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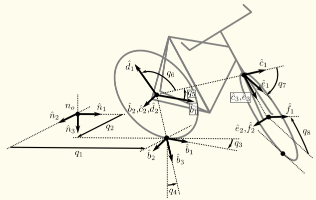
Analytical Dynamics

Holonomic vs. Nonholonomic Constraints



Admin: HW9 Optionally Due Next Wednesday (2nd lowest HW Dropped
Graded on Completion)

Last Time: • Review of Rigid Body Dynamics in 3D

Today:
- Holonomic vs. Nonholonomic Constraints
- Virtual Dispacements consistent w/ constraints
- Generalized Speeds

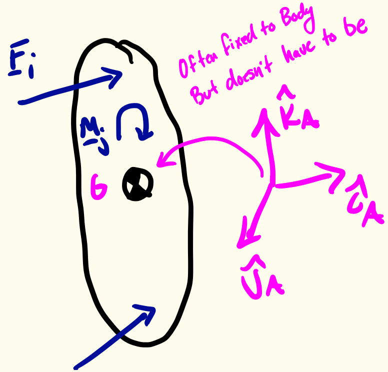
Pre-reqs: (Mostly for 60623)

Null space of a matrix $\mathcal{N}(A) = \{ x \mid Ax = 0 \}$

Span of a set of vectors: $\text{Span}(\{w_1, \dots, w_m\}) = \left\{ \sum_{i=1}^{m} \alpha_i w_i \;\middle|\; \alpha_1, \dots, \alpha_m \in \mathbb{R} \right\}$

Review: Newton's and Euler's Equations

For Any Rigid Body



$$m \underline{a}_G = \sum \underline{F}_i = \underline{F}_{net}$$

$$\dot{\underline{H}}_G = \sum_j \underline{M}_j + \sum_i (\underline{r}_i - \underline{r}_G) \times \underline{F}_i = \underline{M}_{G,net}$$

$${}^A\{\underline{H}_G\} = {}^A\underline{\underline{I}}_G \; {}^A\{\underline{\omega}\}$$

without a subscript denotes angular velocity of the Body

$${}^A\{\dot{\underline{H}}_G\} = \frac{d}{dt}\,{}^A\{\underline{H}_G\} + {}^A\{\underline{\omega}_A\} \times {}^A\{\underline{H}_G\}$$

Angular velocity of frame A (Same as $\underline{\omega}$ if A Body fixed)

$${}^A\underline{\underline{I}}_G = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix}$$

if A aligned w/ principal Axes

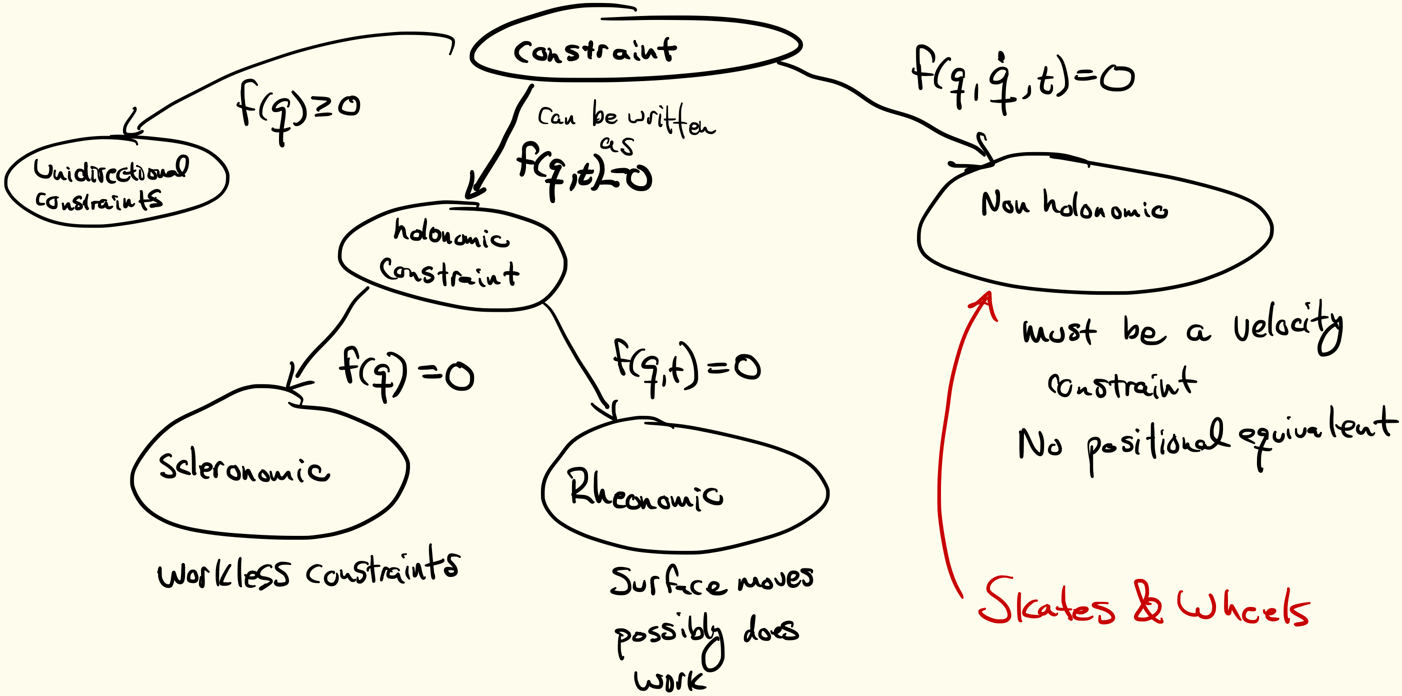
Euler's Equation: In any frame A (Derivation on next Slide)

(whether A is body fixed or not, whether inertial or not)

$${}^A\{\underline{M}_{G,net}\} = {}^A\underline{\underline{I}}_G \; {}^A\{\dot{\underline{\omega}}\} + {}^A\{\underline{\omega}\} \times {}^A\underline{\underline{I}}_G \; {}^A\{\underline{\omega}\}$$

Always angular velocity of the Body. I was wrong regarding using $\underline{\omega}_A$ here in class.

Constraint Classification



Example: Consider cylindrical coordinates $q = [r, \theta, z]^T$

Constraint: $f(q, \dot{q}) = \dot{r} c_\theta - r s_\theta \dot{\theta} + \dot{z} = 0$ Is it holonomic?

$$\frac{d}{dt}\left[r c_\theta + z \right] = 0$$

$$r c_\theta + z = C$$

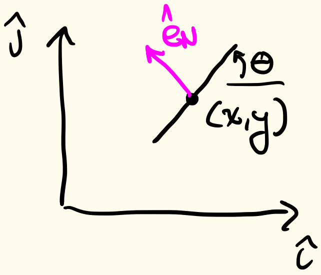
$\boxed{\text{holonomic}}$

$$\boxed{g(r, \theta, z) = r c_\theta + z - C = 0}$$

Scleronomic

Example: Frog riding unicycle

Top View



Generalized coordinates

$$q = \begin{bmatrix} x \\ y \\ \theta \end{bmatrix}$$

$$\underline{V} = \dot{x}\hat{i} + \dot{y}\hat{j} \qquad \hat{e}_N = -\hat{i}S_\theta + \hat{j}C_\theta$$

Velocity constraint: Identify if the constraint is holonomic.

$$0 = \hat{e}_N \cdot \underline{V} = -\dot{x}S_\theta + \dot{y}C_\theta$$

Is there an equivalent constraint

$$f(q) = 0$$

No. Frog can drive to any x, y, θ over time, so no equivalent configuration constraint.

Holonomic vs. Nonholonomic: How can you check?

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Consider two different vector fields $\dot{q} = \omega_1(q)$ $\quad \dot{q} = \omega_2(q)$

Defn: The Lie bracket of two vector fields ω_1 and ω_2 is given by

$$[\omega_1, \omega_2] = \frac{\partial \omega_2}{\partial q}\omega_1 - \frac{\partial \omega_1}{\partial q}\omega_2$$

Theorem: Consider constraints $A(q)\dot{q} = 0$ $\quad A(q) \in \mathbb{R}^{m \times n}$ and let $\omega_1(q), \ldots, \omega_{n-m}(q)$ be such that

$A(q)$ full rank

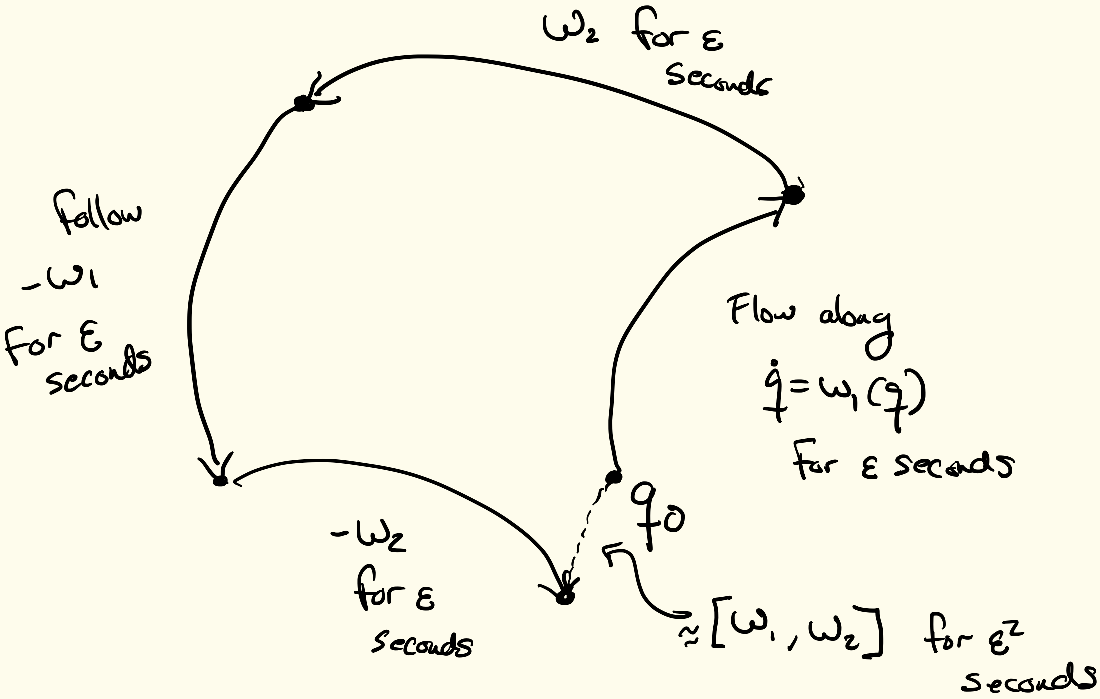
$$\mathcal{N}(A(q)) = \text{span}(\{\omega_1(q), \ldots, \omega_{n-m}(q)\})$$

$\Rightarrow A(q)\dot{q} = 0$ is holonomic iff

$\boxed{\text{Frobenius Theorem}}$

$$[\omega_i(q), \omega_j(q)] \in \mathcal{N}(A(q)) \quad \forall i,j \in \{1, \ldots, n-m\}$$

Flow Box Interpretation of Lie Brackets:



(Only accurate for small ε. Approximation error of order ε^3)

Example: $q = \begin{bmatrix} x \\ y \\ \theta \end{bmatrix}$

$$-s_\theta \dot{x} + c_\theta \dot{y} = 0$$

① Write as $A(q)\dot{q} = 0$

$$A(q) = \begin{bmatrix} -s_\theta & c_\theta & 0 \end{bmatrix}$$

② Find $\omega_1(q)$ & $\omega_2(q)$ s.t. they span $\mathcal{N}(A)$

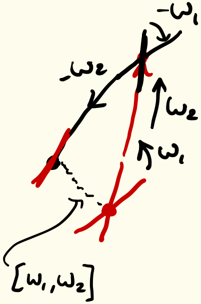
$\omega_2(q) = \begin{bmatrix} c_\theta \\ s_\theta \\ 0 \end{bmatrix}$ $\quad A\omega_1 = 0 \checkmark \quad$ $\omega_1(q) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

③ Check that Lie brackets $[\omega_i, \omega_j] \in \mathcal{N}(A)$

$$[\omega_1, \omega_2] = \frac{\partial \omega_2}{\partial q}\omega_1 - \cancelto{0}{\frac{\partial \omega_1}{\partial q}}\omega_2 = \begin{bmatrix} \frac{\partial \omega_2}{\partial x} & \frac{\partial \omega_2}{\partial y} & \frac{\partial \omega_2}{\partial \theta} \end{bmatrix}\omega_1 = \begin{bmatrix} 0 & 0 & -s_\theta \\ 0 & 0 & c_\theta \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} -s_\theta \\ c_\theta \\ 0 \end{bmatrix}}_{\omega_3}$$

$A\omega_3 = 1$

$[\omega_1, \omega_2] \notin \mathcal{N}(A(q)) \Rightarrow$ $A\dot{q} = 0$ is nonholonomic



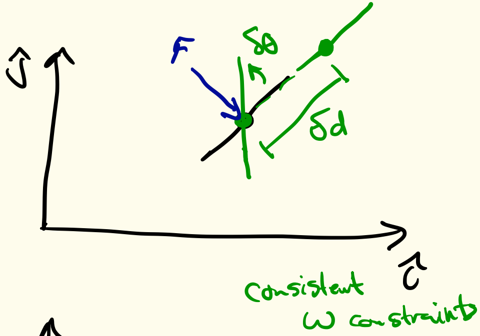
Virtual Displacements with Nonholonomic Constraints

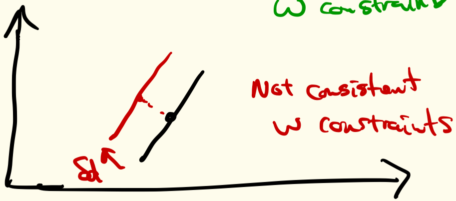
Given a constraint in the form

$$A(\underline{q},t)\,\dot{\underline{q}} + c(\underline{q},t) = 0$$

Its Pfaffian form is given by

$$A(\underline{q},t)\,d\underline{q} + c(\underline{q},t)\,dt = 0$$

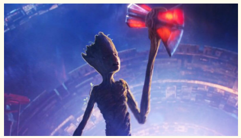




Virtual displacement $\delta\underline{q}$ is said to be consistent w/ constraints if

$$A(\underline{q},t)\,\delta\underline{q} = 0$$

Axiom: Constraint forces do no virtual work under virtual displacements consistent w/ constraints.



Dynamics with Nonholonomic Constraints

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Suppose constraint
$A(q,t)\dot{q}=0$

- **D'Alemberts Principle**

$$0=\sum_k (Q_k - Q_{k_{inertial}})\delta q_k$$

$\forall \delta q$ consistent w/ constraints
Still works!

$\forall \delta q$ s.t. $A(q,t)\delta q=0$

OR w/ Lagrange Multipliers

$$\underline{Q}_{inertial} = \underline{Q} + A^T \underline{\lambda}$$

for some $\underline{\lambda}$
s.t. $A\ddot{q} + \dot{A}\dot{q} = 0$

- **Lagrange's Equations:**

$$\frac{d}{dt}\left(\nabla_{\dot{q}} \mathcal{L}\right) - \nabla_q \mathcal{L} = \underline{Q}_{nc} + A^T \underline{\lambda}$$

for some λ
s.t. $A\ddot{q} + \dot{A}\dot{q} = 0$

Generalized Speeds for the Rolling Disk

- Generalized coordinates

$$q = [x, y, \theta, \phi]^T$$

- Generalized speeds

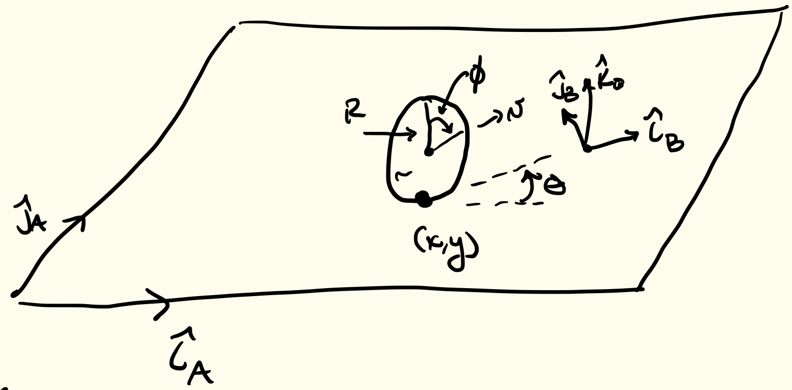
$$U = [\dot{\theta}, \dot{\phi}]^T$$

if I give you these two numbers & q that is enough to specify all velocities for this system.

$$U = [v, \dot{\theta}]$$

Also valid

All constraints are accounted for w/ this choice of generalized speeds.



More next time: We'll use generalized speeds to derive a minimal set of EOM for systems with nonholonomic constraints.

Summary

Holonomic: $f(q,t)=0$

Non holonomic: Not holonomic. $f(q,\dot{q},t)=0$

more specifically

$A(q)\dot{q}=0$ in this class

How to tell if Holonomic: 60623

Let $\operatorname{span}\{\omega_1(q), \ldots, \omega_{n-m}(q)\} = \mathcal{N}(A(q))$

Holonomic iff $[\omega_i, \omega_j] \in \mathcal{N}(A) \quad \forall q$

Inclusion in Dynamics: Use Lagrange Multipliers 60623

$$\frac{d}{dt}\left(\nabla_{\dot{q}}\mathcal{L}\right) - \nabla_q \mathcal{L} = \underline{Q} + A^T \underline{\lambda}$$

for some $\underline{\lambda}$

s.t. $A\ddot{q} + \dot{A}\dot{q} = 0$