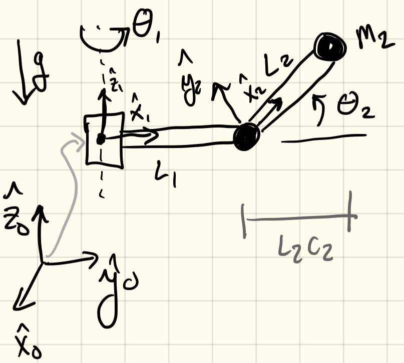
Lecture 29 - Lagrangian Dynamics

- Exam 2 returned Wednesday in Class

Today:

- Quick Review of RNEA Results
  - $\Rightarrow$ Structure of the dynamic equations
- Introduce Lagrangian formulation
  - Simple Example
  - Comparisson with RNEA

Example: 2R Manip



From Before:

$$\tau_1 = \left[m_1 L_1^2 + m_2 (L_1 + L_2 c_2)^2\right] \ddot{\theta}_1 - 2(L_1 + L_2 c_2) m_2 L_2 s_2 \dot{\theta}_1 \dot{\theta}_2$$

$$\tau_2 = m_2 L_2 \left[(L_1 + L_2 c_2) s_2 \dot{\theta}_1^2 + L_2 \ddot{\theta}_2 + g c_2 \right]$$

Structure of RNEA Solution:

$$\tau = M(\theta)\ddot{\theta} + V(\theta,\dot{\theta}) + G(\theta)$$

$M(\theta)$: Mass matrix $(n \times n)$

$V(\theta,\dot{\theta})$: Coriolis & Centripetal Terms $(n \times 1)$ $\quad \underbrace{\dot{\theta}_i \dot{\theta}_j}_{\text{coriolis}} \quad \underbrace{\dot{\theta}_i^2}_{\text{centripetal}}$

$G(\theta)$: Gravitatial term $(n \times 1)$

$$\tau_1 = m_1 L_1^2 \ddot{\theta}_1 + (L_1 + c_2 L_2)^2 m_2 \ddot{\theta}_1 - 2m_2(L_1 + c_2 L_2) L_2 s_2 \dot{\theta}_1 \dot{\theta}_2$$

$$\tau_2 = m_2 L_2 \left[(L_1 + L_2 c_2) s_2 \dot{\theta}_1^2 + L_2 \ddot{\theta}_2 + g c_2 \right]$$

$$M(\theta) = \begin{bmatrix} m_1 L_1^2 + (L_1 + c_2 L_2)^2 m_2 & 0 \\ 0 & m_2 L_2^2 \end{bmatrix}$$

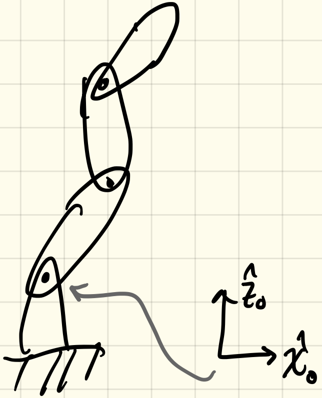
$$V(\theta,\dot{\theta}) = \begin{bmatrix} -2m_2(L_1 + L_2 c_2) L_2 S_2 \dot{\theta}_1 \dot{\theta}_2 \\ m_2 L_2 (L_1 + L_2 c_2) S_1 \dot{\theta}_1^2 \end{bmatrix}$$

$$G(\theta) = \begin{bmatrix} 0 \\ m_2 L_2 g c_2 \end{bmatrix}$$

Sometimes, Individual Components are useful in control.

Example: What torque is reqd. to hold in place?

$\downarrow g$



$$M(\theta)\ddot{\theta} + V(\theta,\dot{\theta}) + G(\theta) = \tau$$

(A) $\tau = J^T \begin{bmatrix} 0 \\ 0 \\ mg \end{bmatrix}$

This answer is correct if $J = \dfrac{\partial\, {}^0 p_{com}}{\partial \theta}$ (Jacobian of the center of mass)

(B) $\tau = G(\theta)$

(C) $\tau = M(\theta) g$

(d) $\tau = G(\theta) + \overline{V(\theta, 0)}$

$= 0$ since every term includes $\dot{\theta}_i^2$ or $\dot{\theta}_i \dot{\theta}_j$

(e) A, B, & D

Alternate Method: Lagrangian Equations of Motion

Potential Energy: $U(\theta)$

Kinetic Energy: $K(\theta, \dot{\theta})$

Lagrangian: $\mathcal{L} = K - U$ (Analytical Dynamics)

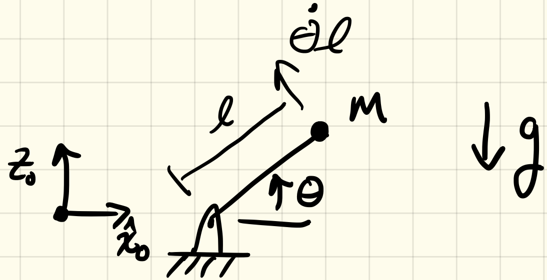
$$\boxed{\tau = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} - \frac{\partial \mathcal{L}}{\partial \theta}}$$

$$\frac{\partial \mathcal{L}}{\partial \theta} = \begin{bmatrix} \partial \mathcal{L}/\partial \theta_1 \\ \vdots \\ \partial \mathcal{L}/\partial \theta_n \end{bmatrix}$$

Lagrangian: Simple Example

$$\mathcal{L} = K - U$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} - \frac{\partial \mathcal{L}}{\partial \theta}$$



$$K = \frac{1}{2}m\|v\|^2 = \frac{1}{2}ml^2\dot{\theta}^2$$

$$U = mgl\sin\theta \qquad \mathcal{L} = \frac{1}{2}ml^2\dot{\theta}^2 - mgl\sin\theta$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = ml^2\dot{\theta}$$

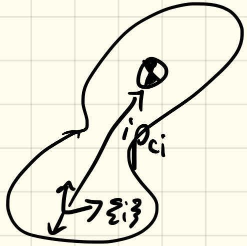
$$\tau = \frac{d}{dt}\left[ml^2\dot{\theta}\right] + mglc_\theta$$

$$\frac{\partial \mathcal{L}}{\partial \theta} = -mglc_\theta$$

$$\boxed{\tau = ml^2\ddot{\theta} + mglc_\theta}$$

$$\tau - mglc_\theta = ml^2\ddot{\theta}$$

Lagrangian: General Case $\quad (m_i , {}^{c_i}I)$



- Velocity Prop

$$ {}^{i}\omega_i = {}^{i}R_{i-1}\, {}^{i-1}\omega_{i-1} + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_i \end{bmatrix} $$

$$ {}^{i}v_i = {}^{i}R_{i-1}\left[{}^{i-1}v_{i-1} + {}^{i-1}\omega_{i-1} \times {}^{i-1}p_i \right] + \begin{bmatrix} 0 \\ 0 \\ \dot{d}_i \end{bmatrix} $$

$$ {}^{i}v_{c_i} = {}^{i}v_i + {}^{i}\omega_i \times {}^{i}p_{c_i} $$

$$ K_i = \frac{1}{2} m_i \, \| {}^{i}v_{c_i} \|^2 + \frac{1}{2}\, {}^{i}\omega_i^T \, {}^{c_i}I \, {}^{i}\omega_i \qquad K = \sum K_i $$

- Kinematic Prop

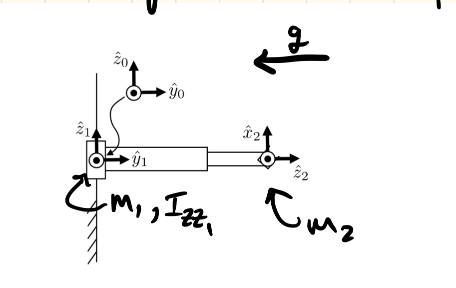
  - $ {}^{0}T_i = {}^{0}T_{i-1}\, {}^{i-1}T_i = \begin{bmatrix} {}^{0}R_i & {}^{0}p_i \\ 0 & 1 \end{bmatrix} $
  - $ {}^{0}p_{c_i} = {}^{0}p_i + {}^{0}R_i \, {}^{i}p_{c_i} $

  - $ U_i = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} {}^{0}p_{c_i} \cdot m_i g \qquad U = \sum U_i $

    (if gravity in $-\hat{z}_0$)

Example: RP Manipulator

① $\mathcal{L} = K - U$ ② $\tau = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) - \left(\partial \mathcal{L} / \partial \theta\right)$



Body 1: $U_1 = 0$ $K_1 = \frac{1}{2} \cancel{m_1 \|{}^1 v_{c_1}\|^2} + \frac{1}{2} {}^1\omega_1^T \, {}^{c_1}I \, {}^1\omega_1$

$${}^1\omega_1 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix}$$

$$\frac{1}{2} {}^1\omega_1^T \, {}^{c_1}I \, {}^1\omega_1 = \begin{bmatrix} 0 & 0 & \dot{\theta}_1 \end{bmatrix} \begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & I_{zz_1} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} = \frac{1}{2} I_{zz_1} \dot{\theta}_1^2$$

$$K_1 = \frac{1}{2} I_{zz_1} \dot{\theta}_1^2$$

Body 2: Potential

$$U_2 = m_2 c_1 d_2 g$$

Kinetic for Body 2:

$${}^2v_{c_2} = \begin{bmatrix} 0 \\ -d_2\dot{\theta}_1 \\ \dot{d}_2 \end{bmatrix}$$

TO BE CONTINUED

Recap:

- RNEA
  - Solution Structure $\tau = M(\theta)\ddot{\theta} + V(\theta, \dot{\theta}) + G(\theta)$
- Lagrangian Technique
  - Same equations
  - Same Structure