Modal Logics with Composition on Finite Forests: Expressivity and Complexity

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LICS 2020

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‘99 Logic of Bunched Implications (BI) [P. O’Hearn, D. Pym]

Resources composition: $\varphi \bullet \psi$:

 $\models \varphi \bullet \psi$ *iff*  can be split into  and  s.t.  $\models \varphi$ and  $\models \psi$.

Reasoning on resources, locally

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Ambient Logic

$$\varphi | \psi$$

Verification of Concurrent Systems
specified in Ambient Calculus

Separation Logic

$$\varphi * \psi$$

Verification of programs
manipulating pointers

Reasoning on resources, locally

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Ambient Logic

φ

Verification of C
specified in A

Separation Logic

$\varphi * \psi$

on of programs
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Our goal:

Compare | and $*$, in terms of complexity and expressive power.

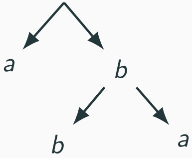
Ambient Logic and Separation Logic

Ambient Logic

Model: *Information trees*

$$
\begin{aligned}
T := {} & 0 && \text{empty tree} \\
\mid {} & \mathtt{n}[T] && \\
\mid {} & T \mathbin{\boldsymbol{|}} T && \text{union of trees}
\end{aligned}
$$

e.g. $a[0] \mathbin{\boldsymbol{|}} b\big[b[0] \mathbin{\boldsymbol{|}} a[0]\big]$



Separation Logic

Ambient Logic and Separation Logic

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$+$ congruence $\equiv$

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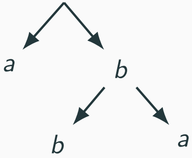

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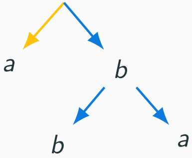

Chop operator:

$T \models \varphi | \psi$ iff there are T_1 and T_2 such that

$T \equiv T_1 | T_2$, $T_1 \models \varphi$ and $T_2 \models \psi$.

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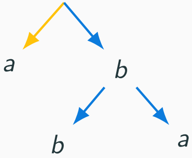

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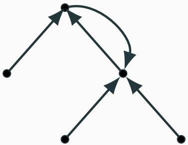
Model: *Memory state* (s, h)

A heap $h \in \mathbb{H}$: finite functional graph



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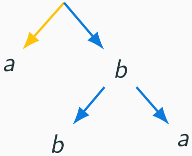

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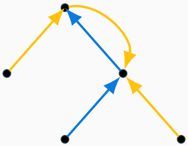

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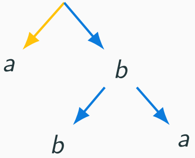

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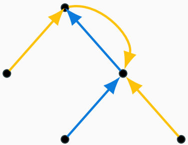

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Star operator:

$(s, h) \models \varphi * \psi$ iff there are h_1 and h_2 such that
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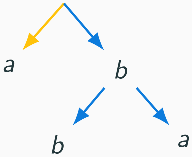

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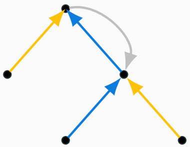

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Comparing composition operators: Chop vs. Star

We introduce,

- ML($|$) : Modal Logic (ML) extended with $\varphi | \psi$ from Ambient Logic
- ML($*$)

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For ML($|$) and ML($*$) interpreted on finite forests:

	ML($|$)	ML($*$)
Expressive Power	Graded Modal Logic (GML)	$<$ GML
Complexity (SAT)	$\textsc{AExp}_{\textsc{pol}}$-complete	$\textsc{Tower}$-complete

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$\textsc{AExpTime}$, with polynomially many alternations

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$$\underbrace{2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}}_{\text{elem}}$$

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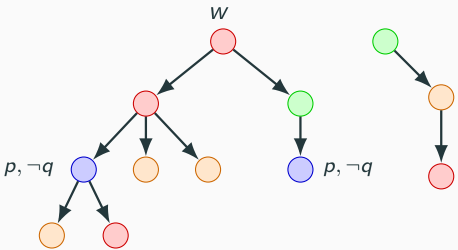
Results transfer to known fragments/extensions of Ambient Logic and Separation Logic.

ML($|$) and ML($*$): two ways to chop a tree

The models:

- Atomic propositions: p, q, ...
- Kripke-style finite forest: $(\mathfrak{M}, w)$



Standard ML**:**

$$\varphi \;:=\; p \mid \top \mid \varphi \wedge \psi \mid \neg\varphi \mid \Diamond\varphi$$

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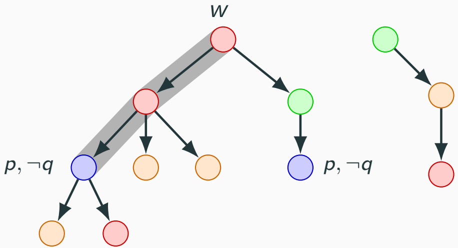

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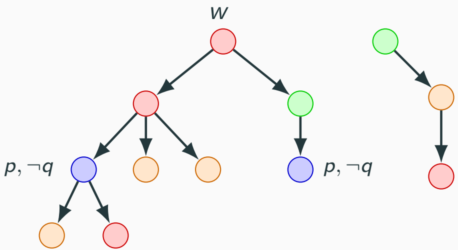

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ML($*$) = ML + $\varphi * \psi$

$\mathfrak{M}, w \models \varphi * \psi$ iff there are $\mathfrak{M}_1$ and $\mathfrak{M}_2$, such that

- $\mathfrak{M} = \mathfrak{M}_1 + \mathfrak{M}_2$,
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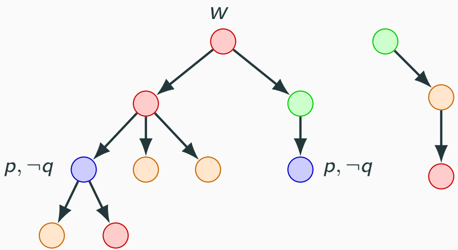

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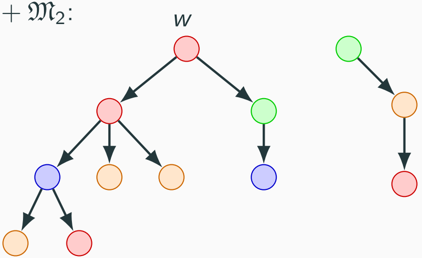

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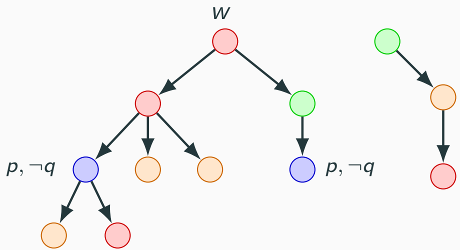

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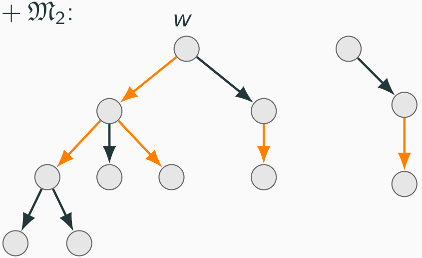

Arrows of $\mathfrak{M}$ are arbitrarily split between $\mathfrak{M}_1$ and $\mathfrak{M}_2$.

$\mathsf{ML}(|)$ and $\mathsf{ML}(*)$: two ways to chop a tree

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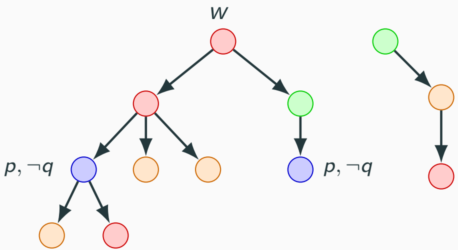

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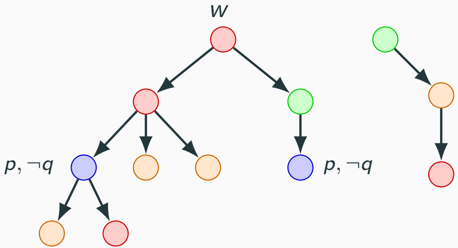

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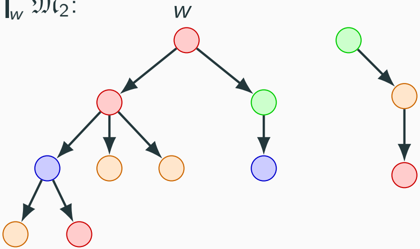

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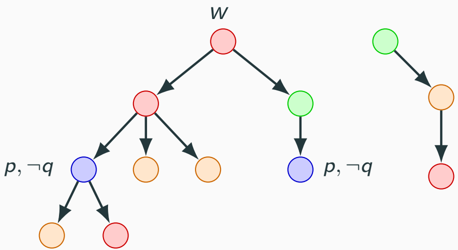

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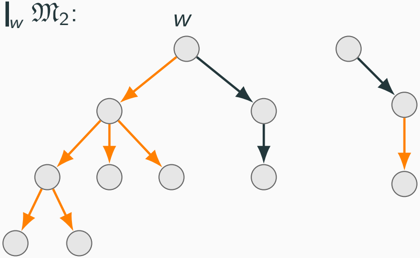

Subtrees rooted in a children of w are preserved.

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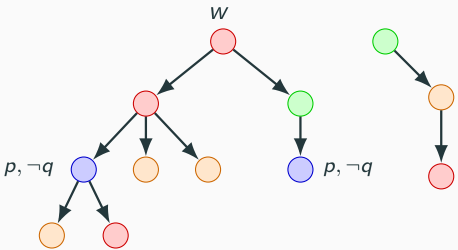

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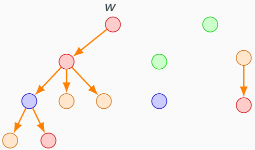
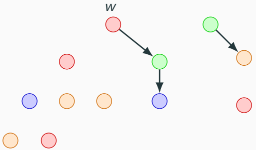


$\mathfrak{M}_1, w \models \varphi$



$\mathfrak{M}_2, w \models \psi$

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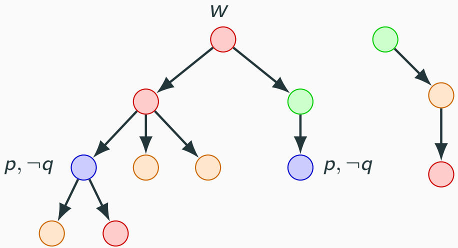

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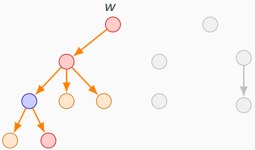
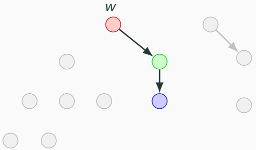


$\mathfrak{M}_1, w \models \varphi$



$\mathfrak{M}_2, w \models \psi$

ML(|) vs ML($*$): preliminary analysis

- Both ML(|) and ML($*$) are captured by MSO on finite forests

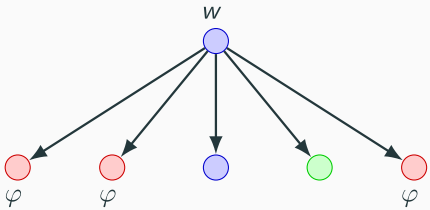
$\Rightarrow$ their SAT problem is in TOWER

$\mathsf{ML}(\mid)$ vs $\mathsf{ML}(*)$: preliminary analysis

- Both $\mathsf{ML}(\mid)$ and $\mathsf{ML}(*)$ are captured by MSO on finite forests

$\Rightarrow$ their SAT problem is in $\textsc{Tower}$

$$(\Diamond\varphi) \mid (\Diamond\varphi) \mid \Diamond\varphi$$



ML($|$) vs ML($*$): preliminary analysis

- Both ML($|$) and ML($*$) are captured by MSO on finite forests

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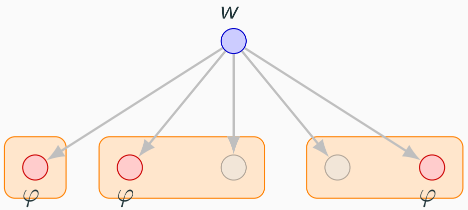

ML($|$) vs ML($*$): preliminary analysis

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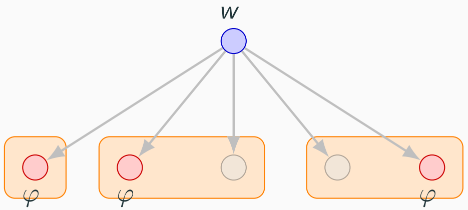
$$\Diamond_{\geq 3}\,\varphi \;\equiv\; (\Diamond\varphi)\,|\,(\Diamond\varphi)\,|\,\Diamond\varphi$$

Graded modal logic (GML): ML $+$ $(\Diamond_{\geq k}\varphi)_{k\in\mathbb{N}}$

$\mathfrak{M}, w \models \Diamond_{\geq k}\varphi$ iff w has at least k children satisfying φ.



ML($|$) **is as expressive as** GML

Show that GML is closed under the operator $|$:

Consider $\varphi_1 | \varphi_2$ such that φ_1 and φ_2 are in GML. Find γ in GML s.t. $\gamma \equiv \varphi_1 | \varphi_2$.

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Some ingredients:

$$(\Diamond_{\geq 3}\, \varphi \wedge \neg \Diamond_{\geq 7}\, \varphi) \,|\, \Diamond_{\geq 2}\, \varphi \;\equiv\; \Diamond_{\geq 3+2}\, \varphi$$

ML(|) is as expressive as GML

Show that GML is closed under the operator $\mathbin{|}$:

Consider $\varphi_1 \mathbin{|} \varphi_2$ such that φ_1 and φ_2 are in GML. Find γ in GML s.t. $\gamma \equiv \varphi_1 \mathbin{|} \varphi_2$.

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- **Idea:** we see $\Diamond_{\geq k}\varphi$ as the expression $\mathtt{x}_\varphi \geq k$ of Presburger Arithmetic (PA)

- Translation in PA:

$$\underbrace{\exists \mathtt{y}_\varphi \exists \mathtt{z}_\varphi}_{\text{there are } \mathfrak{M}_1, \mathfrak{M}_2} \;.\; \underbrace{\mathtt{x}_\varphi = \mathtt{y}_\varphi + \mathtt{z}_\varphi}_{\mathfrak{M} = \mathfrak{M}_1 \mathbin{|}_w \mathfrak{M}_2} \wedge \underbrace{\mathtt{y}_\varphi \geq 3 \wedge \neg \mathtt{y}_\varphi \geq 7}_{\mathfrak{M}_1, w \models \Diamond_{\geq 3}\varphi \wedge \neg\Diamond_{\geq 7}\varphi} \wedge \underbrace{\mathtt{z}_\varphi \geq 2}_{\mathfrak{M}_2, w \models \Diamond_{\geq 2}\varphi}$$

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ML($|$) is as expressive as GML

Show that GML is closed under the operator $|$:

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Some ingredients:

$$\Diamond_{\geq k_1}\varphi \,|\, \Diamond_{\geq k_2}\psi$$

If $\varphi \wedge \psi$ is satisfiable, then the value of $\mathtt{x}_\varphi$ could depend on the value of $\mathtt{x}_\psi$.

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- **Idea:** manipulate the formula so that $\varphi = \psi$, or $\varphi \wedge \psi$ unsatisfiable
- Cost: one exponential blow-up

ML($\mid$) is as expressive as GML

Show that GML is closed under the operator $\mid$:

Consider $\varphi_1 \mid \varphi_2$ such that φ_1 and φ_2 are in GML. Find γ in GML s.t. $\gamma \equiv \varphi_1 \mid \varphi_2$.

Some ingredients:

$$\Diamond_{\geq k_1}\varphi \mid \Diamond_{\geq k_2}\psi$$

If $\varphi \land \psi$ is satisfiable, then the value of $\mathtt{x}_\varphi$ could depend on the value of $\mathtt{x}_\psi$.

- **Idea:** manipulate the formula so that $\varphi = \psi$, or $\varphi \land \psi$ unsatisfiable
- Cost: one exponential blow-up

From the translation: ML($\mid$) has an exp-size small-model property.

SAT(ML(|)) is $\mathrm{AExp}_{\mathrm{POL}}$-complete

AExpTime, with polynomially many alternations

SAT(ML(|)) is in $\mathrm{AExp}_{\mathrm{POL}}$:

- See φ in ML(|) as a formula from MSO (on finite forests)
- Guess a finite forest $(\mathfrak{M}, w)$ of size exponential in $|\varphi|$
- Check whether $\mathfrak{M}, w \models \varphi$

SAT(ML($\mid$)) is AExp$_{\text{POL}}$-complete

$\textsc{AExpTime}$, with polynomially many alternations

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- See φ in ML($\mid$) as a formula from MSO (on finite forests)
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- Check whether $\mathfrak{M}, w \models \varphi$

SAT(ML($\mid$)) is AExp$_{\text{POL}}$-hard:

- Every formula of Team Logic can be translated to a ML($\mid$) formula of with modal depth 1
- Team Logic is $\mathrm{AExp}_{\mathrm{POL}}$-complete [Hannula et al., TOCL'18]

A closer look on $\mathsf{ML}(*)$

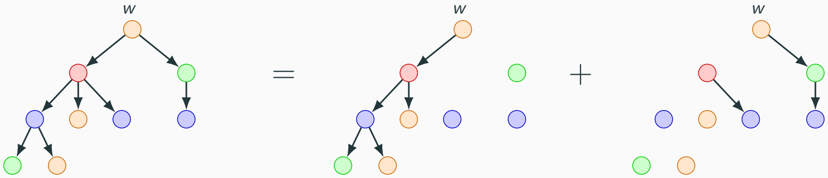
$$(\Diamond\varphi) \mathbin{|} (\Diamond\varphi) \Rightarrow \Diamond_{\geq 2}\varphi$$

A closer look on ML($*$)

$$(\Diamond\Diamond_{=1}\bigcirc) \,\boldsymbol{|}\, (\Diamond\Diamond_{=1}\bigcirc) \Rightarrow \Diamond_{\geq 2}\Diamond_{=1}\bigcirc$$

A closer look on ML($*$)

$$(\Diamond\Diamond_{=1}\bigcirc) * (\Diamond\Diamond_{=1}\bigcirc) \cancel{\Rightarrow \Diamond_{\geq 2}\Diamond_{=1}\bigcirc}$$



A closer look on ML($*$)

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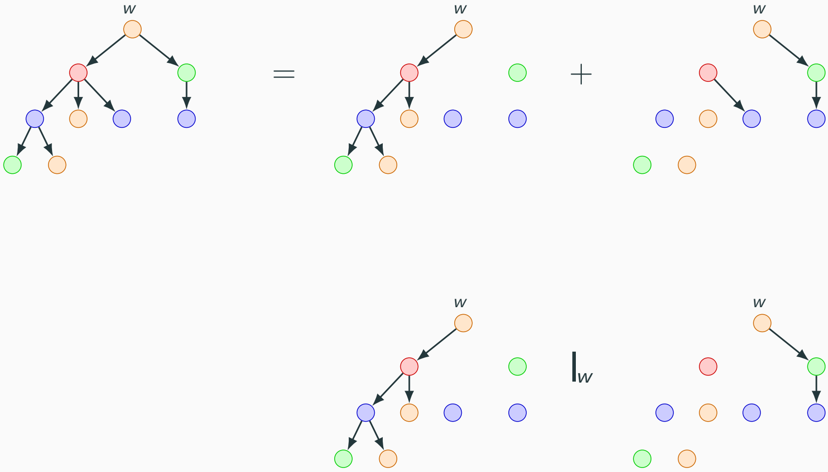

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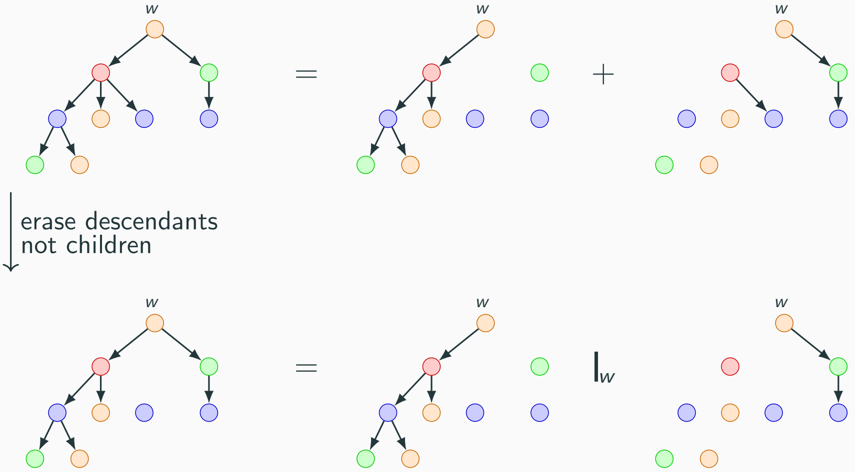

A closer look on ML($*$)

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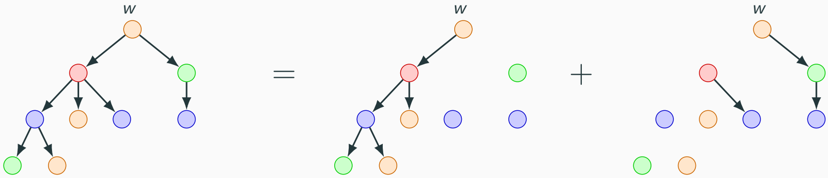
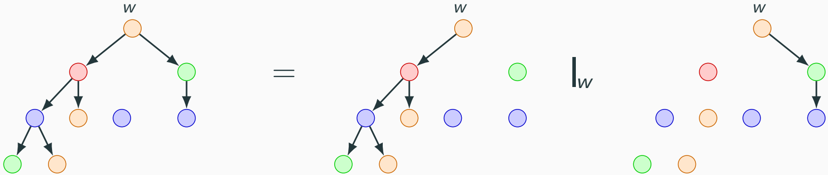


$\downarrow$ erase descendants
not children

$$\boxed{\varphi * \Box \perp}$$

$$\boxed{\varphi * \psi \equiv (\varphi \mid \psi) * \Box \perp}$$



ML($*$) **is at most as expressive as** GML

Show that GML is closed under the "operator" $(\_) * \Box \perp$.

ML($*$) **is at most as expressive as** GML

Show that GML is closed under the "operator" $(\_) * \Box \bot$.

Some ingredients:

We rely on g-bisimulation [see De Rijke,'00]

$$\mathfrak{M}, w \approx^{P}_{m,k} \mathfrak{M}', w \text{ iff for every } \varphi \in \text{GML}[m,k,P]\ (\mathfrak{M}, w \models \varphi \text{ iff } \mathfrak{M}', w' \models \varphi)$$

(m maximal modal depth, k maximal coefficient for $\Diamond_{\geq k}$, P finite set of atomic propositions)

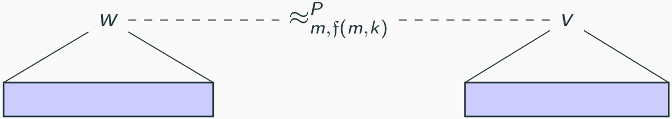
ML($*$) **is at most as expressive as** GML

Show that GML is closed under the "operator" $(\_) * \Box \bot$.

Some ingredients:

(modal depth, coefficient $\Diamond_{\geq k}$)

To show: there is a function $\mathfrak{f} : \mathbb{N}^2 \to \mathbb{N}$ such that



ML($*$) **is at most as expressive as** GML

Show that GML is closed under the "operator" $(\_) * \Box\bot$.

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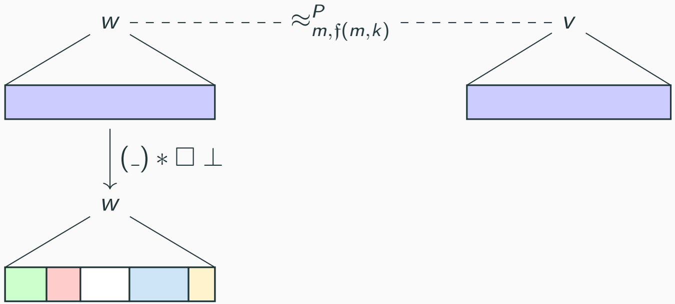
To show: there is a function $\mathfrak{f} : \mathbb{N}^2 \to \mathbb{N}$ such that



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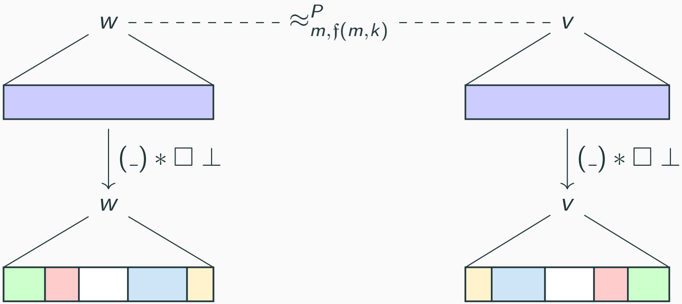

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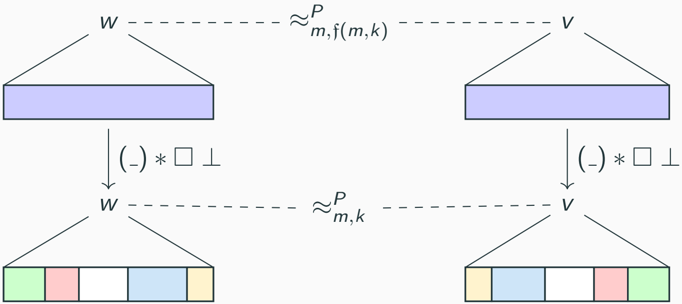

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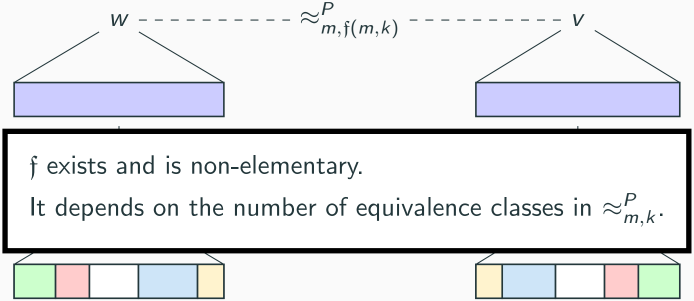

$\mathfrak{f}$ exists and is non-elementary.

It depends on the number of equivalence classes in $\approx^P_{m,k}$.

$\mathfrak{f}$ is a non-elementary function. Can we do better?

No: we can characterise worlds of "type k" with a ML($*$) formula of size exponential in k

- Type k nodes used for Quantified ML (QML) [Bednarczyk, Demri, LICS'19]

$$\underbrace{2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}}_{k} \text{ children}$$

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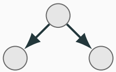

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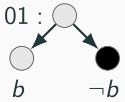

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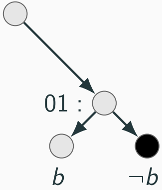

- every child is of type 1
- every child encode a different number
- every number is encoded by a child

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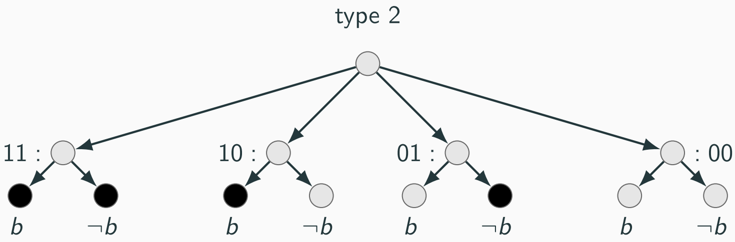

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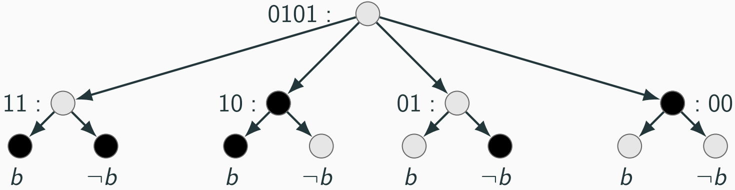

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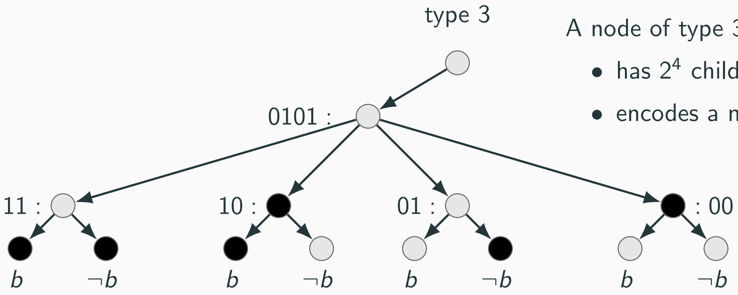

A node of type 3

- has 2^4 children
- encodes a number in $[0, 2^{(2^4)} - 1]$

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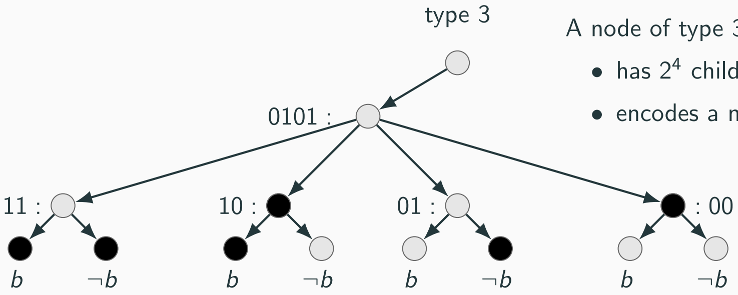


A node of type 3

- has 2^4 children
- encodes a number in $[0, 2^{(2^4)} - 1]$

A node of type k, has $\mathfrak{f}(k)$ children, where $\mathfrak{f}(1) = 2$ and $\mathfrak{f}(k+1) = 2^{\mathfrak{f}(k)}$

$\mathfrak{f}$ is a non-elementary function. Can we do better?

No: we can characterise worlds of "type k" with a ML($*$) formula of size exponential in k

- Type k nodes used for Quantified ML (QML) [Bednarczyk, Demri, LICS'19]

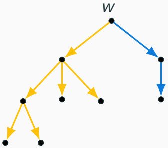
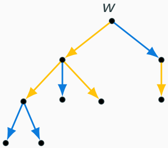
Issues:

- The operator $*$ is less powerful than the 2nd-order quantification of QML
- The operator $*$ breaks the encoding of numbers (if not handled correctly)

Tower-hardness of SAT(ML($*$)):

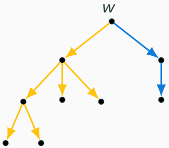
- Uniform reduction from the k-$\textsc{NExpTime}$ version of the tiling problem, for $k \geq 2$

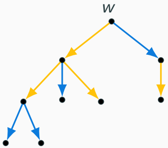
Recap

	 ML($\mid$)	 ML($\ast$)
Expressive Power	GML	$\leq$ GML
Complexity (SAT)	AEXP$_{\mathrm{POL}}$-complete	TOWER-complete

There is more...

Recap



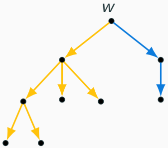
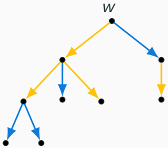


	ML($\mid$)	ML($*$)
Expressive Power	GML	$<$ GML
Complexity (SAT)	$\textsc{AExp}_{\textsc{pol}}$-complete	$\textsc{Tower}$-complete

There is more...

$\Diamond_{=2}\Diamond_{=1}\top$ cannot be expressed in ML($*$) (proof via EF-games)

Recap

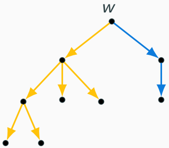
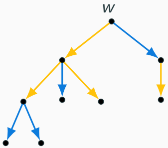
	 ML(‖)	 ML($*$)
Expressive Power	GML	$<$ GML
Complexity (SAT)	$\textsc{AExp}_{\textsc{Pol}}$-complete	$\textsc{Tower}$-complete

There is more...

SAT(ML(‖)) shows that

- Quantifier-free Intensional fragment of Ambient Logic is $\textsc{AExp}_{\textsc{Pol}}$-complete
- Ambient Logic in [Calcagno et al., TLDI'03] is $\textsc{AExp}_{\textsc{Pol}}$-hard

Recap

	 ML($\mid$)	 ML($*$)
Expressive Power	GML	$<$ GML
Complexity (SAT)	$\textsc{AExp}_{\textsc{Pol}}$-complete	$\textsc{Tower}$-complete

There is more...

SAT(ML($*$)) shows that MSL($*, \Diamond^{-1}$) [Demri, Fervari, AIML'18] is $\textsc{Tower}$-complete.

Thanks for your attention.

Modal Logics with Composition on Finite Forests: Expressivity and Complexity

Bartosz Bednarczyk, Stéphane Demri, Raul Fervari, **Alessio Mansutti**