A geometric procedure for Presburger arithmetic

Alessio Mansutti

joint work with **Dmitry Chistikov** and **Christoph Haase**

Presburger arithmetic (PA)

The first-order theory of $\langle \mathbb{Z}, 0, 1, +, \leq \rangle$

"Every integer is either even or odd"

$$\forall \mathtt{x}\, \exists \mathtt{y} : \mathtt{x} = 2\mathtt{y} \vee \mathtt{x} = 2\mathtt{y} + 1$$

Why Presburger arithmetic?
Wide range of applications in verification, program synthesis, compiler optimisation, games...

M. Presburger

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Several applications require to reason on the **set of solutions** of a formula.

Example [Verdoolaege et al., Algorithmica 48, 2007]**:**
"How many times is a loop executed?" $\rightsquigarrow f(\mathbf{y}) = \#\{\mathbf{x} \in \mathbb{Z}^d \mid A \cdot \mathbf{x} \geq B \cdot \mathbf{y} + \mathbf{c}\}$

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M. Presburger

Theorem (Ginsburg and Spanier,'66)

Sets definable in Presburger arithmetic coincide with the family of ***semilinear sets****.*

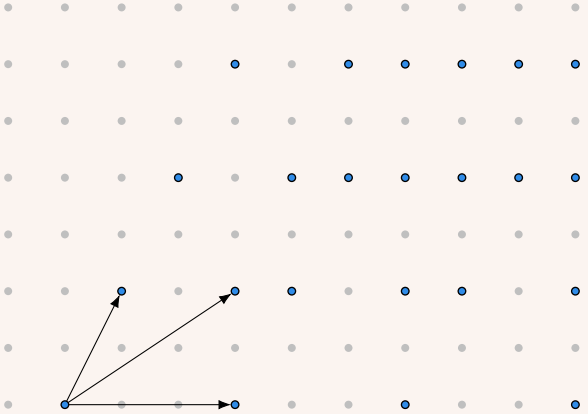
Semilinear sets

Arithmetic progression

$b + i \cdot p$, where $i \in \mathbb{N}$

b base point, p period

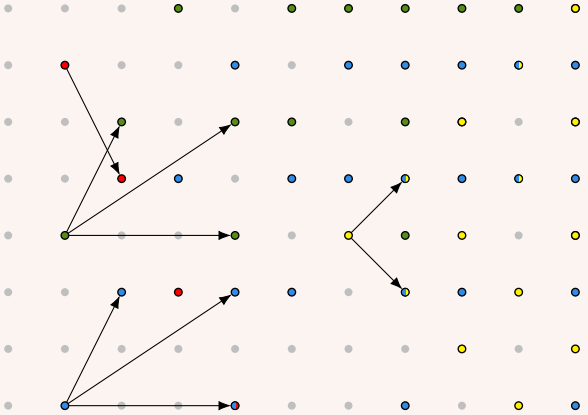
Semilinear sets



Linear set
(arithmetic progression
in multiple dimensions)

$L(b, P)$, where b base
and $P = \{p_1, \dots, p_n\}$ periods

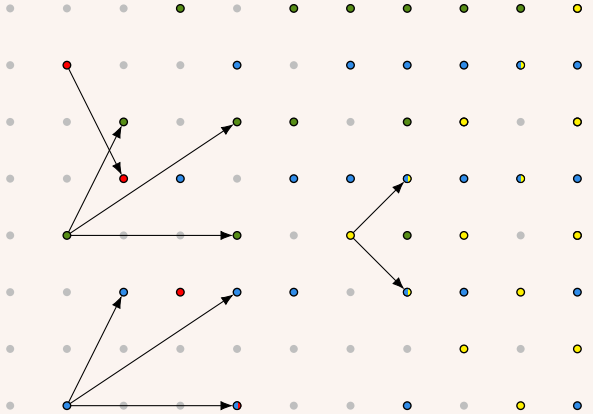
Semilinear sets



Semilinear set
(finite union of
linear sets)

$\bigcup_{i \in I} L(b_i, P_i)$,
where I finite set of indices

Semilinear sets

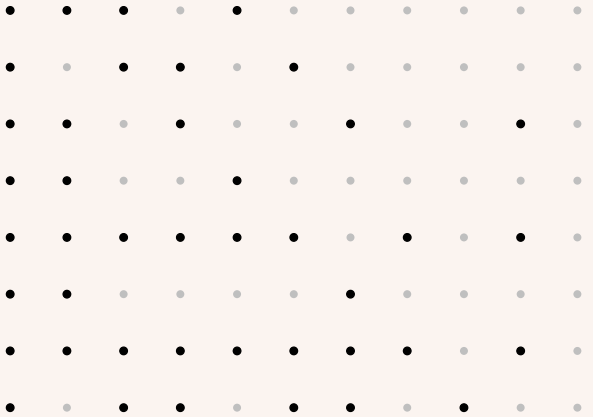


Ginsburg & Spanier, '66

The set of solutions of a system of linear inequalities over $\mathbb{Z}$ is semilinear. Semilinear sets are closed under

- union
- projection
- complementation

Semilinear sets



Ginsburg & Spanier, '66

The set of solutions of a system of linear inequalities over $\mathbb{Z}$ is semilinear. Semilinear sets are closed under

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Main problem: How to compute the complement of a semilinear set **optimally**?

A geometric procedure for Presburger arithmetic

In this talk: We discuss an optimal algorithm to complement semilinear sets.

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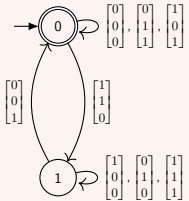
Quantifier elimination
[Presburger, '29]

$$\exists x : \varphi(x, \mathbf{y})$$
$$\equiv$$
$$\psi(\mathbf{y})$$

3EXPTIME

Automata
[Büchi, '60]



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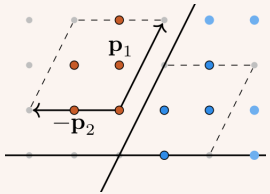

TOWER

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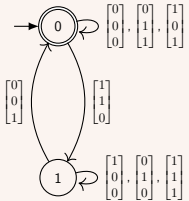

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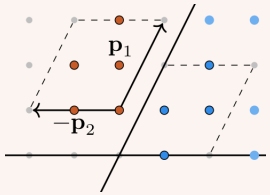
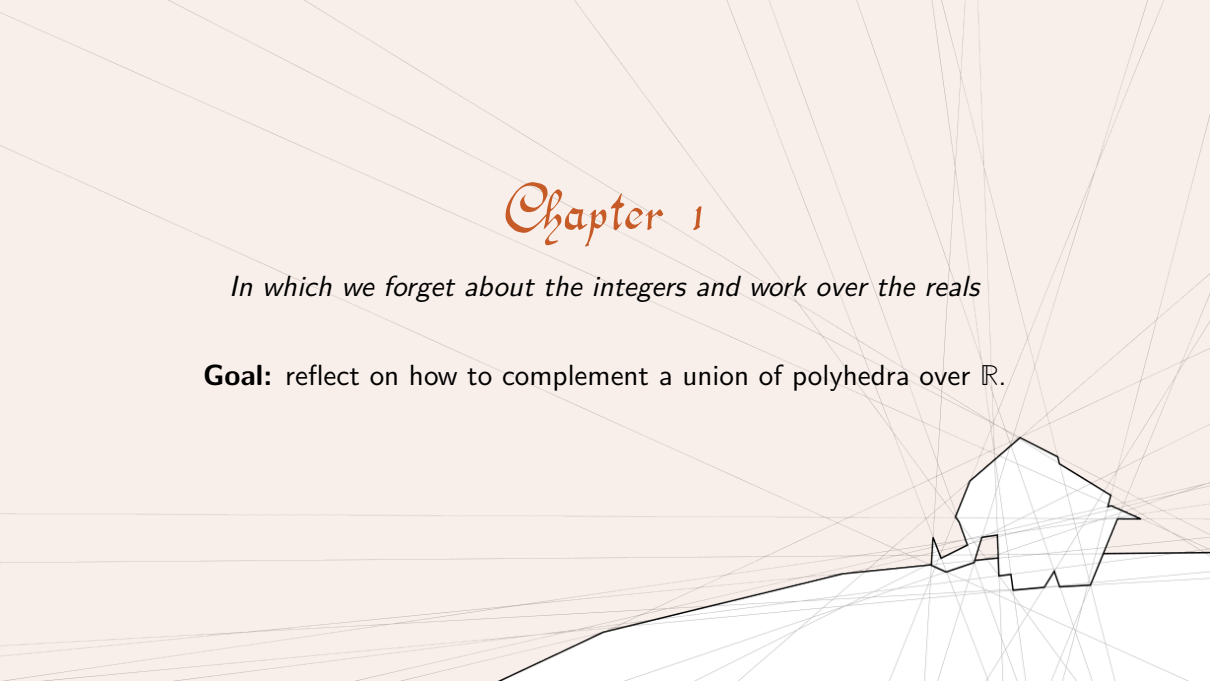


3EXPTIME

Chapter 1

In which we forget about the integers and work over the reals

Goal: reflect on how to complement a union of polyhedra over $\mathbb{R}$.



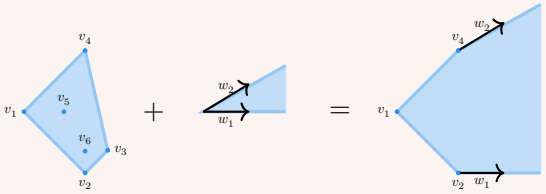
The geometry of a system of inequalities over the **reals**

Theorem (Minkowski-Weyl theorem (1897, 1935))

Consider $S \subseteq \mathbb{R}^d$. The two following statements are equivalent:

(H) $S = \{\mathbf{x} \in \mathbb{R} : A \cdot \mathbf{x} \leq \mathbf{b}\}$ *for some matrix* $A \in \mathbb{Q}^{n \times d}$ *and vector* $\mathbf{b} \in \mathbb{Q}^d$

(V) $S = \mathit{conv}\, V + \mathit{cone}\, W$ *for some finite sets* $V, W \subseteq \mathbb{Q}^d$.



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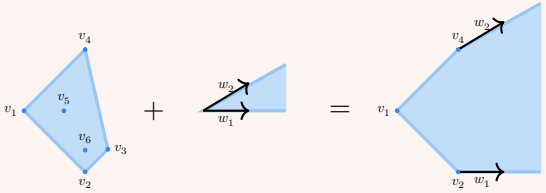

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Cost of switching:

bitsize of numbers:

$$\langle \texttt{output} \rangle \leq \mathsf{poly}(d) \cdot \langle \texttt{input} \rangle$$

amount of numbers (with repetitions):

$$\#(\texttt{output}) \leq \#(\texttt{input}) \uparrow d$$

$$(a \uparrow b \uparrow c \;=\; (a+2)^{(b+2)^{\mathsf{poly}(c)}})$$

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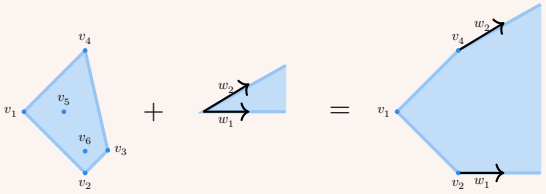

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amount of numbers (with repetitions):

$$\#[A \,|\, \mathbf{b}] \leq (\#V + \#W) \uparrow d$$

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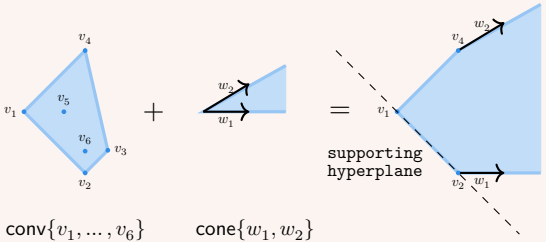

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Cutting $\mathbb{R}^d$ using hyperplanes

Question: In how many parts can we cut $\mathbb{R}^d$ using n hyperplanes?

$\Phi(d, n) :=$ maximal number of parts, in dimension d and using n hyperplanes.

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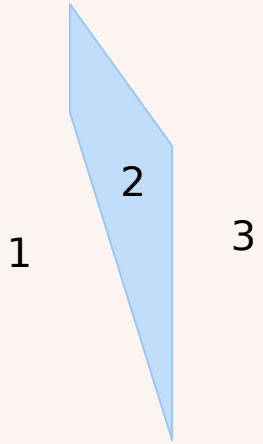

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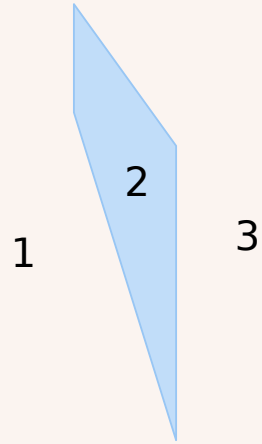

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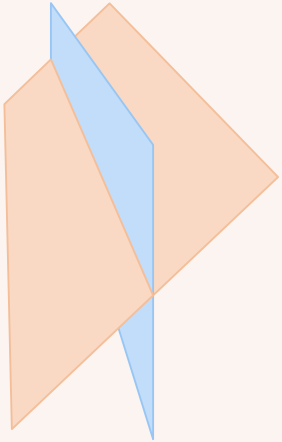

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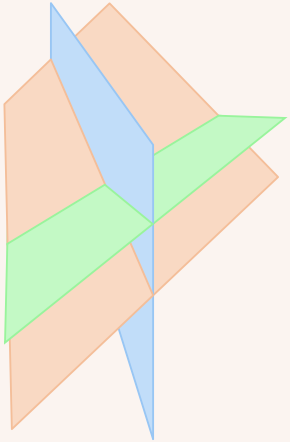

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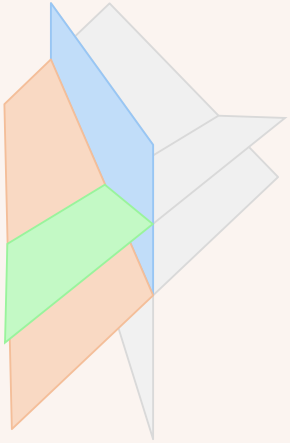

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The $(n+1)$-th hyperplane cuts the parts it intersects in three.

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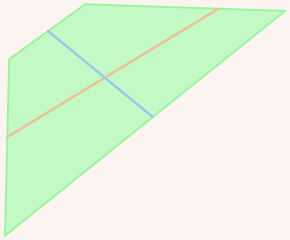

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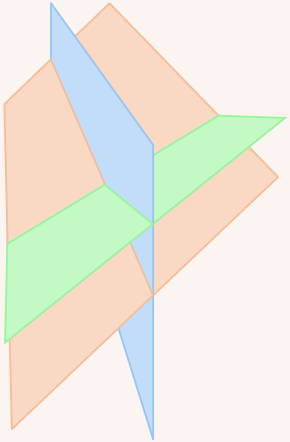

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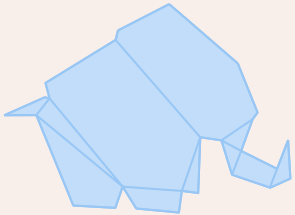
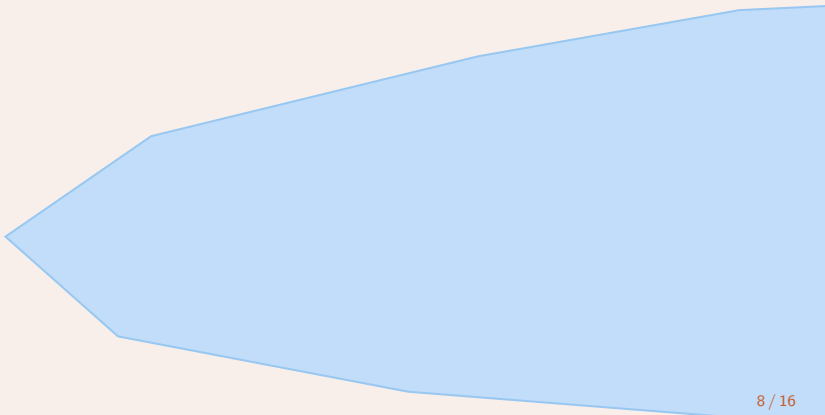
$$\Rightarrow \quad \Phi(d,n) \leq (2n)^d + 1$$

"n^d cutting lemma"

Carving out a union of convex polyhedra using splitters

Question: How do we compute the complement of $\bigcup_{i \in I} (\operatorname{conv} V_i + \operatorname{cone} W_i)$?





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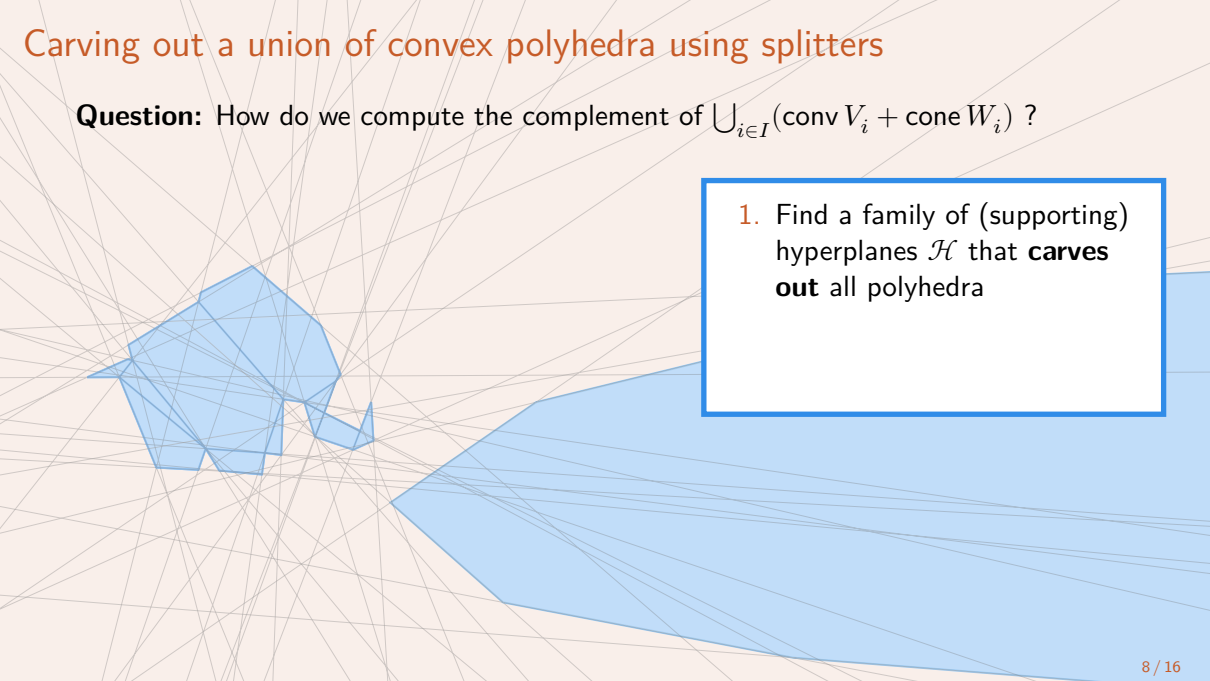

1. Find a family of (supporting) hyperplanes $\mathcal{H}$ that **carves out** all polyhedra

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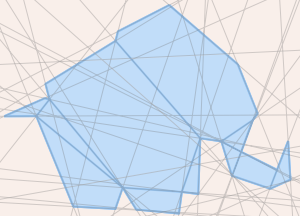

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Question: How do we compute the complement of $\bigcup_{i \in I} (\operatorname{conv} V_i + \operatorname{cone} W_i)$?

An **($\mathbb{R}$-)splitter** for a family of convex polyhedra $\mathcal{P}$ is a family of convex polyhedra $\mathcal{R}$ s.t.

1. $\bigcup \mathcal{R} = \mathbb{R}^d$ and $\mathcal{R}$ is closed under taking faces
2. for every $R_1, R_2 \in \mathcal{R}$, the set $R_1 \cap R_2$ is either $\emptyset$ or a face of both R_1 and R_2
3. for every $R \in \mathcal{R}$ and $P \in \mathcal{P}$ the set $R \cap P$ is either $\emptyset$ or a face of R

Claim: Given $\mathcal{R}$ it is easy to compute $\mathbb{R}^d \setminus \bigcup \mathcal{P}$.

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Theorem

Every $\mathcal{P} = \{\mathit{conv}\, V_i + \mathit{cone}\, W_i\}_{i\in I}$ *has an* $\mathbb{R}$*-splitter* $\mathcal{R} = \{\mathit{conv}\, C_j + \mathit{cone}\, Q_j\}_{j\in J}$ *s.t.*

$$\langle \mathcal{R} \rangle \le \mathit{poly}(d) \cdot \langle \mathcal{P} \rangle \qquad \text{and} \qquad \#\mathcal{R} \le (2 \cdot \#\mathcal{P}) \uparrow d.$$

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Theorem `Minkowski-Weyl theorem + $n^d$ cutting lemma`

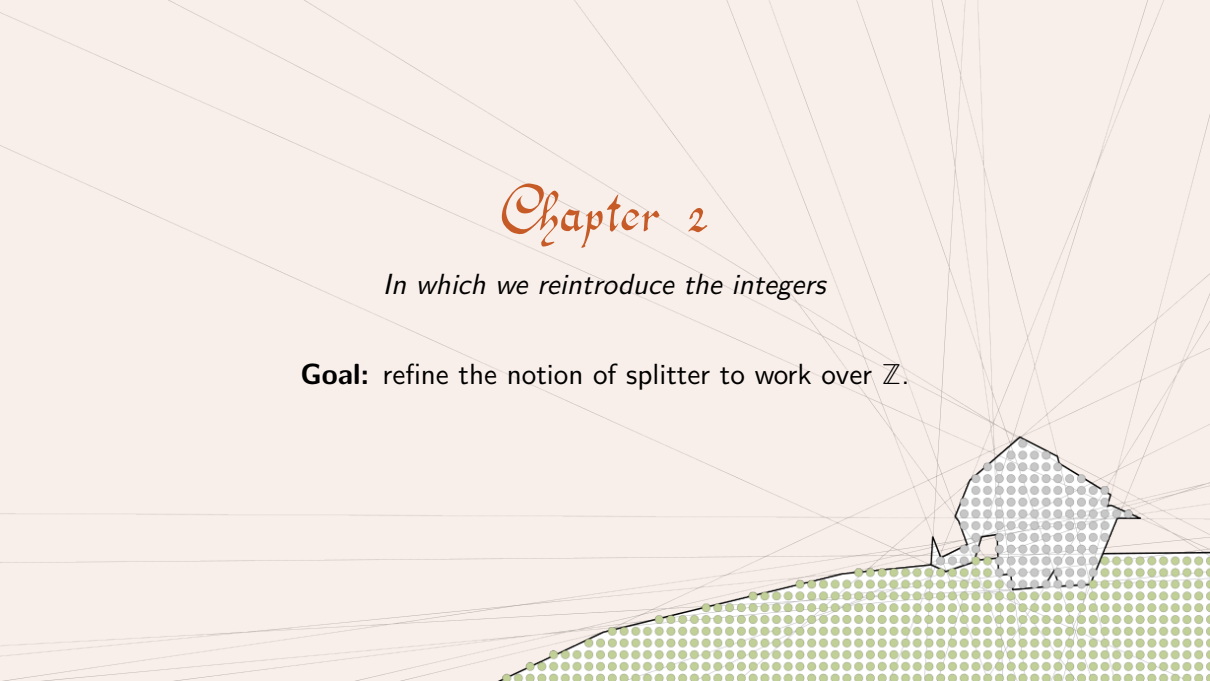
Every $\mathcal{P} = \{\operatorname{conv} V_i + \operatorname{cone} W_i\}_{i\in I}$ *has an* $\mathbb{R}$*-splitter* $\mathcal{R} = \{\operatorname{conv} C_j + \operatorname{cone} Q_j\}_{j\in J}$ *s.t.*

$$\langle \mathcal{R} \rangle \le \mathit{poly}(d) \cdot \langle \mathcal{P} \rangle \qquad \text{and} \qquad \#\mathcal{R} \le (2 \cdot \#\mathcal{P}) \uparrow d.$$

Chapter 2

In which we reintroduce the integers

Goal: refine the notion of splitter to work over $\mathbb{Z}$.



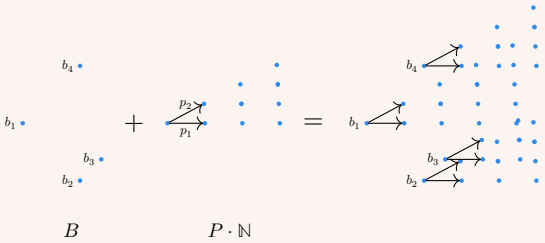
The geometry of a system of inequalities over the **integers**

Theorem (von zur Gathen & Sieveking, '78)

Consider $S \subseteq \mathbb{Z}^d$. Then, below (H) implies (V), but not vice versa:

(H) $S = \{\mathbf{x} \in \mathbb{Z}^d : A \cdot \mathbf{x} \leq \mathbf{c}\}$ *for some* $A \in \mathbb{Z}^{n \times d}$ *and* $\mathbf{c} \in \mathbb{Z}^m$

(V) $S = \bigcup_{\mathbf{b} \in B} L(\mathbf{b}, P)$ *for some finite sets* $B, P \subseteq \mathbb{Z}^d$.



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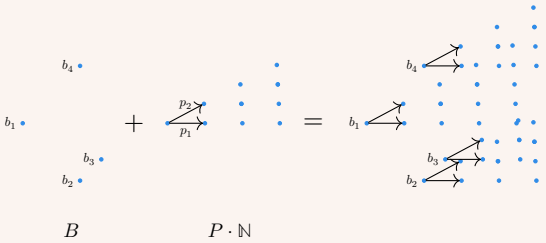

Cost of switching: (H) $\rightarrow$ (V)

$\langle B \rangle, \langle P \rangle \leq \mathsf{poly}(d) \cdot \langle [A|\mathbf{c}] \rangle$

$\#P \leq \#[A|\mathbf{c}] \uparrow d$

$\#B \leq 2 \uparrow (d \cdot \langle B \rangle)$

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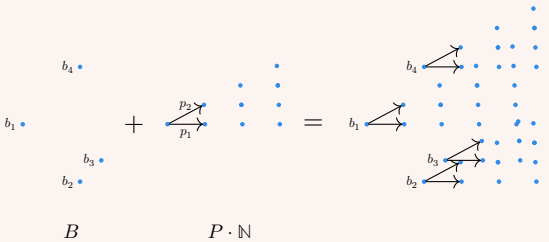

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We call $L(B, P) \stackrel{\text{def}}{=} \bigcup_{\mathbf{b} \in B} L(\mathbf{b}, P)$ an hybrid linear set.

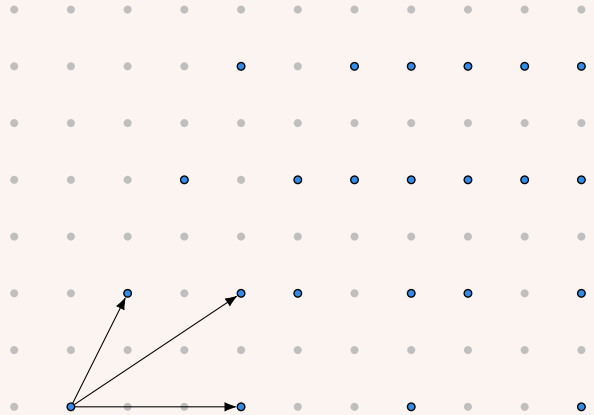
Semilinear sets (again)

Arithmetic progression

$b + i \cdot p$, where $i \in \mathbb{N}$

b base point, p period

Semilinear sets (again)

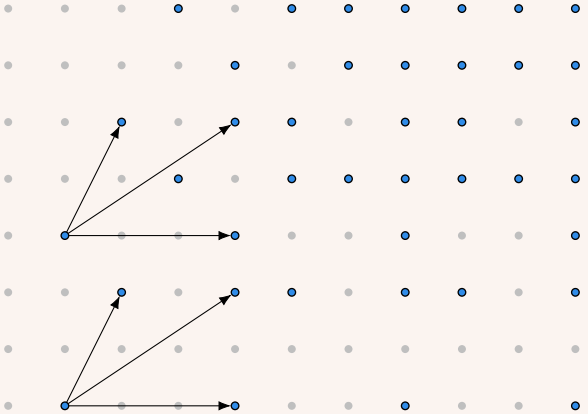


Linear set
(arithmetic progression
in multiple dimensions)

$L(\mathbf{b}, P)$

$\mathbf{b}$ base, P periods

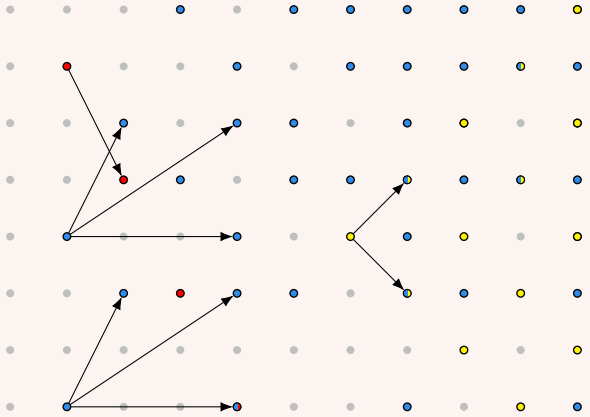
Semilinear sets (again)



Hybrid linear set
(finite union of linear sets having the same periodic behaviour)

$L(B, P)$
B bases, P periods

Semilinear sets (again)



Semilinear set
(finite union of
hybrid linear sets)

$$\bigcup_{i \in I} L(B_i, P_i)$$

I finite set of indices

$\mathbb{Z}$-splitters

An **($\mathbb{R}$-)splitter** for a family of convex polyhedra $\mathcal{P}$ is a family of convex polyhedra $\mathcal{R}$ s.t.

1. $\bigcup \mathcal{R} = \mathbb{R}^d$ and $\mathcal{R}$ is closed under taking faces
2. for every $R_1, R_2 \in \mathcal{R}$, the set $R_1 \cap R_2$ is either $\emptyset$ or a face of both R_1 and R_2
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Claim: Given $\mathcal{R}$ it is easy to compute $\mathbb{R}^d \setminus \bigcup \mathcal{P}$.

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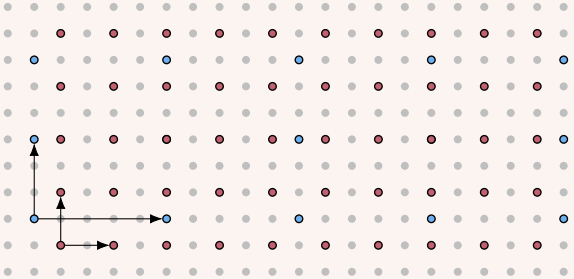

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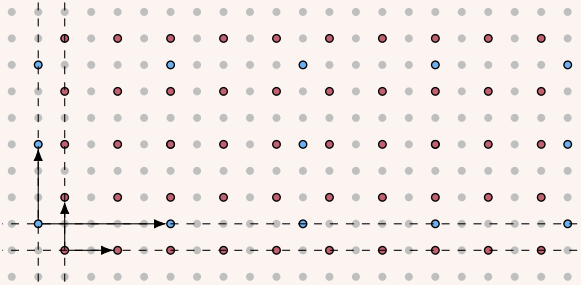

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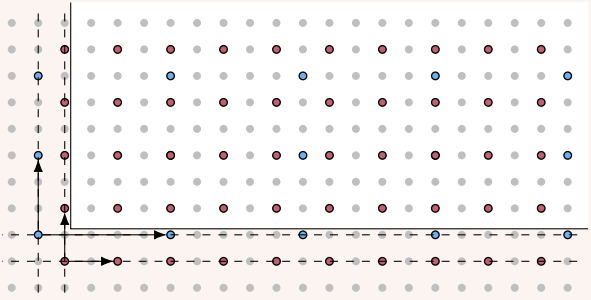
The region Z characterises a portion of $\mathbb{Z}^d$



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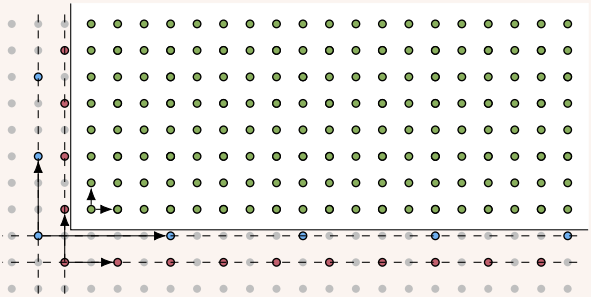

Not all hybrid linear sets agree with our goal!

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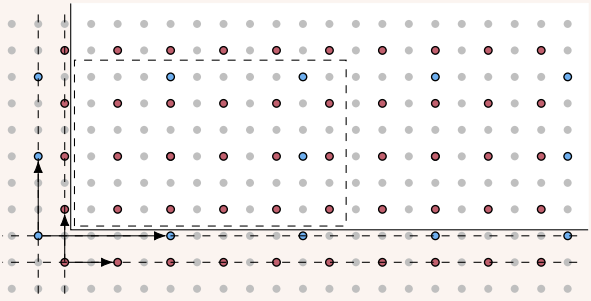

Idea: find C and Q such that

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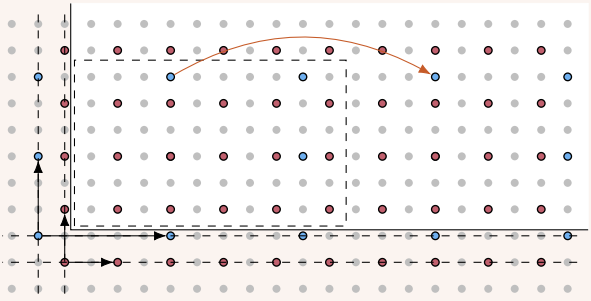

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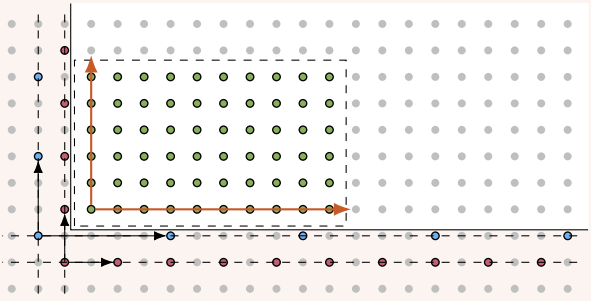

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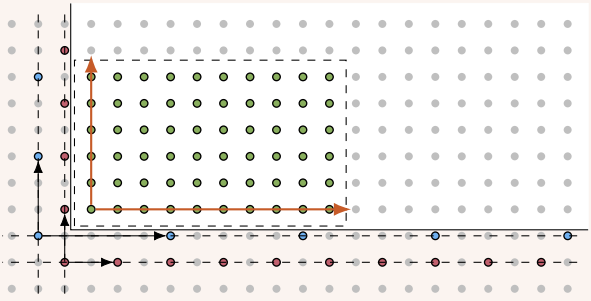
Claim: Given $\mathcal{Z}$ it is easy to compute $\mathbb{Z}^d \setminus \bigcup \mathcal{P}$.



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Theorem

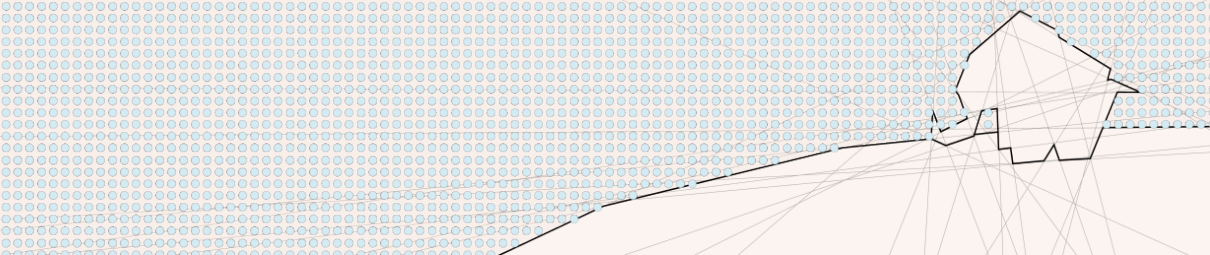
Every $\mathcal{P} = \{L(B_i, P_i)\}_{i \in I}$ has a $\mathbb{Z}$-splitter $\mathcal{Z} = \{L(C_j, Q_j)\}_{j \in J}$ such that

$$\langle \mathcal{Z} \rangle \leq \textit{poly}(d) \cdot \#I \cdot \langle \mathcal{P} \rangle \qquad \textit{and} \qquad \#\Big(\bigcup_{j \in J} Q_j\Big) \leq (\#I \cdot \max_{i \in I} \#P_i) \uparrow d\,.$$

Chapter 3

Complementation

Goal: Complementation.



The procedure

Input: a semilinear set M
Output: the complement $\mathbb{Z}^d \setminus M$ as a semilinear set

1. update M to $\bigcup_{i \in I} L(B_i, P_i)$ where each P_i is proper
2. compute $\mathcal{Z} = \{L(C_j, Q_j)\}_{j \in J}$ $\mathbb{Z}$-splitter for $\{L(B_i, P_i)\}_{i \in I}$
3. **for** $j \in J$ **do**
4. $\quad E_j \leftarrow C_j \setminus M$
5. $\mathcal{Q} \leftarrow \{Q_j\}_{j \in J}$
6. **for** $Q \in \mathcal{Q}$ **do**
7. $\quad \mathcal{E}_Q \leftarrow \{E_j : j \in J, Q_j = Q\}$
8. **return** $\bigcup_{Q \in \mathcal{Q}} L\Big(\big(\bigcup \mathcal{E}_Q\big), Q\Big)$

→ Discrete Carathéodory's theorem [Chistikov and Haase, ICALP'16]

→ Complement M region-wise

→ Merge hybrid linear sets having the same period sets

$L(B, P) \cup L(C, P) = L(B \cup C, P)$

Running time and bounds on the complement

Theorem

Let $M = \bigcup_{i \in I} L(B_i, P_i) \subseteq \mathbb{Z}^d$. *Then,* $\mathbb{Z}^d \setminus M = N = \bigcup_{j \in J} L(C_j, Q_j)$ *s.t.*

$$\#J \leq (\#I \cdot \max_{i \in I} \#P_i) \uparrow d \qquad \langle N \rangle \leq \textit{poly}(d) \cdot \#I \cdot \langle M \rangle \qquad \textit{and} \qquad Q_j \textit{ is proper.}$$

The family $\{(C_j, Q_j)\}_{j \in J}$ *can be computed in time* $2 \uparrow (\langle M \rangle \cdot \#I) \uparrow d$.

Running time and bounds on the complement

Theorem

Let $M = \bigcup_{i \in I} L(B_i, P_i) \subseteq \mathbb{Z}^d$. *Then,* $\mathbb{Z}^d \setminus M = N = \bigcup_{j \in J} L(C_j, Q_j)$ *s.t.*

$$\#J \leq (\#I \cdot \max_{i \in I} \#P_i) \uparrow d \qquad \langle N \rangle \leq \mathit{poly}(d) \cdot \#I \cdot \langle M \rangle \qquad \text{and} \qquad Q_j \text{ is proper.}$$

The family $\{(C_j, Q_j)\}_{j \in J}$ *can be computed in time* $2 \uparrow (\langle M \rangle \cdot \#I) \uparrow d$.

Theorem (A geometric procedure for Presburger arithmetic)

There is an algorithm that given Φ *in PA, computes a family* $\{(B_i, P_i)\}_{i \in I}$ *such that* $[\![\Phi]\!] = \bigcup_{i \in I} L(B_i, P_i)$. *The algorithm runs in time* $2 \uparrow \langle \Phi \rangle \uparrow d \uparrow h$ *and ensures:*

$$\#I \leq 2 \uparrow d \uparrow h \qquad \langle B_i \rangle, \langle P_i \rangle \leq \langle \Phi \rangle \uparrow d \uparrow h \qquad \text{and} \qquad P_i \text{ proper,}$$

where h *is the depth of the formula and* d *is the number of its variables.*

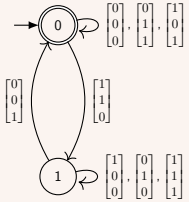
Concluding remarks

Quantifier elimination
[Presburger, ‘29]

$$\exists x : \varphi(x, \mathbf{y})$$
$$\equiv$$
$$\psi(\mathbf{y})$$

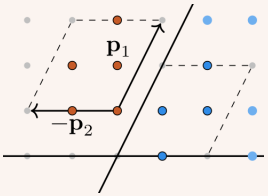
3EXPTIME

Automata
[Büchi, ‘60]



3EXPTIME

Geometry
[Ginsburg and Spanier, ‘66]



3EXPTIME

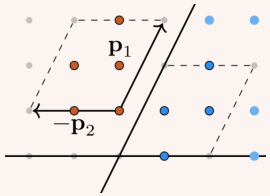
Concluding remarks

Key notion: $\mathbb{Z}$-splitters.

A decomposition of $\mathbb{Z}^d$ with many interesting properties.

Geometry

[Ginsburg and Spanier, '66]



3EXPTIME

Concluding remarks

Key notion: $\mathbb{Z}$-splitters.

A decomposition of $\mathbb{Z}^d$ with many interesting properties.

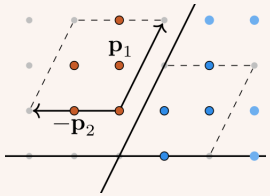
From our procedure:

$$[\![\Phi]\!] = \bigcup_{i \in I} L(B_i, P_i) \text{ where } \#I \leq 2 \uparrow d \uparrow h.$$

From this bound we conclude that the VC dimension of PA is doubly exponential.

For details see the paper: *"Geometric decision procedures and the VC dimension of linear arithmetic theories"*, Chistikov, Haase, Mansutti, LICS'22

Geometry
[Ginsburg and Spanier, '66]



3EXPTIME