

# Existential Presburger Arithmetic with Divisibility Constraints: A tour

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Trends in Arithmetic Theories – ICALP'24

# Presburger arithmetic (PrA)

The first-order theory of  $\langle \mathbb{Z}; 0, 1, +, \leq \rangle$

*“Every integer is either even or odd”*

$$\forall x \exists y : x = 2y \vee x = 2y + 1$$

## Why Presburger arithmetic?

Wide range of applications in verification,  
program synthesis, compiler optimisation...

- SAT of the existential fragment is in NP
- Full theory is decidable in 2EXPSPACE

### A Survival Guide to Presburger Arithmetic

Christoph Haase, University of Oxford, UK

The first-order theory of the integers with addition and order, commonly known as Presburger arithmetic, has been a central topic in mathematical logic and computer science for almost 90 years. Presburger arithmetic has been the starting point for numerous lines of research in automata theory, model theory and discrete geometry. In formal verification, Presburger arithmetic is the first-choice logic to represent and reason about systems with infinitely many states. This article provides a broad yet concise overview over the history, decision procedures, extensions and geometric properties of Presburger arithmetic.

#### 1. A VERY SHORT HISTORY OF PRESBURGER ARITHMETIC

Around the 1920s of the last millennium, David Hilbert together with his doctoral student Wilhelm Ackermann began to pursue what is nowadays known as *Hilbert's program*. The goal of this program was to create a formal system that would allow for providing solid foundations for all of mathematics. The means to achieve this goal was to use mathematical logic as an unambiguous language in which all mathematical statements could be formalised and manipulated according to a well-defined axiomatic system. In addition to asking for consistency and completeness, Hilbert also required that it should be possible to verify or falsify the truth of any given mathematical statement in a finite number of steps within this formal system. This requirement gave rise to the *Entscheidungsproblem* (*decision problem*) that was introduced by Hilbert and Ackermann in their book *Grundzüge der Theoretischen Logik* (*Principles of Mathematical Logic*) published in 1928, see [Hilbert and Ackermann 1950] for an English translation. The Entscheidungsproblem demands an algorithm that given a sentence in first-order logic together with a finite number of axioms allows for deciding whether that sentence is valid, i.e., holds in any structure satisfying the given axioms.

After studying the *Principles of Mathematical Logic* and related work, Alfred Tarski approached his student Mojżesz Presburger and asked him to investigate the completeness of a particular theory capturing a limited fragment of number theory. A couple of months later, Presburger showed in his Master's thesis the completeness



Fig. 1. Presburger's student card from the University of Warsaw, Poland.

## Existential Presburger Arithmetic with Divisibility constraints (EPAD)

**EPAD:** existential first-order theory of the structure  $\langle \mathbb{Z}; 0, 1, +, |, \leq \rangle$ .

$$x \mid y \iff \text{there is } k \in \mathbb{Z} \text{ such that } k \cdot x = y$$

*A meaningless example:*  $\varphi(x, y) := \exists w : (x + y) \mid w \wedge (2w \leq 5x + y \vee \neg(w \mid (y + 2)))$

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## Goals for this tutorial

- Familiarise with EPAD (proof of NP-hardness for fixed number of variables)
- Give an overview on applications of EPAD (automata theory, word equations)
- Present the main technique to decide EPAD (with some new improvement)

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- 2021:** M. Starchak gives a quantifier-elimination-style procedure for EPAD.

# Part 1

*What can we say and do with EPAD?*

*(plot of  $y | x$ )*

## A few (meaningful) examples

**Modular arithmetic:**

$$z \mid x - y$$

$$x \equiv y \pmod{z}$$

$$\exists r : 1 \leq r \leq f(x) - 1 \wedge (g(x) \equiv r \pmod{f(x)})$$

$$\neg(f(x) \mid g(x))$$

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## A few (meaningful) examples

Modular arithmetic

### EPAD does not have a polynomial small-model property

Consider the family of formulae  $\{\varphi_n\}_{n \in \mathbb{N}}$  given by

$$\varphi_n(x) := \exists x_1, \dots, x_{n+1} : x \geq x_{n+1} \wedge x_1 \geq 2 \wedge \bigwedge_{i=1}^n \underbrace{(x_i \mid x_{i+1} \wedge x_i + 1 \mid x_{i+1})}_{\text{implies } x_{i+1} > x_i^2}.$$

The smallest  $x$  satisfying  $\varphi_n(x)$  is bigger than  $2^{2^n}$ .

$$z \mid x \wedge z \mid y \wedge \exists w : x \mid w \wedge y \mid (w + z)$$

$$\gcd(x, y) = z$$

$$x \mid y \wedge x + 1 \mid y$$

$$\exists k : y = k(x^2 + x) \wedge k \neq 0$$

## How to express bounded squaring

How can we characterise  $y = x^2 \wedge 1 \leq x \leq b$ , where  $b \in \mathbb{N}$  ?

$$1 \leq x \leq b \wedge 1 \leq y \leq b^2 \wedge b^2 \mid y - x^2$$

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$$1 \leq x \leq b \wedge 1 \leq y \leq b^2 \wedge \exists k \in \mathbb{Z} : y = x^2 + k \cdot b^2$$

  $k$  forced to be 0

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**Idea:** there is nothing special about  $b^2$ , we just need  $f: \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$  such that

- $f(x, y) \geq b^2$ , for every  $x \in [1, b]$  and  $y \in [1, b^2]$
- $f(x, y) \mid y - x^2$  is expressible in EPAD (intent: cancel  $x^2$  using  $f$ )

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**Solution:**<sup>1</sup>  $f(x, y) := x + b^2$

$$x + b^2 \mid y - x^2 \quad \text{iff} \quad x + b^2 \mid y - x^2 + x(x + b^2) \quad \text{iff} \quad x + b^2 \mid y + b^2 x$$

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<sup>1</sup>In Lipshitz's '81 paper,  $f(x, y) := (x + b)(x + b + 1)$  is used instead

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**Final formula:**  $1 \leq x \leq b \wedge 1 \leq y \leq b^2 \wedge x + b^2 \mid y + b^2 x$ .

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# How to express bounded squaring

NP-hardness for 2 variables, 4 inequalities, and 2 divisibility constraints

Quadratic residue problem [Manders and Aldeman, STOC'76]

**Input:**  $a, b, m \in \mathbb{N}$  given in binary.

**Question:** Is there  $x \in [1, b]$  such that  $x^2 \equiv a \pmod{m}$  ?

The quadratic residue problem is NP-complete. Reduction to EPAD:

$$\exists x \exists y : 1 \leq x \leq b \wedge 1 \leq y \leq b^2 \wedge x + b^2 \mid y + b^2 x \wedge m \mid y - a .$$

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## Revisiting Parameter Synthesis for One-Counter Automata

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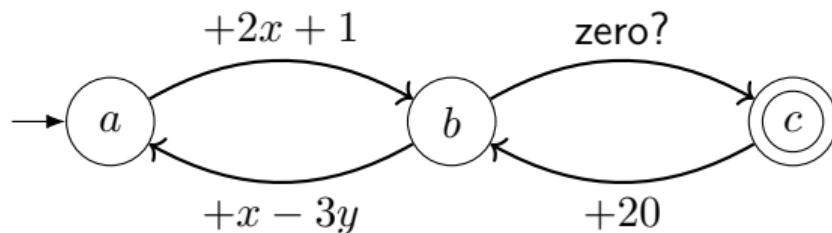
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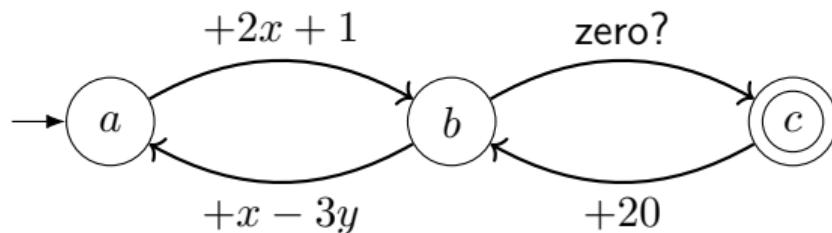
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## Parametric one-counter automata (POCA):



## Parameter synthesis problem:

*Input:* A POCA  $\mathcal{A}$  with initial state  $a$ , and an  $\omega$ -regular property  $P$  (e.g. finite reachability, Büchi, coBüchi, LTL languages...).

*Question:* Is there a valuation of the parameters (e.g.  $x, y$ ) over  $\mathbb{Z}$  such that in all runs starting at  $(a, 0)$ , the property  $P$  holds.

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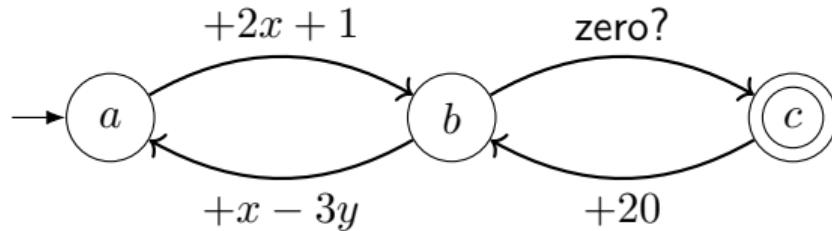
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**Parametric one-counter automata (POCA):**Let us try to obtain **one** run that reaches  $c$  :

$$\exists K \in \mathbb{N} : 0 + K \cdot (2x + 1 + x - 3y) + 2x + 1 = 0$$

initial value  
of the counter

final value  
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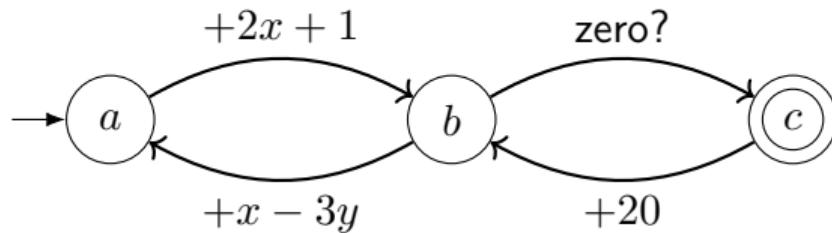
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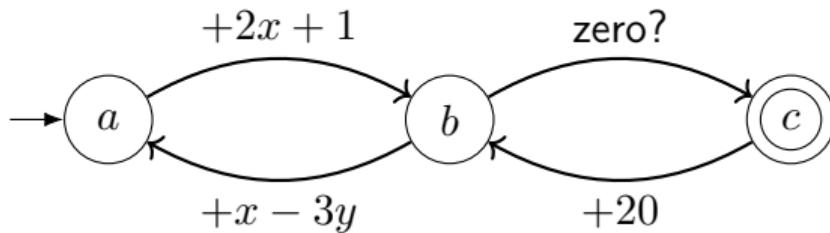
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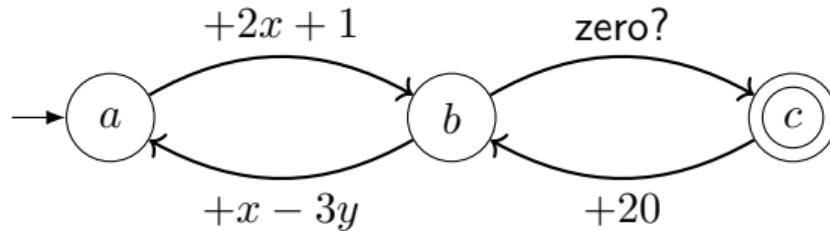
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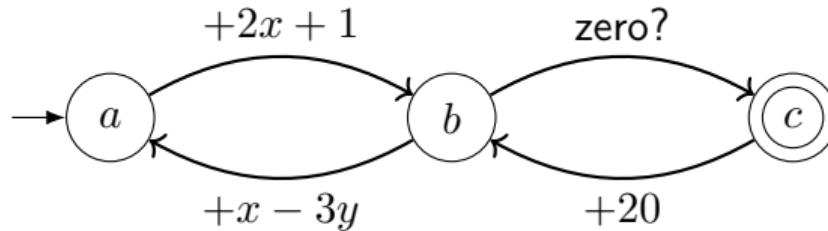
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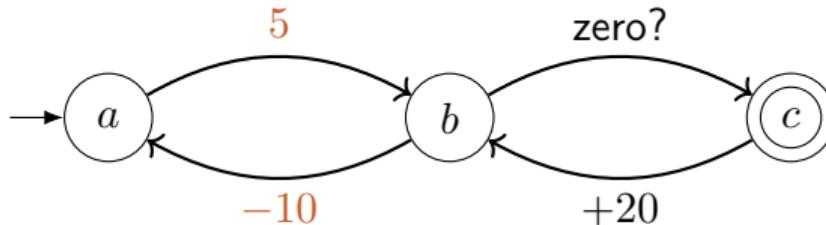
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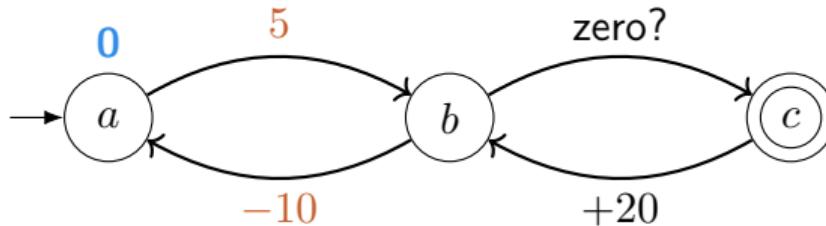
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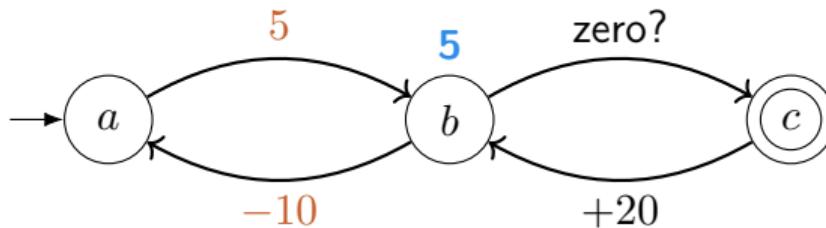
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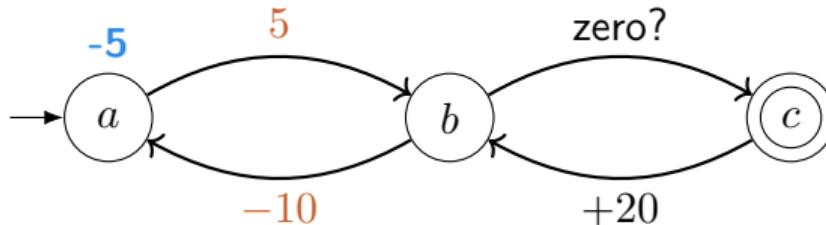
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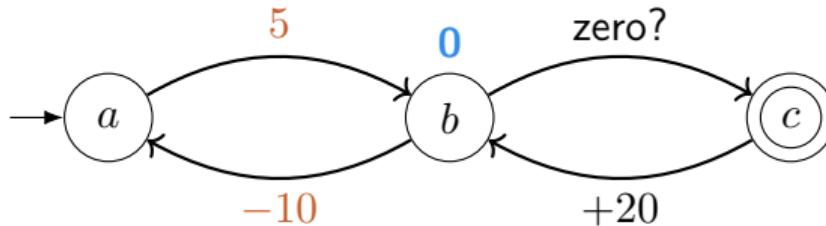
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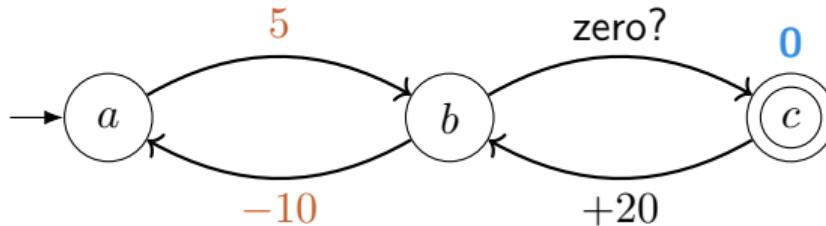
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However, synthesis requires to test the given property against **all** runs!



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Translation into BIL, a decidable fragment of  $\forall\exists$ PAD:

$$\forall \mathbf{x} \in \mathbb{N} \bigvee_{i \in I} \left( \varphi_i(\mathbf{x}) \wedge \exists \mathbf{y} \in \mathbb{N} \bigwedge_{j \in J_i} (f_j(\mathbf{x}) \mid g_j(\mathbf{x}, \mathbf{y}) \wedge f_j(\mathbf{x}) > 0) \right)$$

where every  $\varphi_i$  is a formula from PrA.

## From BIL to ∀PAD

### Theorem (Extended Chinese Remainder Theorem (CRT))

For  $i \in [1, k]$ , let  $a_i, r_i \in \mathbb{Z}$  and  $m_i \in \mathbb{Z} \setminus \{0\}$ .

The univariate system of divisibilities  $\left\{ m_i \mid (a_i \cdot x - r_i) \quad i \in [1, k] \right.$

has a solution iff so does  $\begin{cases} \gcd(a_i, m_i) \mid r_i & i \in [1, k] \\ \gcd(a_i \cdot m_j, a_j \cdot m_i) \mid (a_i \cdot r_j - a_j \cdot r_i) & i, j \in [1, k]. \end{cases}$

## From BIL to ∀PAD

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1. remove  $\mathbf{y}$  using the CRT. This introduces  $\gcd(h_1(\mathbf{x}), \dots, h_m(\mathbf{x})) \mid f(\mathbf{x}) \dots$
2. ...which can be rewritten as  $\forall u : \bigwedge_{i=1}^m u \mid h_i(\mathbf{x}) \implies u \mid f(\mathbf{x})$ .

## Applying EPAD:

### QUADRATIC WORD EQUATIONS WITH LENGTH CONSTRAINTS, COUNTER SYSTEMS, AND PRESBURGER ARITHMETIC WITH DIVISIBILITY

ANTHONY W. LIN<sup>a</sup> AND RUPAK MAJUMDAR<sup>b</sup>

**Word equations:** A word equations problem is a system  $E$

$$w_1 = w_2, \quad w_3 = w_4, \quad \dots, \quad w_{2k-1} = w_{2k} \qquad \text{(e.g. } x \cdot x \cdot b = y \cdot a \cdot z\text{)}$$

where each  $w_i$  is a word in  $(\Sigma \cup X)^*$ , with  $\Sigma$  finite alphabet and  $X$  set of variables.

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**Problem:** Is there a substitution  $\sigma: X \rightarrow \Sigma^*$  satisfying  $E$  and  $\varphi$ ?

# Applying EPAD:

## QUADRATIC WORD EQUATIONS WITH LENGTH CONSTRAINTS, COUNTER SYSTEMS, AND PRESBURGER ARITHMETIC WITH DIVISIBILITY

ANTHONY W. LIN<sup>a</sup> AND RUPAK MAJUMDAR<sup>b</sup>

**Open problem:** Is solving word equations with length constraints decidable?

**Quadratic fragment:** In the word equations, each variable occur at most twice.

This problem can be solved with EPAD.

- translate the equations in a counter-automata-like device.
- the reachability relation of these devices can be expressed in EPAD.

**Problem:** Is there a substitution  $\sigma: X \rightarrow \Sigma^*$  satisfying  $E$  and  $\varphi$ ?

# Problems that are interreducible to EPAD

## Reachability in Succinct and Parametric One-Counter Automata

Christoph Haase, Stephan Kreutzer, Joël Ouaknine, and James Worrell  
Oxford University Computing Laboratory, UK

## Simultaneous Rigid *E*-Unification and Related Algorithmic Problems

Anatoli Degtyarev

Computing Science Department  
Uppsala University

Yuri Matiyasevich

Steklov Institute of Mathematics  
St.Petersburg Division

Andrei Voronkov

Computing Science Department  
Uppsala University

# Part 2

*How to solve EPAD.*

*(plot of  $\gcd(x, y) = 1$ )*

## Searching solutions – a roadmap

Consider the problem of finding a solution for a system of inequalities with divisibilities:

$$A \cdot \mathbf{x} \leq \mathbf{b} \wedge \bigwedge_{i=1}^n f_i(\mathbf{x}) \mid g_i(\mathbf{x})$$

**Lipshitz's algorithm in a nutshell:**

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1. Remove the system of inequalities  $A \cdot \mathbf{x} \leq \mathbf{b}$   
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1. Remove the system of inequalities  $A \cdot \mathbf{x} \leq \mathbf{b}$   
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2. Translate the system of divisibilities into an equisatisfiable **increasing system**  
(notion inspired by the Chinese Remainder Theorem (CRT); “tautology-driven”)

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(why? Increasing systems enjoy a local-to-global property)

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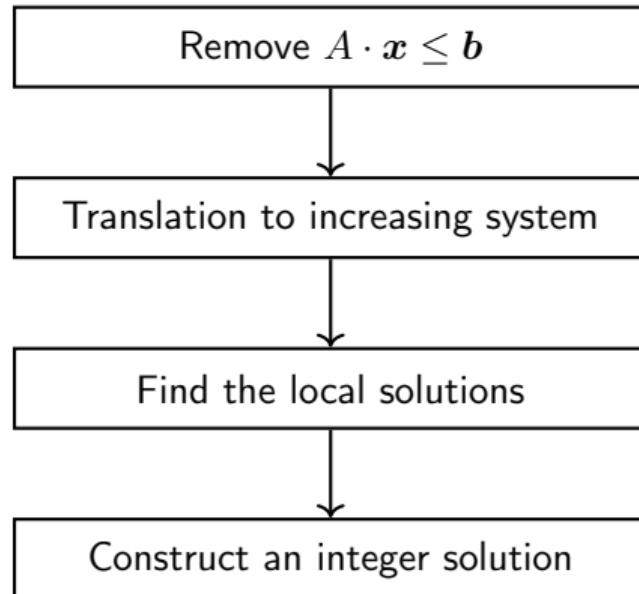
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(construction of the integer solution uses the CRT)

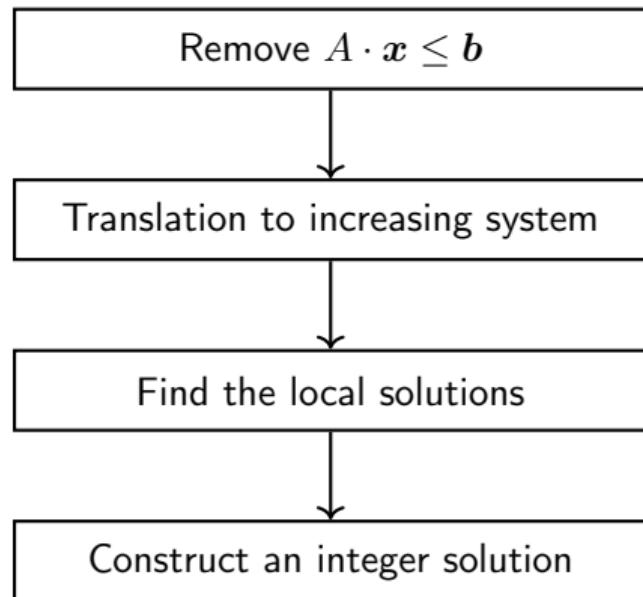
## Algorithms for EPAD – search vs. decision

Search algorithm:

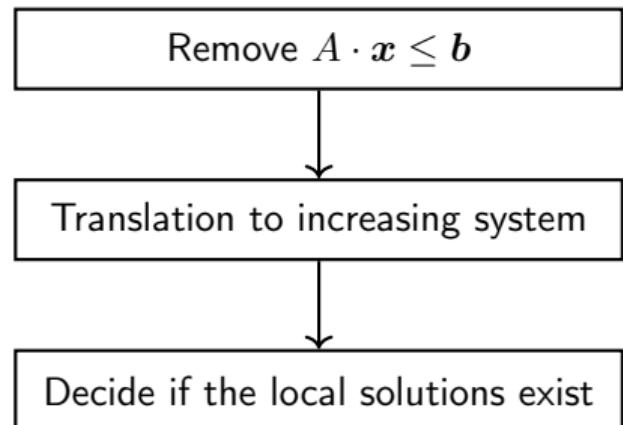


# Algorithms for EPAD – search vs. decision

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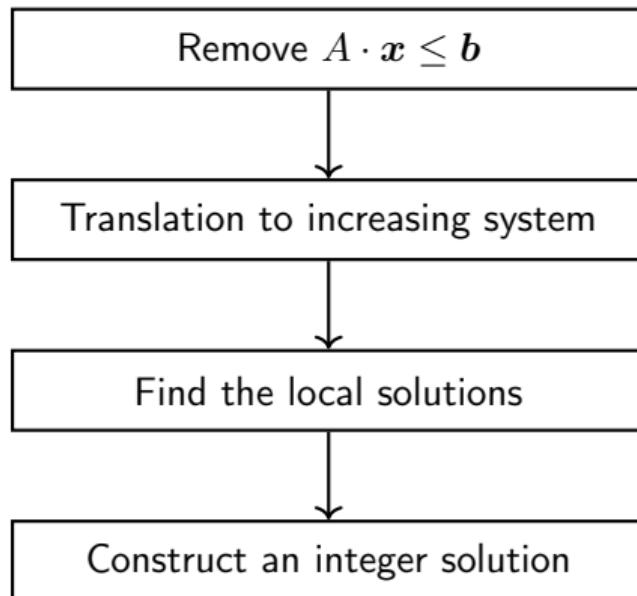


Decision algorithm:



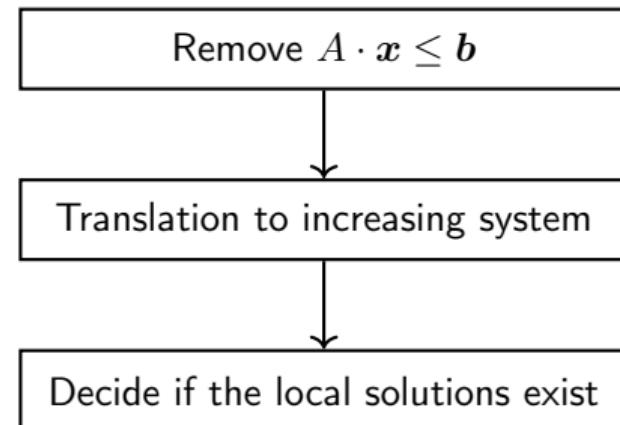
# Algorithms for EPAD – search vs. decision

Search algorithm:



(F)NP

Decision algorithm:



NEXP

DEXP vs. NP

DEXP

■ : optimal   ■ : we don't know

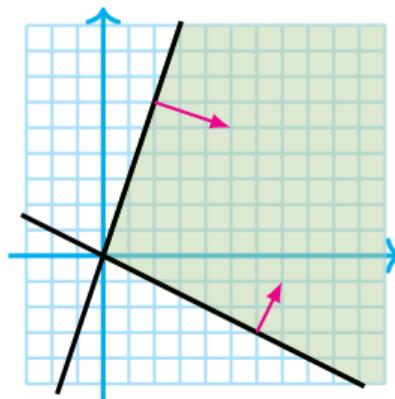
# 1: Remove the system of inequalities (NP)

## Theorem (von zur Gathen and Sieveking, '78)

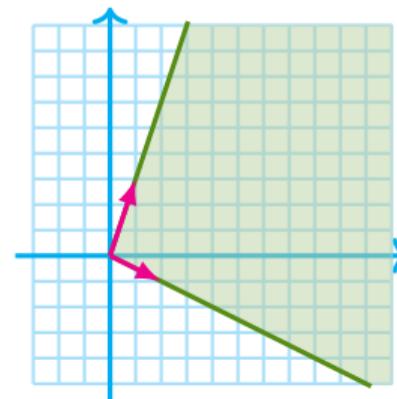
Let  $\Phi(\mathbf{x}) := A \cdot \mathbf{x} \leq \mathbf{b} \wedge C \cdot \mathbf{x} = \mathbf{d}$ , with  $\mathbf{x}$  vector of  $d$  variables. Then,

$$\{\mathbf{x} \in \mathbb{Z}^d : \Phi(\mathbf{x})\} = \bigcup_{j=1}^k \{\mathbf{u}_j + E_j \cdot \mathbf{y} : \mathbf{y} \in \mathbb{N}^{d-r}\},$$

where  $r := \text{rank}(C)$ ,  $\mathbf{u}_j \in \mathbb{Z}^d$  and  $E_j = [\mathbf{p}_{j,1}, \dots, \mathbf{p}_{j,d-r}] \in \mathbb{Z}^{d \times (d-r)}$ .



≡



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$$A \cdot \mathbf{x} \leq \mathbf{b} \wedge \bigwedge_{i=1}^n f_i(\mathbf{x}) | g_i(\mathbf{x}) \rightsquigarrow \bigvee_{j=1}^k \left( \mathbf{y} \geq \mathbf{0} \wedge \bigwedge_{i=1}^n f_i(\mathbf{u}_j + E_j \cdot \mathbf{y}) | g_i(\mathbf{u}_j + E_j \cdot \mathbf{y}) \right)$$

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(implicit) equalities in  $A \cdot \mathbf{x} \leq \mathbf{b} \Rightarrow$  number of variables decreases

## 2: Find an equisatisfiable increasing system – idea

### Theorem (Extended Chinese Remainder Theorem (CRT))

For  $i \in [1, k]$ , let  $a_i, r_i \in \mathbb{Z}$  and  $m_i \in \mathbb{Z} \setminus \{0\}$ .

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By iterating the CRT one can decide multivariate system of divisibility constraints  $\bigwedge_{i=1}^k f_i(\mathbf{x}) \mid g_i(\mathbf{x})$  where the  $f_i(\mathbf{x})$  are constant polynomials.

We would like to use the CRT for arbitrary systems of divisibilities.

**Main problem:** a variable can occur in both sides of a divisibility.

## 2: Find an equisatisfiable increasing system – definition

The system  $\Phi := \bigwedge_{i=1}^k f_i(\mathbf{x}) \mid g_i(\mathbf{x})$  implies further divisibilities following the rules:

$$\frac{f \mid f}{f \mid f} \qquad \frac{f \mid g \quad a \in \mathbb{Z}}{f \mid a \cdot g} \qquad \frac{f \mid g \quad f \mid h}{f \mid g + h} \qquad \frac{f \mid a \cdot g \quad g \mid h \quad a \in \mathbb{Z}}{f \mid a \cdot h}$$

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$\Phi(\mathbf{x})$  is said to be **increasing** whenever there is an ordering  $x_1 \prec \dots \prec x_d$  of the variables in  $\mathbf{x}$  such that, for every  $f \mid g$  implied by  $\Phi$ ,

(leading variable of  $f$ )  $\prec$  (leading variable of  $g$ )  
or     $f \mid g$  is a trivial divisibility of the form  $f \mid a \cdot f$ .

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**Examples:**

$$x + 1 \mid y - 2$$

is increasing for  $x \prec y$ , but **not** for  $y \prec x$

$$x + 1 \mid y - 2 \wedge x + 1 \mid x + y$$

is **not** increasing (it implies  $x + 1 \mid x + 2$ )

## 2: Find an equisatisfiable increasing system – computation (NEXP)

**Input:** a system  $\Phi := \bigwedge_{i=1}^k f_i(\mathbf{x}) \mid g_i(\mathbf{x})$

**Output:** an increasing system that is equisatisfiable with  $\Phi$  over  $\mathbb{N}$

1. if  $\Phi$  is increasing, then return a suitable order for the variables  
(check in polynomial time in the size of  $\Phi$ ; based on iterated Kernel computations)
2. if  $\Phi$  is not increasing,

### Proposition

$\Phi \models \bigvee_{i=1}^n h_i = 0$  for some finite set  $\{h_1, \dots, h_n\}$  of non-constant linear polynomials.

- 2.1 guess  $i \in [1, n]$  and apply the Theorem by von zur Gathen and Sieveking

$$\{\mathbf{x} \in \mathbb{N}^d : h_i(\mathbf{x}) = 0\} = \bigcup_{j=1}^k \{\mathbf{u}_j + E_j \cdot \mathbf{y} : \mathbf{y} \in \mathbb{N}^{d-1}\}$$

- 2.2 guess  $j \in [1, k]$ , substitute  $\mathbf{x}$  by  $\mathbf{u}_j + E_j \cdot \mathbf{y}$  and goto 1. (less variables!)

## 2: Find an equisatisfiable increasing system – computation (NEXP)

### Proposition

$\Phi \models \bigvee_{i=1}^n h_i = 0$  for some non-constant linear polynomials  $h_1, \dots, h_n$ .

**Proof by example:** The system  $2x + 1 \mid -x + 5$  is not increasing.

Assume  $x \geq 1$ . By definition, we are searching for  $c \in \mathbb{Z}$  such that

$$c \cdot (2x + 1) = -x + 5.$$

We see that  $c$  can be bounded in  $[-3, 3]$ :

$$|c| \leq \frac{|-x+5|}{|2x+1|} \leq \frac{6x}{2x} \leq 3.$$

Therefore, in this case  $\{h_1, \dots, h_n\} = \{c \cdot (2x + 1) + x - 5 : c \in [-3, 3]\}$ .

2.2 guess  $j \in [1, n]$ , substitute  $\omega$  by  $\omega_j + \Sigma_j \cdot g$  and goto 1. (less variables.)

## Searching solutions – what we have seen so far

Consider the problem of finding a solution for a system of inequalities with divisibilities:

$$A \cdot \mathbf{x} \leq \mathbf{b} \wedge \bigwedge_{i=1}^n f_i(\mathbf{x}) \mid g_i(\mathbf{x})$$

**Lipshitz's algorithm in a nutshell:**

- ✓ Remove the system of inequalities  $A \cdot \mathbf{x} \leq \mathbf{b}$
- ✓ Translate the system of divisibilities into an equisatisfiable **increasing system**
- 3. Find solutions over the  $p$ -adic integers for a finite set of primes  
(why? Increasing systems enjoy a local-to-global property)
- 4. Construct an integer solution from the  $p$ -adic solutions  
(construction of the integer solution uses the CRT)

### 3-4: Appeal to the local-to-global property – why?

**Input:** an increasing system  $\Phi := \bigwedge_{i=1}^k f_i(\mathbf{x}) \mid g_i(\mathbf{x})$  with respect to  $x_1 \prec \cdots \prec x_d$

**A sketch of the algorithm we want:**

1. Take  $x$  to be the smallest variable in  $\prec$
2. Consider the set  $S$  of all non-trivial divisibilities  $f \mid g$  in  $\Phi$  with  $\text{LV}(g) = x$   
(because of increasingness,  $f$  is constant and  $g = a \cdot x + r$  with  $a, r \in \mathbb{Z}$ )
3. Apply the CRT on  $S$ , finding a value  $v$  for  $x$
4. Replace  $x$  with  $v$ . If  $\Phi$  contains further variables, goto 1.

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**Problem:** Take for instance the following system increasing for  $x \prec y$ :

$$2 \mid x + 1 \wedge 5 \mid x + 5y$$

CRT gives  $x = 1$ , but  $5 \mid 1 + 5y$  is unsatisfiable. However,  $x = 5$  works!

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A little further on...

We need prophets foretelling us what values we can pick.

These prophets are the **local solutions**.

- They add congruences for the variable we are eliminating...
- ...so that satisfiability is preserved after variable replacement.
- The local solutions exist if and only if  $\Phi$  is satisfiable.

$$x \equiv 0 \pmod{5}$$



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(because of increasingness,  $f$  is constant and  $g = a \cdot x + r$  with  $a, r \in \mathbb{Z}$ )
3. Add further congruences to  $S$  accordingly to the **local solutions**
4. Apply the CRT on  $S$ , finding a value  $v$  for  $x$
5. Replace  $x$  with  $v$ . If  $\Phi$  contains further variables, goto 1.

### 3-4: Appeal to the local-to-global property – overview

Let  $p$  be a prime number. The  $p$ -adic valuation  $v_p: \mathbb{Z} \rightarrow \mathbb{N} \cup \{\infty\}$  is defined as

$$v_p(\ell) := \begin{cases} \infty & \text{if } \ell = 0 \\ k & \text{unique such that } p^k \mid \ell \text{ and } p^{k+1} \nmid \ell \end{cases}$$

For every  $\mathbf{x} \in \mathbb{Z}^d$ ,  $f(\mathbf{x}) \mid g(\mathbf{x})$  if and only if  $\forall p \in \mathbb{P}: v_p(f(\mathbf{x})) \leq v_p(g(\mathbf{x}))$ .

### 3-4: Appeal to the local-to-global property – overview

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#### Theorem (Local-to-global property – Lipshitz, 1978)

Suppose  $\bigwedge_{i=1}^n f_i(\mathbf{x}) \mid g_i(\mathbf{x})$  increasing. There is a finite set of primes  $P$  such that

$$\exists \mathbf{x} : \bigwedge_{i=1}^n f_i(\mathbf{x}) \mid g_i(\mathbf{x}) \quad \text{if and only if} \quad \forall p \in P \exists \mathbf{x} : \bigwedge_{i=1}^n v_p(f_i(\mathbf{x})) \leq v_p(g_i(\mathbf{x})).$$

The right-to-left direction “is” the algorithm in the previous slide!

### 3: Find the local solutions for a finite set of primes (DEXP, NP)

Given  $\Phi := \bigwedge_{i=1}^k f_i(\mathbf{x}) \mid g_i(\mathbf{x})$ , define  $\mathbb{P}(\Phi)$  as the set of primes  $p$  such that

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#### Lemma (Défossez, Haase, M., Pérez, 2024)

For every prime  $p \notin \mathbb{P}(\Phi)$ ,  $\bigwedge_{i=1}^n v_p(f_i(\mathbf{x})) \leq v_p(g_i(\mathbf{x}))$  has a solution  $\mathbf{x} \mapsto [0, p - 1]$ .

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#### The complexity of building $\mathbb{P}(\Phi)$ and the local solutions:

- $\mathbb{P}(\Phi)$  can be built in polynomial time with an oracle for factoring.
- Given  $p \in \mathbb{P}(\Phi)$ ,  $\exists \mathbf{x} \bigwedge_{i=1}^n v_p(f_i(\mathbf{x})) \leq v_p(g_i(\mathbf{x}))$  can be solved in NP...
- ...but the certificate represents a solution succinctly. Decompression in DEXP.

### 3: Find the local solutions for a finite set of primes (DEXP, NP)

Given  $\Phi := \bigwedge_{i=1}^k f_i(x) \mid g_i(x)$ , define  $\mathbb{P}(\Phi)$  as the set of primes  $p$  such that

This concludes the algorithm for the **decision** problem!

1. eliminate the system of inequalities
2. put the system of divisibilities in increasing form
3. decide the existence of a local solution for every prime in  $\mathbb{P}(\Phi)$

**Overall complexity:** NEXP

- $\mathbb{P}(\Phi)$  can be built in polynomial time with an oracle for factoring.
- Given  $p \in \mathbb{P}(\Phi)$ ,  $\exists x \bigwedge_{i=1}^n v_p(f_i(x)) \leq v_p(g_i(x))$  can be solved in NP...
- ...but the certificate represents a solution succinctly. Decompression in DEXP.

#### 4: Construct an integer solution (DEXP)

##### Theorem (Lechner, Ouaknine, Worrell, 2015)

Consider an increasing system  $\Phi$  having a local solution  $\mathbf{b}_p$  for every  $p \in \mathbb{P}(\Phi)$ . Then,  $\Phi$  has a solution  $\mathbf{a}$  over  $\mathbb{N}$  of bit size  $\langle \mathbf{a} \rangle \leq (\langle \Phi \rangle + \max\{\langle \mathbf{b}_p \rangle : p \in \mathbb{P}(\Phi)\})^{\text{poly}(d)}$ .

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1. Take  $x$  to be the smallest variable in  $\prec$
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3. Add further congruences to  $S$  accordingly to the local solutions
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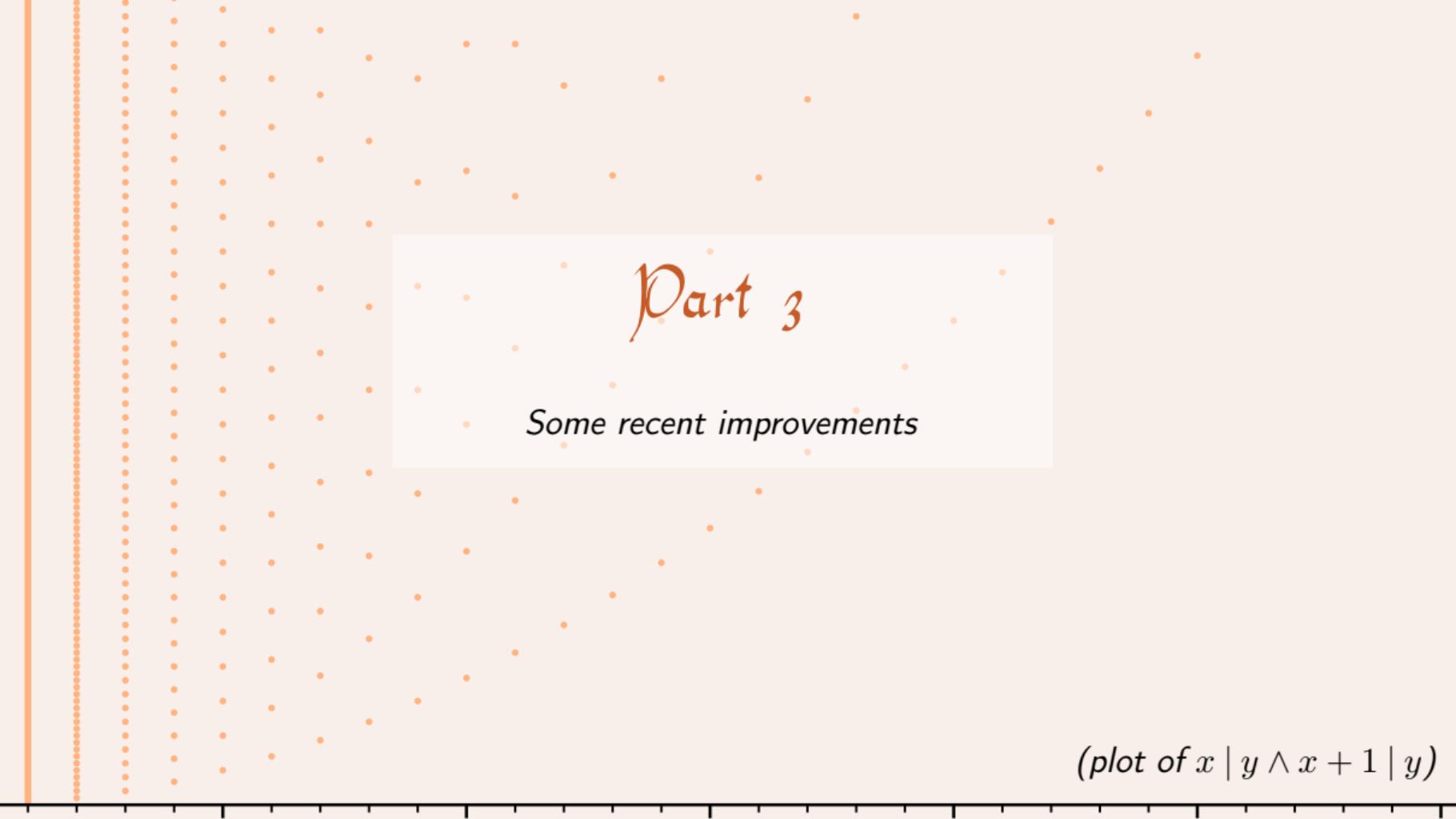
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**Disclaimer:** This is not the full picture!

- During the procedure further primes needs to be added to  $\mathbb{P}(\Phi)$ .
- Key: after replacing  $x$ , the new system satisfies the hypothesis of the theorem.



## Part 3

*Some recent improvements*

(plot of  $x \mid y \wedge x + 1 \mid y$ )

$r$ -increasing systems: grouping “independent variables”

$$f_1(\mathbf{z}) \mid g_1(\mathbf{z}) + a \cdot x$$

$$f_2(\mathbf{z}) \mid g_2(\mathbf{z}) + b \cdot y$$

Lipshitz's procedure requires ordering  $x$  and  $y$ . What happens if we relax this condition?

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### Definition ( $r$ -increasingness)

A system of divisibilities  $\Phi$  is said to be  $r$ -increasing whenever there is a partition  $(X_1, \dots, X_r)$  of its variables such that  $\Phi$  is increasing for every total order  $\prec$  having the property that given  $x \prec y$ , we have  $x \in X_i$  and  $y \in X_j$  for some  $1 \leq i \leq j \leq r$ .

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The smallest  $x$  satisfying  $\varphi_n(x)$  is bigger than  $2^{2^n}$ .

$\varphi_n(x)$  is  $(n + 2)$ -increasing.

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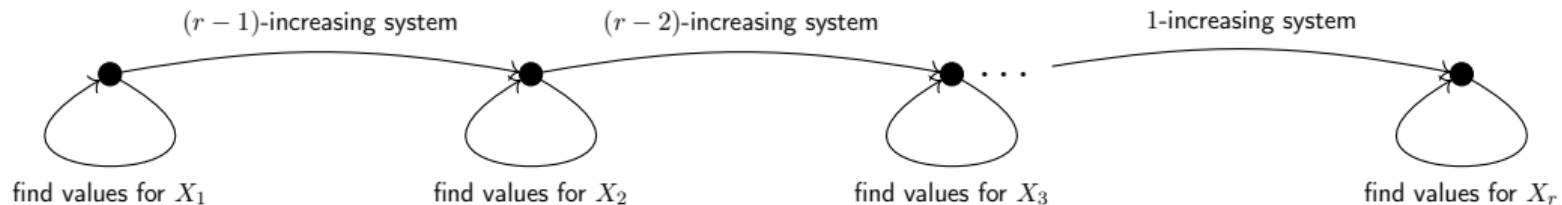
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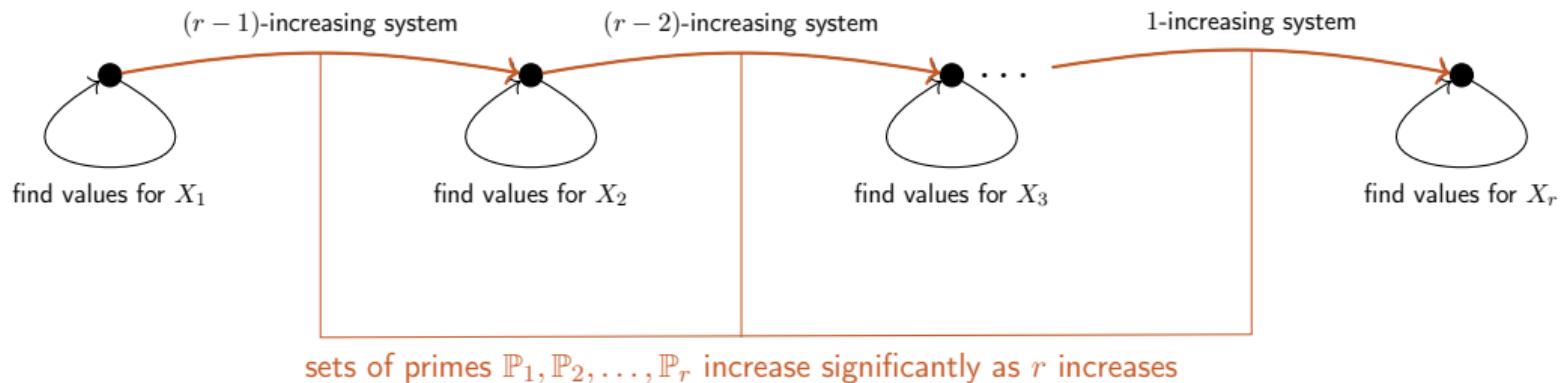


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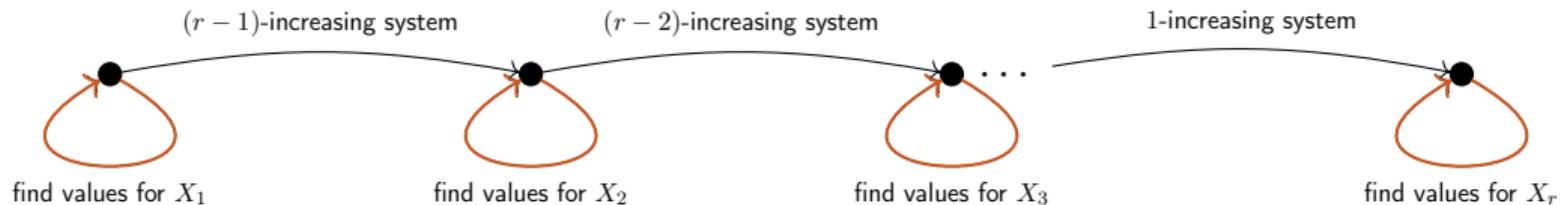


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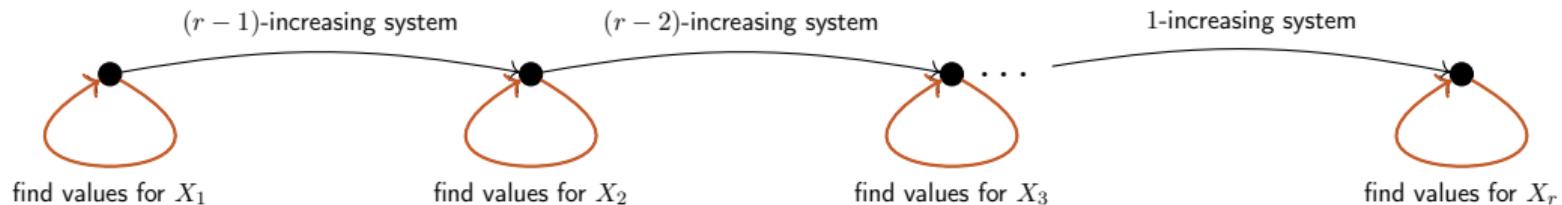
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Let  $d \in \mathbb{N}$  and consider a finite set of coprime positive integers  $M \cup Q$  such that  $M \cap Q = \emptyset$ ,  $Q$  set of primes, and  $d < \min(Q)$ . the system of constraints

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**Conclusion:** We can rely on an abstract version of [Brun's pure sieve](#) in order to prove our theorem. Off-the-shelf, this sieve already establishes that the system

$$x \not\equiv 0 \pmod{p} \quad p \in P,$$

with  $P$  set of primes, admits a solution of size at most  $\#P^{7 \cdot \ln \ln(\#P+1)}$ .

## A corollary: IP-GCD has a polynomial small-model property

minimise  $\mathbf{c}^\top \cdot \mathbf{x}$

(or maximise; variables over  $\mathbb{Z}$ )

subject to  $A \cdot \mathbf{x} \leq \mathbf{b}$

$\gcd(f_i(\mathbf{x}), g_i(\mathbf{x})) \sim_i d_i \quad i \in [1, k], \text{ where } \sim_i \in \{=, \neq, \leq, \geq\}, d_i \in \mathbb{N}$

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### Theorem (Défossez, Haase, M., Pérez, 2024)

If an instance of IP-GCD is feasible then it has a solution (and an optimal solution, if one exists) of polynomial bit length. Hence, IP-GCD feasibility is NP-complete.

- polynomial-time translation into EPAD
- ad-hoc way to construct a polynomial-size 3-increasing system
- ad-hoc way for constructing polynomial-size local solutions

## Bits to take away

### On expressiveness...

EPAD does not have a polynomial small-model property

Consider the family of formulae  $\{\varphi_n\}_{n \in \mathbb{N}}$  given by

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Consider

NP-hardness for 2 variables, 4 inequalities, and 2 divisibility constraints

$\varphi_n($

**Quadratic residue problem [Manders and Alderman, STOC'76]**

**Input:**  $a, b, m \in \mathbb{N}$  given in binary.

**Question:** Is there  $x \in [1, b]$  such that  $x^2 \equiv a \pmod{m}$  ?

The s

The quadratic residue problem is NP-complete. Reduction to EPAD:

$$\exists x \exists y : 1 \leq x \leq b \wedge 1 \leq y \leq b^2 \wedge x + b^2 \mid y + b^2 x \wedge m \mid y - a .$$

# Bits to take away

## On applications...

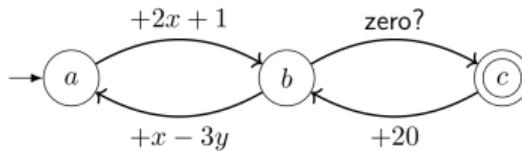
### Applying EPAD:

#### Revisiting Parameter Synthesis for One-Counter Automata

Guillermo A. Pérez  University of Antwerp – Flanders Make, Belgium

Ritam Raha  University of Antwerp, Belgium  
LaBRI, University of Bordeaux, France

Parametric one-counter automata (POCA):



# Bits to take away

On applications...

## Applying EPAD:

### Revisiting Parameter Synthesis for One-Counter Automata

Guillermo A. Pérez  University of Antwerp – Flanders Make, Belgium

Ritam Raha  University of Antwerp, Belgium  
LaBRI, University of Bordeaux, France

### QUADRATIC WORD EQUATIONS WITH LENGTH CONSTRAINTS, COUNTER SYSTEMS, AND PRESBURGER ARITHMETIC WITH DIVISIBILITY

ANTHONY W. LIN <sup>a</sup> AND RUPAK MAJUMDAR <sup>b</sup>

### Reachability in Succinct and Parametric One-Counter Automata

Christoph Haase, Stephan Kreutzer, Joël Ouaknine, and James Worrell  
Oxford University Computing Laboratory, UK

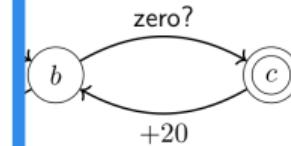
### Simultaneous Rigid *E*-Unification and Related Algorithmic Problems

Anatoli Degtyarev  
Computing Science Department  
Uppsala University

Yuri Matiyasevich  
Steklov Institute of Mathematics  
St.Petersburg Division

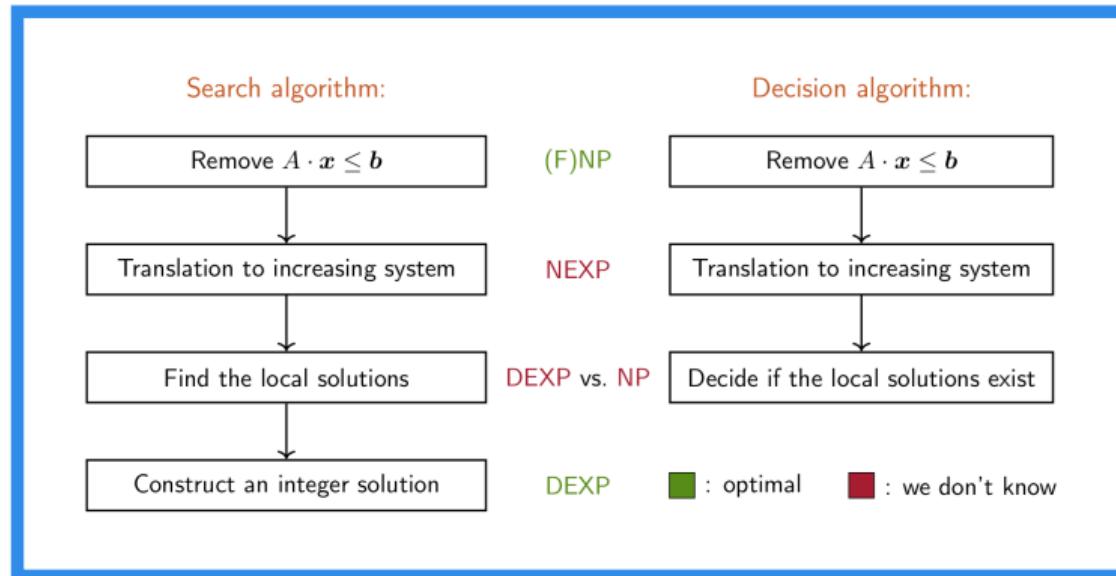
Andrei Voronkov  
Computing Science Department  
Uppsala University

CA):



# Bits to take away

## On procedures...



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## On procedures...

