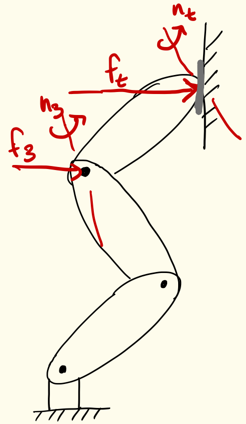
Lecture 19 - Static Force Analysis & Numeric IK

From Last time:



Rigid Transformation of Velocities

$$\begin{bmatrix} {}^t\omega_t \\ {}^t v_t \end{bmatrix} = {}^tX_3 \begin{bmatrix} {}^3\omega_3 \\ {}^3 v_3 \end{bmatrix}$$

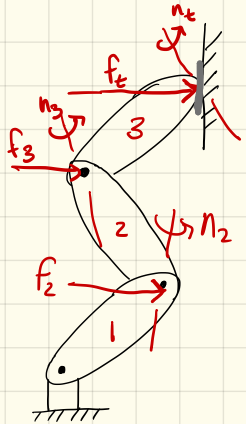
Rigid Transformation of Forces

$$\begin{bmatrix} {}^3n_3 \\ {}^3 f_3 \end{bmatrix} = {}^tX_3^T \begin{bmatrix} {}^tn_t \\ {}^t f_t \end{bmatrix}$$

Today:

- Finish up Static Force Analysis
- Intro to numerical IK

Static Force Propagation:



Force balance for body 3

$$^3f_3 = {}^3R_t\, {}^tf_t$$

Moment Balance For Body 3

$$^3n_3 = {}^3R_t\, {}^tn_t + {}^3p_t \times {}^3R_t\, {}^tf_t$$

$$\begin{bmatrix} ^3n_3 \\ ^3f_3 \end{bmatrix} = \begin{bmatrix} ^3R_t & S(^3p_t)\, ^3R_t \\ 0 & ^3R_t \end{bmatrix} \begin{bmatrix} ^tn_t \\ ^tf_t \end{bmatrix}$$

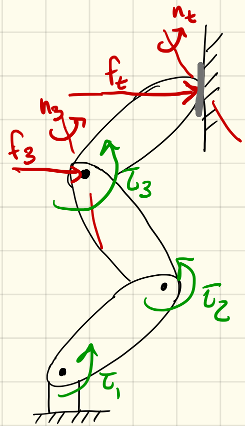
$$= \begin{bmatrix} ^tR_3 & 0 \\ -^tR_3 S(^3p_t) & ^tR_3 \end{bmatrix}^T \begin{bmatrix} ^tn_t \\ ^tf_4 \end{bmatrix} = {}^tX_3^T \begin{bmatrix} ^tn_t \\ ^tf_t \end{bmatrix}$$

$$\begin{bmatrix} ^2n_2 \\ ^2f_2 \end{bmatrix} = {}^3X_2^T \begin{bmatrix} ^3n_3 \\ ^3f_3 \end{bmatrix} = {}^tX_2^T \begin{bmatrix} ^tn_t \\ ^tf_t \end{bmatrix}$$

Static Joint Torque analysis

$$\tau_3 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} {}^3n_3 = \begin{bmatrix} 0 & 0 & 1 & \vdots & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} {}^3n_3 \\ \hline {}^3f_3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 & \vdots & 0 & 0 & 0 \end{bmatrix} {}^tX_3^T \begin{bmatrix} {}^tn_t \\ {}^tf_t \end{bmatrix} = \Bigg(\underbrace{{}^tX_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ \hline 0 \\ 0 \\ 0 \end{bmatrix}}_{\text{3rd column of } {}^tJ_t} \Bigg)^T \begin{bmatrix} {}^tn_t \\ \hline {}^tf_t \end{bmatrix}$$



$$\begin{bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{bmatrix} = \begin{bmatrix} \text{— 1st column of } {}^tJ_t \text{ —} \\ \text{— 2nd column of } {}^tJ_t \text{ —} \\ \text{— 3rd column of } {}^tJ_t \text{ —} \end{bmatrix} \begin{bmatrix} {}^tn_t \\ {}^tf_t \end{bmatrix} = {}^tJ_t^T \begin{bmatrix} {}^tn_t \\ {}^tf_t \end{bmatrix} = {}^0J_t^T \begin{bmatrix} {}^0n_t \\ \hline {}^0f_t \end{bmatrix}$$

Uses For The Jacobian

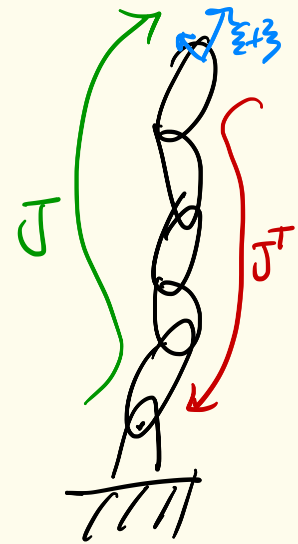
① Task-space (tool) velocity Kinematics

$$\begin{bmatrix} {}^t\omega_t \\ {}^t v_t \end{bmatrix} = {}^tJ_t \dot{\Theta}$$

(Forward problem)

② Static Force analysis

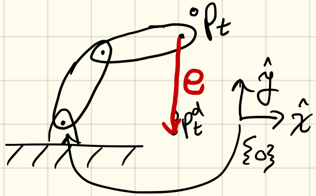
$$\tau = {}^tJ_t^T \begin{bmatrix} {}^t n_t \\ {}^t f_t \end{bmatrix}$$



③ Numerical Inverse Kinematics

J^{-1} plays a role

Numerical IK:



$$^0p_t = \begin{bmatrix} l_1 c_1 + l_2 c_{12} \\ l_1 s_1 + l_2 s_{12} \end{bmatrix} \qquad {}^0J_t^{\,v} = \begin{bmatrix} -l_1 s_1 - l_2 s_{12} & -l_2 s_{12} \\ l_1 c_1 + l_2 c_{12} & l_2 c_{12} \end{bmatrix}$$

$$l_1 = l_2 = 1m \qquad {}^0p_t^d = \begin{bmatrix} 1.5\,m \\ 0\,m \end{bmatrix}$$

- $^0p_t(\theta) = {}^0p_t^d$ Solve for θ
- $0 = {}^0p_t^d - {}^0p_t(\theta)$ Solve for θ

Workings of numerical IK:

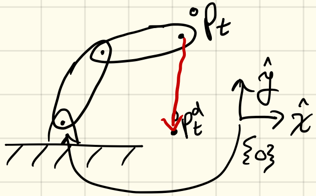
Sequence of $\theta^{(0)}, \ldots, \theta^{(i)}, \ldots$ $\quad e^{(i)} = {}^0p_t^d - {}^0p_t(\theta^{(i)})$

with $\|e^{(0)}\| > \|e^{(1)}\| > \ldots > \|e^{(i)}\|$

Goal: $\|e^{(i)}\| \to 0$ as $i \to \infty$

In practice, stop when $\|e^{(i)}\| <$ tolerance

Newton's Method: Vanilla IK



infinitesimal $\rightarrow$ $d\,{}^0p_t = {}^0J_t^{\mathcal{N}}\, d\theta$

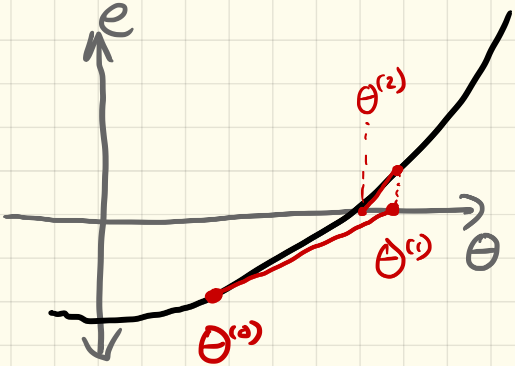
Jacobian provides a linearization of Forward Kin

non infinitesimal $\rightarrow$ $\Delta p \approx {}^0J_t^{\mathcal{N}}\, \Delta\theta$

Newtons Step:

$$e^{(i)} = {}^0J_t^{\mathcal{N}}\, \Delta\theta$$

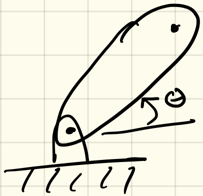
$$\Rightarrow \theta^{(i+1)} = \theta^{(i)} + \left({}^0J_t^{\mathcal{N}}\right)^{-1} e^{(i)}$$



- When $\|e^{(i)}\|$ is small

$$\|e^{(i+1)}\| \approx \|e^{(i)}\|^2$$

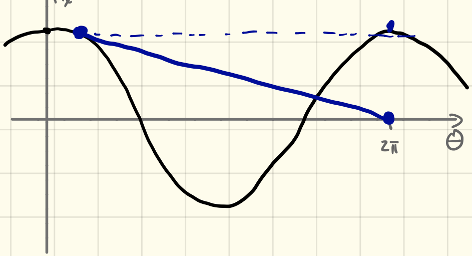
Issues with the Vanilla Newton:



$P_x = \cos\theta = 0$

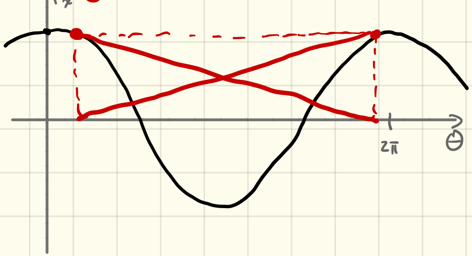
① You can get stuck @ points of zero slope

$\theta^{(0)} = 0.162$



② You can get stuck in cycles

$\theta^{(0)} = .166 = \theta^{(2)}$ $\quad \theta^{(1)} = 2\pi - .166$



- We can avoid both issues by $\|e^{(i)}\|$ decreases with each iteration.

- We can accomplish this by moving in the direction of the Newton Step but take a smaller step size.