Lecture 4 - Combining Position & Orientation

<u>Announcements</u>: • Office Hours 4:30-6:00 Debart 242

• HW1 due Friday @ 8:20AM

<u>Goals</u> <u>For</u> <u>today</u>:

• Finishing Up Orientation
  - Euler Angles & Angle Axis

• Representing position & Orientation
  - Homogeneous Transforms
  - Composition & Properties

4x4 Matrix
Analog of
Rotation matrices

Representing Rotations with Euler Angles:

seq. of rot. about earth fixed axes ①

$$ {}^{A}R_B = R_z(\alpha)\, R_y(\beta)\, R_x(\gamma) $$

seq. abt. body-fixed axes ②

① ⓐ Rotation of γ about $\hat{x}_A$ axis
ⓑ " " β about $\hat{y}_A$ axis
ⓒ " " α about $\hat{z}_A$ axis

(γ, β, α)
X-Y-Z Fixed angles for ${}^{A}R_B$
X → roll, Y → Pitch, Z → Yaw
[Fig. 2.17]

② ⓐ Rotation of α about $\hat{z}_B$
ⓑ " " β about $\hat{y}_{B'}$
ⓒ " " γ about $\hat{x}_{B''}$

(α, β, γ)
Z, Y, X Euler Angles

Note: Other sequences are equally valid (e.g., Z-Y-Z, Z-X-Y)

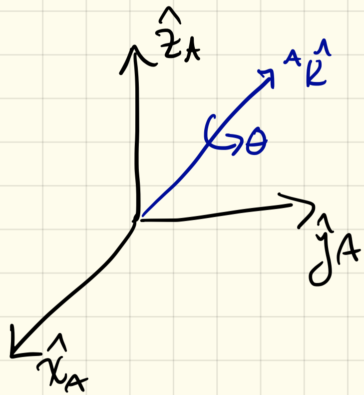
Note: Any rotation can be expressed via 3 angles in a valid convention
(see 2.74 in book, or video example)

A Final Representation of Orientation: Angle Axis

- We have only rotated about coordinate axes so far...
- You can rotate about any axis

"Rotation about ${}^A\hat{k}$ by θ" $= R_{{}^A\hat{k}}(\theta)$ $\boxed{\text{HW 2}}$

- In fact any rotation ${}^A R_B$ can be express in this form for some $({}^A\hat{k}, \theta)$



Combining Rotation & Translation:

$$^{A}p_C = {}^{A}p_B + {}^{A}R_B \, {}^{B}p_C$$

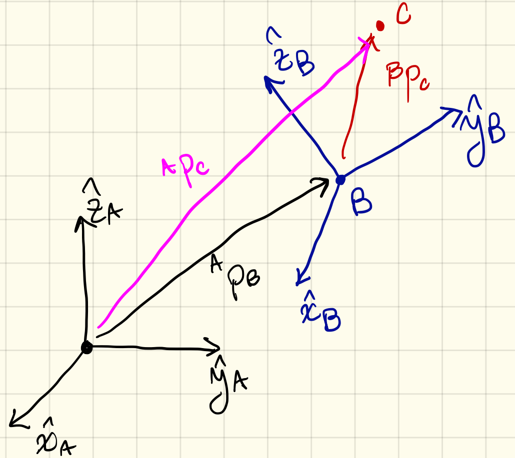
Homogeneous Transformation

$$^{A}T_B = \left[\begin{array}{c|c} ^{A}R_B & ^{A}p_B \\ \hline 0 \;\; 0 \;\; 0 & 1 \end{array}\right]$$

Homogeneous Coordinates

Homogeneous coordinates of C in {A}

$$^{A}T_B \underbrace{\begin{bmatrix} ^{B}p_C \\ \hline 1 \end{bmatrix}}_{\text{Homogeneous coordinates of C in \{B\}}} = \begin{bmatrix} ^{A}R_B \, {}^{B}p_C + {}^{A}p_B \\ \hline 1 \end{bmatrix} = \begin{bmatrix} ^{A}p_C \\ \hline 1 \end{bmatrix}$$



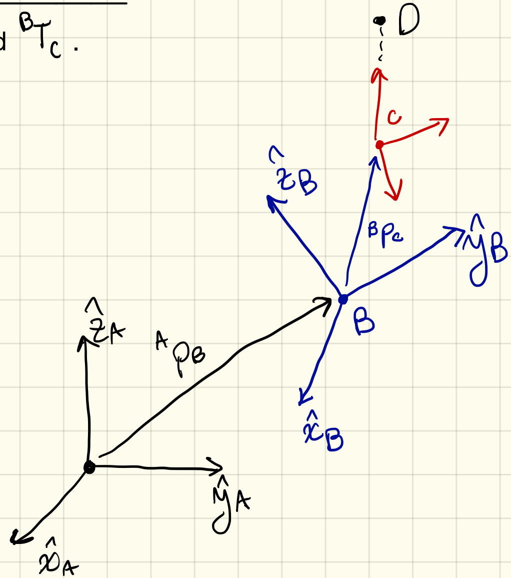
Chaining Homogeneous Transforms:

$$^{A}T_{B} = \begin{bmatrix} ^{A}R_{B} & ^{A}p_{B} \\ 0 & 1 \end{bmatrix} \qquad ^{B}T_{C} = \begin{bmatrix} ^{B}R_{C} & ^{B}p_{C} \\ 0 & 1 \end{bmatrix}$$

$$^{A}T_{\cancel{B}}\,^{\cancel{B}}T_{C} = \begin{bmatrix} ^{A}R_{\cancel{B}}\,^{\cancel{B}}R_{C} & ^{A}R_{B}\,^{B}p_{C} + {}^{A}p_{B} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} ^{A}R_{C} & ^{A}p_{C} \\ 0 & 1 \end{bmatrix} = {}^{A}T_{C}$$

Example: Suppose a point D in frame {C} 3 units along $\hat{z}_C$. You are given $^{A}T_{B}$ and $^{B}T_{C}$.
What are the homogenous coordinates of point D in frame {A}.
Express your answer using the information given.

$$\begin{bmatrix} ^{A}p_{D} \\ \hline 1 \end{bmatrix} = {}^{A}T_{C} \begin{bmatrix} ^{C}p_{D} \\ \hline 1 \end{bmatrix} = {}^{A}T_{B}\,^{B}T_{C} \begin{bmatrix} 0 \\ 0 \\ 3 \\ \hline 1 \end{bmatrix}$$



Properties of Homogeneous Transforms:

$${}^{A}T_{B} = \begin{bmatrix} {}^{A}R_{B} & {}^{A}P_{B} \\ 0 & 1 \end{bmatrix}$$

① $${}^{A}T_{B}\,{}^{B}T_{A} = {}^{A}T_{A} = I \Rightarrow {}^{B}T_{A} = {}^{A}T_{B}^{-1}$$

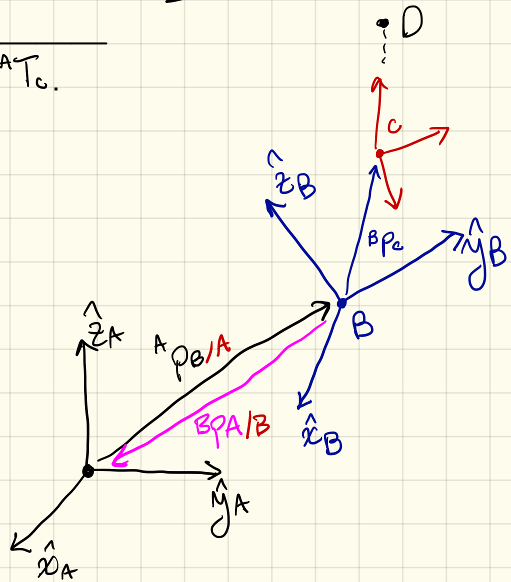
② $${}^{A}T_{B}^{T} \neq {}^{A}T_{B}^{-1}$$

③ $${}^{B}T_{A} = \begin{bmatrix} {}^{B}R_{A} & {}^{B}P_{A/B} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} {}^{B}R_{A} & -{}^{B}R_{A}\,{}^{A}P_{B/A} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} {}^{A}R_{B}^{T} & -{}^{A}R_{B}^{T}\,{}^{A}P_{B/A} \\ 0 & 1 \end{bmatrix} = {}^{A}T_{B}^{-1}$$

Example: Suppose a point D in frame {C} 3 units along $\hat{z}_C$. You are given ${}^{A}T_{B}$ and ${}^{A}T_{C}$.
What are the homogenous coordinates of point D in frame {B}.
Express your answer using the information given.

$${}^{B}T_{C}\begin{bmatrix} 0 \\ 0 \\ 3 \\ \hline 1 \end{bmatrix} = {}^{B}T_{\cancel{A}}\,{}^{\cancel{A}}T_{C}\begin{bmatrix} 0 \\ 0 \\ 3 \\ \hline 1 \end{bmatrix}$$

$$= {}^{A}T_{B}^{-1}\,{}^{A}T_{C}\begin{bmatrix} 0 \\ 0 \\ 3 \\ \hline 1 \end{bmatrix}$$

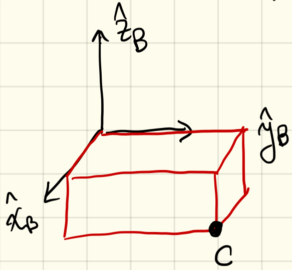


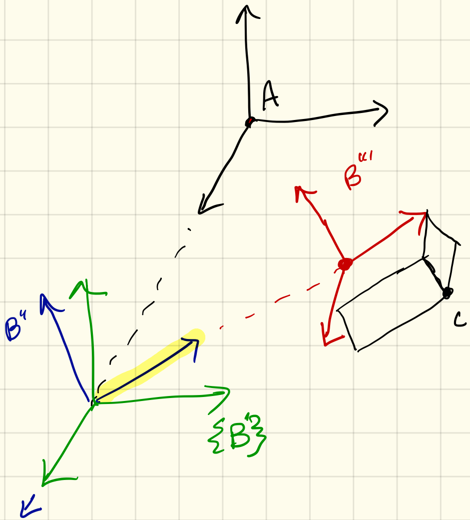
Consider a block with width 4m along $\hat{y}_B$, height 1m along $\hat{z}_B$, depth 2m along $\hat{x}_B$. (Shown below) Suppose the block frame {B} is initially aligned with a fixed frame {A}. Apply the following transformations to {B} in the order given below. Following these transformations, what are the homogeneous coordinates of point C on the block?

1) Translation along $\hat{x}_A$ by 3m

2) Rotation about $\hat{x}_B$ By 30 degrees

3) Translation along $\hat{y}_B$ By 2m





$$^{A}T_{B'} = \begin{bmatrix} I & \begin{matrix} 3 \\ 0 \\ 0 \end{matrix} \\ 0 & 1 \end{bmatrix}$$

$$^{B''}T_{B'''} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$^{B'}T_{B''} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c30^\circ & -s30^\circ & 0 \\ 0 & s30^\circ & c30^\circ & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$^{B''}P_C = \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}$$

$${}^{A}T_{B'} = \begin{bmatrix} I & \begin{matrix} 3 \\ 0 \\ 0 \end{matrix} \\ 0 & 1 \end{bmatrix}$$

$${}^{B'}T_{B''} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c30^\circ & -s30^\circ & 0 \\ 0 & s30^\circ & c30^\circ & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^{B''}T_{B'''} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^{B''}p_c = \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} {}^{A}p_c \\ 1 \end{bmatrix} = {}^{A}T_{B'}\,{}^{B'}T_{B''}\,{}^{B''}T_{B''}\begin{bmatrix} {}^{B''}p_c \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 \\ 3\sqrt{3} - 1/2 \\ 3 + \sqrt{3}/2 \\ 1 \end{bmatrix}$$