Weierstrass sets on finite graphs

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University of Warwick

Goal: tropical analogues of Weierstrass semigroups

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Theorem (Weierstrass gap theorem)

$$|\mathbb{N} \setminus H(P)| = g$$

numerical semigroup = cofinite submonoid of $\mathbb{N}$

Question (Hurwitz 1893)

Which numerical semigroups are Weierstrass semigroups?

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Recent work of Cotterill, Pflueger, Zhang (2022) certfies Weierstrass-realizability of some numerical semigroups.

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graph := finite connected multigraph with no loops

simple graph := graph with no multiple edges

Divisor theory on graphs

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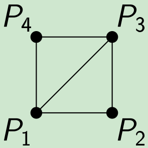
$$D = 3P_1 + P_2 + P_3$$



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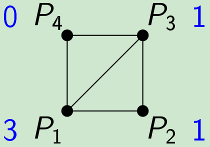

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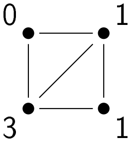

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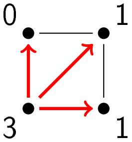

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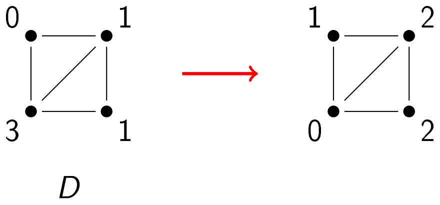

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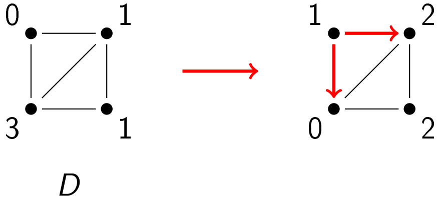

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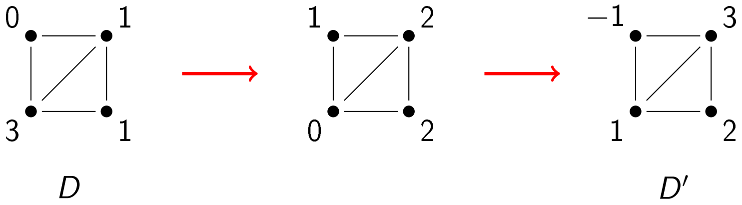

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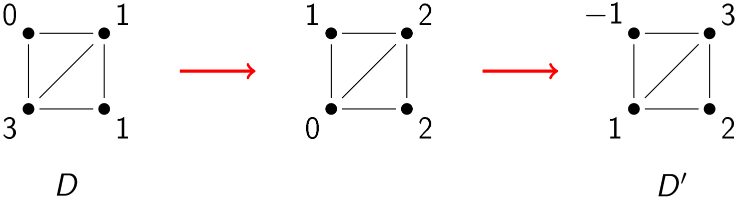
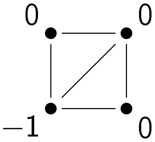


$D \sim D'$

Principal divisors

Let $f : V(G) \to \mathbb{Z}$



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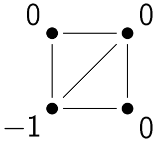

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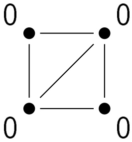

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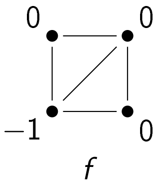

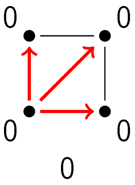

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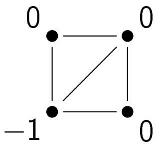

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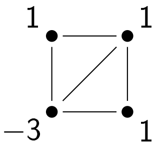

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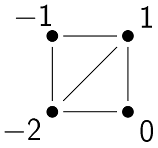

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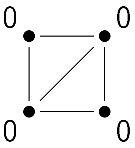

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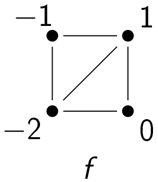

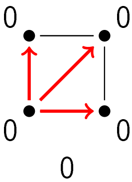

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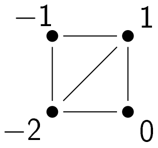

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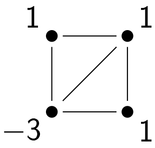

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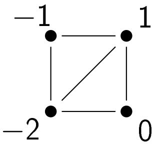

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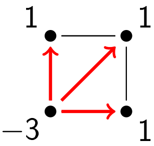

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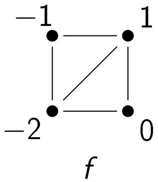

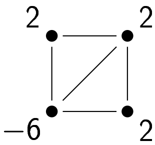

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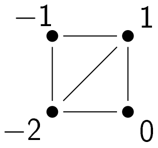

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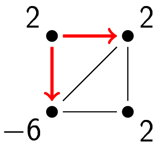

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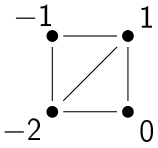

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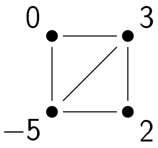

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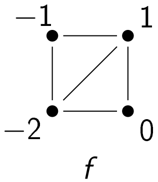

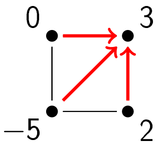

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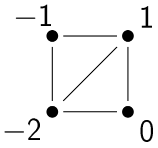

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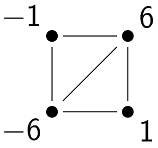

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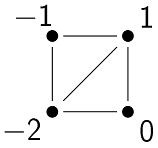

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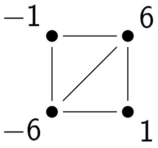

Δf

Two divisors $D, D' \in \mathrm{Div}(G)$ are **linearly equivalent** if

$$D - D' = \Delta f \quad \text{for some } f : V(G) \to \mathbb{Z}.$$

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Denote by $\mathrm{Div}_+^d(G)$ the set of effective of divisors of degree d.

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The **rank** of D is -1 if $|D| = \emptyset$, otherwise

$$r(D) = \max\{d \in \mathbb{N} : |D - E| \neq \emptyset, \forall E \in \mathrm{Div}_+^d(G)\}.$$

Weierstrass sets

Recall (for curves):

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H(P) &= \{n \in \mathbb{N} : \exists f \in K(X) \text{ regular on } X \setminus \{P\}, \mathrm{ord}_P(f) = -n\} \\
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Definition (Kang, Matthews, Peachey 2020)

Let G be a graph and let $P \in V(G)$.

Rank Weierstrass set:

$$H_r(P) = \{n \in \mathbb{N} : r(nP) > r((n-1)P)\}$$

Functional Weierstrass set:

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For curves: $H_r(P) = H_f(P) = H(P)$,

For graphs: $H_f(P) \setminus H_r(P)$ can be arbitrarily large!

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The **genus** of a graph G is $g = |E(G)| - |V(G)| + 1$.

Lemma (Tropical Weierstrass Gap Theorem)

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Not true for $H_f(P)$.

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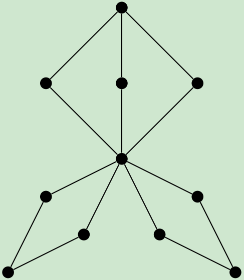
$H_f(P)$ is a semigroup, $H_r(P)$ is *not*.

Example

Consider the following graph G



It is the vertex gluing of $K_{2,3}$ and two copies of $K_{2,2}$.
Let $P \in V(G)$ be the vertex of degree 7. Then

$$H_r(P) = \{0, 3, 5, 7\} \cup (8 + \mathbb{N}).$$

Note that $H_r(P)$ is not a semigroup $6 = 3 + 3 \notin H_r(P)$.

This result was conjectured by Kang, Matthews and Peachey:

Theorem (B. 2022)

Let G be a simple graph. For every $P \in V(G)$

$$H_r(P) \subseteq H_f(P)$$

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Let K_{n+1} be the complete graph on $n+1$ vertices.

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$H_r(P) \subseteq H_f(P)$ and $|\mathbb{N} \setminus H_r(P)| = g(K_{n+1})$ imply:

Corollary

For every $P \in V(K_{n+1})$ $\qquad H_r(P) = H_f(P) = \langle n, n+1 \rangle$.

Let $K_{m,n}$ be the complete bipartite graph.

Proposition

For every $P \in V(K_{m,n})$

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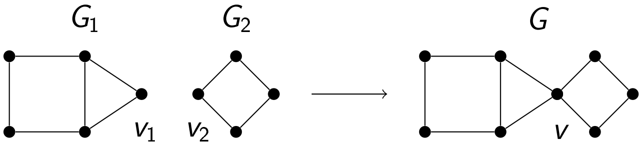
Question

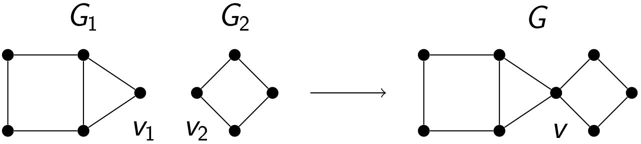
Under which conditions on G we have $H_r(P) = H_f(P)$?

Vertex gluing



Vertex gluing: the graph G obtained by identifying v_1 and v_1

Vertex gluing



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Proposition

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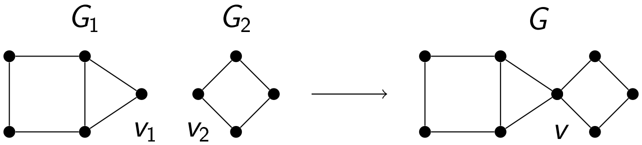

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Theorem (B. 2022)

Functional Weierstrass sets of graphs	$\longleftrightarrow$	*submonoids of* $\mathbb{N}$
Functional Weierstrass sets of simple graphs	$\longleftrightarrow$	*numerical semigroups*

Fix $P \in V(G)$, let $\lambda_P : \mathbb{N} \to \mathbb{N}$ defined by

$$\lambda_P(k) = \min\{n \in \mathbb{N} : r(nP) = k\}.$$

Note that λ_P completely determines $H_r(P)$ and vice versa.

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Theorem (B. 2022)

Let $e_1 \geq e_2 \geq \cdots \geq e_n \geq 0$ be integers, and set $s_i = \sum_{j=1}^{i} e_j$. There exists a simple graph G with a vertex $P \in V(G)$ such that

$$H_r(P) = \{0, s_1, \ldots, s_{n-1}\} \cup (s_n + \mathbb{N})$$

Thank you very much!