Geometric decision procedures and the VC dimension of linear arithmetic theories

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Abstract

This paper resolves two open problems on linear integer arithmetic (LIA), also known as Presburger arithmetic:

(1) We give a 3EXPTIME geometric decision procedure for LIA, i.e., a procedure based on manipulating semilinear sets. This matches the running time of the best quantifier elimination and automata based procedures.

(2) Building upon (1), we give a doubly exponential upper bound on the Vapnik-Chervonenkis (VC) dimension of sets definable in LIA, proving a conjecture of Nguyen and Pak [*Combinatorica* 39, pp. 923–932, 2019].

Those results partially rely on an analysis of sets definable in linear real arithmetic (LRA), and analogous problems for LRA are also considered. At the core of these developments are new decomposition results for semilinear and $\mathbb{R}$-semilinear sets, the latter being the sets definable in LRA. These results yield new algorithms to compute the complement of ($\mathbb{R}$-)semilinear sets that do not cause a non-elementary blowup when repeatedly combined with procedures for other Boolean operations and projection. The existence of such an algorithm for semilinear sets has been a long-standing open problem.

CCS Concepts: • **Theory of computation → Logic**; **Machine learning theory**.

Keywords: semilinear sets, convex polyhedra, linear real arithmetic, linear integer arithmetic, Presburger arithmetic, VC dimension

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1 Introduction

Linear arithmetic theories are first-order theories over the domains $\mathbb{R}$ or $\mathbb{Z}$ and the signature $\langle 0, 1, +, \leq \rangle$ whose constant and relation symbols are interpreted in their natural semantics. Those theories are commonly referred to as *linear real arithmetic (LRA)* and *linear integer arithmetic (LIA)*, respectively. Linear integer arithmetic is also known as *Presburger arithmetic*. The expressiveness and computational complexity of (fragments of) these theories have been studied for decades. From the perspective of computational logic, an appealing aspect of those arithmetic theories is that there is a troika of decision procedure paradigms for both LRA and LIA: quantifier elimination procedures [10, 26, 28, 34], automata-based procedures [3, 4, 9, 35], and geometric (generator-based) procedures [6, 13].

The main focus of this paper is on the algorithmic and descriptional complexity of Boolean operations on sets definable in LRA and LIA and their geometric properties. The fact that both LRA and LIA admit quantifier elimination (in case of LIA in the extended structure with additional unary divisibility predicates $c \mid \cdot$ for all $c > 0$ [28]) immediately enables us to understand the geometry of the sets they define. For LRA, a quantifier-free formula is a Boolean combination of linear inequalities, and hence the sets definable in LRA are finite unions of copolyhedra, which are convex polyhedra possibly with some faces removed. We refer to such sets as $\mathbb{R}$-*semilinear sets*. The sets definable in LIA are commonly known as *semilinear sets*, which are finite unions of intersections of a convex polyhedron with an integer lattice.

Semilinear sets admit a generator representation as finite unions of *hybrid linear sets*. Given finite sets of *generators* $B, P \subseteq \mathbb{Z}^d$, the hybrid linear set $M \subseteq \mathbb{Z}^d$ in dimension d generated by B and P is the set $L(B,P) := \{\boldsymbol{b} + \lambda_1 \cdot \boldsymbol{p}_1 + \cdots \lambda_k \cdot \boldsymbol{p}_k : \boldsymbol{b} \in B, k \geq 0, \lambda_i \in \mathbb{N}, \boldsymbol{p}_i \in P\}$ [6, 13]. We call the hybrid linear sets constituting a semilinear set its *components*, and we denote by $\|M\|$ the maximum of two and the absolute value of the largest constant appearing in any of the generators. Analogously, by the Minkowski–Weyl theorem, rational convex polyhedra also admit a generator representation $K(V, W) \subseteq \mathbb{R}^d$ as the sum of the convex hull of a finite set $V \subseteq \mathbb{Q}^d$ and a cone generated by another finite $W \subseteq \mathbb{Q}^d$. The generator representation of $\mathbb{R}$-semilinear sets are finite unions of copolyhedra, i.e., sets of the form $K(V, W) \setminus (K(V_1, W_1) \cup \cdots K(V_n, W_n))$, where all $K(V_i, W_i)$ are faces of $K(V, W)$. Since the components of $\mathbb{R}$-semilinear sets and semilinear sets are easily seen to be definable in LRA and LIA, respectively, it follows that they are effectively closed under projection along the coordinate axes and under all Boolean operations. Consequently, it is not difficult to see that ($\mathbb{R}$-)semilinear sets in generator representation can be employed in *generator-based (geometric) decision procedures* for LRA and LIA, respectively. Given a formula $\Phi(\boldsymbol{x}) \equiv \exists \boldsymbol{y}_1 \forall \boldsymbol{y}_2 \cdots Q_n \boldsymbol{y}_n \Psi(\boldsymbol{x}, \boldsymbol{y}_1, \ldots \boldsymbol{y}_n)$ of LRA or LIA in prenex normal form, the idea is to transform Ψ into an ($\mathbb{R}$-)semilinear set and to then perform repeated complementation and projection steps to eliminate all $\boldsymbol{y}_i$ until a generator-representation of all $\boldsymbol{x}$ satisfying Φ is obtained.

Algorithms for Boolean operations on ($\mathbb{R}$-)semilinear sets given in generator representation have been investigated over the last 40 years. For the reals, generator-based decision procedures have seen early successes. E. Sontag [32] gave such a decision procedure for LRA with optimal complexity-theoretic upper bounds, in particular for a fixed number of quantifier alternations. Over the integers, however, the situation has been less satisfactory. D.T. Huynh [14, 15] was the first to use a geometric approach to show that the inclusion problem for explicitly given semilinear sets is in Π_2^{P} by establishing that the complement of a semilinear set, if non-empty, contains an element of polynomially bounded bit-size. E. Kopczyński [19] generalised this result and furthermore showed that for (implicitly defined) semilinear sets $M, N \subseteq \mathbb{Z}^d$ in fixed dimension d, generated by n components each, only a subset of points of bit-size $O(n+\log(\|M\|+\|N\|))$ needs to be explored to decide $M \subseteq N$, which is hence in Π_2^{P} in this setting. Continuing this line of research, D. Chistikov and C. Haase [6] established that the bit-size of the largest constant in the generator representation of the complement of M can be bounded by $O((n \cdot d^4) + \log\|M\|)$. Furthermore, S. Beier et al. [2] analysed the growth of constants in S. Ginsburg and E.H. Spanier's seminal paper on the relationship between Presburger arithmetic and semilinear sets [12]. All the constructions and algorithms in this line of work so far lead to a non-elementary blow-up for repeated complementation of any given semilinear set. It has been a widely open problem whether there exists a complementation algorithm which, when interleaved with intersection and projection operations, results in an elementary procedure.

The first main contribution of this paper is to affirmatively settle this open problem in a general setting. We establish that a term in an algebra over semilinear sets consisting of Boolean operations and projection operations can be evaluated in triply exponential time. A consequence of this result is a generator-based decision procedure for Presburger arithmetic whose running time matches the 3EXPTIME upper bounds of quantifier elimination and automata-based approaches [9, 26]. At the heart of our algorithm lies an algorithm for constructing a *splitter* for a semilinear set $M \subseteq \mathbb{Z}^d$. In a nutshell, this is a partition of $\mathbb{Z}^d$ into simple disjoint parts, such that inside each of them the set is easy to complement. These parts are all hybrid linear sets of the form $L(B, P)$. We control the number of periods P: for all of these parts combined, this number is upper bounded (in any fixed dimension d) by a polynomial of the same number for the original set M. Iteration of this bound leads to the aforementioned triply-exponential bound.

Our second main contribution concerns the *VC dimension* of Presburger arithmetic. The VC dimension is a core concept in computational learning theory and gives an upper bound on the sample complexity required to train a binary classifier, see e.g. [18]; we refer the reader to Section 6 for a formal definition. In the context of giving bounds on the VC dimension of neural networks, M. Karpinski and A. Macintyre were the first to propose a systematic study of the VC dimension of first-order theories [16, 17] and related concepts. This line of research has led to deep results in model theory, for instance, that every quasi-o-minimal structure has linear VC density [1]. Recently, D. Nguyen and I. Pak established polynomial upper bounds on the VC dimension of LIA with a fixed number of variables [25], building upon a geometric approach for inferring properties of Boolean operations on semilinear sets [24]. The authors of [25] conjecture that it should be possible to establish a doubly-exponential upper bound on the VC dimension of full LIA, but also remark that this is unlikely to be achieved by analysing quantifier-elimination procedures because of known lower bounds on the growth of formula sizes of such procedures [33]. The second main contribution of this paper is to prove this conjecture, establishing singly- and doubly-exponential upper bounds on the VC dimension of LRA and LIA, respectively. The basis for these upper bounds are the bounds on the size and structure of ($\mathbb{R}$-)semilinear sets established in the first part of the paper.

2 Preliminaries

Let $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{Q}$, and $\mathbb{R}$ denote the set of integers, non-negative integers, rationals, and reals, respectively. We write $\mathbb{Q}_+$ and $\mathbb{R}_+$ to denote the non-negative part of $\mathbb{Q}$ and $\mathbb{R}$, respectively. Given $a, b \in \mathbb{Z}$, we define $[a, b] := \{a, a+1, \ldots, b\}$. We denote by $\mathbf{0}$ and $\mathbf{1}$ the null vector and the vector with all components equal to one, respectively, in any finite dimension.

For an arbitrary set S, we write $\#S$ for its cardinality. If S is infinite, then we write $\#S = \infty$. For sets of numbers or vectors S and T, we use the *Minkowski sum notation*: $S + T := \{s + t : s \in S, t \in T\}$. We omit curly brackets when S (or alternatively T) is a singleton, and thus abbreviate $\{s\} + T$ with $s + T$. For a finite set of vectors $P = \{\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n\}$ over a numerical domain, e.g. $P \subseteq \mathbb{R}^d$, we assume a lexicographic ordering on their elements and thus sometimes treat P as matrix whose column vectors are its elements. Then, for instance, for $P \subseteq \mathbb{R}^d$ and $\boldsymbol{\lambda} \in \mathbb{R}^{\#P}$, the notation $P \cdot \boldsymbol{\lambda}$ denotes the product of the matrix P with $\boldsymbol{\lambda}$, and given a set $Q \subseteq \mathbb{R}^{\#P}$, $P \cdot Q := \{P \cdot \boldsymbol{\lambda} : \boldsymbol{\lambda} \in Q\}$.

We write $\operatorname{rank} A$ for the rank of the matrix A, i.e., its maximal number of linearly independent columns (equiv., rows).

The binary and natural logarithm functions are denoted by $\log(\cdot)$ and $\ln(\cdot)$, respectively.

Linear algebra and geometry. The *linear span*, *cone*, *convex hull* and *affine hull* of a set $S \subseteq \mathbb{R}^d$ are defined as:

$$\operatorname{span} S \coloneqq \left\{T \cdot \boldsymbol{\lambda} \,:\, T \subseteq S,\, \#T \in \mathbb{N},\, \boldsymbol{\lambda} \in \mathbb{R}^{\#T}\right\},$$

$$\operatorname{cone} S \coloneqq \left\{T \cdot \boldsymbol{\lambda} \,:\, T \subseteq S,\, \#T \in \mathbb{N},\, \boldsymbol{\lambda} \in \mathbb{R}_+^{\#T}\right\},$$

$$\operatorname{conv} S \coloneqq \left\{T \cdot \boldsymbol{\lambda} \,:\, T \subseteq S,\, \#T \in \mathbb{N},\, \boldsymbol{\lambda} \in \mathbb{R}_+^{\#T},\, \boldsymbol{\lambda} \cdot \mathbf{1} = 1\right\},$$

$$\operatorname{aff} S \coloneqq \left\{T \cdot \boldsymbol{\lambda} \,:\, T \subseteq S,\, \#T \leq d+1,\, \boldsymbol{\lambda} \in \mathbb{R}^{\#T},\, \boldsymbol{\lambda} \cdot \mathbf{1} = 1\right\}.$$

A subset of $\mathbb{R}^d$ is an *affine subspace* if it coincides with its affine hull; such sets have the form $\boldsymbol{v} + T$ where T is a linear subspace of $\mathbb{R}^d$. The *dimension* of an affine subspace A is the minimal number of vectors that spans it, i.e., the smallest k such that $A = \operatorname{aff} G$ and $\#G = k$. The dimension of an arbitrary non-empty set $S \subseteq \mathbb{R}^d$, written $\dim S$, is the dimension of the affine subspace $\operatorname{aff} S$; and $\dim \emptyset \coloneqq -1$. A *hyperplane* in $\mathbb{R}^d$ is an affine subspace of dimension $d-1$. A (rational) hyperplane is the set of solutions of a single equation $\boldsymbol{a} \cdot \boldsymbol{x} = c$, having *coefficients* $\boldsymbol{a} \in \mathbb{Q}^d$ and *constant* $c \in \mathbb{Q}$. We call sets of linearly independent vectors in $\mathbb{R}^d$ *proper*.

Polyhedral geometry. We refer the reader to [30, Ch. 7–8] and [27] for further background on the following concepts.

Let $S \subseteq \mathbb{R}^d$ be the set of solutions of a system of linear inequalities $\mathfrak{S}\colon A \cdot \boldsymbol{x} \leq \boldsymbol{c}$, $S \neq \emptyset$. Such sets are called (closed) *convex polyhedra*. A row $\boldsymbol{a} \cdot \boldsymbol{x} \leq c$ of $\mathfrak{S}$ called an *implicit inequality* of $\mathfrak{S}$ whenever $\boldsymbol{a} \cdot \boldsymbol{x} = c$ holds for every $x \in S$.

The *lineality space* and the *characteristic cone* of $S \subseteq \mathbb{R}^d$ are defined as

$$\operatorname{lin.space} S = \{\boldsymbol{x} \in \mathbb{R}^d : A \cdot \boldsymbol{x} = \mathbf{0}\}$$

$$\operatorname{char.cone} S = \{\boldsymbol{x} \in \mathbb{R}^d : A \cdot \boldsymbol{x} \leq 0\}.$$

Given a nonzero vector $\boldsymbol{w} \in \mathbb{R}^d$ such that $\delta = \max\{\boldsymbol{w} \cdot \boldsymbol{x} : A \cdot \boldsymbol{w} \leq \boldsymbol{c}\}$ is finite, the hyperplane $\{\boldsymbol{x} : \boldsymbol{w} \cdot \boldsymbol{x} = \delta\}$ is called a *supporting hyperplane* of S. A set $F \subseteq \mathbb{R}^d$ is a *face* of S if $F = S$ or if F is the intersection of S with a supporting hyperplane of S. This means that $F \subseteq S$ is a face whenever there is some $\boldsymbol{w} \in \mathbb{R}^d$ for which F is the (non-empty) set of all points in S attaining $\max\{\boldsymbol{w} \cdot \boldsymbol{x} : \boldsymbol{x} \in S\}$, provided that this maximum is finite (possibly $\boldsymbol{w} = \mathbf{0}$). Alternatively, a face F of S is a nonempty subset of S such that $F = \{\boldsymbol{x} \in S : A' \cdot \boldsymbol{x} = \boldsymbol{c}'\}$ for some subsystem $A' \cdot \boldsymbol{x} \leq \boldsymbol{c}'$ of $\mathfrak{S}$.

Note that if $\mathfrak{S}$ has rational coefficients and right-hand sides then all vectors $\boldsymbol{w}$ in the paragraph above lie in $\mathbb{Q}^d$.

A *facet* of S is a face of S of dimension $\dim S - 1$. We call F a *minimal face* whenever F is a face and $F = \{\boldsymbol{x} \in \mathbb{R}^d : A' \cdot \boldsymbol{x} = \boldsymbol{c}'\}$ for some subsystem $A' \cdot \boldsymbol{x} \leq \boldsymbol{c}'$ of $\mathfrak{S}$. Minimal faces are exactly maximal affine subspaces contained in S.

Let $C = \operatorname{char.cone} S$ and $t = \dim(\operatorname{lin.space} C)$. A *minimal proper face* of C is a face of C of dimension $t+1$. Equivalently, F is a minimal proper face of C whenever

$$F = \{\boldsymbol{x} \in \mathbb{R}^d : A' \cdot \boldsymbol{x} = \mathbf{0},\, \boldsymbol{a} \cdot \boldsymbol{x} \leq 0\},$$

$\dim F = t+1$ and $\operatorname{lin.space} C = \{\boldsymbol{x} \in \mathbb{R}^d : A' \cdot \boldsymbol{x} = \mathbf{0}, \boldsymbol{a} \cdot \boldsymbol{x} = 0\}$, where $A' \cdot \boldsymbol{x} \leq \boldsymbol{c}'$ is a subsystem of $\mathfrak{S}$ and $\boldsymbol{a}$ is a row of A.

For finite sets $V, W \in \mathbb{Q}^d$, define

$$K(V, W) \coloneqq \operatorname{conv} V + \operatorname{cone} W.$$

The Minkowski–Weyl theorem (see, e.g., [30, Chap. 8]) states that a set is a convex polyhedron if and only if it is equal to $K(V, W)$ for some $V, W \in \mathbb{Q}^d$. A set $C \subseteq \mathbb{R}^d$ is a *finitely generated shifted (rational) cone* whenever $C = K(\boldsymbol{v}, W)$ for some $\boldsymbol{v} \in \mathbb{Q}^d$ and finite $W \subseteq \mathbb{Q}^d$.

Given a system of inequalities $\mathfrak{S}$ with the set of solutions S, we say that a set of hyperplanes H *carves out* S whenever for every row $\boldsymbol{a} \cdot \boldsymbol{x} \leq c$ of $\mathfrak{S}$ there is a hyperplane $h \in H$ with $h\colon \boldsymbol{a} \cdot \boldsymbol{x} = c$. Two comments are in order:

- S can be obtained by intersecting $\mathbb{R}^d$ with a subset of the half-spaces induced by H. Each $h\colon \boldsymbol{a} \cdot \boldsymbol{x} = c$ in H induces two half-spaces: $\boldsymbol{a} \cdot \boldsymbol{x} \geq c$ and $\boldsymbol{a} \cdot \boldsymbol{x} \leq c$;
- $\operatorname{aff} S$ can be obtained by intersecting $\mathbb{R}^d$ with a subset of the hyperplanes in H. This follows from the fact that $\operatorname{aff} S$ is the set of vectors satisfying all implicit equalities in $\mathfrak{S}$, see [30, Ch. 8].

We postulate that $H = \emptyset$ carves out $\mathbb{R}^d$.

Semilinear sets and $\mathbb{R}$-semilinear sets. Fix a natural number $d \geq 1$. A set $S \subseteq \mathbb{Z}^d$ is a (d-dimensional) *linear set* if it is of the form $S = L(\boldsymbol{b}, P) \coloneqq \boldsymbol{b} + P \cdot \mathbb{N}^{\#P}$ for some *base* $\boldsymbol{b} \in \mathbb{Z}^d$ and a finite set of *periods* $P \subseteq \mathbb{Z}^d$. The set S is *hybrid linear* if it is of the form $S = L(B, P) \coloneqq B + P \cdot \mathbb{N}^{\#P}$, where $Q, P \subseteq \mathbb{Z}^d$ are finite sets. Notice that $L(B, P) = \bigcup_{\boldsymbol{b} \in B} L(\boldsymbol{b}, P)$, and thus every hybrid linear set is a union of finitely many linear sets sharing the same periods P. A *semilinear set* is a finite union of linear sets, i.e., represented as $\bigcup_{i \in I} L(B_i, P_i)$, where I is a finite set of indices.

A set $S \subseteq \mathbb{R}^d$ is an $\mathbb{R}$-*semilinear set* whenever it is of the form $S = \bigcup_{i \in I} K(V_i, W_i) \setminus \left(\bigcup_{j \in J_i} K(V_j, W_j)\right)$, where for all $i \in I$ and $j \in J_i$, the polyhedron $K(V_j, W_j)$ is a face of $K(V_i, W_i)$. Every component $K(V_i, W_i) \setminus \bigcup_{j \in J_i} K(V_j, W_j)$ is the set of solutions of a system $A \cdot \boldsymbol{x} \leq \boldsymbol{c} \wedge B \cdot \boldsymbol{x} < \boldsymbol{d}$.

Magnitude and encoding of numbers. In this paper, the *(infinity) norm* of a vector $\boldsymbol{v} = (v_1, \dots, v_d) \in \mathbb{R}^d$ is defined as $\|\boldsymbol{v}\| \coloneqq \max\{2, |v_i| : i \in [1, d]\}$. This non-standard definition is for technical and presentational convenience only as it, e.g., prevents multiplication by zero terms when deriving bounds on the size of objects. For a matrix A, $\|A\|$ is the maximum norm of its columns. Similarly, for a finite set $W \subseteq \mathbb{R}^d$, we define $\|W\| \coloneqq \max\{\|\boldsymbol{v}\| : \boldsymbol{v} \in W\}$.

For finite representations of infinite sets, we extend the notation $\|\cdot\|$ to refer to the maximum infinity norm of all numbers appearing in the representation of that set. For instance, given a semilinear set $M = \bigcup_{i \in I} L(B_i, P_i)$, we write $\|M\| \coloneqq \max\{\|B_i\|, \|P_i\| : i \in I\}$.

Following [30, Sec. 3.2], given a rational number $\frac{p}{q}$ where p and $q \geq 1$ are relatively prime integers, we write $\langle \frac{p}{q} \rangle$ to denote the real number

$$\langle \tfrac{p}{q} \rangle \coloneqq 1 + \lceil \log_2(|p| + 1) \rceil + \lceil \log_2(|q| + 1) \rceil.$$

Intuitively, $\frac{p}{q}$ can be encoded using $O(\langle \frac{p}{q} \rangle)$ bits. We extend this notation to integers, rational matrices, and finite sets of rational matrices. We set $\langle n \rangle \coloneqq \langle \frac{n}{1} \rangle = \lceil \log_2(|n| + 1) \rceil + 2$ for every integer n. Given a matrix $A \in \mathbb{Q}^{n \times d}$, we denote $\langle A \rangle \coloneqq \max\{\langle A[i, j] \rangle : i \in [1, n], j \in [1, d]\}$, where $A[i, j]$ is the rational number appearing in the i-th row and j-th column of A. Notice that the number of bits required to encode A is $O(n \cdot d \cdot \langle A \rangle)$. Finally, $\langle S \rangle \coloneqq \max_{A \in S} \langle A \rangle$.

Our convention for the infinity norm for finite representations of infinite sets extends to the notation $\langle \cdot \rangle$. For instance, given a set $S = K(V, W) \subseteq \mathbb{R}^d$ with $V, W \subseteq \mathbb{Q}^d$, we write $\langle S \rangle \coloneqq \max\{\langle V \rangle, \langle W \rangle\}$.

3 Operations on polyhedra and their representation

In this section, we identify a set of technical tools from polyhedral geometry that we use for establishing our main results. We mostly recall such results and provide bounds on algorithms and descriptional complexity when necessary. Due to space constraints, proofs are relegated to Appendix A.

Change of representation over $\mathbb{R}$. We need to move between the representations of polyhedra as solutions to systems of (in)equalities and as sets of the form $K(V, W)$.

Proposition 3.1. *Let $S = K(V, W)$, with $V, W \subseteq \mathbb{Q}^d$ finite sets. There is a system of linear inequalities $\mathfrak{S} \colon A \cdot \boldsymbol{x} \leq \boldsymbol{c}$ whose solutions define S, and such that*

- *$A \in \mathbb{Q}^{n \times d}$ with $n \leq (\#V + \#W)^d + 2d$;*
- *$\langle A \rangle, \langle \boldsymbol{c} \rangle \leq O(d^2) \cdot \langle S \rangle$.*

This system can be computed in time $(\#V + \#W)^d \cdot \mathrm{poly}(d, \langle S \rangle)$.

Further results of this kind are discussed in the Appendix. These are all underpinned by the fact that Gaussian elimination over $\mathbb{Q}$ in dimension d can be carried out in time polynomial in the size of the matrix and bit size of its entries; see, e.g., [30, Sec. 3.3].

Membership and representation results over $\mathbb{Z}$ and $\mathbb{N}$. The following lemma gives an algorithm to decide membership in a semilinear set.

Lemma 3.2. *Let $\boldsymbol{v} \in \mathbb{Z}^d$ and $M = L(B, P) \subseteq \mathbb{Z}^d$. Deciding $\boldsymbol{v} \in M$ can be performed in time $\mathrm{poly}(d^d, \langle \boldsymbol{v} \rangle, (\|B\| + \#P \cdot \|P\|)^d)$.*

We recall a discrete version of Carathéodory's Theorem.

Proposition 3.3 ([6, Prop. 5])**.** *Let $S = L(B, P) \subseteq \mathbb{Z}^d$ be a hybrid linear set. Then, $S = \bigcup_{i \in I} L(C_i, Q_i)$ where*

- *$\#I \leq (\#P)^d$; $\max_{i \in I} \|C_i\| \leq \|B\| + (\#P \cdot \|P\|)^{O(d)}$; and*
- *for all $i \in I$, $Q_i \subseteq P$ and Q_i is proper.*

This representation can be computed in time

$$O(\#B \cdot (d \cdot \#P \cdot \|P\|)^{(d+1)}).$$

The following lemma shows that when represented as a hybrid linear set, the set of non-negative solutions of a homogeneous system of linear equations has few periods. Its proof uses techniques from Domenjoud [8].

Lemma 3.4. *Let $S \subseteq \mathbb{N}^d$ be the set of all non-negative integer solutions of $\mathfrak{S} \colon A \cdot \boldsymbol{x} = 0$, with $A \in \mathbb{Z}^{n \times d}$. Then $S = L(B, P)$ such that $\langle B \rangle, \langle P \rangle \leq O(n \cdot d^3 \cdot \langle A \rangle)$, $\#P \leq d^{(k+1)}$, where $k = \operatorname{rank} A$; and B, P are computable in time $\mathrm{poly}(d^{k+1}, \|A\|^{n \cdot k^3})$.*

An equivalence relation induced by hyperplanes. Let $H = \{h_1, \ldots, h_n\}$ be a set of n rational hyperplanes given by equations $h_i : \boldsymbol{a}_i \cdot \boldsymbol{x} = c_i$ in d variables. Below, let $\sim_H \subseteq \mathbb{R}^d \times \mathbb{R}^d$ be the equivalence relation defined as

$$\boldsymbol{x}_1 \sim_H \boldsymbol{x}_2 \text{ iff for all } i \in [1, n], \operatorname{sgn}(\boldsymbol{a}_i \cdot \boldsymbol{x}_1 - c_i) = \operatorname{sgn}(\boldsymbol{a}_i \cdot \boldsymbol{x}_2 - c_i),$$

where $\operatorname{sgn} \colon \mathbb{R} \to \{-1, 0, 1\}$ is the *sign function*: $\operatorname{sgn}(0) \coloneqq 0$, $\operatorname{sgn}(r) \coloneqq -1$ for $r < 0$, and $\operatorname{sgn}(r) \coloneqq 1$ for $r > 0$.

We recall a folklore result on the number of regions induced by H on $\mathbb{R}^d$ (cf. [23, Ch. 6]).

Proposition 3.5. *For every set H of n hyperplanes in $\mathbb{R}^d$, the relation $\sim_H$ has at most $(2n)^d + 1$ many equivalence classes.*

Polyhedral complexes and triangulations. A *polyhedral complex* is a finite set $\mathcal{R}$ of convex polyhedra that satisfies the following two properties:

- for every face R' of every $R \in \mathcal{R}$, $R' \in \mathcal{R}$;
- for every $R_1, R_2 \in \mathcal{R}$, either $R_1 \cap R_2 = \emptyset$ or $R_1 \cap R_2$ is a face of both R_1 and R_2.

Definitions and statements in the remainder of this section are only required for our construction of $\mathbb{Z}$-splitters (Sec. 4.3) and can be skipped at the first reading.

A *generalised m-dimensional simplex* is a set of the form $T = K(V, W)$, $V, W \subseteq \mathbb{Q}^d$ such that $\#V + \#W = m + 1$ and $\dim(\operatorname{aff} T) = m$. For example, two-dimensional generalised simplices are triangles, closed half-infinite strips, and closed infinite sectors.

A *triangulation* is a polyhedral complex $\mathcal{T}$ where every $T \in \mathcal{T}$ is a generalised simplex and all $T \in \mathcal{T}$ that are not a face of any $T' \in \mathcal{T} \setminus \{T\}$ have the same dimension m. The latter T are called the set of maxima in $\mathcal{T}$, and we write $\dim \mathcal{T} = m$. Given $S \subseteq \mathbb{R}^d$, we say that $\mathcal{T}$ is a *triangulation of* S whenever $S = \bigcup_{T \in \mathcal{T}} T$.

Proposition 3.6. *Every polyhedron $K(V, W)$ has a triangulation $\mathcal{T}$ such that for all $T \in \mathcal{T}$, $T = K(V', W')$ for some $V' \subseteq V$ and $W' \subseteq W$. This $\mathcal{T}$ can be found in time $(\#V + \#W)^{O(d)} \cdot \mathrm{poly}(d, \langle V \rangle + \langle W \rangle)$.*

Let $S \subseteq \mathbb{R}^d$ be a polyhedron given by the real solutions of $\mathfrak{S} \colon A \cdot \boldsymbol{x} \leq \boldsymbol{b}$, where $A \in \mathbb{Q}^{m \times d}$ and $\boldsymbol{b} \in \mathbb{Q}^m$. A *half-opening* of S is a set of the form $S^{\mathrm{op}} = \{\boldsymbol{x} \in S : A' \cdot \boldsymbol{x} < \boldsymbol{b}'\}$ for some subsystem $A' \cdot \boldsymbol{x} \leq \boldsymbol{b}'$ of $\mathfrak{S}$.

The following proposition is a version of [6, Lem. 10], strengthening it slightly to accommodate for the case of non-integer vectors in the set of base vectors V.

Proposition 3.7. *Let $S = K(V, W)$ with $V \subseteq \mathbb{Q}^d$ and $W \subseteq \mathbb{Z}^d$, where $\#V + \#W \le d + 1$ and W is proper. Then $S^{\text{op}} \cap \mathbb{Z}^d = L(B, W)$ for any half-opening S^{op} of S, and $\|B\| \le \|V\| + 2d \cdot \|W\|$. The set B can be computed in time $(\|V\| + \|W\| + 2)^{O(d)}$.*

Let $\mathcal{T}$ be a triangulation of some polyhedron $P \subseteq \mathbb{R}^d$. We define the *maximal half-opening* $\mathcal{T}^{\text{op}}$ of $\mathcal{T}$ as the smallest set containing all finite $T \in \mathcal{T}$ and for every infinite $T \in \mathcal{T}$ given by $\mathfrak{T}\colon A \cdot \boldsymbol{x} \le \boldsymbol{c}$ the half-opening given by $\mathfrak{T}'\colon A \cdot \boldsymbol{x} < \boldsymbol{c}$.

Intuitively, if P is a generalised simplex itself, then $\mathcal{T}^{\text{op}}$ contains, for each face F of P, the relative interior of that face (i.e., the set difference of F and all its proper sub-faces). This extends to the general case, enumerating all $F \in \mathcal{T}$. We remark that in the definition above, since each $T \in \mathcal{T}$ is a generalised simplex, lin.space $T = \{\boldsymbol{0}\}$, so T is finite if and only if T is a minimal face of $\mathcal{T}$.

Proposition 3.8. *Let $\mathcal{T}$ be a triangulation and $\mathcal{T}^{\text{op}}$ be its maximal half-opening. Then $\bigcup_{T^{\text{op}} \in \mathcal{T}^{\text{op}}} T^{\text{op}} = \bigcup_{T \in \mathcal{T}} T$ and for all distinct $T_1^{\text{op}}, T_2^{\text{op}} \in \mathcal{T}^{\text{op}}$, $T_1^{\text{op}} \cap T_2^{\text{op}} = \emptyset$.*

4 Splitters

In this section we present geometric constructions that are at the core of our main results. We start with the setting of linear real arithmetic and later will move on to integers.

Let us first fix $M = \bigcup_{i \in I} K(V_i, W_i) \subseteq \mathbb{R}^d$. Our overall goal here is to characterise the complement $\overline{M}$ of the set M as a union of polyhedra. To do this, we construct a partition of $\mathbb{R}^d$ induced by M, in the sense captured by the following definition, and study the descriptional and computational complexity of this construction.

Given a family $\mathcal{P}$ of polyhedra in $\mathbb{R}^d$, a *splitter* for $\mathcal{P}$ is any polyhedral complex $\mathcal{R} = \{R_1, \dots, R_m\}$ that satisfies the following two properties:

(S1) for all $R \in \mathcal{R}$ and for all $P \in \mathcal{P}$, the set $R \cap P$ is either empty or equal to a face of R; and

(S2) $R_1 \cup \dots \cup R_m = \mathbb{R}^d$.

We remark that, as every polyhedron is a face of itself, condition (S1) is satisfied if in particular $R \subseteq P$. Abusing notation slightly, we will talk about splitters for a *union* of convex polyhedra (making the *family* of polyhedra implicit).

In the theorem below, recall that $\langle M \rangle := \max_{i \in I} \langle K(V_i, W_i) \rangle$.

Theorem 4.1 (splitters for unions of polyhedra)**.** *Given any set of the form $M = \bigcup_{i \in I} K(V_i, W_i) \subseteq \mathbb{R}^d$, there exists a splitter $\mathcal{R} = \{R_1, \dots, R_m\}$ for M that has the following properties:*

1. *given $j \in J := [1, m]$, $R_j = K(C_j, Q_j)$ where $C_j, Q_j \subseteq \mathbb{Q}^d$ and $\langle C_j \rangle, \langle Q_j \rangle \le O(d^5) \cdot \langle M \rangle$; and*
2. *$m, \#(\bigcup_{j \in J} C_j), \#(\bigcup_{j \in J} Q_j) \le (\#I \cdot \max_{i \in I}(\#V_i + \#W_i) + d)^{O(d^2)}$.*

Moreover, $\mathcal{R}$ can be computed from M in time

$$(\#I \cdot \max_{i \in I}(\#V_i + \#W_i + 1))^{O(d^2)} \cdot \operatorname{poly}(\langle M \rangle)$$

in generator representation.

We describe the proof idea for Theorem 4.1 in Section 4.1.

While Theorem 4.1 will be useful to us when we deal with operations on $\mathbb{R}$-semilinear sets, it is not sufficient as it is to support our constructions for sets inside $\mathbb{Z}^d$. Intuitively, the reason for this is that, in the presence of polyhedra $K(V_i, W_i)$ with large $\#V_i$, the splitter for the union M will have to "respect" many hyperplanes that pass through various subsets of V_i. These hyperplanes will be irrelevant if we view the set $M = \bigcup_{i \in I} K(V_i, W_i)$ as an overapproximation of the semilinear set $\bigcup_{i \in I} L(V_i, W_i)$: the latter is simply the union of many sets of the form $L(\boldsymbol{v}, W_i)$, and, roughly speaking, there is no "interaction" between different elements of the same set V_i. This motivates the following refinement of Theorem 4.1.

Theorem 4.2 (splitters for unions of cones)**.** *Given any set of the form $N = \bigcup_{i \in I} \bigcup_{\boldsymbol{v} \in V_i} K(\boldsymbol{v}, W_i) \subseteq \mathbb{R}^d$, with I and $V_i, W_i \subseteq \mathbb{Q}^d$ finite sets, $i \in I$, there exists a splitter $\mathcal{R}' = \{R'_1, \dots, R'_t\}$ for N that has the following properties:*

1. *for each $j \in J := [1, t]$, $R'_j = K(E_j, F_j)$ where $E_j \subseteq \mathbb{Q}^d$, $F_j \subseteq \mathbb{Z}^d$, $\langle E_j \rangle \le O(d^5) \cdot \langle N \rangle$, $\langle F_j \rangle \le O(d^6) \cdot \max_{i \in I} \langle W_i \rangle$;*
2. *$\#(\bigcup_{j \in J} F_j) \le (\#I \cdot \max_{i \in I} \#W_i + d)^{O(d^2)}$; and*
3. *$t, \#(\bigcup_{j \in J} E_j) \le (\#I \cdot (\max_{i \in I} \#V_i) \cdot \max_{i \in I}(1 + \#W_i) + d)^{O(d^2)}$.*

Moreover, $\mathcal{R}'$ can be computed from N in time

$$(\#I \cdot \max_{i \in I}(\#V_i + \#W_i + 1))^{O(d^2)} \cdot \operatorname{poly}(\langle N \rangle)$$

in generator representation.

While some of the bounds of Theorem 4.2 follow directly from the previous theorem, there are several differences, of which we highlight one. The upper bound on $\#(\bigcup_{j \in [1,t]} F_j)$ is independent of the cardinality and norms of sets V_i; controlling the number of cone generators is crucial for the triply exponential running time bound of our decision procedure for Presburger arithmetic.

Moving further from $\mathbb{R}$ to $\mathbb{Z}$, let us fix a semilinear set $M = \bigcup_{i \in I} L(B_i, P_i) \subseteq \mathbb{Z}^d$. We will need an integer analogue of splitters, partitioning $\mathbb{Z}^d$ into disjoint regions that are in some sense induced by M.

Given a family $\mathcal{M} = \{L(B_i, P_i)\}_{i \in I}$ of hybrid linear sets in $\mathbb{Z}^d$, a $\mathbb{Z}$-*splitter* for $\mathcal{M}$ is any family of sets $\mathcal{Z} = \{Z_1, \dots, Z_m\}$ that satisfies the following four properties:

(Z1) for all $Z_j \in \mathcal{Z}$, $Z_j = L(C_j, Q_j)$, where $C_j, Q_j \subseteq \mathbb{Z}^d$ and Q_j is proper;

(Z2) for all $i \in I$, $\boldsymbol{b} \in B_i$ and for all $Z_j \in \mathcal{Z}$, either $Z_j \subseteq K(\boldsymbol{b}, P_i)$ or $Z_j \cap K(\boldsymbol{b}, P_i) = \emptyset$;

(Z3) for all $i \in I$, $\boldsymbol{b} \in B_i$ and for all $Z_j \in \mathcal{Z}$, if $Z_j \subseteq K(\boldsymbol{b}, P_i)$ then $Q_j \subseteq L(\boldsymbol{0}, P_i)$;

(Z4) $Z_1 \cup \dots \cup Z_m = \mathbb{Z}^d$, where the union is disjoint.

As above, we will abuse notation and talk about $\mathbb{Z}$-splitters for a semilinear set M, implying the family of hybrid linear sets that are components of M in a given representation. We show the following theorem:

Theorem 4.3 (splitters for semilinear sets). *Given any set of the form $M = \bigcup_{i\in I} L(B_i, P_i) \subseteq \mathbb{Z}^d$, there exists a $\mathbb{Z}$-splitter $\mathcal{Z} = \{Z_1, \ldots, Z_m\}$ for the family $\{L(B_i, P_i)\}_{i\in I}$ that has the following properties:*

1. $\|C_j\| \leq \|M\|^{O(d^{10})\cdot \#I}$ *and* $\|Q_j\| \leq (\max_{i\in I}\|P_i\|)^{O(d^{10})\cdot\#I}$ *for all* $j \in J := [1, m]$*:*
2. $\#(\bigcup_{j\in J} Q_j) \leq (\#I \cdot \max_{i\in I} \#P_i + d)^{O(d^2)}$.
3. $m \leq (\#I \cdot \max_{i\in I} \#B_i \cdot \max_{i\in I}(1 + \#P_i))^{O(d^3)}$.

Moreover, $\mathcal{Z}$ can be computed from M in time

$$(\max_{i\in I}(\#B_i + \#P_i) + \|M\|)^{O(d^{11})\cdot\#I}.$$

Observe that while $\|Q_j\|$ may be exponential in $\#I$, the number of different vectors across all Q_j is comparably small and bounded, in any fixed dimension d, by a polynomial in $\#I \cdot \max_{i\in I} \#P_i$. The proof of Theorem 4.3 invokes the construction of splitter for a union of cones (Theorem 4.2), decomposes each atomic polyhedron further and intersects parts of the decomposition with $\mathbb{Z}^d$.

Importantly, Theorem 4.3 benefits from refined bounds of Theorem 4.2 to control the cardinality of $\bigcup_{j\in J} Q_j$. This turns out possible even though the dependency of $\|Q_j\|$ on $\#I$ is exponential, roughly speaking because (Z3) effectively forces an intersection of up to $\#I$ hybrid linear sets. As mentioned previously, our upper bound on the total number of periods (coming from the splitters) is the key to an elementary decision procedure.

4.1 Splitters for unions of polyhedra: sketch

We sketch the proof of Theorem 4.1. Let $i \in I$. We start by considering a set of hyperplanes $\mathcal{H}(V_i, W_i)$ that carves out $K(V_i, W_i)$, of which we characterise the descriptional complexity. By Proposition 3.1, there exists such a set of hyperplanes, denote it by $\mathcal{H}(V_i, W_i)$, respecting the following bounds:

$$\begin{aligned} \#\mathcal{H}(V_i, W_i) &\leq (\#V_i + \#W_i)^d + 2d, \\ \langle\mathcal{H}(V_i, W_i)\rangle &\leq O(d^2)\cdot\langle K(V_i, W_i)\rangle. \end{aligned} \tag{$*$}$$

The left-hand side of the second equation in ($*$) refers to the maximum $\langle\cdot\rangle$ measure of numbers appearing in the linear equations defining these hyperplanes. The set of hyperplanes in $\mathcal{H}(M) := \bigcup_{i\in I}\mathcal{H}(V_i, W_i)$ divides $\mathbb{R}^d$ into regions that we call *atomic polyhedra*. More precisely, for $\mathcal{H}(M) = \{h_1, \ldots, h_k\}$ with $h_i\colon \boldsymbol{a}_i\cdot\boldsymbol{x} = c_i$, an atomic polyhedron R induced by $\mathcal{H}(M)$ associates to every h_i a set H_i that is the set of solutions to $\boldsymbol{a}_i\cdot\boldsymbol{x} \sim c_i$ for some $\sim\, \in \{\leq, =, \geq\}$ such that R is the intersection $R = \bigcap_{1\leq i\leq m} H_i$. Let $R_1, \ldots, R_m$ be the family of all atomic polyhedra induced by $\mathcal{H}(M)$.

We first show that polyhedra $R_1, \ldots, R_m$ form a polyhedral complex. Indeed, by our definition of atomic polyhedra, $\{R_1, \ldots, R_m\}$ is closed under taking faces of polyhedra. This is due to the characterisation of faces using systems of equations, recalled in Sec. 2. To see the closure under intersection, whenever for some $j, k \in [1, m]$ the set $R_j \cap R_k$ is non-empty and different from R_j, we note that $R_j \cap R_k$ can be obtained as the intersection of R_j with some hyperplanes specified by equations of the form $\boldsymbol{a}_i\cdot\boldsymbol{x} = \boldsymbol{c}_i$, and thus forms a face of R_j. Thus, $\{R_1, \ldots, R_m\}$ is a polyhedral complex.

We next show that this family is a splitter for $\{K(V_i, W_i)\}_{i\in I}$. Property (S2), i.e., the equality $R_1 \cup \cdots \cup R_m = \mathbb{R}^d$, is immediate. Property (S1) follows from the definition of atomic polyhedra and the definition of $\mathcal{H}(M)$. Indeed, take some R_j and $P_i = K(V_i, W_i)$. Suppose $R_j \cap P_i$ is nonempty and different from R_j; we show that it must be a face of R_j and contained in a facet of P_i. We know that, for each $h \in \mathcal{H}(M)$, all points of R_j lie on the same side of h, in the non-strict sense; or possibly even on h itself. Recall that $P_i = K(V_i, W_i)$ is a convex polyhedron and, as such, is the set of solutions to a conjunction of affine inequalities, all represented by constraints of the form $\boldsymbol{a}_i\cdot\boldsymbol{x} \sim c_i$, for $\sim\, \in \{\leq, =, \geq\}$ and some $h_i\colon \boldsymbol{a}_i\cdot\boldsymbol{x} = c_i$ with $h_i \in \mathcal{H}(M)$. Therefore, each of these inequalities either is valid for all points of the set R_j, or is violated at all these points, or restricts R_j to some non-empty face. Since an intersection of faces must be a face itself, we conclude that $R_j \cap P_i$ is a face of R_j.

Towards checking the satisfaction of the two properties required by Theorem 4.1, note that from Proposition 3.5 together with Eqs. ($*$) we conclude that the number m of atomic polyhedra is bounded by $(\#I\cdot\max_{i\in I}(\#V_i + \#W_i))^{O(d^2)}$, as required by Property (2). What is left is to study the descriptional complexity of atomic polyhedra: prove that they satisfy Property (1) and show that the sets of bases and periods required to describe all atomic polyhedra satisfy the bound in Property (2). We leave this part out for space reasons.

4.2 Splitters for unions of cones: idea

The proof of Theorem 4.2 is left out for space reasons. Our construction refines the analysis of Theorem 4.1 for the case of sets N, where polyhedra are cones, but big groups of these cones share periods. This analysis is possible because, intuitively, supporting hyperplanes for cones from a union $\bigcup_{\boldsymbol{v}\in V_i} K(\boldsymbol{v}, W_i)$ are translates of one another.

4.3 Splitters for semilinear sets: sketch

We sketch the proof of Theorem 4.3. We start with a set $M = \bigcup_{i\in I} L(B_i, P_i)$. We first apply Theorem 4.2 to obtain a splitter for the union of cones $\bigcup_{i\in I}\bigcup_{\boldsymbol{b}\in B_i} K(\boldsymbol{b}, P_i)$. Let the obtained splitter be $\mathcal{A} = \{R_1, \ldots, R_t\}$ with $R_j = K(E_j, F_j)$. We further split these polyhedra to obtain a $\mathbb{Z}$-splitter. For simplicity, in what follows we denote $F := \bigcup_{j\in[1,t]} F_j$, recalling that

$F \subseteq \mathbb{Z}^d$ by condition 1 in Theorem 4.2. The set F enjoys the bound from condition 2 in the same theorem.

In order to satisfy the property (Z3) in the definition of $\mathbb{Z}$-splitter, we scale each vector in F using the lemma below.

Lemma 4.4. *For every $\boldsymbol{p} \in F$, there is an integer $\lambda \geq 0$ with $\langle \lambda \rangle \leq \#I \cdot O(d^{10}) \cdot \max_{i \in I} \langle P_i \rangle$ and such that, for all $i \in I$, if $\boldsymbol{p} \in \operatorname{cone}(P_i)$ then $\lambda \cdot \boldsymbol{p} \in L(\boldsymbol{0}, P_i)$. This λ can be computed in time*

$$((\#I)^2 + \sum_{i \in I} (\#P_i)^{d+1}) \cdot \operatorname{poly}(d, \max_{i \in I} \langle P_i \rangle, \langle \boldsymbol{p} \rangle).$$

Below, we write $\widehat{F}$ for the set $\{\lambda_{\boldsymbol{p}} \cdot \boldsymbol{p} : \boldsymbol{p} \in F\}$, where $\lambda_{\boldsymbol{p}}$ is the integer obtained from Lemma 4.4 for the vector $\boldsymbol{p}$. The following lemma partitions $\mathbb{Z}^d$ into hybrid linear sets with periods from $\widehat{F}$. Taking the splitter $\mathcal{A}$ computed above, we let $\mathcal{A}_k = \{A \in \mathcal{A} : \dim A \leq k\}$ and notice that $\mathcal{A} = \mathcal{A}_d$.

Lemma 4.5. *For each $k \leq d$, there is a finite collection C_k of subsets of $\mathbb{R}^d$ such that*

(i) all sets in C_k are pairwise disjoint;
(ii) $\#C_k \leq \#\mathcal{A}_k \cdot (\#I \cdot \max_{i \in I} \#B_i \cdot \max_{i \in I}(1 + \#P_i) + d)^{O(d^3)}$;
(iii) $\bigcup_{A \in \mathcal{A}_k} A = \bigcup_{C \in C_k} C$;
(iv) for every $C \in C_k$, we have $C \cap \mathbb{Z}^d = L(D, Q)$ where $\|D\| \leq \|M\|^{O(d^{10}) \cdot \#I}$, $Q \subseteq \widehat{F}$, and Q is proper; and
(v) for every $C \in C_k$ there is $A \in \mathcal{A}_k$ such that $C \subseteq A$.

All the generator sets D and Q together can be computed in time

$$(\max_{i \in I}(\#B_i + \#P_i) + \|M\|)^{O(d^{11}) \cdot \#I}.$$

Proof (sketch). We use induction on $k \leq d$. In the induction base case $k = 0$, we set $C_0 = \mathcal{A}_0$, the latter being a finite set of points. For the induction step, C_{k+1} contains C_k and is further populated as follows. For every $A \in \mathcal{A}_{k+1} \setminus \mathcal{A}_k$, let $\mathcal{T}$ be a triangulation of A, and let $\mathcal{T}^{\mathrm{op}}$ be the maximal half-opening of $\mathcal{T}$. We add to C_{k+1} every $T^{\mathrm{op}} \in \mathcal{T}^{\mathrm{op}}$ that is not fully contained in some $A' \in \mathcal{A}_k$. We show that the resulting set has the desired properties.

Property (v) follows directly by definition of C_{k+1}.

For Property (iii), recall that for every polyhedron $A \subseteq \mathbb{Q}^d$, every triangulation $\mathcal{T}$ of A and every half-opening $\mathcal{T}^{\mathrm{op}}$ of $\mathcal{T}$, by Proposition 3.8, $A = \bigcup_{T^{\mathrm{op}} \in \mathcal{T}^{\mathrm{op}}} T^{\mathrm{op}}$. So, $A \subseteq \bigcup_{C \in C_{k+1}} C$, and Property (iii) follows.

For Property (iv), recall that every atomic polyhedron A has a representation $K(E_j, F_j)$, where $F_j \subseteq F \subseteq \mathbb{Z}^d$, and $E_j \subseteq \mathbb{Q}^d$ is such that $\langle E_j \rangle \leq O(d^5) \cdot \langle M \rangle$. By definition of $\widehat{F}$ and following Lemma 4.4, there is $F'_j \subseteq \widehat{F} \subseteq \mathbb{Z}^d$ such that $\#F'_j = \#F_j$, $\langle F'_j \rangle \leq \#I \cdot O(d^{10}) \cdot \max_{i \in I} \langle P_i \rangle$ and $A = K(E_j, F'_j)$. Then, by computing the triangulation of $K(E_j, F'_j)$ with Proposition 3.6 and by applying Proposition 3.7, Property (iv) follows. An observation: when applying Proposition 3.7, we consider the infinity norm of bases and period, instead of their bit-length. The relation between bit-length and infinity norm is simple: for every rational number $\frac{p}{q}$ with p and $q \geq 1$ relatively prime integers, if $\langle \frac{p}{q} \rangle \leq \alpha$ then $\|\frac{p}{q}\| \leq 2^{O(\alpha)}$, as $\|\frac{p}{q}\| \leq \|p\| \leq 2^{O(\langle p \rangle)}$ and $\langle p \rangle \leq \langle \frac{p}{q} \rangle$.

We give the proofs of Property (ii) and Property (i) in Appendix B.3. ◻

Once Lemma 4.5 is in place, we can prove Theorem 4.3 by picking $\mathcal{Z} = \{C \cap \mathbb{Z}^d : C \in C_d\}$.

5 Semilinear and $\mathbb{R}$-semilinear expressions

In this section, we define an algebra of ($\mathbb{R}$-)semilinear sets comprising all Boolean operations with projections along the coordinate axes and show that expressions in this algebra can be evaluated in doubly- and triply-exponential time over the reals and integers, respectively. To this end, consider the following grammar

$$s ::= a \mid \pi_D(s) \mid \bar{s} \mid s \cap s \mid s \cup s,$$

where a are *atoms* to be defined below, and D can be any finite subset of positive integers.

A *semilinear expression* is an expression from the above grammar where atoms are hybrid linear sets. Whenever possible, we endow a semilinear expression s with a *dimension* $d \in \mathbb{N}$, below $s\colon d$, given by the typing rules

$$\frac{L(B,P) \subseteq \mathbb{Z}^d}{L(B,P)\colon d} \qquad \frac{s\colon d \quad D \subseteq [1,d]}{\pi_D(s)\colon d - \#D} \qquad \frac{s\colon d}{\bar{s}\colon d} \qquad \frac{s_1\colon d \quad s_2\colon d}{s_1 \oplus s_2\colon d}$$

where $\oplus \in \{\cap, \cup\}$. Expressions that comply with the type assertions above are *well-defined*, and we restrict ourselves subsequently to well-defined expressions.

Every well-defined semilinear expression $s\colon d$ evaluates to a subset $[\![s]\!] \subseteq \mathbb{Z}^d$, following the standard semantics where the symbols $\cup$, $\cap$ and $\bar{\cdot}$ denote the Boolean operations union, intersection and complement, respectively, and $\pi_D(\cdot)$ is the function projecting away the coordinates indexed by all $i \in D$. By convention, the coordinates of of $\mathbb{Z}^d$ are indexed 1 through d.

Analogously, we define $\mathbb{R}$-*semilinear expressions* in which atoms are closed convex polyhedra given as $K(V, W)$.

For an ($\mathbb{R}$-)semilinear expression s, we denote by $n_{\mathrm{p}}(s)$, $d(s)$ and $h(s)$ the *maximal number of periods*, *maximal dimension* and *height* of s: for semilinear expressions, $n_{\mathrm{p}}(s)$ is the maximal cardinality of P of a hybrid linear $L(B,P)$ appearing as an atom of s, and for $\mathbb{R}$-semilinear sets, $n_{\mathrm{p}}(s)$ is the maximal cardinality of $\#(V \cup W)$ of a convex polyhedron $K(V,W)$ appearing as an atom of s; $d(s)$ is the maximal dimension of atoms in s; $h(s)$ is the maximum nesting depth of operations appearing in s. We write $\langle s \rangle$ for the maximal $\langle a \rangle$ of an atom a appearing in s.

Theorem 5.1. *There is an algorithm that, given a well-defined semilinear expression s, computes a family of pairs $\{(B_i, P_i)\}_{i \in I}$ such that $[\![s]\!] = \bigcup_{i \in I} L(B_i, P_i)$. The algorithm ensures*

$$\#I \leq n^{d^{O(h)}}, \quad \langle B_i \rangle, \langle P_i \rangle \leq (\langle s \rangle + n)^{d^{O(h)}}, \quad P_i \text{ proper},$$

where $n = n_{\mathrm{p}}(s) + 2$, $d = d(s)$ and $h = h(s)$. Moreover, the running time of the algorithm is $\exp((\langle s\rangle + n)^{d^{O(h)}})$.

As a corollary, we obtain a geometric decision procedure for Presburger arithmetic that matches the optimal running time of quantifier-elimination and automata-based decision procedures [9, 26]. Given a formula of LIA or LRA whose atomic formulas are linear inequalities $\boldsymbol{a}\cdot\boldsymbol{x} \le b$, we write $\langle\Phi\rangle$ for the maximal $\langle c\rangle$ for a coefficient or constant c appearing in an atomic formula of Φ; $d(\Phi)$ for the maximal number of free variables appearing in a subformula of Φ; and $h(\Phi)$ for the maximum nesting depth of Boolean connectives and quantifications appearing in Φ.

Corollary 5.2. *There is an algorithm that, given a formula Φ of LIA, computes a family of pairs $\{(B_i, P_i)\}_{i\in I}$ such that $[\![\Phi]\!] = \bigcup_{i\in I} L(B_i, P_i)$. The algorithm ensures that*

$$\#I \le 2^{d^{O(h)}}, \quad \langle B_i\rangle, \langle P_i\rangle \le (\langle\Phi\rangle + 2)^{d^{O(h)}}, \quad P_i \text{ proper},$$

where $d = d(\Phi)$ and $h = h(\Phi)$. Moreover, the running time of the algorithm is $\exp((\langle\Phi\rangle + 2)^{d^{O(h)}})$.

We establish analogous results for $\mathbb{R}$-semilinear expressions and formulas of LRA where the running time and bounds on the constants is one exponential lower.

Theorem 5.3. *There is an algorithm that, given a well-defined $\mathbb{R}$-semilinear expression s, computes a family of triples $\{(U_k, Y_k, \{(U_\ell, Y_\ell)\}_{\ell\in L_k})\}_{k\in K}$ such that*

$$[\![s]\!] = \textstyle\bigcup_{k\in K}(K(U_k, Y_k) \setminus \bigcup_{\ell\in L_k} K(U_\ell, Y_\ell)).$$

The algorithm ensures, wlog. assuming $d, \langle s\rangle \ge 2$, that

$$\#K, \#L_k, \#U_k, \#Y_k, \#U_\ell, \#Y_\ell \le n^{d^{O(h)}};\ \langle U_k\rangle, \langle Y_k\rangle \le d^{O(h)}\langle s\rangle,$$

where $n = n_{\mathrm{p}}(s) + 2$, $d = d(s)$ and $h = h(s)$. Moreover, the running time of the algorithm is $\langle s\rangle^{O(d)} \cdot n^{d^{O(h)}}$.

Interestingly enough, the bound on the number $\#K$ of components of $[\![s]\!]$ derived in Theorem 5.3 is doubly exponential, exactly as in the case of LIA. While this may on the first sight seem surprising, it turns out that there is a matching lower bound. It is known from [20, Lecture 23, p. 146] that there is a formula I_n of size linear in $n \in \mathbb{N}$ that defines the set of integers in the interval $[0, 2^{2^n} - 1]$. The only way to represent this formula as an $\mathbb{R}$-semilinear set is $\bigcup_{i\in[0,2^{2^n}-1]} K(\{i\}, \emptyset)$.

From Theorem 5.3 we establish the following corollary.

Corollary 5.4. *There is an algorithm that, given a formula Φ of LRA, computes a family of triples $\{(U_k, Y_k, \{(U_\ell, Y_\ell)\}_{\ell\in L_k})\}_{k\in K}$ such that*

$$[\![\Phi]\!] = \textstyle\bigcup_{k\in K}(K(U_k, Y_k) \setminus \bigcup_{\ell\in L_k} K(U_\ell, Y_\ell)).$$

The algorithm ensures, wlog. assuming $d, \langle\Phi\rangle \ge 2$,

$$\#K, \#L_k, \#U_k, \#Y_k, \#U_\ell, \#Y_\ell \le 2^{d^{O(h)}};\ \langle U_k\rangle, \langle Y_k\rangle \le d^{O(h)}\langle\Phi\rangle,$$

where $d = d(\Phi)$ and $h = h(\Phi)$. Moreover, the running time of the algorithm is $\langle\Phi\rangle^{O(d)} \cdot 2^{d^{O(h)}}$.

5.1 Evaluating semilinear expressions

We will now provide an analysis of the operations required to evaluate a semilinear expression, which in turn enables us to prove Theorem 5.1. Thanks to the notion of $\mathbb{Z}$-splitters and the bounds derived in Theorem 4.3, we can provide a complementation procedure for semilinear sets that, when combined with further algorithms for other Boolean operations and projection, allows for evaluating a semilinear expression in triply-exponential time. Below, we give a complementation procedure for semilinear sets with proper sets of periods, which has to be combined with Proposition 3.3 to apply to arbitrary semilinear sets.

Lemma 5.5. *There is an algorithm that given a set $M = \bigcup_{i\in I} L(B_i, P_i) \subseteq \mathbb{Z}^d$, where each P_i is proper, computes a family of pairs $\{(D_j, Q_j)\}_{j\in J}$ such that $\overline{M} = \bigcup_{j\in J} L(D_j, Q_j)$, and*

- *$\#J \le ((\#I + 1)\cdot d)^{O(d^3)}$ and each Q_j is proper; and*
- *$\langle D_j\rangle, \langle Q_j\rangle \le \#I \cdot O(d^{10}) \cdot \langle M\rangle$.*

Moreover, the running time of the algorithm is

$$(\max_{i\in I}(\#B_i + \#P_i) + \|M\|)^{O(d^{11})\cdot\#I}.$$

The algorithm is simple to state:

1: $\mathcal{Z} = \{Z_1, \ldots, Z_m\} \leftarrow \mathbb{Z}$-splitter for $\{L(B_i, P_i)\}_{i\in I}$, where $Z_j = L(C_j, Q_j)$
2: **for** $j \in J \coloneqq [1, m]$ **do**
3: $\quad E_j \leftarrow C_j \setminus M$
4: $\mathcal{Q} \leftarrow \{Q_j : j \in J\}$
5: **for** $Q \in \mathcal{Q}$ **do**
6: $\quad \mathcal{E}_Q \leftarrow \{E_j : j \in J, Q_j = Q\}$
7: $\quad$ **output** $(\bigcup_{E\in\mathcal{E}_Q} E, Q)$

In Line 1 of the algorithm, the $\mathbb{Z}$-splitter is computed according to Theorem 4.3. The set E_j in Line 3 can be computed by deciding membership in M of every $\boldsymbol{v} \in C_j$ according to Lemma 3.2 and discarding such $\boldsymbol{v}$ accordingly. The running time of the overall algorithm is easily seen to have the same order of magnitude as that for the $\mathbb{Z}$-splitter.

Note that the algorithm returns the set $\{(\bigcup_{E\in\mathcal{E}_Q} E, Q)\}_{Q\in\mathcal{Q}}$ for which we have

$$\textstyle\bigcup_{j\in J} L(E_j, Q_j) = \bigcup_{Q\in\mathcal{Q}} L(\bigcup_{E\in\mathcal{E}_Q} E, Q).$$

The difference between the set on the left hand side and the one on the right hand side is that the latter groups together base points from hybrid linear sets sharing the same set of periods, i.e., if on the left hand side we have $L(E_1, Q) \cup L(E_2, Q)$, then on the right hand side we have $L(E_1 \cup E_2, Q)$.

For every $j \in J$, let $F_j \coloneqq C_j \cap M$. So,

$$\mathbb{Z}^d = \textstyle\bigcup_{j\in J} L(C_j, Q_j) = \left(\bigcup_{j\in J} L(E_j, Q_j)\right) \cup \left(\bigcup_{j\in J} L(F_j, Q_j)\right).$$

To establish the correctness of the algorithm, it suffices to show that $\overline{M} = \bigcup_{j\in J} L(E_j, Q_j)$. We do so by proving that, given $j \in J$, both $L(E_j, Q_j) \cap M = \emptyset$ and $L(F_j, Q_j) \subseteq M$ hold. This relies on conditions (Z2) and (Z3) from the definition of $\mathbb{Z}$-splitters; see Appendix C.

We now turn towards the proof of Theorem 5.1, for which the complementation procedure established in Lemma 5.5 is the key. Informally, the algorithm to construct $[\![s]\!]$ as a semilinear set starting from a semilinear expression s works bottom up, beginning from the atoms of s. When considering an expression $s = s_1 \cup s_2$, $s = s_1 \cap s_2$, $s = \overline{s_1}$ or $s = \pi_D(s_1)$, the algorithm first computes semilinear sets $[\![s_1]\!]$ and $[\![s_2]\!]$, and then computes $[\![s]\!]$ according to the type of the operator. Whenever needed, e.g., before a complementation step, the algorithm uses Proposition 3.3 to make all periods of the semilinear sets $[\![s_1]\!]$ and $[\![s_2]\!]$ proper. To compute the complement $[\![\overline{s_1}]\!]$, the algorithm appeals to Lemma 5.5.

For the intersection of $[\![s_1]\!] = \bigcup_{j\in J} L(C_j, Q_j)$ and $[\![s_2]\!] = \bigcup_{k\in K} L(D_k, R_k)$, the algorithm first distributes the intersection over the unions of the two semilinear sets, obtaining $[\![s_1 \cap s_2]\!] = \bigcup_{(j,k)\in J\times K}(L(C_j, Q_j) \cap L(D_k, R_k))$, and then computes a semilinear set equivalent to each $L(C_j, Q_j) \cap L(D_k, R_k)$ following the lemma below, which makes the construction of [6, Thm. 6] effective.

Lemma 5.6. *Let $M = L(C, Q) \subseteq \mathbb{Z}^d$ and $N = L(D, R) \subseteq \mathbb{Z}^d$. Then $M \cap N = L(B, P)$ such that*

- $\langle B\rangle, \langle P\rangle \leq O(d \cdot (\#Q + \#R)^3 \cdot \max\{\langle M\rangle, \langle N\rangle\}^2)$; *and*
- $\#P \leq (\#Q + \#R)^{(d+1)}$.

Moreover, B and P are computable in time

$$(\#Q + \#R)^{O(d)} \cdot \max\{\|M\|, \|N\|\}^{O(d^4)}.$$

The cases for union and projection are not difficult, and only recalled in the lemma below for completeness.

Lemma 5.7. *Let $M_k = \bigcup_{i\in I_k} L(B_i, P_i) \subseteq \mathbb{Z}^d$, with $k \in \{1, 2\}$ and $I_1 \cap I_2 = \emptyset$, and let $D \subseteq [1, d]$. We have,*

- $M_1 \cup M_2 = \bigcup_{i\in I_1\cup I_2} L(B_i, P_i)$;
- $\pi_D(M_1) = \bigcup_{i\in I_1} L(\pi_D(B_i), \pi_D(P_i))$.

Moreover, $M_1 \cup M_2$ (resp. $\pi_D(M_1)$) can be computed in time $O(\max_{k\in\{1,2\}}(\sum_{i\in I_k} \#(B_i \cup P_i) \cdot \langle M_k\rangle))$ (resp. with $k = 1$).

The bounds and running time in Theorem 5.1 are established with an induction on the height of the input semilinear expression, together with the bounds and running times of the various operations, established in Lemmas 5.5 to 5.7. We rely on Proposition 3.3 to make period sets proper whenever needed, for instance before complementing a semilinear set.

In a nutshell, notice that complementation is the most expensive of the operations. For a sequence of h nested complementations (possibly interleaved with negations), first estimate the number of hybrid linear sets in the output, $\#J$. Assuming $\#I \geq 2$, this will be

$$(((\#I)^e)^e \ldots)^e, \qquad \text{where } e = O(d^3 \log d)$$

and there are h exponentiations in total. Therefore, $\#J \leq (\#I)^{d^{O(h)}}$. Other description size and running time bounds then rely on this key estimate. As Lemma 5.5 relies on each input $L(B_i, P_i)$ having linear independent P_i, we can use Proposition 3.3 so that we initially have $\#I \leq (n_{\mathrm{p}}(s))^d$.

Finally, to establish Corollary 5.2, given a formula of Presburger arithmetic, we first translate it into a semilinear expression: disjunctions, conjunction and negations become unions, intersections and complements, respectively; a sequence of quantifiers $\exists x_1 \cdots \exists x_k$ is translated into a projection $\pi_D(\cdot)$ where D contains k indices for the variables $x_1, \ldots, x_k$ (assuming an enumerations across all variables in the formula). Each inequality $\boldsymbol{a} \cdot \boldsymbol{x} \leq c$ is translated into a hybrid linear set thanks to the following lemma.

Lemma 5.8. *Let $S \subseteq \mathbb{Z}^d$ be given by the integer solutions of a linear inequality $\boldsymbol{a} \cdot \boldsymbol{x} \leq c$, $\boldsymbol{a} \in \mathbb{Z}^{1\times d}$ and $c \in \mathbb{Z}$. Then, $S = L(B, P)$ such that $\#P \leq 2d-1$ and $\langle B\rangle, \langle P\rangle \leq O(d^4(\langle \boldsymbol{a}\rangle + \langle c\rangle))$. Moreover, B and P can be computed in time $(\|\boldsymbol{a}\| + |c|)^{\mathrm{poly}(d)}$.*

The statement of Corollary 5.2 then follows by an application of Theorem 5.1.

5.2 Evaluating $\mathbb{R}$-semilinear sets

Analogously to the previous section, the algorithm for evaluating $\mathbb{R}$-semilinear expressions required by Theorem 5.3 can be obtained from algorithms for Boolean operations and projections on $\mathbb{R}$-semilinear sets. Due to space constraints, we only provide the relevant statements, all proofs are relegated to Appendix C. It is worth mentioning that due to $\mathbb{R}$-semilinear sets being constituted by copolyhedra, projection is not a trivial operation as it is for semilinear sets.

Lemma 5.9. *There is an algorithm that given an $\mathbb{R}$-semilinear set $M = \bigcup_{i\in I}(K(V_i, W_i) \setminus \bigcup_{j\in J_i} K(V_j, W_j)) \subseteq \mathbb{R}^d$ computes a family of triples $\{(U_k, Y_k, \{(U_\ell, Y_\ell)\}_{\ell\in L_k})\}_{k\in K}$ such that*

$$\overline{M} = \textstyle\bigcup_{k\in K}(K(U_k, Y_k) \setminus \bigcup_{\ell\in L_k} K(U_\ell, Y_\ell)).$$

The algorithm ensures, for every $k \in K$ and $\ell \in K_\ell$,

- $\#U_k, \#Y_k, \#U_\ell, \#Y_\ell \leq (\#I \cdot \max_{i\in I}(\#V_i + \#W_i) + d)^{O(d^2)}$;
- $\langle U_k\rangle, \langle Y_k\rangle, \langle U_\ell\rangle, \langle Y_\ell\rangle \leq O(d^5 \cdot \langle M\rangle)$; *and*
- $\#K, \#L_k \leq (\#I \cdot \max_{i\in I}(\#V_i + \#W_i) + d)^{O(d^2)}$.

Moreover, the running time of the algorithm is

$$\mathrm{poly}(\#I, \max_{i\in I} \#J_i, (\max_{i\in I}(\#V_i + \#W_i) + d)^{d^3}, \langle M\rangle).$$

Lemma 5.10. *Let $M_k = \bigcup_{i\in I_k}(K(V_i, W_i) \setminus \bigcup_{j\in J_i} K(V_j, W_j))$, with $k \in \{1, 2\}$ and $I_1 \cap I_2 = \emptyset$. We have*

$$M_1 \cap M_2 = \textstyle\bigcup_{k\in K}(K(U_k, Y_k) \setminus \bigcup_{\ell\in L_k} K(U_\ell, Y_\ell))$$

where, given $\#\mathcal{P}$ to be the maximal cardinality of a set in $\{V_i, W_i, V_j, W_j : i \in I_1 \cup I_2, j \in J_i\}$, we have

- $\#K \leq \#I_1 \cdot \#I_2$ *and* $\#L_k \leq 2 \cdot \max_{i\in I_1\cup I_2} \#J_i$;
- $\#U_k, \#Y_k, \#U_\ell, \#Y_\ell \leq (2(\#\mathcal{P} + d))^{O(d^2)}$; *and*
- $\langle U_k\rangle, \langle Y_k\rangle, \langle U_\ell\rangle, \langle Y_\ell\rangle \leq O(d^4) \cdot \max_{k\in\{1,2\}}\langle M_k\rangle$.

Moreover, all sets U_j, Y_j, U_ℓ and Y_ℓ can be computed in time

$$\mathrm{poly}(\#I_1, \#I_2, \max_{i\in I_1\cup I_2} \#J_i, \max_{k\in\{1,2\}}\langle M_k\rangle, (\#\mathcal{P} + d)^{d^2}).$$

Lemma 5.11. *Let* $M = \bigcup_{i \in I}(K(V_i, W_i) \setminus \bigcup_{j \in J_i} K(V_j, W_j))$ *be an* $\mathbb{R}$*-semilinear set, and let* $D \subseteq [1, d]$*. We have,*

$$\pi_D(M) = \bigcup_{i \in I}(K(\pi_D(V_i), \pi_D(W_i)) \setminus \bigcup_{\ell \in L_i} K(U_\ell, Y_\ell))$$

$\mathbb{R}$*-semilinear set where for every* $i \in I$ *and* $\ell \in L_i$*,*

- $\#L_i \leq (\#V_i + \#W_i + 2d)^{d^2}$*;*
- $\#U_\ell, \#Y_\ell \leq 2(\#V_i + \#W_i + 2d)^{d^2}$*;*
- $\langle U_\ell \rangle, \langle Y_\ell \rangle \leq O(d^4) \cdot \max(\langle V_i \rangle, \langle W_i \rangle)$*.*

Moreover, such a representation can be computed in time

$$\mathrm{poly}(\#I, \max_{i \in I}(\#J_i + 1), \langle M \rangle, (\#\mathcal{P} + d)^{d^2}),$$

where $\#\mathcal{P} \coloneqq \max_{i \in I, j \in J_i}(\#V_i, \#W_i, \#V_j, \#W_j)$*.*

Lemma 5.12. *Let* $M_k = \bigcup_{i \in I_k}(K(V_i, W_i) \setminus \bigcup_{j \in J_i} K(V_j, W_j))$*, with* $k \in \{1, 2\}$ *and* $I_1 \cap I_2 = \emptyset$*. We have,*

$$M_1 \cup M_2 = \bigcup_{i \in I_1 \cup I_2}(K(V_i, W_i) \setminus \bigcup_{j \in J_i} K(V_j, W_j)),$$

which can be computed in time $\max_{k \in \{1,2\}} O(\#I_k \cdot \#\mathcal{P}) \cdot \langle M_k \rangle$*, where* $\#\mathcal{P} \coloneqq \max_{i \in I_1 \cup I_2, j \in J_i}(\#V_i, \#W_i, \#V_j, \#W_j)$*.*

6 The VC dimension of LRA and LIA

We recall the notion of VC dimension [18, Ch. 3]. Let $(X, \mathcal{F})$ be a *set system* consisting of a set X and a family $\mathcal{F}$ of subsets of X. We say that $\mathcal{F}$ *shatters* $A \subseteq X$ if for every $A' \subseteq A$ there is $S \in \mathcal{F}$ such that $S \cap A = A'$. The largest cardinality k of some $A \subseteq X$ shattered by $\mathcal{F}$ is the *VC dimension* of $\mathcal{F}$, written as $\mathrm{VC}(\mathcal{F}) = k$, which may be infinite.

Given a first-order formula $\Phi(\boldsymbol{x}; \boldsymbol{y})$, in the structure $\mathcal{M}$ with universe M, open in $n + m$ free variables $\boldsymbol{x}$ and $\boldsymbol{y}$ partitioned into groups of $n \geq 1$ *objects* $\boldsymbol{x}$ and $m \geq 1$ *parameters* $\boldsymbol{y}$, for a given $\boldsymbol{y} \in M^m$ we define

$$\mathcal{S}_{\boldsymbol{y}} \coloneqq \{\boldsymbol{x} \in M^n : \mathcal{M} \models \Phi(\boldsymbol{x}, \boldsymbol{y})\}.$$

We associate to $\Phi(\boldsymbol{x}; \boldsymbol{y})$ the family $\mathcal{S}_\Phi \coloneqq \{\mathcal{S}_{\boldsymbol{y}} : \boldsymbol{y} \in M^m\}$. The formula Φ *"does not have the independence property"* (Φ is NIP for short) if $\mathcal{S}_\Phi$ has finite VC dimension. More generally, the structure $\mathcal{M}$ is NIP if every partitioned first-order formula in $\mathcal{M}$ is NIP.

In this section, we establish upper bounds on the VC dimension for both LRA and LIA. We recall that the length of a formula from LIA or LRA is the number of symbols required to write it down, assuming that numbers are encoded in binary.

Theorem 6.1. *Every formula* Φ *of LRA has VC dimension that is at most exponential in the length of* Φ*.*

Theorem 6.2. *Every formula* Φ *of LIA has VC dimension that is at most doubly exponential in the length of* Φ*.*

These upper bounds have simple matching lower bounds. For LRA, it is known from [20, Lec. 23] that there is a formula $\mathrm{div}_n(x, y)$, of length polynomial in $n \in \mathbb{N}$, that is satisfied whenever $x, y \in \mathbb{N}$ with $0 \leq x \leq y < 2^{2^n}$, and x divides y. By considering x as an object and y as a parameter, $\mathrm{div}_{(n+1)}(x, y)$ shatters the set $\mathbb{P}_{2^n}$ of prime numbers below 2^n. By the prime number theorem, $\#\mathbb{P}_{2^n}$ is $\Theta(2^{n - \log n})$, i.e., it is exponential in the length of $\mathrm{div}_{(n+1)}(x, y)$, and the product of the primes in $\mathbb{P}_{2^n}$ is less than $2^{2^{n+1}}$. Then, each subset $\{p_1, \dots, p_k\} \subseteq \mathbb{P}_{2^n}$ is obtained through the parameter $y = \prod_{i=1}^{k} p_i$. Similarly, for LIA, [20, Lec. 24] defines a formula $\mathrm{div}_n(x, y)$, of length polynomial in $n \in \mathbb{N}$, that is satisfied whenever x divides y and $0 \leq x \leq y \leq \ell_n$, where ℓ_n is the product of all primes below 2^{2^n}; thus $\ell_n \leq 2^{c 2^{2^n}}$ for some constant $c > 0$. With x as object and y as parameter, this formula shatters the set of all primes below 2^{2^n}.

6.1 The VC dimension of linear real arithmetic

To derive an upper bound on the VC dimension of LRA, we consider the analogous problem of bounding the VC dimension of an $\mathbb{R}$-semilinear set. Similarly to the definition of VC dimension for a first-order theory, given a set $M \subseteq \mathbb{R}^{n+m}$ with $n+m$ dimensions, where the first $n \geq 1$ dimensions are called *objects* and the last $m \geq 1$ are called *parameters*, we define sets $\mathcal{S}_{\boldsymbol{y}} \coloneqq \{\boldsymbol{x} \in \mathbb{R}^n : (\boldsymbol{x}, \boldsymbol{y}) \in M\}$ for a particular choice $\boldsymbol{y} \in \mathbb{R}^m$ of the m parameters, and associate M to the family $\mathcal{S}_M = \{\mathcal{S}_{\boldsymbol{y}} : \boldsymbol{y} \in \mathbb{R}^m\}$. Define $\mathrm{VC}(M) \coloneqq \mathrm{VC}(\mathcal{S}_M)$.

Whenever M is an $\mathbb{R}$-semilinear set, we show that its VC dimension is polynomial in the dimension d and logarithmic in the number of its components and the maximum cardinality of its generators.

Theorem 6.3. *Let* $M = \bigcup_{i \in I}\left(K(V_i, W_i) \setminus \bigcup_{j \in J_i} K(V_j, W_j)\right)$ *be an* $\mathbb{R}$*-semilinear set of dimension* $d = n + m$ *partitioned into* $n \geq 1$ *objects and* $m \geq 1$ *parameters. Then,* $\mathrm{VC}(M)$ *is bounded by* $6 \cdot (d+1)^2 \cdot \log\left(\#I \cdot d \cdot \max_{i \in I}(\#V_i + \#W_i + 1)\right)$*.*

Thanks to Corollary 5.4, Theorem 6.3 suffices to prove the upper bound to the VC dimension of LRA in Theorem 6.1.

The key insight that leads to this result is depicted in Figure 1. Pick a set V of objects, in the figure $V = \{v_1, v_2\}$. Each object yields a hyperplane h that, when intersected with M, generates an $\mathbb{R}$-semilinear set. We project all these intersections coming from the different objects in V on the parameter space $\mathbb{R}^m$, and build a set $\mathcal{H}$ of hyperplanes that carves out all the convex polyhedra appearing in the $\mathbb{R}$-semilinear set resulting from this projection. The hyperplanes in $\mathcal{H}$ divide the parameter space into regions with a fundamental property: every two parameters $\boldsymbol{y}_1$ and $\boldsymbol{y}_2$ belonging to the same region satisfy $\mathcal{S}_{\boldsymbol{y}_1} \cap V = \mathcal{S}_{\boldsymbol{y}_2} \cap V$. This implies that, if M shatters V, the set $\mathcal{H}$ divides $\mathbb{R}^m$ into at least $2^{\#V}$ regions. By relying on Proposition 3.5, we show that the number of these regions is at most $\#V^d \cdot \alpha$, where α is a quantity that depends on the descriptional complexity of M. As $f(n) = 2^n$ grows faster than $g(n) = n^d \cdot c$, this allows us to derive an upper bound on the maximal set V that $\mathcal{S}_M$ shatters.

Let us now formalise the ideas above. Consider the set $\mathcal{H}(M) \coloneqq \bigcup_{i \in I} \mathcal{H}(V_i, W_i)$, where $\mathcal{H}(V_i, W_i)$ is a set of hyperplanes of dimension $n + m - 1$ carving out $K(V_i, W_i)$. In

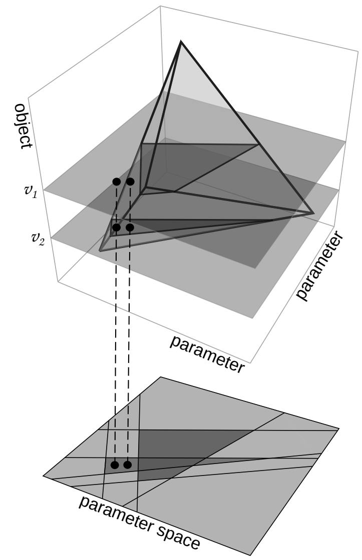

Figure 1. A tetrahedron intersected with hyperplanes coming from selecting two objects v_1 and v_2 (above). Supporting hyperplanes of these intersections split the parameter space into regions (below). Parameters $\boldsymbol{y}_1$ and $\boldsymbol{y}_2$ belonging to the same region satisfy $\mathcal{S}_{\boldsymbol{y}_1} \cap \{v_1, v_2\} = \mathcal{S}_{\boldsymbol{y}_2} \cap \{v_1, v_2\}$.

M, for every $i \in I$ and $j \in J_i$, the polyhedron $K(V_j, W_j)$ is a face of $K(V_i, W_i)$, and so $\mathcal{H}(M)$ also carves out $K(V_j, W_j)$. By Proposition 3.1, $\#\mathcal{H}(M) \leq \#I \cdot (2d + \max_{i \in I}(\#V_i + \#W_i)^d)$.

Assume that $\mathcal{S}_M$ shatters a set $V = \{\boldsymbol{v}_1, \dots, \boldsymbol{v}_k\} \subseteq \mathbb{R}^n$ of size $k \geq 1$. We derive an upper bound on k. For each $\boldsymbol{v} \in V$, we define the m dimensional affine subspace

$$A_{\boldsymbol{v}} \coloneqq \{(\boldsymbol{v}, \boldsymbol{y}) : \boldsymbol{y} \in \mathbb{Z}^m\} = \{\boldsymbol{v}\} \times \mathbb{R}^m.$$

By looking at the topmost part of Figure 1, these affine subspaces correspond to the two hyperplanes in light grey.

We construct a set $\mathcal{H}_{\boldsymbol{v}}$ of non-trivial intersections between $A_{\boldsymbol{v}}$ and hyperplanes in $\mathcal{H}(M)$. First, observe that for every $h \in \mathcal{H}(M)$ one of the following holds:

- $\emptyset \neq (A_{\boldsymbol{v}} \cap h) \neq A_{\boldsymbol{v}}$: In this case, $\dim(A_{\boldsymbol{v}} \cap h) = m - 1$;
- $A_{\boldsymbol{v}} \subseteq h$: In this case, $\dim(A_{\boldsymbol{v}} \cap h) = \dim(A_{\boldsymbol{v}}) = m$; or
- $A_{\boldsymbol{v}} \cap h = \emptyset$: In this case, $\dim(A_{\boldsymbol{v}} \cap h) = -1$.

The dimensional analysis of the second and third cases is trivial. For the dimensional analysis of the first case, pick $\boldsymbol{w} \in A_{\boldsymbol{v}} \cap h$ and $\boldsymbol{u} \in A_{\boldsymbol{v}} \setminus h$. Consider the sets $A_{\boldsymbol{v}} - \boldsymbol{w} \coloneqq A_{\boldsymbol{v}} + \{-\boldsymbol{w}\}$ and $h - \boldsymbol{w}$, which are subspaces of $\mathbb{R}^d$. We have

$$\dim(\operatorname{aff}(\{\boldsymbol{w}, \boldsymbol{u}\} + h)) = \dim(\operatorname{aff}(\{\boldsymbol{0}, \boldsymbol{u} - \boldsymbol{w}\} + (h - \boldsymbol{w})).$$

Since $\boldsymbol{u} - \boldsymbol{w}$ does not belong to $h - \boldsymbol{w}$, we conclude that $\operatorname{aff}(\{\boldsymbol{0}, \boldsymbol{u} - \boldsymbol{w}\} + (h - \boldsymbol{w})) = 1 + (n + m - 1) = d$, which in turn implies $\dim((A_{\boldsymbol{v}} - \boldsymbol{w}) + (h - \boldsymbol{w})) = \dim(A_{\boldsymbol{v}} + h) = d$. Given that for every two subspaces U and W we have

$$\dim(U \cap W) = \dim U + \dim W - dim(U + W),$$

we obtain $\dim((A_{\boldsymbol{v}} - \boldsymbol{w}) \cap (h - \boldsymbol{w})) = \dim(A_{\boldsymbol{v}} \cap h) = m - 1$.

Define $\mathcal{H}_{\boldsymbol{v}} \coloneqq \{A_{\boldsymbol{v}} \cap h : \dim(A_{\boldsymbol{v}} \cap h) = m - 1, h \in \mathcal{H}(M)\}$. Notice that each $A_{\boldsymbol{v}} \cap h$ in $\mathcal{H}_{\boldsymbol{v}}$ is a hyperplane for the affine subspace $A_{\boldsymbol{v}}$ and that all points in $A_{\boldsymbol{v}} \cap h$ are of the form $(\boldsymbol{v}, \boldsymbol{y}) \in \mathbb{R}^{n+m}$, for some $\boldsymbol{y} \in \mathbb{R}^m$.

We define an equivalence relation $\sim_{\mathcal{H}}$ of finite index, on the parameter space $\mathbb{R}^m$, such that $\boldsymbol{y}_1 \sim_{\mathcal{H}} \boldsymbol{y}_2$ implies $\mathcal{S}_{\boldsymbol{y}_1} \cap V = \mathcal{S}_{\boldsymbol{y}_2} \cap V$. Intuitively, $\sim_{\mathcal{H}}$ splits $\mathbb{R}^m$ as shown in the bottom-most part of Figure 1. We first build analogous relations a single $\boldsymbol{v} \in V$. Let $\boldsymbol{v} = (v_1, \dots, v_n)$ and $\mathcal{H}_{\boldsymbol{v}} = \{A_{\boldsymbol{v}} \cap h_1, \dots, A_{\boldsymbol{v}} \cap h_r\}$, where each hyperplane h_i is given by $\mathfrak{h}_i \colon \boldsymbol{a}_i \cdot \boldsymbol{x} = c_i$. Let $\sim_{\mathcal{H}_{\boldsymbol{v}}} \subseteq \mathbb{R}^m \times \mathbb{R}^m$ be the equivalence relation given by:

$$\boldsymbol{y}_1 \sim_{\mathcal{H}_{\boldsymbol{v}}} \boldsymbol{y}_2 \text{ iff for every } i \in [1, r], \\ \operatorname{sgn}(\boldsymbol{a}_i \cdot (\boldsymbol{v}, \boldsymbol{y}_1) - c_i) = \operatorname{sgn}(\boldsymbol{a}_i \cdot (\boldsymbol{v}, \boldsymbol{x}_2) - c_i).$$

Lemma 6.4. *Consider $\boldsymbol{y}_1, \boldsymbol{y}_2 \in \mathbb{R}^m$ such that $\boldsymbol{y}_1 \sim_{\mathcal{H}_{\boldsymbol{v}}} \boldsymbol{y}_2$. Then, $(\boldsymbol{v}, \boldsymbol{y}_1) \in M$ if and only if $(\boldsymbol{v}, \boldsymbol{y}_2) \in M$.*

This lemma follows from the definition of $A_{\boldsymbol{v}}$ and $\mathcal{H}(M)$.

We define $\sim_{\mathcal{H}} \coloneqq \bigcap_{\boldsymbol{v} \in V} \sim_{\mathcal{H}_{\boldsymbol{v}}}$, which enjoys three properties discussed in the following lemma, and that are proved by Lemma 6.4 together with Proposition 3.5 plus the fact that $\mathcal{H}(M)$ carves out each polyhedron $K(V_i, W_i)$ in M.

Lemma 6.5. *Let $M = \bigcup_{i \in I} M_i \subseteq \mathbb{R}^{n+m}$ be an $\mathbb{R}$-semilinear set, where $M_i = \big(K(V_i, W_i) \setminus \bigcup_{j \in J_i} K(V_j, W_j)\big)$ and the $d = n + m$ dimensions are partitioned into $n \geq 1$ objects and $m \geq 1$ parameters. Let $V \subseteq \mathbb{R}^n$ be a finite set of objects. There is an equivalence relation $\sim_{\mathcal{H}}$ satisfying:*

1. *given $i \in I$, $\boldsymbol{y}_1, \boldsymbol{y}_2 \in \mathbb{R}^m$ and $\boldsymbol{v} \in V$, if $\boldsymbol{y}_1 \sim_{\mathcal{H}} \boldsymbol{y}_2$, then $(\boldsymbol{v}, \boldsymbol{y}_1) \in M_i$ if and only if $(\boldsymbol{v}, \boldsymbol{y}_2) \in M_i$;*
2. *given $i \in I$, $\boldsymbol{y}_1, \boldsymbol{y}_2 \in \mathbb{R}^m$ and $\boldsymbol{v} \in V$, if $\boldsymbol{y}_1 \sim_{\mathcal{H}} \boldsymbol{y}_2$, then $(\boldsymbol{v}, \boldsymbol{y}_1) \in \operatorname{aff} M_i$ if and only if $(\boldsymbol{v}, \boldsymbol{y}_2) \in \operatorname{aff} M_i$;*
3. *the number of equivalence classes of $\sim_{\mathcal{H}}$ is bounded by $k^d \cdot 2^d \cdot \#I^d \cdot \big(2d + \max_{i \in I}(\#V_i + \#W_i)^d\big)^d + 1$.*

By definition, Property (1) of Lemma 6.5 implies that, for every two parameters $\boldsymbol{y}_1 \sim_{\mathcal{H}} \boldsymbol{y}_2$, we have $\mathcal{S}_{\boldsymbol{y}_1} \cap V = \mathcal{S}_{\boldsymbol{y}_2} \cap V$. Then, Property (3) implies that $\#\{\mathcal{S}_{\boldsymbol{y}} \cap V : \boldsymbol{y} \in \mathbb{R}^m\}$ is bounded by $k^d \cdot 2^d \cdot \#I^d \cdot \big(2d + \max_{i \in I}(\#V_i + \#W_i)^d\big)^d + 1$. Since we are assuming that $\mathcal{S}_M$ shatters V, it follows that

$$2^k \leq k^d \cdot 2^d \cdot \#I^d \cdot \big(2d + \max_{i \in I}(\#V_i + \#W_i)^d\big)^d + 1. \qquad (\dagger)$$

We conclude that proof of Theorem 6.3 by analysing the inequality above. First, we show a general result for similar system of inequalities, which we later instantiate to our case.

Lemma 6.6. *Consider $k, d, \alpha \in \mathbb{R}$ satisfying*

$$k^d \cdot \alpha \geq 2^k, \quad d \geq 1, \quad \alpha \geq 2, \quad k \geq 1.$$

Then, $k \leq 2 \cdot (\log \alpha + d \log(2 \cdot d))$.

Plugging Equation (†) in Lemma 6.6 yields the following bound on k, which implies Theorem 6.3.

Lemma 6.7. *Consider k from Equation (†). We have,*

$$k \leq 6(d+1)^2 \cdot \log\big(\#I \cdot d \cdot \max_{i \in I}(\#V_i + \#W_i + 1)\big).$$

6.2 The VC dimension of Presburger arithmetic

We now move to Presburger arithmetic, and adapt the proof technique used for Theorem 6.1 in order to establish Theorem 6.2, as well as upper bounds to the VC dimension of semilinear sets. Throughout this section, given a set $M \subseteq \mathbb{Z}^{n+m}$ of dimension $n+m$, where the first $n \geq 1$ dimensions are called *objects* and the last $m \geq 1$ are called *parameters*, we define sets $\mathcal{S}_{\boldsymbol{y}} := \{\boldsymbol{x} \in \mathbb{Z}^n : (\boldsymbol{x}, \boldsymbol{y}) \in M\}$ for a particular choice $\boldsymbol{y} \in \mathbb{Z}^m$ of the m parameters, and associate M to the family $\mathcal{S}_M = \{\mathcal{S}_{\boldsymbol{y}} : \boldsymbol{y} \in \mathbb{Z}^m\}$. Then, $\mathrm{VC}(M) := \mathrm{VC}(\mathcal{S}_M)$.

We establish an upper bound on the VC dimension of a semilinear set $M \subseteq \mathbb{Z}^d$: it is exponential, but only in d.

Theorem 6.8. *Let $M = \bigcup_{i \in I} L(B_i, P_i)$ be a semilinear set in dimension $d = n + m$ partitioned into $n \geq 1$ objects and $m \geq 1$ parameters. Then, $\mathrm{VC}(M) \leq \alpha \log \alpha$ with α in*

$$O(\#I \cdot (\max_{i \in I} \#P_i + 1)^{d+1} (d+1)^6 \log((\max_{i \in I} \|B_i\|) \cdot \max_{i \in I} \|P_i\|^2)).$$

To prove Theorem 6.8, the main issue is handling the discrete behaviour of the various hybrid-linear sets. Here, we employ a simple yet effective approach that consists in first establishing an upper bound to the VC dimension of a hybrid-linear set with proper period set, and then compute the VC dimension of a semilinear set by relying on Proposition 3.3 together with the following proposition by Sauer and Shelah [29, 31] (see also [25, Proof of Thm. 4]).

Proposition 6.9. *Let $S_1, \ldots, S_t \subseteq \mathbb{Z}^{n+m}$ be any t sets with $\mathrm{VC}(\mathcal{S}_{S_i}) = k_i$. If T is any Boolean combination of $S_1, \ldots, S_t$, then $\mathrm{VC}(T)$ is bounded by $O((\sum_{i=1}^t k_i) \cdot \log(\sum_{i=1}^t k_i))$.*

We start by deriving an upper bound on the VC dimension of a hybrid-linear set with proper period set. To do so, we require the notion of integer lattices.

A set $S \subseteq \mathbb{Z}^d$ is an *(integer) lattice* whenever it is of the form $S = \Lambda(P) := P \cdot \mathbb{Z}^{\#P}$ with $P \subseteq \mathbb{Z}^d$ proper. Notice that for our purposes we do not require $\dim S$ to be d.

Below, let us fix $L = L(B, P) \subseteq \mathbb{Z}^{n+m}$, with P proper, having $d = n + m$ dimensions partitioned into $n \geq 1$ objects and $m \geq 1$ parameters. Let us assume that $\mathcal{S}_L$ shatters a set $V = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k\} \subseteq \mathbb{Z}^n$ of size $k \geq 1$. Following the lemma below, the strategy to bound k becomes clear: it is sufficient to add to the strategy employed in Section 6.1 an analysis on how the VC dimension increases in the presence of the integer lattice $\Lambda(P)$.

Lemma 6.10. *For P proper, $L(\boldsymbol{b}, P) = K(\boldsymbol{b}, P) \cap (\boldsymbol{b} + \Lambda(P))$.*

Proof. As P is proper, every $\boldsymbol{v} \in \mathbb{R}^d$ has at most one $\boldsymbol{\lambda} \in \mathbb{R}^{\#P}$ s.t. $\boldsymbol{v} = \boldsymbol{b} + P \cdot \lambda$. Then, the lemma follows as $\mathbb{N} = \mathbb{R}_+ \cap \mathbb{Z}$. □

We consider the $\mathbb{R}$-semilinear set $\bigcup_{\boldsymbol{b} \in B} K(\boldsymbol{b}, P)$ and, by Lemma 6.5, construct an equivalence relation $\sim_{\mathcal{H}}$ such that:

1. given $\boldsymbol{b} \in B$, $\boldsymbol{y}_1, \boldsymbol{y}_2 \in \mathbb{R}^m$ and $\boldsymbol{v} \in V$, if $\boldsymbol{y}_1 \sim_{\mathcal{H}} \boldsymbol{y}_2$, then $(\boldsymbol{v}, \boldsymbol{y}_1) \in K(\boldsymbol{b}, P)$ if and only if $(\boldsymbol{v}, \boldsymbol{y}_2) \in K(\boldsymbol{b}, P)$;
2. given $\boldsymbol{b} \in B$, $\boldsymbol{y}_1, \boldsymbol{y}_2 \in \mathbb{R}^m$ and $\boldsymbol{v} \in V$, if $\boldsymbol{y}_1 \sim_{\mathcal{H}} \boldsymbol{y}_2$, then $(\boldsymbol{v}, \boldsymbol{y}_1) \in \operatorname{aff} K(\boldsymbol{b}, P)$ if and only if $(\boldsymbol{v}, \boldsymbol{y}_2) \in \operatorname{aff} K(\boldsymbol{b}, P)$;
3. the number of equivalence classes of $\sim_{\mathcal{H}}$ is bounded by $k^d \cdot 2^{2d} \cdot \#B^d \cdot (d+1)^{d^2} + 1$ (given that $\#P \leq d$).

Given two parameters $\boldsymbol{y}_1, \boldsymbol{y}_2 \in \mathbb{Z}^m$, we write $\boldsymbol{y}_1 \sim_\Lambda \boldsymbol{y}_2$ whenever $(\boldsymbol{0}, \boldsymbol{y}_1) - (\boldsymbol{0}, \boldsymbol{y}_2) \in \Lambda(P)$, with $\boldsymbol{0} \in \mathbb{Z}^n$. It is easy to see that $\sim_\Lambda$ is an equivalence relation. The following two lemmas show how to refine $\sim_{\mathcal{H}}$ to account for the lattice $\Lambda(P)$.

Lemma 6.11. *Let $\boldsymbol{y}_1, \boldsymbol{y}_2 \in \mathbb{Z}^n$ such that $\boldsymbol{y}_1 \,(\sim_{\mathcal{H}} \cap \sim_\Lambda)\, \boldsymbol{y}_2$, and let $\boldsymbol{v} \in V$. Then, $(\boldsymbol{v}, \boldsymbol{y}_1) \in L$ if and only if $(\boldsymbol{v}, y_2) \in L$.*

Proof. We show the left to right direction, as the other direction follows by symmetry. Suppose $(\boldsymbol{v}, \boldsymbol{y}_1) \in L$, and thus $(\boldsymbol{v}, \boldsymbol{y}_1) \in L(\boldsymbol{b}, P)$ for some $\boldsymbol{b} \in B$. Since P is proper, from Lemma 6.10, $(\boldsymbol{v}, \boldsymbol{y}_1) \in K(\boldsymbol{b}, P)$ and $(\boldsymbol{v}, \boldsymbol{y}_1) \in (\boldsymbol{b} + \Lambda(P))$; and to conclude the proof it suffices to show that $(\boldsymbol{v}, \boldsymbol{y}_2) \in K(\boldsymbol{b}, P)$ and $(\boldsymbol{v}, \boldsymbol{y}_2) \in (\boldsymbol{b} + \Lambda(P))$. The former, i.e. $(\boldsymbol{v}, \boldsymbol{y}_2) \in K(\boldsymbol{b}, P)$, follows directly from $\boldsymbol{y}_1 \sim_{\mathcal{H}} \boldsymbol{y}_2$. The second follows from $\boldsymbol{y}_1 \sim_\Lambda \boldsymbol{y}_2$ and $(\boldsymbol{v}, \boldsymbol{y}_1) \in (\boldsymbol{b} + \Lambda(P))$. Indeed:

$$\begin{aligned} \boldsymbol{y}_1 \sim_\Lambda \boldsymbol{y}_2 &\iff (\boldsymbol{0}, \boldsymbol{y}_1) - (\boldsymbol{0}, \boldsymbol{y}_2) \in \Lambda(P) \\ &\iff (\boldsymbol{v}, \boldsymbol{y}_1) - (\boldsymbol{v}, \boldsymbol{y}_2) \in \Lambda(P). \end{aligned}$$

Thus, $(\boldsymbol{v}, \boldsymbol{y}_2) - (\boldsymbol{v}, \boldsymbol{y}_1) = P \cdot \boldsymbol{\lambda}$ for some $\boldsymbol{\lambda} \in \mathbb{Z}^d$, and by $(\boldsymbol{v}, \boldsymbol{y}_1) \in (\boldsymbol{b} + \Lambda(P))$ there is $\boldsymbol{\mu} \in \mathbb{Z}^d$ such that $(\boldsymbol{v}, \boldsymbol{y}_1) = \boldsymbol{b} + P \cdot \boldsymbol{\mu}$. Hence $(\boldsymbol{v}, \boldsymbol{y}_2) = \boldsymbol{b} + P \cdot (\boldsymbol{\lambda} + \boldsymbol{\mu}) \in (\boldsymbol{b} + \Lambda(P))$. □

Lemma 6.12. *Let E be an equivalence class of $\sim_{\mathcal{H}}$. Either*

- *for every $\boldsymbol{y} \in E$, $\mathcal{S}_{\boldsymbol{y}} \cap V = \emptyset$; or*
- *the relation $\sim_\Lambda$ partitions $E \cap \mathbb{Z}^m$ into at most $(2d \cdot \|P\|)^d$ equivalence classes.*

Assuming $\dim \operatorname{span}(P) = d$, this lemma follows from the fact that the number of equivalence classes in $\sim_\Lambda$ is $|\det P|$ [21, Lem. 2.3.14]. Appendix D contains a self-contained proof of Lemma 6.12 that does not require $\dim \operatorname{span}(P) = d$.

Together, Lemmas 6.11 and 6.12 allows us to derive a bound on the number of distinct intersections $\mathcal{S}_{\boldsymbol{y}} \cap V$ across all parameters $\boldsymbol{y} \in \mathbb{Z}^m$.

Lemma 6.13. *The cardinality of $\{\mathcal{S}_{\boldsymbol{y}} \cap V : \boldsymbol{y} \in \mathbb{Z}^m\}$ is bounded by $k^d \cdot 2^{2d+2} \cdot (d+1)^{d^2+1} \cdot (\#B \cdot \|P\|)^d$.*

Since we are assuming that $\mathcal{S}_L$ shatters V, Lemma 6.13 yields a bound on the VC dimension of L by Lemma 6.6.

Lemma 6.14. *The VC dimension of a set $L(B, P) \subseteq \mathbb{Z}^d$ with P proper is bounded by $6 \cdot (d+1)^4 \log\big((d+1) \cdot \#B \cdot \|P\|\big)$.*

Finally, we apply Proposition 6.9 to extend Lemma 6.14 to semilinear sets with proper period sets.

Lemma 6.15. *The VC dimension of a set $\bigcup_{i \in I} L(B_i, P_i)$ where each P_i is proper is bounded by $\alpha \log \alpha$ where*

$$\alpha := 6 \cdot \#I \cdot (d+1)^4 \log\big((d+1) \cdot \#B \cdot \|P\|\big).$$

Theorem 6.8 follows by Lemma 6.15 and Proposition 3.3. Together with Theorem 5.1, this result show a doubly exponential bound on the VC dimension of semilinear expressions. Theorem 6.2 follows from Lemma 6.15 and Corollary 5.2.

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A Appendix to Section 3

Lemma 3.2. *Let $\boldsymbol{v} \in \mathbb{Z}^d$ and $M = L(B, P) \subseteq \mathbb{Z}^d$. Deciding $\boldsymbol{v} \in M$ can be performed in time $\operatorname{poly}(d^d, \langle \boldsymbol{v} \rangle, (\|B\| + \#P \cdot \|P\|)^d)$.*

Proof. Compute a decomposition $M = \bigcup_{i \in I} L(C_i, Q_i)$ such that all Q_i are linearly independent subsets of P, $\#I \leq (\#P)^d$ and $\|C_i\| \leq \|B\| + (\#P \cdot \|P\|)^d$ in time $O(\#B \cdot (d \cdot \#P \cdot \|P\|)^{d+1})$ according to Proposition 3.3. Iterate through all $i \in I$ and $\boldsymbol{c} \in C_i$. In every step, consider the system of equations $\mathfrak{S} \colon \boldsymbol{v} = \boldsymbol{c} + Q_i \cdot \boldsymbol{\lambda}, \boldsymbol{\lambda} \geq 0$. By [11], feasibility of $\mathfrak{S}$ is decidable in time $\operatorname{poly}(d^d, \langle \boldsymbol{v} \rangle, \langle B \rangle, \#P, \langle P \rangle)$. Overall, this algorithm requires $\operatorname{poly}(d^d, (\|B\| + \#P \cdot \|P\|)^d)$. □

Proposition 3.1. *Let $S = K(V, W)$, with $V, W \subseteq \mathbb{Q}^d$ finite sets. There is a system of linear inequalities $\mathfrak{S} \colon A \cdot \boldsymbol{x} \leq \boldsymbol{c}$ whose solutions define S, and such that*

- *$A \in \mathbb{Q}^{n \times d}$ with $n \leq (\#V + \#W)^d + 2d$;*
- *$\langle A \rangle, \langle \boldsymbol{c} \rangle \leq O(d^2) \cdot \langle S \rangle$.*

This system can be computed in time $(\#V + \#W)^d \cdot \operatorname{poly}(d, \langle S \rangle)$.

Proof. This is a consequence of the construction described by Schrijver [30, Thm. 10.2].

If S is full-dimensional, the system $\mathfrak{S}$ has one inequality per facet of S (which are maximal faces of S different from S). All entries of A and $\boldsymbol{c}$ are $d \times d$ minors (subdeterminants) in the matrix formed by attaching an extra row of 1s and 0s to columns of V, W (treated as matrices). The upper bound on n holds because there are at most $(\#V + \#W)^d$ many $d \times d$ minors. Each of these entries can be computed in time polynomial in d and $\langle S \rangle$. Checking whether an obtained inequality is valid (should be included in $\mathfrak{S}$) is a polynomial-time check against each element of V, W.

We remark here that the (bit) size measure is defined differently by Schrijver: the size of a vector is the sum of sizes of its components, plus one [30, Sec. 3.2]. The upper bound on the size of individual entries of A and $\boldsymbol{c}$, obtained in the proof of the theorem, is $O(d)$ multiplied by the vertex complexity (which is the minimum, over all V, W such that $S = K(V, W)$, of the maximum (bit) size of elements of $V \cup W$). This translates into $O(d) \cdot d \cdot \langle S \rangle \leq \operatorname{poly}(d \cdot \langle S \rangle)$.

To complete the proof, we discuss the case where S is not full-dimensional. As again explained by Schrijver, the one extra complication is to find at most d equalities that define the set aff S. This corresponds to finding a fundamental system of solutions for a system of equations where the coefficient matrix is again formed from V, W, and the right-hand sides are 0s and 1s. Each fundamental solution corresponds to a hyperplane, specified by an equation with rational coefficients; this can be written as a conjunction of two non-strict inequalities. The size bounds for coefficients can be taken from [30, Cor. 3.2d], and the computation can be done by Gaussian elimination, as described in [30, Sec. 3.3]. □

Proposition A.1. *Let $S \subseteq \mathbb{R}^d$ be a non-empty affine subspace given by the real solutions of $\mathfrak{S} : A \cdot \boldsymbol{x} = \boldsymbol{c}$, with $A \in \mathbb{Q}^{n \times d}$ and $\boldsymbol{c} \in \mathbb{Q}^n$. Then, $S = \boldsymbol{b} + \operatorname{span}(\boldsymbol{p}_1, \cdots, \boldsymbol{p}_t)$ where $t = \dim S$, $\boldsymbol{b}, \boldsymbol{p}_i \in \mathbb{Q}^d$ and $\langle \boldsymbol{b} \rangle, \langle \boldsymbol{p}_i \rangle \leq O(d^2(\langle A \rangle + \langle \boldsymbol{c} \rangle))$. In particular, $\boldsymbol{b}$ and all $\boldsymbol{p}_i$ can be computed in time $\operatorname{poly}(n \cdot d \cdot (\langle A \rangle + \langle \boldsymbol{c} \rangle))$.*

Proof. The numerical statement is paraphrased from [30, Cor. 3.2d]. We remark that the (bit) size measures are defined differently by Schrijver. On the one hand, his facet complexity φ corresponds to the *sum* of sizes of the coefficients in the affine constraint. As such, we can bound this quantity from above by $(d+1) \cdot (\langle A \rangle + \langle \boldsymbol{c} \rangle)$. On the other hand, Schrijver shows a bound of the form $O(d\varphi)$ on the size of individual components of generators $\boldsymbol{b}, \boldsymbol{p}_1, \ldots, \boldsymbol{p}_t$. Combining the two bounds gives $O(d^2(\langle A \rangle + \langle \boldsymbol{c} \rangle))$.

Gaussian elimination can be used to find the representation, as described in [30, Sec. 3.3]. We remark that while the number of rows, n, enters the running time bound, the size of encoding of the computed generators is independent of it. □

Proposition A.2 ([30, Thm. 8.5])**.** *Let $S = \{\boldsymbol{x} \in \mathbb{R}^d : A \cdot \boldsymbol{x} \leq \boldsymbol{c}\}$ be a non-empty polyhedron, where $A \in \mathbb{Q}^{n \times d}$ and $\boldsymbol{c} \in \mathbb{Q}^n$. Then*

$$S = \operatorname{conv}\{\boldsymbol{x}_F : F \text{ a minimal face of } S\} + \operatorname{cone}\{\boldsymbol{y}_F : F \text{ minimal proper face of } \operatorname{char.cone} S\} + \operatorname{span}\{\boldsymbol{z}_1, \ldots, \boldsymbol{z}_t\},$$

where

- *every $\boldsymbol{x}_F$ is an arbitrary element of F,*
- *every $\boldsymbol{y}_F$ is an arbitrary element of $F \setminus \operatorname{lin.space}(S)$, and*
- *$\boldsymbol{z}_1, \ldots, \boldsymbol{z}_t$ form the basis of $\operatorname{lin.space}(S)$, $t = \dim \operatorname{lin.space}(S)$.*

Given A and $\boldsymbol{c}$, such a representation with $\langle \boldsymbol{x}_F \rangle, \langle \boldsymbol{y}_F \rangle, \langle \boldsymbol{z}_i \rangle = O(d^2) \cdot (\langle A \rangle + \langle \boldsymbol{c} \rangle)$ can be computed in time $n^d \operatorname{poly}(d, \langle A \rangle + \langle \boldsymbol{c} \rangle)$.

Proof. This proposition follows [30, Thm. 8.5]. We sketch the proof of the running time bound as this is not discussed explicitly by Schrijver.

Firstly, we can find a basis of lin.space(S) and compute t using Proposition A.1, relying on the fact that lin.space $= \{\boldsymbol{x} \in \mathbb{R}^d : A \cdot \boldsymbol{x} = \boldsymbol{0}\}$.

Secondly, we can identify all minimal faces of S as follows. We recall that every minimal face is a translation (shift) of the lineality space; if the latter is $\{\boldsymbol{0}\}$, then these are vertices of S. As discussed in Section 2, minimal faces have the form $\{\boldsymbol{x} \in \mathbb{R}^d : A' \cdot \boldsymbol{x} = \boldsymbol{c}'\}$ for subsystems $A' \cdot \boldsymbol{x} \leq \boldsymbol{c}'$ of $A \cdot \boldsymbol{x} \leq \boldsymbol{c}$. Clearly, we should only be looking for subsystems where $\operatorname{rank} A' + t = d$, so we can restrict ourselves to A' with $d - t$ rows; there are at most n^d of them. This rank check is then polynomial in d and $\langle A' \rangle$. For each subsystem of this form, the set of its real solutions F is an affine subspace, indeed a translate of lin.space(S) and is therefore either included in S or disjoint from it. By the definition of the lineality space, to check this inclusion it is sufficient to check whether any concrete point from F belongs to S. Proposition A.1 gives us such a point in time poly$(()d, \langle A \rangle + \langle \boldsymbol{c} \rangle)$ and with $\langle \boldsymbol{x}_F \rangle = O(d^2) \cdot (\langle A \rangle + \langle \boldsymbol{c} \rangle)$.

Finally, we would like to find the generators of all minimal proper faces of char.cone S; if lin.space$(S) = \{\boldsymbol{0}\}$ then these are the edges of this characteristic cone. Again as discussed in Section 2, these minimal proper faces have the form $F = \{\boldsymbol{x} \in \mathbb{R}^d : A' \cdot \boldsymbol{x} = \boldsymbol{0},\ \boldsymbol{a} \cdot \boldsymbol{x} \leq 0\}$, where $\dim F = t + 1$ and lin.space $C = \{\boldsymbol{x} \in \mathbb{R}^d : A' \cdot \boldsymbol{x} = \boldsymbol{0}, \boldsymbol{a} \cdot \boldsymbol{x} = 0\}$, where $A' \cdot \boldsymbol{x} \leq \boldsymbol{c}'$ is a subsystem of $A \cdot \boldsymbol{x} \leq \boldsymbol{c}$ and $\boldsymbol{a}$ is a row of A. So, we enumerate d-tuples of rows of A, written as a sub-matrix A' and extra row $\boldsymbol{a}$, such that

$$\dim\{\boldsymbol{x} \in \mathbb{R}^d : A' \cdot \boldsymbol{x} = 0\} >$$
$$\dim\{\boldsymbol{x} \in \mathbb{R}^d : A' \cdot \boldsymbol{x} = 0, \boldsymbol{a} \cdot \boldsymbol{x} = 0\} = t.$$

For every such system, we can solve the system $A \cdot \boldsymbol{x} = 0$, $\boldsymbol{a} \cdot \boldsymbol{x} = -1$ and choose its solution as y_F; $\langle \boldsymbol{y}_F \rangle = O(d^2) \cdot (\langle A \rangle + \langle \boldsymbol{c} \rangle)$ The running time bound is the same as above. ◻

Note that $\operatorname{span}\{\boldsymbol{z}_1, \dots, \boldsymbol{z}_t\} = \operatorname{cone}\{\boldsymbol{z}_1, \dots, \boldsymbol{z}_t, -\boldsymbol{z}_1, \dots, -\boldsymbol{z}_t\}$, and thus Proposition A.2 allows to represent S as a set $K(V, W)$.

Proposition 3.3 ([6, Prop. 5])**.** *Let $S = L(B, P) \subseteq \mathbb{Z}^d$ be a hybrid linear set. Then, $S = \bigcup_{i \in I} L(C_i, Q_i)$ where*

- *$\#I \leq (\#P)^d$; $\max_{i \in I} \|C_i\| \leq \|B\| + (\#P \cdot \|P\|)^{O(d)}$; and*
- *for all $i \in I$, $Q_i \subseteq P$ and Q_i is proper.*

This representation can be computed in time

$$O(\#B \cdot (d \cdot \#P \cdot \|P\|)^{(d+1)}).$$

Proof. We follow the construction in [6, Prop. 5], which iterates over all linearly independent subsets of P, of which there are at most $(\#P)^d$ many, to obtain all Q_i. For every Q_i, the set E_i of all points lying in the intersection of the fundamental parallelepiped of Q_i and $L(\boldsymbol{0}, P)$ are then computed, and every C_i is set to $C_i := B + E_i$. Hence all C_i, Q_i can be computed in time $O(\#B \cdot (\#P)^d \cdot (d \cdot \|P\|)^{(d+1)})$. ◻

Lemma 3.4. *Let $S \subseteq \mathbb{N}^d$ be the set of all non-negative integer solutions of $\mathfrak{S}\colon A \cdot \boldsymbol{x} = 0$, with $A \in \mathbb{Z}^{n \times d}$. Then $S = L(B, P)$ such that $\langle B \rangle, \langle P \rangle \leq O(n \cdot d^3 \cdot \langle A \rangle)$, $\#P \leq d^{(k+1)}$, where $k = \operatorname{rank} A$; and B, P are computable in time* $\operatorname{poly}(d^{k+1}, \|A\|^{n \cdot k^3})$.

Proof. Let $B_1, \dots, B_m$ be all submatrices of A consisting of at most $k+1$ column vectors of A such that $\boldsymbol{0} \in \operatorname{conv}(B_i)$ for every $i \in [1, m]$. Clearly, $m \leq \binom{d}{k+1} \leq d^{(k+1)}$, and every system

$$\mathfrak{T}_i \colon B_i \cdot \boldsymbol{z} = \boldsymbol{0},\ \boldsymbol{z} \geq \boldsymbol{0},\ \boldsymbol{z} \cdot \boldsymbol{1} = 1$$

has a solution $\boldsymbol{z}_i \in \mathbb{Q}^{\#B_i}$, and let $\boldsymbol{v}_i \in \mathbb{Q}^d$ be obtained from $\boldsymbol{z}_i$ by inserting additional zero components into $\boldsymbol{z}_i$ so that $\boldsymbol{v}_i$ is non-zero at those components corresponding to columns of B_i in A. It follows from [30, Thm. 10.3] that

$$\langle \boldsymbol{v}_i \rangle = \langle z_i \rangle \leq O(n \cdot k^2 \cdot \langle A \rangle)\,.$$

Notice that every $\mathfrak{T}_i$ is an instance of linear programming, and hence $\boldsymbol{v}_i \in \mathbb{Q}^d$ can be computed in time $\operatorname{poly}(n \cdot k \cdot \langle A \rangle)$.

Let $V = \{\boldsymbol{v}_1, \dots, \boldsymbol{v}_m\}$ and let P be obtained from V by making every $\boldsymbol{v}_i$ integral by multiplying through with the denominators of the components of every $\boldsymbol{v}_i$. It follows that

$$\langle P \rangle \leq O(n \cdot k^3 \cdot \langle A \rangle)\,.$$

Moreover, P is computable in time $\operatorname{poly}(d^{(k+1)} \cdot n \cdot k^3 \cdot \langle A \rangle)$.

By [8], $\operatorname{cone}(P)$ contains all real solutions to $\mathfrak{S}$. It remains to include in B all integral solutions that are not in $L(\boldsymbol{0}, P)$. The smallest such solutions all lie in the fundamental parallelepiped F formed by P, i.e.,

$$F := \{\lambda_1 \cdot \boldsymbol{p}_1 + \cdots + \lambda_j \cdot \boldsymbol{p}_j : \boldsymbol{p}_i \in P \lambda_i \in \mathbb{Q}, 0 \leq \lambda_i < 1\}.$$

It follows that $\|B\| \leq O(d^{(k+1)} \cdot \|P\|)$ and thus

$$\langle B \rangle \leq (k+1)\log(d) + O(n \cdot k^3 \cdot \langle A \rangle).$$

Using a dynamic programming approach, B can thus be computed in time $\mathrm{poly}(d^{k+1} \cdot 2^{n \cdot k^3 \cdot \langle A \rangle})$, and hence B and P can overall be computed within the same time bounds. $\square$

Proposition 3.7. *Let $S = K(V, W)$ with $V \subseteq \mathbb{Q}^d$ and $W \subseteq \mathbb{Z}^d$, where $\#V + \#W \leq d+1$ and W is proper. Then $S^{\mathrm{op}} \cap \mathbb{Z}^d = L(B, W)$ for any half-opening S^{op} of S, and $\|B\| \leq \|V\| + 2d \cdot \|W\|$. The set B can be computed in time $(\|V\| + \|W\| + 2)^{O(d)}$.*

Proof. We reproduce the argument from [6, Lem. 10] verbatim. Whereas Chistikov and Haase assume $V \subseteq \mathbb{Z}^d$ and $\dim \mathrm{aff}\, K(V, W) = \#V + \#W - 1$, both assumptions are in fact unnecessary as we demonstrate below. We will adjust the description size bound accordingly and describe how an appropriate set B can be computed.

Suppose $V = \{\boldsymbol{v}_1, \dots, \boldsymbol{v}_k\}$ and $W = \{\boldsymbol{w}_1, \dots, \boldsymbol{w}_r\}$ with $k + r \leq d + 1$. First note that a vector $\boldsymbol{x} \in \mathbb{Z}^d$ belongs to S iff there exist numbers $\lambda_1, \dots, \lambda_k \in \mathbb{Q}_{\geq 0}$ and $\mu_1, \dots, \mu_r \in \mathbb{Q}_{\geq 0}$ with $\sum_{i=1}^k \lambda_i = 1$ such that

$$\boldsymbol{x} = \sum_{i=1}^k \lambda_i \boldsymbol{v}_i + \sum_{j=1}^r \mu_j \boldsymbol{w}_j = \left(\sum_{i=1}^k \lambda_i \boldsymbol{v}_i + \sum_{j=1}^r (\mu_j - \lfloor \mu_j \rfloor) \boldsymbol{w}_j \right) + \sum_{j=1}^r \lfloor \mu_j \rfloor \boldsymbol{w}_j .$$

Define

$$A \quad = \quad \mathbb{Z}^d \ \cap \ \left\{ \sum_{i=1}^k \lambda_i \boldsymbol{v}_i + \sum_{j=1}^r (\rho_j + \mu_j - \lfloor \mu_j \rfloor)\, \boldsymbol{w}_j : \lambda_1, \dots, \lambda_k \in \mathbb{Q}_{\geq 0}, \sum_{i=1}^k \lambda_i = 1, \rho_1, \dots, \rho_r \in \{0,1\}, \mu_1, \dots, \mu_r \in \mathbb{Q}_{\geq 0} \right\}$$

and observe that $S \cap \mathbb{Z}^d = L(A, W)$. Now let S^{op} be the half-opening of S obtained by cutting off some ℓ supporting hyperplanes, i.e., hyperplanes that contain faces of S: every such hyperplane contains at least one point of S, and the rest of S lies to one side of the hyperplane. Suppose these ℓ hyperplanes are of the form $\boldsymbol{a}_s \cdot \boldsymbol{x} = b_s$, $1 \leq s \leq \ell$.

We now claim that $S^{\mathrm{op}} \cap \mathbb{Z}^d = L(B, W)$ where $B = A \cap \{\boldsymbol{x} \in \mathbb{Q}^d : \boldsymbol{a}_s \cdot \boldsymbol{x} < b_s, 1 \leq s \leq \ell\}$. Indeed, first note that $L(B, W) \subseteq S^{\mathrm{op}} \cap \mathbb{Z}^d$. This inclusion holds because all directions $\boldsymbol{w}_1, \dots, \boldsymbol{w}_r \in W$ satisfy $\boldsymbol{a}_s \cdot \boldsymbol{w}_j \leq 0$, $1 \leq s \leq \ell$, and all vectors $\boldsymbol{v} \in B$ satisfy $\boldsymbol{a}_s \cdot \boldsymbol{v} < b_s$. Next, observe that, conversely, $S^{\mathrm{op}} \cap \mathbb{Z}^d \subseteq L(B, W)$. Indeed, take any vector $\boldsymbol{x} \in S^{\mathrm{op}} \cap \mathbb{Z}^d$; since $S^{\mathrm{op}} \subseteq S$, it can be written as $\boldsymbol{x} = \boldsymbol{y} + \boldsymbol{z}$ where $\boldsymbol{y}, \boldsymbol{z} \in \mathbb{Z}^d$,

$$\boldsymbol{y} = \sum_{i=1}^k \lambda_i \boldsymbol{v}_i + \sum_{j=1}^r (\mu_j - \lfloor \mu_j \rfloor) \boldsymbol{w}_j \qquad \text{and} \qquad \boldsymbol{z} = \sum_{j=1}^r \lfloor \mu_j \rfloor \boldsymbol{w}_j$$

for some $\lambda_1, \dots, \lambda_k \in \mathbb{Q}_{\geq 0}$, $\sum_{i=1}^k \lambda_i = 1$, and $\mu_1, \dots, \mu_r \in \mathbb{Q}_{\geq 0}$. We remark that this is where the original argument seems to rely on B being subset of $\mathbb{Z}^d$. In fact, this is unnecessary: if $\boldsymbol{x} \in S$, then it can be written as a convex combination of vectors from V plus a nonnegative combination of vectors from W. Rounding the rational coefficients in the latter combination down to the nearest integers, we obtain vectors $\boldsymbol{y}$ and $\boldsymbol{z}$ as above. Since we started from $\boldsymbol{x} \in \mathbb{Z}^d$, and since all the coefficients of vectors from W are now integers, and $W \subseteq \mathbb{Z}^d$ by assumption, we conclude that $\boldsymbol{z} \in \mathbb{Z}^d$ and therefore $\boldsymbol{y} \in \mathbb{Z}^d$ too.

We now proceed with the remainder of the proof. If $\boldsymbol{y} \in B$, then $\boldsymbol{x} \in L(B, W)$, so assume otherwise. Note that $\boldsymbol{y} \in A$ with $\rho_1 = \ldots = \rho_r = 0$ and $A \subseteq S$. Assume that $\boldsymbol{a}_s \cdot \boldsymbol{y} = b_s$, $1 \leq s \leq t$, and $\boldsymbol{a}_s \cdot \boldsymbol{y} < b_s$, $t < s \leq \ell$. Then for each $s = 1, \dots, t$, since $\boldsymbol{y} + \boldsymbol{z} \in S^{\mathrm{op}}$ and $\boldsymbol{a}_s \cdot \boldsymbol{y} = b_s$, there exists a $\boldsymbol{w}_j \in W$ such that $\boldsymbol{a}_s \cdot \boldsymbol{w}_j < 0$ and $\lfloor \mu_j \rfloor \geq 1$. Therefore, decreasing μ_j by 1 and setting $\rho_j = 1$ instead, we make sure that the newly obtained vector $\boldsymbol{y}'$ satisfies $\boldsymbol{a}_s \cdot \boldsymbol{y}' < b_s$. Repeating this procedure at most once for each direction $\boldsymbol{w}_j$, we obtain a new representation $\boldsymbol{x} = \boldsymbol{y}' + \boldsymbol{z}'$ where $\boldsymbol{y}' \in A$ and $z' \in L(\boldsymbol{0}, W)$. Therefore, we conclude that $S^{\mathrm{op}} \cap \mathbb{Z}^d = L(B, W)$, and the upper bound on the magnitude of elements holds by our choice of B and A. To compute the set B, we remark that it suffices to enumerate all vectors in $\mathbb{Z}^d$ up to the computed norm bound and, for each of them, to check whether it is included in the given S^{op}. $\square$

Proposition 3.5. *For every set H of n hyperplanes in $\mathbb{R}^d$, the relation $\sim_H$ has at most $(2n)^d + 1$ many equivalence classes.*

Proof. Let us denote by $\Phi(n, d)$ the maximal number of *regions* that can be obtained form a family J of n hyperplanes of $\mathbb{R}^d$, were each region corresponds to an equivalence class of $\sim_J$. We show $\Phi(n, d) \leq (2n)^d + 1$ by induction on (n, d).

First, we establish base cases for $n, d \in \{0, 1\}$. When $n = 0$ or $d = 0$, clearly $\Phi(n, d) = 1$ by definition, and the above inequality is satisfied. For $d = 1$ we have a line and n points in it, and so $\Phi(n, 1) \leq 2n + 1$, as expected. For $n = 1$ we have one hyperplane dividing $\mathbb{R}^d$, and thus $\Phi(1, d) = 3$, which again satisfies the above inequality.

In the inductive step, we consider $n \geq 1$, $d \geq 1$ and show that $\Phi(n+1, d+1) \leq (2(n+1))^{d+1} + 1$. We start by considering a family K of n hyperplanes. By definition, the relation $\sim_K$ has at most $\Phi(n, d+1)$ many equivalence classes. Suppose we want to insert another hyperplane h in K. The n hyperplanes in K divide the (d dimensional) hyperplane h into at most $\Phi(n, d)$ regions (simply consider h as the ambient space). Moreover, h splits each region in at most 3 parts, following the $\mathrm{sgn}(.)$ function. One of these three parts is already accounted for in $\Phi(n, d+1)$, and thus the total increase in the number of equivalence classes of $\sim_K$ caused by inserting h is $2 \cdot \Phi(n, d)$. We derive the inequality

$$\Phi(n+1, d+1) \leq \Phi(n, d+1) + 2 \cdot \Phi(n, d).$$

By recalling that $\Phi(n+1, d+1) \leq (2n)^{d+1} + 1$ and $\Phi(n, d) \leq (2n)^d + 1$ by induction hypothesis, is relatively straightforward to conclude that $\Phi(n+1, d+1) \leq (2(n+1))^{d+1} + 1$. ◻

Proposition 3.6. *Every polyhedron $K(V, W)$ has a triangulation $\mathcal{T}$ such that for all $T \in \mathcal{T}$, $T = K(V', W')$ for some $V' \subseteq V$ and $W' \subseteq W$. This $\mathcal{T}$ can be found in time $(\#V + \#W)^{O(d)} \cdot \mathrm{poly}(d, \langle V \rangle + \langle W \rangle)$.*

Proof. The existence statement is taken verbatim from [6, Lem. 8]. The algorithmic part relies on the triangulation of a convex cone C in $\mathbb{Q}^{d+1}$ generated by $\#V + \#W$ vectors. Our description summarises the approach taken by [22, Section 2.5]. We find a so-called *regular* triangulation by lifting the generators to $\mathbb{Q}^{d+2}$ by attaching an extra coordinate ("*height*"). The height coordinates are chosen non-negative. We then inspect the obtained cone $C' = \mathrm{cone}(G)$, $\#G = \#V + \#W$, in $\mathbb{Q}^{d+2}$ by looking at all its facets "visible from below", i.e., those that minimise some linear functional $f \colon \mathbb{Q}^{d+2} \to \mathbb{Q}$, $f(\boldsymbol{x}) = \boldsymbol{a} \cdot \boldsymbol{x}$, with a positive coefficient at the height coordinate. Projecting every facet of this kind back to $\mathbb{Q}^{d+1}$, we obtain a family of finitely generated cones in $\mathbb{Q}^{d+1}$. Their union is equal to C; it is known that collecting all (sub)faces of these cones will result in a triangulation of C if all the facets in question in $\mathbb{Q}^{d+2}$ are simplicial, i.e., no k-dimensional facet contains more than k vectors from G.

To verify the last condition, we proceed as follows. The cone C' has at most $(\#G)^{O(d)}$ facets; for each of them, say $\mathrm{cone}\{\boldsymbol{g}_1, \ldots, \boldsymbol{g}_k\}$, we would need to ensure that none of the other generators of C' belong to the linear span of $\boldsymbol{g}_1, \ldots, \boldsymbol{g}_k$. This can be done by writing down and computing an at most $(d+3) \times (d+3)$ determinant; see, e.g., [22, Prop. 2.2.4] where all the entries are components of vectors of G.

Importantly, we need to find an assignment of heights that ensures that all facets are simplicial. By the determinant argument above, the set of all height assignments that make a fixed facet "bad" forms a hyperplane in $\mathbb{Q}^N$, where $N = \#G$. Notice that if a facet is k-dimensional, $k \leq d+2$, then at least one of the coefficients in the equation that defines the hyperplane is indeed nonzero; that is to say, it is indeed a hyperplane.

Finding a suitable assignment of heights amounts then to picking a vector from $\mathbb{R}^N_+$ that avoids these $N^{O(d)}$ hyperplanes. A practical algorithm could be to pick this vector at random; we describe a simple trick to do this deterministically. Observe that each of the $N^{O(d)}$ equations in question has at most $d+3$ nonzero coefficients. These coefficients are rational numbers; multiplying them by the least common denominator leads to an equation where the coefficients are integers not exceeding some $B \in \mathbb{Z}$ in absolute value, for $\log B \leq O(d) \cdot (\langle V \rangle + \langle W \rangle)$. Denote the height variables by $\eta_1, \ldots, \eta_N$; by picking $\eta^*_j = (B+1)^{j-1}$ we ensure that the vector $\boldsymbol{\eta}^* = (\eta^*_1, \ldots, \eta^*_N)$ lies in none of the hyperplanes. Observe that $\langle \boldsymbol{\eta}^* \rangle \leq N \cdot O(d) \cdot (\langle V \rangle + \langle W \rangle)$.

It remains to upper-bound the running time of the entire procedure. Enumerating all possible $\#G)^{O(d)}$ facets of C', we check if all generators of C' lie on the same side of the relevant hyperplane; that is, that the hyperplane is indeed a facet. If so, we check if this facet is visible from below, looking at the coefficients in the defining hyperplane. The biggest integers our procedure takes on input are components of $\boldsymbol{\eta}^*$, bounded as above; so the overall running time is $N^{O(d)} \cdot \mathrm{poly}(d, \langle V \rangle + \langle W \rangle)$. ◻

B Appendix to Section 4

Section B.1 is devoted to proving Theorem 4.1, and Sec. B.2 to proving Theorem 4.2. Finally, Section 4.3 proves Theorem 4.3.

B.1 Proof of Theorem 4.1: missing details: Descriptional complexity of atomic polyhedra

We begin this section by stating a simple lemma with upper bounds on the descriptional complexity of hyperplanes that carve out given polyhedra $K(V_i, W_i)$. The bounds are the same as in Sec. 4.1 and follow directly from Proposition 3.1.

Lemma B.1. *For each $i \in I$, there is a set of hyperplanes $\mathcal{H}(V_i, W_i)$ carving out $K(V_i, W_i)$ and such that*

1. *$\#\mathcal{H}(V_i, W_i) \leq (\#V_i + \#W_i)^d + 2d$; and*
2. *$\langle \mathcal{H}(V_i, W_i) \rangle \leq O(\langle K(V_i, W_i) \rangle \cdot d^2)$.*

Given i, a set of equations that describe these hyperplanes can be computed in time $(\#V_i + \#W_i)^d \cdot \mathrm{poly}(d) \cdot (\langle V_i \rangle + \langle W_i \rangle)^{O(1)}$.

The inequality of the second bullet point in this lemma refers to the maximum $\langle \cdot \rangle$ measure of numbers appearing in the linear equations defining the hyperplanes $\mathcal{H}(V_i, W_i)$.

We aim at defining finite sets $V_0(M) \subseteq Q^d$ and $D_0(M) \subseteq \mathbb{Q}^d$ of points whose convex and conic combinations are sufficient to describe all atomic polyhedra induced by $\mathcal{H}(M)$.

Let first consider a set $V_0(M)$ containing, for every minimal face F of every atomic polyhedron, a point $\boldsymbol{x}_F$ in F that is small enough in the sense of the measure $\langle \cdot \rangle$. The existence of such a set is the subject of our next lemma.

Lemma B.2. *There exists a set $V_0(M)$ that contains, for every minimal face F of every atomic polyhedron, a point $\boldsymbol{x}_F$ in F, and satisfies the following bounds:*

- $\#V_0(M) \leq (\#I \cdot \max_{i \in I}(\#V_i + \#W_i))^{O(d^2)}$*; and*
- $\langle V_0(M) \rangle \leq O(d^5 \cdot \langle M \rangle)$.

Such a set can be computed in time $(\#I \cdot \max_{i \in I}(\#V_i + \#W_i + 1))^{O(d^2)} \cdot \mathrm{poly}(\langle M \rangle)$.

Proof. As already discussed, the number of atomic polyhedra is bounded by $(\#I \cdot \max_{i \in I}(\#V_i + \#W_i))^{O(d^2)}$. Then, the first bound of the lemma follows from the fact that the family of atomic polyhedra is closed under taking faces.

To prove the second bound of the lemma, consider an atomic polyhedron R given by a system $\mathfrak{R} : A \cdot \boldsymbol{x} \leq \boldsymbol{c}$, where each inequality from $\mathfrak{R}$ arises from some hyperplane in $\mathcal{H}(M)$, i.e. for every hyperplane $\boldsymbol{a} \cdot \boldsymbol{x} = c$ in $\mathcal{H}(M)$, at least one among $\boldsymbol{a} \cdot \boldsymbol{x} \leq c$ and $-\boldsymbol{a} \cdot \boldsymbol{x} \leq -c$ appears as a row of $\mathfrak{R}$. By Lemma B.1, $\langle A \rangle, \langle \boldsymbol{c} \rangle \leq O(d^2 \cdot \max_{i \in I} \langle K(V_i, W_i) \rangle)$. As every minimal face F can be characterised as the set of real solutions of $\mathfrak{R}' : A' \cdot \boldsymbol{x} = \boldsymbol{c}'$ where $A' \cdot \boldsymbol{x} \leq \boldsymbol{c}'$ is a subsystem of $\mathfrak{R}$, by Proposition A.1, the vector $\boldsymbol{x}_F$ can be chosen so that $\langle \boldsymbol{x}_F \rangle \leq O(d^3(\langle A \rangle + \langle \boldsymbol{c} \rangle)) \leq O(d^5 \cdot \langle M \rangle)$. The running time bound follows as well; we remark that $+1$ under the maximum ensures that the bound is at least $2^{O(d^2)}$, dominating all $\mathrm{poly}(d)$ factors. ◻

To define the set $D_0(M)$, we consider the set $\mathcal{H}_0(M)$ obtained from $\mathcal{H}(M)$ by shifting each of its hyperplanes so that it passes through the origin. Formally,

$$\mathcal{H}_0(M) \coloneqq \{\boldsymbol{a} \cdot \boldsymbol{x} = 0 : (\boldsymbol{a} \cdot \boldsymbol{x} = c) \in \mathcal{H}(M)\}.$$

Then, $D_0(M) \subseteq \mathbb{Q}^d$ is defined as the smallest set such that:

- $D_0(M)$ contains a basis of $\mathbb{R}^d$;
- if $\boldsymbol{v} \in D_0(M)$ then $-\boldsymbol{v} \in D_0(M)$; and
- for every $\{h_1, \dots, h_r\} \subseteq \mathcal{H}_0(M)$ where $1 \leq r \leq d - 1$, there are $\boldsymbol{v}_1, \dots, \boldsymbol{v}_{d-r} \in D_0(M)$ such that
$$\mathrm{span}\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_{d-r}\} = \textstyle\bigcap_{1 \leq i \leq r} h_i.$$

The last condition states that $D_0(M)$ contains, for every non-empty intersection of hyperplanes from $\mathcal{H}_0(M)$, vectors that span this intersection.

In the lemma below, $\langle \mathcal{H}_0(M) \rangle$ is the maximum number of bits required to encode a coefficient appearing in a hyperplane of $\mathcal{H}_0(M)$, and it is thus bounded by $\langle \mathcal{H}(M) \rangle = \max_{i \in I} . \langle \mathcal{H}(V_i, W_i) \rangle$.

Lemma B.3. *The set $D_0(M)$ satisfies the following bounds:*

- $\#D_0(M) \leq O(\#\mathcal{H}_0(M)^d + d)$*; and*
- $\langle D_0(M) \rangle \leq O(d^3 \langle \mathcal{H}_0(M) \rangle)$.

and can be constructed in time $(\#\mathcal{H}_0(M)^d + d)^{O(d)} \cdot \langle \mathcal{H}_0(M) \rangle^{O(1)}$.

Proof. For every $\{h_1, \dots, h_r\} \subseteq \mathcal{H}_0(M)$ with $r < d$, consider the corresponding system of r equations $\mathfrak{S} \colon A \cdot \boldsymbol{x} = \boldsymbol{0}$. Whenever this system has a solution, by Proposition A.1, there there are $\boldsymbol{w}_1, \dots, \boldsymbol{w}_t \in \mathbb{Q}^d$, with $t = \dim(\bigcap_{1 \leq i \leq r} h_i) \leq d - 1$, and $\langle \boldsymbol{w}_i \rangle \leq O(d^3 \langle A \rangle)$ such that $\mathrm{span}\{\boldsymbol{w}_1, \dots, \boldsymbol{w}_t\} = \bigcap_{1 \leq i \leq r} h_i$.

For all $\{h_1, \dots, h_r\} \subseteq \mathcal{H}_0(M)$, we stipulate that $D_0(M)$ contains these vectors $\boldsymbol{w}_1, \dots, \boldsymbol{w}_t$, a base of $\mathbb{R}^d$, and is closed under taking opposites. Overall, as $r < d$, we conclude that $D_0(M)$ has at most $2 \cdot (\#\mathcal{H}_0(M)^d + d)$ elements. The bound on the running time follows from Proposition A.1 too. ◻

We are now ready to describe each atomic polyhedron as a combination of elements in $V_0(M)$ and $D_0(M)$:

Lemma B.4. *Let $R \subseteq \mathbb{R}^d$ be an atomic polyhedron induced by $\mathcal{H}(M)$. Then, $R = K(V, W)$ with $V \subseteq V_0(M)$ and $W \subseteq D_0(M)$.*

Proof. Let R be an atomic polyhedron given by a system $\mathfrak{R} \colon A \cdot \boldsymbol{x} \leq \boldsymbol{c}$, where each inequality from $\mathfrak{R}$ arises from some hyperplane in $\mathcal{H}(M)$. By Proposition A.2,

$$R = \mathrm{conv}\{\boldsymbol{x}_F : F \text{ a minimal face of } R\} + \mathrm{cone}\{\boldsymbol{y}_F : F \text{ minimal proper face of } \mathrm{char.cone}\, R\} + \mathrm{span}\{\boldsymbol{z}_1, \dots, \boldsymbol{z}_t\},$$

where every $\boldsymbol{x}_F$ is an arbitrary element of F, every $\boldsymbol{y}_F$ is an arbitrary element of $F \setminus \mathrm{lin.space}(R)$, and $\boldsymbol{z}_1, \dots, \boldsymbol{z}_t$, with $t = \dim(\mathrm{lin.space}(R))$, are generators of $\mathrm{lin.space}(R)$. By definition, each point $\boldsymbol{x}_F$ can be picked from $V_0(M)$. Then, let us set $V = \{\boldsymbol{x}_F : F \text{ a minimal face of } R\} \subseteq V_0(M)$.

Next, we show that every $\boldsymbol{y}_F$ can be chosen from $D_0(M)$. To this end, let F be a minimal proper face of $\mathrm{char.cone}(R)$ given by $\mathfrak{F} \colon A' \cdot \boldsymbol{x} = \boldsymbol{0}, \boldsymbol{a} \cdot \boldsymbol{x} \leq 0$ and let t be the dimension of the lineality space of R. Now $D_0(M)$ contains a set $U = \{\boldsymbol{v}_1, \dots, \boldsymbol{v}_{t+1}\}$ of linearly independent vectors generating the set of solutions of $A' \cdot \boldsymbol{x} = \boldsymbol{0}$. Clearly, we have $\boldsymbol{a} \cdot \boldsymbol{v}_i \neq 0$ for some $\boldsymbol{v}_i$ as otherwise U would not be linearly independent. Thus, we can choose y_F as either $\boldsymbol{v}_i$ or $-\boldsymbol{v}_i$, both of which are contained in $D_0(M)$. Finally, by definition, $D_0(M)$ also contains a basis of the lineality space of R. Observing that $\operatorname{span} Z = \operatorname{cone}(Z \cup -Z)$, the statement follows. $\square$

Together, Lemmas B.1 to B.4 imply that each atomic polyhedra satisfies the properties (1) and (2). For the combination of description size and running time bounds, we use inequalities $\#\mathcal{H}_0(M) \leq \#\mathcal{H}(M)$ and $\langle \mathcal{H}_0(M) \rangle \leq \langle \mathcal{H}(M) \rangle$. This completes the proof of Theorem 4.1.

B.2 Proof of Theorem 4.2

Let the union of cones N be given. By Theorem 4.1, there is a splitter $\mathcal{R} = \{R_1, \dots, R_t\}$ for $\{K(\boldsymbol{v}, W_i)\}_{\boldsymbol{v} \in V_i, i \in I}$ that has the following properties:

1. for each $j \in J := [1, t]$, $R_j = K(E_j, F_j)$ where $E_j, F_j \subseteq \mathbb{Q}^d$ and $\langle E_j \rangle \leq O(d^5) \cdot \langle N \rangle$;
2. $t, \#(\bigcup_{j \in J} E_j) \leq (\#I \cdot (\max_{i \in I} \#V_i) \cdot \max_{i \in I}(1 + \#W_i) + d)^{O(d^2)}$.

This splitter can be found in time $(\sum_{i \in I} \#V_i \cdot \max_{i \in I}(\#W_i + 2))^{O(d^2)} \cdot \operatorname{poly}(\langle N \rangle)$. Here, we omitted the bounds that Theorem 4.1 yields on the cardinality and bit-length of F_j, as they can be greatly improved. Indeed, tracing back the construction of F_j, we see that, by Lemma B.4 and B.3, $F_j \subseteq D_0(N)$ where $D_0(N)$ is a set of rational vectors satisfying the following bounds:

- $\#D_0(N) \leq O(\#\mathcal{H}_0(N)^d + d)$; and
- $\langle D_0(N) \rangle \leq O(d^3 \langle \mathcal{H}_0(N) \rangle)$,

where $\mathcal{H}_0(N)$ is the set containing all the supporting hyperplanes of the cones $K(\boldsymbol{v}, W_i)$ (for all $i \in I$ and $\boldsymbol{v} \in V_i$), shifted to the origin. As such, $\mathcal{H}_0(N) = \mathcal{H}_0(N')$ where $N' = \bigcup_{i \in I} K(\boldsymbol{0}, W_i)$; the key point being that $\mathcal{H}_0(N)$ does not depend on the base points of the set N.

Lemma B.5. *The set $\mathcal{H}_0(N)$ satisfies the following bounds:*

1. *$\#\mathcal{H}_0(N) \leq \#I \cdot ((\max_{i \in I} \#W_i + 1)^d + 2d)$; and*
2. *$\langle \mathcal{H}_0(N) \rangle \leq O(d^2 \cdot \max_{i \in I} \langle W_i \rangle)$;*

and can be constructed in time $\#I \cdot \max_{i \in I}((\#W_i + 1)^d \cdot \operatorname{poly}(d \cdot \langle W_i \rangle))$.

Proof. Let $S = K(\boldsymbol{0}, W)$ be one of the cones that generate the set N' defined above. We consider a set $\mathcal{H}(S)$ of hyperplanes that carves out S and derive $\#\mathcal{H}(S) \leq ((\#W + 1)^d + 2d)$ and $\langle \mathcal{H}(S) \rangle \leq O(\langle W \rangle \cdot d^2)$, which is sufficient to show the lemma. Both bounds follow by Proposition 3.1, which yields a system of inequalities $\mathfrak{S} \colon A \cdot \boldsymbol{x} \leq \boldsymbol{c}$ defining S, such that $A \in \mathbb{Q}^{n \times d}$ with $n \leq (\#W + 1)^d + 2d$ and $\langle A \rangle \leq O(\langle W \rangle \cdot d^2)$. Since $\boldsymbol{0}$ is the vertex of the cone S, we have $A\boldsymbol{0} = \boldsymbol{c}$ and so $\boldsymbol{c} = \boldsymbol{0}$. Then, $\mathcal{H}(S)$ is given by the set of those hyperplanes $\boldsymbol{a} \cdot \boldsymbol{x} = 0$ such that $\boldsymbol{a} \cdot \boldsymbol{x} \leq 0$ is a row of $\mathfrak{S}$. The running time bound is again from Proposition 3.1. $\square$

Lemma B.5 implies bounds on $\#(\bigcup_{j \in [1,t]} F_j)$ and $\langle F_j \rangle$ that do not depend on the base points of N. We force integrality of these vectors by multiplying each vector in $D_0(N)$ by the least common multiple of its denominators. Overall, the following bounds are observed:

Lemma B.6. *The sets F_j used to describe the atomic polyhedra induced by $\mathcal{H}(N)$ can be chosen so that $F_j \subseteq \mathbb{Z}^d$ and*

- *$\langle F_j \rangle \leq O(d^6 \cdot \max_{i \in I} \langle W_i \rangle)$ and*
- *$\#(\bigcup_{j \in [1,t]} F_j) \leq (\#I \cdot \max_{i \in I} \#W_i + d)^{O(d^2)}$*

and can be computed in time $\#I^{O(d)} \cdot (\max_{i \in I} \#W_i + 2)^{O(d^2)} \cdot \max_{i \in I} \langle W_i \rangle^{O(1)}$.

Theorem 4.2 now follows from Lemma B.2, Lemma B.6, Lemma B.5, and Lemma B.4, combined with the bounds from the beginning of the present subsection.

B.3 Proof of Theorem 4.3: missing details

Proof of Lemma 4.4. Consider $i \in I$ such that $\boldsymbol{p} \in \operatorname{cone}(P_i)$. Recall that Carathéodory's theorem states that, for all $\boldsymbol{v} \in \mathbb{Q}^d$ and $W \subseteq \mathbb{Q}^d$, if $\boldsymbol{v} \in \operatorname{cone}(W)$ then there is a linearly independent subset $W' \subseteq W$ such that $\boldsymbol{v} \in \operatorname{cone}(W')$. Hence, there is a set $P_i' \subseteq P_i$ of linearly independent vectors such that $\boldsymbol{p} \in \operatorname{cone}(P_i')$. Notice that $\#P_i' \leq d$. We consider the system of equations $P_i' \cdot \boldsymbol{x} = \boldsymbol{p}$, which by Proposition A.1 has a solution $\boldsymbol{x}_i \in \mathbb{Q}^{\#P_i'}$ such that $\langle \boldsymbol{x}_i \rangle \leq O(d^3(\langle P_i' \rangle + \langle \boldsymbol{p} \rangle))$. Since P_i' is a set of linearly independent vectors, $\boldsymbol{x}_i$ is the unique solution to the system of equations $P_i' \cdot \boldsymbol{x} = \boldsymbol{p}$. Then, since $\boldsymbol{p} \in \operatorname{cone}(P_i)$, all entries of $\boldsymbol{x}_i$ are nonnegative rational numbers. Multiplying through by all denominators of $\boldsymbol{x}_i$, it follows that $\lambda_i \cdot \boldsymbol{p} \in L(\boldsymbol{0}, P_i)$ for some $\lambda_i \in \mathbb{N}$ such that $\langle \lambda_i \rangle \leq O(d^4(\langle P_i' \rangle + \langle \boldsymbol{p} \rangle)) \leq O(d^{10} \max_{i \in I} \langle P_i \rangle)$, where the last inequality follows from Lemma B.6, as $\boldsymbol{p} \in F$. For all $i \in I$ such that $\boldsymbol{p} \notin \operatorname{cone}(P_i)$, set $\lambda_i = 1$. Defining λ as $\prod_{i \in I} \lambda_i$ yields the desired properties.

The running time of the computation can be upper-bounded as follows. To find the subset P_i', we can try enumerate subsets of P_i of cardinality at most d; their number does not exceed $(\#P_i)^{d+1}$. For each subset, we need to check linear independence and then solve a system of linear equations; these operations are polynomial by, e.g., Proposition A.1. Multiplying k integers of n bits each takes time $O(k \log k \cdot n^2) = O(k^2 n^2)$. We do this once for $k = d$ denominators (possibly for each $i \in I$), and then for $k = \#I$ hybrid linear sets. This completes the proof.

Proof of Lemma 4.5: missing details. For Property (ii), given $A \in \mathcal{A}_{k+1} \setminus \mathcal{A}_k$ such that $A = K(E_j, F_j')$, we recall that the triangulation $\mathcal{T}$ of A is a set of generalised $(k+1)$-dimensional simplices of the form $K(V, W)$ where $\#V + \#W = k + 2$ and, again by Proposition 3.7, $V \subseteq E_j$ and $W \subseteq F_j$. This means that

$$\#\mathcal{T} \leq (\#E_j + \#F_j')^{O(d)} \leq (\#I \cdot \max_{i\in I} \#B_i \cdot \max_{i\in I}(1 + \#P_i) + d)^{O(d^3)},$$

where the last inequality follows directly from the bounds $\#E_j \leq (\#I \cdot (\max_{i\in I} \#B_i) \cdot \max_{i\in I}(1 + \#P_i) + d)^{O(d^2)}$ and $\#F_j' \leq (\#I \cdot \max_{i\in I} \#P_i + d)^{O(d^2)}$ (see the definition of splitters and Theorem 4.2). This implies that, when considering all atomic polyhedra in $\mathcal{A}_{k+1} \setminus \mathcal{A}_k$, at most $\#(\mathcal{A}_{k+1} \setminus A_k) \cdot (\#I \cdot \max_{i\in I} \#B_i \cdot \max_{i\in I}(1+\#P_i)+d)^{O(d^3)}$ generalised $(k+1)$-dimensional simplices are introduced by the triangulation. By induction hypothesis we have $\#\mathcal{C}_k \leq \#\mathcal{A}_k \cdot (\#I \cdot \max_{i\in I} \#B_i \cdot \max_{i\in I}(1 + \#P_i) + d)^{O(d^3)}$, which is enough to conclude that $\#\mathcal{C}_{k+1} \leq \#\mathcal{A}_{k+1} \cdot (\#I \cdot \max_{i\in I} \#B_i \cdot \max_{i\in I}(1 + \#P_i) + d)^{O(d^3)}$, as $\mathcal{A}_{k+1} = \mathcal{A}_k \cup (\mathcal{A}_{k+1} \setminus \mathcal{A}_k)$.

To establish Property (i), first recall that by Proposition 3.8 for every T_1^{op} and T_2^{op} in a maximal half-opening of a triangulation we have $T_1^{\text{op}} \cap T_2^{\text{op}} = \emptyset$, and by induction hypothesis $C \cap C' = \emptyset$ for every two distinct $C, C' \in \mathcal{C}_k$. Therefore, it is sufficient to establish $C \cap C' = \emptyset$ for every two distinct C and C' in $\mathcal{C}_{k+1}$ such that C belongs to the maximal half-opening of the triangulation $\mathcal{T}$ of some $A \in \mathcal{A}_{k+1} \setminus \mathcal{A}_k$ and C' is a subset of an atomic polyhedron $A' \in \mathcal{A}_{k+1}$ different form A. Notice that, by our choice of $\mathcal{C}_{k+1}$, the set C is not fully contained in any atomic polyhedron from $\mathcal{A}_k$. Therefore, in particular, C is not fully contained in any facet G of A since G must belong to to $\mathcal{A}_k$, because faces of atomic polyhedra are atomic polyhedra themselves. However, since C is taken from the maximal half-opening of A, it is equal to the relative interior of some face of A; the only possible scenario is then that C is disjoint from all the facets of A (being the relative interior of A itself).

To prove $C \cap C' = \emptyset$ for the set C' as described above, recall that $C' \subseteq A'$ for some other atomic polyhedron A'. Every non-empty intersection of atomic polyhedra is also an atomic polyhedron (as they form a polyhedral complex), and so is in particular the set $A \cap A'$. Since $\dim A' < k + 1$ and $\dim A = k + 1$, their intersection must be contained in some facet G of A. But then $C \cap C' \subseteq (C \cap A) \cap A' = C \cap G$, and the set on the right is empty by the argument above.

It remains to estimate the running time for the computation described above. For each of the t atomic polyhedra, we compute a triangulation by Proposition 3.6 and then compute the intersection of elements of its maximal half-opening with $\mathbb{Z}^d$ using Proposition 3.7. The former operation takes time

$$(\#E_j + \#F_j')^{O(d)} \cdot \text{poly}(d, \langle E_j \rangle + \langle F_j' \rangle),$$

and the latter time

$$(\|C_j\| + \|Q_j\| + 2)^{O(d)}.$$

The total time is thus upper bounded by

$$\#\mathcal{T} \cdot (\#(\bigcup_{j\in[1,t]} F_j) + \#(\bigcup_{j\in[1,t]} E_j) + \|C_j\|)^{O(d)} = (\max_{i\in I}(\#B_i + \#P_i) + \|M\|)^{O(d^{11}) \cdot \#I}.$$

We remark that this time dominates the computation of the set $\widehat{F}$ by Lemma 4.4.

Lemma B.7. *There are sets $Z_1, \ldots, Z_m$ partitioning $\mathbb{Z}^d$, with $m \leq (\#I \cdot \max_{i\in I} \#B_i \cdot \max_{i\in I}(1+\#P_i)+d)^{O(d^3)}$, s.t. for all $j \in [1, m]$:*

- *$Z_j = L(C_j, Q_j)$ where $C_j, Q_j \subseteq \mathbb{Z}^d$, and Q_j is proper;*
- *$\|C_j\| \leq \|M\|^{O(d^{10}\#I)}$ and $Q_j \subseteq \widehat{F}$;*
- *for all $i \in I$, $\boldsymbol{b} \in B_i$, $Z_j \subseteq K(\boldsymbol{b}, P_i)$ or $Z_j \cap K(\boldsymbol{b}, P_i) = \emptyset$.*

All these sets can be computed in time $(\max_{i\in I}(\#B_i + \#P_i) + \|M\|)^{O(d^{11}) \cdot \#I}$.

Proof. The statement follows from Lemma 4.5 by setting every Z_i to some $C \cap \mathbb{Z}^d$ with $C \in \mathcal{C}_d$. Indeed, the fact that $Z_1, \ldots, Z_m$ is a partition of $\mathbb{Z}^d$ follows from Properties (i) and (iii) for the case $k = d$. The upper bound on m follows from Property (ii) together with the fact that $\#\mathcal{A}_d = \#\mathcal{A} = t \leq (\#I \cdot (\max_{i\in I} \#B_i) \cdot \max_{i\in I}(1 + \#P_i) + d)^{O(d^2)}$. The three bullet points in Lemma B.7

follow from Properties (iv) and (v) together with property (S1) of the splitter $\{R_1, \dots, R_t\}$. The last of these points requires an argument, which we now provide.

Consider a set Z_j that arose as the intersection of $\mathbb{Z}^d$ with some $C \in \mathcal{C}_{k+1}$. This C got added to $\mathcal{C}_{k+1}$ as an element of the maximal half-opening of some atomic polyhedron $R \in \mathcal{A}_{k+1} \setminus \mathcal{A}_k$. In particular, $C \subseteq R$. Now take some $K(\boldsymbol{b}, P_i)$ for $\boldsymbol{b} \in B_i$. By property (S1) of the splitter, the intersection $G = R \cap K(\boldsymbol{b}, P_i)$ is either empty or equal a face of R. If $G =$ or $G = R$, then C (and so Z_j) is disjoint from resp. contained in $K(\boldsymbol{b}, P_i)$, as required; so it remains to consider the scenario in which G is a proper face of R. Observe that if $C \subseteq G$, then $Z_j \subseteq C \subseteq K(\boldsymbol{b}, P_i)$. Similarly, if $C \cap G = \emptyset$, then $C \cap K(\boldsymbol{b}, P_i) = \emptyset$. The only alternative is that $C \cap G$ and $C \setminus G$ are both non-empty. As G is a proper face of R, there is a linear function $f(\boldsymbol{x}) = \boldsymbol{a} \cdot \boldsymbol{x}$ such that G is exactly the subset of R on which f attains its maximum over R; which we denote by f^*. Since $C \cap G$ and $C \setminus G$ are both non-empty, f takes on the value f^* on at least one point of C, inside G, and moreover it takes on at least one other value, lower than f^* — this is because $C \subseteq R$. Therefore, f^* is also the maximum of f over C, attained on $C \cap G \neq C$. But this is a contradiction with our choice of C, as it is an element from the maximum half-opening of R and thus cannot include any boundary points (that is to say, C must be equal to the relative interior of some face of R; so all proper sub-faces of that face must have been removed to obtain C). ◻

It remains to observe that the sets $Z_1, \dots, Z_m$ derived in Lemma B.7 satisfy all the conditions of Theorem 4.3. In particular, property ((Z3)) and the bound on the infinity norm of the periods Q_j hold directly by definition of $\widehat{F}$ and by Lemma 4.4.

C Appendix to Section 5

Lemma 5.5. *There is an algorithm that given a set $M = \bigcup_{i \in I} L(B_i, P_i) \subseteq \mathbb{Z}^d$, where each P_i is proper, computes a family of pairs $\{(D_j, Q_j)\}_{j \in J}$ such that $\overline{M} = \bigcup_{j \in J} L(D_j, Q_j)$, and*

- *$\#J \leq ((\#I + 1) \cdot d)^{O(d^3)}$ and each Q_j is proper; and*
- *$\langle D_j \rangle, \langle Q_j \rangle \leq \#I \cdot O(d^{10}) \cdot \langle M \rangle$.*

Moreover, the running time of the algorithm is

$$(\max_{i \in I}(\#B_i + \#P_i) + \|M\|)^{O(d^{11}) \cdot \#I}.$$

Proof. Let us first discuss the correctness of this algorithm, and then analyse the descriptional complexity of the family it returns. By definition and Theorem 4.3, the $\mathbb{Z}$-splitter $\mathcal{Z}$ is a partition of $\mathbb{Z}^d$ satisfying the following properties for every $j \in J \coloneqq [1, m]$,

1. $Z_j = L(C_j, Q_j)$, where $C_j, Q_j \subseteq \mathbb{Z}^d$, and Q_j is proper;
2. for all $i \in I$, $\boldsymbol{b} \in B_i$, $Z_j \subseteq K(\boldsymbol{b}, P_i)$ or $Z_j \cap K(\boldsymbol{b}, P_i) = \emptyset$;
3. if $Z_j \subseteq K(\boldsymbol{b}_i, P_i)$ then $Q_j \subseteq L(\boldsymbol{0}, P_i)$;
4. $\|C_j\| \leq \|M\|^{O(d^{10}\#I)}$ and $\|Q_j\| \leq (\max_{i \in I}\|P_i\|)^{O(d^{10}\#I)}$;
5. $\#\left(\bigcup_{j \in J} Q_j\right) \leq ((\#I + 1) \cdot d)^{O(d^2)}$.

For the last property, notice that we relied on the fact that each P_i is a proper set, and thus $\#P_i \leq d$.

For every $j \in J$, let $E_j \coloneqq C_j \setminus M$ and $F_j \coloneqq C_j \cap M$. So,

$$\mathbb{Z}^d = \bigcup_{j \in J} L(C_j, Q_j) = \left(\bigcup_{j \in J} L(E_j, Q_j)\right) \cup \left(\bigcup_{j \in J} L(F_j, Q_j)\right).$$

To establish the correctness of the algorithm, it suffices to show that $\overline{M} = \bigcup_{j \in J} L(E_j, Q_j)$. We do so by proving that, given $j \in J$, both $L(E_j, Q_j) \cap M = \emptyset$ and $L(F_j, Q_j) \subseteq M$ hold.

The latter is straightforward: let $\boldsymbol{v} \in L(F_j, Q_j)$ and thus there is $\boldsymbol{y} \in F_j$ and $\boldsymbol{\lambda} \in \mathbb{N}^{\#Q_j}$ such that $\boldsymbol{v} = \boldsymbol{y} + Q_j \cdot \boldsymbol{\lambda}$. By definition of F_j, there is $i \in I$ and $\boldsymbol{b} \in B_i$ such that $\boldsymbol{y} \in L(\boldsymbol{b}, P_i)$. From $F_j \subseteq C_j$, and the properties (2) and (3) of $\mathcal{Z}$, we conclude that $Q_j \subseteq L(\boldsymbol{0}, P_i)$. Hence, $\boldsymbol{v} \in L(\boldsymbol{b}, P_i)$.

Let us now move to $L(E_j, Q_j) \cap M = \emptyset$. Consider $j \in J$, $i \in I$ and $\boldsymbol{b} \in B_i$. We show $L(E_j, Q_j) \cap L(\boldsymbol{b}, P_i) = \emptyset$. Below, let $Q_j = \{\boldsymbol{q}_1, \dots, \boldsymbol{q}_\ell\}$ and $\boldsymbol{P}_i = \{\boldsymbol{p}_1, \dots, \boldsymbol{p}_r\}$. *Ad absurdum*, assume there is $\boldsymbol{v} \in \mathbb{Z}^d$ in both $L(E_j, Q_j)$ and $L(\boldsymbol{b}, P_i)$. Therefore,

- $\boldsymbol{v} = \boldsymbol{y} + Q_j \cdot \boldsymbol{\lambda}$ for some $\boldsymbol{y} \in E_j$, $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_\ell) \in \mathbb{N}^\ell$;
- $\boldsymbol{v} = \boldsymbol{b} + P_i \cdot \boldsymbol{\mu}$ for some $\boldsymbol{\mu} = (\mu_1, \dots, \mu_r) \in \mathbb{N}^r$.

Moreover, from $L(E_j, Q_j) \subseteq Z_j$ and the properties (2) and (3) of $\mathcal{Z}$, we derive

- $L(E_j, Q_j) \subseteq K(\boldsymbol{b}, P_i)$;
- for all $k \in [1, \ell]$, $\boldsymbol{q}_k = P_i \cdot \boldsymbol{\xi}_k$ for some $\boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_\ell \in \mathbb{N}^r$.

From the latter, we obtain

$$\boldsymbol{v} \;=\; \boldsymbol{y} + Q_j \cdot \boldsymbol{\lambda} \;=\; \boldsymbol{y} + \textstyle\sum_{k=1}^{\ell} \lambda_k \cdot P_i \cdot \boldsymbol{\xi}_k,$$

and therefore $\boldsymbol{y} = \boldsymbol{b} + P_i \cdot (\boldsymbol{\mu} - \sum_{k=1}^{\ell} \lambda_k \cdot \boldsymbol{\xi}_k)$. By definition of $\boldsymbol{\mu}, \boldsymbol{\lambda}$ and each $\boldsymbol{\xi}_k$, the vector $\boldsymbol{\mu} - \sum_{k=1}^{\ell} \lambda_k \cdot \boldsymbol{\xi}_k$ belongs to $\mathbb{Z}^r$. We argue that, more precisely, this vector belongs to $\mathbb{N}^r$, leading to $\boldsymbol{y} \in L(\boldsymbol{b}, P_i) \subseteq M$, in contradiction with $\boldsymbol{y} \in E_j$. Indeed, from $L(E_j, Q_j) \subseteq K(\boldsymbol{b}, P_i)$ and $\boldsymbol{y} \in E_j$, we have that there is $\boldsymbol{\kappa} \in \mathbb{R}_+^r$ such that $\boldsymbol{y} = \boldsymbol{b} + P_i \cdot \boldsymbol{\kappa}$. However, by hypothesis, P_i is proper, and thus $\boldsymbol{\kappa} = \boldsymbol{\mu} - \sum_{k=1}^{\ell} \lambda_k \cdot \boldsymbol{\xi}_k \in (\mathbb{Z}^r \cap \mathbb{R}_+^r) = \mathbb{N}^r$. This ends the proof of correctness of the algorithm.

The bounds on the descriptional complexity of the family $\{(C_j, Q_j)\}_{j \in J}$ are straightforward: the bounds on $\langle C_j \rangle$ and $\langle Q_j \rangle$ follow directly from property (3) of $\mathcal{Z}$, together with $\langle S \rangle = O(\log\|S\|)$ for every finite set $S \subseteq \mathbb{Z}^d$. From property (4) together with the fact that each Q_j is a proper set, we derive $\#J = \#\boldsymbol{Q} \leq ((\#I + 1) \cdot d)^{O(d^3)}$. ◻

Lemma 5.6. *Let $M = L(C, Q) \subseteq \mathbb{Z}^d$ and $N = L(D, R) \subseteq \mathbb{Z}^d$. Then $M \cap N = L(B, P)$ such that*

- $\langle B \rangle, \langle P \rangle \leq O(d \cdot (\#Q + \#R)^3 \cdot \max\{\langle M \rangle, \langle N \rangle\}^2)$*; and*
- $\#P \leq (\#Q + \#R)^{(d+1)}$.

Moreover, B and P are computable in time

$$(\#Q + \#R)^{O(d)} \cdot \max\{\|M\|, \|N\|\}^{O(d^4)}.$$

Proof. By [6, Thm. 6], $M \cap N = L(B', P')$ such that $\|B\| \leq ((\#Q + \#R) \cdot \max\{\|M\|, \|N\|\})^{O(d)}$. To compute B', we iterate over all points with norm bounded by $\|B\| \leq ((\#Q + \#R) \cdot \max\{\|M\|, \|N\|\})^{O(d)}$. To every such point $\boldsymbol{v} \in \mathbb{Z}^d$, we invoke Lemma 3.2 to test whether $\boldsymbol{v} \in M \cap N$ in time $\mathrm{poly}(d^d, \langle \boldsymbol{v} \rangle, (\|C\| + \#Q \cdot \|Q\|)^d, (\|D\| + \#R \cdot \|R\|)^d)$. Overall, constructing B' takes time $\mathrm{poly}(d^d, (\max\{\|C\|, \|D\|\} + \max\{\#Q, \#R\} \cdot \max\{\|Q\|, \|R\|\})^d)$.

To compute P, consider the system of equations $\mathfrak{S}\colon\, Q \cdot \boldsymbol{x} - R \cdot \boldsymbol{y} \,=\, \boldsymbol{0}$. By Lemma 3.4, the solutions of $\mathfrak{S}$ are generated by $L(E, S)$ such that $\langle E \rangle, \langle S \rangle \leq O(d \cdot (\#Q + \#R)^3 \cdot \max\{\langle M \rangle, \langle N \rangle\})$ and $\#S \leq (\#Q + \#R)^{(d+1)}$, and E, S is computable in time $\mathrm{poly}((\#Q + \#R)^{(d+1)} \cdot \max\{\|M\|, \|N\|\}^{d^4})$. Let E' and S' be the projection of E and S, respectively, onto the first $\#Q$ components, and set $B := B' + Q \cdot E'$ and $P := Q \cdot S'$. We have $\langle B \rangle, \langle P \rangle \leq O(d \cdot (\#Q + \#R)^3 \cdot \max\{\langle M \rangle, \langle N \rangle\}^2)$ and B and P are computable in time $\mathrm{poly}((\#Q + \#R)^{(d+1)} \cdot \max\{\|M\|, \|N\|\}^{d^4})$. ◻

Lemma 5.8. *Let $S \subseteq \mathbb{Z}^d$ be given by the integer solutions of a linear inequality $\boldsymbol{a} \cdot \boldsymbol{x} \leq c$, $\boldsymbol{a} \in \mathbb{Z}^{1 \times d}$ and $c \in \mathbb{Z}$. Then, $S = L(B, P)$ such that $\#P \leq 2d - 1$ and $\langle B \rangle, \langle P \rangle \leq O(d^4(\langle \boldsymbol{a} \rangle + \langle c \rangle))$. Moreover, B and P can be computed in time $(\|\boldsymbol{a}\| + |c|)^{\mathrm{poly}(d)}$.*

Proof. We look at the set S' of solutions of $\boldsymbol{a} \cdot \boldsymbol{x} \leq c$ over the reals, which correspond to an half-space. By Proposition A.2

$$S' \,\cancel{=}\mathrm{conv}\{\boldsymbol{x}_F : F \text{ a minimal face of } S'\} + \mathrm{cone}\{\boldsymbol{y}_F : F \text{ minimal proper face of } \mathrm{char.cone}\, S'\} + \mathrm{span}\{\boldsymbol{z}_1, \dots, \boldsymbol{z}_t\},$$

where

- every $\boldsymbol{x}_F$ is an arbitrary element of F,
- every $\boldsymbol{y}_F$ is an arbitrary element of $F \setminus \mathrm{lin.space}(S')$, and
- $\boldsymbol{z}_1, \dots, \boldsymbol{z}_t$ are generators of $\mathrm{lin.space}(S')$,
  with $t = \dim \mathrm{lin.space}(S')$.

Since $\boldsymbol{a} \cdot \boldsymbol{x} \leq c$ is just one inequality, it has only one minimal face F, that is $\boldsymbol{a} \cdot \boldsymbol{x} = c$, its characteristic cone has only one minimal proper face G, that is $\boldsymbol{a} \cdot \boldsymbol{x} \leq 0$, and its lineality space is $\boldsymbol{a} \cdot \boldsymbol{x} = 0$ and has dimension $d - 1$. Notice that, then, the equivalence above simplifies to

$$S' = \boldsymbol{x}_F + \mathrm{span}\{\boldsymbol{z}_1, \dots, \boldsymbol{z}_{d-1}\} + \mathrm{cone}\{\boldsymbol{y}_G\},$$

where $\boldsymbol{z}_1, \dots, \boldsymbol{z}_{d-1}$ are generators of $\mathrm{lin.space}(S')$, and $\boldsymbol{y}_F$ is an arbitrary element of $G \setminus \mathrm{lin.space}(S')$. By definition, the set $\boldsymbol{x}_F + \mathrm{span}\{\boldsymbol{z}_1, \dots, \boldsymbol{z}_{d-1}\}$ corresponds to the set of solutions of $\boldsymbol{a} \cdot \boldsymbol{x} = c$, and thus all $\boldsymbol{x}_F, \boldsymbol{z}_1, \dots, \boldsymbol{z}_{d-1}$ can be computed in time $\mathrm{poly}(d \cdot (\langle \boldsymbol{a} \rangle + \langle c \rangle))$ by relying on Proposition A.1, and $\langle \boldsymbol{x}_F \rangle, \langle \boldsymbol{z}_1 \rangle, \dots, \langle \boldsymbol{z}_{d-1} \rangle \leq O(d^2(\langle \boldsymbol{a} \rangle + \langle c \rangle))$. Let $V = \{\boldsymbol{v}_1, \dots, \boldsymbol{v}_{d-1}\}$ be obtained from $\{\langle \boldsymbol{z}_1 \rangle, \dots, \langle \boldsymbol{z}_{d-1} \rangle\}$ by making every $\boldsymbol{z}_i$ integral by multiplying through with the denominators of the components of every $\boldsymbol{z}_i$. It follows that $\langle V \rangle \leq O(d^3(\langle \boldsymbol{a} \rangle + \langle c \rangle))$.

For $\boldsymbol{y}_G$, consider a base $(\boldsymbol{e}_1, \dots \boldsymbol{e}_d)$ of $\mathbb{Z}^d$. As $\mathrm{lin.space}(S')$ has dimension $d - 1$, there must be must be $i \in [1, d]$ such that both $\boldsymbol{e}_i$ and $-\boldsymbol{e}_i$ are not in $\mathrm{lin.space}(S')$. One among $\boldsymbol{e}_i$ and $-\boldsymbol{e}_i$ is included in G, and can thus be picked as $\boldsymbol{y}_G$. Finding such a vector can be done in time $\mathrm{poly}(d \cdot (\langle \boldsymbol{a} \rangle + \langle c \rangle))$.

Define $P := \{\boldsymbol{y}_G\} \cup V \cup -V$. We have $S' = K(\boldsymbol{x}_F, P)$, and $S = S' \cap \mathbb{Z}^d$. Let B to be the set of points lying in the intersection of $\mathbb{Z}^d$ with the shifted fundamental parallelepiped

$$\boldsymbol{x}_F + \{\lambda_1 \cdot \boldsymbol{p}_1 + \dots + \lambda_j \cdot \boldsymbol{p}_j : \boldsymbol{p}_i \in P, \lambda_i \in \mathbb{Q}, 0 \leq \lambda_i < 1\}.$$

which can be computed in time $O((\|V\|+1)^d)$, i.e. an upper bound to the volume of the parallelepiped. Notice that $\boldsymbol{y}_G$ being a coordinate $\boldsymbol{e}_i$ of $\mathbb{Z}^d$ (or its negation) that does not lie in lin.space(S'), is enough to conclude that B is non-empty. We have $\langle B\rangle \leq O(d^4(\langle \boldsymbol{a}\rangle + \langle c\rangle))$, and $S = L(B, P)$. ◻

C.1 Proof of Theorem 5.1

Since the algorithm for complementation of semilinear sets described in Lemma 5.5 takes in input and returns as output semilinear sets with proper period set, let us first slightly update our results for intersection and projection of semilinear sets to have a similar properties. This can be done in a straightforward way by using Proposition 3.3.

For the projection of a semilinear set, we apply Proposition 3.3 to the output of Lemma 5.7:

Lemma C.1. *Let $M = \bigcup_{i \in I_k} L(B_i, P_i) \subseteq \mathbb{Z}^d$ where each P_i is proper, and let $D \subseteq [1, d]$. Then,*

$$\pi_D(M) = \bigcup_{j \in J} L(C_j, Q_j) \subseteq \mathbb{Z}^{d-\#D},$$

where each Q_j is proper and moreover,

- $\#J \leq d^d \cdot \#I$;
- $\max_{j \in J}\langle C_i\rangle \leq d(\langle M\rangle + \log d)$; *and*
- *for all $j \in J$ there is $i \in I$ such that $Q_j \subseteq P_i$.*

Moreover, the sets C_j and D_j can be computed in time $\text{poly}(\#I, d^d, \|M\|^d)$.

Proof. Apply Lemma 5.7 followed by Proposition 3.3, the latter being applied to all hybrid-linear sets. In computing the bounds resulting from Proposition 3.3, we recall that $\langle S\rangle = O(\|S\|)$ and $\|S\| = 2^{O(\langle S\rangle)}$ for every finite set $S \subseteq \mathbb{Z}^d$. ◻

Similarly, the case of intersection follows from Lemma 5.6. and Proposition 3.3:

Lemma C.2. *Let $M_k = \bigcup_{i \in I_k} L(B_i, P_i) \subseteq \mathbb{Z}^d$, with $k \in \{1, 2\}$, $I_1 \cap I_2 = \emptyset$ and every P_i proper ($i \in I_1 \cup I_2$). We have,*

$$M_1 \cap M_2 = \bigcup_{j \in J} L(C_j, Q_j),$$

where each Q_j is proper and moreover,

- $\#J \leq \#I_1 \cdot \#I_2 \cdot (2d)^{O(d^2)}$ *and*
- $\max_{j \in J}(\langle C_j\rangle, \langle D_j\rangle) \leq O(d^5 \cdot \max(\langle M\rangle, \langle N\rangle)^2)$.

Moreover, the sets C_j and D_j can be computed in times

$$\#I_1 \cdot \#I_2 \cdot \text{poly}((2d)^d \cdot \max\{\|M\|_1, \|M\|_2\}^{d^4}).$$

Proof. For every $(i, k) \in I_1 \times I_2$, we consider the intersection $L(B_i, P_i) \cap L(B_k, P_k)$, and compute the equivalent hybrid linear set $L_{i,k}$ following Lemma 5.6. We then apply Proposition 3.3 to each $L_{i,k}$, obtaining a semilinear set $S_{i,k}$ with proper period sets. We have

$$M \cap N = \bigcup_{(i,k) \in I_1 \times I_2} S_{i,k}.$$

Running time and bounds on the resulting sets follow directly from Lemma 5.6. and Proposition 3.3. In computing the bounds after applying Proposition 3.3, we recall that $\langle S\rangle = O(\|S\|)$ and $\|S\| = 2^{O(\langle S\rangle)}$ for every finite set $S \subseteq \mathbb{Z}^d$. ◻

We are now ready to prove Theorem 5.1.

Theorem 5.1. *There is an algorithm that, given a well-defined semilinear expression s, computes a family of pairs $\{(B_i, P_i)\}_{i \in I}$ such that $[\![s]\!] = \bigcup_{i \in I} L(B_i, P_i)$. The algorithm ensures*

$$\#I \leq n^{d^{O(h)}}, \quad \langle B_i\rangle, \langle P_i\rangle \leq (\langle s\rangle + n)^{d^{O(h)}}, \quad P_i \text{ proper},$$

where $n = n_{\mathrm{p}}(s) + 2$, $d = d(s)$ and $h = h(s)$. Moreover, the running time of the algorithm is $\exp((\langle s\rangle + n)^{d^{O(h)}})$.

Proof. The algorithm simply applies bottom up the procedures given by Lemma C.1, Lemma C.2 and Lemma 5.5, together with the straightforward procedure for union of semilinear sets.

For the sake of clarity, let us consider a constant $\lambda \geq 2$ that upper bound all constants hidden by the Big O notation of Lemma C.2 and Lemma 5.5, i.e., whenever $O(n)$ appears in one of these lemmas, the actual value is at most $\lambda \cdot (n+1)$.

Below we consider the case where the semilinear expression s has, as atoms, semilinear sets with period proper, instead of (arbitrary) hybrid-linear sets. Afterwards, we derive the bounds for the (standard) semilinear expressions featuring hybrid-linear sets by applying Proposition 3.3. So, below, assume that s has atoms

$$\bigcup_{i\in I_1} L(B_i, P_i)\,,\ \ldots\ ,\ \bigcup_{i\in I_\ell} L(B_\ell, P_\ell), \qquad\qquad \ell \geq 1,$$

with every P_i proper. Recall that $d(s)$ stands for the maximal dimension of atoms in s, hence $\#P_i \leq d(s)$, and that $h(s)$ stands for the maximum number of nested operations appearing in s. We define

$$\#\mathcal{I} \coloneqq \max_{j\in[1,\ell]} \#I_j; \qquad\qquad \langle\mathcal{G}\rangle \coloneqq \max_{i\in\bigcup_j I_j}(\langle B_i\rangle, \langle P_i\rangle).$$

We show that the family of pairs $\{(C_j, Q_j)\}_{j\in J}$ computed by the algorithm and such that $[\![s]\!] = \bigcup_{j\in J} L(C_j, Q_j)$ satisfies:

(i) $\#J \leq \Omega(s)$ where $\Omega(s)$ is the function defined as $\Omega(s) \coloneqq ((\#\mathcal{I}+1)\cdot d(s))^{\lambda^{2(h(s)+1)}\cdot(d(s)^3+1)^{h(s)+1}}$;

(ii) $\max_{j\in J}(\langle C_j\rangle, \langle Q_j\rangle) \leq \lambda^{2(2^{h(s)}-1)} \cdot \Omega(s)^{2^{h(s)}-1} \cdot \langle\mathcal{G}\rangle^{2^{h(s)}} \cdot d(s)^{10(2^{h(s)}-1)}$;

(iii) each period set Q_j is proper (hence $\#Q_j \leq d(s)$).

The proof is by induction on the height of s, with standard induction hypothesis stating that the inequalities above holds for every strict subexpression of s, and by cases on the type of the operator appearing in the root of s (seeing the expression as a tree). Below, let $h \coloneqq h(s)$ and $d \coloneqq d(s)$. The base case where s is an atom, i.e. the semilinear set $\bigcup_{i\in I_1} L(B_i, P_i)$, is trivial. Here are the inductive steps:

case: $s = s_1 \cup s_2$**.** By induction hypothesis, $[\![s_1]\!] = \bigcup_{k\in K_1} L(D_k, R_k)$ and $[\![s_2]\!] = \bigcup_{k\in K_2} L(D_k, R_k)$ where

- for all $r \in \{1,2\}$, $\#K_r \leq ((\#\mathcal{I}+1)\cdot d(s_r))^{\lambda^{2(h(s_r)+1)}\cdot(d(s_r)^3+1)^{h(s_r)+1}}$;
- for all $r \in \{1,2\}$, $\max_{k\in K_r}(\langle D_k\rangle, \langle R_k\rangle) \leq \lambda^{2(2^{h(s_r)}-1)} \cdot \Omega(s_r)^{2^{h(s_r)}-1} \cdot \langle\mathcal{G}\rangle^{2^{h(s_r)}} \cdot d(s_r)^{10(2^{h(s_r)}-1)}$;
- for every $k \in K_1 \cup K_2$, the period set R_k is proper (as required by property (iii)).

By definition, $\max(d(s_1), d(s_2)) \leq d(s) = d$ and $\max(h(s_1), h(s_2)) = h(s) - 1 = h - 1$, and so $\max(\Omega(s_1), \Omega(s_2)) \leq \Omega(s)$. Then, $[\![s]\!] = \bigcup_{j\in K_1\cup K_2} L(D_j, R_j)$, which satisfies the expected properties:

(i) $\#(K_1 \cup K_2) \leq 2\cdot((\#\mathcal{I}+1)\cdot d)^{\lambda^{2h}\cdot(d^3+1)^h} \leq ((\#\mathcal{I}+1)\cdot d)^{2\cdot\lambda^{2h}\cdot(d^3+1)^h} \leq ((\#\mathcal{I}+1)\cdot d)^{\cdot\lambda^{2(h+1)}\cdot(d^3+1)^{h+1}}$;

(ii) $\max_{j\in K_1\cup K_2}(\langle D_j\rangle, \langle R_j\rangle) \leq \lambda^{2(2^h-1)} \cdot \Omega(s)^{2^h-1} \cdot \langle\mathcal{G}\rangle^{2^h} \cdot d^{10(2^h-1)}$.

case: $s = \pi_D(s')$**.** By induction hypothesis, $[\![s']\!] = \bigcup_{k\in K} L(D_k, R_k)$ where

- $\#K \leq ((\#\mathcal{I}+1)\cdot d(s'))^{\lambda^{2(h(s')+1)}\cdot(d^3+1)^{h(s')+1}}$;
- $\max_{k\in K}(\langle D_k\rangle, \langle R_k\rangle) \leq \lambda^{2(2^{h(s')}-1)} \cdot \Omega(s')^{2^{h(s')}-1} \cdot \langle\mathcal{G}\rangle^{2^{h(s')}} \cdot d(s')^{10(2^{h(s')}-1)}$;
- for every $j \in J_1 \cup J_2$, the period set R_j is proper.

By definition, $d(s') = d(s) = d$ and $d(s') = h(s) - 1 = h - 1$, and so $\Omega(s') \leq \Omega(s)$. We apply Lemma C.1, and conclude that $[\![s]\!] = \bigcup_{j\in J} L(C_j, Q_j)$ where each Q_j is proper (as required by property (iii)) and moreover

- $\#J \leq d^d \cdot \#K$;
- $\max_{j\in J}\langle C_i\rangle \leq d(\langle[\![s']\!]\rangle + \log d)$; and
- for all $j \in J$ there is $k \in K$ such that $Q_j \subseteq R_k$.

Hence, all the properties expected from $[\![s]\!]$ are satisfied:

(i) $\#J \leq d^d \cdot \#K \leq d^d \cdot ((\#\mathcal{I}+1)\cdot d)^{\lambda^{2h}\cdot(d^3+1)^h} \leq ((\#\mathcal{I}+1)\cdot d)^{\lambda^{2(h+1)}\cdot(d^3+1)^{h+1}}$;

(ii.a) from $\bigcup_{j\in J} Q_j \subseteq \bigcup_{k\in K} R_k$ we get $\max_{j\in J}\langle Q_j\rangle \leq \lambda^{2(2^h-1)} \cdot \Omega(s)^{2^h-1} \cdot \langle\mathcal{G}\rangle^{2^h} \cdot d^{10(2^h-1)}$;

(ii.b) $\max_{j\in J}\langle C_i\rangle \leq d(\langle[\![s']\!]\rangle + \log d) \leq d\left(\lambda^{2(2^{h-1}-1)} \cdot \Omega(s')^{2^{h-1}-1} \cdot \langle\mathcal{G}\rangle^{2^{h-1}} \cdot d^{10(2^{h-1}-1)} + \log d\right)$

$$\leq \lambda^{2(2^h-1)} \cdot \Omega(s)^{2^h-1} \cdot \langle\mathcal{G}\rangle^{2^h} \cdot d^{10(2^h-1)}.$$

case: $s = s_1 \cap s_2$**.** By induction hypothesis, $[\![s_1]\!] = \bigcup_{k\in K_1} L(D_k, R_k)$ and $[\![s_2]\!] = \bigcup_{k\in K_2} L(D_k, R_k)$ where

- for all $r \in \{1,2\}$, $\#K_r \leq ((\#\mathcal{I}+1)\cdot d(s_r))^{\lambda^{2(h(s_r)+1)}\cdot(d(s_r)^3+1)^{h(s_r)+1}}$;
- for all $r \in \{1,2\}$, $\max_{k\in K_r}(\langle D_k\rangle, \langle R_k\rangle) \leq \lambda^{2(2^{h(s_r)}-1)} \cdot \Omega(s_r)^{2^{h(s_r)}-1} \cdot \langle\mathcal{G}\rangle^{2^{h(s_r)}} \cdot d(s_r)^{10(2^{h(s_r)}-1)}$;
- for every $k \in K_1 \cup K_2$, the period set R_k is proper.

By definition, $\max(d(s_1), d(s_2)) \leq d(s) = d$ and $\max(h(s_1), h(s_2)) = h(s) - 1 = h - 1$, and so $\max(\Omega(s_1), \Omega(s_2)) \leq \Omega(s)$. We apply Lemma C.2, and conclude that $[\![s]\!] = \bigcup_{j\in J} L(C_j, Q_j)$ where each Q_j is proper (as required by property (iii)) and moreover,

- $\#J \leq (\#K_1 \cdot \#K_2 \cdot (2d)^{\lambda(d^2+1)})$ and

- $\max_{j \in J}(\langle C_j \rangle, \langle Q_j \rangle) \leq \lambda(d^5 \cdot \max(\langle [\![s_1]\!] \rangle, \langle [\![s_2]\!] \rangle)^2 + 1)$.

Hence, all the properties expected from $[\![s]\!]$ are satisfied:

(i) $\#J \leq \#K_1 \cdot \#K_2 \cdot 2^{\lambda(d^2+1)} \leq \Big(((\#\mathcal{I}+1) \cdot d)^{\lambda^{2h} \cdot (d^3+1)^h}\Big)^2 \cdot 2^{\lambda(d^2+1)} \leq \Big(((\#\mathcal{I}+1) \cdot d)^{\lambda^{2h} \cdot (d^3+1)^h}\Big)^3$

$\leq ((\#\mathcal{I}+1) \cdot d)^{3 \cdot \lambda^{2h} \cdot (d^3+1)^h} \leq ((\#\mathcal{I}+1) \cdot d)^{\lambda^{2h+2} \cdot (d^3+1)^h} \leq ((\#\mathcal{I}+1) \cdot d)^{\lambda^{2(h+1)} \cdot (d^3+1)^{h+1}}$;

(ii)
$$\begin{aligned}\max_{j \in J}(\langle C_j \rangle, \langle Q_j \rangle) &\leq \lambda(d^5 \cdot (\lambda^{2(2^{h-1}-1)} \cdot \Omega(s_r)^{2^{h-1}-1} \cdot \langle \mathcal{G} \rangle^{2^{h-1}} \cdot d^{10(2^{h-1}-1)})^2 + 1)\\ &\leq \lambda^{2 \cdot 2(2^{h-1}-1)+2} \cdot \Omega(s_r)^{2 \cdot 2^{h-1}-1} \cdot \langle \mathcal{G} \rangle^{2 \cdot 2^{h-1}} \cdot d^{2 \cdot 10(2^{h-1}-1)+5}\\ &\leq \lambda^{2(2^h-1)} \cdot \Omega(s)^{2^h-1} \cdot \langle \mathcal{G} \rangle^{2^h} \cdot d^{10(2^h-1)}.\end{aligned}$$

case: $s = \overline{s'}$**.** By induction hypothesis, $[\![s']\!] = \bigcup_{k \in K} L(D_k, R_k)$ where

- $\#K \leq ((\#\mathcal{I}+1) \cdot d(s'))^{\lambda^{2(h(s')+1)} \cdot (d(s')^3+1)^{h(s')+1}}$;
- $\max_{k \in K}(\langle D_k \rangle, \langle R_k \rangle) \leq \lambda^{2(2^{h(s')}-1)} \cdot \Omega(s')^{2^{h(s')}-1} \cdot \langle \mathcal{G} \rangle^{2^{h(s')}} \cdot d(s')^{10(2^{h(s')}-1)}$;
- for every $j \in J_1 \cup J_2$, the period set R_j is proper.

By definition, $d(s') = d(s) = d$ and $d(s') = h(s) - 1 = h - 1$, and so $\Omega(s') \leq \Omega(s)$. We apply Lemma 5.5, and conclude that $[\![s]\!] = \bigcup_{j \in J} L(C_j, Q_j)$ where each Q_j is proper (as required by property (iii)) and moreover

- $\#J \leq ((\#K+1) \cdot d)^{O(d^3)}$; and
- $\max_{j \in J}(\langle C_j \rangle, \langle Q_j \rangle) \leq O(\#K \cdot d^{10} \cdot \max_{k \in K}(\langle D_k \rangle, \langle R_k \rangle))$.

Hence, all the properties expected from $[\![s]\!]$ are satisfied:

(i) $\#J \leq ((\#K+1) \cdot d)^{O(d^3)} \leq \Big(\big(((\#\mathcal{I}+1) \cdot d)^{\lambda^{2h} \cdot (d^3+1)^h} + 1\big) \cdot d\Big)^{\lambda(d^3+1)} \leq \Big(2 \cdot d \cdot ((\#\mathcal{I}+1) \cdot d)^{\lambda^{2h} \cdot (d^3+1)^h}\Big)^{\lambda(d^3+1)}$

$\leq \Big(((\#\mathcal{I}+1) \cdot d)^{\lambda^{2h} \cdot (d^3+1)^h + 1}\Big)^{\lambda(d^3+1)} \leq ((\#\mathcal{I}+1) \cdot d)^{(\lambda^{2h+1} \cdot (d^3+1)^h) \cdot (\lambda(d^3+1))} \leq ((\#\mathcal{I}+1) \cdot d)^{\lambda^{2(h+1)} \cdot (d^3+1)^{h+1}}$;

Notice that this implies $\#K \leq \#J \leq \Omega(s)$, which we use for the following bound on the magnitude of C_j and Q_j.

(ii)
$$\begin{aligned}\max_{j \in J}(\langle C_j \rangle, \langle Q_j \rangle) &\leq \lambda \cdot \#K \cdot d^{10} \cdot \Big(\lambda^{2(2^{h-1}-1)} \cdot \Omega(s)^{2^{h-1}-1} \cdot \langle \mathcal{G} \rangle^{2^{h-1}} \cdot d^{10(2^{h-1}-1)}\Big) + 1\\ &\leq \lambda^2 \cdot \Omega(s) \cdot d^{10} \cdot \Big(\lambda^{2(2^{h-1}-1)} \cdot \Omega(s)^{2^{h-1}-1} \cdot \langle \mathcal{G} \rangle^{2^{h-1}} \cdot d^{10(2^{h-1}-1)}\Big)\\ &\leq \lambda^{2(2^h-1)} \cdot \Omega(s)^{2^h-1} \cdot \langle \mathcal{G} \rangle^{2^h} \cdot d^{10(2^h-1)}.\end{aligned}$$

To recap, we have just shown that the semilinear expressions s where atoms are semilinear sets with period sets that are proper is such that $[\![s]\!] = \bigcup_{j \in J} L(C_j, Q_j)$ where each Q_j is proper, and moreover,

- $\#J \leq \Omega(s) \coloneqq ((\#\mathcal{I}+1) \cdot d(s))^{\lambda^{2(h(s)+1)} \cdot (d(s)^3+1)^{h(s)+1}}$;
- $\max_{j \in J}(\langle C_j \rangle, \langle Q_j \rangle) \leq \lambda^{2(2^{h(s)}-1)} \cdot \Omega(s)^{2^{h(s)}-1} \cdot \langle \mathcal{G} \rangle^{2^{h(s)}} \cdot d(s)^{10(2^{h(s)}-1)}$.

Since $(d(s)+1) \geq 2$ and $\lambda \geq 2$ is a constant, we have $\lambda \leq (d(s)+1)^{\log \lambda} \leq (d(s)+1)^{O(1)}$. Hence, the two bounds above can be rewritten as

- $\#J \leq ((\#\mathcal{I}+1) \cdot d(s))^{(d(s)^3+1)^{O(h(s))}} \leq (\#\mathcal{I}+1)^{(d(s)+1)^{O(h(s))}}$;
- $\max_{j \in J}(\langle C_j \rangle, \langle Q_j \rangle) \leq (\#\mathcal{I}+1)^{(d(s)+1)^{O(h(s))}} \cdot \langle \mathcal{G} \rangle^{2^{h(s)}}$.

Let us now consider a (standard) semilinear expression s' with hybrid-linear sets as atoms. We first apply Proposition 3.3 in order to translate every hybrid-linear set with non-empty period set appearing in s' into an equivalent semilinear set represented with period sets that are proper. After this step, for each of these semilinear sets $\bigcup_{i \in I} L(B_i, P_i)$, we have that $\#I \leq \max(n_{\mathrm{p}}(s')^{d(s')}, 1)$ and $\max_{i \in I}(\langle B_i \rangle, \langle P_i \rangle) \leq O(d(s') \cdot (\langle s' \rangle + \log(n_{\mathrm{p}}(s')+1)))$, where we recall that $n_{\mathrm{p}}(s')$ is the maximal cardinality of a set of periods appearing in an atom of s', and $\langle s' \rangle$ stands for the maximal $\langle a \rangle$ of an atom a appearing in s. Then, from the above bounds, as well as the definition of $\#\mathcal{I}$ and $\langle \mathcal{G} \rangle$, we conclude that $[\![s']\!] = M \coloneqq \bigcup_{j \in J} L(C_j, Q_j)$ such that

- $\#J \leq (n_{\mathrm{p}}(s')^{d(s')} + 2)^{(d(s')+1)^{O(h(s'))}} \leq (n_{\mathrm{p}}(s') + 2)^{(d(s')+1)^{O(h(s'))}}$;
- $\max_{j \in J}(\langle C_j \rangle, \langle Q_j \rangle) \leq \big(n_{\mathrm{p}}(s')+2\big)^{(d(s')+1)^{O(h(s'))}} \cdot (\langle s' \rangle + \log(n_{\mathrm{p}}(s')+1))^{2^{h(s')}} \leq (\langle s' \rangle + n_{\mathrm{p}}(s') + 2)^{(d(s')+1)^{O(h(s'))}}$.

These are the bounds given in the statement of the lemma.

We now move to the running time of the procedure. Consider the semilinear expression s' above, together with the semilinear set M computed by the procedure. Since each operator of semilinear expressions are either unary or binary, the overall number of operators in s' is at most $2^{h(s')}$. This means that, to obtain an upper bound to the running time of the whole procedure, it is sufficient to see what is the running time of applying $2^{h(s')}$ operations to semilinear sets that have bounds as the one established for s' above. Let $d = d(s')$, $h = h(s')$, $n = n_{\mathrm{p}}(s')$. We have:

- for $\pi_D(.)$, by Lemma C.1, the running time is

$$\mathrm{poly}(\#J, d^d, \|M\|^d) \leq \mathrm{poly}((n+2)^{(d+1)^{O(h)}}, d^d, 2^{(\langle s'\rangle+n+2)^{(d+1)^{O(h)}}}) \leq 2^{(\langle s'\rangle+n+2)^{(d+1)^{O(h)}}}.$$

- for $\cap$, by Lemma C.2, the running time is

$$\#J \cdot \#J \cdot \mathrm{poly}((2d)^d \cdot \max\{\|M\|, \|M\|\}^{d^4}) \leq 2^{(\langle s'\rangle+n+2)^{(d+1)^{O(h)}}}.$$

- for $\overline{(.)}$, by Lemma 5.5 the running time is

$$(\max_{j\in J}(\#C_j + \#Q_j) + \|M\|)^{O(d^{11})\cdot\#I} \leq (\max_{j\in J}(\|C_j\|^d + d) + \|M\|)^{O(d^{11})\cdot\#I}$$
$$\leq \left(2^{(\langle s'\rangle+n+2)^{(d+1)^{O(h)}}}\right)^{O(d^{11})\cdot(n+2)^{(d+1)^{O(h)}}} \leq 2^{(\langle s'\rangle+n+2)^{(d+1)^{O(h)}}}.$$

Therefore, M is computed in time $2^h \cdot 2^{(\langle s'\rangle+n+2)^{(d+1)^{O(h)}}} \leq 2^{(\langle s'\rangle+n+2)^{(d+1)^{O(h)}}}$, i.e., $\exp((\langle s'\rangle + n + 2)^{(d+1)^{O(h)}})$. □

C.2 Towards Theorem 5.3: the complexity of Boolean operations and projection on $\mathbb{R}$-semilinear sets

Lemma 5.9. *There is an algorithm that given an $\mathbb{R}$-semilinear set $M = \bigcup_{i\in I}(K(V_i, W_i) \setminus \bigcup_{j\in J_i} K(V_j, W_j)) \subseteq \mathbb{R}^d$ computes a family of triples $\{(U_k, Y_k, \{(U_\ell, Y_\ell)\}_{\ell\in L_k})\}_{k\in K}$ such that*

$$\overline{M} = \textstyle\bigcup_{k\in K}(K(U_k, Y_k) \setminus \bigcup_{\ell\in L_k} K(U_\ell, Y_\ell)).$$

The algorithm ensures, for every $k \in K$ and $\ell \in K_\ell$,

- *$\#U_k, \#Y_k, \#U_\ell, \#Y_\ell \leq (\#I \cdot \max_{i\in I}(\#V_i + \#W_i) + d)^{O(d^2)}$;*
- *$\langle U_k\rangle, \langle Y_k\rangle, \langle U_\ell\rangle, \langle Y_\ell\rangle \leq O(d^5 \cdot \langle M\rangle)$; and*
- *$\#K, \#L_k \leq (\#I \cdot \max_{i\in I}(\#V_i + \#W_i) + d)^{O(d^2)}$.*

Moreover, the running time of the algorithm is

$$\mathrm{poly}(\#I, \max_{i\in I} \#J_i, (\max_{i\in I}(\#V_i + \#W_i) + d)^{d^3}, \langle M\rangle).$$

Proof. Here is the algorithm:

1: compute a splitter $\mathcal{R} = \{R_1, \dots, R_m\}$ for $\{K(V_i, W_i)\}_{i\in I}$
2: **let** $P = \emptyset$
3: **for** $R \in \mathcal{R}$ **do**
4:     **if** for all $i \in I$, if $R \subseteq K(V_i, W_i)$ then there is $j \in J_i$, $R \subseteq K(V_j, W_j)$ **then**
5:         add R to P
6: **let** $T = \emptyset$
7: **for** $R \in P$ **do**
8:     **let** $K(C, Q) = R$
9:     **let** $N = \{(U, Y) : S \in \mathcal{R}, S \text{ face of } R, S \neq R, S = K(U, Y)\}$
10:     add (C, Q, N) to T
11: **return** T

In line 1, the splitter is computed according to Theorem 4.1.

Let $\{(U_k, Y_k, \{(U_\ell, Y_\ell)\}_{\ell\in L_k})\}_{k\in K}$ be the output of the algorithm on M. First of all, notice that all the sets U_k, Y_k, U_ℓ and Y_ℓ come from the splitter of $\{K(V_i, W_i)\}_{i\in I}$ computed in line 1. Then, the bounds given to these sets are direct from Theorem 4.1.

Let us now discuss the correctness of the algorithm, and show that

$$\overline{M} = \textstyle\bigcup_{k\in K}(K(U_k, Y_k) \setminus \bigcup_{\ell\in L_k} K(U_\ell, Y_\ell)).$$

($\supseteq$): consider $\boldsymbol{x} \in \mathbb{R}^d$ such that there is $k \in K$, $\boldsymbol{x} \in K(U_k, Y_k) \setminus \bigcup_{\ell\in L_k} K(U_\ell, Y_\ell)$. Since $\mathcal{R}$ is a polyhedral complex, from line 9 we conclude that $\boldsymbol{x}$ does not belong to any face of $K(U_k, Y_k)$ that is not $K(U_k, Y_k)$ itself, i.e., $\boldsymbol{x}$ is in the *interior* of $K(U_k, Y_k)$. Moreover, notice that $K(U_k, Y_k)$ belongs to P (see line 7). *Ad absurdum*, let us suppose $\boldsymbol{x} \in M$, and thus that there is $i \in I$, $\boldsymbol{x} \in K(V_i, W_i) \setminus \bigcup_{j\in J_i} K(V_j, W_j)$. We split the proof into two cases:

- case: $K(U_k, Y_k) \subseteq K(V_i, W_i)$. Then, by $K(U_k, Y_k) \in P$, we deduce from line 4 that there is $j \in J_i$ such that $K(V_j, W_j)$. This is contradictory, as $\boldsymbol{x} \in K(V_j, W_j)$ implies $\boldsymbol{x} \notin K(V_i, W_i) \setminus \bigcup_{j\in J_i} K(V_j, W_j)$;

- case: $K(U_k, Y_k) \not\subseteq K(V_i, W_i)$. From $\boldsymbol{x} in K(V_i, W_i) \setminus \bigcup_{j \in J_i} K(V_j, W_j)$ we conclude that $K(U_k, Y_k) \cap K(V_i, W_i) \neq \emptyset$. Since $K(U_k, Y_k) \not\subseteq K(V_i, W_i)$, by definition of splitter we conclude that $K(U_k, Y_k) \cap K(V_i, W_i)$ is inside a facet of $K(U_k, Y_k)$. This is again contradictory, as it means that $\boldsymbol{x}$ is inside a facet of $K(U_k, Y_k)$, and thus not in its interior.

We deduce that $\boldsymbol{x} \notin M$, and thus $\boldsymbol{x} \in \overline{M}$.

($\subseteq$): suppose $\boldsymbol{x} \in \overline{M}$. *Ad absurdum*, suppose $\boldsymbol{x} \notin \bigcup_{k \in K}(K(U_k, Y_k) \setminus \bigcup_{\ell \in L_k} K(U_\ell, Y_\ell))$. Let us show that then there is $R \in \mathcal{R}$ such that $\boldsymbol{x} \in R$ and $R \notin P \coloneqq \{K(U_k, Y_k) : k \in K\}$ (P defined by line 4 and line 5 of the algorithm). Let δ be the smallest dimension of a polyhedron $K(U_k, Y_k) \in P$ for which $\boldsymbol{x} \in K(U_k, Y_k)$. Since $\boldsymbol{x} \notin \bigcup_{k \in K}(K(U_k, Y_k) \setminus \bigcup_{\ell \in L_k} K(U_\ell, Y_\ell))$, then there is $\ell \in J_k$ such that $\boldsymbol{x} \in K(U_\ell, Y_\ell)$. As each $K(U_\ell, Y_\ell)$ is a face of $K(U_k, Y_k)$ that is distinct from $K(U_k, Y_k$, the dimension of $K(U_\ell, Y_\ell)$ is less than δ. Since $\boldsymbol{x} \in K(U_\ell, Y_\ell)$, by definition of δ we obtain that $K(U_\ell, Y_\ell) \notin P$.

We now know that there is $R \in \mathcal{R} \setminus P$ such that $\boldsymbol{x} \in R$. Let μ be the smallest dimension of such a polyhedron, and notice that $\mu < \delta$. Pick $R \in \mathcal{R} \setminus P$ such that $\boldsymbol{x} \in R$ and $\dim R = \mu$. From line 4 of the algorithm, $R \in \mathcal{R} \setminus P$ implies that there is $i \in I$ such that $R \subseteq K(V_i, W_i)$ and for all $j \in J_i$, $R \not\subseteq K(V_j, W_j)$. So, $\boldsymbol{x} \in K(V_i, W_i)$ and, since $\boldsymbol{x} \in \overline{M}$, we must have $\boldsymbol{x} \in K(V_j, W_j)$ for some $j \in J_i$. So, $R \cap K(V_j, W_j) \neq \emptyset$. From $R \not\subseteq K(V_j, W_j)$ and the property of splitter, we obtain that $R \cap \subseteq K(V_j, W_j)$ is a subset of a facet F of R, and $F \in \mathcal{R}$. This is however contradictory: we have $\boldsymbol{x} \in F$ and $\dim F < \mu$, and so if $F \in P$ then we contradict the minimality of δ, and if $F \notin P$ we contradict the minimality of μ. Therefore, $\boldsymbol{x} \in \bigcup_{k \in K}(K(U_k, Y_k) \setminus \bigcup_{\ell \in L_k} K(U_\ell, Y_\ell))$.

What is left is to discuss the complexity of the procedure. Excluding the computation of the splitter, which by Theorem 4.1 takes time

$$(\#I \cdot \max_{i \in I}(\#V_i + \#W_i + 1))^{O(d^2)} \cdot \mathrm{poly}(\langle M \rangle),$$

the only operation that we use is essentially the inclusion of polyhedra, see 4 and 9, that we compute at most $O(\#\mathcal{R} \cdot \#I(1 + \max_{i \in I} \#J_i))$ times in 4 and $O((\#\mathcal{R})^2)$ in 9. To check $K(V, W) \subseteq K(U, Y)$, one can equivalently check whether $\dim(K(V, W) \cap K(U, Y) = \dim K(V, W)$. To this end, one first computes systems $\mathfrak{S} \colon A \cdot \boldsymbol{x} \leq \boldsymbol{c}$ and $\mathfrak{T} \colon B \cdot \boldsymbol{x} \leq \boldsymbol{d}$ of inequalities whose solutions are $S \coloneqq K(V, W)$ and $T \coloneqq K(U, Y)$, respectively. By Proposition 3.1 this can be done in time

$$(\#V + \#W)^d \cdot \mathrm{poly}(d \cdot \langle S \rangle) + (\#U + \#Y)^d \cdot \mathrm{poly}(d \cdot \langle T \rangle),$$

and moreover

- $A \in \mathbb{Q}^{n \times d}$ with $n \leq (\#V + \#W)^d + 2d$, and $\langle A \rangle, \langle \boldsymbol{c} \rangle \leq O(d^2) \cdot \langle S \rangle$;
- $B \in \mathbb{Q}^{m \times d}$ with $m \leq (\#U + \#Y)^d + 2d$, and $\langle B \rangle, \langle \boldsymbol{d} \rangle \leq O(d^2) \cdot \langle T \rangle$.

Then, we compute the affine hulls of the systems $\mathfrak{S}$ and $\mathfrak{S} \wedge \mathfrak{T} \colon A \cdot \boldsymbol{x} \leq \boldsymbol{c} \wedge B \cdot \boldsymbol{x} \leq \boldsymbol{d}$. This can be done in time polynomial in the input [30, see Remark page 170]. Afterwards, we compute the dimension of the two affine hulls by computing the rank of the resulting matrices. This can be done by Gaussian elimination, again in polynomial time on $\mathrm{poly}(\langle A \rangle, \langle B \rangle, \langle \boldsymbol{c} \rangle, \langle \boldsymbol{d} \rangle, n, m, d)$. Overall, checking whether $K(V, W) \subseteq K(U, Y)$ can be done in time

$$\mathrm{poly}(d, \langle K(V, W) \rangle, \langle K(U, Y) \rangle, (\#V + \#W)^d, (\#V + \#W)^d).$$

In the algorithm, these inclusions are checked on polyhedra from either $\mathcal{R}$ or M. By recalling that, for $K(C, Q) \in \mathcal{R}$ we have

1. $\langle C \rangle, \langle Q \rangle \leq O(d^5 \cdot \langle M \rangle)$; and
2. $\#C$ and $\#Q$ are bounded by $(\#I \cdot \max_{i \in I}(\#V_i + \#W_i) + d)^{O(d^2)}$.

each inclusion check done by the algorithm can be performed in time $\mathrm{poly}(d, \langle M \rangle, (\max_{i \in I}(\#V_i + \#W_i) + d)^{O(d^3)})$, which leads to the following complexity bounds for lines 2–11 of the algorithm:

$$\begin{aligned} & O(\#\mathcal{R} \cdot \#I(1 + \max_{i \in I} \#J_i)) \cdot O(\#\mathcal{R}^2) \cdot \mathrm{poly}(d, \langle M \rangle, (\max_{i \in I}(\#V_i + \#W_i) + d)^{O(d^3)}) \\ \leq\, & \mathrm{poly}(d, \langle M \rangle, (\max_{i \in I}(\#V_i + \#W_i) + d)^{O(d^3)}) \\ \leq\, & \mathrm{poly}(s\#I, \max_{i \in I} \#J_i, (\max_{i \in I}(\#V_i + \#W_i) + d)^{d^3}, \langle M \rangle). \end{aligned}$$

This class exceeds the running time required to compute the splitter, and it is thus an upper bound to the running time of the whole procedure. ◻

Lemma 5.10. *Let $M_k = \bigcup_{i \in I_k}(K(V_i, W_i) \setminus \bigcup_{j \in J_i} K(V_j, W_j))$, with $k \in \{1, 2\}$ and $I_1 \cap I_2 = \emptyset$. We have*

$$M_1 \cap M_2 = \textstyle\bigcup_{k \in K}(K(U_k, Y_k) \setminus \bigcup_{\ell \in L_k} K(U_\ell, Y_\ell))$$

where, given $\#\mathcal{P}$ to be the maximal cardinality of a set in $\{V_i, W_i, V_j, W_j : i \in I_1 \cup I_2, j \in J_i\}$, we have

- *$\#K \leq \#I_1 \cdot \#I_2$ and $\#L_k \leq 2 \cdot \max_{i \in I_1 \cup I_2} \#J_i$;*
- *$\#U_k, \#Y_k, \#U_\ell, \#Y_\ell \leq (2(\#\mathcal{P} + d))^{O(d^2)}$; and*

- $\langle U_k \rangle, \langle Y_k \rangle, \langle U_\ell \rangle, \langle Y_\ell \rangle \le O(d^4) \cdot \max_{k \in \{1,2\}} \langle M_k \rangle$.

Moreover, all sets U_j, Y_j, U_ℓ and Y_ℓ can be computed in time

$$\mathrm{poly}(\#I_1, \#I_2, \max_{i \in I_1 \cup I_2} \#J_i, \max_{k \in \{1,2\}} \langle M_k \rangle, (\#\mathcal{P} + d)^{d^2}).$$

Proof. We distribute intersection under unions, resulting in $M_1 \cap M_2 = \bigcup_{(i_1,i_2) \in I_1 \times I_2} P_{i_1,i_2}$ where each P_{i_1,i_2} is

$$\bigcap_{k \in \{1,2\}} (K(V_{i_k}, W_{i_k}) \setminus \bigcup_{j \in J_{i_k}} K(V_j, W_j))$$

Consider each P_{i_1,i_2}. We apply Proposition 3.1 on the polyhedra $S_1 \coloneqq K(V_{i_1}, W_{i_1})$ and $S_2 \coloneqq K(V_{i_2}, W_{i_2})$, obtaining systems of inequalities $\mathfrak{S}_k \colon A_k \cdot \boldsymbol{x} \le \boldsymbol{c}_k$ defining S_k ($k \in \{1,2\}$). Moreover,

- $A_k \in \mathbb{Q}^{n_k \times d}$ with $n_k \le (\#V_{i_k} + \#W_{i_k})^d + 2d$;
- $\langle A_k \rangle, \langle \boldsymbol{c}_k \rangle \le O(d^2) \cdot \langle S_k \rangle$;
- $\mathfrak{S}_k$ is computed in time $(\#V_{i_k} + \#W_{i_k})^d \cdot \langle S_k \rangle \cdot \mathrm{poly}(d)$.

Let $\mathfrak{S} \colon A_1 \cdot \boldsymbol{x} \le \boldsymbol{c}_1, A_2 \cdot \boldsymbol{x} \le \boldsymbol{c}_2$, which below we simply write as $\mathfrak{S} \colon A \cdot \boldsymbol{x} \le \boldsymbol{c}$. Set $n \coloneqq n_1 + n_2$, i.e., n is the number of rows of A. By Proposition A.2, the set S of solutions of $\mathfrak{S}$ as

$$S = \mathrm{conv}\{\boldsymbol{x}_F : F \text{ a minimal face of } S\} + \mathrm{cone}\{\boldsymbol{y}_F : F \text{ minimal proper face of } \mathrm{char.cone}\, S\} + \mathrm{span}\{\boldsymbol{z}_1, \dots, \boldsymbol{z}_t\},$$

where

- every $\boldsymbol{x}_F$ is an arbitrary element of F,
- every $\boldsymbol{y}_F$ is an arbitrary element of $F \setminus \mathrm{lin.space}(S)$, and
- $\boldsymbol{z}_1, \dots, \boldsymbol{z}_t$ form the bases of $\mathrm{lin.space}(S)$, with $t = \dim \mathrm{lin.space}(S)$.

Combined, all the vectors $\boldsymbol{x}_F, \boldsymbol{y}_F$ and $\boldsymbol{z}_1, \dots, \boldsymbol{z}_t$ can be computed in time $n^d \cdot \mathrm{poly}(d, \langle A \rangle + \langle \boldsymbol{c} \rangle) \le \mathrm{poly}((\#\mathcal{R} + d)^{d^2} \cdot \langle \mathcal{R} \rangle)$, where

- $\#\mathcal{R} \coloneqq \max_{i \in I_1 \cup I_2}(\#V_i, \#W_i)$; and
- $\langle \mathcal{R} \rangle \coloneqq \max_{i \in I_1 \cup I_2}(\langle V_i \rangle, \langle W_i \rangle)$.

Moreover, $\langle \boldsymbol{x}_F \rangle, \langle \boldsymbol{y}_F \rangle$ and $\langle \boldsymbol{z}_1 \rangle, \dots, \langle \boldsymbol{z}_t \rangle$ are in $O(d^4) \cdot \langle \mathcal{R} \rangle$. So, $S = K(U, Y)$, where $U \coloneqq \{\boldsymbol{x}_F : F \text{ a minimal face of } S\}$ and $Y \coloneqq \{\boldsymbol{y}_F : F \text{ minimal proper face of } \mathrm{char.cone}\, S\} \cup \{\boldsymbol{z}_1, \dots, \boldsymbol{z}_t, -\boldsymbol{z}_1, \dots, -\boldsymbol{z}_t\}$.

Notice that, overall, the number of vectors $\boldsymbol{x}_F$ is bounded by $n^d \le (2(\#\mathcal{R} + d))^{d^2}$, as they corresponds to distinct faces of S and thus obtained from $\mathfrak{S}$ by setting at most d inequalities as equalities. Similarly, the number of vectors $\boldsymbol{y}_F$ is also bounded by n^d. By definition, $P_{i_1,i_2} \subseteq S$. In order to define P_{i_1,i_2}, we now need to remove from S the faces that arise from the polyhedra in $Q \coloneqq \{K(V_j, W_j) : j \in J_{i_1} \cup J_{i_2}\}$.

Consider a $K(V_j, W_j) \in Q$. Applying Proposition 3.1 we obtain a systems of inequalities $\mathfrak{T}_j \colon B_j \cdot \boldsymbol{x} \le \boldsymbol{d}_j$ defining $K(V_j, W_j)$. Moreover,

- $B_j \in \mathbb{Q}^{n_j \times d}$ with $n_j \le (\#V_j + \#W_j)^d + 2d$;
- $\langle B_j \rangle, \langle \boldsymbol{d}_j \rangle \le O(d^2) \cdot \max(\langle V_j \rangle, \langle W_j \rangle)$;
- $\mathfrak{T}_j$ is computed in time $(\#V_j + \#W_j)^d \cdot \max(\langle V_j \rangle, \langle W_j \rangle) \cdot \mathrm{poly}(d)$.

We consider the system of inequalities $\mathfrak{S} \wedge \mathfrak{T}_j$ given by $A \cdot \boldsymbol{x} \le \boldsymbol{c} \wedge B_j \cdot \boldsymbol{x} \le \boldsymbol{d}_j$. By definition of $\mathbb{R}$-semilinear set, $K(V_j, W_j)$ is a face of either S_1 or S_2, and so whenever nonempty, the set of solutions of $\mathfrak{S} \wedge \mathfrak{T}_j$ characterise a face of S. This face does not appear in the intersection of P_{i_1,i_2}. Therefore, we have

$$P_{i_1,i_2} = S \setminus \bigcup \{T_j : T_j \neq \emptyset, T_j \text{ solutions of } \mathfrak{S} \wedge \mathfrak{T}_j, j \in J_{i_1} \cup J_{i_2}\}.$$

Finding whether $\mathfrak{S} \wedge \mathfrak{T}_j$ has a solution over $\mathbb{R}$ can be done in polynomial time in the dimension of the system and its entries, i.e., in our case, in time $\mathrm{poly}((2(d + \#\mathcal{P}))^d \cdot \langle \mathcal{P} \rangle)$, where

- $\#\mathcal{P} \coloneqq \max_{i \in I_1 \cup I_2, j \in J_i}(\#V_i, \#W_i, \#V_j, \#W_j)$; and
- $\langle \mathcal{P} \rangle \coloneqq \max_{i \in I_1 \cup I_2, j \in J_i}(\langle V_i \rangle, \langle W_i \rangle, \langle V_j \rangle, \langle W_j \rangle)$.

Once a solution for $\mathfrak{S} \wedge \mathfrak{T}_j$ has been found, we apply Proposition A.2 to obtain a representation T_j of the set of solutions of $\mathfrak{S} \wedge \mathfrak{T}_j$ as

$$T_j = \mathrm{conv}\{\boldsymbol{x}'_F : F \text{ a minimal face of } T_j\} + \mathrm{cone}\{\boldsymbol{y}'_F : F \text{ minimal proper face of } \mathrm{char.cone}\, T_j\} + \mathrm{span}\{\boldsymbol{z}'_1, \dots, \boldsymbol{z}'_t\},$$

where

- every $\boldsymbol{x}'_F$ is an arbitrary element of F,
- every $\boldsymbol{y}'_F$ is an arbitrary element of $F \setminus \mathrm{lin.space}(T_j)$, and

- $z_1', \dots, z_t'$ form the basis of lin.space(T_j),
  $t = \dim \text{lin.space}(T_j)$.

Again following Proposition A.2, this representation of T_j is such that $\langle x_F' \rangle, \langle y_F' \rangle, \langle z_i' \rangle \leq O(d^4) \cdot \langle \mathcal{P} \rangle$, and it can be computed in time $\text{poly}((\langle \mathcal{P} \rangle + 1) \cdot (\#\mathcal{P} + 1)^{d^2})$. So, $T_j = K(U_j, Y_j)$, where $U \coloneqq \{x_F' : F \text{ a minimal face of } T_j\}$ and $Y \coloneqq \{y_F' : F \text{ minimal proper face of char.cone } T_j\} \cup \{z_1, \dots, z_t, -z_1, \dots, -z_t\}$.

Recall that n and n_j are the number of rows in the systems $\mathfrak{S}$ and $\mathfrak{T}_j$, respectively. Overall, the number of vectors x_F' is bounded by $(n + n_j)^d \leq (2(\#\mathcal{P} + d))^{d(d+1)}$, as they corresponds to distinct faces of S and thus obtained from $\mathfrak{S}$ by setting at most d inequalities as equalities. Similarly, the number of vectors y_F' is also bounded by $(n + n_j)^d$.

We derived that P_{i_1,i_2} is a set of the form $K(U, Y) \setminus \bigcup_{j \in J'} K(U_j, Y_j)$ where

- $\#J' \leq \#J_{i_1} + \#J_{i_2}$,
- $\#U, \#V \leq n^d + 2d \leq (2(\#\mathcal{R} + d))^{d^2}$,
- $\langle U \rangle, \langle V \rangle \leq O(d^4) \cdot \langle \mathcal{R} \rangle$,
- $\#U_j, \#V_j \leq (2(\#\mathcal{P} + d))^{d(d+1)} + 2d$,
- $\langle U_j \rangle, \langle V_j \rangle \leq O(d^4) \cdot \langle \mathcal{P} \rangle$.

Applying this construction for every $i_1 \in I_1$ and $i_2 \in I_2$ yields the required set equivalent to $M_1 \cap M_2$. Let us now look at the running time of the algorithm.

For every $i_1 \in I_1$ and $i_2 \in I_2$,

- computing $\mathfrak{S}$ takes time $(2\#\mathcal{R})^d \langle \mathcal{R} \rangle \cdot \text{poly}(d)$
- computing $K(U, Y)$, i.e. the set of solutions of $\mathfrak{S}$, takes time $\text{poly}((\#\mathcal{R} + d)^{d^2} \cdot \langle \mathcal{R} \rangle)$,
- computing each $\mathfrak{T}_j$ takes time $(2\#\mathcal{P})^d \langle \mathcal{P} \rangle \cdot \text{poly}(d)$. There are $\#J_{i_1} + \#J_{i+2}$ many such systems.
- computing whether $\mathfrak{S} \wedge \mathfrak{T}_j$ has a solution takes time $\text{poly}((2(d + \#\mathcal{P}))^d \cdot \langle \mathcal{P} \rangle)$,.
- computing $K(U_j, Y_j)$, i.e. the set of solutions of $\mathfrak{S} \wedge \mathfrak{T}_j$ takes time $\text{poly}((\langle \mathcal{P} \rangle + 1) \cdot (\#\mathcal{P} + 1)^{d^2})$.

We conclude that computing $M_1 \cap M_2$ can be done in time

$$\#I_1 \cdot \#I_2 \cdot \max_{i \in I_1 \cup I_2} (3\#J_i) \cdot \text{poly}((\langle \mathcal{P} \rangle + 1) \cdot (\#\mathcal{P} + d)^{d^2}). \qquad \square$$

Lemma 5.11. *Let $M = \bigcup_{i \in I} (K(V_i, W_i) \setminus \bigcup_{j \in J_i} K(V_j, W_j))$ be an $\mathbb{R}$-semilinear set, and let $D \subsetneq [1, d]$. We have,*

$$\pi_D(M) = \textstyle\bigcup_{i \in I} (K(\pi_D(V_i), \pi_D(W_i)) \setminus \bigcup_{\ell \in L_i} K(U_\ell, Y_\ell))$$

$\mathbb{R}$-semilinear set where for every $i \in I$ and $\ell \in L_i$,

- $\#L_i \leq (\#V_i + \#W_i + 2d)^{d^2}$;
- $\#U_\ell, \#Y_\ell \leq 2(\#V_i + \#W_i + 2d)^{d^2}$;
- $\langle U_\ell \rangle, \langle Y_\ell \rangle \leq O(d^4) \cdot \max(\langle V_i \rangle, \langle W_i \rangle)$.

Moreover, such a representation can be computed in time

$$\text{poly}(\#I, \max_{i \in I} (\#J_i + 1), \langle M \rangle, (\#\mathcal{P} + d)^{d^2}),$$

where $\#\mathcal{P} \coloneqq \max_{i \in I, j \in J_i} (\#V_i, \#W_i, \#V_j, \#W_j)$.

Proof. Without loss of generality, we assume $D = [1, m]$ with $r \leq d$, i.e. that the projection is on the first m coordinates or $\mathbb{R}^d$. We distribute projection under unions, leading to

$$\pi_D(M) = \textstyle\bigcup_{i \in I} \pi_D(K(V_i, W_i) \setminus \bigcup_{j \in J_i} K(V_j, W_j)).$$

For $i \in I$, below we consider the component $\pi_D(N_i)$ where $N_i \coloneqq K(V_i, W_i) \setminus \bigcup_{j \in J_i} K(V_j, W_j)$.

As a corollary of Furier-Motzkin quantifier elimination method [30, Ch. 12], eliminating variables from a system of inequalities $A \cdot \boldsymbol{x} \leq \boldsymbol{b} \wedge C \cdot \boldsymbol{x} < \boldsymbol{d}$ leads to a similar system of both strict and weak inequalities. This directly implies that

$$\pi_D(N_i) = \pi_D(K(V_i, W_i)) \setminus \bigcup G$$

where G is a set of faces of $\pi_D(K(V_i, W_i))$. Not, let F be a face of $\pi_D(K(V_i, W_i))$ for which $(\mathbb{R}^m \times F) \cap N_i \neq \emptyset$. Clearly, F cannot be contained in G, as otherwise $\pi_D(N_i)$ will loose all points in F, and in particular the point $\boldsymbol{x} \in F$ such that $(\mathbb{R}^m \times \{\boldsymbol{x}\}) \cap N_i \neq \emptyset$. At the same time, given a face F' such that $(\mathbb{R}^m \times F') \cap N_i = \emptyset$, we have $\pi_D(N_i) = \pi_D(N_i) \setminus F'$. Therefore,

$$\pi_D(N_i) = \pi_D(K(V_i, W_i)) \setminus \bigcup H$$

where H is the set of faces of $\pi_D(K(V_i, W_i))$ for which we have $(\mathbb{R}^m \times F) \cap N_i = \emptyset$.

By definition, $\pi_D(K(V_i, W_i)) = K(\pi_D(V_i), \pi_D(W_i))$. We now aim at representing the faces in H as convex polyhedra of the form $K(V, W)$. By proposition Proposition 3.1, $K(\pi_D(V_i), \pi_D(W_i))$ is the set of solutions of a system of inequalities $\mathfrak{S}\colon A \cdot \boldsymbol{y} \leq \boldsymbol{c}$, such that

- $A \in \mathbb{Q}^{n \cdot (d - \#D)}$ with $n \leq (\#V_i + \#W_i)^d + 2d$;
- $\langle A \rangle, \langle \boldsymbol{c} \rangle \leq O(d^2) \cdot \max(\langle V_i \rangle, \langle W_i \rangle)$.

Each face F of $K(\pi_D(V_i), \pi_D(W_i))$ corresponds to a nonempty set of the form

$$F = \{\boldsymbol{x} \in S : A' \cdot \boldsymbol{y} = \boldsymbol{c}'\},$$

for some subsystem $A' \cdot \boldsymbol{y} \leq \boldsymbol{c}'$ of $\mathfrak{S}$. As each row $\boldsymbol{a} \cdot \boldsymbol{y} \leq c$ of $A \cdot \boldsymbol{y} \leq \boldsymbol{c}$ corresponds to the hyperplane $\boldsymbol{a} \cdot \boldsymbol{y} = c$ of $\mathbb{R}^{d - \#D}$, each subsystem that needs to be considered in order to produce a face can be picked to have $d - m$ equalities (i.e. $2(d - m)$ rows). That is, $K(\pi_D(V_i), \pi_D(W_i))$ has faces $F_1, \dots, F_r$ with $r \leq n^d \leq ((\#V_i + \#W_i)^d + 2d)^d$, where face F_ℓ is defined as the set of solutions of a system of inequalities

$$\mathfrak{S}_\ell : A \cdot \boldsymbol{y} \leq \boldsymbol{c} \wedge A' \cdot \boldsymbol{y} = \boldsymbol{c}',$$

where $A' \cdot \boldsymbol{y} \leq \boldsymbol{c}'$ is a subsystem of $\mathfrak{S}$.

We now want to test whether $(\mathbb{R}^m \times F_\ell) \cap N_i = \emptyset$. To do so, we fist construct a system of both weak and strict inequalities characterising N_i. Again applying Proposition 3.1, $K(V_i, W_i)$ is the set of solutions of a system of inequalities $\mathfrak{T}\colon B \cdot (\boldsymbol{x}, \boldsymbol{y}) \leq \boldsymbol{d}$, where $\boldsymbol{x}$ is a vector of $\#D = m$ many variables, such that

- $B \in \mathbb{Q}^{n \cdot (d - m)}$ with $n \leq (\#V_i + \#W_i)^d + 2d$;
- $\langle B \rangle, \langle \boldsymbol{d} \rangle \leq O(d^2) \cdot \max(\langle V_i \rangle, \langle W_i \rangle)$.

Similarly, for each $j \in J_i$, we construct a system of inequalities $\mathfrak{T}_j\colon C_j \cdot (\boldsymbol{x}, \boldsymbol{y}) \leq \boldsymbol{h}_j$ characterising $K(V_j, W_j)$. Then, N_i is the set of solutions of the following combination of systems $\mathfrak{T}, \mathfrak{T}_1, \dots, \mathfrak{T}_{\#J_i}$:

$$B \cdot (\boldsymbol{x}, \boldsymbol{y}) \leq \boldsymbol{d} \wedge \textstyle\bigwedge_{j \in J_i} C_j \cdot (\boldsymbol{x}, \boldsymbol{y}) > \boldsymbol{h}_j.$$

In order to test whether $(R^m \times F_\ell) \cap N_i = \emptyset$, it is sufficient to check for the infeasibility of the system

$$A \cdot \boldsymbol{y} \leq \boldsymbol{c} \wedge A' \cdot \boldsymbol{y} = \boldsymbol{c}' \wedge B \cdot (\boldsymbol{x}, \boldsymbol{y}) \leq \boldsymbol{d} \wedge \textstyle\bigwedge_{j \in J_i} C_j \cdot (\boldsymbol{x}, \boldsymbol{y}) > \boldsymbol{h}_j,$$

over the reals, which can be done in polynomial time in the dimension of the system and its entries, i.e. in time

$$\mathrm{poly}((\#V_i + \#W_i)^d + (\#J_i + 1) \cdot (d + \max_{j \in J_i}(\#V_j + \#W_j)^d) \cdot \langle N_i \rangle).$$

For every face F_ℓ for which the system above is found infeasible, we rely on Proposition A.2 to compute from $\mathfrak{S}_\ell$ the set representation of F_ℓ as

$$F_\ell = \mathrm{conv}\{\boldsymbol{x}_F : F \text{ a minimal face of } F_\ell\} + \mathrm{cone}\{\boldsymbol{y}_F : F \text{ minimal proper face of } \mathrm{char.cone}\, F_\ell\} + \mathrm{span}\{\boldsymbol{z}_1, \dots, \boldsymbol{z}_t\},$$

where

- every $\boldsymbol{x}_F$ is an arbitrary element of F,
- every $\boldsymbol{y}_F$ is an arbitrary element of $F \setminus \mathrm{lin.space}(F_\ell)$, and
- $\boldsymbol{z}_1, \dots, \boldsymbol{z}_t$ form the basis of $\mathrm{lin.space}(F_\ell)$,
  $t = \dim \mathrm{lin.space}(F_\ell)$.

Moreover,

- $\langle \boldsymbol{x}_F \rangle, \langle \boldsymbol{y}_F \rangle, \langle \boldsymbol{z}_i \rangle \leq O(d^4) \cdot \max(\langle V_i \rangle, \langle W_i \rangle)$,
- this representation can be computed in time $((\#V_i, +\#W_i)^d + 2d)^d \cdot \mathrm{poly}(d, \max(\langle V_i \rangle, \langle W_i \rangle)))$.

We have $F_\ell = K(U_\ell, Y_\ell)$ where $U_\ell \coloneqq \{\boldsymbol{x}_F : F \text{ a minimal face of } F_\ell\}$ and $Y_\ell \coloneqq \{\boldsymbol{y}_F : F \text{ minimal proper face of } \mathrm{char.cone}\, F_\ell\} \cup \{\boldsymbol{z}_1, \dots, \boldsymbol{z}_t, -\boldsymbol{z}_1, \dots, -\boldsymbol{z}_t\}$.

Notice that, overall, the number of vectors $\boldsymbol{x}_F$ is bounded by $n^d \leq ((\#V_i, +\#W_i)^d + 2d)^d$, as they corresponds to distinct faces of F_ℓ and thus obtained from $\mathfrak{S}_\ell$ by setting at most d inequalities as equalities. Similarly, the number of vectors $\boldsymbol{y}_F$ is also bounded by $n^d \leq ((\#V_i, +\#W_i)^d + 2d)^d$.

We add each set $K(U_\ell, Y_\ell)$ as above to the set H, concluding the characterisation of $\pi_D(N_i)$.

To summarise, we concluded that

$$\pi_D(N_i) = \pi_D(K(V_i, W_i)) \setminus \bigcup_{\ell \in L_i} K(U_\ell, Y_\ell)$$

where

- $\#L_i \leq ((\#V_i + \#W_i)^d + 2d)^d$.

- $\#U_\ell, \#Y_\ell \leq 2d + ((\#V_i, +\#W_i)^d + 2d)^d$
  $\leq 2(\#V_i, +\#W_i + 2d)^{d^2}$; and
- $\langle U_\ell \rangle, \langle Y_\ell \rangle \leq O(d^4) \cdot \max(\langle V_i \rangle, \langle W_i \rangle)$.

Let us now look at the running time of computing this representation of $\pi_D(N_i)$.

- the set $K(\pi_D(V_i), \pi_D(W_i))$ can be computed in time $O(d \cdot (\#V_i + \#W_i))$;
- the systems $\mathfrak{S}$ and $\mathfrak{T}$ can be computed in time $(\#V_i + \#W_i)^d \cdot \text{poly}(d \cdot \max(\langle V_i \rangle, \langle W_i \rangle))$, by Proposition 3.1;
- each system $\mathfrak{S}_\ell$ can be computed in time $O((\#V_i + \#W_i)^d + d)$, as corresponds to simply selecting at most d rows from $\mathfrak{S}$. Recall that the number of faces F_ℓ is bounded by $((\#V_i + \#W_i)^d + 2d)^d$;
- each system $\mathfrak{T}_j$ can be computed in time $(\#V_j + \#W_j)^d \cdot \text{poly}(d \cdot \max(\langle V_j \rangle, \langle W_j \rangle))$, again by Proposition 3.1. There are $\#J_i$ many such systems;
- The infeasibility of $\mathfrak{T}_j$ is checked in time

$$\text{poly}((\#V_i + \#W_i)^d + (\#J_i + 1) \cdot (d + \max_{j \in J_i}(\#V_j + \#W_j)^d) \cdot \langle N_i \rangle).$$

- Each $K(U_\ell, Y_\ell)$ is computed from $\mathfrak{S}_\ell$ in time

$$((\#V_i, +\#W_i)^d + 2d)^d \cdot \text{poly}(d, \max(\langle V_i \rangle, \langle W_i \rangle))).$$

Therefore, computing $\pi_D(N_i)$ can be done in time:

$$\text{poly}((\#V_i + \#W_i + 2d)^{d^2} \cdot (\#J_i + 1) \cdot \max_{j \in J_i}(\#V_j + \#W_j)^d \cdot \langle N_i \rangle).$$

The bounds and running time in the statement of the lemma follows by computing each $\pi_D(N_i)$ for every $i \in I$. ◻

C.3 Proof of Theorem 5.3

Let us recall the statements of the lemma we proved in the previous appendix, together with the straightforward lemma for union of $\mathbb{R}$-semilinear sets.

Lemma 5.9. *There is an algorithm that given an $\mathbb{R}$-semilinear set $M = \bigcup_{i \in I}(K(V_i, W_i) \setminus \bigcup_{j \in J_i} K(V_j, W_j)) \subseteq \mathbb{R}^d$ computes a family of triples $\{(U_k, Y_k, \{(U_\ell, Y_\ell)\}_{\ell \in L_k})\}_{k \in K}$ such that*

$$\overline{M} = \textstyle\bigcup_{k \in K}(K(U_k, Y_k) \setminus \bigcup_{\ell \in L_k} K(U_\ell, Y_\ell)).$$

The algorithm ensures, for every $k \in K$ and $\ell \in K_\ell$,

- *$\#U_k, \#Y_k, \#U_\ell, \#Y_\ell \leq (\#I \cdot \max_{i \in I}(\#V_i + \#W_i) + d)^{O(d^2)}$;*
- *$\langle U_k \rangle, \langle Y_k \rangle, \langle U_\ell \rangle, \langle Y_\ell \rangle \leq O(d^5 \cdot \langle M \rangle)$; and*
- *$\#K, \#L_k \leq (\#I \cdot \max_{i \in I}(\#V_i + \#W_i) + d)^{O(d^2)}$.*

Moreover, the running time of the algorithm is

$$\text{poly}(\#I, \max_{i \in I} \#J_i, (\max_{i \in I}(\#V_i + \#W_i) + d)^{d^3}, \langle M \rangle).$$

Lemma 5.10. *Let $M_k = \bigcup_{i \in I_k}(K(V_i, W_i) \setminus \bigcup_{j \in J_i} K(V_j, W_j))$, with $k \in \{1, 2\}$ and $I_1 \cap I_2 = \emptyset$. We have*

$$M_1 \cap M_2 = \textstyle\bigcup_{k \in K}(K(U_k, Y_k) \setminus \bigcup_{\ell \in L_k} K(U_\ell, Y_\ell))$$

where, given $\#\mathcal{P}$ to be the maximal cardinality of a set in $\{V_i, W_i, V_j, W_j : i \in I_1 \cup I_2, j \in J_i\}$, we have

- *$\#K \leq \#I_1 \cdot \#I_2$ and $\#L_k \leq 2 \cdot \max_{i \in I_1 \cup I_2} \#J_i$;*
- *$\#U_k, \#Y_k, \#U_\ell, \#Y_\ell \leq (2(\#\mathcal{P} + d))^{O(d^2)}$; and*
- *$\langle U_k \rangle, \langle Y_k \rangle, \langle U_\ell \rangle, \langle Y_\ell \rangle \leq O(d^4) \cdot \max_{k \in \{1,2\}} \langle M_k \rangle$.*

Moreover, all sets U_j, Y_j, U_ℓ and Y_ℓ can be computed in time

$$\text{poly}(\#I_1, \#I_2, \max_{i \in I_1 \cup I_2} \#J_i, \max_{k \in \{1,2\}} \langle M_k \rangle, (\#\mathcal{P} + d)^{d^2}).$$

Lemma 5.11. *Let $M = \bigcup_{i \in I}(K(V_i, W_i) \setminus \bigcup_{j \in J_i} K(V_j, W_j))$ be an $\mathbb{R}$-semilinear set, and let $D \subseteq [1, d]$. We have,*

$$\pi_D(M) = \textstyle\bigcup_{i \in I}(K(\pi_D(V_i), \pi_D(W_i)) \setminus \bigcup_{\ell \in L_i} K(U_\ell, Y_\ell))$$

$\mathbb{R}$-semilinear set where for every $i \in I$ and $\ell \in L_i$,

- *$\#L_i \leq (\#V_i + \#W_i + 2d)^{d^2}$;*
- *$\#U_\ell, \#Y_\ell \leq 2(\#V_i + \#W_i + 2d)^{d^2}$;*

- $\langle U_\ell \rangle, \langle Y_\ell \rangle \leq O(d^4) \cdot \max(\langle V_i \rangle, \langle W_i \rangle)$.

Moreover, such a representation can be computed in time

$$\mathrm{poly}(\#I, \max_{i \in I}(\#J_i + 1), \langle M \rangle, (\#\mathcal{P} + d)^{d^2}),$$

where $\#\mathcal{P} \coloneqq \max_{i \in I, j \in J_i}(\#V_i, \#W_i, \#V_j, \#W_j)$.

Lemma 5.12. *Let* $M_k = \bigcup_{i \in I_k}(K(V_i, W_i) \setminus \bigcup_{j \in J_i} K(V_j, W_j))$, *with* $k \in \{1, 2\}$ *and* $I_1 \cap I_2 = \emptyset$. *We have,*

$$M_1 \cup M_2 = \textstyle\bigcup_{i \in I_1 \cup I_2}(K(V_i, W_i) \setminus \bigcup_{j \in J_i} K(V_j, W_j)),$$

which can be computed in time $\max_{k \in \{1,2\}} O(\#I_k \cdot \#\mathcal{P}) \cdot \langle M_k \rangle$, *where* $\#\mathcal{P} \coloneqq \max_{i \in I_1 \cup I_2,\, j \in J_i}(\#V_i, \#W_i, \#V_j, \#W_j)$.

We move to the proof of Theorem 5.3.

Theorem 5.3. *There is an algorithm that, given a well-defined* $\mathbb{R}$*-semilinear expression* s*, computes a family of triples* $\{(U_k, Y_k, \{(U_\ell, Y_\ell)\}_{\ell \in L_k})\}_{k \in K}$ *such that*

$$[\![s]\!] = \textstyle\bigcup_{k \in K}(K(U_k, Y_k) \setminus \bigcup_{\ell \in L_k} K(U_\ell, Y_\ell)).$$

The algorithm ensures, wlog. assuming $d, \langle s \rangle \geq 2$, *that*

$$\#K, \#L_k, \#U_k, \#Y_k, \#U_\ell, \#Y_\ell \leq n^{d^{O(h)}};\ \langle U_k \rangle, \langle Y_k \rangle \leq d^{O(h)} \langle s \rangle,$$

where $n = n_{\mathrm{p}}(s) + 2$, $d = d(s)$ *and* $h = h(s)$. *Moreover, the running time of the algorithm is* $\langle s \rangle^{O(d)} \cdot n^{d^{O(h)}}$.

Proof. The proof of Theorem 5.3 is by induction on the height of an $\mathbb{R}$-semilinear expression, and follows exactly the same ideas as in the proof of Theorem 5.1.

For the sake of simplicity, let us consider extended $\mathbb{R}$-semilinear expressions where atoms can be arbitrary $\mathbb{R}$-semilinear sets instead of just polyhedra of the form $K(V, W)$. For a $\mathbb{R}$-semilinear expression s we write $\#(s)$ for the maximal number of components $\#I$, $\#K_i$, $\#V_i$, $\#W_i$, $\#V_k$, $\#W_k$ appearing in an atom $\bigcup_{i \in I}(K(V_i, W_i) \setminus \bigcup_{k \in K_i} K(V_k, W_k))$ of s. Briefly, consider $M = \bigcup_{i \in I} K(V_i, W_i) \setminus \bigcup_{j \in J_i} K(V_j, W_j)$. From the bounds of the lemma above, notice that, whenever an operation $\oplus \in \{\overline{(\cdot)}, \pi_D\}$ is applied to M we obtain

$$\oplus(M) = \bigcup_{k \in K} K(U_k, Y_k) \setminus \bigcup_{\ell \in L_k} K(U_\ell, Y_\ell)$$

where

- $\#K, \#L_k, \#U_k, \#Y_k, \#U_\ell, \#Y_\ell \leq (\#I^2 \max_{i \in I}(\#V_i + \#W_i) + 2d)^{O(d^2)} \cdot 2 \cdot \max_{i \in I} \#J_i$;
- $\langle U_k \rangle, \langle Y_k \rangle, \langle U_\ell \rangle, \langle Y_\ell \rangle \leq O(d^5 \langle M \rangle)$.

The upper bounds above also the $\mathbb{R}$-semilinear set resulting from $M_1 \cup M_2$ and $M_1 \cap M_2$, modulo that $\#I$ and $\langle M \rangle$ above must be updated to the maximal number of components in either M_1 and M_2, and to $\max(\langle M_1 \rangle, \langle M_2 \rangle)$, respectively. Hence, in order obtain the upper bounds in the statement of the theorem it suffices to assume that s is a $\mathbb{R}$-semilinear set of height h that only contains unary operations. Let us consider a constant $\lambda \geq 2$ that upper bound all the constants hidden by the Big O notation, that is whenever $O(n)$ appears in one of the bounds above, the actual value is at most $\lambda(n+1)$. By induction on s, we prove that

- $\#K, \#L_k, \#U_k, \#Y_k, \#U_\ell, \#Y_\ell \leq (4\#(s)^4(d+1))^{\lambda^{4(h+1)}(d^2+1)^h}$;
- $\langle U_k \rangle, \langle Y_k \rangle, \langle U_\ell \rangle, \langle Y_\ell \rangle \leq \lambda^{5h} d^{5h}(\langle s \rangle + 1)$.

The base case for $h = 0$ is trivial. For the induction step, given $s = \overline{s'}$ or $s = \pi_D(s')$, by induction hypothesis we have

$$[\![s']\!] = \bigcup_{p \in P} K(A_p, B_p) \setminus \bigcup_{q \in Q_p} K(A_q, B_q)$$

where

- $\#P, \#Q_p, \#A_p, \#B_p, \#A_q, \#B_q \leq (4\#(s')^4(d+1))^{\lambda^{4(h-1+1)}(d^2+1)^{h-1}s}$
- $\langle A_p \rangle, \langle B_p \rangle, \langle A_q \rangle, \langle B_q \rangle \leq \lambda^{5(h-1)} d^{5(h-1)}(\langle s' \rangle + 1)$.

We have $\langle s' \rangle \leq \langle s \rangle$ and $\#s' \leq \#s$. Then,

$$[\![s]\!] = \bigcup_{k \in K} K(U_k, Y_k) \setminus \bigcup_{\ell \in L_k} K(U_\ell, Y_\ell)$$

where

- $\#K, \#L_k, \#U_k, \#Y_k, \#U_\ell, \#Y_\ell$ are bounded by

$$
\begin{aligned}
&(2\left(\left(4\#(s')^4(d+1)\right)^{\lambda^{4(h-1+1)}(d^2+1)^{h-1}}\right)^3 + 2d)^{\lambda(d^2+1)} \cdot 2 \cdot \left(\left(4\#(s')^4(d+1)\right)^{\lambda^{4(h-1+1)}(d^2+1)^{h-1}s}\right)\\
\leq &(4\left(\left(4\#(s)^4(d+1)\right)^{\lambda^{4(h-1+1)}(d^2+1)^{h-1}}\right)^4 (d+1))^{\lambda(d^2+1)}\\
\leq &\left(\left(4\#(s)^4(d+1)\right)^{4\lambda^{4(h-1+1)}(d^2+1)^{h-1}+1}\right)^{\lambda(d^2+1)} \leq (4\#(s)^4(d+1))^{5\lambda^{4h+1}(d^2+1)^h}\\
\leq &(4\#(s)^4(d+1))^{5\lambda^{4h+1}(d^2+1)^h} \leq (4\#(s)^4(d+1))^{\lambda^{4(h+1)}(d^2+1)^h}.
\end{aligned}
$$

- $\langle A_p\rangle, \langle B_p\rangle, \langle A_q\rangle, \langle B_q\rangle \leq \lambda(d^5(\lambda^{5(h-1)}d^{5(h-1)}(\langle s\rangle+1))+1) \leq \lambda^{5h}d^{5h}(\langle s\rangle+1)$.

Therefore, for an $\mathbb{R}$-semilinear expression s with polyhedra $K(V,W)$ as atoms, $n = n_{\mathrm{p}}(s)+2$, $h(s)=h$, $d(s)=d$, we have

$$[\![s]\!] = \bigcup_{k\in K} K(U_k, Y_k) \setminus \bigcup_{\ell\in L_k} K(U_\ell, Y_\ell)$$

where

$$\#K, \#L_k, \#U_k, \#Y_k, \#U_\ell, \#Y_\ell \leq n^{d^{O(h)}}; \qquad \langle U_k\rangle, \langle Y_k\rangle \leq (2d)^{O(h)}(\langle s\rangle+1),$$

An $\mathbb{R}$-semilinear expression with height h has at most 2^h operations. For the running time, we can estimate the running time of 2^h many operations applied on a $\mathbb{R}$-semilinear set with bounds as the ones just above. From Lemmas 5.9 to 5.12, all Boolean operations and projection are exponential in d and polynomial in the other parameters. Hence, the algorithm that translates an $\mathbb{R}$-semilinear expression into an equivalent $\mathbb{R}$-semilinear set runs in time $2^h \cdot \mathrm{poly}((2d)^{O(h)}(\langle s\rangle+1), n^{d^{O(h)}})^{O(d)}$, which is in $(\langle s\rangle+1)^{O(d)} \cdot n^{d^{O(h)}}$. ◻

Corollary 5.4. *There is an algorithm that, given a formula Φ of LRA, computes a family of triples $\{(U_k, Y_k, \{(U_\ell, Y_\ell)\}_{\ell\in L_k})\}_{k\in K}$ such that*

$$[\![\Phi]\!] = \textstyle\bigcup_{k\in K}(K(U_k,Y_k) \setminus \bigcup_{\ell\in L_k} K(U_\ell, Y_\ell)).$$

The algorithm ensures, wlog. assuming $d, \langle\Phi\rangle \geq 2$,

$$\#K, \#L_k, \#U_k, \#Y_k, \#U_\ell, \#Y_\ell \leq 2^{d^{O(h)}};\ \langle U_k\rangle, \langle Y_k\rangle \leq d^{O(h)}\langle\Phi\rangle,$$

where $d = d(\Phi)$ and $h = h(\Phi)$. Moreover, the running time of the algorithm is $\langle\Phi\rangle^{O(d)} \cdot 2^{d^{O(h)}}$.

Proof. To prove this corollary, we first need to translate a formula Φ from LRA to an $\mathbb{R}$-semilinear expression. As in the case of Presburger, Boolean connectives and quantifiers can be replaced with Boolean operators and projection from semilinear expressions, without modifying the height of the formula. Each atomic formula, i.e. a single linear inequality, is translated into a polyhedron $K(V,W)$ via Proposition A.2. More precisely, we compute the set S of solutions of an atomic formula $\boldsymbol{a}\cdot\boldsymbol{x} \leq c$ as

$$S = \mathrm{conv}\{\boldsymbol{x}_F : F \text{ a minimal face of } S\} + \mathrm{cone}\{\boldsymbol{y}_F : F \text{ minimal proper face of } \mathrm{char.cone}\, S\} + \mathrm{span}\{\boldsymbol{z}_1,\dots,\boldsymbol{z}_t\},$$

where

- every $\boldsymbol{x}_F$ is an arbitrary element of F,
- every $\boldsymbol{y}_F$ is an arbitrary element of $F \setminus \mathrm{lin.space}(S)$, and
- $\boldsymbol{z}_1,\dots,\boldsymbol{z}_t$ are generators of $\mathrm{lin.space}(S)$,
  with $t = \dim \mathrm{lin.space}(S)$.

Since $\boldsymbol{a}\cdot\boldsymbol{x} \leq c$ is just one inequality, it has only one minimal face F, that is $\boldsymbol{a}\cdot\boldsymbol{x} = c$, its characteristic cone has only one minimal proper face G, that is $\boldsymbol{a}\cdot\boldsymbol{x} \leq 0$, and its lineality space is $\boldsymbol{a}\cdot\boldsymbol{x} = 0$ and has dimension $d-1$. Notice that, then, the equivalence above simplifies to

$$S = \boldsymbol{x}_F + \mathrm{span}\{\boldsymbol{z}_1,\dots,\boldsymbol{z}_{d-1}\} + \mathrm{cone}\{\boldsymbol{y}_G\},$$

where $\boldsymbol{z}_1,\dots,\boldsymbol{z}_{d-1}$ are generators of $\mathrm{lin.space}(S)$, and $\boldsymbol{y}_F$ is an arbitrary element of $G \setminus \mathrm{lin.space}(S)$. Set $V := \{\boldsymbol{x}_F\}$ and $W := \{\boldsymbol{y}_G, \boldsymbol{z}_1,\dots,\boldsymbol{z}_{d-1}, -\boldsymbol{z}_1,\dots,\boldsymbol{z}_{d-1}\}$. We have $S = K(V,W)$, where $\#V = 1$ and $\#W \leq 2d-1$. By Proposition A.2, $\langle V\rangle, \langle W\rangle \leq O(d^2)\cdot(\langle\boldsymbol{a}\rangle + \langle c\rangle)$, and V and W can be computed in time $\mathrm{poly}(d, \langle\boldsymbol{a}\rangle + \langle c\rangle)$.

Once the formula is translated into an $\mathbb{R}$-semilinear expression, Corollary 5.4 follows directly from Theorem 5.3. Here, notice that the bound $\#I \leq n^{d^{O(h)}}$ written in Theorem 5.3 simplifies to $2^{d^{O(h)}}$ in the case of Corollary 5.4 as each polyhedron $K(V,W)$ obtained from an atomic formula $\boldsymbol{a}\cdot\boldsymbol{x} \leq c$ is such that $\#V + \#W = 2d$. ◻

D Appendix to Section 6

Lemma 6.4. *Consider $\boldsymbol{y}_1, \boldsymbol{y}_2 \in \mathbb{R}^m$ such that $\boldsymbol{y}_1 \sim_{\mathcal{H}_{\boldsymbol{v}}} \boldsymbol{y}_2$. Then, $(\boldsymbol{v}, \boldsymbol{y}_1) \in M$ if and only if $(\boldsymbol{v}, \boldsymbol{y}_2) \in M$.*

Proof. Let $h \in \mathcal{H}(M)$ given by $\mathfrak{h} \colon \boldsymbol{a} \cdot \boldsymbol{x} = c$. We show that $\operatorname{sgn}(\boldsymbol{a} \cdot (\boldsymbol{v}, \boldsymbol{y}_1) - c) = \operatorname{sgn}(\boldsymbol{a} \cdot (\boldsymbol{v}, \boldsymbol{y}_2) - c)$. This is enough to conclude that, for any polyhedron S carved out by $\mathcal{H}(M)$, we have $(\boldsymbol{v}, \boldsymbol{y}_1) \in S$ if and only if $(\boldsymbol{v}, \boldsymbol{y}_2) \in S$. Since for every $i \in I$ and $j \in J_i$, $\mathcal{H}(M)$ carves out both $K(V_i, W_i)$ and $K(V_j, W_j)$, this implies $(\boldsymbol{v}, \boldsymbol{y}_1) \in M$ iff $(\boldsymbol{v}, \boldsymbol{y}_2) \in M$.

We consider three cases, depending on whether $A_{\boldsymbol{v}} \subseteq h$, $A_{\boldsymbol{v}} \cap h = \emptyset$ or $\emptyset \neq (A_{\boldsymbol{v}} \cap h) \neq A_{\boldsymbol{v}}$ holds. In the first case we have $\operatorname{sgn}(\boldsymbol{a} \cdot (\boldsymbol{v}, \boldsymbol{y}_1) - c) = \operatorname{sgn}(\boldsymbol{a} \cdot (\boldsymbol{v}, \boldsymbol{x}_2) - c) = 0$. In the second case, $A_{\boldsymbol{v}}$ is parallel to h, and it is not included in it. Hence, there is a sign $s \in \{+1, -1\}$ for which $\operatorname{sgn}(\boldsymbol{a} \cdot \boldsymbol{x} - c) = s$ is achieved by all points in $\boldsymbol{x} \in A_{\boldsymbol{v}}$. Again, this implies that $\operatorname{sgn}(\boldsymbol{a} \cdot (\boldsymbol{v}, \boldsymbol{y}_1) - c) = \operatorname{sgn}(\boldsymbol{a} \cdot (\boldsymbol{v}, \boldsymbol{x}_2) - c)$. Lastly, suppose $\emptyset \neq (A_{\boldsymbol{v}} \cap h) \neq A_{\boldsymbol{v}}$. In this case $A_{\boldsymbol{v}} \cap h$ belongs to $\mathcal{H}_{\boldsymbol{v}}$, and therefore from the assumption $\boldsymbol{y}_1 \sim_{\mathcal{H}_{\boldsymbol{v}}} \boldsymbol{y}_2$ we conclude that $\operatorname{sgn}(\boldsymbol{a} \cdot (\boldsymbol{v}, \boldsymbol{y}_1) - c) = \operatorname{sgn}(\boldsymbol{a} \cdot (\boldsymbol{v}, \boldsymbol{x}_2) - c)$. ◻

Lemma 6.5. *Let $M = \bigcup_{i \in I} M_i \subseteq \mathbb{R}^{n+m}$ be an $\mathbb{R}$-semilinear set, where $M_i = \big(K(V_i, W_i) \setminus \bigcup_{j \in J_i} K(V_j, W_j)\big)$ and the $d = n + m$ dimensions are partitioned into $n \geq 1$ objects and $m \geq 1$ parameters. Let $V \subseteq \mathbb{R}^n$ be a finite set of objects. There is an equivalence relation $\sim_{\mathcal{H}}$ satisfying:*

1. *given $i \in I$, $\boldsymbol{y}_1, \boldsymbol{y}_2 \in \mathbb{R}^m$ and $\boldsymbol{v} \in V$, if $\boldsymbol{y}_1 \sim_{\mathcal{H}} \boldsymbol{y}_2$, then $(\boldsymbol{v}, \boldsymbol{y}_1) \in M_i$ if and only if $(\boldsymbol{v}, \boldsymbol{y}_2) \in M_i$;*
2. *given $i \in I$, $\boldsymbol{y}_1, \boldsymbol{y}_2 \in \mathbb{R}^m$ and $\boldsymbol{v} \in V$, if $\boldsymbol{y}_1 \sim_{\mathcal{H}} \boldsymbol{y}_2$, then $(\boldsymbol{v}, \boldsymbol{y}_1) \in \operatorname{aff} M_i$ if and only if $(\boldsymbol{v}, \boldsymbol{y}_2) \in \operatorname{aff} M_i$;*
3. *the number of equivalence classes of $\sim_{\mathcal{H}}$ is bounded by $k^d \cdot 2^d \cdot \#I^d \cdot \big(2d + \max_{i \in I}(\#V_i + \#W_i)^d\big)^d + 1$.*

Proof. Consider $\sim_{\mathcal{H}} = \bigcap_{\boldsymbol{v} \in V} \sim_{\mathcal{H}_{\boldsymbol{v}}}$. follows directly from Lemma 6.4 and by definition of $\sim_{\mathcal{H}}$.

For Property (2), as $\mathcal{H}(V_i, W_i)$ carves out $K(V_i, W_i)$, there are hyperplanes $h_1, \dots, h_t$, with h_j given by $\mathfrak{h}_j \colon \boldsymbol{a}_j \cdot \boldsymbol{x} = c_j$, such that $\bigcap_{j=1}^t h_j = \operatorname{aff} K(V_i, W_i) = \operatorname{aff} M_i$. Consider $\boldsymbol{y}_1, \boldsymbol{y}_2 \in \mathbb{R}^m$ and $\boldsymbol{v} \in V$ such that $\boldsymbol{y}_1 \sim_{\mathcal{H}} \boldsymbol{y}_2$ and $(\boldsymbol{v}, \boldsymbol{y}_1) \in \operatorname{aff} M_i$. From $(\boldsymbol{v}, \boldsymbol{y}_1) \in \operatorname{aff} M_i$ and $\boldsymbol{y}_1 \sim_{\mathcal{H}} \boldsymbol{y}_2$ we derive that for all $j \in [1.t]$, $\operatorname{sgn}(\boldsymbol{a}_j \cdot (\boldsymbol{v}, \boldsymbol{y}_2) - c_j) = 0$. Hence, $(\boldsymbol{v}, \boldsymbol{y}_2) \in \bigcap_{j=1}^t h_j = \operatorname{aff} M_i$.

To show Property (3), consider $\boldsymbol{v} \in V$. Each element of $\mathcal{H}_{\boldsymbol{v}}$ corresponds to a hyperplane for the affine subspace $A_{\boldsymbol{v}}$. From the upper bound on $\mathcal{H}(M)$, we get

$$\#\mathcal{H}_{\boldsymbol{v}} \leq \#I \cdot (2d + \max_{i \in I}(\#V_i + \#W_i)^d).$$

Therefore, the set $\bigcup_{\boldsymbol{v} \in V} \mathcal{H}_{\boldsymbol{v}}$ has at most

$$\alpha \coloneqq k \cdot \#\mathcal{H}_{\boldsymbol{v}} \leq \#I \cdot (2d + \max_{i \in I}(\#V_i + \#W_i)^d)$$

elements. Each element corresponds to one hyperplane in $\mathbb{R}^m$, and the equivalence classes of $\sim_{\mathcal{H}}$ are induced by these hyperplanes. Then, directly from Proposition 3.5, we conclude that $\sim_{\mathcal{H}}$ has at most $(2\alpha)^d + 1$ equivalence classes. ◻

Lemma 6.6. *Consider $k, d, \alpha \in \mathbb{R}$ satisfying*

$$k^d \cdot \alpha \geq 2^k, \quad d \geq 1, \quad \alpha \geq 2, \quad k \geq 1.$$

Then, $k \leq 2 \cdot (\log \alpha + d \log(2 \cdot d))$.

Proof. Following a simple manipulation, one shows that the inequality $k^d \cdot \alpha \geq 2^k$ holds if and only if $k - d \cdot \log k \leq \log \alpha$. We derive the upper bound on k required by the lemma by means of the Lambert W function. To this end, we manipulate the inequality $k - d \cdot \log k \leq \log \alpha$ as follows:

$$\begin{gathered}
k - d \cdot \log k \leq \log \alpha, \\
\frac{\ln 2}{d} \cdot k - \ln k \leq \frac{\ln 2}{d} \cdot \log \alpha, \\
-\frac{\ln 2}{d} \cdot k + \ln k \geq -\frac{\ln 2}{d} \cdot \log \alpha, \\
e^{-\frac{\ln 2}{d} \cdot k + \ln k} \geq e^{-\frac{\ln 2}{d} \cdot \log \alpha}, \\
k \cdot e^{-\frac{\ln 2}{d} \cdot k} \geq e^{-\frac{\ln 2}{d} \cdot \log \alpha}, \\
-\frac{\ln 2}{d} \cdot k \cdot e^{-\frac{\ln 2}{d} \cdot k} \leq -\frac{\ln 2}{d} \cdot e^{-\frac{\ln 2}{d} \cdot \log \alpha}.
\end{gathered}$$

Notice that the left hand side of the last inequality in the series of equivalent formulae above is of the form $y \cdot e^y$.

For a guide on the Lambert W function, see [7]. Briefly, the Lambert W function is a multivalued function $W_k(z)$, where $k \in \mathbb{Z}$ and z is a complex number, with the property that given a feasible equation $ye^y = x$, we have $y = W_k(x)$ for some $k \in \mathbb{Z}$.

In the case of real numbers, $ye^y = x$ can be solved only if $x \geq -\frac{1}{e}$, and moreover it is sufficient to considers the functions W_0 and W_{-1}. Here, we are interested in the function W_{-1}, which has the following properties:

- $W_{-1}(x)$ is only defined for $-\frac{1}{e} \leq x < 0$;
- W_{-1} is monotonically decreasing;
- W_{-1} is always negative.

To use W_{-1}, we must check that both the left and right hand side of the inequality $-\frac{\ln 2}{d} \cdot k \cdot e^{-\frac{\ln 2}{d} \cdot k} \leq -\frac{\ln 2}{d} \cdot e^{-\frac{\ln 2}{d} \cdot \log \alpha}$ belong to the interval $[-\frac{1}{e}, 0) \subseteq \mathbb{R}$, which follows directly from the hypothesis $k \geq 1$, $d \geq 1$ and $\alpha \geq 2$.

Then, since W_{-1} is monotonically decreasing, by applying it to both sides of the last inequality we get

$$\begin{aligned}
-\frac{\ln 2}{d} \cdot k &\geq W_{-1}\left(-\frac{\ln 2}{d} \cdot e^{-\frac{\ln 2}{d} \cdot \log \alpha}\right), \\
k &\leq -\frac{d}{\ln 2} \cdot W_{-1}\left(-\frac{\ln 2}{d} \cdot e^{-\frac{\ln 2}{d} \cdot \log \alpha}\right), \\
k &\leq -\frac{d}{\ln 2} \cdot W_{-1}\left(-e^{-(\frac{\ln 2}{d} \cdot \log \alpha + \ln \frac{d}{\ln 2})}\right).
\end{aligned}$$

Notice that, in the right hand side of the last inequality both $-\frac{d}{\ln 2}$ and $W_{-1}(\dots)$ are negative numbers. Hence, to obtain a upper bound on k we can rely on the lower bound for W_{-1} established by Chatzigeorgiou in [5, Thm. 1]: for all $u > 0$,

$$-1 - \sqrt{2u} - u < W_{-1}(-e^{-(u+1)}).$$

Let us define $u \coloneqq \frac{\ln 2}{d} \cdot \log \alpha + \ln \frac{d}{\ln 2} - 1$, which is greater than 0 since $\ln \frac{d}{\ln 2} > 1$ and $\frac{\ln 2}{d} \cdot \log \alpha > 0$ (recall that $d \geq 1$).

Notice that $u + 1 \geq \sqrt{2u}$ for all $u \geq 0$, which we use to simplify the analysis below. We conclude the proof by applying the aforementioned bound from [5]:

$$\begin{aligned}
k &\leq -\frac{d}{\ln 2} \cdot (-1 - \sqrt{2u} - u) = \frac{d}{\ln 2} \cdot (1 + \sqrt{2u} + u) \\
&\leq \frac{d}{\ln 2} \cdot 2 \cdot (u + 1) \\
&\leq \frac{d}{\ln 2} \cdot 2 \cdot \left(\frac{\ln 2}{d} \cdot \log \alpha + \ln \frac{d}{\ln 2}\right) \\
&\leq 2 \cdot (\log \alpha + \frac{d}{\ln 2} \cdot \ln \frac{d}{\ln 2}) \\
&\leq 2 \cdot (\log \alpha + d(\log d - \log \ln 2)) \\
&\leq 2 \cdot (\log \alpha + d \log(2 \cdot d)).
\end{aligned}$$

□

Lemma 6.7. *Consider k from Equation (†). We have,*

$$k \leq 6(d+1)^2 \cdot \log\left(\#I \cdot d \cdot \max_{i \in I}(\#V_i + \#W_i + 1)\right).$$

Proof. Let us set $\alpha \coloneqq 2^d \cdot \#I^d \cdot \left(2d + \max_{i \in I}(\#V_i + \#W_i)^d\right)^d + 1$. Since $d \geq 2$, $k \geq 1$ and $\alpha \geq 2$, Lemma 6.6 can be applied and yields the , it suffices to check that , which is clearly the case. So $k \leq 2 \cdot (\log \alpha + d \log(2 \cdot d))$, which we manipulate further as follows:

$$\begin{aligned}
k &\leq 2 \cdot (\log \alpha + d \log(2 \cdot d)) \\
&\leq 2\Big(\log\left(2^d \#I^d \left(2d + \max_{i \in I}(\#V_i + \#W_i)^d\right)^d + 1\right) + d \log(2d)\Big) \\
&\leq 3 \cdot (d+1)^2 \cdot \log\left(2 \cdot \#I \cdot \left(2d + \max_{i \in I}(\#V_i + \#W_i)\right)\right) \\
&\leq 6 \cdot (d+1)^2 \cdot \log\left(\#I \cdot d \cdot \max_{i \in I}(\#V_i + \#W_i + 1)\right).
\end{aligned}$$

□

Lemma 6.12. *Let E be an equivalence class of $\sim_{\mathcal{H}}$. Either*

- *for every $\boldsymbol{y} \in E$, $\mathcal{S}_{\boldsymbol{y}} \cap V = \emptyset$; or*
- *the relation $\sim_\Lambda$ partitions $E \cap \mathbb{Z}^m$ into at most $(2d \cdot \|P\|)^d$ equivalence classes.*

Assuming $\dim \operatorname{span}(P) = d$, the second bullet point in the lemma above follows from the fact that the number of equivalence classes in $\sim_\Lambda$ correspond to the number of integer points in the fundamental parallelepiped of P, which is bounded by $|\det P|$, see [21, Lem. 2.3.14]. Below, we give a self-contained proof that does not require $\dim \operatorname{span}(P) = d$.

Proof. We start by considering two complementary cases:

1. for all $\boldsymbol{y} \in E$, $\boldsymbol{v} \in V$ and $\boldsymbol{b} \in B$, $(\boldsymbol{v}, \boldsymbol{y}) \notin (\boldsymbol{b} + \operatorname{aff} P)$; or
2. there is $\boldsymbol{y} \in E$, $\boldsymbol{v} \in V$ and $\boldsymbol{b} \in B$, $(\boldsymbol{v}, \boldsymbol{y}) \in (\boldsymbol{b} + \operatorname{aff} P)$.

In the first case, clearly for every $\boldsymbol{y} \in E$ there is no $\boldsymbol{v} \in V$ such that $(\boldsymbol{v}, \boldsymbol{y}) \in L$, and therefore $\mathcal{S}_{\boldsymbol{y}} \cap V = \emptyset$.

In the second case, we show that $\sim_\Lambda$ partitions $E \cap \mathbb{Z}^m$ into at most $(2d \cdot \|P\|)^d$ equivalence classes. Let $\boldsymbol{b}$ and $\boldsymbol{v} \in V$ such that there is $\boldsymbol{y} \in E$ satisfying $(\boldsymbol{v}, \boldsymbol{y}) \in (\boldsymbol{b} + \operatorname{aff} P)$. Noticing that $(\boldsymbol{b} + \operatorname{aff} P) = \operatorname{aff} K(\boldsymbol{b}, P)$, from the Property (2) of $\sim_{\mathcal{H}}$ we conclude that for every $\boldsymbol{y} \in E$ we have $(\boldsymbol{v}, \boldsymbol{y}) \in (\boldsymbol{b} + \operatorname{aff} P)$.

Consider the equivalence relation $\approx$ on $\mathbb{Z}^d$ given by

$$\boldsymbol{z}_1 \approx \boldsymbol{z}_2 \text{ if and only if } \boldsymbol{z}_1 - \boldsymbol{z}_2 \in \Lambda(P).$$

To conclude the proof, it is sufficient to show that $\approx$ divides $(\boldsymbol{b} + \operatorname{aff} P) \cap \mathbb{Z}^d$ into at most $(2d \cdot \|P\|)^d$ equivalence classes, as indeed we have that $\boldsymbol{y}_1 \sim_\Lambda \boldsymbol{y}_2$ implies $(\boldsymbol{v}, \boldsymbol{y}_1) \approx (\boldsymbol{v}, \boldsymbol{y}_2)$, for every $\boldsymbol{y}_1, \boldsymbol{y}_2 \in \mathbb{Z}^m$ and $\boldsymbol{v} \in \mathbb{Z}^n$. Notice that, for every $\boldsymbol{z} \in \mathbb{Z}^d$, $\boldsymbol{z}_1 \approx \boldsymbol{z}_2$ if and only if $\boldsymbol{z} + \boldsymbol{z}_1 \approx \boldsymbol{z} + \boldsymbol{z}_2$. Therefore, it is sufficient to establish the result for $\boldsymbol{b} = \boldsymbol{0}$, i.e. we show that $\approx$ divides $(\operatorname{aff} P) \cap \mathbb{Z}^d$ into at most $(2d \cdot \|P\|)^d$ equivalence classes.

Increasing any given element $\boldsymbol{z} \in \mathbb{Z}^d$ by an arbitrary period $\boldsymbol{p} \in P$ lead to an element in the same equivalence class as $\boldsymbol{z}$, i.e. $\boldsymbol{z} \approx \boldsymbol{z} + \boldsymbol{p}$. Hence, the number η of equivalence classes induced by $\approx$ on $(\operatorname{aff} P) \cap \mathbb{Z}^d$ is bounded by the number of integral points in the polytope $C \coloneqq \{P \cdot \boldsymbol{\lambda} : \boldsymbol{\lambda} \in [0,1]^{\#P}\} \subseteq \mathbb{R}^d$. As the magnitude of each point in C is bounded by $\#P \cdot \|P\|$, the set C is contained in the hypercube $[-\#P \cdot \|P\|, \#P \cdot \|P\|]^d$. As P is proper, this implies $\eta \leq (2 \cdot d \cdot \|P\|)^d$. □

Lemma 6.13. *The cardinality of $\{\mathcal{S}_{\boldsymbol{y}} \cap V : \boldsymbol{y} \in \mathbb{Z}^m\}$ is bounded by $k^d \cdot 2^{2d+2} \cdot (d+1)^{d^2+1} \cdot (\#B \cdot \|P\|)^d$.*

Proof. By Property (3), the relation $\sim_{\mathcal{H}}$, divides $\mathbb{R}^m$ into at most $k^d \cdot 2^{2d} \cdot \#B^d \cdot (d+1)^{d^2} + 1$ equivalence classes. Let E be one these classes. By Lemma 6.12, either every $\boldsymbol{y} \in E$ achieves $\mathcal{S}_{\boldsymbol{y}} \cap V = \emptyset$, or $\sim_\Lambda$ partitions $E \cap \mathbb{Z}^m$ into at most $(2 \cdot d \cdot \|P\|)^d$ many equivalence classes. Let E' be one of these equivalence classes. For every $\boldsymbol{y}_1, \boldsymbol{y}_2 \in \mathbb{Z}^m$ we have $\boldsymbol{y}_1 \,(\sim_{\mathcal{H}} \cap \sim_\Lambda)\, \boldsymbol{y}_2$, and thus by Lemma 6.11 we derive that $\mathcal{S}_{\boldsymbol{y}_1} \cap V = \mathcal{S}_{\boldsymbol{y}_2} \cap V$. We conclude that the cardinality of $\{\mathcal{S}_{\boldsymbol{y}} \cap V : \boldsymbol{y} \in \mathbb{Z}^m\}$ is bounded by $(2 \cdot d \cdot \|P\|)^d \cdot (k^d \cdot 2^{2d} \cdot \#B^d \cdot (d+1)^{d^2} + 1) \leq k^d \cdot 2^{2d+2} \cdot (d+1)^{d^2+1} \cdot (\#B \cdot \|P\|)^d$. □

Lemma 6.14. *The VC dimension of a set $L(B,P) \subseteq \mathbb{Z}^d$ with P proper is bounded by $6 \cdot (d+1)^4 \log\big((d+1) \cdot \#B \cdot \|P\|\big)$.*

Proof. From Lemma 6.13, given V shattered by $L(B,P)$ with $\#V = k$, we have

$$2^k \leq k^d \cdot 2^{2d+2} \cdot (d+1)^{d^2+1} \cdot (\#B \cdot \|P\|)^d.$$

Set $\alpha \coloneqq 2^{2d+2} \cdot (d+1)^{d^2+1} \cdot (\#B \cdot \|P\|)^d$. Then, by Lemma 6.6, $k \leq 2(\log \alpha + d \cdot \log(2d))$, which we further manipulate:

$$\begin{aligned} k &\leq 2(\log \alpha + d \cdot \log(2d)) \\ &\leq 2\big(\log(2^{2d+2} \cdot (d+1)^{d^2+1} \cdot (\#B \cdot \|P\|)^d) + d \log(2d)\big) \\ &\leq 6 \cdot (d+1)^4 \log\big((d+1) \cdot \#B \cdot \|P\|\big). \end{aligned}$$

□