10/

Systems of Particles

2D Rigid-Bodies

Admin:

- Exam 1 corrections Due next Monday 10/7
- HW 3 to appear later this week

Goals for Today:

3D In Class

Supplemental Content Available on Panopto

- Go over Exam 1
- Systems of Particles & 2D Rigid Bodies
  - Should be all review from Undergrad / Mechanics II
  - If we don't get to it, I'll add it online for viewing @ home

On Deck:

- The fundamentals of Analytical Mechanics
  - No more worrying about constraint forces
  - No relative accelerations (until after exam II)

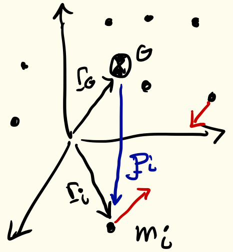
Systems of Particles:

- N particles mass m_i

  Total mass $m = \sum_{i=1}^{N} m_i$

  Center of mass $\underline{r}_G = \frac{1}{m} \sum_{i=1}^{N} m_i \underline{r}_i$

  $\underline{r}_i = \underline{r}_G + \underline{\rho}_i \qquad \left(\sum_{i=1}^{N} m_i \underline{\rho}_i = 0 \right)$



F_i' total internal constraint force

F_i net external force applied onto particle i

- CoM velocity $\underline{V}_G = \dot{\underline{r}}_G = \frac{1}{m} \sum_{i=1}^{N} m_i \underline{V}_i$

- Linear momentum $\underline{p} = \sum m_i v_i = m \underline{V}_G$

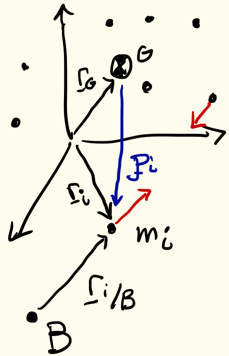
- Forces $m_i a_i = F_i + F_i' \Rightarrow m \underline{a}_G = \sum m_i a_i = \sum_{i=1}^{N} \underbrace{F_i'}_{} + F_i = \sum F_i$

all constraint forces occur in equal and opposite pairs.

Systems of Particles: Angular Momentum

$$H_B = \sum_{i=1}^{N} r_{i/B} \times m_i v_i$$

$$= \sum (r_{G/B} + \rho_i) \times m_i (v_G + \dot{\rho}_i)$$



↓ Derivation on next Slide

$$= \underbrace{r_{G/B} \times m v_G}_{\text{moment of the linear momentum of the CoM}} + \underbrace{\sum_{i=1}^{N} \rho_i \times m_i \dot{\rho}_i}_{\substack{\text{Angular momentum about the CoM} \\ := H_G}}$$

$$\dot{H}_B = \underbrace{\sum_{i=1}^{N} r_{i/B} \times F_i}_{:= M_B} + m v_G \times v_B$$

Derivation on Next Slide

Interesting Cases

① B stays fixed

$$\Rightarrow \dot{H}_B = M_B$$

② B = G

$$\Rightarrow \dot{H}_G = M_G$$

$$\underline{H}_B = \sum_{i=1}^{N} \big(\underbrace{\underline{r}_{G/B}}_{(a)} + \underbrace{\underline{\rho}_i}_{(b)}\big) \times m_i \big(\underbrace{\underline{v}_G}_{(c)} + \underbrace{\dot{\underline{\rho}}_i}_{(d)}\big)$$

$$= \underbrace{\sum_{i=1}^{N} m_i \underline{r}_{G/B} \times \underline{v}_G}_{(a)\times(c)} + \underbrace{\cancelto{=0}{\left[\sum_{i=1}^{N} m_i \underline{\rho}_i\right]} \times \underline{v}_G}_{(b)\times(c)} + \underbrace{\underline{r}_{G/B} \times \cancelto{=0}{\left[\sum_{i=1}^{N} m_i \dot{\underline{\rho}}_i\right]}}_{(a)\times(d)} + \underbrace{\sum_{i=1}^{N} \underline{\rho}_i \times \left[m_i \dot{\underline{\rho}}_i\right]}_{(b)\times(d)}$$

$$= \underline{r}_{G/B} \times \left[m \underline{v}_G\right] + \sum_{i=1}^{N} \underline{\rho}_i \times m_i \dot{\underline{\rho}}_i$$

$$\dot{H}_B = \frac{d}{dt} \sum_{i=1}^{N} r_{i/B} \times m_i v_i = \sum_{i=1}^{N} (v_i - v_B) \times m_i v_i + r_{i/B} \times m_i a_i$$

Cancels out since equal & opposite pairs act along the same line in space

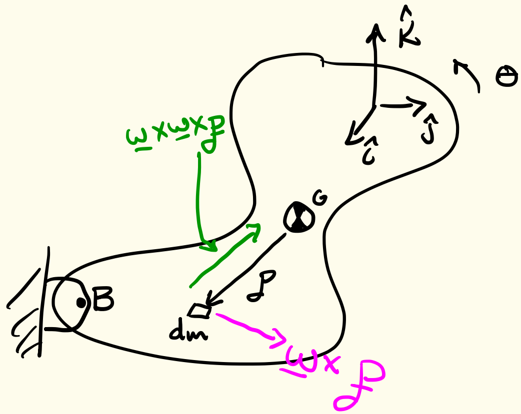
$$= \sum_{i=1}^{N} -v_B \times (m_i v_i) + r_{i/B} \times \big(\overbrace{F_i'} + F_i\big)$$

$$= m v_G \times v_B + \sum_{i=1}^{N} r_{i/B} \times F_i$$

Special Case: 2D Planar Body

Ang. Vel & acceleration: $\underline{\omega} = \hat{K}\dot{\theta}$ $\quad \underline{\alpha} = \hat{K}\ddot{\theta}$

Velocity & Acceleration: $\underline{V}_G$ & $\underline{a}_G$



Velocity & Acceleration of mass dm

$$\underline{V} = \underline{V}_G + \underline{\omega}\times\underline{\rho}$$

$$\underline{a} = \underline{a}_G + \underline{\alpha}\times\underline{\rho} + \underline{\omega}\times\underline{\omega}\times\underline{\rho}$$

$$= \underline{a}_G + \underline{\alpha}\times\underline{\rho} - \dot{\theta}^2\underline{\rho}$$

$$\underline{M}_G = \int_{Body}\underline{\rho}\times d\underline{F} = \int_{Body}\underline{\rho}\times(\cancel{\underline{a}_G} + \underline{\alpha}\times\underline{\rho} - \cancel{\dot{\theta}^2\underline{\rho}})\,dm = \int_{Body}\underline{\rho}\times(\underline{\alpha}\times\underline{\rho})\,dm$$

$$= \int_{Body}|\underline{\rho}|^2 dm\,\ddot{\theta}\hat{K}$$

$:= I_G$ Rotational Inertia About CoM

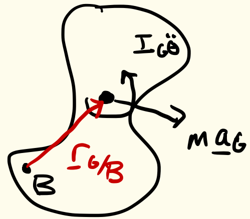
Suppose B fixed in space

$$\underline{M}_B = \dot{\underline{H}}_B = I_G\ddot{\theta}\hat{K} + \underline{r}_{G/B}\times m\,\underline{a}_G$$

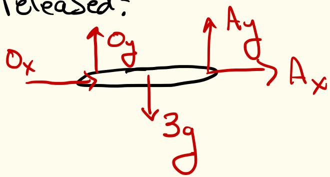
$\underline{\alpha}\times\underline{r}_{G/B} - \underline{r}_{G/B}\dot{\theta}^2$

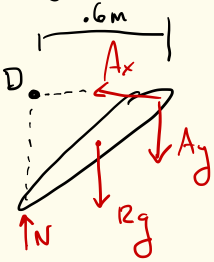
$$= \underbrace{(I_G + |r_{G/B}|^2 m)}_{I_B}\ddot{\theta}\hat{K}$$

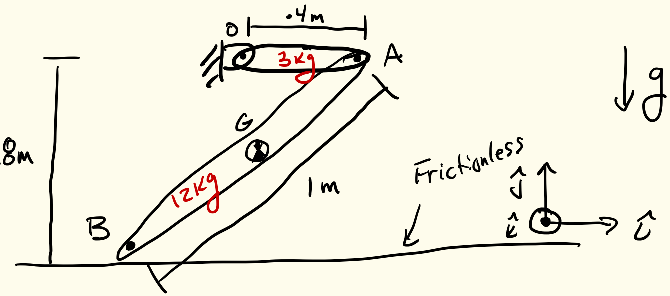
I_B rotational inertia about B



Example: Released from Rest. What are the angular velocities of the rods at the moment the system is released?







① **Rod OA**

$$I_o = \frac{1}{3}mL^2 = 0.16\ kgm^2$$

② **Moment balance @ O:**

$$I_o\alpha_1 = .4A_y - .6g$$

2 eqns, 2 unknowns Solve for A_y, α_2

③ Rod AB

$$\underline{a}_A = .4\alpha_1 \hat{j}$$

$$\underline{a}_B = \ddot{x}\,\hat{i}$$

$$\underline{a}_B = \underline{a}_A + \underline{\alpha}_2 \times \underline{r}_{B/A}$$

$$= \uparrow .4\alpha_1 + \odot\alpha_2 \times \begin{pmatrix} .6 \\ .8 \end{pmatrix}$$

$$= \uparrow .4\alpha_1 + \downarrow .6\alpha_2 + \overrightarrow{.8\alpha_2}$$

$$= (.4\alpha_1 - .6\alpha_2)\hat{j} + .8\alpha_2\hat{i}$$

$$\boxed{\ddot{x} = .8\alpha_2} \qquad \boxed{\alpha_1 = 1.5\alpha_2}$$

$$\underline{a}_G = \underline{a}_B + \underline{\alpha}_2 \times \underline{r}_{G/B}$$

$$= .4\alpha_2\hat{i} + .3\alpha_2\hat{j}$$

④ **Moment Balance @ D:** $I_G = \frac{1}{12}(12kg)\cdot 1^2 = 1\ kgm^2$

$$\underline{M}_D = \left[I_G\alpha_2\hat{k} + \underline{r}_{G/D} \times m\,\underline{a}_G\right] = \left[-.6A_y - 3.6g\right]\hat{k}$$

⑤ Solve @ home

$$\boxed{\alpha_2 = 10.12\ rad/s^2} \qquad \boxed{\alpha_1 = 15.18\ rad/s^2}$$

Summary:

Systems of Particles in 3D:

- $m \dot{\underline{v}}_G = \sum_i \underline{F}_i$
- $\dot{H}_B = \underbrace{\sum \underline{r}_{i/B} \times F_i}_{:= M_B}$ if B fixed OR $B = G$

Rigid Bodies in 2D:

$$\dot{H}_B = \underline{r}_{G/B} \times m\, \underline{a}_G + I_G \ddot{\theta} \hat{K} \quad (B \text{ fixed OR } B = G)$$

Special case: Pure rotation about fixed point B

$$\dot{H}_B = \left(I_G + md^2\right) \ddot{\theta} \hat{K} \quad d = |\underline{r}_{G/B}|$$