admissible)

Equivalent patterns

Denote with $\mathscr{S}(p)$ the set of all numerical semigroups that admits a pattern p.

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Proposition (B. 2019)

- *If p induce q then $\mathrm{ad}(p) \leq \mathrm{ad}(q)$.*
- *If p is equivalent to q then $\mathrm{ad}(p) = \mathrm{ad}(q)$.*

Boolean patterns

The pattern $p = \sum a_i x_i$ is Boolean if $a_i \in \{\pm 1\}$.

Proposition (Bras-Amóros, García-Sánchez)

All Boolean patterns p *with* $\mathrm{ad}(p) = i$ *are equivalent (*$i \in \{0, 1, 2\}$*).*

Arf pattern

The Arf pattern is $x_1 + x_2 - x_3$.

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Set $p = \sum a_i x_i$ and $b_i = \sum_{j \leq i} a_j$.

Theorem (B. 2019)

- *The Arf pattern induces every strongly admissible pattern.*
- *p is equivalent to the Arf pattern if and only if $\mathrm{ad}(p) = 2$ and there exists i s.t. $b_i = 2$.*

Head, Center, Tail

Every pattern p can be decomposed uniquely into the sum of three patterns (called respectively the head, center and tail)

$$p = H_p + C_p + T_p$$

such that the coefficients of H_p are positive, the sum of the coefficients of C_p is zero and $\mathrm{ad}(T_p) > 1$.

Patterns on the numerical duplication

Let $E \subseteq S$ be a semigroup ideal and $d \in S$ an odd integer. Recall that if $p = \sum a_i x_i$ we set $b_i = \sum_{j \leq i} a_j$. We note also that $E \cup \{0\}$ form a numerical semigroups, and we set $c(E) = c(E \cup \{0\})$.

We say that $S \bowtie^d E$ admits a pattern p eventually with respect to d if there exists $d' \in \mathbb{N}$ such that $S \bowtie^d E$ admits p for every $d \geq d'$.

Patterns on the numerical duplication

Theorem (B. 2019)

The numerical duplication $S \bowtie^d E$ admits a monic pattern $p = \sum a_i x_i$ eventually with respect to d if one of the following cases occurs

1. $\mathrm{ad}(p) = 1$, $S \bowtie^d E = \mathbb{N}$.
2. $\mathrm{ad}(p) = 2$, *for all* $i \in \{1, \ldots, n\}$
$$\begin{cases} (b_i - 1)/2 \in E - E & b_i \text{ odd} \\ b_i/2 \geq c(E) - \min(E) & b_i \text{ even} \end{cases}$$
and $S \bowtie^d E$ *admits* $x_1 + T_p$.
3. $\mathrm{ad}(p) \geq 3$ *and* $p'(S) \subseteq E - E$.

Thank you for your attention!