Lecture 18 - The Jacobian & Static Force Analysis

- Jacobian for Velocity Analysis

① $\begin{bmatrix} {}^A\omega_B \\ {}^A v_B \end{bmatrix} = {}^A J_B \, \dot{\Theta}$

Jacobian for the velocities of $\{B\}$ expressed in $\{A\}$

Frame being described

Frame of expression

② $\begin{bmatrix} {}^C\omega_C \\ {}^C v_C \end{bmatrix} = {}^C X_A \begin{bmatrix} {}^A\omega_A \\ {}^A v_A \end{bmatrix}$ when $\{C\}$ and $\{A\}$ rigidly attached

Today:

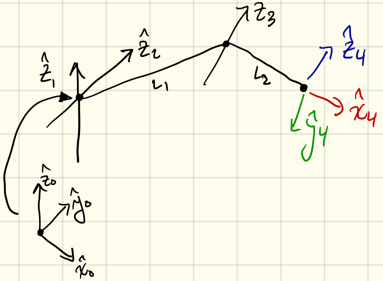
- Jacobian wrap-up
  - Spatial reasoning
  - Changing Frames
- Statics analysis for end-effector forces

$$\begin{bmatrix} {}^4\omega_4 \\ {}^4v_4 \end{bmatrix} = {}^4J_4\,\dot{\theta} = \begin{bmatrix} {}^4v_{4,1} & {}^4v_{4,2} & {}^4v_{4,3} \end{bmatrix}\dot{\theta} \qquad {}^tv_{t,i} = {}^tX_i \begin{bmatrix} 0\\0\\1\\ \hline 0\\0\\0 \end{bmatrix} = \begin{bmatrix} {}^iR_t^T \begin{bmatrix}0\\0\\1\end{bmatrix} \\ \hline -{}^iR_t^T \left[{}^iP_t \times \begin{bmatrix}0\\0\\1\end{bmatrix}\right] \end{bmatrix}$$

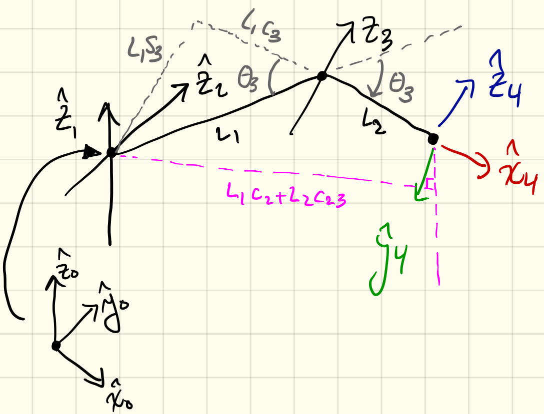
$${}^1T_4 = \begin{bmatrix} c_{23} & -s_{23} & 0 & L_2c_{23}+L_1c_2 \\ 0 & 0 & 1 & 0 \\ -s_{23} & -c_{23} & 0 & -L_2s_{23}-L_1s_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad {}^2T_4 = \begin{bmatrix} c_3 & -s_3 & 0 & L_1+L_2c_3 \\ s_3 & c_3 & 0 & L_2s_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad {}^3T_4 = \begin{bmatrix} 1 & 0 & 0 & L_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Compute the Second Column of the Jacobian ${}^4J_4$: ${}^4v_{4,2}$

$${}^4v_{4,2} = {}^4X_2 \begin{bmatrix}0\\0\\1\\0\\0\\0\end{bmatrix} = \begin{bmatrix} {}^2R_4^T \begin{bmatrix}0\\0\\1\end{bmatrix} \\ {}^2R_4^T \left(\begin{bmatrix}0\\0\\1\end{bmatrix} \times {}^2P_4 \right) \end{bmatrix} = \begin{bmatrix} 0\\0\\1\\ \hline L_1s_3 \\ L_2+L_1c_3 \\ 0 \end{bmatrix}$$



Spatial Reasoning:



Second column of ${}^4J_4$

$$\begin{bmatrix} {}^4\omega_4 \\ {}^4v_4 \end{bmatrix}\Bigg|_{\dot{\theta}_2=1,\ \dot{\theta}_1=\dot{\theta}_3=0} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ \hline L_1 S_3 \\ L_2 + L_1 C_3 \\ 0 \end{bmatrix}$$

Warm Up: Third column of the Jacobian ${}^4J_4$

$$\begin{bmatrix} {}^4\omega_4 \\ {}^4v_4 \end{bmatrix}\Bigg|_{\dot{\theta}_3=1,\ \dot{\theta}_1=\dot{\theta}_2=0} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ \hline 0 \\ L_2 \\ 0 \end{bmatrix}$$

Since $\hat{z}_3$ and $\hat{z}_4$ aligned

Since rotation about $\hat{z}_3$ creates velocity along $\hat{y}_4$

First Column of ${}^4J_4$:

$$\begin{bmatrix} {}^4\omega_4 \\ {}^4v_4 \end{bmatrix}\Bigg|_{\dot{\theta}_1=1,\ \dot{\theta}_2=\dot{\theta}_3=0} = \begin{bmatrix} -S_{23} \\ -C_{23} \\ 0 \\ \hline 0 \\ 0 \\ L_1 C_2 + L_2 C_{23} \end{bmatrix}$$

when $\theta_{23} = 0^\circ$, ${}^4\omega_4 = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$

when $\theta_{23} = 90^\circ$, ${}^4\omega_4 = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$

Changing Frames of Expression:

$$ {}^{4}J_4 \overset{?}{\Longleftrightarrow} {}^{0}J_4 \qquad {}^{0}R_4 \, , \, {}^{0}P_4 $$

$$ {}^{0}X_4 \, , \, {}^{4}X_0 $$

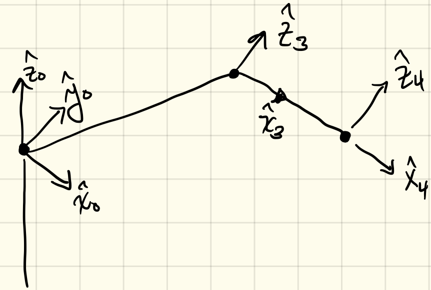
$$ \begin{bmatrix} {}^{0}\omega_4 \\ {}^{0}v_4 \end{bmatrix} = {}^{0}J_4 \dot{\Theta} $$

$$ = \begin{bmatrix} {}^{0}R_4 \, {}^{4}\omega_4 \\ {}^{0}R_4 \, {}^{4}v_4 \end{bmatrix} = \begin{bmatrix} {}^{0}R_4 & 0 \\ 0 & {}^{0}R_4 \end{bmatrix} \begin{bmatrix} {}^{4}\omega_4 \\ {}^{4}v_4 \end{bmatrix} = \begin{bmatrix} {}^{0}R_4 & 0 \\ 0 & {}^{0}R_4 \end{bmatrix} {}^{4}J_4 \dot{\Theta} $$

$$ \boxed{ {}^{0}J_4 = \begin{bmatrix} {}^{0}R_4 & 0 \\ 0 & {}^{0}R_4 \end{bmatrix} {}^{4}J_4 } $$

Can we also change the frame of description?

what is ${}^4\bar{J}_4$?



(a) $\begin{bmatrix} {}^4R_3 & 0 \\ 0 & {}^4R_3 \end{bmatrix} {}^3\bar{J}_3 = {}^4\bar{J}_3$

(b) ${}^4X_3 \, {}^3\bar{J}_3$ ✓

(c) $(6\times 6)\cdot(6\times n)\cdot(6\times 6)$ ✗

${}^4X_3 \, {}^3\bar{J}_3 \, {}^3X_4$

(d) None of the above

- Yes when two frames are rigidly attached to the same body.

$$\begin{bmatrix} {}^4\omega_4 \\ {}^4v_4 \end{bmatrix} = {}^4\bar{J}_4 \dot{\theta}$$

$$= {}^4X_3 \begin{bmatrix} {}^3\omega_3 \\ {}^3v_3 \end{bmatrix} = {}^4X_3 \, {}^3\bar{J}_3 \dot{\theta}$$

Static Force Propagation

Force Balance on Body 3:

$$ {}^3f_3 = {}^3R_t \, {}^tf_t $$

Moment Balance on Body 3 about origin $\{3\}$:

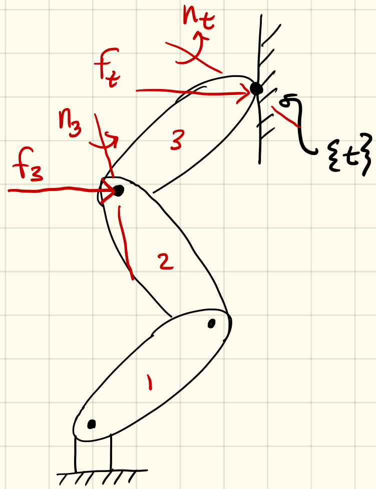
$$ {}^3n_3 = {}^3R_t \, {}^tn_t + {}^3P_t \times {}^3R_t \, {}^tf_t $$

f_t: Force applied by tool onto the wall

n_t: Moments applied by tool onto the wall (about $\{t\}$)

f_i: Force applied at $\{i\}$ onto body $\{i\}$

n_i: Moment about $\{i\}$ onto body $\{i\}$



$$ \begin{bmatrix} {}^3n_3 \\ {}^3f_3 \end{bmatrix} = \begin{bmatrix} {}^3R_t & S({}^3P_t)\,{}^3R_t \\ 0 & {}^3R_t \end{bmatrix} \begin{bmatrix} {}^tn_t \\ {}^tf_t \end{bmatrix} $$

$$ = \begin{bmatrix} {}^tR_3 & 0 \\ -{}^tR_3 S({}^3P_t) & {}^tR_3 \end{bmatrix}^T \begin{bmatrix} {}^tn_t \\ {}^tf_t \end{bmatrix} = {}^tX_3^T \begin{bmatrix} {}^tn_t \\ {}^tf_t \end{bmatrix} $$