A geometric procedure for Presburger arithmetic

Alessio Mansutti

joint work with **Dmitry Chistikov** and **Christoph Haase**

advanced reasoning in arithmetic theories

Presburger arithmetic (PA)

The first-order theory of $\langle \mathbb{Z}, 0, 1, +, \leq \rangle$

"Every integer is either even or odd"

$$\forall \mathrm{x}\, \exists \mathrm{y} : \mathrm{x} = 2\mathrm{y} \vee \mathrm{x} = 2\mathrm{y} + 1$$

Why Presburger arithmetic?

Wide range of applications in verification, program synthesis, compiler optimisation, games...

M. Presburger

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Several applications require to reason on the **set of solutions** of a formula.

Example [Verdoolaege et al., Algorithmica 48, 2007]:
"How many times is a loop executed?" $\rightsquigarrow f(\mathbf{y}) = \#\{\mathbf{x} \in \mathbb{Z}^d \mid A \cdot \mathbf{x} \geq B \cdot \mathbf{y} + \mathbf{c}\}$

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$$\forall x \, \exists y : x = 2y \vee x = 2y + 1$$

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M. Presburger

Theorem (Ginsburg and Spanier,'66)

Sets definable in Presburger arithmetic coincide with the family of ***semilinear sets****.*

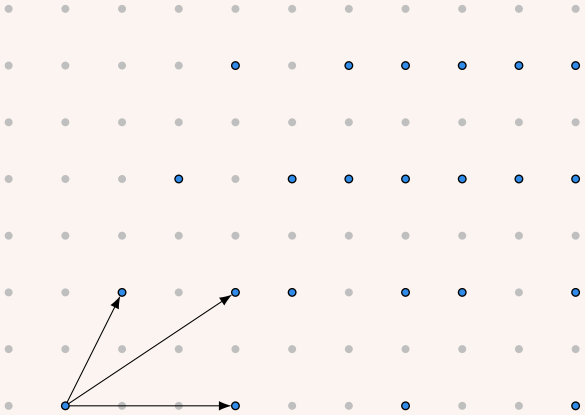
Semilinear sets

Arithmetic progression

$b + i \cdot p$, where $i \in \mathbb{N}$

b base point, p period

Semilinear sets

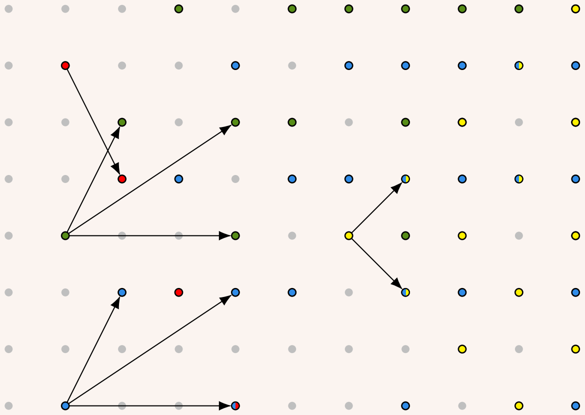


Linear set

(arithmetic progression
in multiple dimensions)

$L(b, P)$, where b base
and $P = \{p_1, \ldots, p_n\}$ periods

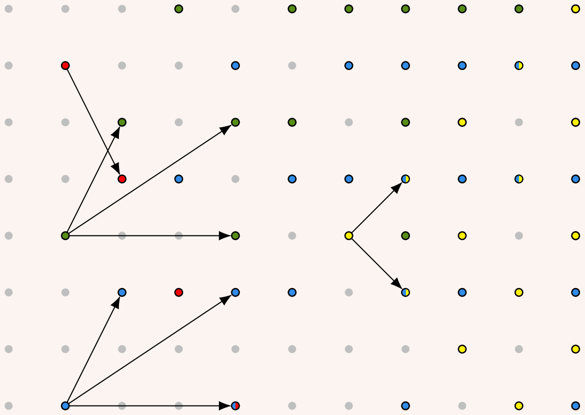
Semilinear sets



Semilinear set
(finite union of
linear sets)

$\bigcup_{i \in I} L(b_i, P_i)$,
where I finite set of indices

Semilinear sets

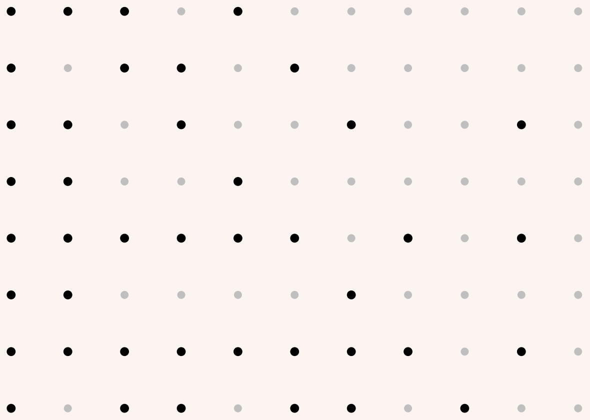


Ginsburg & Spanier, '66

The set of solutions of a system of linear inequalities over $\mathbb{Z}$ is semilinear. Semilinear sets are closed under

- union
- projection
- complementation

Semilinear sets



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The set of solutions of a system of linear inequalities over $\mathbb{Z}$ is semilinear. Semilinear sets are closed under

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- projection
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Main problem: How to compute the complement of a semilinear set **optimally**?

A geometric procedure for Presburger arithmetic

In this talk: We discuss an optimal algorithm to complement semilinear sets.

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Quantifier elimination

[Presburger, '29]

$$\exists x : \varphi(x, \mathbf{y})$$

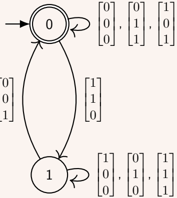
$$\Big\downarrow \equiv$$

$$\psi(\mathbf{y})$$

3EXPTIME

Automata

[Büchi, '60]



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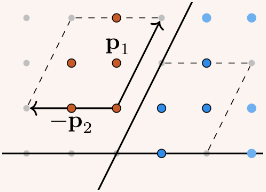
Geometry

[Ginsburg and Spanier, '66]



TOWER

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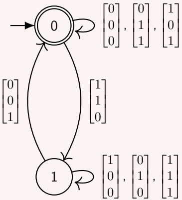

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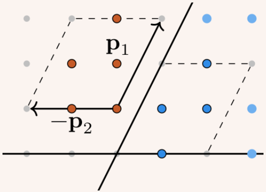
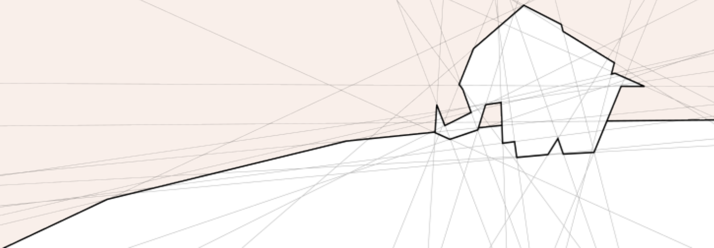


3EXPTIME

Chapter 1

In which we forget about the integers and work over the reals

Goal: reflect on how to complement a union of polyhedra over $\mathbb{R}$.



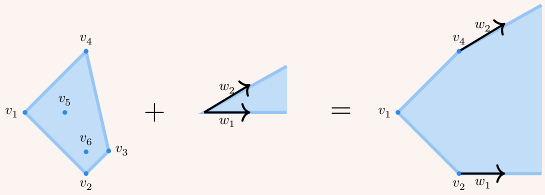
First ingredient: The geometry of a system of inequalities over the **reals**

Theorem (Minkowski-Weyl theorem (1897, 1935))

Consider $S \subseteq \mathbb{R}^d$. The two following statements are equivalent:

(H) $S = \{\mathbf{x} \in \mathbb{R} : A \cdot \mathbf{x} \leq \mathbf{b}\}$ *for some matrix* $A \in \mathbb{Q}^{n \times d}$ *and vector* $\mathbf{b} \in \mathbb{Q}^d$

(V) $S = \textit{conv}\, V + \textit{cone}\, W$ *for some finite sets* $V, W \subseteq \mathbb{Q}^d$.



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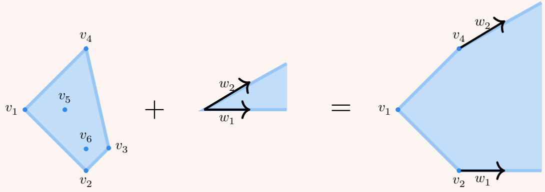

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Cost of switching:

bitsize of numbers:

$$\langle \texttt{output} \rangle \leq \text{poly}(d) \cdot \langle \texttt{input} \rangle$$

amount of numbers (with repetitions):

$$\#(\texttt{output}) \leq \#(\texttt{input}) \uparrow d$$

$$(a \uparrow b \uparrow c \;=\; (a+2)^{(b+2)^{\text{poly}(c)}})$$

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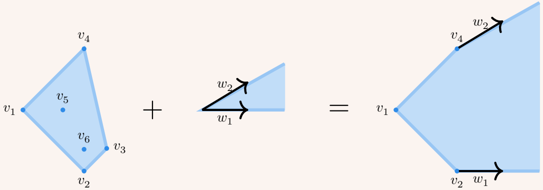

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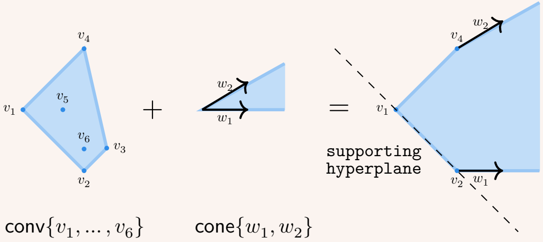

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Second ingredient: Cutting $\mathbb{R}^d$ using hyperplanes

Question: In how many parts can we cut $\mathbb{R}^d$ using n hyperplanes?

$\Phi(d,n) :=$ maximal number of parts, in dimension d and using n hyperplanes.

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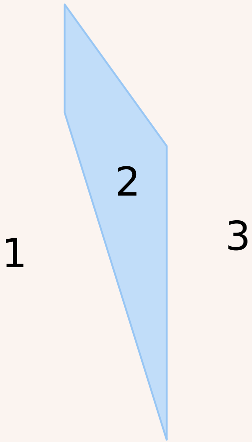

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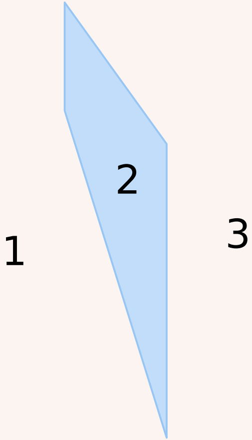

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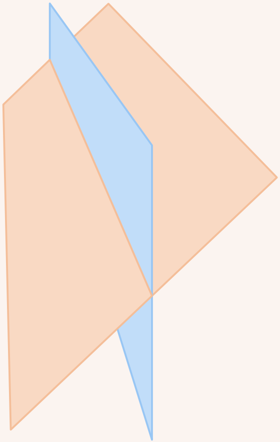

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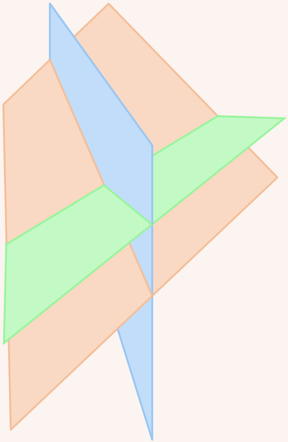

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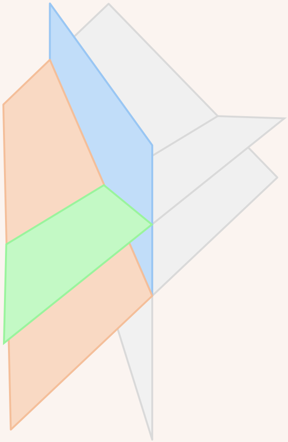

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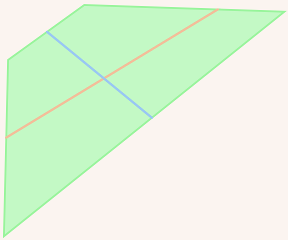

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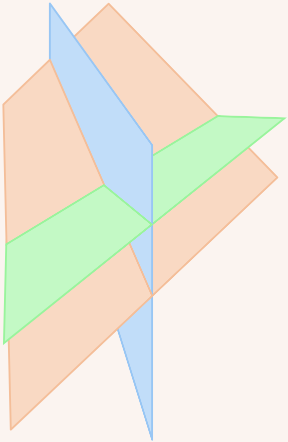

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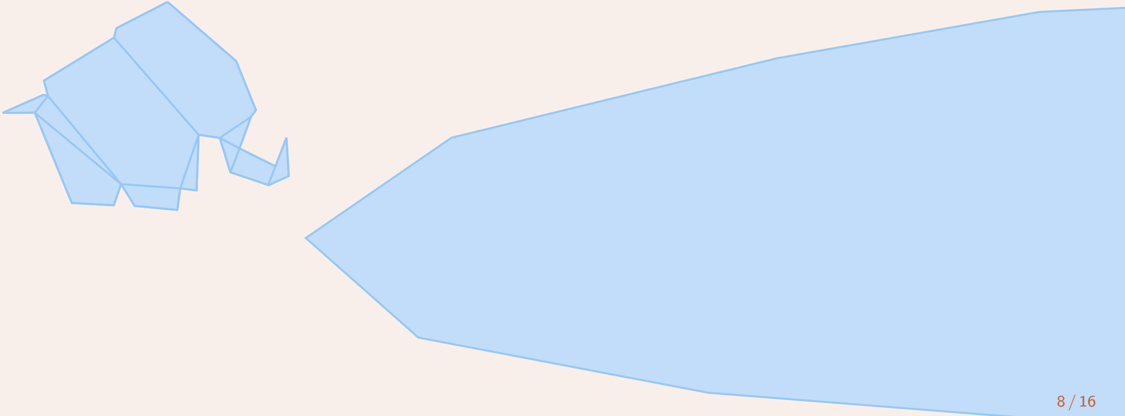
$$\Rightarrow \ \Phi(d,n) \leq (2n)^d + 1$$

"n^d cutting lemma"

Carving out a union of convex polyhedra using splitters

Question: How do we compute the complement of $\bigcup_{i\in I}(\operatorname{conv} V_i + \operatorname{cone} W_i)$?



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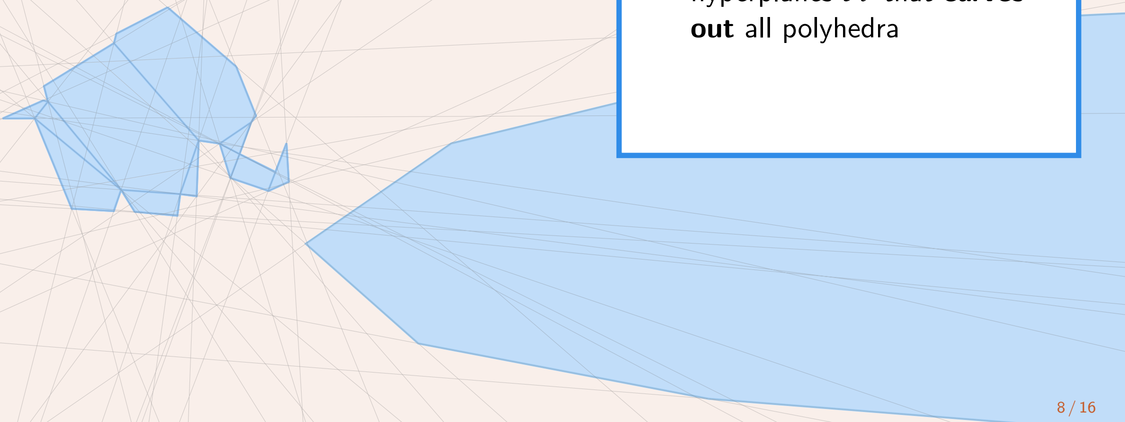

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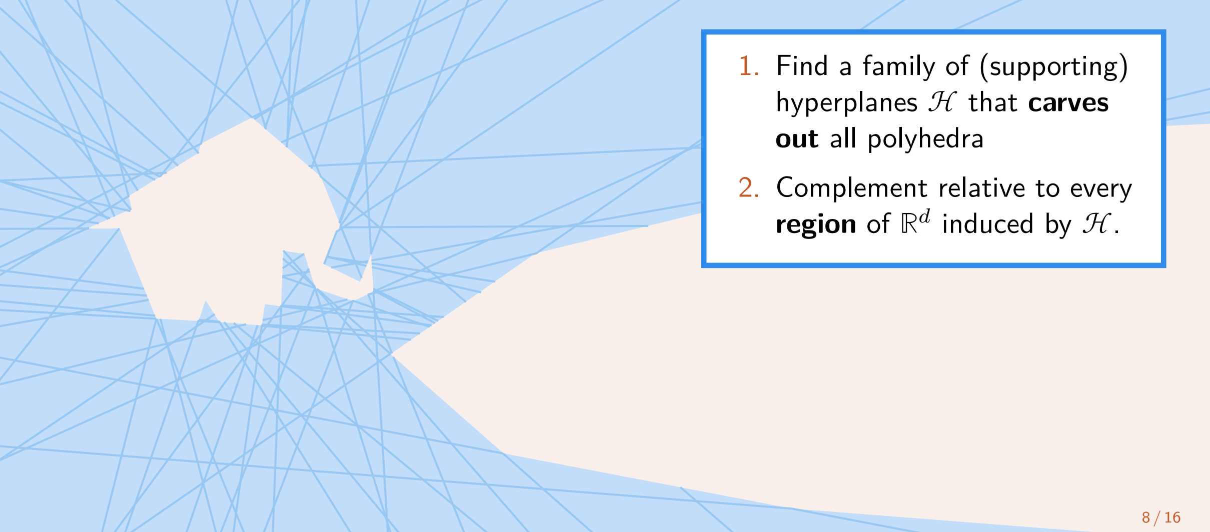

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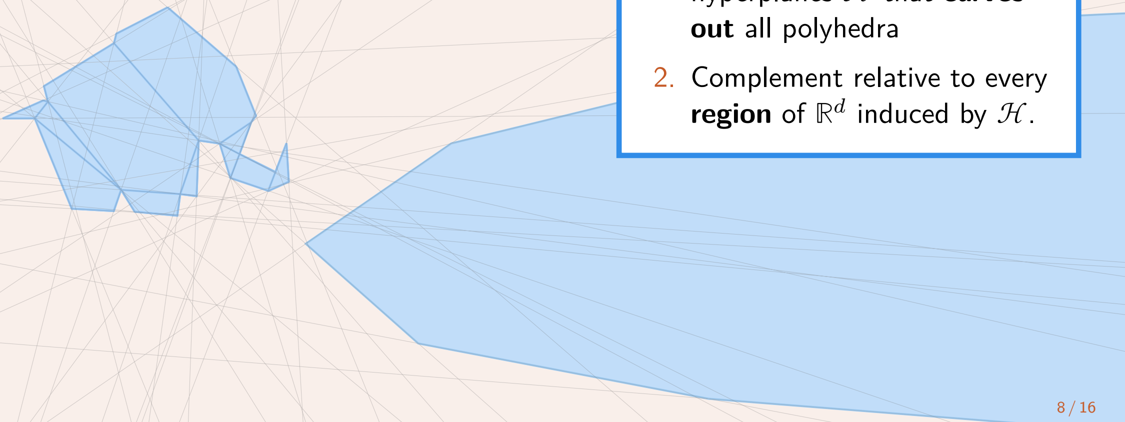

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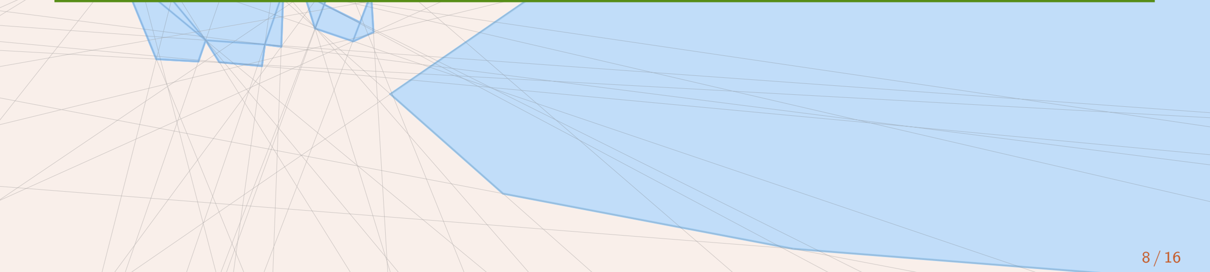
An **($\mathbb{R}$-)splitter** for a family of convex polyhedra $\mathcal{P}$ is a family of convex polyhedra $\mathcal{R}$ s.t.

1. $\bigcup \mathcal{R} = \mathbb{R}^d$ and $\mathcal{R}$ is closed under taking faces
2. for every $R_1, R_2 \in \mathcal{R}$, the set $R_1 \cap R_2$ is either $\emptyset$ or a face of both R_1 and R_2
3. for every $R \in \mathcal{R}$ and $P \in \mathcal{P}$ the set $R \cap P$ is either $\emptyset$ or a face of R

Claim: Given $\mathcal{R}$ it is easy to compute $\mathbb{R}^d \setminus \bigcup \mathcal{P}$.



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Theorem

Every $\mathcal{P} = \{\operatorname{conv} V_i + \operatorname{cone} W_i\}_{i \in I}$ has an $\mathbb{R}$-splitter $\mathcal{R} = \{\operatorname{conv} C_j + \operatorname{cone} Q_j\}_{j \in J}$ s.t.

$$\langle \mathcal{R} \rangle \leq \textit{poly}(d) \cdot \langle \mathcal{P} \rangle \qquad \textit{and} \qquad \#\mathcal{R} \leq (2 \cdot \#\mathcal{P}) \uparrow d.$$

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Theorem `Minkowski-Weyl theorem` + n^d `cutting lemma`

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Chapter 2

In which we reintroduce the integers

Goal: refine the notion of splitter to work over $\mathbb{Z}$.

The geometry of a system of inequalities over the **integers**

Theorem (von zur Gathen & Sieveking, '78)

Consider $S \subseteq \mathbb{Z}^d$. Then, below (H) implies (V), but not vice versa:

(H) $S = \{\mathbf{x} \in \mathbb{Z}^d : A \cdot \mathbf{x} \leq \mathbf{c}\}$ *for some* $A \in \mathbb{Z}^{n \times d}$ *and* $\mathbf{c} \in \mathbb{Z}^m$

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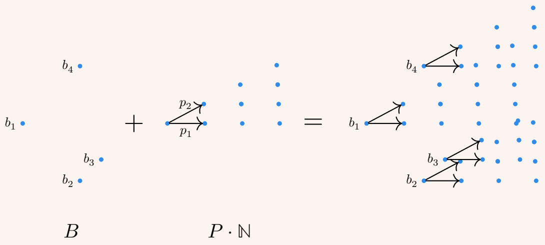

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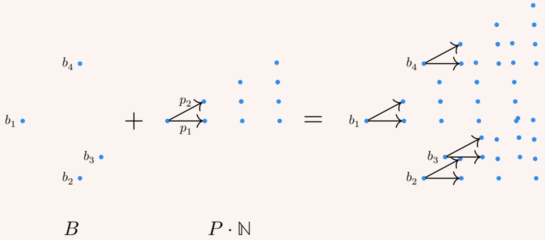

Cost of switching: (H) $\to$ (V)

$\langle B \rangle, \langle P \rangle \leq \mathrm{poly}(d) \cdot \langle [A|\mathbf{c}] \rangle$

$\#P \leq \#[A|\mathbf{c}] \uparrow d$

$\#B \leq 2 \uparrow (d \cdot \langle B \rangle)$

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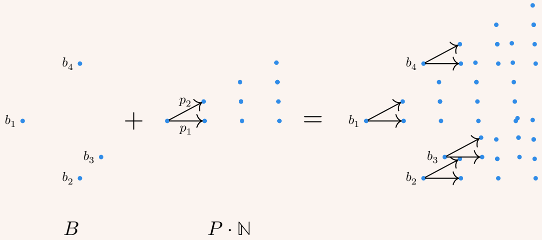
(H) $S = \{\mathbf{x} \in \mathbb{Z}^d : A \cdot \mathbf{x} \leq \mathbf{c}\}$ *for some* $A \in \mathbb{Z}^{n \times d}$ *and* $\mathbf{c} \in \mathbb{Z}^m$

(V) $S = \bigcup_{\mathbf{b} \in B} L(\mathbf{b}, P)$ *for some finite sets* $B, P \subseteq \mathbb{Z}^d$.



Cost of switching: (H) $\to$ (V)

$\langle B \rangle, \langle P \rangle \leq \text{poly}(d) \cdot \langle [A|\mathbf{c}] \rangle$

$\#P \leq \#[A|\mathbf{c}] \uparrow d$

$\#B \leq 2 \uparrow (d \cdot \langle B \rangle)$

We call $L(B, P) \stackrel{\text{def}}{=} \bigcup_{\mathbf{b} \in B} L(\mathbf{b}, P)$ an hybrid linear set.

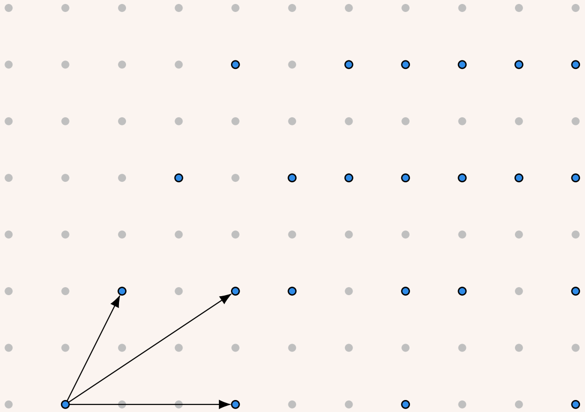
Semilinear sets (again)

Arithmetic progression

$b + i \cdot p$, where $i \in \mathbb{N}$

b base point, p period

Semilinear sets (again)



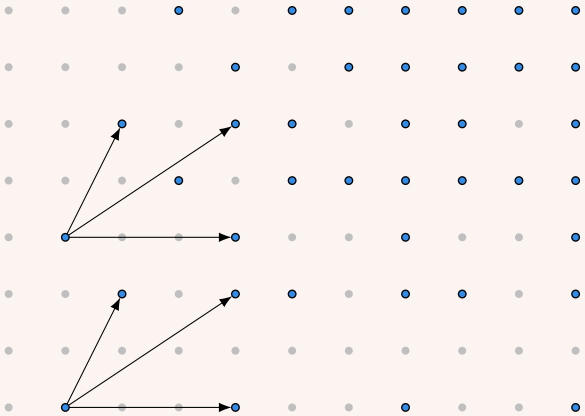
Linear set

(arithmetic progression in multiple dimensions)

$L(\mathbf{b}, P)$

$\mathbf{b}$ base, P periods

Semilinear sets (again)



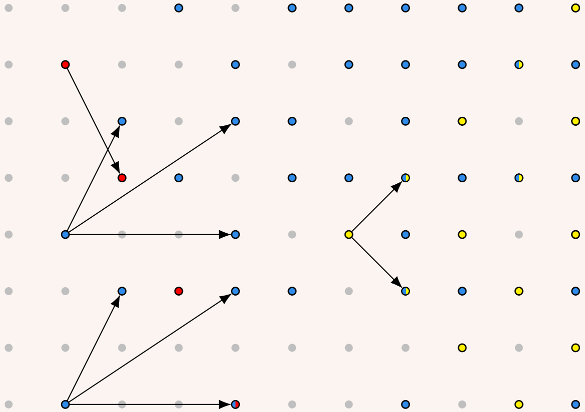
Hybrid linear set

(finite union of linear sets having the same periodic behaviour)

$L(B, P)$

B bases, P periods

Semilinear sets (again)



Semilinear set
(finite union of
hybrid linear sets)

$$\bigcup_{i \in I} L(B_i, P_i)$$

I finite set of indices

$\mathbb{Z}$-splitters

An **($\mathbb{R}$-)splitter** for a family of convex polyhedra $\mathcal{P}$ is a family of convex polyhedra $\mathcal{R}$ s.t.

1. $\bigcup \mathcal{R} = \mathbb{R}^d$ and $\mathcal{R}$ is closed under taking faces
2. for every $R_1, R_2 \in \mathcal{R}$, the set $R_1 \cap R_2$ is either $\emptyset$ or a face of both R_1 and R_2
3. for every $R \in \mathcal{R}$ and $P \in \mathcal{P}$ the set $R \cap P$ is either $\emptyset$ or a face of R

Claim: Given $\mathcal{R}$ it is easy to compute $\mathbb{R}^d \setminus \bigcup \mathcal{P}$.

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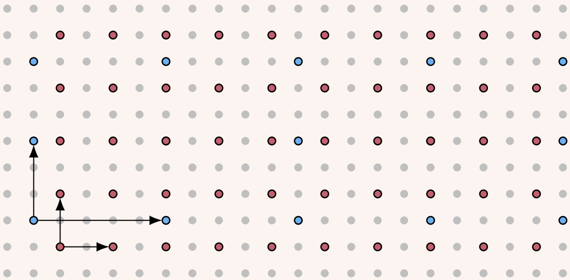

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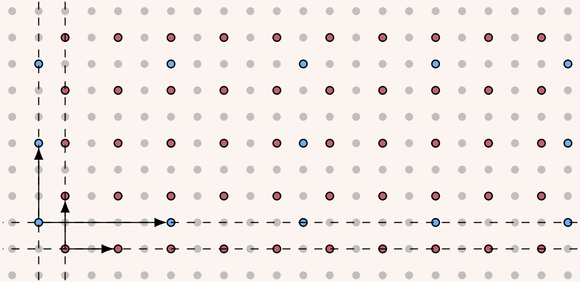

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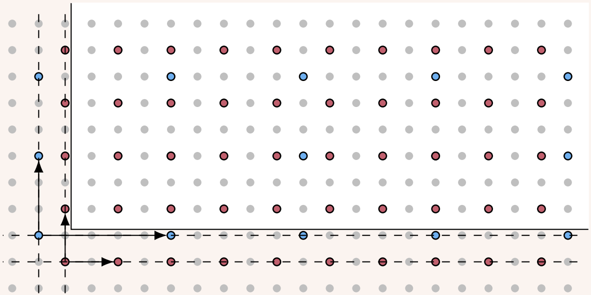
The region Z characterises a portion of $\mathbb{Z}^d$



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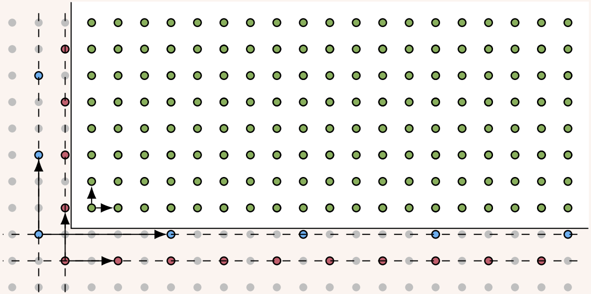

Not all hybrid linear sets agree with our goal!

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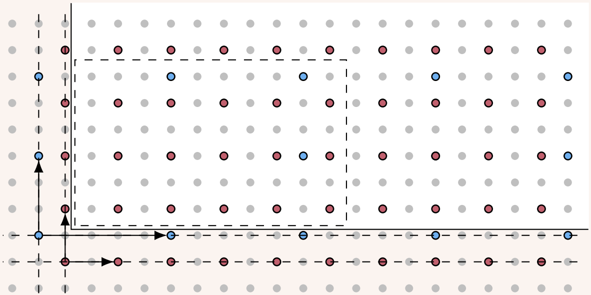

Idea: find C and Q such that

- $Z = L(C, Q)$
- $L(C, Q) \cap L(\mathbf{b}, P_i) = L(C \cap L(\mathbf{b}, P_i), Q)$

  for every $i \in I$ and $\mathbf{b} \in B_i$

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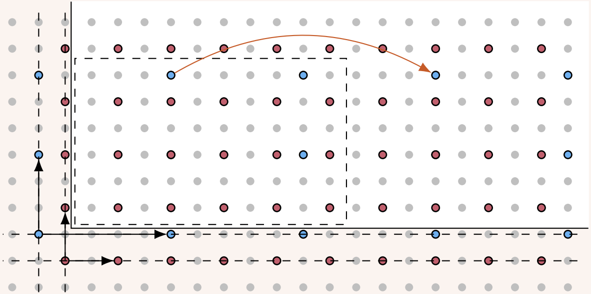

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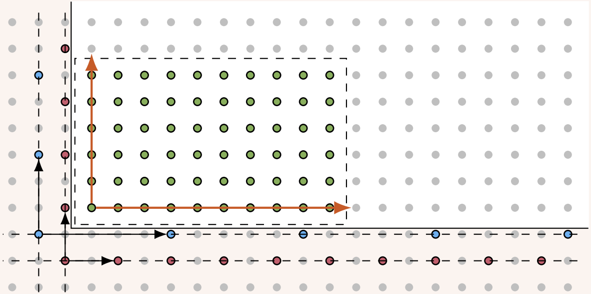

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$\Rightarrow \ Z \setminus \bigcup \mathcal{P} = L(C \setminus \bigcup \mathcal{P}, Q)$

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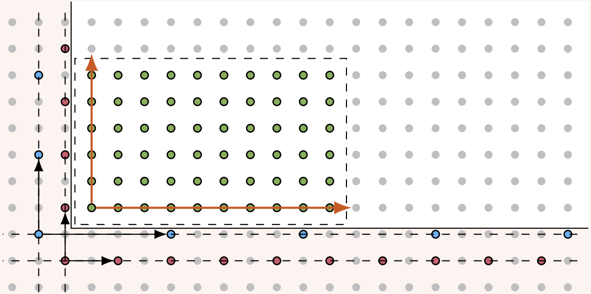
3. **for every $j \in J$, the set Q_j is proper (i.e. made of linearly independent vectors)**
4. **if $Z_j \subseteq (\mathbf{b} + \text{cone } P_i)$ then $Q_j \subseteq L(\mathbf{0}, P_i)$** $\quad \forall i \in I,\ \mathbf{b} \in B_i,\ j \in J$

Claim: Given $\mathcal{Z}$ it is easy to compute $\mathbb{Z}^d \setminus \bigcup \mathcal{P}$.



Idea: find C and Q such that

- $Z = L(C, Q)$
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$\Rightarrow$ $Z \setminus \bigcup \mathcal{P} = L(C \setminus \bigcup \mathcal{P}, Q)$

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Claim: Given $\mathcal{Z}$ it is easy to compute $\mathbb{Z}^d \setminus \bigcup \mathcal{P}$.

Theorem

Every $\mathcal{P} = \{L(B_i, P_i)\}_{i \in I}$ has a $\mathbb{Z}$-splitter $\mathcal{Z} = \{L(C_j, Q_j)\}_{j \in J}$ such that

$$\langle \mathcal{Z} \rangle \leq \textit{poly}(d) \cdot \#I \cdot \langle \mathcal{P} \rangle \qquad \textit{and} \qquad \#(\bigcup_{j \in J} Q_j) \leq (\#I \cdot \max_{i \in I} \#P_i) \uparrow d\,.$$

Chapter 3

Complementation

Goal: Complementation.

The procedure

Input: a semilinear set M

Output: the complement $\mathbb{Z}^d \setminus M$ as a semilinear set

1. update M to $\bigcup_{i \in I} L(B_i, P_i)$ where each P_i is proper
2. compute $\mathcal{Z} = \{L(C_j, Q_j)\}_{j \in J}$ $\mathbb{Z}$-splitter for $\{L(B_i, P_i)\}_{i \in I}$
3. **for** $j \in J$ **do**
4. $\quad E_j \leftarrow C_j \setminus M$
5. $\mathcal{Q} \leftarrow \{Q_j\}_{j \in J}$
6. **for** $Q \in \mathcal{Q}$ **do**
7. $\quad \mathcal{E}_Q \leftarrow \{E_j : j \in J, Q_j = Q\}$
8. **return** $\bigcup_{Q \in \mathcal{Q}} L\left(\left(\bigcup \mathcal{E}_Q\right), Q\right)$

Discrete Carathéodory's theorem [Chistikov and Haase, ICALP'16]

Complement M region-wise

Merge hybrid linear sets having the same period sets

$L(B, P) \cup L(C, P) = L(B \cup C, P)$

Concluding remarks

Quantifier elimination

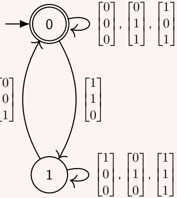
[Presburger, ‘29]

$$\exists x : \varphi(x, \mathbf{y}) \overset{\equiv}{\rightsquigarrow} \psi(\mathbf{y})$$

3EXPTIME

Automata

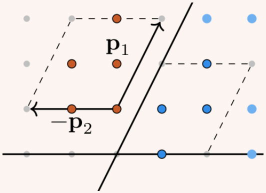
[Büchi, ‘60]



3EXPTIME

Geometry

[Ginsburg and Spanier, ‘66]



3EXPTIME

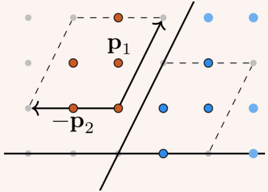
Concluding remarks

Key notion: $\mathbb{Z}$-splitters.

A decomposition of $\mathbb{Z}^d$ with many interesting properties.

Geometry

[Ginsburg and Spanier, '66]



3EXPTIME

Concluding remarks

Key notion: $\mathbb{Z}$-splitters.

A decomposition of $\mathbb{Z}^d$ with many interesting properties.

From our procedure:

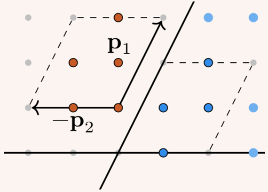
$$[\![\Phi]\!] = \bigcup_{i \in I} L(B_i, P_i) \text{ where } \#I \leq 2 \uparrow d \uparrow h.$$

From this bound we conclude that the VC dimension of PA is doubly exponential.

For details see the paper: *"Geometric decision procedures and the VC dimension of linear arithmetic theories"*, Chistikov, Haase, Mansutti, LICS'22

Geometry

[Ginsburg and Spanier, '66]



3EXPTIME