Lecture 21: "Stability Criteria" for legged robots

So Far:
- Simple models
- Stability of limit cycles

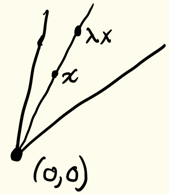
Today: Classical notions of "stability" for legged robots
- Static & Dynamic Feasibility: Contact Wrench Cone
- Zero moment Point
- Capture Point

On Deck:
- Using these criteria for planning w/ simple models
- Model Predictive Control
- Heirarchical Control

Background:

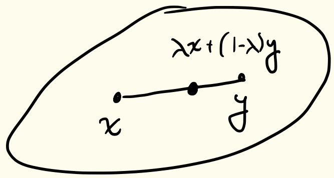
① A set $C \subseteq \mathbb{R}^n$ is a cone if

$$x \in C \Rightarrow \lambda x \in C \quad \forall \lambda \geq 0$$



② A set $C \subseteq \mathbb{R}^n$ is convex if

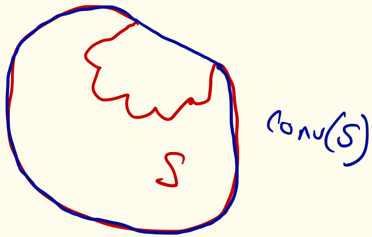
$$x, y \in C \Rightarrow \lambda x + (1-\lambda) y \in C \quad \forall \lambda \in [0,1]$$



③ A set is called a convex cone if it is convex and a cone

④ The convex hull of a set $S \subseteq \mathbb{R}^n$ is given by

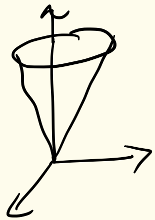
$$\operatorname{conv}(S) = \{ \lambda x + (1-\lambda) y \mid x, y \in S \text{ and } \lambda \in [0,1] \}$$



$\operatorname{conv}(S)$ is the smallest convex set that contains S.

Examples:

① $\{(f_x, f_y, f_z) \mid \sqrt{f_x^2+f_y^2} \le \mu f_z\}$ $\mu>0$



Friction cone ✓ convex ✓

Friction cone is a convex cone

$\{x \mid x \in \mathbb{R}^n, \sqrt{x_1^2+x_2^2+\dots+x_{n-1}^2} \le x_n\} \subseteq \mathbb{R}^n$ Second-order cone

② $\{X \in \mathbb{R}^{n\times n} \mid X = X^T, X \succeq 0\} = S$

cone?

λX ✓ it is a cone since λX has all non-neg eig vals

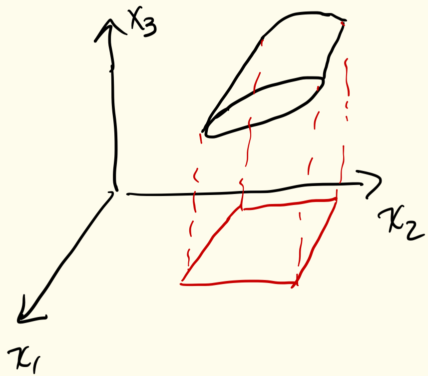
$X, Y \in S$

$z^T(\lambda X + (1-\lambda)Y)z = \lambda z^T X z + (1-\lambda) z^T Y z \ge 0$ $\forall \lambda \in [0,1]$

$\Rightarrow \lambda X + (1-\lambda) Y \succeq 0$

S is convex ✓

Properties of Convex Sets

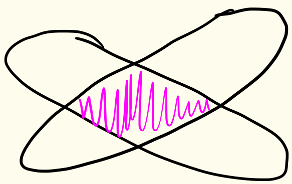


① The projection of a convex set is a convex set

$$C \subseteq \mathbb{R}^{n+m} \qquad C \text{ convex}$$

$$\{ x \mid x \in \mathbb{R}^n,\ y \in \mathbb{R}^m,\ (x,y) \in C \} \text{ is convex}$$

② The intersection of convex sets is convex

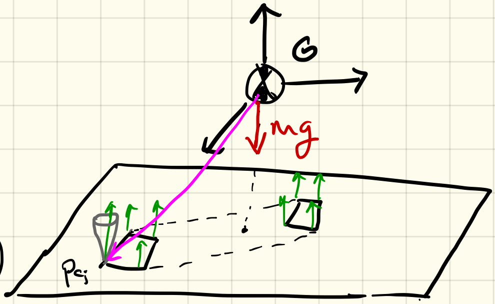


③ Hyperplanes are convex $\{ x \in \mathbb{R}^n \mid a^T x = b \}$ $a \in \mathbb{R}^n,\ b \in \mathbb{R}$

④ Intersection of hyperplanes $\{ x \in \mathbb{R}^n \mid Ax = b \}$ $A \in \mathbb{R}^{m \times n}$ $b \in \mathbb{R}^m$ is convex

Contact Forces Dictate Feasible Motions

Case 1: When stationary on flat ground



- A configuration is statically stable if the CoM is over the convex hull of contact points (support polygon)
- Equivalently there exist forces $f_{c_j} \in \mathbb{R}^3$ at contact vertices that oppose the gravity force

$$-\begin{bmatrix}0\\ mg\end{bmatrix} = \sum_{\text{Contact vertices } c_j} {}^{G}X_{c_j}^{*}\begin{bmatrix}0\\ f_{c_j}\end{bmatrix}$$

and each $f_{c_j} \in$ 🍦

Breaking it down:

$$0 = \sum (p_{c_j} - p_G) \times f_{c_j} = \sum (p_{c_j} \times f_{c_j} - p_G \times f_{c_j}) = \left[\sum p_{c_j} \times f_{c_j}\right] + p_G \times mg$$

$$-mg = \sum f_{c_j}$$

Define: $\mathcal{Y} = \{ (p_G, \{f_{c_j}\}) \mid$ each $f_{c_j} \in$ 🍦, $\sum f_{c_j} = -mg,\ 0 = \sum p_{c_j} \times f_{c_j} + p_G \times mg \}$

$\mathcal{Y}$ is convex

Bretl 2007 T-RO

What about uneven terrain?

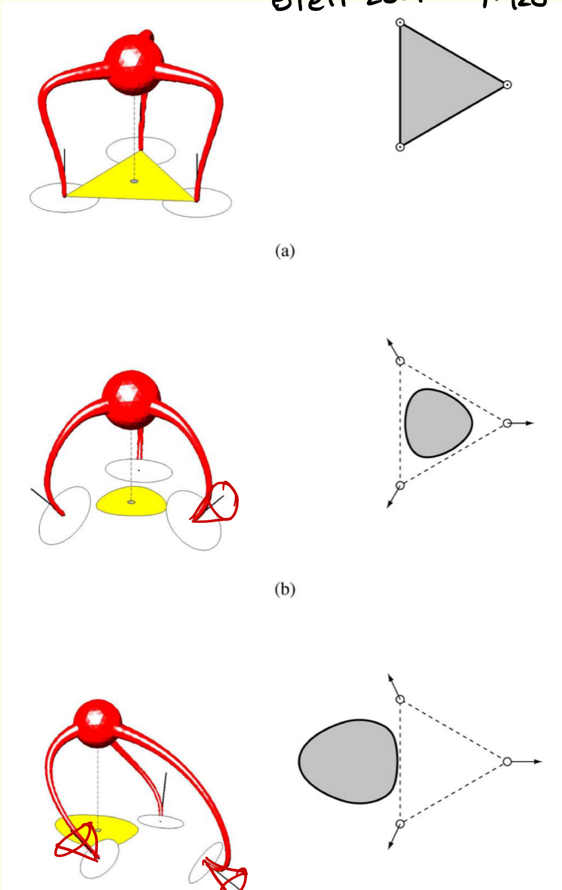
There is a convex set over which the COM must reside.

⇒ Determined by location of contacts

⇒ Determined by contact normals

$$\{ p_G \mid p_G \in \mathbb{R}^3,\ \{f_{c_j} \in \mathbb{R}^3\}\ (p_G, \{f_{c_j}\}) \in \mathcal{Y} \}$$

is convex

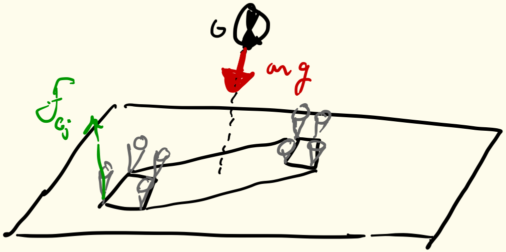


(a)

(b)

(c)

Case 2: When moving



$$h_G = \begin{bmatrix} k_G \\ l_G \end{bmatrix} \begin{matrix} \leftarrow \text{Angular momentum of system as a whole} \\ \leftarrow \text{linear momentum} \end{matrix}$$

- Newton & Euler eqns for the system

$$\dot{h}_G = \text{"net external spatial force on the system"}$$

$$= \begin{bmatrix} 0 \\ mg \end{bmatrix} + \sum_j {}^G X^*_{c_j} \begin{bmatrix} 0 \\ f_{c_j} \end{bmatrix}$$

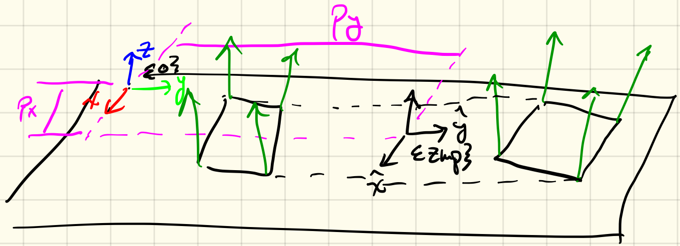
- General Requirement

$$\left(\dot{h}_G - \begin{bmatrix} 0 \\ mg \end{bmatrix}\right) \in \underbrace{\left\{ \sum_j {}^G X^*_{c_j} \begin{bmatrix} 0 \\ f_{c_j} \end{bmatrix} \;\middle|\; \text{each } f_{c_j} \in \right\}}_{\text{Contact wrench cone (CWC)}}$$

From CWC to ZMP: Assume Flat ground

- Express net contact wrench @ {0}

$$ {}^0 f_{net} = {}^0 X_G^* \left[\dot{h}_G - \begin{bmatrix} 0 \\ mg \end{bmatrix} \right] = \begin{bmatrix} n_x \\ n_y \\ n_z \\ f_x \\ f_y \\ f_z \end{bmatrix} $$



- Place a new frame @ point that we call the zero moment point (ZMP)

$$ p_x = \frac{-n_y}{f_z} \qquad p_y = \frac{n_x}{f_z} \quad \Rightarrow \quad {}^{ZMP}X_0^* \, {}^0 f_{net} = \begin{bmatrix} 0 \\ 0 \\ n_z - p_x f_y + p_y f_x \\ f_x \\ f_y \\ f_z \end{bmatrix} $$

$\Rightarrow$ We call this point the ZMP because moments tangential to the contact surface are zero.

$\boxed{\text{If}}$

1. The normal moment $n_z - p_x f_y + p_y f_x$ is sufficiently small
2. Net Force satisfies friction
3. $f_z > 0$
4. Ground is flat

Then

$$ \left(\dot{h}_G - \begin{bmatrix} 0 \\ mg \end{bmatrix} \right) \in CWC $$

iff

ZMP $\in$ Support Polygon

Checking if $f_G = \left(\dot{h}_G - \begin{bmatrix}0\\ mg\end{bmatrix}\right) \in CWC$

Solve the Feasibility Problem

$$\min \quad 0$$

$$\text{s.t.} \quad f_G = \sum^{G} X^{*}_{c_j} \begin{bmatrix}0\\ f_{c_j}\end{bmatrix}$$

each $f_{c_j} \in$ $\quad$ $\text{norm}([f_x, f_y]) \le \mu f_z$

Implementation in CVX

- This problem is a second-order cone problem (SOCP).