Final Review

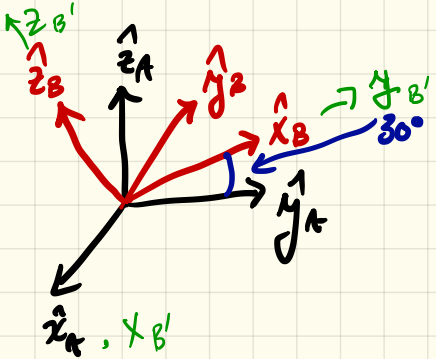
Notes From Session

on 12/8/2018

① Representations of Orientation

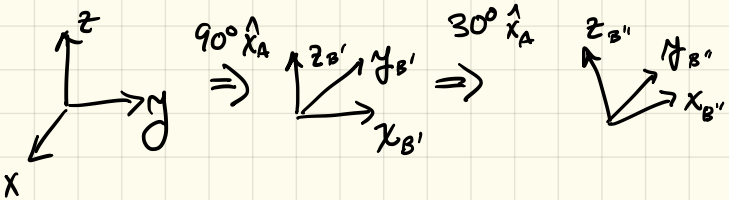
ⓐ Give the orientation of $\{B\}$ with $\overset{\downarrow}{x}$-$\overset{\downarrow}{y}$-$\overset{\downarrow}{z}$ Euler angles

$30°, 0°, 90°$ ✓



Z-Y-X Fixed Angle

$90°, 0°, 30°$



ⓑ Give ${}^A R_B$

$$ {}^A R_B = \left[{}^A\hat{x}_B \quad {}^A\hat{y}_B \quad {}^A\hat{z}_B \right] $$

$$ = \begin{bmatrix} 0 & -1 & 0 \\ c30° & 0 & -s30° \\ s30° & 0 & c30° \end{bmatrix} $$

$\det({}^A R_B) = 1$

$$ {}^{A}T_{B} = \begin{bmatrix} {}^{A}R_{B} & {}^{A}p_{B} \\ 0 & 1 \end{bmatrix} \qquad {}^{A}T_{C} = \begin{bmatrix} {}^{A}R_{C} & {}^{A}p_{C} \\ 0 & 1 \end{bmatrix} $$

<u>Give ${}^{B}T_{C}$:</u>

$$ {}^{B}T_{A} = {}^{A}T_{B}^{-1} = \begin{bmatrix} {}^{A}R_{B}^{T} & -{}^{A}R_{B}^{T}\,{}^{A}p_{B} \\ 0 & 1 \end{bmatrix} $$

$$ {}^{B}T_{C} = {}^{B}T_{A}\,{}^{A}T_{C} = \begin{bmatrix} {}^{A}R_{B}^{T} & -{}^{A}R_{B}^{T}\,{}^{A}p_{B} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} {}^{A}R_{C} & {}^{A}p_{C} \\ 0 & 1 \end{bmatrix} $$

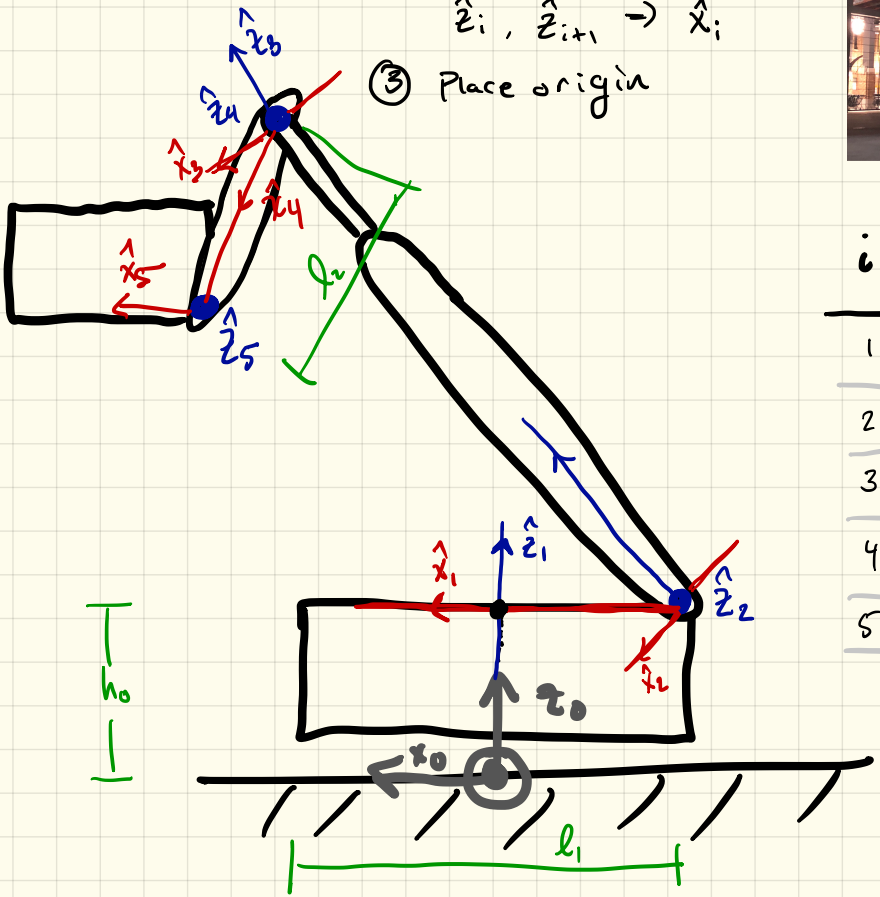
$$ = \begin{bmatrix} {}^{A}R_{B}^{T}\,{}^{A}R_{C} & {}^{A}R_{B}^{T}\,{}^{A}p_{C} - {}^{A}R_{B}^{T}\,{}^{A}p_{B} \\ 0 & 1 \end{bmatrix} $$

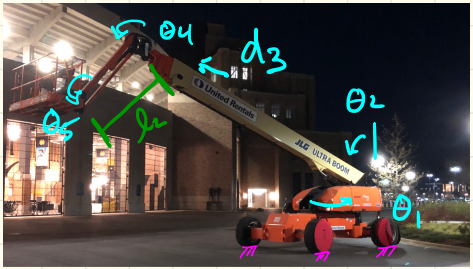
② DH

① Draw joint axes $\hat{z}_i$

② Draw common normal $\hat{z}_i$, $\hat{z}_{i+1}$ → $\hat{x}_i$

③ Place origin



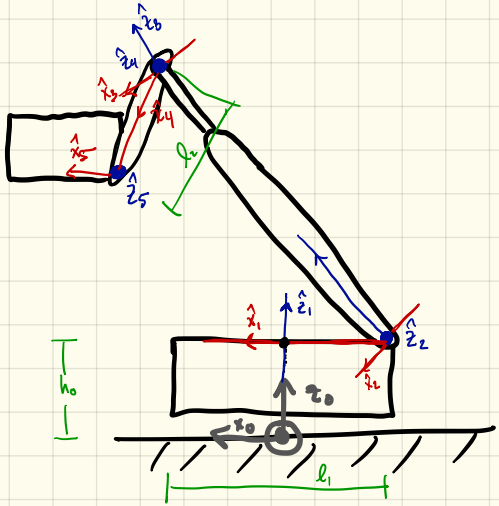


	↺ $\hat{x}_{i-1}$	↑ $\hat{x}_{i-1}$	↑ z_1	↺ z_i
i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	h_0	θ_1
2	$-\pi/2$	$-l_1/2$	0	θ_2
3	$\pi/2$	0	d_3	0
4	$-\pi/2$	0	0	θ_4
5	0	l_2	0	θ_5

As drawn: $\theta_1 = 0°$ $\quad d_3 = 5\,m?$

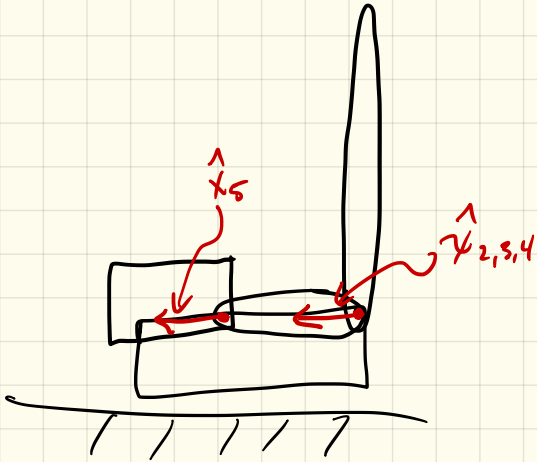
$\theta_2 \approx 45°$ $\quad \theta_4 = 30°$

$\theta_5 = -60°$ ish



Draw in zero configuration

$\theta_1 = 0,\ \theta_2 = 0,\ d_3 = 0,\ \theta_4 = 0,\ \theta_5 = 0$



③ Solve inverse kinematics

$$d_1,\ \theta_2,\ \theta_3,\ \theta_4,\ \theta_5$$

$$
{}^0T_5 = \begin{bmatrix} c_5c_{234} & -s_5c_{234} & s_{234} & a_4\overbrace{c_{234}}^{-r_{23}} + a_2c_2 + a_3c_{23} \\ c_5s_{234} & -s_5s_{234} & -c_{234} & a_4\underbrace{s_{234}}_{r_{13}} + a_2s_2 + a_3s_{23} \\ s_5 & c_5 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\text{given}}
$$

$$\boxed{d_1 = p_z}$$

$$\boxed{\theta_5 = \operatorname{atan2}(r_{31},\ r_{32})}$$

$$\theta_{234} = \operatorname{atan2}(r_{13},\ -r_{23})$$

$$\left.\begin{aligned} a_3c_{23} &= p_x - a_2c_2 + a_4r_{23} \\ a_3s_{23} &= p_y - a_2s_2 - a_4r_{13} \end{aligned}\right\} \text{Square \& add}$$

$$a_3^2 = (p_x + a_4r_{23})^2 + a_2^2c_2^2 - 2a_2c_2(p_x + a_4r_{23})$$
$$+ (p_y - a_4r_{13})^2 + a_2^2s_2^2 - 2a_2s_2(p_y - a_4r_{13})$$

$$a_3^2 - a_2^2 - (p_x + a_4r_{23})^2 - (p_y - a_4r_{13})^2 = \underbrace{-2a_2(p_x + a_4r_{23})}_{A}c_2 \underbrace{- 2a_2(p_y - a_4r_{13})}_{B}s_2$$

$$\boxed{\theta_2 = \operatorname{atan2}(B, A) \pm \operatorname{atan2}(\sqrt{A^2+B^2-C^2},\ C)}$$

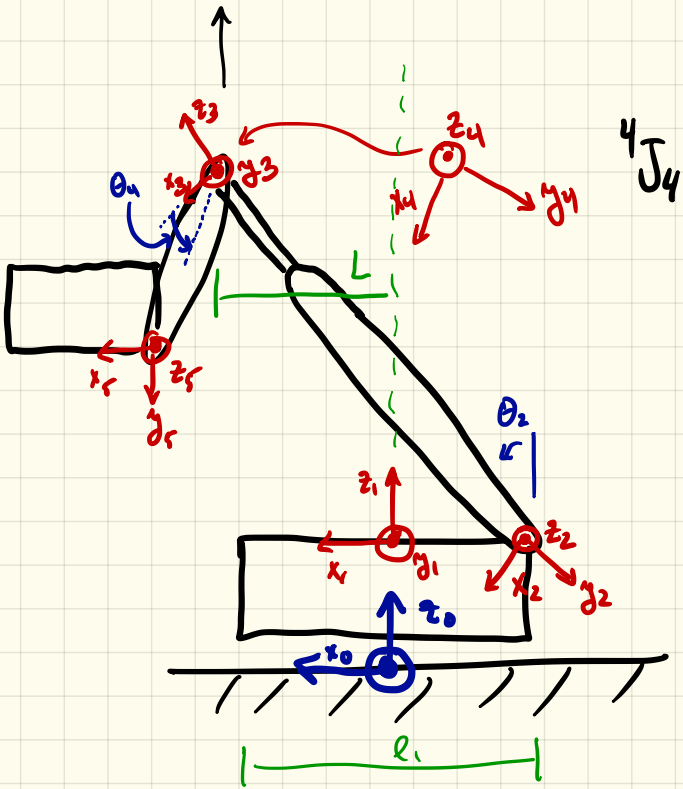
$$\theta_{23} = \operatorname{atan2}\left(\frac{p_y - a_2s_2 - a_4r_{13}}{a_3},\ \frac{p_x - a_2c_2 + a_4r_{23}}{a_3}\right)$$

$$\boxed{\theta_3 = \theta_{23} - \theta_2}$$

$$\boxed{\theta_4 = \theta_{234} - \theta_{23}}$$

④ Jacobian : Give ${}^4J_4$

$$\begin{bmatrix} {}^4\omega_4 \\ {}^4v_4 \end{bmatrix} = {}^4J_4\,\dot{\theta}$$



$$\begin{array}{ccccc} \dot{\theta}_1 = 1 & \dot{\theta}_2 = 1 & \dot{d}_3 = 1 & \dot{\theta}_4 = 1 & \\ \end{array}$$

$${}^4J_4 = \left[\begin{array}{c|c|c|c|c} -s_{24} & 0 & 0 & 0 & 0 \\ -c_{24} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ \hline 0 & d_3 c_4 & -s_4 & 0 & 0 \\ 0 & -d_3 s_4 & -c_4 & 0 & 0 \\ L & 0 & 0 & 0 & 0 \end{array}\right]$$

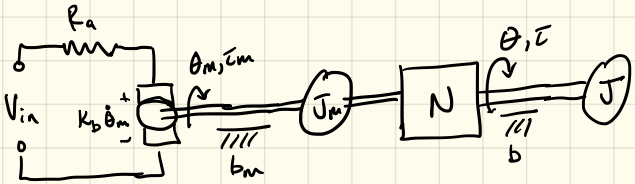
$$L = d_3 s_2 - \frac{l_1}{2}$$

Problem 2: Consider the same motor with gearbox having ratio $N = 10$. Assume the output has damping b, but the motor damping b_m is negligible. The output shaft is hooked up to a load with constant inertia J.

$$V_{in} = R_a\, i_a + k_b\, \dot{\theta}_m \qquad \tau_m = k_\tau i_a$$

$$\tau_m = k_\tau i_a = \left(J_m + \frac{J}{N^2}\right)\ddot{\theta}_m + \frac{b}{N^2}\dot{\theta}_m \qquad \theta = N\theta_m$$

1. [redacted], an input of 15 volts is applied to the motor. The system reaches a steady-state output velocity of $\dot{\theta} = 6$ rad/s. Then, the voltage is removed, and the speed falls to 36.8% (which is e^{-1}) of its steady-state value in 180ms. Find the output damping b and the output inertia J.



$K_\tau = K_b = 0.1\ Nm/A \qquad R_a = 3\Omega \qquad J_m = 10^{-3}\ kg\,m^2$

(a) $$\frac{\dot{\Theta}_m(s)}{V_{in}(s)} = \frac{K_\tau / R_a}{\left(J_m + \frac{J}{N^2}\right)s + \left(\frac{b}{N^2} + \frac{K_b K_\tau}{R_a}\right)}$$

Transfer function @ s=0 gives steady state of step response $\Rightarrow$

$$6\ rads = \frac{K_\tau / R_a}{\frac{b}{N^2} + \frac{K_b K_\tau}{R_a}}\ 15\ V$$

Solve for b $\Rightarrow$ $b = 0.5\ Nms/rad$

(b) $$\frac{\dot{\Theta}_m(s)}{V_{in}(s)} = \frac{*}{\frac{\left(J_m + \frac{J}{N^2}\right)}{\left(\frac{b}{N^2} + \frac{K_b K_\tau}{R_a}\right)}s + 1}$$

time constant of first order decay

$$\Rightarrow \frac{\left(J_m + \frac{J}{N^2}\right)}{\left(\frac{b}{N^2} + \frac{K_b K_\tau}{R_a}\right)} = 180\ ms$$

Solve for J $\Rightarrow$ $J = 0.05\ kg\,m^2$

Final Review

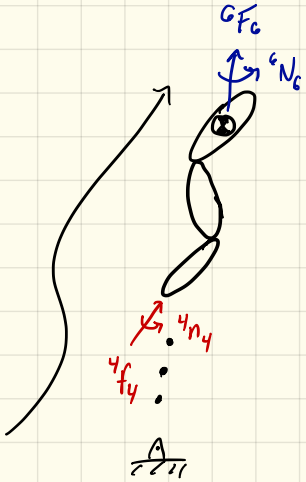
Recursive Newton Euler

RNEA

① Outward Recursion (Body 1 → Body N)

- ${}^{i}\omega_i$, ${}^{i}\dot{\omega}_i$, ${}^{i}\dot{v}_i$, ${}^{i}\dot{v}_{c_i}$

- $\underbrace{{}^{i}F_i = m_i\, {}^{i}\dot{v}_{c_i}}_{\text{Newton Eqn}}$ $\qquad$ $\underbrace{{}^{i}N_i = {}^{c_i}I\, {}^{i}\dot{\omega}_i + {}^{i}\omega_i \times {}^{c_i}I\, {}^{i}\omega_i}_{\text{Euler Eqn}}$



② Inward Recursion

$${}^{i}f_i = {}^{i}F_i + {}^{i}R_{i+1}\, {}^{i+1}f_{i+1}$$

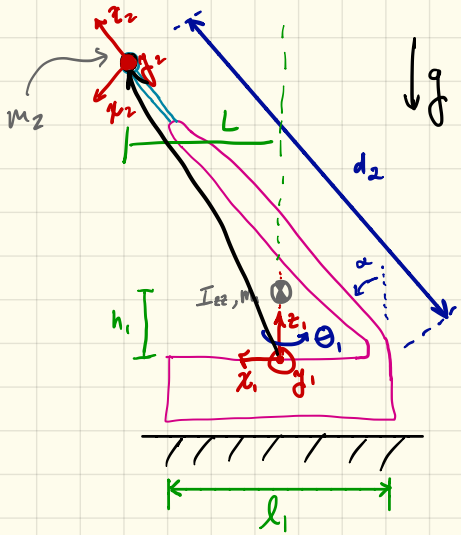
$${}^{i}n_i = {}^{i}N_i + {}^{i}p_{c_i} \times {}^{i}F_i + {}^{i}R_{i+1}\, {}^{i+1}n_{i+} + {}^{i}p_{i+1} \times {}^{i}R_{i+1}\, {}^{i+1}f_{i+1}$$

$$\tau_i = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} {}^{i}n_i \qquad \tau_i = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} {}^{i}f_i$$

Revolute $\qquad$ Prismatic

Example System:

RRPRR



$$ {}^1N_1 = \begin{bmatrix} 0 \\ 0 \\ I_{zz}\ddot{\theta}_1 \end{bmatrix} \qquad {}^1F_1 = \begin{bmatrix} 0 \\ 0 \\ m_1 g \end{bmatrix} \qquad {}^2F_2 = m_2 \begin{bmatrix} s_\alpha g - c_\alpha L\dot{\theta}_1^2 \\ L\ddot{\theta}_1 + 2\dot{\theta}_1\dot{d}_2 s_\alpha \\ c_\alpha g - s_\alpha L\dot{\theta}_1^2 + \ddot{d}_2 \end{bmatrix} $$

② Inward Recursion

$$ {}^2f_2 = {}^2F_2 \Rightarrow \boxed{\tau_2 = m_2\left[\ddot{d}_2 - s_\alpha L\dot{\theta}_1^2 + c_\alpha g\right]} $$

$$ {}^2n_2 = {}^2N_2 = 0 $$

$$ {}^1f_1 = {}^1F_1 + {}^1R_2\,{}^2f_2 \quad * \text{ not needed for } \tau_1 $$

$$ {}^1n_1 = {}^1N_1 + \cancel{{}^1p_1 \times {}^1F_1}^{\,0} + {}^1p_2 \times {}^1R_2\,{}^2f_2 $$

$$ {}^1n_1 = {}^1p_2 \times {}^1R_2\,{}^2f_2 = \uparrow\!\!\!\!\leftarrow \times m_2\left(\overset{*}{\uparrow g} + \underbrace{\odot L\ddot{\theta}_1} + \overset{*}{\rightarrow L\dot{\theta}_1^2} + \underbrace{\odot 2\dot{\theta}_1\dot{d}_2 s_\alpha} + \overset{*}{\uparrow \ddot{d}_2}\right) + \uparrow I_{zz}\ddot{\theta}_1 $$

Same torque about z $\Rightarrow \leftarrow^{L} \times m_2\left(\odot L\ddot{\theta}_1 + 2\dot{\theta}_1\dot{d}_2 s_\alpha\right) + \uparrow I_{zz}\ddot{\theta}_1$

$$ \Rightarrow \uparrow m_2 L^2\ddot{\theta}_1 + 2m_2 L s_\alpha \dot{\theta}_1\dot{d}_2 + I_{zz}\ddot{\theta}_1 $$

$$ \boxed{\tau_1 = \left(m_2 L^2 + I_{zz}\right)\ddot{\theta}_1 + 2m_2 L s_\alpha \dot{\theta}_1\dot{d}_2} $$

Final Review

Lagrangian Dynamics

Lagrangian Steps

1. Kinetic Energy (K) and Potential Energy (U) (as a function $\theta, \dot{\theta}$)

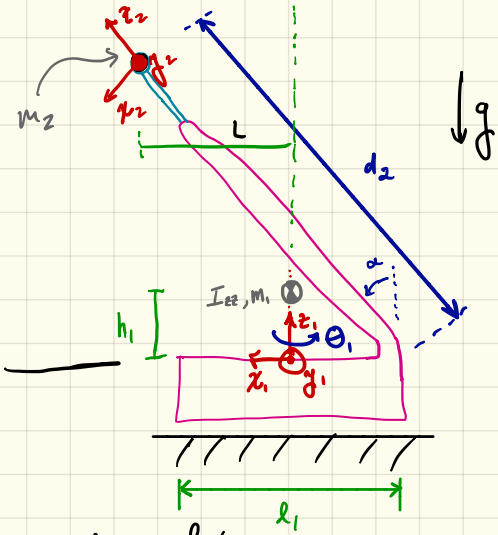
2. $\mathcal{L} = K - U$ "the Lagrangian"

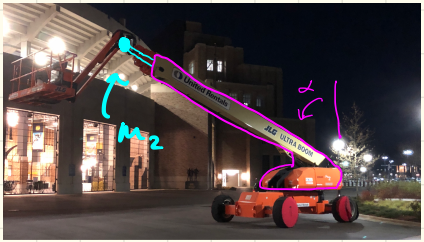
3. $\tau = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} - \frac{\partial \mathcal{L}}{\partial \theta}$

$$\Theta = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} \qquad \frac{\partial \mathcal{L}}{\partial \Theta} = \begin{bmatrix} \partial \mathcal{L}/\partial \theta_1 \\ \partial \mathcal{L}/\partial \theta_2 \\ \partial \mathcal{L}/\partial \theta_3 \end{bmatrix}$$

Example System:

RRPRR





$$L = d_2 s_\alpha - l_1/2$$

$$^2v_{c_2} = \nwarrow \dot{d}_2 + \odot \dot{\theta}_1 L = \begin{bmatrix} 0 \\ L\dot{\theta}_1 \\ \dot{d}_2 \end{bmatrix}$$

① $U_1 = m_1 g h_1$

$K_1 =$ "linear kinetic" $\to 0$ + "rotational kinetic en."

$$= \frac{1}{2}\, {}^1\omega_1^T\, {}^{c_1}\underline{I}\, {}^1\omega_1 = \frac{1}{2} I_{zz} \dot{\theta}_1^2$$

$U_2 = m_2 g d_2 c_\alpha$

$K_2 =$ "Linear kinetic" + "rotational kinetic" $\to 0$

$$= \frac{1}{2} m_2 \| {}^2v_{c_2} \|^2 = \frac{1}{2} m_2 \left(L^2 \dot{\theta}_1^2 + \dot{d}_2^2 \right)$$

$$= \frac{1}{2} m_2 \left((d_2 s_\alpha - l_1/2)^2 \dot{\theta}_1^2 + \dot{d}_2^2 \right)$$

② $$\mathcal{L} = \frac{1}{2} I_{zz} \dot{\theta}_1^2 + \frac{1}{2} m_2 \left((d_2 s_\alpha - l_1/2)^2 \dot{\theta}_1^2 + \dot{d}_2^2 \right) - m_1 g h_1 - m_2 g d_2 c_\alpha$$

$$\mathcal{L} = \frac{1}{2} I_{zz} \dot{\theta}_1^2 + \frac{1}{2} m_2 \left(\left(d_2 s_\alpha - l_1/2 \right)^2 \dot{\theta}_1^2 + \dot{d}_2^2 \right) - m_1 g h_1 - m_2 g d_2 c_\alpha$$

$$\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\theta}} - \frac{\partial \mathcal{L}}{\partial \theta} = \begin{bmatrix} \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\theta}_1} - \frac{\partial \mathcal{L}}{\partial \theta_1} \\ \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{d}_2} - \frac{\partial \mathcal{L}}{\partial d_2} \end{bmatrix}$$

$$\tau_2 = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{d}_2} - \frac{\partial \mathcal{L}}{\partial d_2} = \frac{d}{dt} \left[m_2 \dot{d}_2 \right] - \frac{\partial}{\partial d_2} \left(\frac{1}{2} m_2 \left(d_2 s_\alpha - l_1/2 \right)^2 \dot{\theta}_1^2 - m_2 g d_2 c_\alpha \right)$$

$$= m_2 \ddot{d}_2 - m_2 \underline{\left(d_2 s_\alpha - l_1/2 \right)} s_\alpha \dot{\theta}_1^2 + m_2 g c_\alpha$$

$$\boxed{\tau_2 = m_2 \left(\ddot{d}_2 - L s_\alpha \dot{\theta}_1^2 + g c_\alpha \right)}$$

$$\tau_1 = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\theta}_1} - \frac{\partial \mathcal{L}}{\partial \theta_1} = \frac{d}{dt} \left[I_{zz} \dot{\theta}_1 + m_2 \underline{\left(d_2 s_\alpha - l_1/2 \right)^2} \dot{\theta}_1 \right]$$

$$= I_{zz} \ddot{\theta}_1 + m_2 L^2 \ddot{\theta}_1 + 2 m_2 \left(d_2 s_\alpha - l_1/2 \right) \dot{d}_2 s_\alpha \dot{\theta}_1$$

$$\boxed{\tau_1 = \left[I_{zz} + m_2 L^2 \right] \ddot{\theta}_1 + 2 m_2 L s_\alpha \dot{d}_2 \dot{\theta}_1}$$