On Polynomial-Time Decidability of k-Negations Fragments of FO Theories

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advanced reasoning in arithmetic theories

The Frobenius problem

Given coins in denominations

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what is the largest value c that cannot be generated? Does such a c exist?

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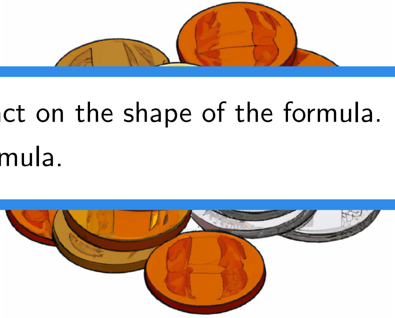
$$\exists c : \neg\mathsf{gen}(c) \land \forall d : d > c \implies \mathsf{gen}(d)$$

$$\mathsf{gen}(x) \coloneqq \exists y_1 \ldots \exists y_k : x = m_1 \cdot y_1 + \cdots + m_k \cdot y_k \land \bigwedge_{i=1}^{k} y_i \geq 0$$

The Frobenius problem

Giver



Input of the problem has little to no impact on the shape of the formula.

The problem statement influences the formula.

what

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Theorem (Nguyen and Pak; SIAM J. Comput. 2022)

Deciding Short PA sentences of the form $\exists^j \forall^k \exists^\ell \Phi$ is NP-hard for some $j, k, \ell \in \mathbb{N}$.

(note: with further alternations the problem climbs the polynomial hierarchy)

What happens for Weak Presburger arithmetic?

Weak PA: first-order theory of $(\mathbb{Z}, 0, 1, +, =)$

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Answer: No! Fixing the number of negations suffices to get P.

In this paper...

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We apply the framework to linear arithmetic theories (e.g. Weak PA)

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The k-negations fragment of a FO theory is the set of all formulae built from

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Let Φ and Ψ be from the j and k-negations fragments of a FO theory. Then,

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Example: $(\exists \boldsymbol{y} : A \cdot \boldsymbol{x} + A' \cdot \boldsymbol{y} = \boldsymbol{b}) \implies (\exists \boldsymbol{z} : C \cdot \boldsymbol{y} + C' \cdot \boldsymbol{z} = \boldsymbol{d})$
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- $\Phi \implies$

For Weak PA, this is essentially as good as it gets:

deciding formulae of the form (a_i, b_i in input)

$$\exists x \forall y : \bigwedge_{i=1}^{n} (a_i \cdot y = x - b_i \implies y = 3 \cdot x + 1)$$

is NP-hard.

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Question 1: normal forms?

Difference Normal form for propositional logic [Hausdorff, 1914]:

$$\Phi_1 - (\Phi_2 - \ldots (\Phi_{n-1} - \Phi_n)) \qquad (\Phi_1 - \Phi_2 := \Phi_1 \wedge \neg\Phi_2)$$

where Φ_i is a negation-free DNF formula.

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$$[\![\exists x(\Phi - \Psi)]\!] = [\![\exists x \Phi]\!] \setminus [\![\forall x(\Psi \mid \Phi)]\!] \qquad [\![\forall x(\Psi_1 - \Psi_2 \mid \Phi)]\!] = [\![\forall x(\Psi_1 \mid \Phi)]\!] \setminus [\![\exists x \Psi_2]\!]$$

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$$[\![\exists x : \Phi_1 - (\Phi_2 - (\Phi_3 - \Phi_4))]\!] = [\![\exists x \Phi_1]\!] \setminus \Big([\![\forall x(\Phi_2 \,|\, \Phi_1)]\!] \setminus \big([\![\exists x \Phi_3]\!] \setminus [\![\forall x(\Phi_4 \,|\, \Phi_3)]\!]\big)\Big)$$

Key point: it suffices to study quantification on negation-free DNF formulae

Question 2: representation for conjunctions of atomic formulae?

mathematical level

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algorithmic level

- Difference normal form partially fixes the representation...
- but we are free to choose the representation for conjunctions of atomic formulae

Example: for Weak PA, we will use shifted lattices $\boldsymbol{v} + \boldsymbol{p}_1 \cdot \mathbb{Z} + \dots \boldsymbol{p}_d \cdot \mathbb{Z}$

Difference normal form + representations

We consider the FO theory of a structure $\mathcal{A} = (A, \sigma, I)$ to be the structure

$$\mathsf{FO}(\mathcal{A}) \coloneqq (\llbracket \mathcal{A} \rrbracket,\ \top,\ \bot,\ \wedge,\ \vee,\ -,\ (\exists, \boldsymbol{X}),\ (\underbrace{\forall(\cdot,\cdot)}_{\text{relative universal quantifier}}, \boldsymbol{X}),\ \implies)$$

- $\llbracket \mathcal{A} \rrbracket \subseteq 2^{[X \to A]}$: set of definable sets

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- $[\![\mathcal{A}]\!] \subseteq 2^{[X \to A]}$: set of definable sets
- let D be a representation of (at least) all conjunctions of atomic formulae
- we represent $[\![\mathcal{A}]\!]$ as diffnf(D), where

$$\text{diffnf}(D) \coloneqq \text{syntactic chains of relative complements of elements in un}(D)$$
$$\text{un}(D) \coloneqq \text{syntactic unions of elements in } D$$

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4. $(\mathrm{diffnf}(D), (\exists, \boldsymbol{X}), (\forall(\cdot,\cdot), \boldsymbol{X}))$ has a UXP signature with parameter depth, where
   - for $(\exists, \boldsymbol{X})$ it suffices to look at inputs from D
   - for $(\forall(\cdot,\cdot), \boldsymbol{X})$ it suffices to look at inputs from $\mathrm{un}(D) \times D$

k-negations fragment in PTIME: the case of Weak PA

$D \coloneqq$ set of tuples $(\boldsymbol{v}, \boldsymbol{p}_1, \dots \boldsymbol{p}_d)$ representing the shifted lattices
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Lemma

The relative universal quantifier $\forall \boldsymbol{x}(\bigvee_{j=1}^{L} X_j, Y)$, where $\bigvee_{j=1}^{L} X_j \in \mathrm{un}(D)$ and $Y \in D$, is equivalent to a combination of $\wedge$, $\vee$, $-$ and $\exists \boldsymbol{x}$ applied to the sets $X_1, \dots, X_L, Y$. This combination can be computed in PTIME when L is fixed.

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Conclusion

- We studied the validity problem for FO formulae with a fixed amount of negations
- Suitable normal form: difference normal form
- Identified a minimal set of computational problems that are sufficient to show that validity is in PTIME; they do not involve complementation
- Applied the framework to Weak PA
- Same technique can be applied to Weak LRA and a few abstract domains