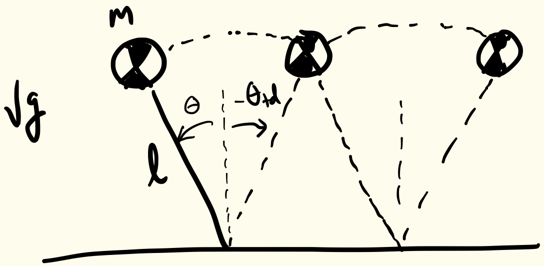
Lecture 19: Simple Models, Limit Cycles, Poincare

1. Simple Models
   - Capture prominent features of a behavior
   - Abstract complexities of target system
   - Capture commonalities across scale (and sometimes species)

2. Today:
   - Discuss a few other Simple Models
   - Characterize Stability of Periodic Behaviors

The Inverted Pendulum Model of Walking



- Single Support

$$ml^2 \ddot{\theta} = mg \sin\theta$$

- Impact when $\theta = \theta_{td}$

$$\dot{\theta}^+ = \dot{\theta}^-$$

$$\theta^+ = -\theta_{td}$$

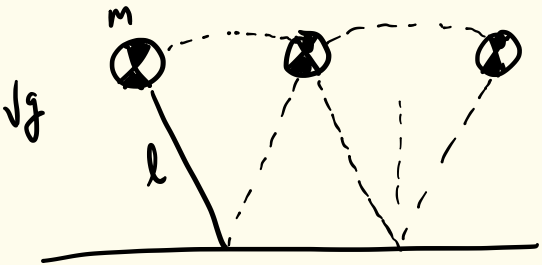
When $\theta = 0$

- tangential acceleration $= 0$
- centripetal acceleration $= \dfrac{(l\dot{\theta})^2}{l} = \dfrac{v^2}{l}$
- Max centipetal accel $= g \Rightarrow \dfrac{v^2}{l} \leq g$

- Exchange of Kinetic & Potential Energy

- Decent physiological explanation of the dynamics of single support

The Inverted Pendulum Model of Walking



- **Defn:** Froude Number $= \frac{v^2}{lg}$

  ⇒ predicts max walking speed = 3 m/s

  ⇒ Nondimensional number

  - Cats, Rats, Dogs, Sheep, Rhinos

    ⇒ Walk to Trot @ $Fr \approx 0.8$

    ⇒ Trot to Gallop @ $Fr \approx 2.5$

Nondimensionalizing Equations

$$ml^2 \frac{d^2\theta}{dt^2} = mgl \sin\theta$$

- Change of variables

$$\sqrt{l/g}\,\bar{t} = t$$

$$\sqrt{l/g}\,d\bar{t} = dt \quad \Rightarrow \quad \frac{d^2\theta}{dt^2} = \frac{d^2\theta}{(d\bar{t})^2} \cdot \frac{g}{l}$$

- $\cancel{mgl} \sin\theta = \cancel{ml^2}\left[\cancel{\frac{g}{l}} \frac{d^2\theta}{(d\bar{t})^2}\right]$

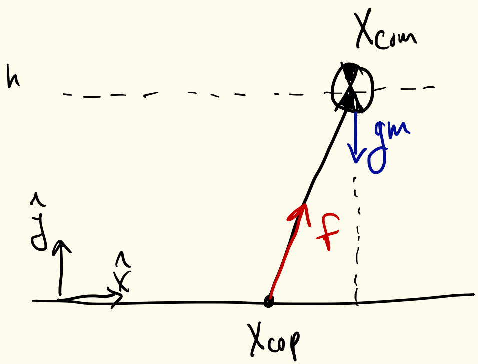
$$\frac{d^2\theta}{d\bar{t}^2} = \sin\theta$$

holds across scales

Linear Inverted Pendulum: LIP

- NOT a linearization of pendulum dynamics
- Assume
  - Leg Length can change
  - CoM stays @ constant height

$f_y = mg$ $\qquad m\ddot{x}_{com} = f_x$ $\qquad \frac{f_y}{f_x} = \frac{h}{x_{com} - x_{cop}} \Rightarrow \cancel{m}\ddot{x}_{com} = \frac{\cancel{m}g}{h}(x_{com} - x_{cop})$



Dynamics:

$$\ddot{x}_{com} = \omega^2 (x_{com} - x_{cop}) \qquad \omega = \sqrt{\frac{g}{h}}$$

Goal: Non dimensionalize the LIP Dynamics

(Non dimensionalize both x_{com} and t)

$$\bar{t} = \omega t \Rightarrow d\bar{t} = \omega^2 t^2$$

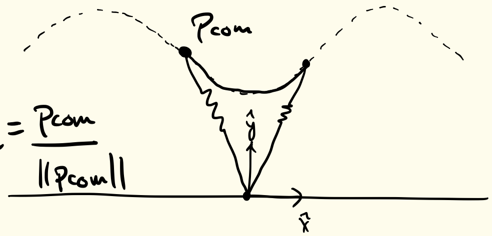
$$\bar{X}_{com} = \frac{x_{com}}{h}$$

$$\bar{X}_{cop} = \frac{x_{cop}}{h}$$

$$\frac{d^2 x_{com}}{dt^2} = \left[\cancel{\omega^2} \frac{d^2 x_{com}}{d\bar{t}^2} = \cancel{\omega^2}(X_{com} - X_{cop})\right] \frac{1}{h}$$

$$\frac{d^2 \bar{x}_{com}}{d\bar{t}^2} = \bar{X}_{com} - \bar{X}_{cop}$$

The Spring-Loaded Inverted pendulum:



$\hat{l} = \frac{p_{com}}{\|p_{com}\|}$

- Flight: Exchange of kinetic & grav potential energy

- Stance: Exchange of kinetic, gravitational potential & spring potential energy

  ⇒ Motivated by observation that we store a good deal of energy in compliant tendons

  ⇒ good abstraction, poor physiological description

Flight: $\ddot{p}_{com} = \begin{bmatrix} 0 \\ -g \end{bmatrix}$

Rest Length

Stance: $m\ddot{p}_{com} = \begin{bmatrix} 0 \\ -mg \end{bmatrix} + \hat{l} f(\|p_{com}\|, l_0)$

Hookean: $f(\|p_{com}\|, l_0) = K(l_0 - \|p_{com}\|)$

Air Spring: $f(l, l_0) = \frac{K}{l_0}\left(\frac{l_0 - l}{l^3}\right)$

Stability Analysis for Gaits:

Consider a system: $\dot{x} = f(x)$

$x(t; x_0)$ solution of the system starting from $x(0) = x_0$

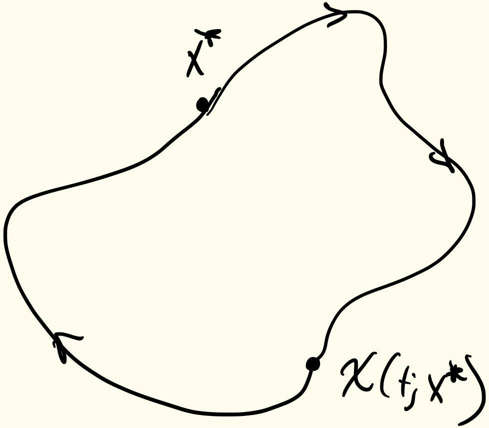
We say the system has a $\boxed{\text{limit cycle}}$ with period $T > 0$ if $\exists x^*$

such that $x(t+T; x^*) = x(t; x^*)$

We will denote an orbit $\mathcal{O}$

$\mathcal{O} = \{x(t; x^*)\}$

where x^* an initial condition that gives rise to a limit cycle



Stability of an Orbit: $\dot{x} = f(x)$ $x \in \mathbb{R}^n$

Denote the distance to the orbit $y \in \mathbb{R}^n$ $\mathcal{O} \subseteq \mathbb{R}^n$

$$d(y, \mathcal{O}) = \min_{x \in \mathcal{O}} \|y - x\|$$

The orbit is stable $\forall \varepsilon > 0$ $\exists \delta > 0$ s.t.

$$d(x_0, \mathcal{O}) < \delta \Rightarrow d(x(t; x_0), \mathcal{O}) < \varepsilon \quad \forall t > 0$$

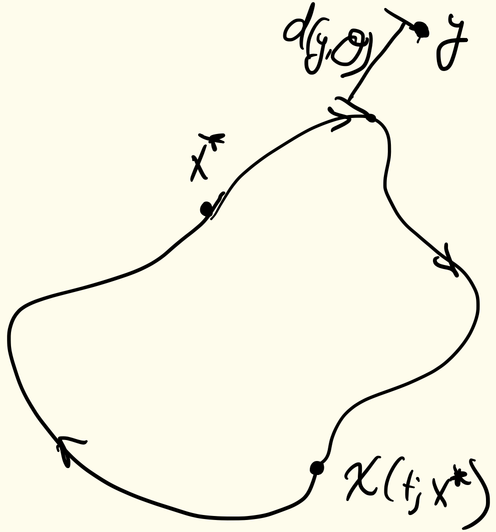
The orbit asymtotically stable if it is stable and $\exists \delta > 0$ s.t

$$d(x_0, \mathcal{O}) < \delta \Rightarrow d(x(t; x_0), \mathcal{O}) \to 0$$

as $t \to \infty$

The orbit is exponentially stable if stable and $\exists \delta > 0, K > 0, \lambda > 0$

$$d(x_0, \mathcal{O}) < \delta \Rightarrow d(x(t; x_0), \mathcal{O}) \le K e^{-\lambda t}$$



Stability of an Orbit: Poincare Section

- Define a surface S transversal to the orbit
- $x \in \mathbb{R}^n$ the surface could be defined by

$$S = \{ x \mid g(x) = 0 \} \quad g: \mathbb{R}^n \to \mathbb{R}$$

w/requirement $\frac{\partial g}{\partial x} \cdot f(x^*) \neq 0$

- States starting near $x^* \in S$ will return to S. So we define the return map

$$P(x): S \to S$$

$$\Rightarrow P(x^*) = x^* \qquad x_{k+1} = P(x_k) \quad (**)$$

① x^* is a stable equilibrium of $(**)$
$\Rightarrow \mathcal{O}$ is stable

② x^* is an asy. stable equilibrium of $(**)$
$\Rightarrow \mathcal{O}$ is asy. stable

③ if x^* is an exponentially stable equilibrium of $(**)$
$\Rightarrow \mathcal{O}$ is exp. stable

③a if $\left.\frac{\partial P}{\partial x}\right|_{x^*}$ has eig vals. λ
$|\lambda_i| < 1$ then $\mathcal{O}$ is exp stable