Lecture 2 : Representations of Orientation

Announcements:

- Project 0 due tonight (midnight) 0.5%
- Wednesday office hours Debart 242 4:30-6:00
- HW 1 online, Due Friday Aug. 31st

Goals for Today:

Representations of Orientation

- Rotation Matrix
- Properties
- Chaining Rotations (maybe)

Representing Position & Orientation

Decribe the location of a point in space: 3 numbers (p_x, p_y, p_z) representing distance to a coordinate origin along 3 mutually $\perp$ axes $\hat{x}, \hat{y}, \hat{z}$.

$$p_B = p_x \hat{x}_A + p_y \hat{y}_A + p_z \hat{z}_A \Rightarrow {}^A p_B = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}$$

Coordinate free

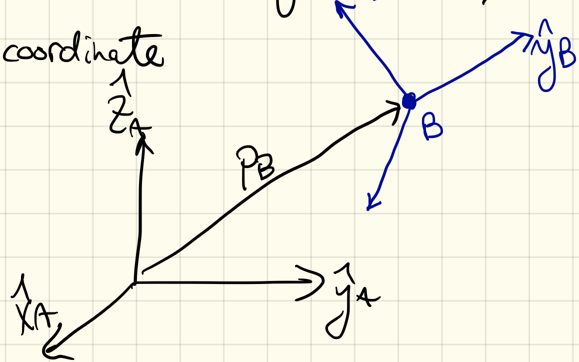
numeric values in coordinates

Coordinate System: origin and 3 coordinate axes "frame" $\{A\}$

Describe orientation of a body: ① Attach a coordinate system $\{B\}$
② Describe $\hat{x}_B, \hat{y}_B, \hat{z}_B$ in some fixed coordinate

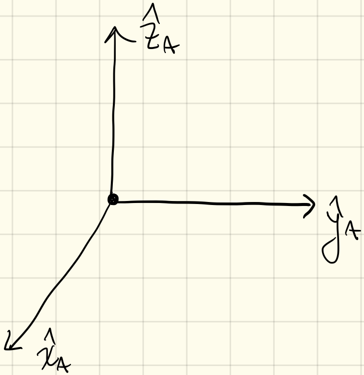
$${}^A R_B = \begin{bmatrix} {}^A\hat{x}_B & {}^A\hat{y}_B & {}^A\hat{z}_B \end{bmatrix}$$

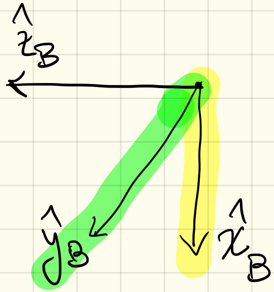
Rotation matrix describes $\{B\}$ relative to $\{A\}$



Rotation Matrices: Simple Case

$$^A R_B = \begin{bmatrix} ^A\hat{x}_B & ^A\hat{y}_B & ^A\hat{z}_B \end{bmatrix}$$





$$\hat{x}_B = -1 \cdot \hat{z}_A \Rightarrow {}^A\hat{x}_B = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \Rightarrow {}^A R_B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix}$$

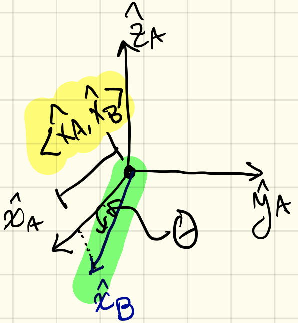
Rotation Matrices General Case:

$${}^{A}R_{B} = \begin{bmatrix} {}^{A}\hat{x}_{B} & {}^{A}\hat{y}_{B} & {}^{A}\hat{z}_{B} \end{bmatrix}$$

$$\hat{x}_{B} = \underline{\langle \hat{x}_{A}, \hat{x}_{B} \rangle} \, \hat{x}_{A} + \underline{\langle \hat{y}_{A}, \hat{x}_{B} \rangle} \, \hat{y}_{A} + \underline{\langle \hat{z}_{A}, \hat{x}_{B} \rangle} \, \hat{z}_{A} \qquad {}^{A}\hat{x}_{B} = \begin{bmatrix} \langle \hat{x}_{A}, \hat{x}_{B} \rangle \\ \langle \hat{y}_{A}, \hat{x}_{B} \rangle \\ \langle \hat{z}_{A}, \hat{x}_{B} \rangle \end{bmatrix}$$

$${}^{A}R_{B} = \begin{bmatrix} \langle \hat{x}_{A}, \hat{x}_{B} \rangle & \langle \hat{x}_{A}, \hat{y}_{B} \rangle & \langle \hat{x}_{A}, \hat{z}_{B} \rangle \\ \langle \hat{y}_{A}, \hat{x}_{B} \rangle & \langle \hat{y}_{A}, \hat{y}_{B} \rangle & \langle \hat{y}_{A}, \hat{z}_{B} \rangle \\ \langle \hat{z}_{A}, \hat{x}_{B} \rangle & \langle \hat{z}_{A}, \hat{y}_{B} \rangle & \langle \hat{z}_{A}, \hat{z}_{B} \rangle \end{bmatrix}$$

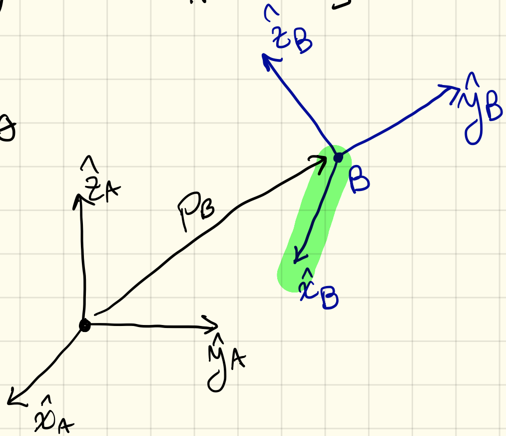
Geometry:



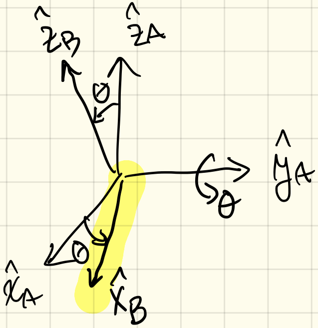
$$\langle \hat{x}_{A}, \hat{x}_{B} \rangle = \|\hat{x}_{A}\| \, \|\hat{x}_{B}\| \cos\theta$$

$$= \cos\theta$$

Rotation $\Leftrightarrow$ Direction cosine matrix



Example 1: Consider two frames {A} and {B} such that {B} results by rotating $\hat{x}_A, \hat{y}_A, \hat{z}_A$ by θ about $\hat{y}_A$. Find ${}^A R_B$.



$$ {}^A R_B = \left[{}^A\hat{x}_B \;,\; {}^A\hat{y}_B \;,\; {}^A\hat{z}_B \right] $$

$$ R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} $$

Similarly: $R_x(\theta)$, $R_z(\theta)$

We'll call each of $R_x(\theta), R_y(\theta), R_z(\theta)$ "elementary" rotation matrices

Properties of Rotation Matricies:

(1) $${}^A R_B^T \, {}^A R_B = \begin{bmatrix} {}^A\hat{x}_B^T \\ {}^A\hat{y}_B^T \\ {}^A\hat{z}_B^T \end{bmatrix} \begin{bmatrix} {}^A\hat{x}_B & {}^A\hat{y}_B & {}^A\hat{z}_B \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$${}^A\hat{x}_B^T \, {}^A\hat{x}_B = \| {}^A\hat{x}_B \|^2 = 1$$

$${}^A\hat{x}_B^T \, {}^A\hat{y}_B = 0 \qquad \hat{x}_B \perp \hat{y}_B$$

$${}^A\hat{y}_B^T \, {}^A\hat{x}_B = 0 \qquad \text{"same"}$$

Rotation Matrix

9 entries
− 6 independent constr.

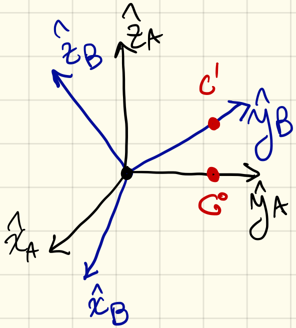
3 Degrees of Freedom

(2) $${}^A R_B^{-1} = {}^A R_B^T = {}^B R_A$$ ← prove for yourself

(3) $$\det({}^A R_B) = 1$$ prove for HW 1

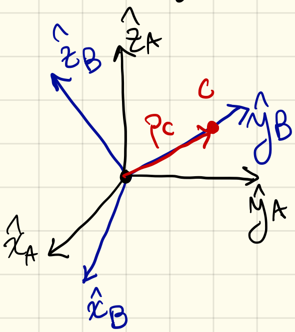
Two More Uses of Rotation Matricies:

① An operator that transforms points in space $\quad R := {}^A R_B$



$${}^A P_{C'} = R\, {}^A P_{C^\circ}$$

② Change of Basis (points don't move



$${}^A P_C = \, ? \, {}^B P_C$$

$${}^A P_C = \begin{bmatrix} {}^B \hat{x}_A^T \, {}^B P_C \\ {}^B \hat{y}_A^T \, {}^B P_C \\ {}^B \hat{z}_A^T \, {}^B P_C \end{bmatrix} = {}^B R_A^T \, {}^B P_C$$

$$= {}^A R_{\cancel{B}} \, {}^{\cancel{B}} P_C$$

Summary:

- Define orientation $\Rightarrow$ Rotation Matrix
  - 9 entries, 6 constraints
- Use Rotation Matrices to operate on vectors, or change basis of a vector