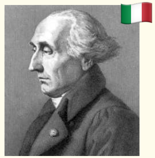
12/

Analytical Dynamics

An Itroduction to the Calculus of Variations


Euler


Lagrange

Admin:

- HW 3 Due Wednesday 10/9 ⟵ 2D Rigid Body Review & Example on Panopto
- HW 4 Due next Wednesday

Last Time:

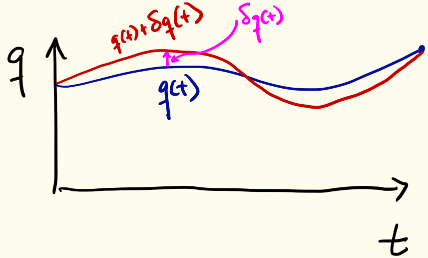
- The total virtual work of constraint forces is zero under any virtual Dispacement that satisfies the constraints
- Virtual work δW is defined during motion. δW for active Forces is zero at Equilibrium.

Next Steps:

Virtual Work over time → ⊕ → Lagrangian Dynamics

Today [Calculus of Variations → ⊕

Lagrangian Dynamics — Wednesday?



Conventional Calculus, Gradients, and Extrema

- Conventional derivative of a function of $y = [y_1, \ldots, y_n]^T$

$$f(y + \underbrace{\delta y}_{\text{Infinitessimal}}) = f(y) + \underbrace{\sum_{i=1}^{n} \frac{\partial f}{\partial y_i}\bigg|_{y} \delta y_i}_{\delta f}$$

- Gradient of f

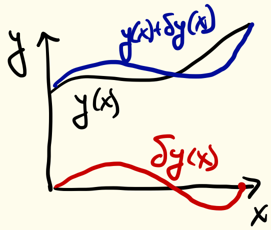
$$\nabla f = \left[\frac{\partial f}{\partial y_1} \cdots \frac{\partial f}{\partial y_n} \right]^T$$

(1) Same dimension as y

(2) Variation of f: $\delta f = f(y + \delta y) - f(y) = \overbrace{\nabla f^T \delta y}^{\text{Inner product of } \nabla f \text{ and } \delta y}$

- When $\nabla f(y^*) = 0$ we say y^* is extremal
  - y^* could be
    - local min/max f
    - Saddle of f
  - In any case, at an extremal $\delta f = 0$ for any δy

Variational Calculus and the Functional Derivative



- Now consider $y(x)$ and a Functional $F[y]$ [Function of a function]

  $x \in [a,b]$

<u>Examples:</u>

$$F[y] = \int_0^3 y(x)^2 \, dx$$

$$F[y] = \int_0^3 [y(x) + \underbrace{y'(x)^2}_{\frac{dy}{dx}}] \, dx$$

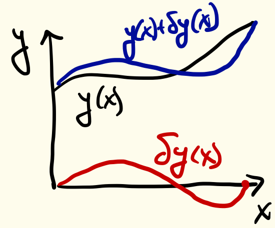
- An infinitessimal variation to $y(x)$ is also a function of x: $\delta y(x)$

- We consider the <u>functional derivative</u> $\frac{\delta F}{\delta y(x)}$ as the <u>function</u> satisfying

$$\delta F = F[y+\delta y] - F[y] = \underbrace{\boxed{\int_a^b \frac{\delta F}{\delta y(x)} \delta y(x) \, dx}}_{\text{contivous analog of } \sum \frac{\partial F}{\partial y_i} \delta y_i}$$

Inner product between two functions $\frac{\delta F}{\delta y(x)}$ and $\delta y(x)$

Example: $F[y] = \int_0^3 y(x)^2\,dx$. Find $\dfrac{\delta F}{\delta y(x)}$



$$\delta F = F[y+\delta y] - F[y] = \int_0^3 \frac{\delta F}{\delta y(x)}\,\delta y(x)\,dx$$

$$= \int_0^3 \left(y(x)+\delta y(x)\right)^2 dx - \int_0^3 y(x)^2\,dx$$

$$= \int_0^3 \left[\cancel{y(x)^2} + 2y(x)\,\delta y(x) - \cancel{y(x)^2}\right] dx$$

$$= \int_0^3 2y(x)\,\delta y(x)\,dx$$

$$\Rightarrow \quad \frac{\delta F}{\delta y(x)} = 2y(x)$$

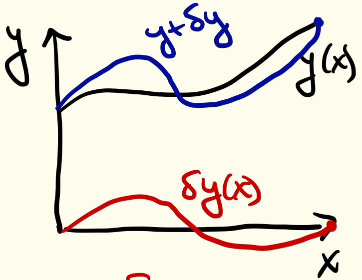
$$\int y(x)\,dx$$

$$\delta F = \int_0^3 y+\delta y - y\,dx$$

$$\int_0^3 \delta y(x)\,dx$$

Functional Derivative Example: More General Case

$$F[y] = \int_a^b L(x, y(x), y'(x))\,dx$$



$$\delta y(a) = \delta y(b) = 0$$

$$F[y+\delta y] = \int_a^b L(x, y(x)+\delta y(x), y'(x)+\delta y'(x))\,dx$$

$$= F[y] + \int_a^b \left[\frac{\partial L}{\partial y}\delta y(x) + \underbrace{\frac{\partial L}{\partial y'}}_{u:=\frac{\partial L}{\partial y'}} \underbrace{\delta y'(x)}_{dv:=d\delta y} \right] dx$$

Integrate By Parts: $\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$

$$= F[y] + \int_a^b \frac{\partial L}{\partial y}\delta y(x)\,dx + \underbrace{\cancel{\frac{\partial L}{\partial y'}\delta y\Big|_a^b}}_{=0 \text{ since endpoints fixed}} - \int_a^b \frac{d}{dx}\left[\frac{\partial L}{\partial y'}\right]\delta y(x)\,dx$$

$$= F[y] + \int_a^b \underbrace{\left[\frac{\partial L}{\partial y} - \frac{d}{dx}\frac{\partial L}{\partial y'}\right]}_{=\frac{\delta F}{\delta y(x)}} \delta y(x)\,dx$$

Extrema of Functionals

Conventional Extrema (Multivariable Case): Infinitesimal variations around y^* do not change $f(y)$

$$\delta f = 0 \quad \forall \delta y \quad \Rightarrow \quad \left.\frac{\partial f}{\partial y_i}\right|_{y^*} = 0 \quad \forall i = \{1, \dots, n\}$$

Extrema of Functionals: " " $F[y]$

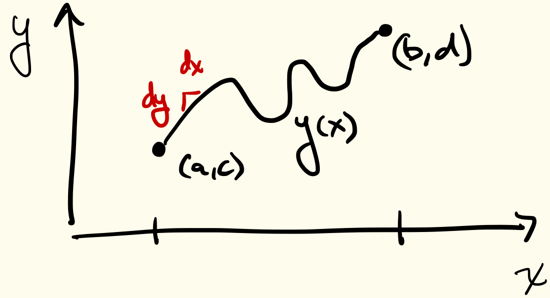
$$\delta F = 0 \quad \forall \delta y \quad \Rightarrow \quad \left.\frac{\delta F}{\delta y(x)}\right|_{y^*} = 0 \quad \forall x \in [a,b]$$

In the case of $F = \int_a^b L(x, y, y')\,dx$ an extremal y requires

$$\boxed{\frac{\partial L}{\partial y} - \frac{d}{dx}\frac{\partial L}{\partial y'} = 0} \quad \forall x$$

Euler-Lagrange Equation

Example: Find shortest path from (a,c) to (b,d)



Arc Length

$$= \int_a^b \sqrt{dx^2 + \left(\frac{dy}{dx}dx\right)^2} = \int_a^b \underbrace{\sqrt{1+y'^2}}_{L(x,y,y')}\,dx$$

Use Euler-Lagrange Equation to find a differential equation that any extremal

y must satisfy:

$$\frac{\partial L}{\partial y} = 0 \qquad \frac{\partial L}{\partial y'} = \frac{y'}{\sqrt{1+y'^2}} \quad \Rightarrow \quad \frac{\partial L}{\partial y} - \frac{d}{dx}\frac{\partial L}{\partial y'} = 0 \quad \Rightarrow \quad \frac{d}{dx}\left[\frac{y'(x)}{\sqrt{1+y'^2_{(x)}}}\right] = 0$$

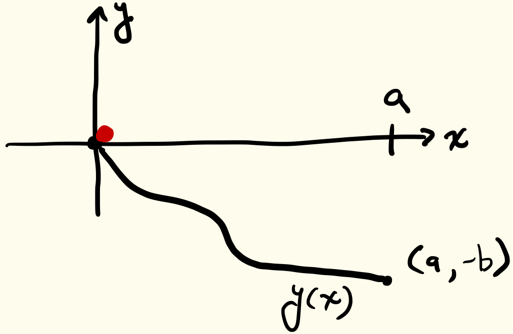
$$\Rightarrow \frac{d}{dx}\frac{y'}{\sqrt{1+y'^2}} = \frac{y''}{(1+y'^2)^{3/2}} = 0 \quad \Rightarrow \quad y''(x) = 0 \quad \Rightarrow \quad \boxed{y'(x) = C}$$

Constant Slope

= Straight Line

MATLAB Example

Brachistochrone Problem



Setup: We place a ball a rest @ (0,0). We want to design a ramp from (0,0) to (a,-b) so that the ball arrives @(a,-b) as soon as possible.

Functional?

$$\frac{1}{2} m v^2 + m g\, y(x) = 0$$

$$\Rightarrow v = \sqrt{-2g\, y(x)} = \sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2} = \sqrt{\left(\frac{dy}{dx}\frac{dx}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2} = \sqrt{1+y'^2}\,\frac{dx}{dt}$$

$$\Rightarrow dt = \sqrt{\frac{1+y'^2}{-2gy}}\, dx \qquad \Rightarrow \text{Time} = \int_0^a \underbrace{\sqrt{\frac{1+y'^2}{-2g\, y(x)}}}_{L(x,y,y')}\, dx = F[y]$$

Brachistochrone Problem

① Euler Lagrange Equation

$$L = \frac{1}{\sqrt{2g}}\sqrt{\frac{1+y'^2}{-y}}$$

$$EL \Rightarrow \quad 0 = \frac{y''}{1+y'^2} + \frac{1}{2y}$$

② Multiply by y' on both sides & integrale

$$\int \frac{y'\,y''\,dx}{1+y'^2} = \int -\frac{y'}{2y}\,dx$$

$$\int \frac{y'\,dy'}{1+y'^2} = \int -\frac{dy}{2y}$$

③ Integrate

$$\ln(1+y'^2) = -\ln(y) + C_1$$

④ $e^{\text{Both Sides}}$

$$y(1+y'^2) = C_2$$

⑤ Smart guess

$$x = R(\theta - \sin\theta)$$

$$y = R(\cos\theta - 1)$$

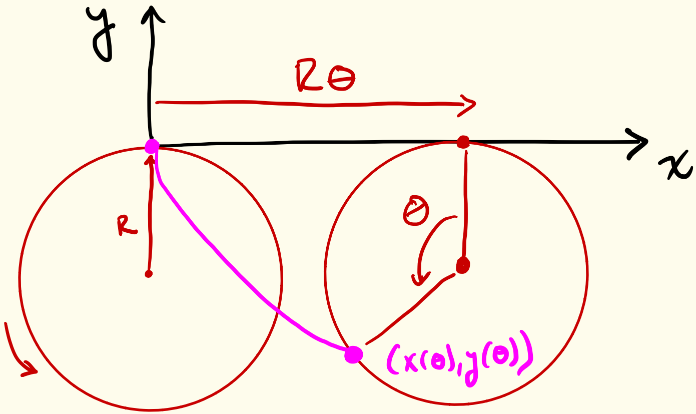
$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{-\cancel{R}\sin\theta}{\cancel{R}(1-\cos(\theta))}$$

$$y(1+y'^2) = R(\cos\theta - 1)\left(1 + \frac{s^2\theta}{(1-c\theta)^2}\right)$$

$$= R(c\theta - 1)\left(\frac{1 + c^2\theta - 2c\theta + s^2\theta}{(1-c\theta)^2}\right)$$

$$= 2R\,\frac{(c\theta - 1)(1 - c\theta)}{(1-c\theta)^2} = -2R$$

Geometric Interpretation



$$x = R(\theta - \sin\theta)$$

$$y = R(\cos\theta - 1)$$

Imagine a bike wheel riding on the ceiling (y=0). Track the point of original contact on the bike wheel as the wheel rolls.

- Curve drawn is called a cycloid.

Summary

- Variational Calculus
  - Extension of Multivariable Calculus to functions of functions
  - Functional Derivative Generalizes the Gradient

$$\frac{\delta F}{\delta y(x)} \quad \text{satisfies} \quad \delta F = \int_a^b \frac{\delta F}{\delta y(x)} \delta y(x)\, dx$$

  - If $F = \int_a^b L(x, y(x), y'(x))$ and endpoints fixed

$$\delta F = \int_a^b \delta L\, dx = \int_a^b \left[\frac{\partial L}{\partial y} - \frac{d}{dx}\frac{\partial L}{\partial y'}\right] \delta y(x)\, dx$$

- Extrema of Functionals:
  - $\frac{\delta F}{\delta y(x)} = 0 \quad \forall x \in [a, b] \qquad (*)$
  - If $F = \int_a^b L(x, y(x), y'(x))\, dx$ and endpoints fixed

    $(*)$ is equivalent to $\frac{d}{dx}\frac{\partial L}{\partial y'} - \frac{\partial L}{\partial y} = 0$ [Euler-Lagrange Equation]