Graded algebras with cyclotomic Hilbert series

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Graded algebras

Graded algebra: $R = k[x_1, \ldots, x_n]/I$ with I homogenoeus
positively graded $\deg(x_i) = e_i > 0$

$$R = \bigoplus_{i \in \mathbb{N}} R_i$$

Definition

The Hilbert series of a graded algebra $R = \oplus_{i \in \mathbb{N}} R_i$ is

$$H(R, t) = \sum_{i \in \mathbb{N}} \dim_k R_i t^i \in \mathbb{Z}[[t]].$$

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$$\frac{k[x,y]}{(x^3 - y^2)} \simeq k[u^2, u^3] = k \oplus 0 \oplus ku^2 \oplus ku^3 \oplus ku^4 \oplus \dots$$

$$H(R,t) = 1 + t^2 + t^3 + t^4 + \cdots = \frac{(1-t^6)}{(1-t^2)(1-t^3)}$$

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$$H(R,t) = \frac{(1-t^6)}{(1-t^2)(1-t^3)} = \frac{1-t+t^2}{1-t}$$

Complete intersections

Definition

A sequence of elements $f_1, \ldots, f_m \in R$ *is a regular sequence if* f_i *is not a zerodivisor in* $R/(f_1, \ldots, f_{i-1})$ *for all* $i \leq m$.

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A graded algebra is a complete intersection if

$$R \simeq k[x_1, \dots, x_n]/(f_1, \dots, f_m)$$

with $f_1, \dots, f_m$ a regular sequence in $k[x_1, \dots, x_n]$.

Complete intersections

Proposition

If $R \simeq k[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$ *is a complete intersection with* $\deg(x_i) = e_i$ *and* $\deg(f_i) = d_i$*, then*

$$H(R, t) = \frac{(1 - t^{d_1}) \ldots (1 - t^{d_m})}{(1 - t^{e_1}) \ldots (1 - t^{e_n})}.$$

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Example

$R = k[x,y]/(x^3 - y^2)$ is a complete intersection with $\deg(x) = 2$, $\deg(y) = 3$ and $\deg(x^3 - y^2) = 6$

$$H(R,t) = \frac{(1-t^6)}{(1-t^2)(1-t^3)}.$$

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Example (Stanley)

$R = k[x,y]/(x^3, xy, y^2)$ standard graded, $H(R,t) = (1+t)^2$ but R is not a complete intersection.

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What about other families?

Numerical semigroups

Definition

A *numerical semigroup* *is a subset* $S \subseteq \mathbb{N}$ *such that:*

- $0 \in S$,
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- $S = \{0, 2, 3, 4, \dots\} = \langle 2, 3 \rangle \subseteq \mathbb{N}$
- $S = \{0, 3, 4, 6, 7, \dots\} = \langle 3, 4 \rangle \subseteq \mathbb{N}$

Semigroup ring

Every numerical semigroup has a unique minimal system of generators $S = \langle n_1, \ldots, n_e \rangle$.

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Semigroup ring: $k[S] = k[u^{n_1}, \dots, u^{n_e}]$

Example

$S = \langle 2, 3 \rangle$, $k[S] = k[u^2, u^3] \simeq k[x, y]/(x^3 - y^2)$.

A conjecture

Conjecture (Ciolan, García-Sánchez, Moree 2016)

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(Ciolan, García-Sánchez, Moree 2016) $\max(\mathbb{N} \setminus S) \leq 70$.

(B., Herrera-Poyatos, Moree, 2021) $k[S]$ with at most two irreducible factors in the numerator of $H(k[S], t)$.

Koszul algebras

Definition

A standard graded algebra R is Koszul if the minimal free resolution of k as an R-module is linear (i.e. $\beta^R_{i,j}(k) = 0$ whenever $i \neq j$).

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If $R = k[x_1, \ldots, x_n]/I$ is standard graded, then

I has a Gröbner basis of quadrics $\Longrightarrow$ R is Koszul $\Longrightarrow$ I is generated by quadrics

Theorem (B., D'Alì 2021)

A Koszul algebra R is cyclotomic if and only if it is a complete intersection.

If the numerator of $H(k[S], t)$ is irreducible, then
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Theorem (B.,D'Alì 2021)

Let R be a standard graded algebra with irreducible h-polynomial. TFAE:

1. *R is cyclotomic*
2. *R is a complete intersection*
3. *R is a hypersurface of degree p, with p prime.*

Thank you for your attention!