23

Analytical Dynamics

D'Alembert Returns (3D Rigid Bodies)

D'Alembert

Euler

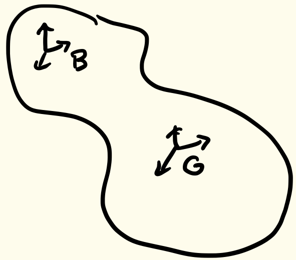
Admin:
- No Class next Monday
- HW 7 due next Tuesday @5PM
- HW 8 due wednesday when you return

Today:
- Euler's Equations wrap up
- Generalized Forces w/ Moments in 3D
- D'Alembert's Equations w/ 3D Rigid Bodies

Final Topics:
- Nonholonomic Constraints (i.e., wheels & skates)
- Generalized Speeds and the Jourdain/Kane equations

Euler's Equation: REVIEW

Assume Frames G and B attached to the Body



$$\dot{\underline{H}}_G = \sum \underline{M}_G$$

$$^G\{\dot{\underline{H}}_G\} = \frac{d}{dt}\,^G\{\underline{H}_G\} + \,^G\{\underline{\omega}\} \times \,^G\{\underline{H}_G\}$$

$$= \frac{d}{dt}\left[^G I_G \,^G\{\underline{\omega}\}\right] + \,^G\{\underline{\omega}\} \times \,^G \underline{I}_G \,^G\{\underline{\omega}_G\}$$

$$= \,^G I_G \,^G\{\dot{\underline{\omega}}\} + \,^G\{\underline{\omega}\} \times \,^G I_G \,^G\{\underline{\omega}\}$$

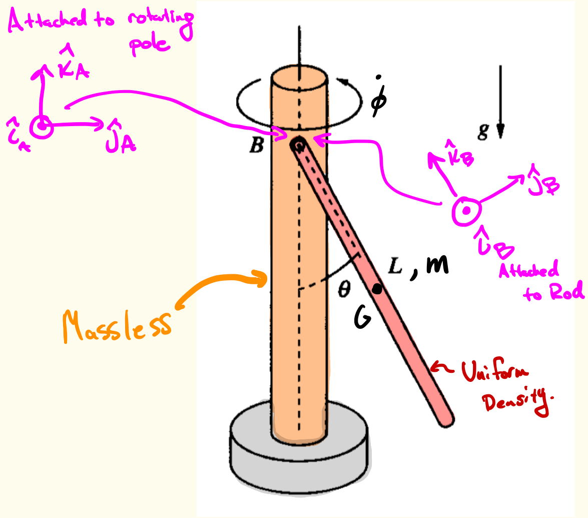
$$\boxed{^G\{\underline{M}_{net,G}\} = \,^G I_G \,^G\{\dot{\underline{\omega}}\} + \,^G\{\underline{\omega}\} \times \,^G I_G \,^G\{\underline{\omega}\}}$$ Euler's Equation

If Rotating Pure About B

$$^B\{\underline{M}_{net,B}\} = \,^B\{\underline{M}_{net,G} + \underline{r}_{G/B} \times \underline{F}_{net}\} = \,^B I_B \,^B\{\dot{\underline{\omega}}\} + \,^B\{\underline{\omega}\} \times \,^B I_B \,^B\{\underline{\omega}\}$$

Verify the EOM Using Euler's Equation

$${}^{B}\{\underline{M}_{B,net}\} = \frac{1}{3} m L^2 \begin{bmatrix} \ddot{\theta} - C_\theta S_\theta \dot{\phi}^2 \\ S_\theta \ddot{\phi} + 2 C_\theta \dot{\phi} \dot{\theta} \\ 0 \end{bmatrix}$$



We must Sample Euler's Equations along carefully Chosen directions to avoid Constraint moments.

(1) $${}^{B}\{\hat{\imath}_B\} \cdot {}^{B}\{\underline{M}_{B,net}\} = \frac{1}{3} m L^2 \left(\ddot{\theta} - C_\theta S_\theta \dot{\phi}^2\right)$$

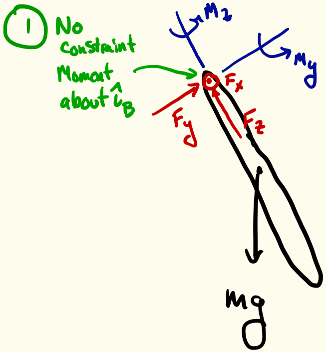
$$\boxed{\underbrace{-mg\frac{L}{2}S_\theta} = \frac{1}{3} m L^2 \left(\ddot{\theta} - C_\theta S_\theta \dot{\phi}^2\right)}$$

Gravity is the only force creating a moment about B in the $\hat{\imath}_B$ direction.

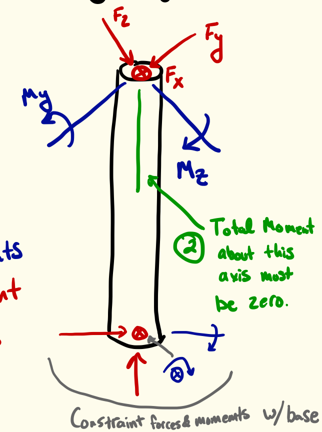
(2) $${}^{B}\{\hat{k}_A\} \cdot {}^{B}\{\underline{M}_{B,net}\} = \begin{bmatrix} 0 \\ S_\theta \\ C_\theta \end{bmatrix} \cdot {}^{B}\{\underline{M}_{B,net}\} = 0$$

$$\boxed{0 = \frac{1}{3} m L^2 \left[S_\theta^2 \ddot{\phi} + 2 C_\theta S_\theta \dot{\phi} \dot{\theta}\right]}$$

Free Body Digram Rod



Free Body Digram Pole



Constraint Moments

Constraint Forces

Euler's Equation when using a frame in line w/ Principal Axes

$$\{\underline{M}_{net}\} = \underline{\underline{I}}\,\{\underline{\dot{\omega}}\} + \{\omega\} \times \underline{\underline{I}}\,\{\omega\}$$

All vectors expressed in a frame aligned w/ principal axes

$$\{\underline{\omega}\} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \qquad \underline{\underline{I}} = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix} \qquad \{\underline{M}_{net}\} = \begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix}$$

$$\begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix} = \begin{bmatrix} I_{xx}\dot{\omega}_x + (I_{zz} - I_{yy})\,\omega_z\omega_y \\ I_{yy}\dot{\omega}_y + (I_{xx} - I_{zz})\,\omega_x\omega_z \\ I_{zz}\dot{\omega}_z + (I_{yy} - I_{xx})\,\omega_x\omega_y \end{bmatrix} \qquad (*)$$

__when:__

$$\underline{M}_{net} = 0 \Rightarrow \{\underline{\omega}\} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ OR } \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \text{ OR } \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

are constant solutions to (*)

Linearization of Dynamics About This solution

$$\omega_x = \omega_x^* + \Delta\omega_x$$
$$\omega_y = \omega_y^* + \Delta\omega_y$$
$$\omega_z = \omega_z^* + \Delta\omega_z$$

$$\frac{d}{dt}\Delta\omega_x = \frac{I_{yy}-I_{zz}}{I_{xx}}\left[\Delta\omega_z\omega_y^* + \omega_z^*\Delta\omega_y\right]$$

$$\frac{d}{dt}\Delta\omega_y = \frac{I_{zz}-I_{xx}}{I_{yy}}\left[\Delta\omega_x\omega_z^* + \omega_x^*\Delta\omega_z\right]$$

$$\frac{d}{dt}\Delta\omega_z = \frac{I_{xx}-I_{yy}}{I_{zz}}\left[\Delta\omega_x\omega_y^* + \omega_x^*\Delta\omega_y\right]$$

Consider $I_{xx} < I_{yy} < I_{zz}$ $\{\omega^*\} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$$\frac{d}{dt}\Delta\omega_y = 0$$

$$\frac{d}{dt}\begin{bmatrix}\Delta\omega_x \\ \Delta\omega_z\end{bmatrix} = \underbrace{\begin{bmatrix} 0 & \frac{I_{yy}-I_{zz}}{I_{xx}} \\ \frac{I_{xx}-I_{yy}}{I_{zz}} & 0 \end{bmatrix}}_{A}\begin{bmatrix}\Delta\omega_x \\ \Delta\omega_z\end{bmatrix}$$

$$\text{trace}(A) = \lambda_1 + \lambda_2 = 0$$

$$\det(A) = \lambda_1\lambda_2$$

$$= \overbrace{\frac{I_{zz}-I_{yy}}{I_{xx}}}^{>0}\ \overbrace{\frac{I_{xx}-I_{yy}}{I_{zz}}}^{<0}$$

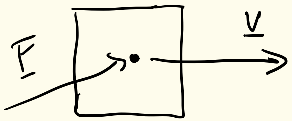
$\Rightarrow$ one eig >0 $\Rightarrow$ unstable

Toward D'Alembert's Principle w/ 3D Rigid Bodies

$$\sum_{i=1}^{N} \left(\underline{F}_i + \underline{F}_i' - \underline{m}\, \underline{a}_i \right) \cdot \delta \underline{r}_i = \sum_{i=1}^{N} \left(\underline{F}_i - m_i \underline{a}_i \right) \cdot \delta \underline{r}_i$$

Key: Virtual work of constraint forces was zero under virtual dispacements consistent w/ constraints

Incremental Work



$$dW = dt\,P = \underline{F} \cdot \underline{v}\, dt = \underline{F} \cdot d\underline{r}$$

Virtual Work

$$\delta W = \underline{F} \cdot \delta \underline{r}$$

Virtual Dispacement

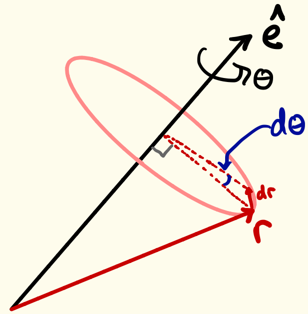
$$\delta \underline{r}_i = \sum_k \frac{\partial \underline{r}_i}{\partial q_k} \delta q_k$$

$$= \sum_k \frac{\partial \underline{v}_i}{\partial \dot{q}_k} \delta q_k$$

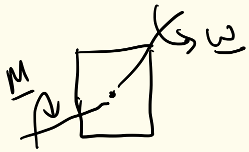
$$\frac{\partial \underline{r}_i}{\partial q_k} = \frac{\partial \underline{v}_i}{\partial \dot{q}_k}$$

Virtual Work : Moments

Recall: Any angular velocity $\underline{\omega} = \hat{e}\,\frac{d\theta}{dt}$ for some axis $\hat{e}$



Incremental Work



$$dW = dt\,P = \underline{M}\cdot\underline{\omega}\,dt = \underline{M}\cdot\overbrace{\hat{e}\,d\theta}^{:=d\underline{\theta}}$$

Virtual Work: $\delta W = \underline{M}\cdot\delta\underline{\theta}$

Virtual Displacement $\delta\underline{\theta}_i = \sum_k \frac{\partial \underline{\omega}_i}{\partial \dot{q}_k}\,\delta q_k$

Aside: variation in a rotation matrix:

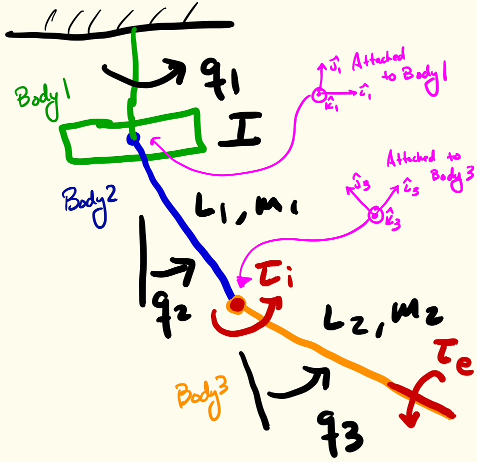
$$\delta\,{}^0R_1 = {}^0\left[\widetilde{\delta\underline{\theta}_1}\right]{}^0R_1$$

Example:

τ_i is an internal torque exerted by body L_1 onto body L_2

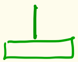
τ_e is an external torque exerted onto body L_2

Give the generalized forces



Moments Applied on Bodies (Ignoring Constraint Moments/Forces)

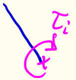
Body 1



No Applied Moments

$\underline{M}_1 = 0$

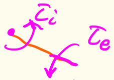
Body 2



Opposite reaction Moment

$\underline{M}_2 = -\tau_i \hat{k}_3$

Body 3



$\underline{M}_3 = \tau_i \hat{k}_3 - \tau_e \hat{j}_3$

When $\delta q_1 \neq 0$ $\quad \delta \underline{\theta}_1 = \delta \underline{\theta}_2 = \delta \underline{\theta}_3 = \hat{j}_1 \delta q_1 \Rightarrow \delta W = (\underline{M}_1 + \underline{M}_2 + \underline{M}_3) \cdot \hat{j}_1 \delta q_1 = -\tau_e \hat{j}_3 \cdot \hat{j}_1 \delta q_1$

$$= -\tau_e \cos(q_2 + q_3)\, \delta q_1$$

$$\boxed{Q_1 = -\tau_e \cos(q_2 + q_3)}$$

When $\delta q_2 \neq 0$ $\quad \delta \underline{\theta}_1 = 0,\ \delta \underline{\theta}_2 = \delta \underline{\theta}_3 = \hat{k}_3 \delta q_2 \Rightarrow \delta W = (\underline{M}_2 + \underline{M}_3) \cdot \hat{k}_3 \delta q_2 = 0$

$$\boxed{Q_2 = 0}$$

When $\delta q_3 \neq 0$ $\quad \delta \underline{\theta}_1 = \delta \underline{\theta}_2 = 0,\ \delta \underline{\theta}_3 = \hat{k}_3 \delta q_3 \Rightarrow \delta W = \underline{M}_3 \cdot \hat{k}_3 \delta q_3 = \tau_i \delta q_3$

$$\boxed{Q_3 = \tau_i}$$

D'Alembert's principle For 3D Rigid Bodies

$$\sum_{i=0}^{N}\left(\underline{F}_i + \cancel{\underline{F}'_i} - m_i \dot{\underline{v}}_{G_i}\right)\cdot \delta \underline{r}_i + \left(\underline{M}_{G,i} + \cancel{\underline{M}'_{G,i}} - \dot{\underline{H}}_{G_i}\right)\cdot \delta \underline{\theta}_i = 0$$

Generalized Forces

$$Q_K = \sum_{i=1}^{N}\left[\underline{F}_i \cdot \frac{\partial \underline{v}_{G_i}}{\partial \dot{q}_K} + \underline{M}_{G,i} \cdot \frac{\partial \underline{\omega}_i}{\partial \dot{q}_K}\right]$$

A bit more detail on next page

Generalized Inertial Forces

$$Q_{K,\text{inertial}} = \sum_{i=1}^{N}\left[m_i \dot{\underline{v}}_{G,i} \cdot \frac{\partial \underline{v}_{G_i}}{\partial \dot{q}_K} + \underbrace{\dot{\underline{H}}_{G_i} \cdot \frac{\partial \underline{\omega}_i}{\partial \dot{q}_K}}_{= \dot{\underline{H}}_{B_i} \cdot \frac{\partial \underline{\omega}_i}{\partial \dot{q}_K}}\right]$$

$= \dot{\underline{H}}_{B_i} \cdot \frac{\partial \underline{\omega}_i}{\partial \dot{q}_K}$ if Body i purely rotating about B_i

If all constraints are captured by generalized coords:

$$Q_K = Q_{K,\text{inertial}} \qquad \forall K$$

Extra Detail: For those MATLAB warriors

Before, with point masses:

$$Q_k = \sum_i \frac{\partial \underline{r}_i}{\partial q_k} \cdot \underline{F}_i \implies \underline{Q} = \begin{bmatrix} Q_1 \\ \vdots \\ Q_n \end{bmatrix} = \sum_i \{J_i\}^T \{F_i\}$$

where

$$J_i = \frac{\partial \underline{r}_i}{\partial \underline{q}}$$

Jacobian of $\underline{r}_i$ w.r.t. to $\underline{q}$

Now, with rigid bodies:

Let

$$\underbrace{J_i^v = \frac{\partial \underline{r}_{G,i}}{\partial \underline{q}} = \frac{\partial \underline{v}_{G,i}}{\partial \dot{\underline{q}}}}_{\text{Linear velocity Jacobian}} \qquad \underbrace{J_i^\omega = \frac{\partial \underline{\omega}_i}{\partial \dot{\underline{q}}}}_{\text{Angular velocity jacobian}}$$

$$\underline{Q} = \begin{bmatrix} Q_1 \\ \vdots \\ Q_n \end{bmatrix} = \sum_{i=1}^{n} \{J_i^v\}^T \{\underline{F}_i\} + \{J_i^\omega\}^T \{\underline{M}_{G,i}\}$$

In MATLAB

```
F2Q = @(F,r) jacobian(r, q)' * F;

M2Q = @(M,w) jacobian(w, q-dot)' * M ;
```

When using F2Q, F and r must be expressed in an inertial frame

When using M2Q, M and w may be expressed in any frame, but that frame must be the same for M and w.

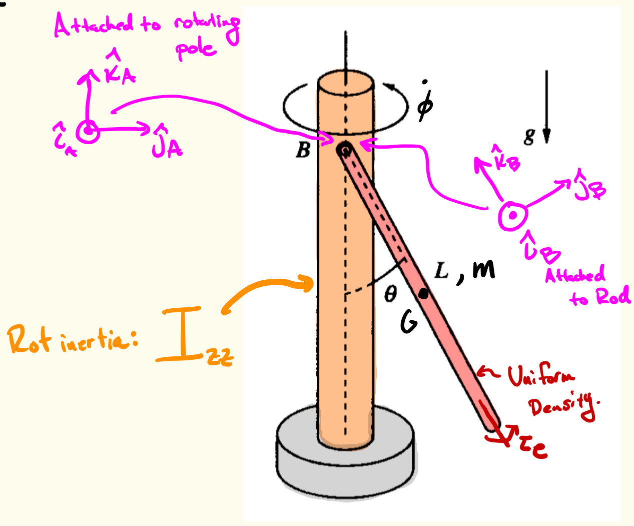
Example: Same as last time, but Pole has Rot. Inertia

Give EOM Via D'Alembert

$$^B\{\underline{\dot{H}}_{rod,B}\} = \frac{1}{3}mL^2 \begin{bmatrix} \ddot\theta - c_\theta s_\theta \dot\phi^2 \\ s_\theta \ddot\phi + 2 c_\theta \dot\theta \dot\phi \\ 0 \end{bmatrix}$$

$$\underline{H}_{pole,B} = I_{zz}\dot\phi \hat{k}_A \qquad \underline{\dot{H}}_{pole,B} = I_{zz}\ddot\phi \hat{k}_A$$

$$\underline{\omega}_{pole} = \dot\phi \hat{k}_A \qquad \underline{\omega}_{rod} = \dot\phi \hat{k}_A + \hat{\imath}_B \dot\theta$$



Generalized Inertial Forces

$$Q_{\phi,inertial} = \underline{\dot{H}}_{rod,B} \cdot \overset{\hat{k}_A}{\frac{\partial \underline{\omega}_{rod}}{\partial \dot\phi}} + \underline{\dot{H}}_{pole,B} \cdot \overset{\hat{k}_A}{\frac{\partial \underline{\omega}_{pole}}{\partial \dot\phi}}$$

$$^B\{\hat{k}_A\} = \begin{bmatrix} 0 \\ s_\theta \\ c_\theta \end{bmatrix}$$

$$= \frac{1}{3}mL^2 s_\theta \left[s_\theta \ddot\phi + 2 c_\theta \dot\theta \dot\phi \right] + I_{zz}\ddot\phi = \boxed{\left(I_{zz} + \frac{1}{3}m(L s_\theta)^2\right)\ddot\phi + \frac{2}{3}mL^2 s_\theta c_\theta \dot\theta \dot\phi}$$

$$\boxed{Q_{\theta,inertial}} = \underline{\dot{H}}_{rod,B} \cdot \overset{\hat{\imath}_B}{\frac{\partial \underline{\omega}_{rod}}{\partial \dot\theta}} + \cancel{\underline{\dot{H}}_{pole,B} \cdot \frac{\partial \underline{\omega}_{pole}}{\partial \dot\theta}}$$

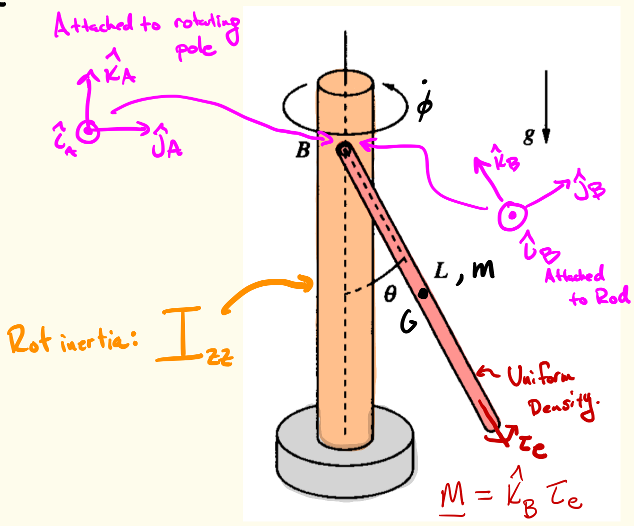
$$\boxed{= \frac{1}{3}mL^2 \left(\ddot\theta - c_\theta s_\theta \dot\phi^2\right)}$$

MATLAB Example online

Example: Same as last time, but Pole has Rot. Inertia

$$^B\{\dot{\underline{H}}_{rod,B}\} = \frac{1}{3} m L^2 \begin{bmatrix} \ddot{\theta} - c_\theta s_\theta \dot{\phi}^2 \\ s_\theta \ddot{\phi} + 2 c_\theta \dot{\theta} \dot{\phi} \\ 0 \end{bmatrix}$$

$$\underline{\omega}_{pole} = \dot{\phi} \hat{k}_A \qquad \underline{\omega}_{rod} = \dot{\phi} \hat{k}_A + \hat{\imath}_B \dot{\theta}$$



Generalized Applied Forces

$$V = -mg \frac{L}{2} c_\theta$$

Missing in Lecture video.

$$Q_\theta = -\frac{\partial V}{\partial \theta} + Q_{\theta,nc} = \boxed{-mg \frac{L}{2} s_\theta} + \cancel{\overbrace{\frac{\partial \underline{\omega}_{Rod}}{\partial \dot{\theta}}}^{\hat{\imath}_B} \cdot \underline{M}}^{\,0}$$

$$Q_\phi = -\frac{\partial V}{\partial \phi} + Q_{\phi,nc} = 0 + \overbrace{\frac{\partial \underline{\omega}_{rod}}{\partial \dot{\phi}}}^{\hat{k}_A} \cdot \underline{M} = \boxed{\cos(\theta)\, \tau_e}$$

EoM:

$$\boxed{\begin{aligned} \tau_e \cos(\theta) &= \left(I_{zz} + \tfrac{1}{3} m (L s_\theta)^2\right) \ddot{\phi} + \tfrac{2}{3} m L^2 s_\theta c_\theta \dot{\theta} \dot{\phi} \\ -mg \tfrac{L}{2} \sin(\theta) &= \tfrac{1}{3} m L^2 \left(\ddot{\theta} - c_\theta s_\theta \dot{\phi}^2\right) \end{aligned}}$$

MATLAB Example online

Summary

- **Euler's Equation in a principal axis Frame**

$$\begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix} = \begin{bmatrix} I_{xx}\dot{\omega}_x + (I_{zz}-I_{yy})\omega_z\omega_y \\ I_{yy}\dot{\omega}_y + (I_{xx}-I_{zz})\omega_x\omega_z \\ I_{zz}\dot{\omega}_z + (I_{yy}-I_{xx})\omega_x\omega_y \end{bmatrix}$$

- **Generalized Forces:**

$$Q_k = \sum_i \underline{M}_{G,i} \cdot \frac{\partial \underline{\omega}_i}{\partial \dot{q}_k} + \underline{F}_i \cdot \frac{\partial \underline{v}_{G_i}}{\partial \dot{q}_k} \qquad \text{(Applied Forces)}$$

$$Q_{k,inertial} = \sum_i \dot{\underline{H}}_{G_i} \cdot \frac{\partial \underline{\omega}_i}{\partial \dot{q}_k} + m_i \underline{a}_{G,i} \cdot \frac{\partial \underline{v}_{G,i}}{\partial \dot{q}_k} \qquad \text{(Inertial Forces)}$$

D'Alembert:

$$\sum_k (Q_k - Q_{k,inertial})\,\delta q_k = 0 \qquad \forall\, \delta q \text{ consistent w/ constraints}$$