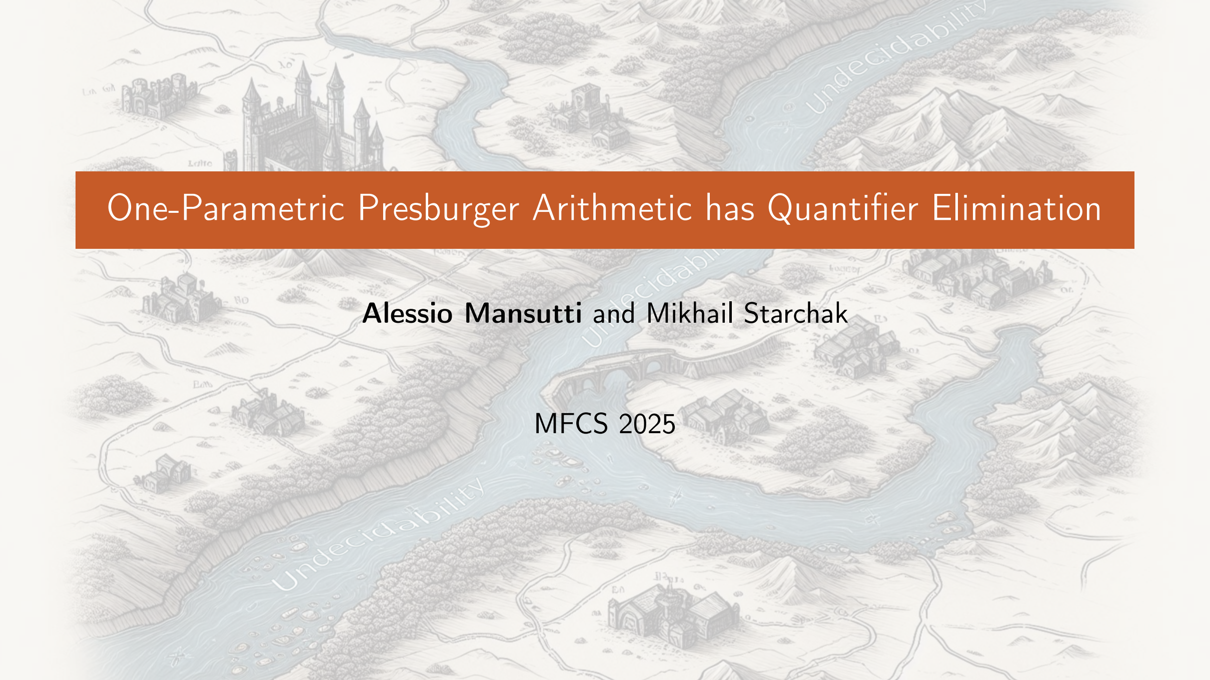

One-Parametric Presburger Arithmetic has Quantifier Elimination

Alessio Mansutti and Mikhail Starchak

MFCS 2025


Undecidability
Undecidability
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Undecidability


Undecidability
Undecidability
Undecidability
Peano arithmetic
ℤ
+
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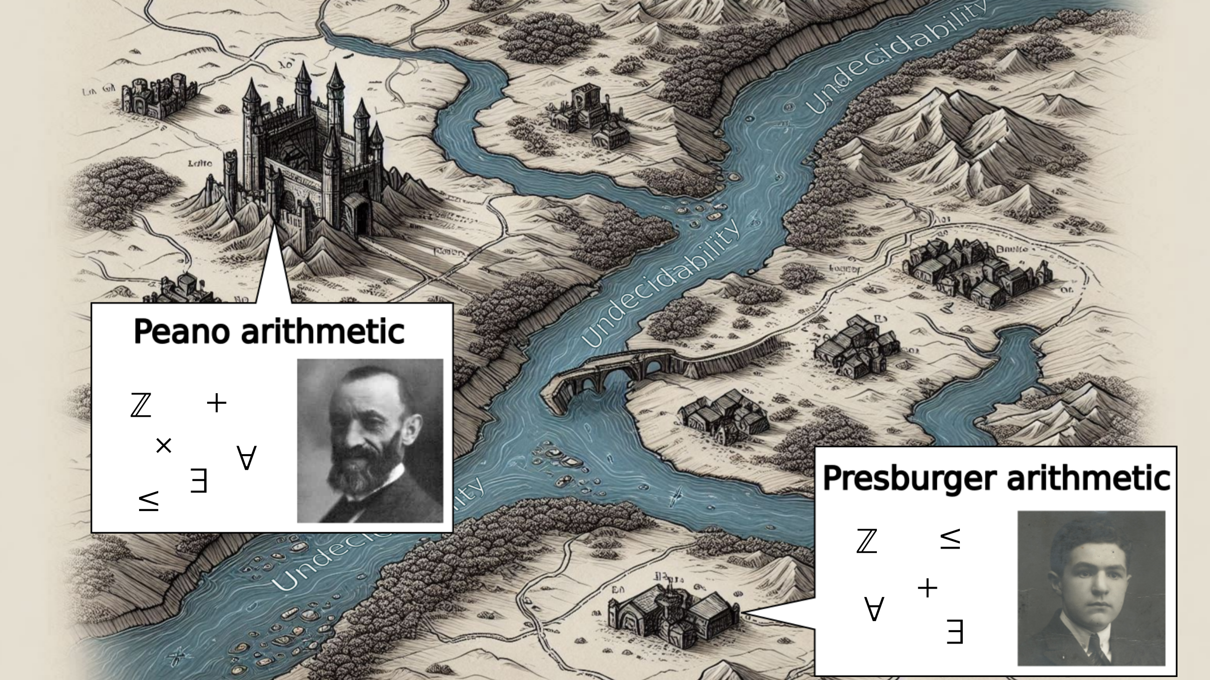
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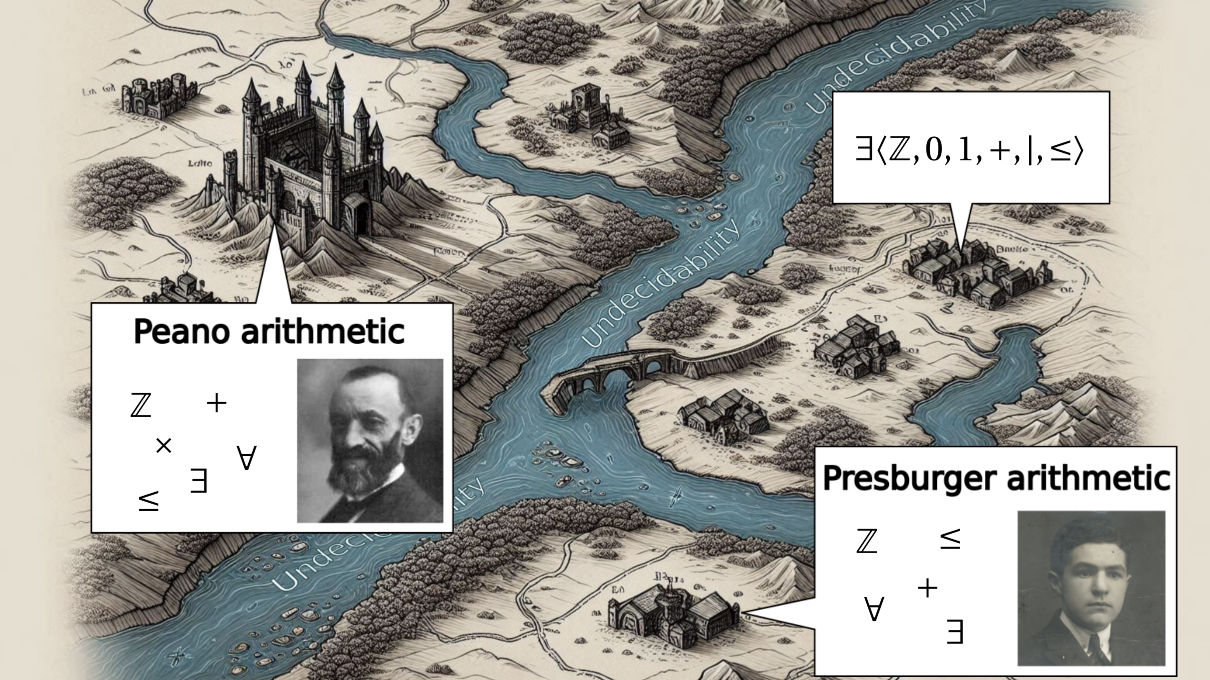

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$\mathbb{Z}$ $+$ $\times$ $\forall$ $\exists$ $\leq$
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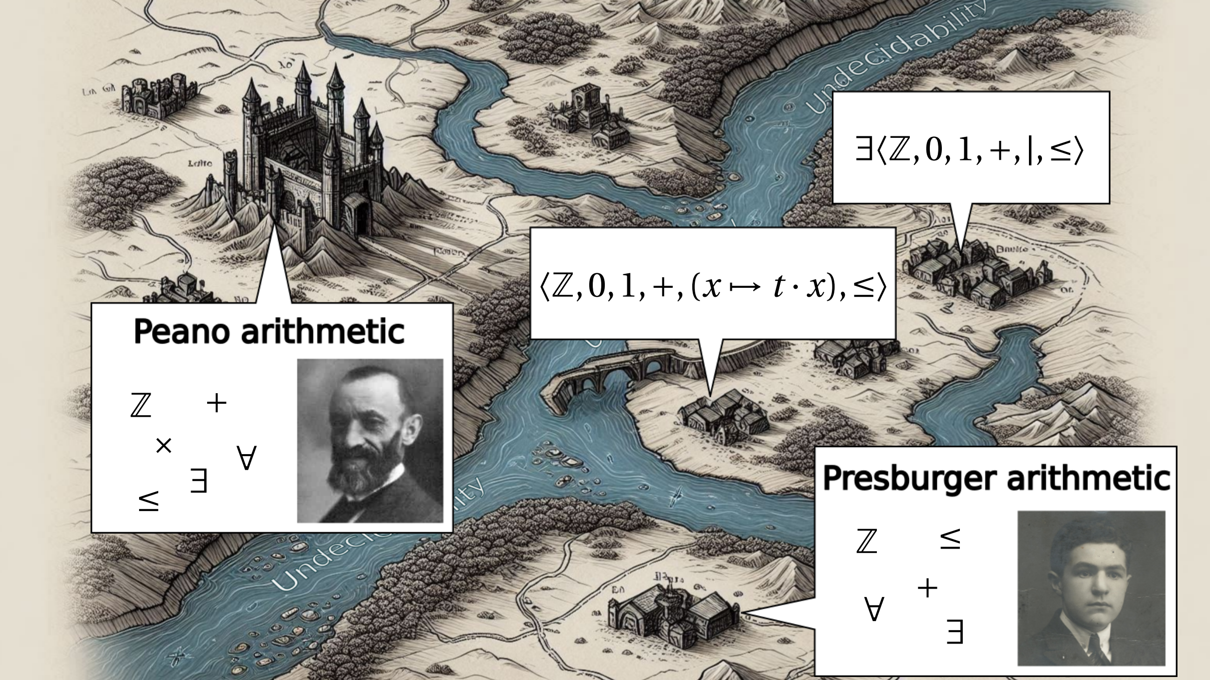
$\mathbb{Z}$ $\leq$ $\forall$ $+$ $\exists$



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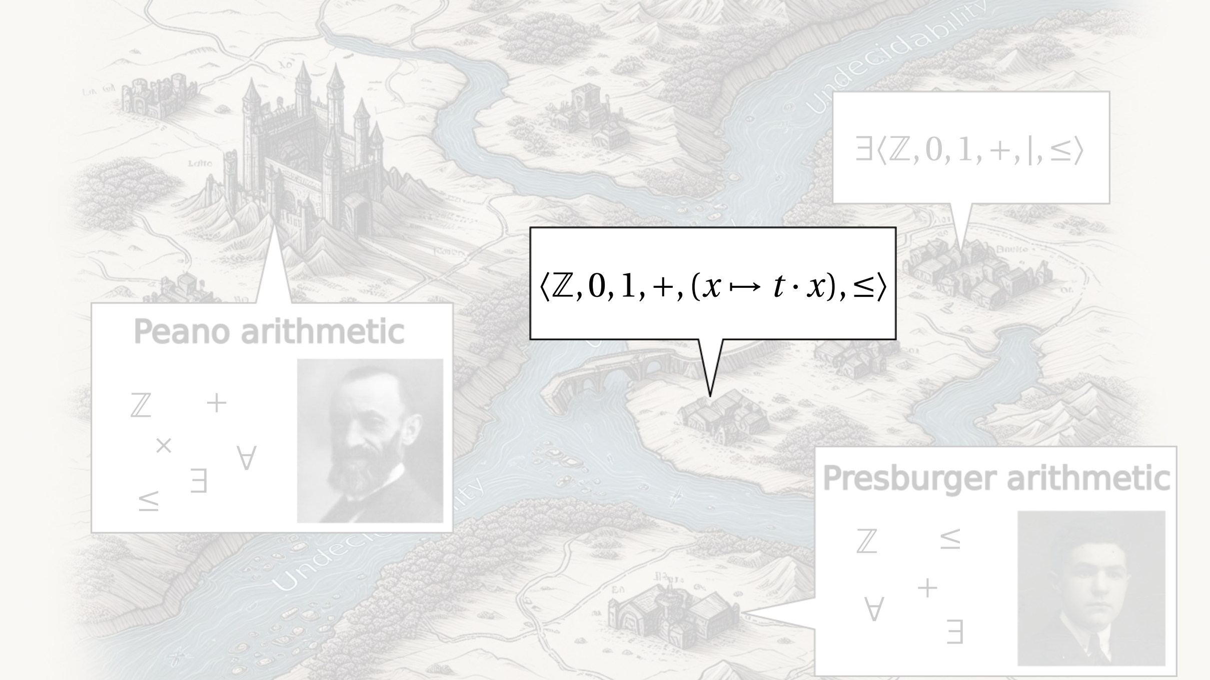
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First-order theory of the structure $\langle \mathbb{Z}, 0, 1, +, (x \mapsto t \cdot x), \leq \rangle$.

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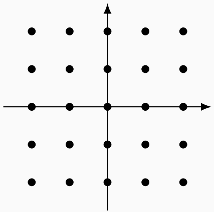

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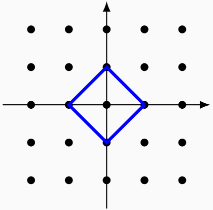

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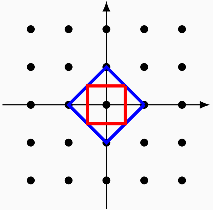

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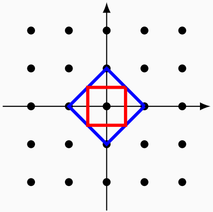

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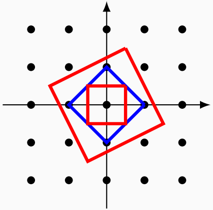
$t = 3$:	$\lvert 2x + 4y\rvert \leq 5 \ \wedge\ \lvert -4x + 2y\rvert \leq 5$	5 solutions



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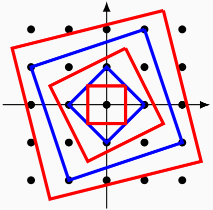
For a fixed $t \geq 0$, this formula:

- has $t^2 - 2t + 2$ solutions when t is odd
- has $t^2 - 2t + 5$ solutions when t is even



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"Chinese Remainder Theorem"

The following formula is valid:

$$t \geq 1 \implies \forall a \forall b \exists x : \quad \begin{aligned} & 0 \leq x < t(t+1) \\ \wedge & \quad\;\; t \mid x - a \\ \wedge & \; t+1 \mid x - b \end{aligned}$$

where $(p(t) \mid \tau) := \exists w \, (w \cdot p(t) = \tau)$.

"For every positive integer t, and for all integers a and b, there is an integer x in the interval $[0..t(t+1)-1]$ that is congruent to a modulo t, and to b modulo $t+1$."

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A formula $\varphi(\boldsymbol{x})$ of 1PPA defines a parametric Presburger family $\{[\![\varphi]\!]_k : k \in \mathbb{Z}\}$, where

$$[\![\varphi]\!]_k : \text{set of solution to } \varphi \text{ after replacing } t \text{ with } k$$

We can ask several questions about φ:

- satisfiability: is $[\![\varphi]\!]_k$ non-empty for some k?
- validity: is $[\![\varphi]\!]_k$ non-empty for every k?
- finiteness: is $[\![\varphi]\!]_k$ non-empty only for finitely many k?

Eventual quasi-polynomials and 1PPA

A function $f : \mathbb{N} \to \mathbb{N}$ is an eventual quasi-polynomial (EQP) whenever there are

- a threshold T and a period P, and
- a family of univariate polynomials $f_0, \ldots, f_{P-1}$

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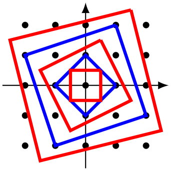
Examples:

$$\left\lfloor \frac{x}{2} \right\rfloor = \begin{cases} \frac{x}{2} & \text{if } x \text{ is even} \\ \frac{x-1}{2} & \text{if } x \text{ is odd} \end{cases}$$



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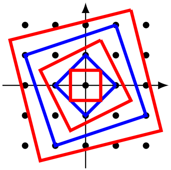

Theorem (Bogart, Goodrick, Woods. *Discrete Analysis 2017*)

Let φ be a 1PPA *formula. The counting function $f(k) := \#[\![\varphi]\!]_k$ is an EQP.*

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$$\Big\downarrow \; \textit{bounded quantifier elimination} \text{ (Weispfenning. } \textit{ISSAC 1997}\text{)}$$

"$\exists y \leq p(t)$" constrains y in $[0..p(t)]$

$$\varphi \;\equiv\; \exists y_1 \leq p_1(t) \; \forall y_2 \leq p_2(t) \; \ldots : \gamma$$

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parsimonious transformation (Chen, Li, Sam. *Trans. Amer. Math. Soc. 2012*)

$\#[\![\varphi]\!]_k = \#[\![\varphi']\!]_k$ for every k

φ' quantifier-free

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Let φ

Proof

In *Discrete Analysis 2017*, Bogart, Goodrick and Woods ask whether the parsimonious transformation can be replaced with quantifier elimination.

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We prove Goodrick's conjecture.

$\#[\![\varphi$

φ' quantifier-free

Our results

Theorem

There is a quantifier elimination procedure for the extension of 1PPA *with the functions:*

- *integer division:* $x \mapsto \left\lfloor \frac{x}{t^d} \right\rfloor$ *one function for each* $d \in \mathbb{N}$, *assumes* $t \neq 0$
- *integer remainder function:* $x \mapsto (x \bmod p)$ *for each* $p \in \mathbb{Z}[t]$
- *divisibility relation:* $p \mid x$ *for each* $p \in \mathbb{Z}[t]$

(The functions $x \mapsto \left\lfloor \frac{x}{p(t)} \right\rfloor$ capture all these functions and relations.)

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Theorem

For the class of all existential formulae of 1PPA*, the following holds:*

Satisfiability: **NP**-*complete*	*Universality:* **coNEXP**-*complete*	*Finiteness:* **coNP**-*complete*

Overview of our procedure

Input: A quantifier-free formula $\varphi(\boldsymbol{x}, \boldsymbol{z})$ from the extended language of 1PPA ($1\mathsf{PPA}^+$).
Output: A quantifier-free formula ψ from $1\mathsf{PPA}^+$ that is equivalent to $\exists \boldsymbol{x}\, \varphi$.

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Step I. Preprocessing: Remove divisions and remainder functions.

$$\cdots + \left\lfloor \frac{\tau}{t^d} \right\rfloor + \cdots \leq 0 \quad \to \quad \exists x \left(\cdots + x + \cdots \leq 0 \land \left(t^d x \leq \tau < t^d (x+1) \right) \right)$$

$$\cdots + (\tau \bmod f) + \cdots \leq 0 \quad \to \quad \exists x \left(\cdots + x + \cdots \leq 0 \land \left(0 \leq x < f - 1 \right) \land \left(f \mid \tau - x \right) \right)$$

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Step II. Bounded quantifier elimination:

$$\exists \boldsymbol{x}' : \varphi'(\boldsymbol{x}', \boldsymbol{z}) \quad \to_\beta \quad \exists \boldsymbol{w} \leq B : \gamma(\boldsymbol{w}, \boldsymbol{z})$$

such that $\exists \boldsymbol{z} \colon \gamma(\boldsymbol{z}, \boldsymbol{z})$ is equivalent to $\bigvee_\beta \exists \boldsymbol{w}_\beta \leq B_\beta : \gamma_\beta(\boldsymbol{w}_\beta, \boldsymbol{z})$

Step II: Bounded quantifier elimination in **NP** (simplified)

Naïve bounded quantifier elimination

Input: $\exists \boldsymbol{x} \colon \varphi(\boldsymbol{x}, \boldsymbol{z})$
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$Q \leftarrow$ empty sequence of bounded quantifiers
$\ell \leftarrow 1$
for x in $\boldsymbol{x}$ and occurring in φ **do**
  $(a \cdot x + \tau \sim 0) \leftarrow$ **guess** an (in)equality in φ featuring x, or $x \leq 0$
  $\tau \leftarrow \tau + w$ with w fresh free variable
  append to Q the quantifier $\exists w \leq$ "$a \cdot \mathit{mod}(\varphi)$"
  $\varphi \leftarrow \varphi[\frac{-\tau}{a} \,/\, x]$
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return $Q\varphi$

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> Consider $\tau_1 \leq a \cdot x \leq \tau_2$ with $a > 0$.
>
> *"between τ_1 and τ_2 there is a multiple of a"*

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"$a \cdot mod(\varphi)$" is a positive polynomial in $\mathbb{Z}[t]$ that upper bounds the product between a and all the divisors appearing in φ.

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Output: $\exists \boldsymbol{w} \leq B : \gamma(\boldsymbol{w}, \boldsymbol{z})$

$Q \leftarrow$ empty sequence of bounded quantifiers
$\ell \leftarrow 1$
for x in $\boldsymbol{x}$ and occurring in φ **do**
  $(a \cdot x + \tau \sim 0) \leftarrow$ **guess** an (in)equality in φ featuring x, or $x \leq 0$
  $\tau \leftarrow \tau + w$ with w fresh free variable
  append to Q the quantifier $\exists w \leq$ "$a \cdot mod(\varphi)$"
  $\varphi \leftarrow \varphi[\frac{-\tau}{a} / x]$
  divide each (in)equality in φ by ℓ
  $\varphi \leftarrow \varphi \land (a \mid \tau)$
  $\ell \leftarrow a$
return $Q\varphi$

$\varphi[\frac{-\tau}{a} / x]$: $-b \cdot \tau + a \cdot \rho = 0$

Step II: Bounded quantifier elimination in $\mathbf{NP}$ (simplified)

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$\varphi[\frac{-\tau}{a} / x]$: $\quad -b \cdot \tau + a \cdot \rho = 0$

Note: $(-b \cdot \tau + a \cdot \rho) = \det \begin{bmatrix} a & \tau \\ b & \rho \end{bmatrix}$

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Bounded quantifier elimination meets Bareiss's algorithm

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Desnanot–Jacobi identity:

$$\begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \\ \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} \end{vmatrix} = a_{11} \cdot \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Overview of our procedure

Input: A quantifier-free formula $\varphi(\boldsymbol{x}, \boldsymbol{z})$ from the extended language of 1PPA ($1\mathrm{PPA}^+$).
Output: A quantifier-free formula ψ from $1\mathrm{PPA}^+$ that is equivalent to $\exists \boldsymbol{x}\, \varphi$.

Step I. Preprocessing: Remove divisions and remainder functions.

Step II. Bounded quantifier elimination: compute $\exists \boldsymbol{w} \leq B : \gamma(\boldsymbol{w}, \boldsymbol{z})$

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Step III. Remove the divisibility relations.

$$f \mid \tau(\boldsymbol{w}) + \sigma(\boldsymbol{z}) \quad \to \quad f \mid \tau(\boldsymbol{w}) + (\sigma(\boldsymbol{z}) \bmod f)$$

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Bounded!

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$$
\begin{aligned}
f \mid \tau(\boldsymbol{w}) + \sigma(\boldsymbol{z}) \quad &\to \quad f \mid \tau(\boldsymbol{w}) + (\sigma(\boldsymbol{z}) \bmod f) \\
&\to \quad \exists w \leq p(t) : f \cdot w = \tau(\boldsymbol{w}) + (\sigma(\boldsymbol{z}) \bmod f)
\end{aligned}
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$$

Step IV. Elimination of bounded quantifiers by "bit blasting".

Step IV: Elimination of bounded quantifiers

$$\exists x \leq t^2 + t - 1 \ \exists z \leq t + 2 \ : \ (t+1) \cdot z = x + (-b \bmod t+1)$$

Assume $t \geq 2$.

Step IV: Elimination of bounded quantifiers

$$\exists x \leq t^2 + t - 1 \; \exists z \leq t + 2 \; : \; (t+1) \cdot z = x + (-b \bmod t + 1)$$

Assume $t \geq 2$. Bit blast:

$$\exists z \leq t + 2 \colon \varphi \quad \to \quad \exists z_0, z_1, z_2 \leq t - 1 \colon \quad 0 \leq z_2 \cdot t^2 + z_1 \cdot t + z_0 \leq t + 2$$
$$\wedge\, \varphi[z_2 \cdot t^2 + z_1 \cdot t + z_0 \,/\, z]$$

Step IV: Elimination of bounded quantifiers

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$$\wedge \varphi[z_2 \cdot t^2 + z_1 \cdot t + z_0 \,/\, z]$$

The equality $(t+1) \cdot z = x - (b \bmod t+1)$ becomes:

$$(t+1) \cdot (z_2 \cdot t^2 + z_1 \cdot t + z_0) = (x_2 \cdot t^2 + x_1 \cdot t + x_0) + (-b \bmod t+1).$$

Step IV: Elimination of bounded quantifiers

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$$\wedge \, \varphi[z_2 \cdot t^2 + z_1 \cdot t + z_0 \, / \, z]$$

The equality $(t+1) \cdot z = x - (b \bmod t+1)$ becomes:

$$-z_2 \cdot t^3 + (x_2 - z_1 - z_2) \cdot t^2 + (x_1 - z_0 - z_1) \cdot t + (x_0 - z_0) + (-b \bmod t+1) = 0.$$

Step IV: Elimination of bounded quantifiers

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Divide by t the maximal subterm with no quantified variables:

$$(-b \bmod t+1) \quad \to \quad \left\lfloor \frac{-b \bmod t+1}{t} \right\rfloor \cdot t + \big((-b \bmod t+1) \bmod t\big)$$

Step IV: Elimination of bounded quantifiers

$$
\begin{aligned}
-z_2 \cdot t^3 + (x_2 - z_1 - z_2) \cdot t^2 + \left(x_1 - z_0 - z_1 + \left\lfloor \frac{-b \bmod t+1}{t} \right\rfloor\right) \cdot t \\
+ (x_0 - z_0) + \big((-b \bmod t + 1) \bmod t\big) = 0
\end{aligned}
$$

Step IV: Elimination of bounded quantifiers

$$-z_2 \cdot t^3 + (x_2 - z_1 - z_2) \cdot t^2 + \left(x_1 - z_0 - z_1 + \left\lfloor \frac{-b \bmod t+1}{t} \right\rfloor\right) \cdot t \\ + (x_0 - z_0) + \big((-b \bmod t+1) \bmod t\big) = 0$$

- $(x_0 - z_0) + \big((-b \bmod t+1) \bmod t\big)$ belongs to $[-t..2 \cdot t]$...
- ...and must be divisibile by t. (This only applies to equalities.)

Step IV: Elimination of bounded quantifiers

$$-z_2 \cdot t^3 + (x_2 - z_1 - z_2) \cdot t^2 + \left(x_1 - z_0 - z_1 + \left\lfloor \frac{-b \bmod t+1}{t} \right\rfloor\right) \cdot t \\ + (x_0 - z_0) + \big((-b \bmod t+1) \bmod t\big) = 0$$

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- ...and must be divisibile by t. (This only applies to equalities.)

Guess $r_0 \in \{-1, 0, 1, 2\}$ and rewrite the equality as

$$-z_2 \cdot t^2 + (x_2 - z_1 - z_2) \cdot t + \left(x_1 - z_0 - z_1 + \left\lfloor \frac{-b \bmod t+1}{t} \right\rfloor\right) + r_0 = 0$$
$$\wedge \ (x_0 - z_0) + \big((-b \bmod t+1) \bmod t\big) = r_0 \cdot t$$

Important: x_0 has only integer coefficients!

Step IV: Elimination of bounded quantifiers

Let's do another iteration:

$$
\begin{aligned}
&-z_2 \cdot t^2 + (x_2 - z_1 - z_2) \cdot t + \left(x_1 - z_0 - z_1 + \left\lfloor \frac{-b \bmod t+1}{t} \right\rfloor\right) + r_0 = 0 \\
\wedge\ &(x_0 - z_0) + \big((-b \bmod t+1) \bmod t\big) = r_0 \cdot t
\end{aligned}
$$

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$$\wedge\ (x_0 - z_0) + \big((-b \bmod t+1) \bmod t\big) = r_0 \cdot t$$

Divide by t!

Step IV: Elimination of bounded quantifiers

Let's do another iteration:

$$-z_2 \cdot t^2 + (x_2 - z_1 - z_2) \cdot t + \left(x_1 - z_0 - z_1 + \left\lfloor \frac{\left\lfloor \frac{-b \bmod t+1}{t} \right\rfloor + r_0}{t} \right\rfloor \cdot t + \left(\left(\left\lfloor \frac{-b \bmod t+1}{t} \right\rfloor + r_0\right) \bmod t\right)\right) = 0$$

$$\wedge\ (x_0 - z_0) + \big((-b \bmod t + 1) \bmod t\big) = r_0 \cdot t$$

Step IV: Elimination of bounded quantifiers

Let's do another iteration:

$$-z_2 \cdot t^2 + (x_2 - z_1 - z_2) \cdot t + \left(x_1 - z_0 - z_1 + \left\lfloor \frac{\left\lfloor \frac{-b \bmod t+1}{t} \right\rfloor + r_0}{t} \right\rfloor \cdot t + \left(\left(\left\lfloor \frac{-b \bmod t+1}{t} \right\rfloor + r_0 \right) \bmod t \right) \right) = 0$$

$$\wedge \ (x_0 - z_0) + \big((-b \bmod t + 1) \bmod t \big) = r_0 \cdot t$$

belongs to $[-2 \cdot t..2 \cdot t]$
so guess $r_1 \in [-2..2]$

Step IV: Elimination of bounded quantifiers

Let's do another iteration:

$$
\begin{aligned}
& -z_2 \cdot t + (x_2 - z_1 - z_2) + \left\lfloor \frac{\left\lfloor \frac{-b \bmod t+1}{t} \right\rfloor + r_0}{t} \right\rfloor + r_1 = 0 \\
\wedge\ & (x_0 - z_0) + \big((-b \bmod t+1) \bmod t\big) = r_0 \cdot t \\
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\end{aligned}
$$

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$$
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& -z_2 \cdot t + (x_2 - z_1 - z_2) + \left\lfloor \frac{\left\lfloor \frac{-b \bmod t+1}{t} \right\rfloor + r_0}{t} \right\rfloor + r_1 = 0 \\
\wedge\ & (x_0 - z_0) + \big((-b \bmod t+1) \bmod t\big) = r_0 \cdot t \\
\wedge\ & (x_1 - z_0 - z_1) + \left(\left(\left\lfloor \frac{-b \bmod t+1}{t} \right\rfloor + r_0\right) \bmod t\right) = r_1 \cdot t
\end{aligned}
$$

Now all variables but z_2 have only integer coefficients!

- Repeat until all quantified variables only occur with integer coefficients.
- Afterwards, call a quantifier-elimination procedure for Presburger arithmetic.

Our results

Theorem

There is a quantifier elimination procedure for the extension of 1PPA *with the functions:*

- *integer division:* $x \mapsto \left\lfloor \frac{x}{t^d} \right\rfloor$ *one function for each* $d \in \mathbb{N}$, *assumes* $t \neq 0$
- *integer remainder function:* $x \mapsto (x \bmod p)$ *for each* $p \in \mathbb{Z}[t]$
- *divisibility relation:* $p \mid x$ *for each* $p \in \mathbb{Z}[t]$

Theorem

For the class of all existential formulae of 1PPA, *the following holds:*

Satisfiability:	Universality:	Finiteness:
NP-complete	**coNEXP**-complete	**coNP**-complete