Analytical Dynamics

The Beginning of the End (i.e., Rigid Bodies in 3D)

1st Order of Business:

- Hamiltonian Wrap Up &
- Aside on Emmy Noether

Noether

"Where there is Symmety one finds laws of nature"

2nd Order of Business:

- Toward Dynamics of 3D Rigid Bodies
- Mass Distribution in 3D
- Rotational Inertia Matrix

Euler
1765

Roadmap to End:

- 3D Rigid Body Dynamics
- Wheels, skates, cats, & astronauts (i.e., nonholonomic constraints)
- D'Alembert's Principle for 3D Bodies & EOM w/ Nonholonomic constraints (Gauss & Jourdain's Principle)

Hamiltonian Dynamics:

$$\mathcal{H}(t, q, \underline{\pi}) = \underline{\pi}^T \dot{q} - \mathcal{L}(t, q, \dot{q})$$

Plug in: $\dot{q} = M^{-1}(\underline{\pi} - \underline{B})$

Properties

① $\mathcal{H} = T_2 + U \qquad U = V - T_0$

$= T + V$ for natural systems (i.e., total energy)

$= \frac{1}{2} \underline{\pi}^T M(q)^{-1} \underline{\pi} + V$

② $\frac{d}{dt}\mathcal{H} = -\frac{\partial \mathcal{L}}{\partial t} + \dot{q}^T \underline{Q}_{nc}$

$= \dot{q}^T \underline{Q}_{nc}$ For natural systems

$= 0$ (for natural systems w/ only cons forces

Applications

- Human-Robot interaction
- Tool to discover integrals of motion

$$\mathcal{H}_1 \overset{F, v}{\rightleftarrows} \mathcal{H}_2$$

Port-Hamiltonian systems

Derivation: Properties of the Hamiltonian

EXTRA : NOT IN CLASS

Recall: $T = \overbrace{\tfrac{1}{2}\dot{\underline{q}}^T M \dot{\underline{q}}}^{T_2} + \underline{\beta}^T \dot{\underline{q}} + T_0$

$$\underline{\pi} = \nabla_{\dot{\underline{q}}} \mathcal{L} = \nabla_{\dot{\underline{q}}} T = M\dot{\underline{q}} + \underline{\beta} \Rightarrow \dot{\underline{q}} = M^{-1}(\underline{\pi} - \underline{\beta})$$

① $$\mathcal{H} = \underline{\pi}^T \dot{\underline{q}} - \mathcal{L} = \pi^T M^{-1}(\pi - \underline{\beta}) - \Big(\overbrace{\tfrac{1}{2}(\underline{\pi}-\underline{\beta})^T M^{-1}(\underline{\pi}-\underline{\beta})}^{T_2} + \underline{\beta}^T M^{-1}(\underline{\pi}-\underline{\beta}) + T_0 - V \Big)$$

$$= (\underline{\pi}-\underline{\beta})^T M^{-1}(\pi-\underline{\beta}) - \tfrac{1}{2}(\underline{\pi}-\underline{\beta})^T M^{-1}(\underline{\pi}-\underline{\beta}) + \underbrace{V - T_0}_{U}$$

$$= \tfrac{1}{2}(\underline{\pi}-\underline{\beta})^T M^{-1}(\underline{\pi}-\underline{\beta}) + U$$

$$= T_2 + U$$

② $$\frac{d}{dt}\mathcal{H}(t,\underline{q},\pi) = \frac{\partial \mathcal{H}}{\partial t} + \sum_{k}^{n}\left[\frac{\partial \mathcal{H}}{\partial q_k}\dot{q}_k + \frac{\partial \mathcal{H}}{\partial \pi_k}\dot{\pi}_k\right]$$

$$\frac{\partial \mathcal{H}}{\partial t} = -\frac{\partial \mathcal{L}}{\partial t} \quad \text{(see last lecture derivation)}$$

$$= -\frac{\partial \mathcal{L}}{\partial t} + \sum_{k=1}^{n}\Big[\underbrace{\frac{\partial \mathcal{H}}{\partial q_k}\frac{\partial \mathcal{H}}{\partial \pi_k}}_{} + \underbrace{\frac{\partial \mathcal{H}}{\partial \pi_k}\Big(-\frac{\partial \mathcal{H}}{\partial q_k}}_{} + Q_{k,nc}\Big)\Big]$$

Terms cancel

$$= -\frac{\partial \mathcal{L}}{\partial t} + \sum_{k=1}^{n}\frac{\partial \mathcal{H}}{\partial \pi_k}Q_{k,nc} = -\frac{\partial \mathcal{L}}{\partial t} + \sum_{k=1}^{n}\dot{q}_k Q_{k,nc}$$

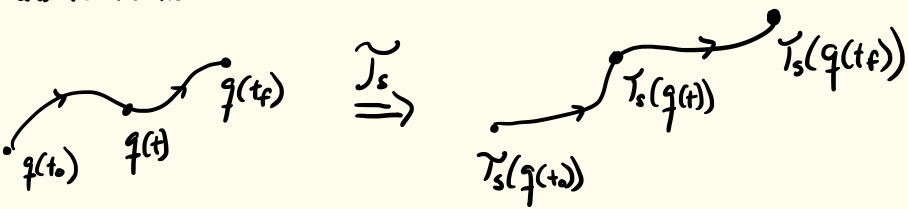
Aside: Emmy Noether

- German Mathemetician

- Did much work unpaid since wemen weren't allowed on the faculty in ~1910s Germany

- **Recall:** If a variable q_i doesn't appear in the Lagrangian (and $Q_{i,nc}=0$) $\Rightarrow$ Π_i is conserved

- **Rough Defn.:** We say that a family of transformations $\mathcal{T}_s(q) \to q'$ $s \in \mathbb{R}$ is a contunous symmetry of the action if the action is unchanged under transformations



Gives Same

$$\int_{t_0}^{t_f} \mathcal{L}\,dt$$

- **Noether's Theorem:** For every continuous symmetry of a system there exists an associated conserved quantity.

Aside: Emmy Noether

- Example: Does your system behave the same before and after translations in space?

  $\Rightarrow$ It's total linear momentum is conserved in those directions

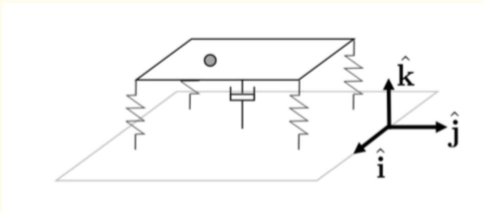
- Example: Is $\int \mathcal{L}$ symmetric under some rotation?

  $\Rightarrow$ total angular momentum is conserved about that axis.

- The laws of nature have as much to do with the geometry of the space upon which they play out as with the objects occupying it.

- Broad Impact: Relativistic Contexts, Quantum, etc.

Symmetry $\longrightarrow$ $\boxed{\text{Noether's Theorem}}$ $\longrightarrow$ Law of Nature



21

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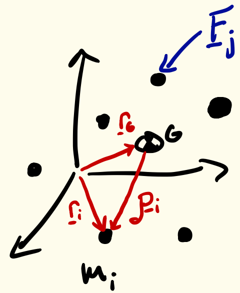
Key Results From Systems of Particles

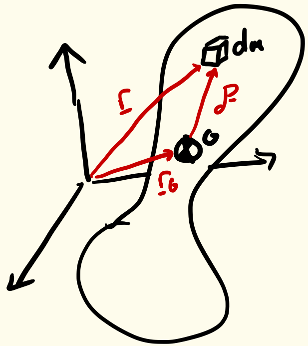
$$m = \sum_i m_i \quad \text{OR} \quad m = \int dm$$

$$\underline{r}_G = \frac{1}{m}\sum_i m_i \underline{r}_i \quad \text{OR} \quad \underline{r}_G = \frac{1}{m}\int \underline{r}\, dm$$



Linear Momentum

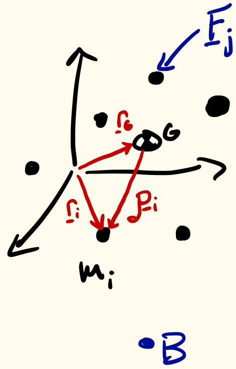
$$\underline{p} = m\underline{v}_G \qquad \dot{\underline{p}} = \sum_i \underline{F}_i = \underline{F}_{net}$$



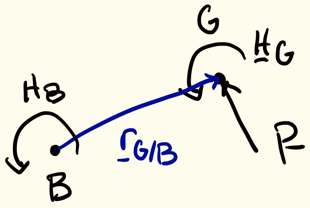
Works in 3D just as well as 2D

Angular Momentum

$$\underline{r}_{i/B} = \underline{r}_i - \underline{r}_B$$



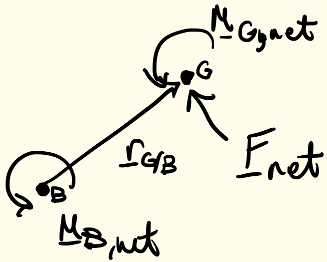
① $\underline{H}_B = \sum_i \underline{r}_{i/B} \times m_i \underline{V}_i = \underline{r}_{G/B} \times \overbrace{m \underline{V}_G}^{\underline{P}} + \overbrace{\sum_i \underline{\rho}_i \times m_i \dot{\underline{\rho}}_i}^{\underline{H}_G}$



$$\underline{H}_B = \underline{r}_{G/B} \times \underline{P} + \underline{H}_G$$

② $\dot{\underline{H}}_B = \sum \underline{M}_{B,net}$ when ① B is fixed in space ② B=G is moving w/ COM

$$\dot{\underline{H}}_G = \sum \underline{M}_G = \underline{M}_{G,net}$$



$$\dot{\underline{H}}_B = \underline{M}_{B,net} = \underline{r}_{G/B} \times \underline{F}_{net} + \underline{M}_{G,net}$$

$$= \underline{r}_{G/B} \times m \underline{a}_G + \dot{\underline{H}}_G$$

Works in 2D & 3D

Angular Momentum:

$$\{\underline{a} \times \underline{b}\} = [\tilde{a}]\{b\}$$

$$\underline{H}_G = \int \underline{\rho} \times dm\, \dot{\underline{\rho}} \qquad \dot{\underline{\rho}} = \underline{\omega} \times \underline{\rho}$$

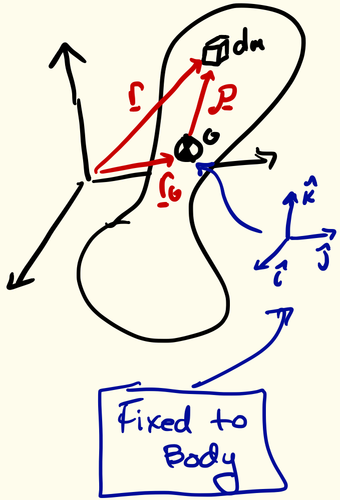
$$= I_G \dot{\theta} \hat{k} \quad \text{for 2D}$$

$\times$ not in 3D

$$= \int -\underline{\rho} \times (\underline{\rho} \times \underline{\omega})\, dm$$

$$\{\underline{H}_G\} = \underbrace{\int [\tilde{\underline{\rho}}][\tilde{\underline{\rho}}]^T dm}\,\{\underline{\omega}\}$$

I_G: Rotational inertia matrix (3x3)



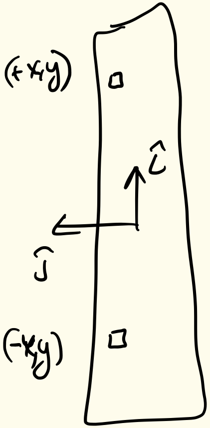
$$\{\rho\} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

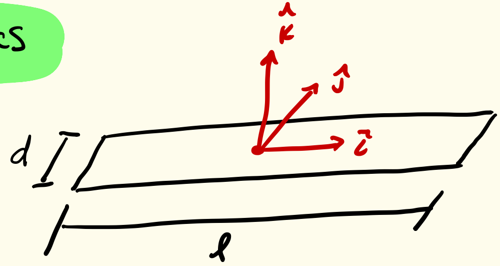
$$\underline{I}_G = \int \begin{bmatrix} y^2+z^2 & -xy & -xz \\ -xy & x^2+z^2 & -yz \\ -xz & -yz & x^2+y^2 \end{bmatrix} dm = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix}$$

Example: Infinitely Thin Sheet mass m

what is I_G?

pollev.com/dynamics





(d) $\frac{1}{12} m \begin{bmatrix} d^2 & 0 & 0 \\ 0 & \ell^2 & 0 \\ 0 & 0 & d^2+\ell^2 \end{bmatrix}$

(e) $\frac{1}{12} m \begin{bmatrix} d^2 & -\ell d & 0 \\ -\ell d & \ell^2 & 0 \\ 0 & 0 & \ell^2+d^2 \end{bmatrix}$

Example: Infinitely Thin Sheet mass m

what is I_G?

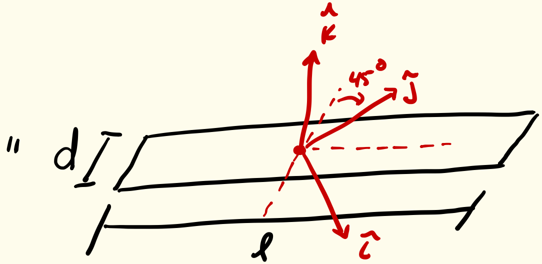
$\alpha =$ "some nonzero value"

$\beta =$ "

$\gamma =$ "

$\eta =$ "

$\varepsilon =$ "



(a) $\cancel{\frac{1}{12} m \begin{bmatrix} \alpha & \beta & 0 \\ -\beta & \alpha & 0 \\ 0 & 0 & \eta \end{bmatrix}}$

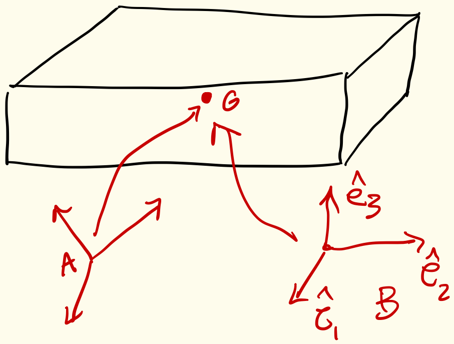
(b) $\frac{1}{12} m \begin{bmatrix} \alpha & \gamma & 0 \\ \gamma & \beta & 0 \\ 0 & 0 & \eta \end{bmatrix}$

(c) $\boxed{\frac{1}{12} m \begin{bmatrix} \alpha & \beta & 0 \\ \beta & \alpha & 0 \\ 0 & 0 & \eta \end{bmatrix}}$

(d) $\cancel{\frac{1}{12} m \begin{bmatrix} \alpha & 0 & \gamma \\ 0 & \alpha & \varepsilon \\ \gamma & \varepsilon & \eta \end{bmatrix}}$

(e) Brain Meltdown

Principal Axes:



- Consider a body w/ rotational inertia matrix ${}^A I_G$

- There is always a frame B s.t.

$$ {}^B I_G = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} $$

λ_1, λ_2, λ_3 are eigenvalues of ${}^A I_G$ (principal moments of Inertia)

- If ${}^A\{\hat{e}_1\}$, ${}^A\{\hat{e}_2\}$, ${}^A\{\hat{e}_3\}$ eig. vecs of ${}^A I_G$ (w.l.o.g. $\hat{e}_3 = \hat{e}_1 \times \hat{e}_2$)

without loss of generality

$$ {}^A R_B = \begin{bmatrix} {}^A\{\hat{e}_1\} & {}^A\{\hat{e}_2\} & {}^A\{\hat{e}_3\} \end{bmatrix} $$

- $\hat{e}_1$, $\hat{e}_2$, $\hat{e}_3$ are what we call the principal axes of the body

$$ {}^B I_G = {}^A R_B^T \, {}^A I_G \, {}^A R_B $$

Summary

- Rotational Inertia Matrix

$$I_G = \int \begin{bmatrix} y^2+z^2 & -xy & -xz \\ -xy & x^2+z^2 & -yz \\ -xz & -yz & x^2+y^2 \end{bmatrix} dm$$

- Relationship w/ Angular Momentum

$$\{\underline{H}_G\} = I_G \{\underline{\omega}\}$$

$$\{\underline{H}_B\} = \{\underline{H}_G\} + [\tilde{\underline{r}}_{G/B}] \times \underline{P}$$