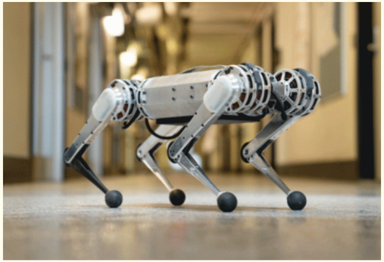
Analytical Dynamics

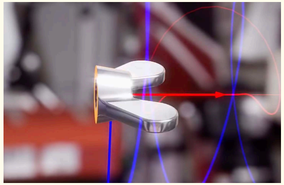
Rigid Body Dynamics

Euler 1765



Admin:
- No Class next Monday
- HW 7 due next Tuesday @5PM

Today:
- Kinetic Energy w/ Rigid Bodies
- Lagrange's Equations w/ Rigid Bodies
- Euler's Equations ($\underline{F} = m\underline{a}$ analog for Rotations)



www.youtube.com/watch?v=1VPfZ_XzisU

On deck:
- Generalized Forces w/ Moments in 3D
- D'Alembert's Equations w/ 3D Rigid Bodies

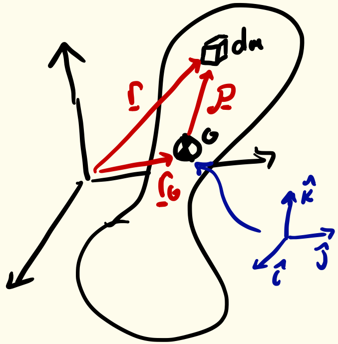
Angular Momentum: REVIEW

Angular velocity $\underline{\omega}$

$$H_G = \int \underline{\rho} \times dm\, \dot{\underline{\rho}} \qquad \dot{\underline{\rho}} = \underline{\omega} \times \underline{\rho}$$

$$\cancel{= I_G \dot{\theta} \hat{k} \text{ for 2D system}}$$

Doesn't work in 3D



$$= \int -\underline{\rho} \times (\underline{\rho} \times \underline{\omega})\, dm$$

$$\{H_G\} = \underbrace{\int [\tilde{\rho}][\tilde{\rho}]^T dm}\,\{\underline{\omega}\}$$

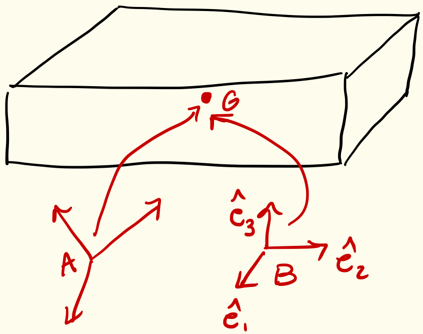
I_G : Rotational Inertia Matrix 3x3

Mass moments of Inertia

Mass products of inertia

$$I_G = \int \begin{bmatrix} y^2+z^2 & -xy & -xz \\ -xy & x^2+z^2 & -yz \\ -xz & -yz & y^2+z^2 \end{bmatrix} dm = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix}$$

Principal Axes: REVIEW



- Consider a body w/ rotational inertia matrix ${}^{A}I_{G}$ Expressed in A Rotational inertia about G

- There Always is a frame B such that

$$ {}^{B}I_{G} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} $$

$\lambda_1, \lambda_2, \lambda_3$ are the eig. values of ${}^{A}I_{G}$ (principal moments of inertia)

Properties: $\lambda_i \geq 0$ (2) $\lambda_i + \lambda_j \geq \lambda_k$ for any i,j,k distinct

- If ${}^{A}\{\hat{e}_1\}, {}^{A}\{\hat{e}_2\}, {}^{A}\{\hat{e}_3\}$ are the eig. vecs of ${}^{A}I_{G}$ (w.l.o.g. $\hat{e}_3 = \hat{e}_1 \times \hat{e}_2$)

$\Rightarrow {}^{A}R_{B} = \left[{}^{A}\{\hat{e}_1\} \; {}^{A}\{\hat{e}_2\} \; {}^{A}\{\hat{e}_3\} \right]$ gives orientation of frame B

$$ {}^{B}I_{G} = {}^{A}R_{B}^{T} \; {}^{A}I_{G} \; {}^{A}R_{B} $$

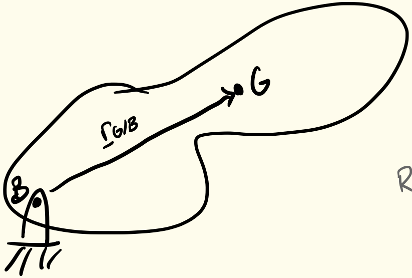
$\hat{e}_1, \hat{e}_2, \hat{e}_3$ are what we call the Principal Axes of the body

Kinetic Energy

$$T = \frac{1}{2} m \|\underline{v}_G\|^2 + \frac{1}{2} \{\underline{\omega}\}^T \underline{I}_G \{\underline{\omega}\}$$ (works all the time)

$$= \frac{1}{2} \{\underline{\omega}\}^T \underline{I}_B \{\underline{\omega}\}$$ (works when rotating purely about B)

Analog of Parallel axis Theorem in 3D:



$$\boxed{\underline{I}_B} = \underline{I}_G + m [\tilde{r}_{G/B}][\tilde{r}_{G/B}]^T \qquad \{r_{G/B}\} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Rotational inertia about B

$$= \underline{I}_G + m \begin{bmatrix} y^2+z^2 & -xy & -xz \\ -xy & x^2+z^2 & -yz \\ -xz & -yz & x^2+y^2 \end{bmatrix}$$

Individual Entries:

$$I_{zz,B} = I_{zz,G} + m(x^2+y^2)$$

$$-I_{xy,B} = -I_{xy,G} - mxy$$

$$\{\underline{H}_B\} = \underline{I}_B \{\underline{\omega}\}$$

when rotating purely about B

Kinetic Energy: Bonus Derivation

$$(*) \quad \sum_i m_i \underline{\rho}_i = 0$$

$$T = \frac{1}{2}\sum_i m_i \|\underline{v}_i\|^2 = \frac{1}{2}\sum_i m_i \|\underline{v}_G + \underline{\omega} \times \underline{\rho}_i\|^2$$

$$= \frac{1}{2}\sum_i m_i \left(\underline{v}_G + \underline{\omega} \times \underline{\rho}_i\right) \cdot \left(\underline{v}_G + \underline{\omega} \times \underline{\rho}_i\right)$$

Sums to zero via (*)

$$= \frac{1}{2}\sum_i m_i \left[\|\underline{v}_G\|^2 + \cancel{2(\underline{\omega} \times \underline{\rho}_i)} \cdot v_G + (\underline{\rho}_i \times \underline{\omega}) \cdot (\underline{\rho}_i \times \underline{\omega})\right]$$

$$= \frac{1}{2} m \|\underline{v}_G\|^2 + \frac{1}{2}\sum m_i \left([\tilde{\underline{\rho}}_i]\{\underline{\omega}\}\right)^T \left([\tilde{\underline{\rho}}_i]\{\underline{\omega}\}\right)$$

$$= \frac{1}{2} m \|\underline{v}_G\|^2 + \frac{1}{2}\{\underline{\omega}\}^T \underbrace{\left[\sum_i m_i [\tilde{\underline{\rho}}_i]^T [\underline{\rho}_i]\right]}_{I_G!} \{\underline{\omega}\}$$

Rotational Inertia about any Point

Bonus Derivation

$$\underline{H}_B = \underline{r}_{G/B} \times m \underline{V}_G + \underline{H}_G$$

$$= \underline{r}_{G/B} \times m \left(\underline{V}_B + \underline{\omega} \times \underline{r}_{G/B} \right) + \underline{H}_G$$

$$= \underline{r}_{G/B} \times m \underline{V}_B + \underbrace{\left(\underline{I}_G + m \left(\underline{r}_{G/B} \times \right) \left(r_{G/B} \times \right)^T \right)}_{:= I_B} \underline{\omega}$$

Linear Momentum

$$\underline{V}_G = \underline{V}_B + \underline{\omega} \times \underline{r}_{G/B}$$

$$\Rightarrow \{\underline{P}\} = m\{\underline{V}_B\} - [\widetilde{m r_{G/B}}]\{\underline{\omega}\}$$

Alltogether:

$$\begin{bmatrix} \{\underline{H}_B\} \\ \{P\} \end{bmatrix} = \overbrace{\begin{bmatrix} I_B & [\widetilde{m \underline{r}_{G/B}}] \\ [\widetilde{m \underline{r}_{G/B}}]^T & m \end{bmatrix}} \begin{bmatrix} \{\underline{\omega}\} \\ \{\underline{V}_B\} \end{bmatrix}$$

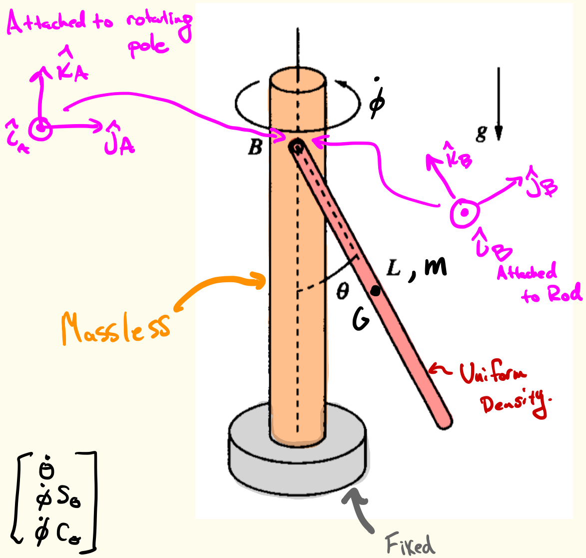
For those grad. Students in robotics: This gives us the Spatial inertia matrix from Optimization - Based Robotics.

Give the Lagrangian

$$q = [\phi, \theta]^T$$

Apply 3D version of Parallel Axis Thm:

$${}^B I_G = \begin{bmatrix} \frac{1}{12}mL^2 & 0 & 0 \\ 0 & \frac{1}{12}mL^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad {}^B I_B = \begin{bmatrix} \frac{1}{3}mL^2 & 0 & 0 \\ 0 & \frac{1}{3}mL^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$



Angular Velocity

$$\underline{\omega}_B = \dot{\theta}\,\hat{i}_B + \dot{\phi}\,\hat{k}_A = \dot{\theta}\,\hat{i}_B + (C_\theta \hat{k}_B + S_\theta \hat{j}_B)\,\dot{\phi} \qquad {}^B\{\underline{\omega}_B\} = \begin{bmatrix} \dot{\theta} \\ \dot{\phi} S_\theta \\ \dot{\phi} C_\theta \end{bmatrix}$$

Kinetic & Potential Energy

$$T = \frac{1}{2}\,{}^B\{\underline{\omega}_B\}^T\,{}^B I_B\,{}^B\{\underline{\omega}\} = \frac{1}{6}mL^2(\dot{\theta}^2 + S_\theta^2\,\dot{\phi}^2)$$

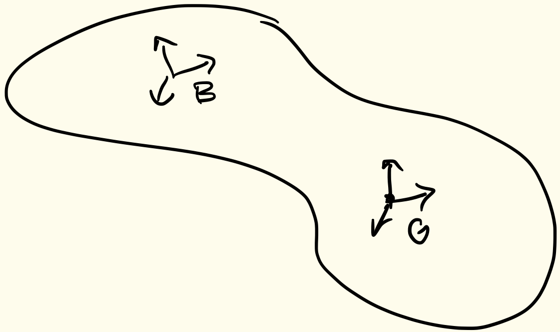
$$V = -\frac{L}{2}mg\,C_\theta \qquad \mathcal{L} = \frac{1}{6}mL^2(\dot{\theta}^2 + S_\theta^2\,\dot{\phi}^2) + \frac{L}{2}mg\,C_\theta$$

EOM

$$\boxed{\begin{aligned} &\frac{1}{3}mL^2\ddot{\theta} - \frac{1}{3}mL^2 S_\theta C_\theta \dot{\phi}^2 + mg\frac{L}{2}S_\theta = 0 \\ &\frac{1}{3}mL^2 S_\theta^2 \ddot{\phi} + \frac{2}{3}mL^2 S_\theta C_\theta \dot{\theta}\dot{\phi} = 0 \end{aligned}}$$

Euler's Equations

Assume G and B rigidly attached to Body



$$\dot{\underline{H}}_G = \sum \underline{M}_G = \underline{M}_{G,net}$$

$$^G\{\dot{\underline{H}}_G\} = \frac{d}{dt}\,^G\{\underline{H}_G\} + {}^G\{\underline{\omega}\} \times {}^G\{\underline{H}_G\}$$

$$= \frac{d}{dt}\left[\underline{\underline{I}}_G\,^G\{\underline{\omega}\}\right] + {}^G\{\underline{\omega}\} \times {}^G\underline{\underline{I}}_G\,^G\{\underline{\omega}\}$$

$$^G\{\underline{H}_G\} = \underline{\underline{I}}_G\,^G\{\underline{\omega}\}$$

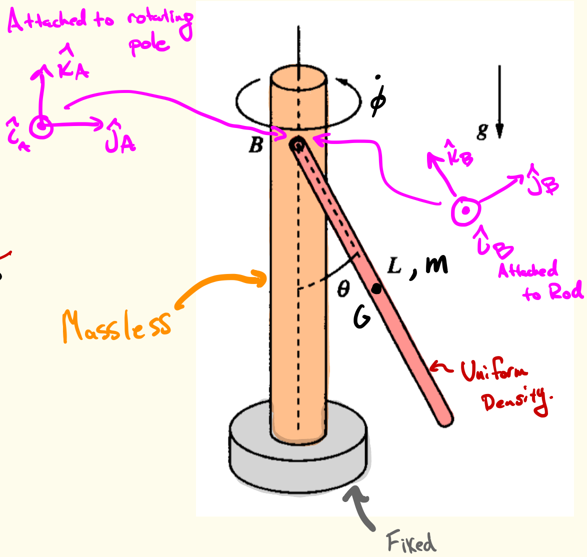
$$\boxed{^G\{\underline{M}_{G,net}\} = {}^G\underline{\underline{I}}_G\,^G\{\dot{\underline{\omega}}\} + {}^G\{\underline{\omega}\} \times {}^G\underline{\underline{I}}_G\,^G\{\underline{\omega}\}}$$

Euler's equation.

If purely rotating about B:

$$^B\{\underline{M}_{net,B}\} = {}^B\underline{\underline{I}}_B\,^B\{\dot{\underline{\omega}}\} + {}^B\{\underline{\omega}\} \times {}^B\underline{\underline{I}}_B\,\{\underline{\omega}\}$$

Verify the EOM Using Euler's Equation



Angular Velocity & Acceleration:

$$\underline{\omega}_B = \dot{\theta}\hat{i}_B + \dot{\phi}\left[C_\theta \hat{k}_B + S_\theta \hat{j}_B\right]$$

$$\dot{\underline{\omega}}_B = \ddot{\theta}\hat{i}_B + \ddot{\phi}\left[C_\theta \hat{k}_B + S_\theta \hat{j}_B\right] + \dot{\phi}\dot{\theta}\left[C_\theta \hat{j}_B - S_\theta \hat{k}_B\right] + \cancel{\underline{\omega}_B \times \underline{\omega}_B}$$

Euler's Equation:

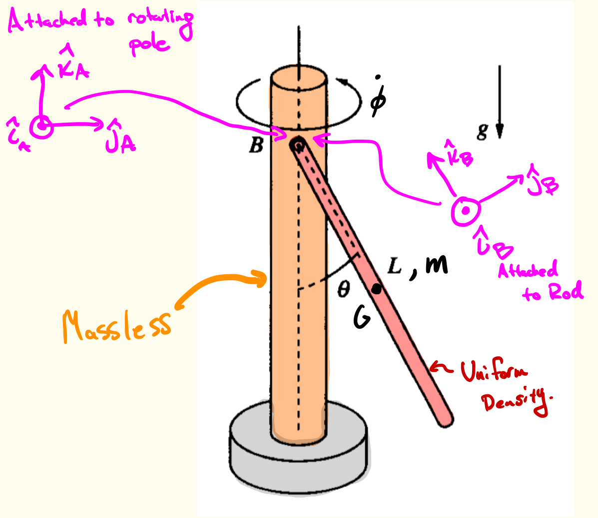
$${}^{B}\{\underline{M}_{B,net}\} = {}^{B}I_B \, {}^{B}\{\dot{\underline{\omega}}_B\} + {}^{B}\{\underline{\omega}_B\} \times {}^{B}I_B \, {}^{B}\{\underline{\omega}_B\}$$

$$= \begin{bmatrix} \frac{1}{3}mL^2 & 0 & 0 \\ 0 & \frac{1}{3}mL^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \ddot{\theta} \\ \ddot{\phi} S_\theta + C_\theta \dot{\theta}\dot{\phi} \\ * \end{bmatrix} + \begin{bmatrix} \dot{\theta} \\ S_\theta \dot{\phi} \\ C_\theta \dot{\phi} \end{bmatrix} \times \begin{bmatrix} \frac{1}{3}mL^2 \dot{\theta} \\ \frac{1}{3}mL^2 S_\theta \dot{\phi} \\ 0 \end{bmatrix}$$

$$= \frac{1}{3}mL^2 \begin{bmatrix} \ddot{\theta} - C_\theta S_\theta \dot{\phi}^2 \\ S_\theta \ddot{\phi} + 2C_\theta \dot{\phi}\dot{\theta} \\ 0 \end{bmatrix}$$

Verify the EOM Using Euler's Equation

$${}^B\{\underline{M}_{B,net}\} = \frac{1}{3} m L^2 \begin{bmatrix} \ddot{\theta} - c_\theta s_\theta \dot{\phi}^2 \\ s_\theta \ddot{\phi} + 2 c_\theta \dot{\phi} \dot{\theta} \\ 0 \end{bmatrix}$$



We must Sample Euler's Equations along carefully Chosen directions to avoid constraint moments.

① $${}^B\{\hat{\imath}_B\} \cdot {}^B\{\underline{M}_{B,net}\} = \frac{1}{3} m L^2 (\ddot{\theta} - c_\theta s_\theta \dot{\phi}^2)$$

$$\boxed{\underbrace{-mg\frac{L}{2} s_\theta}_{} = \frac{1}{3} m L^2 (\ddot{\theta} - c_\theta s_\theta \dot{\phi}^2)}$$

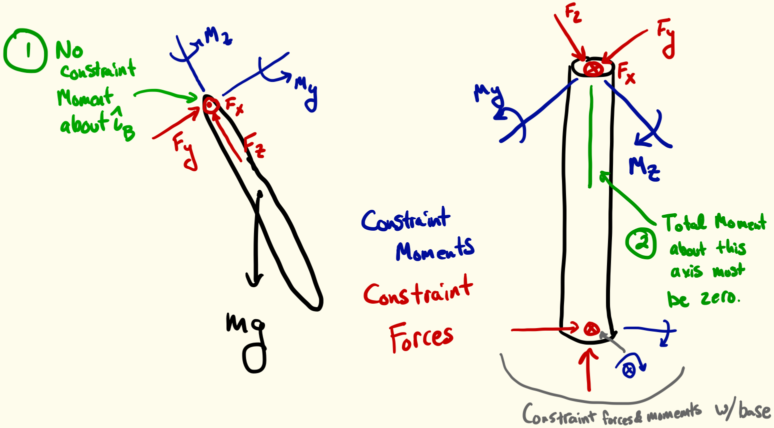
Gravity is the only force creating a moment about B in the $\hat{\imath}_B$ direction.

② $${}^B\{\hat{K}_A\} \cdot {}^B\{\underline{M}_{B,net}\} = \begin{bmatrix} 0 \\ s_\theta \\ c_\theta \end{bmatrix} \cdot {}^B\{\underline{M}_{B,net}\} = 0$$

$$\boxed{0 = \frac{1}{3} m L^2 \left[s_\theta^2 \ddot{\phi} + 2 c_\theta s_\theta \dot{\phi} \dot{\theta} \right]}$$

Free Body Digram Rod

Free Body Digram Pole



Summary

- Rotational Inertia Matrix: Analog of Parallel Axis Theorem

$$I_B = I_G + m[\tilde{r}_{G/B}][\tilde{r}_{G/B}]^T$$

- Angular Momentum

$$\{\underline{H}_G\} = I_G\{\underline{\omega}\}$$

$$\{\underline{H}_B\} = \{\underline{H}_G\} + [\tilde{r}_{G/B}] \times \{\underline{P}\}$$

$$= I_B\{\underline{\omega}\}$$

(when rotating purely about B.)

- Kinetic energy

$$T = \frac{1}{2} m \|V_G\|^2 + \frac{1}{2}\{\underline{\omega}\}^T I_G \{\omega\}$$

$$= \frac{1}{2}\{\underline{\omega}\}^T I_B \{\underline{\omega}\}$$

(when rotating purely about B)

- Euler's Equations

$$\{\underline{M}_{G,net}\} = I_G\{\underline{\dot{\omega}}\} + \{\underline{\omega}\} \times I_G\{\underline{\omega}\}$$

$$\{\underline{M}_{B,net}\} = I_B\{\underline{\dot{\omega}}\} + \{\underline{\omega}\} \times I_B\{\underline{\omega}\}$$ (when rotating purely about B)