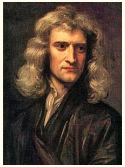
Newtonian Particle Dynamics

Impulse, Momentum, Work, & Energy



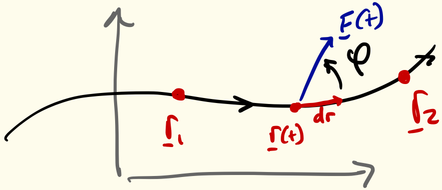
- Admin
  - HW 1: Due next Wednesday
- Last Time: Impulse/ Momentum Principles

$$\underline{p}(t_2) = \underline{p}(t_1) + \int_{t_1}^{t_2} \underline{F}(t)\, dt$$

$$\underline{H}_o(t_2) = \underline{H}_o(t_1) + \int_{t_1}^{t_2} \underline{M}_o(t)\, dt$$

- Today:
  - Incremental Work $\Rightarrow$ Incremental change in Kinetic En.
  - Conservative forces and Total Energy
  - Conditions on Equilibria

Work & Energy:



- Incremental Work done over an incremental distance $d\underline{r}$

$$dW = \underline{F} \cdot d\underline{r} = |\underline{F}||d\underline{r}| \cos\varphi$$

$$= m\underline{a} \cdot d\underline{r}$$

- Kinetic Energy of Particle: $T = \frac{1}{2} m \underline{v} \cdot \underline{v}$

$$dW = m \frac{d\underline{v}}{\cancel{dt}} \cdot \underbrace{\frac{d\underline{r}}{dt}}_{\underline{v}} \cancel{dt} = m \, d\underline{v} \cdot \underline{v} = d\left(\frac{1}{2} m \underline{v} \cdot \underline{v}\right) = dT$$

- Work: $W_{1\to 2} = \int_{r_1}^{r_2} \underbrace{\underline{F} \cdot d\underline{r}}_{dW} = \int_{T_1}^{T_2} dT = T\Big|_{T_1}^{T_2} = T_2 - T_1$

- Work-Energy Thm: $T_1 + W_{1\to 2} = T_2$ only for kinetic energy

Conservative Force (Vector) Fields:

Consider $\underline{F}(\underline{r})$ a vector Field.

$$\underline{F}(x,y,z) = F_x(x,y,z)\,\hat{\imath} + F_y(x,y,z)\,\hat{\jmath} + F_z(x,y,z)\,\hat{k}$$

Def: We say $\underline{F}(\underline{r})$ is conservatie/(exact) if $\underline{F}$ can be expressed as the negative gradient of some scalar function $V(\underline{r})$. That is

$$-\frac{\partial V}{\partial x} = F_x \qquad -\frac{\partial V}{\partial y} = F_y \qquad -\frac{\partial V}{\partial z} = F_z$$

V: Potential energy Function

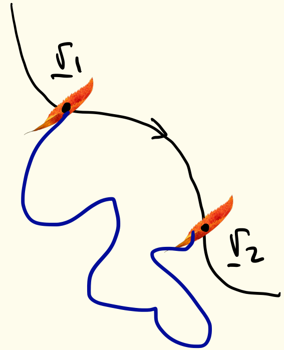
Incremental work:

$$dW = \underline{F}(\underline{r})\cdot d\underline{r} = -\nabla V(\underline{r})\cdot d\underline{r}$$

$$= -\left[\frac{\partial V}{\partial x}\hat{\imath} + \frac{\partial V}{\partial y}\hat{\jmath} + \frac{\partial V}{\partial z}\hat{k}\right]\cdot\left[dx\,\hat{\imath} + dy\,\hat{\jmath} + dz\,\hat{k}\right]$$

$$= -dV = -\frac{\partial V}{\partial x}dx - \frac{\partial V}{\partial y}dy - \frac{\partial V}{\partial z}dz$$

Work Done By a Conservative Force



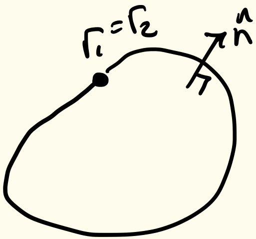
① Along a path: Consider work by gravity

$$\underset{\text{(just from gravity)}}{W_{1\to 2}} = \int_{\underline{r}_1}^{\underline{r}_2} \underline{F}_g(\underline{r}) \cdot d\underline{r} = \int_{V(r_1)}^{V(r_2)} -dV = \Delta T_{\text{due to gravity}}$$

$$= V(\underline{r}_1) - V(\underline{r}_2)$$

- Work done is path independent (depends only on endpoints)

② Around a loop



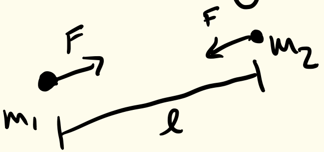
$$W = \oint \underline{F}(\underline{r}) \cdot d\underline{r} = 0 = \int_S \text{curl}(F) \cdot \hat{n}\, dA$$

- Conservative => curl(F) = 0 everwhere
- Opposite holds when F defined everwhere
- curl(F) = 0 same as requiring $\dfrac{\partial F_y}{\partial x} = \dfrac{\partial F_x}{\partial y}$ ...

<u>Reverse:</u> If $\underline{F}(\underline{r})$ is conservative, you can find a $V(\underline{r})$ by line integral

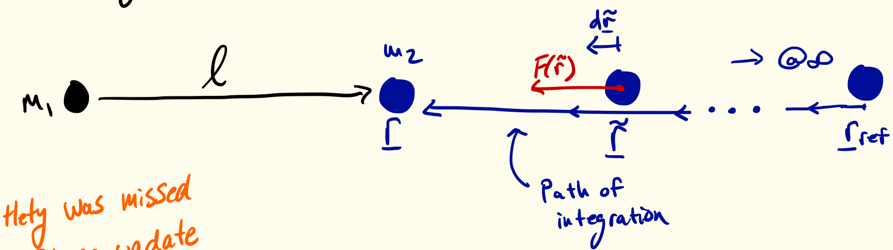
$$V(\underline{r}) = -\int_{\underline{r}_{ref}}^{\underline{r}} F(\tilde{\underline{r}})\cdot d\tilde{\underline{r}}$$

<u>Example:</u> Gravity



$$F = \frac{G m_1 m_2}{\ell^2}$$

Consider reference with particles infinitely far apart.



This subtlety was missed in lecture. Please update your notes.

$$V(\ell) = -\int_{r_{ref}}^{r} \underbrace{F(\tilde{\underline{r}})\cdot d\tilde{\underline{r}}} = -\int_{\infty}^{\ell} \underbrace{\frac{Gm_1m_2}{\tilde{\ell}^2}(-d\tilde{\ell})} = -\int_{\ell}^{\infty} \frac{Gm_1m_2}{\tilde{\ell}^2}\,d\tilde{\ell} = \frac{Gm_1m_2}{\tilde{\ell}}\Bigg|_{\tilde{\ell}=\ell}^{\infty} = \boxed{-\frac{Gm_1m_2}{\ell}}$$

Work always positive since $d\underline{r}$ points ← and $F(\underline{r})$ points ←

We use $-d\tilde{\ell}$ since $d\tilde{\ell}$ is negative due to limits of integration.

Equilibrium & Stability: $\quad V(\underline{r})$

- If a particle reaches a stationary equilibrium: net Force on it is zero.
- If a particle in a potential fied $F = -\nabla V$

Config Stable if when you start close you stay close

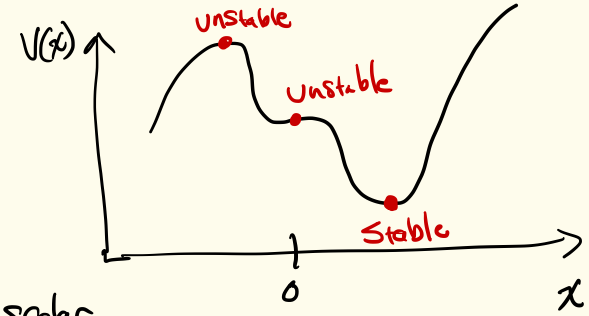
- In equilibrium $\nabla V = 0$
- $\nabla^2 V = \begin{bmatrix} \partial^2 V/\partial x^2 & \partial^2 V/\partial x \partial y & \cdot \\ \partial^2 V/\partial x \partial y & \partial^2 V/\partial y^2 & \cdot \\ \partial^2 V/\partial x \partial z & \partial^2 V/\partial y \partial z & \partial^2 V/\partial z^2 \end{bmatrix}$

  has all positive eigenvalues

  $\Rightarrow$ equilibrium is stable

- Scalar case $V(x)$ $\quad \nabla^2 V = \partial^2 V/\partial x^2$ is scalar

Potential Well in 1D



- if $\nabla^2 V$ has even just one negative eigenvalue

  $\Rightarrow$ equilibrium unstable

- O/w we cant tell from 2nd derivative info alone

Mixing Conservative & Nonconservative Forces:

Consider: $\mathbf{F} = \mathbf{F}_c + \mathbf{F}_{nc}$ conservatie and non conservative parts

Incremental Work: $dW = dW_c + dW_{nc} = -dV + dW_{nc}$

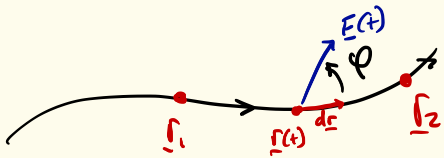
Work:
$$\underbrace{W_{1\to2}}_{T_2 - T_1} = \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F}\cdot d\mathbf{r} = \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F}_c\cdot d\mathbf{r} + \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F}_{nc}\cdot d\mathbf{r}$$
$$= -(V_2 - V_1) + \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F}_{nc}\cdot d\mathbf{r}$$

Rearranging

Change in Kinetic:
$$T_2 + V_2 = T_1 + V_1 + \overbrace{\int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F}_{nc}\cdot d\mathbf{r}}^{W_{nc,1\to2}}$$

Total Energy: $E = T + V \Rightarrow E_2 = E_1 + W_{nc,1\to2}$

Principle of Conservation of Energy: When only conservative forces are present:
$$E_1 = E_2$$



Unifying Conservation of Momentum and Conservation of Energy

- Both are cases of the more general case of an "integral of motion"

- Consider a system w/ n degrees of freedom, $q \in \mathbb{R}^n$ the configuration
  - Equations of motion (eom): n- second order diff. eqs for q.

- <u>Defn</u>: We say a quantity $C(q, \dot{q})$ is an intergral of motion if it is constant along <u>any</u> solution to the eom.

  - often momenta in specific directions
  - Sometimes systems don't have any!

Example: $m\ddot{x} = -Kx$ ① Find an integral of motion

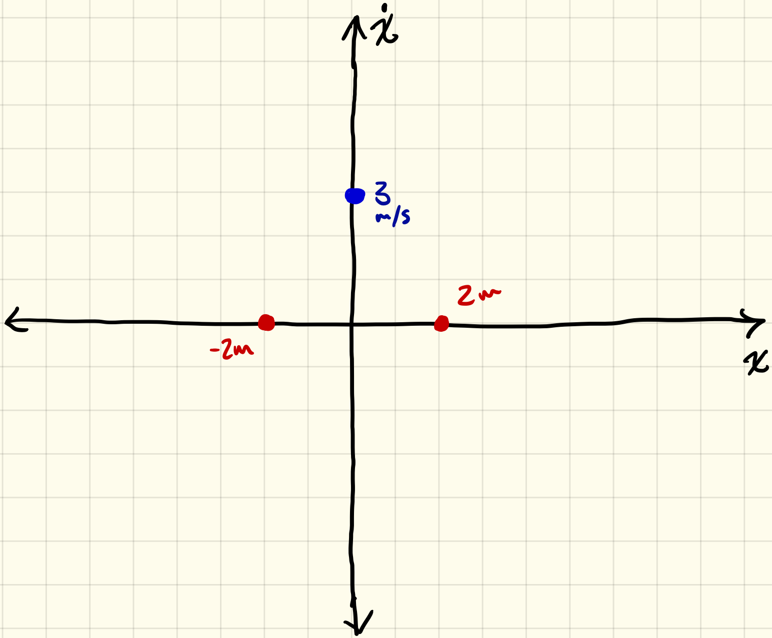
② Sketch the behavior of the system from initial conditions below.

Suppose $K = 9$ N/m, $m = 1$ Kg.

Answer: The integral of motion

$$\boxed{C(x,\dot{x}) = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}Kx^2}$$

TO BE CONTINUED...



Summary

- Conservative Force Field: $\underline{F}(\underline{r}) = -\nabla V(r)$
    - Work done is path independent
    - $\nabla V = 0$ @ Equilibrium, $\nabla^2 V > 0 \Rightarrow$ stable

- Work - Energy Principle

$$T_2 = T_1 + W_{1\to 2} = T_1 + \int_{r_1}^{r_2} \underline{F} \cdot d\underline{r} = T_1 + \int_{t_1}^{t_2} \underline{F} \cdot \underline{v} \, dt$$

$$E_2 = E_1 + W_{nc,1\to 2} = E_1 + \int_{r_1}^{r_2} \underline{F}_{nc} \cdot d\underline{r} = E_1 + \int_{t_1}^{t_2} \underline{F}_{nc} \cdot \underline{v} \, dt$$

- Integral of motion: Constant along trajectories
    - Often energy or momentum