Tropical methods in toric intersection theory

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Motivating example

$$Y = V(x^2 + x + y + 1) \subseteq T$$

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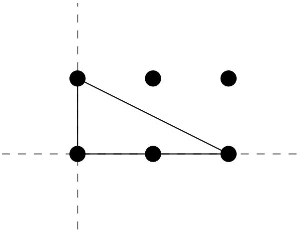

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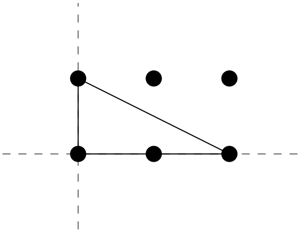
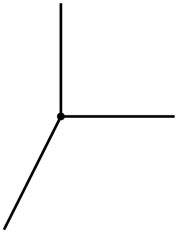


$\mathrm{trop}(Y)$



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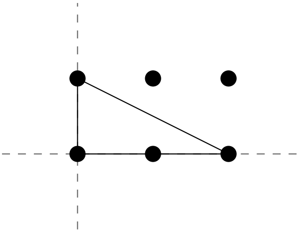

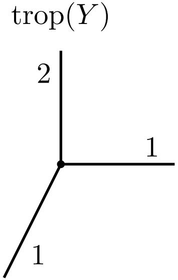

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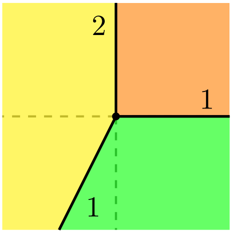

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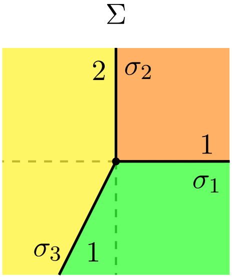

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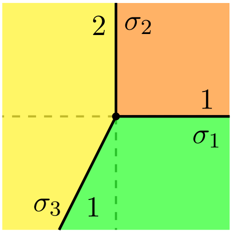
$$X_\Sigma = \mathbb{P}(1,1,2)$$

Σ



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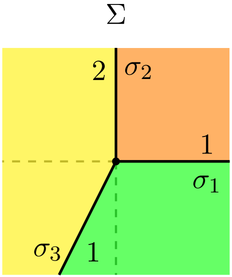

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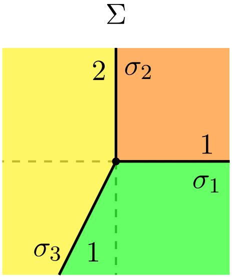
$X_\Sigma = \mathbb{P}(1,1,2)$

$\overline{Y} \cdot V(\sigma_1) = 1 = m(\sigma_1)$
$\overline{Y} \cdot V(\sigma_2) = 2 = m(\sigma_2)$
$\overline{Y} \cdot V(\sigma_3) = 1 = m(\sigma_3)$



How tropical geometry is related with toric intersection theory?

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Can we obtain the intersection numbers $\overline{Y} \cdot V(\sigma)$ from $\mathrm{trop}(Y)$?

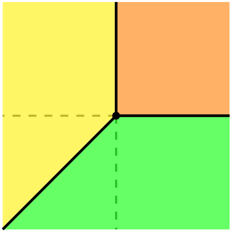
Toric geometry

$$\Sigma \text{ fan} \longrightarrow X_\Sigma \text{ variety}$$

Toric geometry

A **fan** Σ in $\mathbb{R}^n$ is a family of convex polyhedral cones such that

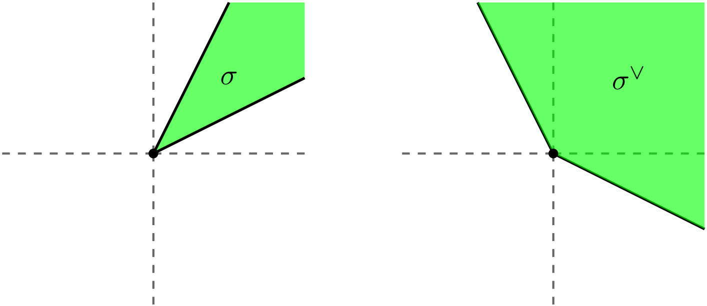
- each face of a cone in Σ is also a cone in Σ,
- the intersection of two cones in Σ is a face of each.



A fan in $\mathbb{R}^2$

Let $\sigma \in \Sigma$ be a cone. The **dual** of σ is

$$\sigma^\vee = \{v \in \mathbb{R}^n : u \cdot v \geq 0, \, \forall u \in \sigma\}.$$



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$$\tau \subseteq \sigma \dashrightarrow U_\tau \subseteq U_\sigma$$

$$\sigma \text{ cone} \dashrightarrow U_\sigma \text{ affine variety}$$

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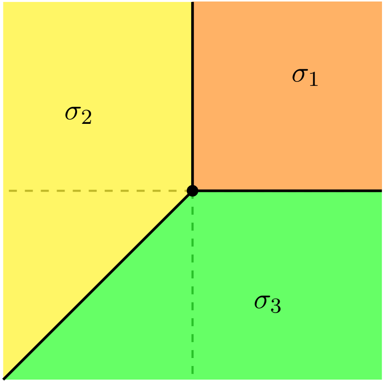
By gluing the affine varieties U_σ, $U_{\sigma'}$ along $U_{\sigma\cap\sigma'}$ we obtain a toric variety X_Σ.

$\mathbb{P}^2$ as a toric variety



$$S_{\sigma_1} = \sigma_1^\vee \cap \mathbb{Z}^2 = \langle (1,0),(0,1) \rangle$$

$$S_{\sigma_2} = \sigma_2^\vee \cap \mathbb{Z}^2 = \langle (-1,0),(-1,1) \rangle$$

$$S_{\sigma_3} = \sigma_3^\vee \cap \mathbb{Z}^2 = \langle (1,-1),(0,-1) \rangle$$

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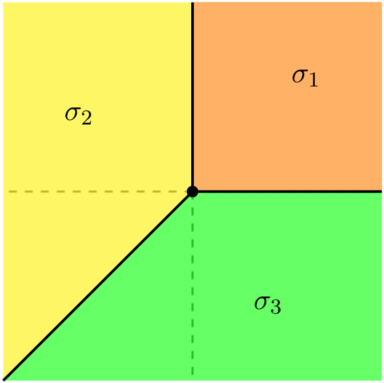

$$\mathbb{C}[S_{\sigma_1}] \simeq \mathbb{C}[x, y]$$

$$\mathbb{C}[S_{\sigma_2}] \simeq \mathbb{C}[x^{-1}, x^{-1}y]$$

$$\mathbb{C}[S_{\sigma_3}] \simeq \mathbb{C}[xy^{-1}, y^{-1}]$$

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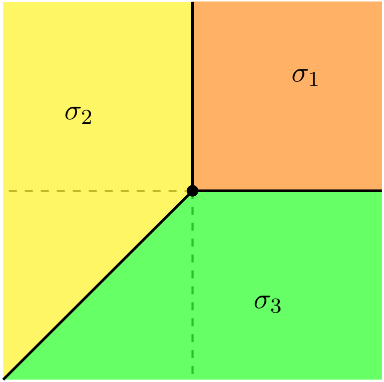

$$U_{\sigma_1} \simeq \mathbb{A}^2$$

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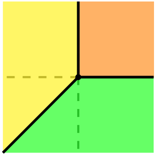
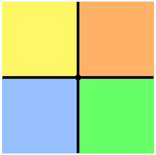
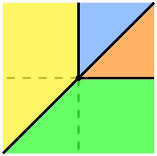


$\mathbb{P}^2$



$\mathbb{P}^1 \times \mathbb{P}^1$



Blow up of $\mathbb{P}^2$ at $(0 : 0 : 1)$

Torus action

The (algebraic) torus $T = (\mathbb{C}^*)^n$ acts on every toric variety X_Σ

$$T \times X_\Sigma \to X_\Sigma$$

the action is algebraic (i.e. it is a morphism of varieties)

Torus action on $\mathbb{P}^2$

Example

$$T = (\mathbb{C}^*)^2 \simeq \{(a : b : 1) : a, b \in \mathbb{C}^*\} \subseteq \mathbb{P}^2,$$

the action is the componentwise multiplication

$$T \times \mathbb{P}^2 \to \mathbb{P}^2 \quad (a : b : 1) \cdot (x : y : z) \mapsto (ax : by : z).$$

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There are 7 orbits

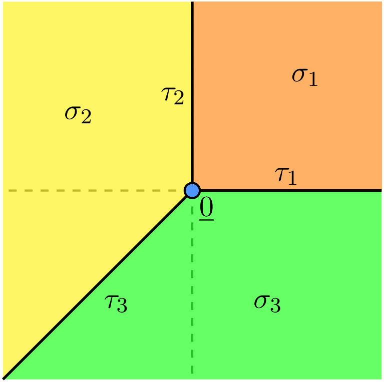
$$\begin{array}{lll} T \cdot (0 : 0 : 1) & T \cdot (0 : 1 : 1) & T \cdot (1 : 1 : 1) \\ T \cdot (1 : 0 : 0) & T \cdot (1 : 0 : 1) & \\ T \cdot (0 : 1 : 0) & T \cdot (1 : 1 : 0) & \end{array}$$

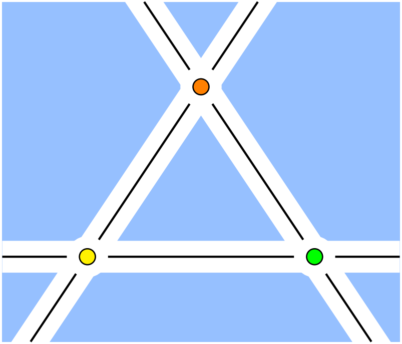
Orbits of $\mathbb{P}^2$

$$O(\sigma_1) = T \cdot (0:0:1) \quad O(\tau_1) = T \cdot (0:1:1) \quad O(\underline{0}) = T \cdot (1:1:1)$$

$$O(\sigma_2) = T \cdot (1:0:0) \quad O(\tau_2) = T \cdot (1:0:1)$$

$$O(\sigma_3) = T \cdot (0:1:0) \quad O(\tau_3) = T \cdot (1:1:0)$$





Orbit-cone correspondence

Theorem (Orbit-cone correspondence)

Let X_Σ be the toric variety

1. *There is a bijection*

$$\{\textit{cones } \sigma \in \Sigma\} \longleftrightarrow \{\textit{orbits } O(\sigma) \subseteq X_\Sigma\}$$
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2. *The orbit closures are union of orbits*

$$V(\sigma) = \overline{O(\sigma)} = \bigcup_{\tau \supseteq \sigma} O(\tau)$$

Intersection theory

X smooth projective variety, $n = \dim X$

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$$\sum_i a_i Y_i \in Z_k(X),\, a_i \in \mathbb{Z} \quad k\text{-}\textbf{cycle}$$

Cycles of codimension 1 are the **divisors**.

Intersection theory

$Y \subseteq X$ irreducible subvariety, $\dim Y = k+1$,
$f \in R(Y)$ rational function

$$\mathrm{div}(f) = \text{"zeros"} - \text{"poles"} \in Z_k(X)$$

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Example

$Y = \mathbb{P}^1 \subseteq \mathbb{P}^2 = X$, defined by $(x:y) \mapsto (x:y:0)$

$$f = \frac{x-1}{x-2} \in R(\mathbb{P}^1), \quad \operatorname{div}(f) = (1:0:0) - (2:0:0) \in Z_0(\mathbb{P}^2)$$

Intersection theory

Two k-cycles $D, C \in Z_k(X)$ are **rationally equivalent** $D \sim C$ if

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Example

Let $X = \mathbb{P}^n$, if D is a hypersurface of degree d and H is any hyperplane, then

$$D \sim dH$$

Therefore $A^1(\mathbb{P}^n) \simeq \mathbb{Z}$

Intersection product

$$A^r(X) \times A^s(X) \to A^{r+s}(X) \quad ([D],[C]) \mapsto [D]\cdot[C]$$

(where $A^r(X) = 0$ if $r > n = \dim X$)

Intuition: $[D]\cdot[C] = [D \cap C]$

Chow Ring

The intersection product makes the direct sum

$$A^*(X) = \bigoplus_{k=0}^{n} A^k(X)$$

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Example

$$A^*(\mathbb{P}^n) \simeq \mathbb{Z}[x]/(x^{n+1})$$

the isomorphism is given by $x \mapsto [H]$, where H is any hyperplane in $\mathbb{P}^n$.

Toric intersection theory

Let Σ be a fan, denote by $\Sigma(k)$ the set of cones of dimension k.

Proposition

The Chow group $A^k(X_\Sigma)$ is generated by the orbit closures

$$\{[V(\sigma)] : \sigma \in \Sigma(k)\}$$

Toric intersection theory

Theorem

Let X_Σ be a smooth projective toric variety of dimensione n with s rays, then

$$A^*(X_\Sigma) \simeq \mathbb{Z}[x_1, \dots, x_s]/(L_\Sigma + \mathrm{SR}(\Sigma))$$

Kronecker duality

For any complete variety X there is a degree homomorphism

$$\deg : A_0(X) \to \mathbb{Z} \quad \deg\left(\sum a_i[P_i]\right) = \sum_i a_i.$$

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Define the homomorphism

$$\mathscr{D}_X : A_k(X) \to \mathrm{Hom}(A^k(X), \mathbb{Z}) \quad \mathscr{D}_X(D)(C) = \deg(D \cdot C)$$

Proposition (Kronecker duality)

If X_Σ is a complete toric variety, then the map $\mathscr{D}_X$ is an isomorphism

$A_k(X_\Sigma)$

Z

$$
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A_k(X_\Sigma) & \longleftrightarrow & \operatorname{Hom}(A^k(X_\Sigma), \mathbb{Z}) \\
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The weight function m makes $\Sigma(k)$ a *balanced fan*.

Tropical compactifications

Definition

Let Y be a subvariety of the torus T. A **tropical compactification** *of Y is its closure $\overline{Y}$ in a toric variety X_Σ with $|\Sigma| = \mathrm{trop}(Y)$.*

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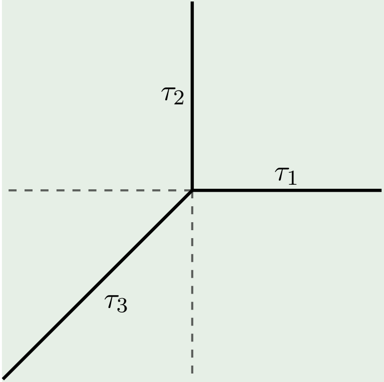
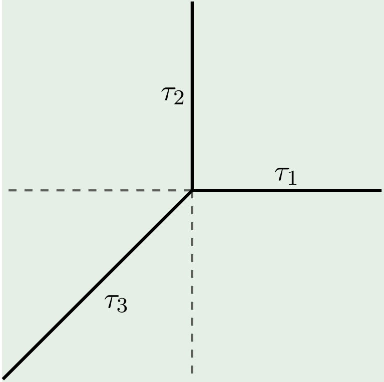
balanced fan $\mathrm{trop}(Y) \longleftrightarrow$ cycle $[\overline{Y}]$

Example



$X_\Sigma = \mathbb{P}^2 \setminus 3$ points

Example



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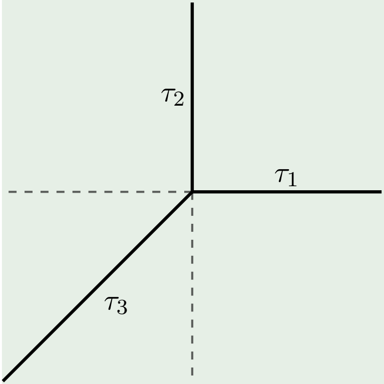

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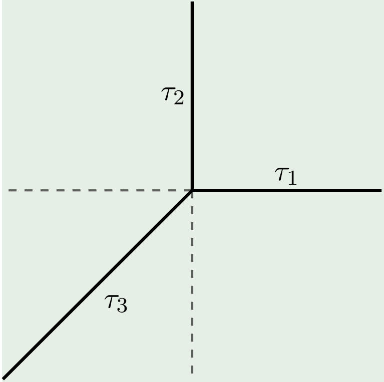

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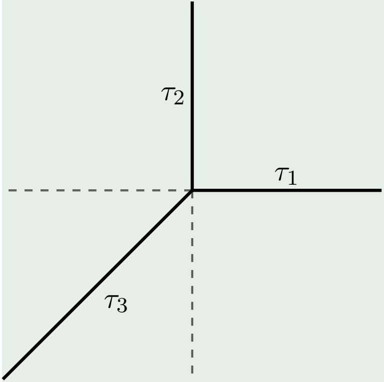

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$m(\tau_i) = 1 = \deg\left([\overline{Y}] \cdot [V(\tau_i)]\right)$

By applying Kronecker duality:
$[Y] = [V(\tau_1)] = [V(\tau_2)] = [V(\tau_3)]$
(as expected)

$\overline{Y}$ tropical compactification

$$\mathrm{trop}(Y) \dashrightarrow [\overline{Y}] \in A^*(X_\Sigma)$$

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What if $\overline{Y}$ is not a tropical compactification?

Idea:

- consider a refinement $\pi : X_{\Sigma'} \to X_\Sigma$ such that the closure $\overline{Y}'$ of Y in $X_{\Sigma'}$ is a tropical compactification

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- Calculate $[\overline{Y}'] \cdot \pi^*[V(\sigma)]$ knowing the products $[\overline{Y}'] \cdot [V(\sigma')]$ from $\mathrm{trop}(Y)$
- Apply Kronecker duality to obtain $[\overline{Y}] \in A^*(X_\Sigma)$

Application (sketch)

$$M_{0,n} \subseteq T, \quad \Sigma = \mathrm{trop}(M_{0,n})$$

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$M_{0,n} \subseteq T, \quad \Sigma = \mathrm{trop}(M_{0,n})$

tropical compactification $\overline{M}_{0,n} \subseteq X_\Sigma$	$=$	*Deligne-Mumford* compactification

- divisors in $\overline{M}_{0,n} \longleftrightarrow$ cycles in X_Σ
- $A^*(\overline{M}_{0,n}) \simeq A^*(X_\Sigma)$

Thank you for your attention!