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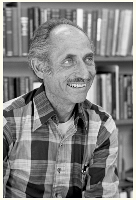
Analytical Dynamics

Jourdain's Principle & Kane's Method

Jourdain 1909



Kane 1961

Admin:
- HW 9 optional, Due Weds (Graded on Completion, 2nd lowest HW Dropped)
- Final: Tues. Dec. 17th 4:15-6:15PM (140 Debartolo)

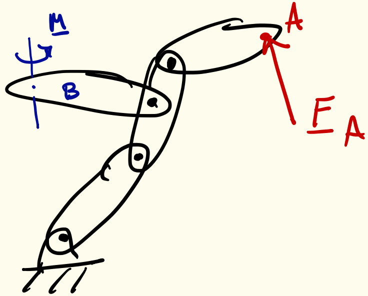
Previously:
- D'Alembert's Principle for 3D Bodies
- Holonomic vs. NonHolonomic constraints

Constraints on mechanism configuration

Constraints on Mechanism Velocities

Today:
- Jourdain's Principle & Virtual Power ($\delta P = \underline{F} \cdot \delta \underline{v}$ rather than $\delta W = \underline{F} \cdot \delta \underline{r}$)
- Use generalized speeds to derive equations of motion w/ Nonholonomic Constraints
- Present Kane's Method for deriving EOM

Generalized Forces REVIEW



- If a force acts at point A on a body it contributes Generalized forces

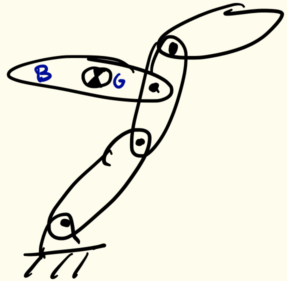
$$Q_K = \frac{\partial \underline{v}_A}{\partial \dot{q}_K} \cdot \underline{F}_A \qquad \underline{Q} = \left\{ \frac{\partial \underline{v}_A}{\partial \dot{q}} \right\}^T \{\underline{F}\}$$

- If a moment acts on body B it contributes GF

$$Q_K = \frac{\partial \underline{\omega}_B}{\partial \dot{q}_K} \cdot \underline{M} \qquad \underline{Q} = \left\{ \frac{\partial \underline{\omega}_B}{\partial \dot{q}} \right\}^T \{\underline{M}\}$$

- The total generalized force is the sum of the contributions from all applied forces.

Generalized Inertial Force REVIEW



- The generalized inertial force from Body B is given by

$$\underline{Q}_{k,\text{inertial}} = \frac{\partial \underline{v}_G}{\partial \dot{q}_k} \cdot (m\underline{a}_G) + \frac{\partial \underline{\omega}_B}{\partial \dot{q}_k} \cdot \dot{\underline{H}}_G$$

$$\underline{Q}_{\text{inertial}} = \left\{\frac{\partial \underline{v}_G}{\partial \dot{q}}\right\}^T \{m\underline{a}_G\} + \left\{\frac{\partial \underline{\omega}_B}{\partial \dot{q}}\right\}^T \{\dot{\underline{H}}_G\}$$

- The total generalized inertial force is the sum over all bodies

- For a system w/ all constraints accounted for in the generalized coordinates

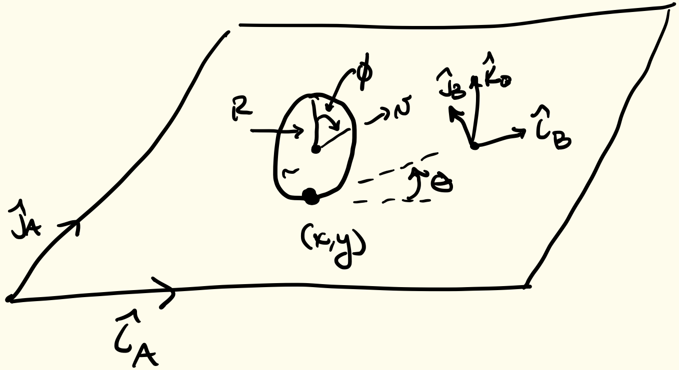
$$\underline{Q} = \underline{Q}_{\text{inertial}}$$

(Key Step: Constraint forces Do no virtual work under virtual displacements to q!)

Generalized Speeds for the Rolling Disk

- Generalized coordinates

$$q = [x, y, \theta, \phi]^T$$



- For the wheel: 2 constraints: x and y velocity of point on wheel that is in contact is zero.

- Generalized speeds: $\underline{U} = [\dot{\theta}, v]^T$

  All constraints accounted for w/ choice of generalized speeds

Generalized Speeds: A set of parameters that uniquely defines the velocity of all particles of a system

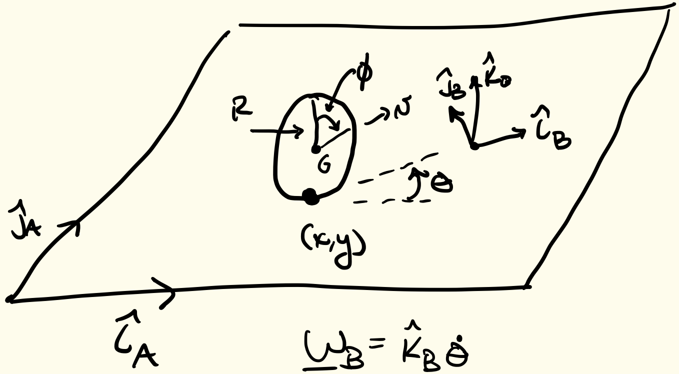
- If we have n generalized coordinates and m independent constraints, then we choose n-m generalized speeds

- Works with holonomic and nonholonomic constraints

Generalized Speeds for the Rolling Disk

Example: Give $\underline{\omega}_{DISK}$ and $\underline{v}_G$ using $\underline{u} = [\dot{\theta}, v]$

Also give $\underline{\dot{\omega}}_{DISK}$ and $\underline{a}_G$



$$\underline{\omega}_B = \hat{k}_B \dot{\theta}$$

$$\dot{\hat{j}}_B = \underline{\omega}_B \times \hat{j}_B = -\dot{\theta}\,\hat{i}_B$$

$$\dot{\hat{i}}_B = \dot{\theta}\,\hat{j}_B$$

Angular:

$$\underline{\omega}_{DISK} = \dot{\theta}\,\hat{k}_B + \dot{\phi}\,\hat{j}_B = \dot{\theta}\,\hat{k}_B + \frac{v}{R}\,\hat{j}_B$$

Linear velocity shows up in angular velocity term.
This is not an issue as:

$\frac{v}{R}$ has units $\frac{m/s}{m} = \frac{1}{s}$

$$\underline{\dot{\omega}}_{DISK} = \ddot{\theta}\,\hat{k}_B + \frac{\dot{v}}{R}\,\hat{j}_B + \frac{v}{R}\,\dot{\hat{j}}_B$$

$$= \ddot{\theta}\,\hat{k}_B + \frac{\dot{v}}{R}\,\hat{j}_B - \frac{v\dot{\theta}}{R}\,\hat{i}_B$$

Linear:

$$\underline{V}_G = v\,\hat{i}_B$$

$$\underline{a}_G = \dot{v}\,\hat{i}_B + v\,\dot{\hat{i}}_B = \dot{v}\,\hat{i}_B + v\dot{\theta}\,\hat{j}_B$$

Jourdain's Principle: (the velocity form of D'Alembert's principle)

Jourdain 1909

- Consider a constrained system of bodies with generalized speeds $\underline{u}$ that account for all constraints
- Consider a small variation $\delta\underline{u}$ to $\underline{u}$
- Jourdain's Principle: The total change in power δP of all constraint forces and moments is zero

(Also known as the principal of virtual power, w/ δP the virtual power)

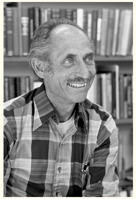
- Implication ⤳ We can use generalized speeds $\underline{u}$ instead of working with $\dot{\underline{q}}$ when computing the generalized active force and generalized Inertial force

For instance:

$$\underset{\text{OLD}}{\underline{Q} = \left\{\frac{\partial \underline{v}_A}{\partial \dot{\underline{q}}}\right\}^T \left\{\underline{F}_A\right\}} \Rightarrow \underset{\text{NEW}}{\underline{Q} = \left\{\frac{\partial \underline{v}_A}{\partial \underline{u}}\right\}^T \left\{\underline{F}_A\right\}}$$

Aside from taking different partials our approch will be the same as before!

Kane's Method For Equations of motion



Kane 1961

① Choose Generalized Speeds $\underline{u}$

② Write out $\underline{V}_G$, $\underline{a}_G$, $\underline{\omega}$ for Each Body (ignore $\underline{V}_G$, $\underline{a}_G$ if Body rotating purely about a point)

③ Write out $\underline{H}_G$, $\dot{\underline{H}}_G$ for each body (or $\underline{H}_B$, $\dot{\underline{H}}_B$ if purely rotating about a point B)

④ Calculate Generalized applied Force contributions

$$Q_k = \frac{\partial \underline{V}_i}{\partial u_k} \cdot \underline{F}_i \qquad \text{AND} \qquad Q_k = \frac{\partial \underline{\omega}_i}{\partial u_k} \cdot \underline{M}_i$$

From each force $\underline{F}_i$ — From each Moment $\underline{M}_i$

(sum up over all forces / Moments)

[Note: No trick to compute $\underline{Q}$ from conservative forces]

⑤ Calculate Generalized Inertial Forces

$$Q_{k,\,inertial} = \sum_i \underbrace{\frac{\partial \underline{V}_{G_i}}{\partial u_k} \cdot (m_i \underline{a}_{G_i}) + \frac{\partial \underline{\omega}_i}{\partial u_k} \cdot \dot{\underline{H}}_{G_i}}$$

OR $\frac{\partial \underline{\omega}_i}{\partial u_k} \cdot \dot{\underline{H}}_{B_i}$ if purely rotating about B_i

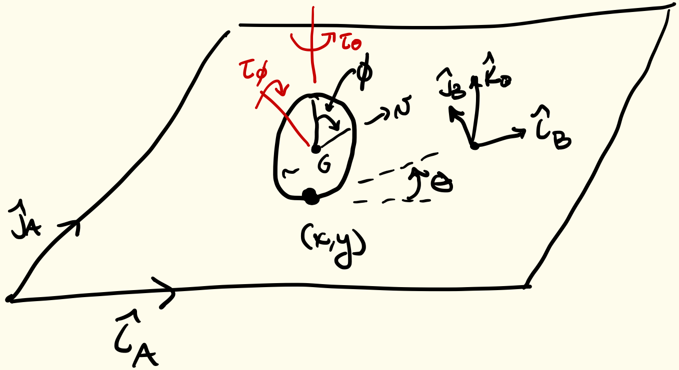
⑥ EOM: $\underline{Q} = \underline{Q}_{inertial}$

EOM for the Rolling Disk (MATLAB online)

① $u = [\dot{\theta}, v]$

② $\underline{v}_G = v\,\hat{i}_B \qquad \underline{a}_G = \dot{v}\,\hat{i}_B + v\dot{\theta}\,\hat{j}_B$

$\underline{\omega}_{DISK} = \dot{\theta}\hat{k}_B + \frac{v}{R}\hat{j}_B$



③ $\underline{H}_G = \frac{1}{2} m R^2 \underbrace{\dot{\phi}}_{= v/R} \hat{j}_B + \frac{1}{4} m R^2 \dot{\theta} \hat{k}_B$

$\dot{\underline{H}}_G = \frac{1}{2} m R \dot{v} \hat{j}_B + \frac{1}{4} m R^2 \ddot{\theta} \hat{k}_B - \frac{1}{2} m R v \dot{\theta} \hat{i}_B$

④ Generalized Force

$Q_{\dot{\theta}} = \underbrace{\frac{\partial \underline{\omega}_{DISK}}{\partial \dot{\theta}}}_{= \hat{k}_B} \cdot \left[\tau_\theta \hat{k}_B + \tau_\phi \hat{j}_B \right] = \tau_\theta \qquad Q_v = \underbrace{\frac{\partial \underline{\omega}_{DISK}}{\partial v}}_{= \hat{j}_B / R} \cdot \left[\tau_\theta \hat{k}_B + \tau_\phi \hat{j}_B \right] = \tau_\phi / R$

⑤ Generalized Inertial Force

$Q_{\dot{\theta}, inertial} = \frac{\partial \underline{\omega}_{DISK}}{\partial \dot{\theta}} \cdot \dot{\underline{H}}_G + \cancelto{0}{\frac{\partial \underline{v}_G}{\partial \dot{\theta}}} \cdot m \underline{a}_G = \frac{1}{4} m R^2 \ddot{\theta}$

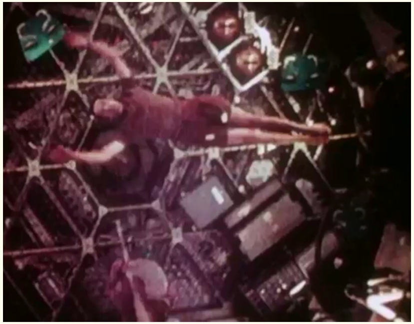
$Q_{v, inertial} = \frac{\partial \underline{\omega}_{DISK}}{\partial v} \cdot \dot{\underline{H}}_G + \underbrace{\frac{\partial \underline{v}_G}{\partial v}}_{= \hat{i}_B} \cdot m \underline{a}_G = \frac{1}{2} m \dot{v} + m\dot{v} = \frac{3}{2} m \dot{v}$

⑥ $$\boxed{\begin{aligned} \tfrac{1}{4} m R^2 \ddot{\theta} &= \tau_\theta \\ \tfrac{3}{2} m \dot{v} &= \tau_\phi / R \end{aligned}}$$

Other Exploits of Thomas Kane

- Conservation of angular momentum is a nonholonomic constraint
- Cats (& Astronauts) can flip over while having zero angular momentum

https://m.youtube.com/watch?v=5TgtW0wDg9E



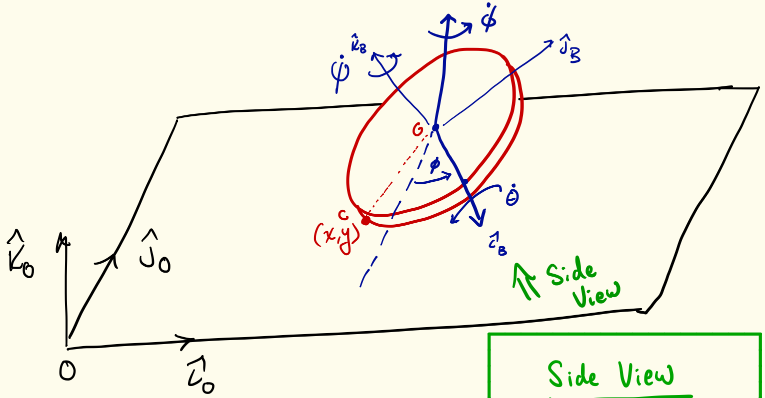
Dropped upside down, the cat shown on this page lands neatly on its feet. The spaceman (in this case a trampolinist in a spacesuit) carefully mimics the cat's motions, a maneuver that may come in handy when he finds himself wafting weightless Out There. The cat's knack is instinctive; to copy it the less gifted spaceman must rely on a series of mathematical equations developed by a professor of applied mechanics.

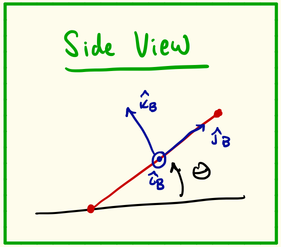
SPACE / A Copycat Astronaut

Example: Thin Coin (MATLAB online)

- $\hat{\imath}_B$ is in the $\hat{\imath}_0, \hat{\jmath}_0$ plane

  ϕ away from $-\hat{\jmath}_0$ about $\hat{k}_0$





Angular Velocity of Disk:

$$\underline{\omega}_{Disk} = \omega_x \hat{\imath}_B + \omega_y \hat{\jmath}_B + \omega_z \hat{k}_B$$

$$= \dot{\theta}\hat{\imath}_B + \dot{\psi}\hat{k}_B + \dot{\phi}(\hat{k}_B C_\theta + \hat{\jmath}_B S_\theta)$$

$$= \dot{\theta}\hat{\imath}_B + \dot{\phi} S_\theta \hat{\jmath}_B + (\dot{\psi} + \dot{\phi} C_\theta)\hat{k}_B$$

$${}^B\underline{\underline{I}}_G = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_1 & 0 \\ 0 & 0 & I_3 \end{bmatrix}$$

Generalized Speeds

Use: $\underline{u} = [\omega_x, \omega_y, \omega_z]^T \qquad \omega_y / S_\theta = \dot{\phi}$

$$\underline{\omega}_B = \underline{\omega}_{Disk}\Big|_{\dot{\psi}=0} = \omega_x \hat{\imath}_B + \omega_y \hat{\jmath}_B + \frac{\omega_y}{\tan\theta}$$

TO MATLAB!

Note: Much harder than what you can expect for the final. But good practice w/ rotations.

Summary: When working with systems w/ non holonomic constraints

1. Choose Generalized Speeds $\underline{u}$

2. Carry out kinematics analysis, describing velocities of all bodies using $\underline{u}$. Calculate $\underline{a}_{G_i}$ for each Body.

3. Calculate $\underline{\dot{H}}_{G_i}$ for each Body

4/5. Generalized Forces:

Type \ From	Forces	Moments	
Applied	$Q_k = \frac{\partial \underline{v}_i}{\partial u_k} \cdot \underline{F}_i$	$Q_k = \frac{\partial \underline{\omega}_i}{\partial u_k} \cdot \underline{M}_i$	(Sum over all Forces/Moments)
Inertial	$Q_{k,inertial} = \frac{\partial \underline{v}_{G_i}}{\partial u_k} \cdot m \underline{a}_{G_i}$	$Q_{k,inertial} = \frac{\partial \underline{\omega}_i}{\partial u_k} \cdot \underline{\dot{H}}_{G_i}$	(Sum over all bodies)

6. EOM: $\underline{Q} = \underline{Q}_{inertial}$

- The high-level idea is the same as D'Alembert's Principle: Project Newton & Euler equations onto directions the system can move $\Rightarrow$ Constraint forces disappear.