Modal Logics and Local Quantifiers: A Zoo in the Elementary Hierarchy

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advanced reasoning in arithmetic theories

Enriching logics with quantification

A way to dynamically update a model and reason on its complex behavior.

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LTL $\rightarrow$ HyperLTL	PSPACE $\rightarrow$ Σ_1^1	[Fortin et al., MFCS'21]

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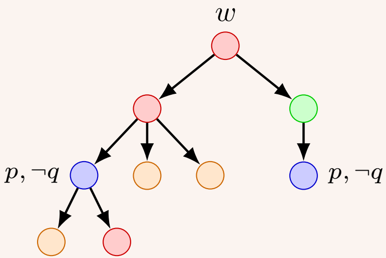
Modal logic and propositional quantifiers

Kripke structure*:

- Atomic propositions: p, q, ...
- worlds and accessibility relation:



* In this work we consider **Kripke trees**

Quantified ML:

$$\varphi, \psi \ := \ p \mid \top \mid \varphi \wedge \psi \mid \neg\varphi \mid \Diamond\varphi \mid \exists p : \varphi$$

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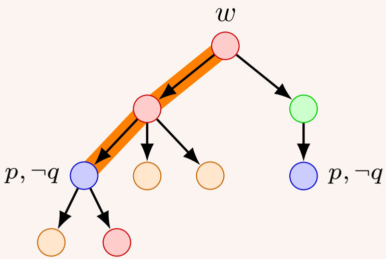

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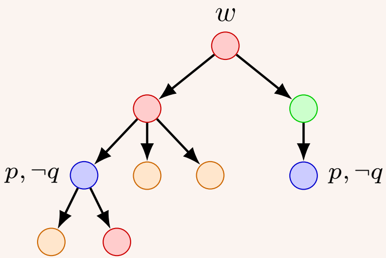

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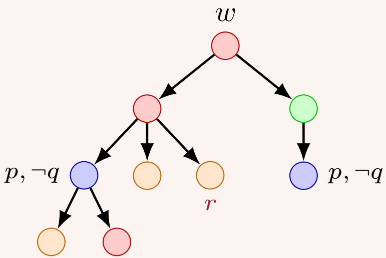

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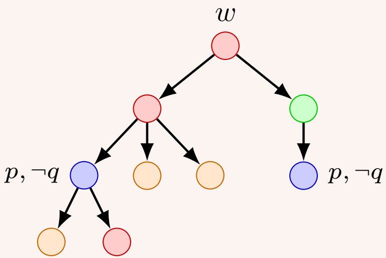

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Second-order semantics:

$\mathfrak{M}, w \models \exists p : \varphi \;\Leftrightarrow$ there is a set of worlds W such that $\mathfrak{M}[p \leftarrow W], w \models \varphi$

QML is very concise!

$$\underbrace{2^{2^{\cdots^{2}}}}_{f(n)}$$

Theorem (Bednarczyk and Demri, LICS'19)

On trees, the SAT problem of second-order quantified modal logic is $\textsc{Tower}$*-hard.*

Proof technique: uniform translation from the $k\textsc{NExpTime}$ version of the tiling problem, for every $k \geq 1$. The translation is exponential in k.

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Unpleasant result:

1. QML on trees subsumes Dependence logic, Team logic, Modal team logic, Ambient logic, Sabotage modal logic (on trees) ...
   but decision procedures for QML are suboptimal for these logics
2. Difficult result to parametrize (e.g. because translation is exponential).

Question: what is the expressiveness of QML on trees?

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Goals of this paper:

- Give a complexity analysis of QML (for both FO and SO semantics) that is more parametric and hierarchical, instead of just TOWER
- Characterise the expressive power of QML.

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Depth of quantification as a parameter?

Local quantifier $\exists^k q$ **:** binds q to worlds accessible in at most k steps.

$$\mathsf{ML}(\exists^k) = \mathsf{ML} + \exists^1 + \cdots + \exists^k$$

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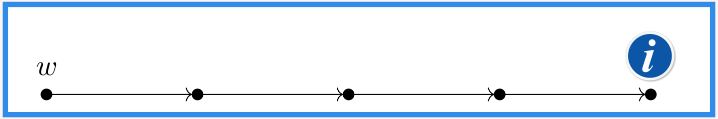

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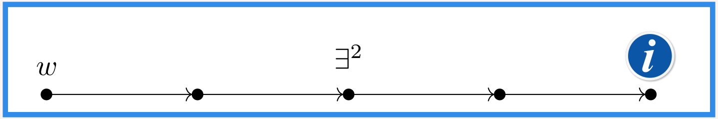

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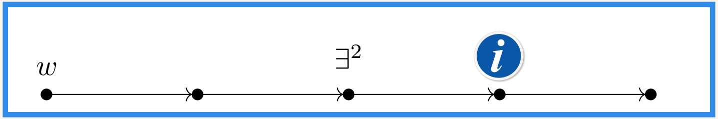

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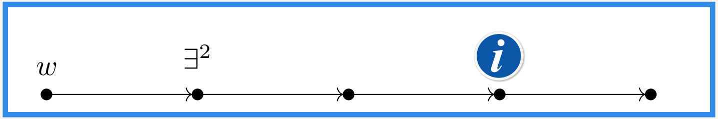

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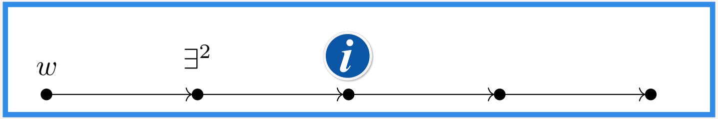

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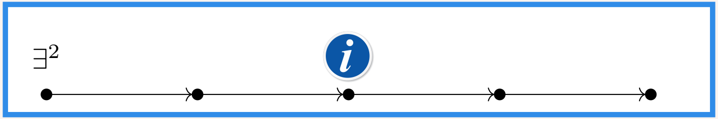

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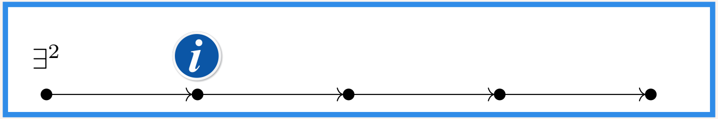

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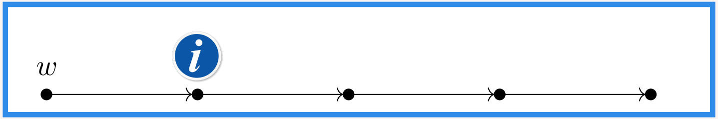

Observation: it is not just the depth of quantification that makes the problem hard, but also the "depth of interaction" between quantifiers.

Depth of interaction via the round-bounded fragment

Syntactic fragment of $\mathsf{ML}(\exists^k)$ (for both FO and SO) such that

$$\mathsf{ML}(\exists^{j+2}) \xrightarrow{\Diamond} \mathsf{ML}(\exists^{j+1}) \qquad\qquad \mathsf{ML}(\exists^{1}) \xrightarrow{\Diamond} \mathsf{ML}(\exists^{k})$$

E.g.: if φ is round-bounded for $\mathsf{ML}(\exists^3)$, then so is $\exists^3 p \,\Diamond \exists^2 q \,\Diamond \exists^1 r \,\Diamond \exists^3 p : \varphi$.

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A hierarchy for QML: $\mathsf{QML} = \bigcup_{k=1}^{\infty}$ (round-bounded $\mathsf{ML}(\exists^k)$)

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Syntactic fragment of ML($\exists^k$) (for both FO and SO) such that

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Theorem

The SAT problem of round-bounded ML($\exists^k_{FO}$) is $(k+1)\mathrm{NExpTime}$-complete. The SAT problem of ML($\exists^k_{FO}$) is $k\mathrm{NExpTime}$-complete for formulae of modal depth $\leq k$.

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k**AExp**$_{\mathbf{Poly}}$: class or problems decidable with an alternating Turing machine running in $k\mathrm{ExpTime}$ and performing polynomially many alternations.

Examples:

- Team logic is $\mathrm{AExp}_{\mathrm{Poly}}$-complete
- Presburger arithmetic is $2\mathrm{AExp}_{\mathrm{Poly}}$-complete.

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Corollary (1)

ML($\exists^1_{FO}$) and ML($\exists^1_{SO}$) are $2\textsc{NExpTime}$-comp. and $2\textsc{AExp}_{\textsc{Poly}}$-comp., respectively.

recall: ML($\exists^2_{FO}$) and ML($\exists^2_{SO}$) are $\textsc{Tower}$-complete.

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Corollary (2)

Quantified S5 [Fine, '70] *is $\mathrm{AE}\textsc{xp}_{\textsc{Poly}}$-comp.; Hybrid logic on trees is $\textsc{Tower}$-comp.*

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Corollary (3)

QML is as expressive as graded modal logic (under both FO and SO semantics).

graded modal logic $=$ ML $+\ \Diamond_{\geq j}$

Upper bounds

Input: A formula φ from $\mathsf{ML}(\exists^k_{SO})$.
Output: A graded modal logic formula ψ equivalent to φ.

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A quantifier-elimination procedure:

From $\exists p : \varphi_{\mathsf{QF}}$ we want to produce an equivalent ψ_{QF}.

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Input:	A formula φ from $\mathsf{ML}(\exists^k_{SO})$.
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A relaxed quantifier-elimination procedure:

- Given $\exists^1 p : \varphi_{\mathsf{GML}}$ we produce an equivalent ψ_{GML}.
- Given $\exists^k p : \varphi$ with φ in $\mathsf{ML}(\exists^{k-1}_{SO})$ we obtain an equivalent formula in $\mathsf{ML}(\exists^{k-1}_{SO})$.

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Complexity remarks:

- at each step, the formula becomes exponentially larger...
- for round-bounded formulae these explosions do not propagate between “rounds”.

Lower bounds

Proposition (Bednarczyk and Demri, LICS'19)

Let $k \geq 1$. There is a formula $\mathrm{type}(k, n)$ *of QML (second-order semantics) that is satisfied only by worlds with* $\mathrm{t}(k, n) \stackrel{\text{def}}{=} \underbrace{2^{2^{\cdot^{\cdot^{\cdot^{2^n}}}}}}_{k}$ *many children. It has size exponential in* k.

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Funny stuff: improvements are based on encoding ripple-carry adders as formulae and leverage a trick by Fisher and Rabin [SIAM-AMS'74] to compress the formula further.

Lower bounds (extra)

Encoding numbers in the Kripke structure: $\mathfrak{M}, w \models \mathrm{type}(\mathrm{k}, \mathrm{n})$ implies

- w has $t(k,n)$ children, all encoding a distinct number in $[0, t(k,n)-1]$
- using its children as indices, w encodes a number $\mathtt{n}_k(w) \in [0, t(k+1,n)-1]$.

In a nutshell, this is done thanks to formulae that express

$$\mathtt{n}_j(x) < \mathtt{n}_j(y) \quad \text{and} \quad \mathtt{n}_j(x)+1 = \mathtt{n}_j(y), \quad j \in [1,k].$$

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Issue: $\mathtt{n}_{j+1}(x) < \mathtt{n}_{j+1}(y)$ and $\mathtt{n}_{j+1}(x){+}1 = \mathtt{n}_{j+1}(y)$ are defined using several times $\mathtt{n}_j(x) < \mathtt{n}_j(y)$ and $\mathtt{n}_j(x){+}1 = \mathtt{n}_j(y)$ and their negation ($\Rightarrow$ exponential blowup in j)

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Solution:

- to subsume both $\mathtt{n}_j(x) < \mathtt{n}_j(y)$ and $\mathtt{n}_j(x){+}1 = \mathtt{n}_j(y)$ with a **single formula**...
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- ...whose negation can be expressed in terms of positive occurrences of itself

Why is this enough? We can use the Fisher and Rabin trick!

$$\varphi(\mathbf{x}) \wedge \varphi(\mathbf{y}) \quad \equiv \quad \forall \mathbf{z} : (\mathbf{z} = \mathbf{x} \vee \mathbf{z} = \mathbf{y}) \Rightarrow \varphi(\mathbf{z})$$

(this trick can be extended to Boolean combinations of positive occurrences of φ)

Lower bounds (extra)

Solut

- t …
- .

Why

Ripple-carry adders!

$+_j(x, y, z, c)$ that holds whenever

- $\mathtt{n}_j(x) + \mathtt{n}_j(y) = \mathtt{n}_j(z)$ and
- $\mathtt{n}_j(c)$ is the sequence of carries obtained when performing $\mathtt{n}_j(x) + \mathtt{n}_j(y)$ in binary, on $\mathtt{t}(j, n)$ many bits.

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$$\mathtt{n}_j(x) < \mathtt{n}_j(z) \;\equiv\; \exists y, c : +_j(x, y, z, c)$$

Lower bounds (extra)

Solut

- t ...n…
- .

Ripple-carry adders!

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Why

$$\mathtt{n}_j(x) + 1 = \mathtt{n}_j(z) \;\equiv\; \exists y, c : +_j(x, y, z, c) \land \mathtt{n}_j(y) = 1$$

(this t)

Lower bounds (extra)

Solut

- t …
- .

Why

(this t)

Ripple-carry adders!

$+_j(x, y, z, c)$ that holds whenever

- $\mathtt{n}_j(x) + \mathtt{n}_j(y) = \mathtt{n}_j(z)$ and
- $\mathtt{n}_j(c)$ is the sequence of carries obtained when performing $\mathtt{n}_j(x) + \mathtt{n}_j(y)$ in binary, on $\mathtt{t}(j, n)$ many bits.

the negation of $+_j(x, y, z, c)$ can be expressed as

$$(\exists w, d : +_j(x, y, w, d) \wedge (\mathtt{n}_j(w) \neq \mathtt{n}_j(z) \vee \mathtt{n}_j(d) \neq \mathtt{n}_j(c))) \vee (\textit{overflow})$$

Conclusion

In the paper:

- we introduce a hierarchy for QML formulae that depend on the “depth of interaction” of quantifiers
- we characterise the complexity of each level of the hierarchy, for both FO and SO
- we characterise the expressive power of QML.

Conclusion

In the paper:

- we introduce a hierarchy for QML formulae that depend on the "depth of interaction" of quantifiers
- we characterise the complexity of each level of the hierarchy, for both FO and SO
- we characterise the expressive power of QML.

This allows us to connect QML with other logics, w.r.t. the right level of the hierarchy.

One exception: Ambient logic is $\mathrm{AE}\textsc{xp}_{\textsc{Poly}}$-comp., but ML($\exists^1_{SO}$) is $2\mathrm{AE}\textsc{xp}_{\textsc{Poly}}$-comp.