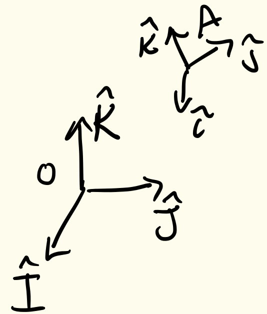
5/

Reference Frames

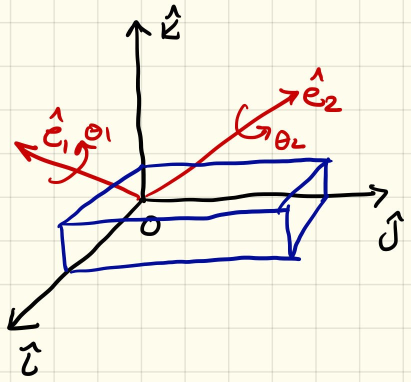
Velocity & Acceleration

- Admin
  - HW 2: Due next wednesday
  - Exam 1: The following wednesday (Newtonian Particle Dynamics in 3D)
- Last time:
  - Rotation matrices: ${}^{O}R_{A} = \left[{}^{O}\{\hat{\imath}\} \;\; {}^{O}\{\hat{\jmath}\} \;\; {}^{O}\{\hat{k}\} \right]$
  - Distinction between a geometric vector $\underline{r}$ and its coordinate representation ${}^{O}\{\underline{r}\}$ or ${}^{A}\{\underline{r}\}$
  - Changing coordinate representation ${}^{O}\{\underline{r}\} = {}^{O}R_{A}\, {}^{A}\{\underline{r}\}$
  - Cross Products: ${}^{O}\{\underline{r}_1 \times \underline{r}_2\} = {}^{O}[\tilde{\underline{r}}_1]\, {}^{O}\{\underline{r}_2\}$ $\quad {}^{O}[\tilde{\underline{r}}_1]^T + {}^{O}[\tilde{\underline{r}}_1] = 0$



- Today:
  - Use these tools to understand velocity and acceleration w/moving frames

Concept Question: Consider two unit vectors $\hat{e}_1$ and $\hat{e}_2$. Consider the following cases.



① The block rotates by angle θ_1 about $\hat{e}_1$ then by angle θ_2 about $\hat{e}_2$

② The block rotates by angle θ_2 about $\hat{e}_2$ then by angle θ_1 about $\hat{e}_1$.

Assume $0 < \theta_1 < 180°$, $0 < \theta_2 < 180°$

True or False: The block end up in the same position for both cases

Ⓐ True (Always)

Ⓑ True if $\hat{e}_1 \perp \hat{e}_2$, False o/w

Ⓒ False (Always)

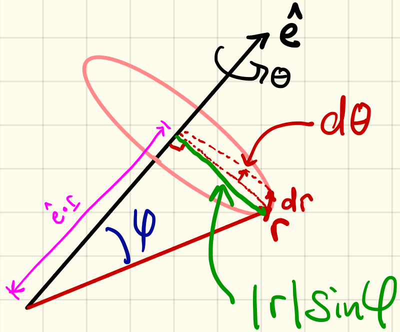
Ⓓ None of the above

True: $\hat{e}_1 \parallel \hat{e}_2$, False o/w

$R_1 R_2 \neq R_2 R_1$ in general

Infinitesimal Rotations:

$$|d\underline{r}| = |r| \sin\varphi \, d\theta$$



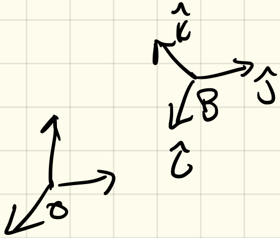
$$|\hat{e} \times \underline{r}| = \cancel{|\hat{e}|}\,|r|\sin\varphi$$

$$\Rightarrow d\underline{r} = \hat{e} \times r \, d\theta$$

$$\Rightarrow \frac{d\underline{r}}{dt} = \underline{v} = \Big(\underbrace{\hat{e}\frac{d\theta}{dt}}\Big) \times r$$

$\underline{\omega}$: angular velocity for rotation about a fixed axis

$$\{\underline{v}\} = [\tilde{\underline{\omega}}]\{r\}$$



Angular Velocities as a rate of change in orientation:

Let $R = {}^{O}R_B$ From Properties:

$$R R^T = I \Rightarrow \dot{R} R^T + R \dot{R}^T = 0$$

Now: $\left[\dot{R} R^T\right]^T = R \dot{R}^T$ so

$$\left[\dot{R} R^T\right] + \left[\dot{R} R^T\right]^T = 0$$

$$\dot{R} R^T = {}^{O}[\tilde{\omega}] \quad \text{for some } \omega$$

$${}^{O}\dot{R}_B = {}^{O}[\tilde{\omega}]\,{}^{O}R_B = {}^{O}[\tilde{\omega}]\left[{}^{O}\{\hat{i}\} \;\; {}^{O}\{\hat{j}\} \;\; {}^{O}\{\hat{k}\}\right]$$
$$= \left[{}^{O}\{\omega \times \hat{i}\} \;\; {}^{O}\{\omega \times \hat{j}\} \;\; {}^{O}\{\omega \times \hat{k}\}\right]$$

Always an instantaneous axis of rotation about which B is spinning and $|\omega|$ the rate of rotation

We define $\underline{\omega}_B := \underline{\omega}$ as the angular velocity of frame B

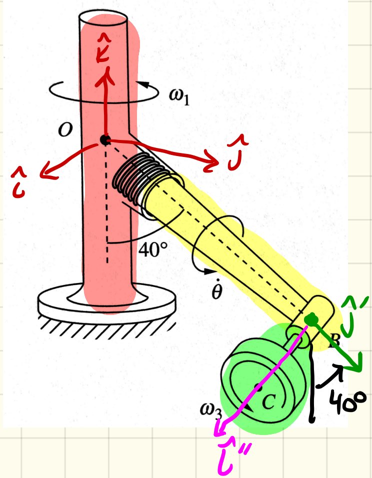
Suppose three frames: O, D, E

we define the relative angular velocity

$$\underline{\omega}_{E/D} = \underline{\omega}_E - \underline{\omega}_D$$

$$\Rightarrow \underline{\omega}_E = \underline{\omega}_D + \underline{\omega}_{E/D}$$

Angular velocities "just add up"



Example: Give the angular velocity of the wheel @ C.

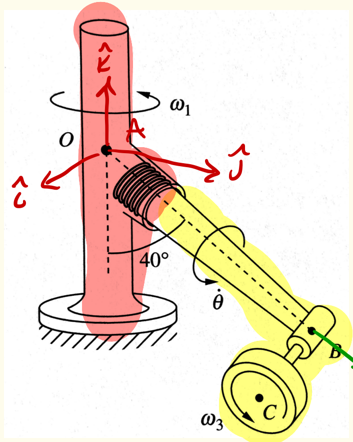
Assume B is in the $\hat{k}, \hat{\jmath}$ and $\overrightarrow{BC}$ in the $\hat{\imath}$ direction.

Give answer w.r.t. $\hat{\imath}, \hat{\jmath}, \hat{k}$

$$\underline{\omega}_C = \omega_1 \hat{k} + \dot{\theta}\hat{\jmath}' + \omega_3 \hat{\imath} \qquad \hat{\jmath}' = -\cos 40° \hat{k} + \sin 40° \hat{\jmath}$$

$$= \omega_3 \hat{\imath} + \dot{\theta}(\sin 40° \hat{\jmath} - \cos 40° \hat{k}) + \omega_1 \hat{k}$$

Angular acceleration: Assume $\omega_3 = 0$. Assume Frame A rotates with red piece:



$$\underline{\omega}_B = \underbrace{\hat{k}\omega_1}_{\underline{\omega}_A} + \underbrace{\hat{\jmath}'\dot{\theta}}_{\underline{\omega}_{B/A}}$$

$$\underline{\omega}_A = \omega_1 \hat{k}$$

$$\dot{\underline{\omega}}_B = \hat{k}\dot{\omega}_1 + \dot{\hat{\jmath}}'\dot{\theta} + \hat{\jmath}'\ddot{\theta}$$

$\dot{\underline{\omega}}_{B/A}$ Total rate of change in $\underline{\omega}_{B/A}$ as observed in an inertial frame

$$= \underbrace{\hat{k}\dot{\omega}_1}_{\dot{\underline{\omega}}_A} + \overbrace{\underbrace{\underline{\omega}_A \times \hat{\jmath}'\dot{\theta}}_{\underline{\omega}_A \times \underline{\omega}_{B/A}} \quad + \quad \underbrace{\hat{\jmath}'\ddot{\theta}}_{\dot{\underline{\omega}}_{B/A,rel}}}$$

$\underline{\omega}_A \times \underline{\omega}_{B/A}$: Accounts for rotation of Frame A

$\dot{\underline{\omega}}_{B/A,rel}$: Accounts for change in $\underline{\omega}_{B/A}$ as observed by someone in frame A.

$$\underline{\omega}_A \times \hat{\jmath}' = -\omega_1 \sin 40^\circ \hat{\imath}$$

(50°)

$$= \hat{k}\dot{\omega}_1 - \omega_1\dot{\theta}\sin 40^\circ \hat{\imath} + \ddot{\theta}\left(-\cos 40^\circ \hat{k} + \sin 40^\circ \hat{\jmath}\right)$$

$$= -\omega_1\dot{\theta}\sin 40^\circ \hat{\imath} + \ddot{\theta}\sin 40^\circ \hat{\jmath} + (\dot{\omega}_1 - \ddot{\theta}\cos 40^\circ)\hat{k}$$

Alternate perspective

$${}^{0}\{\omega_B\} = {}^{0}\{\omega_A\} + {}^{0}R_A\,{}^{A}\{\omega_{B/A}\}$$

$${}^{0}\{\dot{\omega}_B\} = {}^{0}\{\dot{\omega}_A\} + \underbrace{{}^{0}[\tilde{\omega}_A]\,{}^{0}R_A}_{{}^{0}\dot{R}_A}\,{}^{A}\{\omega_{B/A}\} + \underbrace{{}^{0}R_A \frac{d}{dt}\,{}^{A}\{\omega_{B/A}\}}_{{}^{0}\{\dot{\omega}_{B/A,rel}\}}$$

Derivative taken in coordinates of A is equivalent to the change as observed by someone in A.

Baruh calls this strategy the "transport theorem":

Consider any vector $\underline{r}$ and any moving frame $\underline{A}$:

$$\dot{\underline{r}} = \underline{\omega}_A \times \underline{r} + \underbrace{\dot{\underline{r}}_{rel}}$$

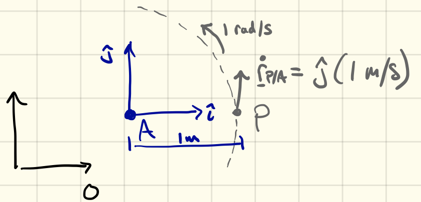
$${}^{A}\{\dot{r}_{rel}\} = \frac{d}{dt}\,{}^{A}\{r\}$$

It works w/ any vector type:

- angular velocity
- position
- linear velocity

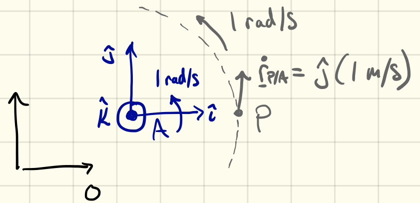
Consider a particle P rotating about A at a radius 1m and @ rate 1 rad/s.

Case 1: Frame A doesn't rotate.



$$\dot{r}_{P/A} = \underbrace{0}_{\omega_A \times r_{P/A}} + \underbrace{(1\ m/s)}_{\dot{r}_{P/A,rel}}\hat{\jmath}$$

Case 2: Frame A rotates @ $(1\ rad/s)\hat{k}$

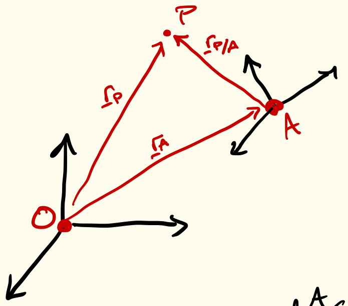


$$\dot{r}_{P/A} = (1\ rad/s\ \hat{k}) \times r_{P/A} + 0$$

$$= \underbrace{(1\ m/s)\hat{\jmath}}_{\omega_A \times r_{P/A}} + \underbrace{0}_{\dot{r}_{P/A,rel}}$$

Velocity and acceleration w/moving frames:

$$\underline{r}_P = \underline{r}_A + \underline{r}_{P/A}$$



$$\underline{v}_P = \underline{v}_A + \dot{\underline{r}}_{P/A}$$

$$= \underline{v}_A + \underbrace{\underline{\omega}_A \times \underline{r}_{P/A} + \underline{v}_{P/A,rel}}_{\dot{r}_{P/A}}$$

$$^A\{\underline{v}_{P/A,rel}\} = \frac{{}^A d}{dt}\{\underline{r}_{P/A}\}$$

$$\underline{a}_P = \underline{a}_A + \ddot{\underline{r}}_{P/A}$$

$$= \underline{a}_A + \dot{\underline{\omega}}_A \times \underline{r}_{P/A} + \underline{\omega}_A \times [\underline{\omega}_A \times \underline{r}_{P/A} + \underline{v}_{P/A,rel}]$$

$$+ \underline{\omega}_A \times \underline{v}_{P/A,rel} + \underline{a}_{P/A,rel}$$

$$^A\{\underline{a}_{P/A,rel}\} = \frac{d^2}{dt^2}\{\underline{r}_{P/A}\}$$

$$= \underline{a}_A + \underbrace{\dot{\underline{\omega}}_A \times \underline{r}_{P/A}}_{\text{tangential}} + \underbrace{\underline{\omega}_A \times \underline{\omega}_A \times \underline{r}_{P/A}}_{\text{centripetal}} + \underbrace{2\,\underline{\omega}_A \times \underline{v}_{P/A,rel}}_{\text{Coriolis}} + \underbrace{\underline{a}_{P/A,rel}}_{\text{relative acceleration}}$$

Summary:

- Angular velocities capture the rate of change in orientation
  - ${}^{0}\dot{R}_B = {}^{0}[\tilde{\underline{\omega}}_B]\, {}^{0}R_B = \left[{}^{0}\{\underline{\omega}_B \times \hat{i}\} \quad {}^{0}\{\underline{\omega}_B \times \hat{j}\} \quad {}^{0}\{\underline{\omega}_B \times \hat{k}\} \right]$
  - Direction of $\underline{\omega}_B$: Instantaneous axis of rotation
  - Magnitude of $\underline{\omega}_B$: Rate of rotation about that axis
- Relative Angular velocity / aceleration

$$\underline{\omega}_B = \underline{\omega}_A + \underline{\omega}_{B/A}$$

$${}^{A}\{\dot{\omega}_{B/A,rel}\} = \frac{d}{dt}\, {}^{A}\{\underline{\omega}_{B/A}\}$$

Represents Derivative of coordinates of $\omega_{B/A}$

$$\dot{\underline{\omega}}_B = \dot{\underline{\omega}}_A + \dot{\underline{\omega}}_{B/A}$$

$$= \dot{\underline{\omega}}_A + \underbrace{\underline{\omega}_A \times \underline{\omega}_{B/A}}_{\text{represents fact that coordinates are moving}} + \underline{\dot{\omega}_{B/A,rel}}$$

Summary

- Relative linear velocity / acceleration

$$\underline{r}_B = \underline{r}_A + \underline{r}_{B/A}$$

$${}^A\{v_{B/A,rel}\} = \frac{d}{dt}\,{}^A\{r_{B/A}\}$$

$$\underline{v}_B = \underline{v}_A + \underbrace{\underline{\omega}_A \times \underline{r}_{B/A} + \underline{v}_{B/A,rel}}_{\dot{\underline{r}}_{B/A}}$$

$$\underline{a}_B = \underline{a}_A + \underbrace{\dot{\underline{\omega}}_A \times \underline{r}_{B/A}}_{\text{tangential}} + \underbrace{\underline{\omega}_A \times \underline{\omega}_A \times \underline{r}_{B/A}}_{\text{centripetal}} + \underbrace{2\,\underline{\omega}_A \times \underline{v}_{B/A,rel}}_{\text{Coriolis}} + \underbrace{\underline{a}_{B/A,rel}}_{\frac{d^2}{dt^2}\,{}^A\{r_{B/A}\}}$$