Lecture 9: The Linear Quadratic Regulator

Last time:

- Optimal Cost to Go
  - Optimal cost to go = min [what I pay now + what I pay Later]
  - Bellman's equation in discrete time
  - Hamilton-Jacobi-Bellman Equation (HJB) in continuous time

Today

- Linear Quadratic Regulator (LQR)
- Robot Dynamics as optimal control

Discrete time Review:

System

$$x[k+1] = f(k, x[k], u[k])$$

$$x[k_0] = x_0$$

Cost Function

$$V(u[\cdot], k_0, x_0) = \sum_{k=0}^{N-1} \ell(k, x[k], u[k]) + \ell_f(x[N])$$

$$x[k_0] = x_0$$

Optimal Cost to Go:

$$V^*(k, x) = \min_{u[\cdot]} V(u[\cdot], k, x)$$

$$= \min_{u} \left[\ell(k, x, u) + V^*(k+1, f(k, x, u)) \right]$$

Bellman's equation

$$V^*(N, x) = \ell_f(x)$$

Value Iteration

- $V^*(N, x) = \ell_f(x)$
- $V^*(k-1, x)$ = "Bellman's Eq"
- Sometimes it converges for infite horizon problems

Optimal control:

$$u^*(k, x) = \arg\min_{u} \left[\ell(k, x, u) + V^*(k+1, f(k, x, u)) \right]$$

Discrete time Review:

System

$$x[k+1] = f(k, x[k], u[k])$$

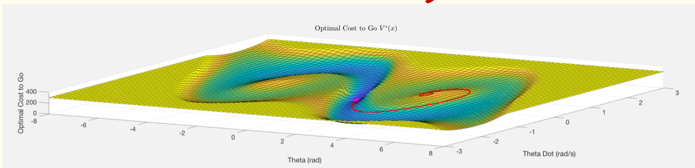
$$x[k_0] = x_0$$

Cost Function

$$V(u[\cdot], k_0, x_0) = \sum_{k=0}^{N-1} \ell(k, x[k], u[k]) + \ell_f(x[N])$$

$$x[k_0] = x_0$$

$V^*(x)$



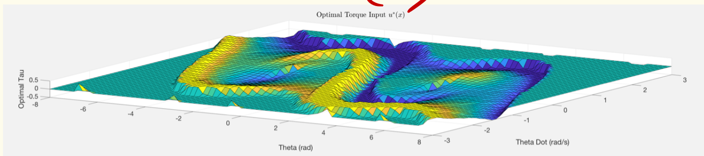
Optimal Cost to Go:

$$V^*(k,x) = \min_{u[\cdot]} V(u[\cdot], k, x)$$

$$= \min_{u} \left[\ell(k,x,u) + V^*(k+1, f(k,x,u)) \right]$$

$$V^*(N,x) = \ell_f(x[N])$$

$U^*(x)$



Optimal Control:

$$u^*(k,x) = \arg\min_{u} \left[\ell(k,x,u) + V^*(k+1, f(k,x,u)) \right]$$

Continuous Time Review:

System

$$\dot{x} = f(t, x, u)$$
$$x(0) = x_0$$

Cost Functional

$$V(u(\cdot), t_0, x_0) = \int_{t_0}^{t_f} \ell(t, x(t), u(t))\,dt + \ell_f(x(t_f))$$
$$x(t_0) = x_0$$

Optimal Cost to Go:

$$V^*(t, x) = \min_{u(\cdot)} V(u(\cdot), t_0, x_0)$$
$$= \min_{u} \left[\ell(t, x, u) + \frac{\partial V^*}{\partial t} + \frac{\partial V^*}{\partial x} f(t, x, u) \right]$$

Hamilton Jacobi Bellman equation

$$V^*(t_f, x) = \ell_f(x)$$

Optimal Control:

$$u^*(t, x) = \underset{u}{\operatorname{argmin}} \left[\ell(t, x, u) + \frac{\partial V^*}{\partial t} + \frac{\partial V^*}{\partial x} f(t, x, u) \right]$$

Continuous Time Review:

System

$$\dot{x} = f(t, x, u)$$
$$x(0) = x_0$$

Cost Functional

$$V(u(\cdot), t_0, x_0) = \int_{t_0}^{t_f} \ell(t, x(t), u(t))\,dt + \ell_f(x(t_f))$$
$$x(t_0) = x_0$$

Optimal Cost to Go:

$$V^*(t, x) = \min_{u(\cdot)} V(u(\cdot), t_0, x_0)$$
$$= \min_{u} \left[\ell(t, x, u) + \frac{\partial V^*}{\partial t} + \frac{\partial V^*}{\partial x} f(t, x, u) \right]$$
$$V^*(t_f, x) = \ell_f(x)$$

Optimal Control:

$$u^*(t, x) = \arg\min_{u} \left[\ell(t, x, u) + \frac{\partial V^*}{\partial t} + \frac{\partial V^*}{\partial x} f(t, x, u) \right]$$

Solvable Case: LQR

$$\dot{x} = Ax + Bu = f(x,u) \qquad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m$$

$$\min_{u(\cdot)} \int_0^\infty [x^T Q x + u^T R u]\, dt \qquad Q = Q^T \succeq 0, \quad R = R^T \succ 0$$

s.t. $x(t_0) = x_0$

- x "small" $\Rightarrow$ good tracking
- u "small" $\Rightarrow$ low effort
- Q, R handle this tradeoff

① Aim to Solve for $V^*(x)$

$$0 = \min_u \left[\ell(x,u) + \frac{\partial V^*}{\partial x} f(x,u) \right]$$

② Guess

$$V^*(x) = x^T P x \qquad P \succeq 0$$

③ Find $u^*(x)$:

$$0 = \min_u \Big[\overbrace{x^T Q x + u^T R u + 2x^T P [Ax + Bu]}^{g(x,u)} \Big]$$

$$\Rightarrow \nabla_u g = 2Ru + 2B^T P x = 0 \Rightarrow \underline{u^*(x) = -R^{-1} B^T P x}$$

④ Plug into HJB

$$0 = x^T Q x + u^{*T} R u^* + 2x^T P [Ax + Bu^*]$$

$$= x^T Q x + x^T P B R^{-1} B^T P x + 2x^T P A x - 2x^T P B R^{-1} B^T P x$$

$$= x^T [Q - PBR^{-1}B^T P + PA + A^T P] x$$

Solvable Case: LQR

$$\dot{x} = Ax + Bu \qquad x \in \mathbb{R}^n, \; u \in \mathbb{R}^m$$

$$\min_{u(\cdot)} \int_0^\infty [x^T Q x + u^T R u] \, dt \qquad Q = Q^T \succeq 0, \quad R = R^T \succ 0$$

$$\text{s.t. } x(t_0) = x_0$$

Result: $V^* = x^T P x$ satisfies HJB if $P \in \mathbb{R}^{n \times n}$ satisfies

$$0 = Q - PBR^{-1}B^TP + PA + A^TP \quad (*)$$

- Quadratic in P. Lots of solutions :-(
- Continuous-time Algebraic Ricatti eqn. (CARE)

Let: $Q = C^TC$ for some C

Thm: If (A,B) is controllable and (C,A) is observable, then there is a unique P that

1) Satisfies $*$ (CARE)
2) $u^* = -R^{-1}B^TPx$ results in a stable equilibrium @ $x = 0$ $(P \succeq 0)$

How to solve it?

- Start w $P=0$, Simulate $-\dot{P} = Q - PBR^{-1}B^TP + PA + A^TP$ until it converges
- In MATLAB: [K,P] = lqr(A,B,Q,R)

Application to nonlinear systems:

1. Let $\dot{x} = f(x, u)$ with $\bar{x}, \bar{u}$ an equilibrium (i.e., $0 = f(\bar{x}, \bar{u})$)

2. Construct a linearization of f about $\bar{x}, \bar{u}$

$$A = \left.\frac{\partial f}{\partial x}\right|_{x=\bar{x},\, u=\bar{u}} \qquad B = \left.\frac{\partial f}{\partial u}\right|_{x=\bar{x},\, u=\bar{u}}$$

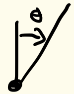
$$\dot{x} = A(x - \bar{x}) + B(u - \bar{u}) + \text{h.o.t.}$$

3. Construct a linearized controller via lqr P, K

$$u = \bar{u} - K(x - \bar{x})$$

4. The resulting controller is guaranteed to provide a locally asymptotically stable equilibrium @ $\bar{x}$!

Intuition into Linearization: Pendulum



$$\dot{x}_1 = x_2$$

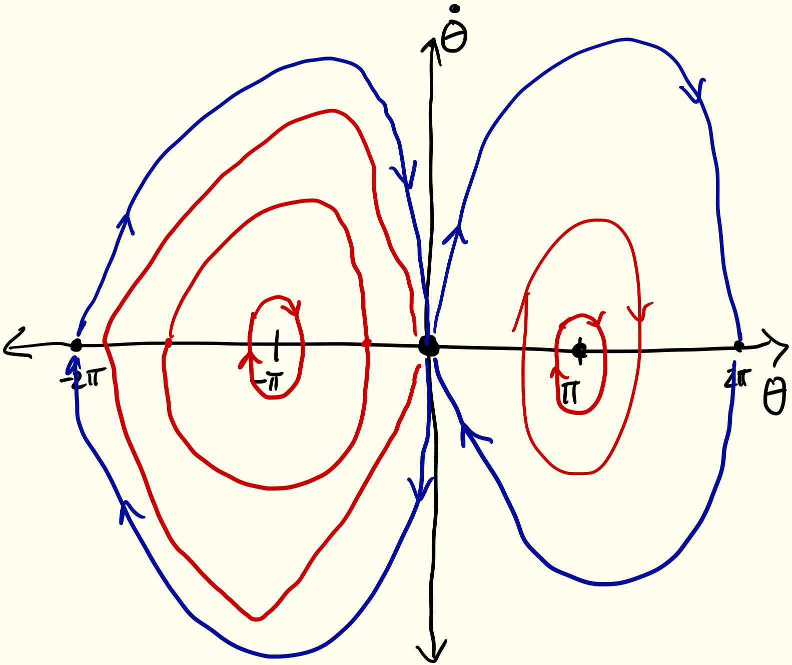
$$\dot{x}_2 = \sin(x_1) + u$$

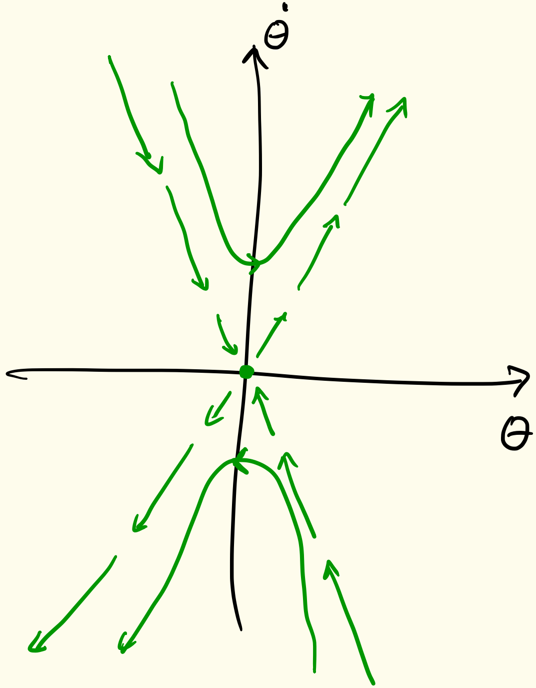
$x_1 = \theta$

$x_2 = \dot{\theta}$

$$\frac{d}{dt}\underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{x} = A x$$

$$A = \frac{\partial f}{\partial x}\bigg|_{\theta=0,\dot{\theta}=0}$$





Discrete Time LQR:

$$\min_{u[\cdot], x[\cdot]} \left[\sum_{i=1}^{N} x_i^T Q x_i + u_i^T R u_i \right]$$

$$\text{s.t. } x_i = A x_{i-1} + B u_i$$

Value Iteration: $V^*(i, x) = x^T P_i x \quad P_N = Q$

$$P_{i-1} = Q + A^T \left[P_i - P_i B (B^T P_i B + R)^{-1} B^T P_i \right] A$$

$$u_i^*(x) = -(B^T P_i B + R)^{-1} B^T P_i A x_{i-1}$$

Infinite Horizon: $V^*(x) = x^T P x$

$$P = Q + A^T (P - PB (B^T P B + R)^{-1} B^T P) A$$

Discrete-time Algebraic Riccatti Eqn. (DARE)

Derivation: LTV case

$$V(u[\,], k_0, x_0) = \sum_{k=k_0+1}^{N} x_{k-1}^T Q_{k-1} x_{k-1} + u_k^T R_k u_k \; + \; x_N^T Q_N x_N \qquad x_k = A_k x_{k-1} + B_k u_k$$

Guess: $V^*(k,x) = x^T P_k x \qquad P_N = Q_N$

Bellman Eq:
$$V^*(k-1, x_{k-1}) = \min_{u_k}\left[x_{k-1}^T Q_{k-1} x_{k-1} + u_k^T R_k u_k + V^*(k, A_k x_{k-1} + B_k u_k)\right]$$
$$= \min_{u_k}\left[x_{k-1}^T Q_{k-1} x_{k-1} + u_k^T R_k u_k + (A_k x_{k-1} + B_k u_k)^T P_k (A_k x_{k-1} + B_k u_k)\right]$$
$$= \min_{u_k}\left[x_{k-1}^T \left[Q_{k-1} + A_k^T P_k A_k\right] x_{k-1} + u_k^T (R_k + B_k^T P_k B_k) u_k + 2 x_{k-1}^T A_k^T P_k B_k u_k\right] \qquad (*)$$

$u_k^*(x)$ must minimize the above, which requires:
$$0 = 2(R_k + B_k^T P_k B_k) u_k^* + 2 B_k^T P_k A_k x_{k-1}$$
$$\Rightarrow u_k^* = -\underbrace{(R_k + B_k^T P_k B_k)^{-1} B_k^T P_k A_k}_{:= L_k} x_{k-1}$$

Rewrite $*$:

Extra term added and subtracted for multi-dimensional analog of "completing the square"

$$V^*(k-1, x_{k-1}) = \min_{u_k}\Big[x_{k-1}^T\left[Q_{k-1} + A_k^T P_k A_k - L_k^T (R_k + B_k^T P_k B_k) L_k\right] x_{k-1} + x_{k-1}^T L_k^T (R_k + B_k^T P_k B_k) L_k x_{k-1}$$
$$+ u_k^T (R_k + B_k^T P_k B_k) u_k + 2 x_{k-1}^T L_k^T (R_k + B_k^T P_k B_k) u_k\Big]$$

$$= \min_{u_k}\left[x_{k-1}^T\left[Q_{k-1} + A_k^T P_k A_k - L_k^T (R_k + B_k^T P_k B_k) L_k\right] x_{k-1} + (u_k + L_k x_{k-1})^T (R_k + B_k^T P_k B_k)(u_k + L_k x_{k-1})\right]$$

Derivation: LTV case

From Before:

$$u_k^* = -\underbrace{(R_k + B_k^T P_k B_k)^{-1} B_k^T P_k A_k}_{:= L_k} x_{k-1}$$

$$V^*(k-1, x_{k-1}) = \min_{u_k} \left[x_{k-1}^T \left[Q_{k-1} + A_k^T P_k A_k - L_k^T (R_k + B_k^T P_k B_k) L_k \right] x_{k-1} + (u_k + L_k x_{k-1})^T (R_k + B_k^T P_k B_k)(u_k + L_k x_{k-1}) \right]$$

Evaluate $V^*(k-1, x_{k-1})$ using u_k^*: ( ) terms are zero!

$$V^*(k-1, x_{k-1}) = x_{k-1}^T \left[Q_{k-1} + A_k^T P_k A_k - A_k^T P_k B_k (R_k + B_k^T P_k B_k)^{-1} B_k^T P_k A_k \right] x_{k-1}$$

$$= x_{k-1}^T P_{k-1} x_{k-1}$$

$$\Rightarrow \boxed{P_{k-1} = Q_{k-1} + A_k^T \left[P_k - P_k B_k (R_k + B_k^T P_k B_k)^{-1} B_k P_k \right] A_k}$$

Summary:

- LQR: The optimal policy is linear

	Control law	Cost to go $x^T P x$, P satisfies
Continuous ∞ Horizon	$u^* = -R^{-1}B^T P x$	$0 = Q + A^T P + PA - PBR^{-1}B^T P$
Discrete ∞ Horizon	$u^* = -(R+B^T PB)^{-1}B^T PAx$	$P = Q + A^T[P - PB(R+B^T PB)^{-1}B^T P]A$

- LQR Based on Linearizations is an effective way to <u>locally</u> Stabilize equilibrium points of nonlinear systems.