14/

Analytical Dynamics

Lagrangian Dynamics

D'Alembert

Lagrange

Admin:

- HW 4 Due Wednesday
- Jacobians got you down? Supplemental video on Panopto.

Today:

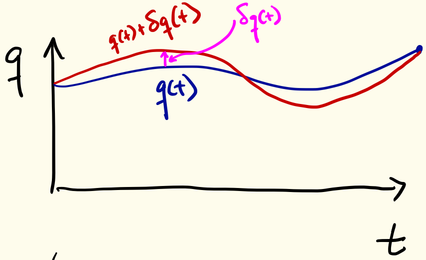
D'Alembert's Principle over time + Calculus of Variations → Lagrangian Dynamics

- Review "Consistent with Constraints"
- Derive Lagrangian dynamics from D'Alembert's principle
- Examples

Conventions

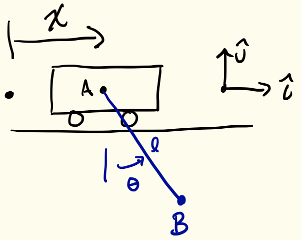
- Masses $m_1, \dots, m_i, \dots, m_N$ at positions $\underline{r}_1, \dots, \underline{r}_i, \dots, \underline{r}_N$
- Generalized coordinates

$$q_1, \dots, q_k, \dots, q_n$$



What is does it mean for virtual displacements to “be consistent with constraints”?

Case 1: All constraint accounted for by choice of GC



When is $\delta \underline{q}$ consistent w/ constraints? Always (no conditions on $\delta \underline{q}$)

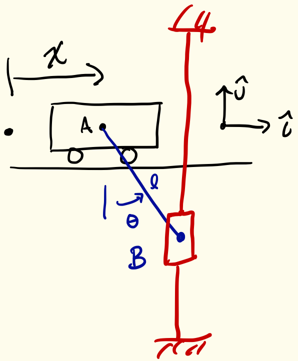
$\underline{r}_A = x\hat{i}$

$\delta \underline{r}_A = \delta x \, \hat{i}$

When are $\delta \underline{r}_A$ and $\delta \underline{r}_B$ consistent w/constraints?

$$\left.\begin{aligned} \delta \underline{r}_A &= \frac{\partial \underline{r}_A}{\partial x}\delta x + \cancel{\frac{\partial \underline{r}_A}{\partial \theta}\delta\theta} \\ \delta \underline{r}_B &= \frac{\partial \underline{r}_B}{\partial x}\delta x + \frac{\partial \underline{r}_B}{\partial \theta}\delta\theta \end{aligned}\right\} \text{ for some } \delta x, \delta\theta$$

Case 2: Supplemental constraints on GC



When are $\delta \underline{r}_A$, $\delta \underline{r}_B$ consistent w/constraints?

$$\left.\begin{aligned} \delta \underline{r}_A &= \frac{\partial \underline{r}_A}{\partial x}\delta x + \cancel{\frac{\partial \underline{r}_A}{\partial \theta}\delta\theta} \\ \delta \underline{r}_B &= \frac{\partial \underline{r}_B}{\partial x}\delta x + \frac{\partial \underline{r}_B}{\partial \theta}\delta\theta \\ \delta \underline{r}_B \cdot \hat{i} &= 0 \end{aligned}\right\} \text{ for some } \delta x, \delta\theta$$

When is $\delta \underline{q}$ consistent w/ constraints: $\left(\frac{\partial \underline{r}_B}{\partial x}\cdot\hat{i}\right)\delta x + \left(\frac{\partial \underline{r}_B}{\partial \theta}\cdot\hat{i}\right)\delta\theta = 0$

Summary

- **Generalized Forces:**

	Always	when: $\underline{F}_i = \overbrace{\underline{F}_{i,c}}^{\text{Conservative}} + \overbrace{\underline{F}_{i,nc}}^{\text{Non-Conservative}}$
One-By-One	$Q_k = \underbrace{\sum_{i=1}^{N} \frac{\partial \underline{r}_i}{\partial q_k} \cdot \underline{F}_i}_{\text{Generalized Force Associated with } q_k}$ ← Active Force	$Q_k = -\frac{\partial V}{\partial q_k} + \sum_{i=1}^{N} \frac{\partial \underline{r}_i}{\partial q_k} \cdot \underline{F}_{i,nc}$
All Together	$Q = \begin{bmatrix} Q_1 \\ \vdots \\ Q_n \end{bmatrix} = \sum_{i=1}^{N} \{J_i\}^T \{\underline{F}_i\}$	$Q = -\nabla_q V + \sum_{i=1}^{N} \{J_i\}^T \{\underline{F}_{i,nc}\}$

- **D'Alembert's Principle**

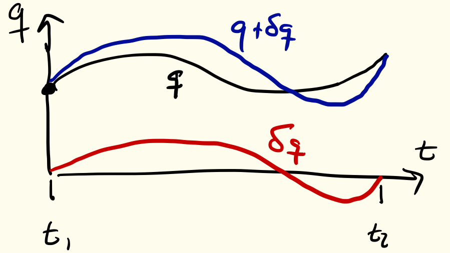
$$\sum_{i=1}^{N} (\underline{F}_i - m_i \underline{a}_i) \cdot \delta \underline{r}_i = 0 \Rightarrow \delta W = \sum_{i=1}^{N} m_i \ddot{\underline{r}}_i \cdot \delta \underline{r}_i$$

($\forall\, \delta r_1, \ldots, \delta r_N$ consistent w/ constraints)

$$\sum_{k=1}^{n} Q_k \delta q_k = \sum_{k=1}^{n} \underbrace{\left[\sum_{i=1}^{N} \frac{\partial \underline{r}_i}{\partial q_k} \cdot \overbrace{(m_i \ddot{\underline{r}}_i)}^{\text{Inertial Force}} \right]}_{\text{Generalized Inertial Force Associated w/ } q_k} \delta q_k$$

($\forall\, \delta q$ consistent w/ constraints)

From D'Alembert to Lagrange (1 of 3)



$$0 = \delta W - \boxed{\sum_{i=1}^{N} m_i \ddot{\underline{r}}_i \cdot \delta \underline{r}_i}$$

$$m_i \ddot{\underline{r}}_i \cdot \delta \underline{r}_i = m_i \left[\ddot{\underline{r}}_i \cdot \delta \underline{r}_i + \delta\left(\tfrac{1}{2} \dot{\underline{r}}_i \cdot \dot{\underline{r}}_i\right) - \underbrace{\delta\left(\tfrac{1}{2} \dot{\underline{r}}_i \cdot \dot{\underline{r}}_i\right)}_{\delta T_i / m_i} \right]$$

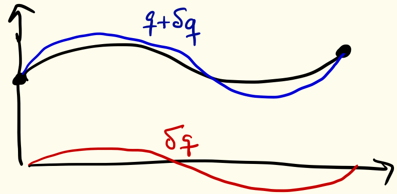
$$= m_i \left[\ddot{\underline{r}}_i \cdot \delta \underline{r}_i + \dot{\underline{r}}_i \cdot \delta \dot{\underline{r}}_i \right] - \delta T_i$$

$$= \frac{d}{dt}\left[m_i \dot{\underline{r}}_i \cdot \delta \underline{r}_i \right] - \delta T_i$$

$$0 = \int_{t_1}^{t_2} \left[\left[\delta W(t) + \delta T(t) - \sum \frac{d}{dt}\left[m_i \dot{\underline{r}}_i \cdot \delta \underline{r}_i \right] \right] \right] dt$$

$$= \int_{t_1}^{t_2} \left[\delta W + \delta T \right] dt - \cancelto{=0}{m_i \dot{\underline{r}}_i \cdot \delta \underline{r}_i \Big|_{t_1}^{t_2}}$$

From D'Alembert to Lagrange (2 of 3)



$$\int_{t_1}^{t_2} \left[\delta T + \delta W\right] dt = 0$$

Using Virtual work: $\delta W = -\delta V + \delta W_{nc}$

$$\int_{t_1}^{t_2} \left[\delta T - \delta V + \delta W_{nc}\right] dt = 0$$

Hamilton's principle

$\forall \delta q$ consistent w/constraints

Special case: All conservative forces, no constraints on δq

$$\int_{t_1}^{t_2} (\delta T - \delta V)\, dt = 0 \Rightarrow \boxed{\delta \underbrace{\int_{t_1}^{t_2} (T - V)\, dt}_{\text{Action Integral "}S\text{"}} = 0}$$

principle of least action

Principle of Least Action

Out of all paths nature could take it picks the one that minimizes $\int_{t_1}^{t_2} (T-V)\,dt$!

① Actually principle of Stationary action (minimum can't be verified but in practice it is the case.)

② <u>Criticism:</u> But it only works for conservative systems?

<u>Response:</u> Energy is always consered if dealing with a closed system.
(e.g., a damper creates heat that leads to kinetic energy of surrounding air. So in that case a damper can be viewed as conservative if system boundaries are suitably redrawn.)

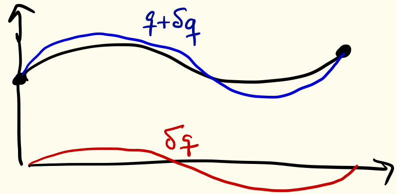
Summary from Last Week

- Variational Calculus
    - Extension of Multivariable Calculus to functions of functions
    - If $F = \int_a^b L(x, y(x), y'(x))\,dx$ and endpoints fixed

$$\delta F = \int_a^b \delta L\, dx = \int_a^b \left[\frac{\partial L}{\partial y} - \frac{d}{dx}\frac{\partial L}{\partial y'}\right] \delta y(x)\, dx$$

Last time		Today
y	$\longrightarrow$	q
x	$\longrightarrow$	t

From D'Alembert to Lagrange (3 of 3)



Hamiltons Principle:

$$0=\int_{t_1}^{t_2}\left[\delta T-\delta V+\sum_{k} Q_{k,nc}\,\delta q_k\right]dt$$

Defn: The Lagrangian $\mathcal{L}=T-V$ depends on $q, \dot{q}$, sometimes t

$$\int_{t_1}^{t_2}\delta \mathcal{L}\,dt=\int_{t_1}^{t_2}\sum_{k=1}^{n}\left[\frac{\partial\mathcal{L}}{\partial q_k}-\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial\dot{q}_k}\right]\delta q_k(t)\,dt$$

Lagrange's form of Hamilton's principle

$$0=\int_{t_1}^{t_2}\sum_{k}\left[\frac{\partial\mathcal{L}}{\partial q_k}-\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{q}_k}+Q_{k,nc}\right]\delta q_k(t)\,dt$$

$\forall\,\delta q(t)$ satisfying constraints

If no constraints on GC:

$$Q_{k,nc}=\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{q}_k}-\frac{\partial\mathcal{L}}{\partial q_k}$$

Lagrangian Equations of motion.

Lagrange Steps:

⓪ Pick GC

① Keypoints & Kinematics

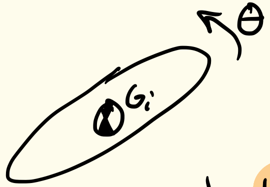
$\underline{r}_i(q)$ for particles

$\underline{r}_{G_i}(q)$, $\theta_i(q)$ for rigid Bodies

② Energy T & V

③ Non conservative Generalized Forces

④ $Q_{k,nc} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_k} - \frac{\partial \mathcal{L}}{\partial q}$

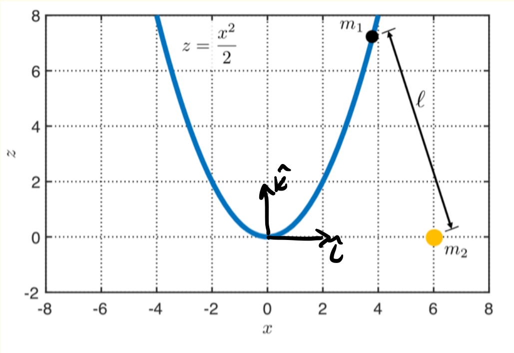


CoM velocity

$$T_i = \frac{1}{2} m_i \left|\dot{\underline{r}}_{G_i}\right|^2 + \frac{1}{2} I_G \dot{\theta}^2$$

Rotational Inertia about the CoM

Example: EOM for Prob 3 on Exam 1



(0) Pick GC: $q = x$

(1) $\underline{r}_1 = x\hat{\imath} + \frac{x^2}{2}\hat{k}$ $\quad \dot{\underline{r}}_1 = \dot{x}\hat{\imath} + x\dot{x}\hat{k}$

(2) Energy

$$T = \tfrac{1}{2}m_1|\dot{\underline{r}}_1|^2 = \tfrac{1}{2}m_1\left[\dot{x}^2 + x^2\dot{x}^2\right] = \tfrac{1}{2}m_1\dot{x}^2(1+x^2)$$

$$V = -\frac{G m_1 m_2}{\ell} = \frac{-G m_1 m_2}{\sqrt{(x-6)^2 + x^4/4}}$$

(3) Generalized non-cons. forces

$$Q_{nc} = 0$$

(4) Turn the crank $\mathcal{L} = T - V$

$$\frac{\partial \mathcal{L}}{\partial \dot{x}} = m_1\dot{x}(1+x^2) \qquad \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} = m_1\ddot{x}(1+x^2) + 2m_1\dot{x}^2 x$$

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{\partial T}{\partial x} - \frac{\partial V}{\partial x} = m_1\dot{x}^2 x + \frac{\partial}{\partial x}\frac{G m_1 m_2}{\ell} = m_1\dot{x}^2 x - \frac{G m_1 m_2}{\ell^3}\left[x - 6 + \frac{x^3}{2}\right]$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} - \frac{\partial \mathcal{L}}{\partial x} = 0 \Rightarrow \boxed{m_1\left[\ddot{x}(1+x^2) + x\dot{x}^2\right] + \frac{G m_1 m_2}{\ell^3}\left[x - 6 + \frac{x^3}{2}\right] = 0}$$

MATLAB Example

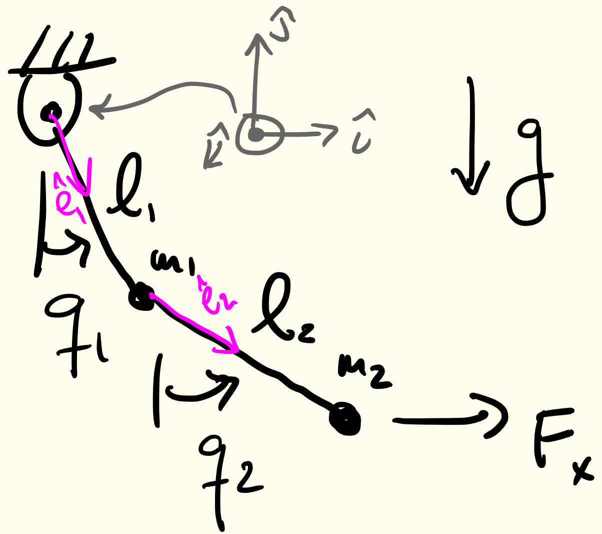
① Keypoints & Kinematics

$\underline{r}_i(q)$ for particles

② Energy T & V

③ Nonconservative Generalized Forces

④ $Q_{nc} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{\partial \mathcal{L}}{\partial q}$



CODE ONLINE AND PANOPTO VIDEO

Three MATLAB Examples of Lagrangian are Online:

Supplemental Video for the Double Pendulum on Panopto

Summary

- Hamilton's Principle:

$$\int_{t_1}^{t_2} [\delta T + \delta W]\, dt = 0 \quad \Rightarrow \quad \int_{t_1}^{t_2} \left[\delta \mathcal{L} + \sum_k Q_{k,nc}\, \delta q_k \right] dt = 0 \qquad (\delta q \text{ sat. constr.})$$

Special Case: Principle of Least Action $\quad \delta \int_{t_1}^{t_2} \mathcal{L}\, dt = 0 \quad (\delta q \text{ unc.})$

(no non-conservative forces)

- Lagrange's Equations:

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_k} - \frac{\partial \mathcal{L}}{\partial q_k} = Q_{k,nc} \qquad k = 1, \ldots, n \qquad (\delta q \text{ unc.})$$