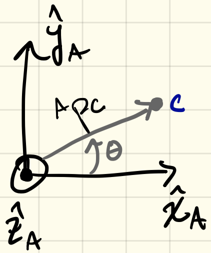
# Applications and Common Tricks for Reasoning Spatially

# Tricks for planar rotations:




$\|{}^A p_c\| = 1$

$${}^A p_c = \begin{bmatrix} \cos\theta \\ \hline \sin\theta \\ \hline 0 \end{bmatrix}$$

when $\theta$ small $^A p_c$ mostly along $+\hat{x}_A$

when $\theta$ small $^A p_c$ slightly along $+\hat{y}_A$

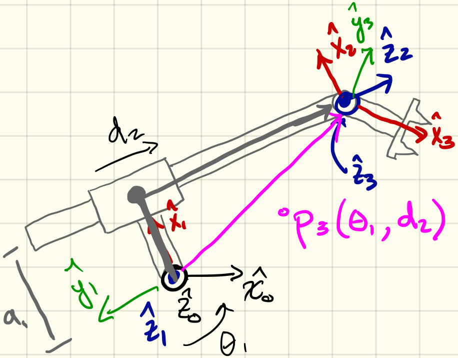
$${}^A p_c = \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix}$$

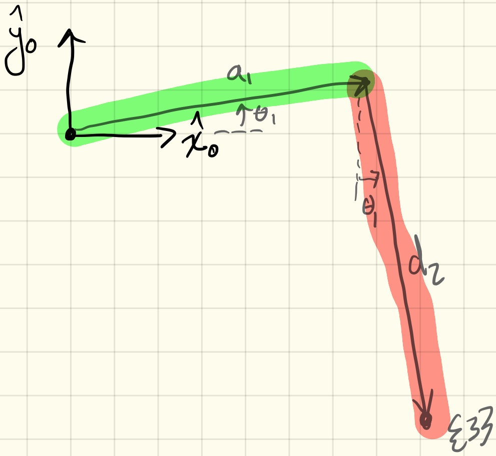
when $\theta = 0$ $^A p_c$ along $+\hat{x}_A$

when $\theta = 90°$ $^A p_c$ along $+\hat{y}_A$

- We could view c as rotating with $\theta$ in a c.c.w. direction ①
- or we could view $\{A\}$ rotating with $\theta$ in a c.w. direction ②

## Example: Stationary Reference Frame, rotating vector (in expressing a vector in {0})

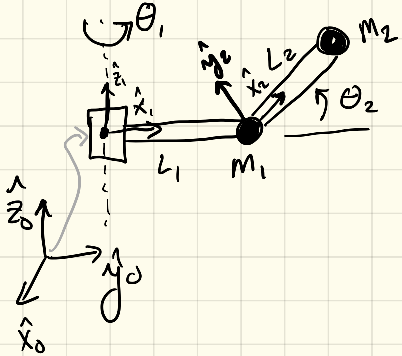



$${}^0P_3 = \begin{bmatrix} a_1 c_1 + d_2 s_1 \\ a_1 s_1 - d_2 c_1 \\ 0 + 0 \end{bmatrix}$$

# Example: Rotating Reference Frame, Fixed Vector

$\left[ \{i\}\ (\omega_i) \text{ express in some frame } \{j\} \quad j > i \right]$

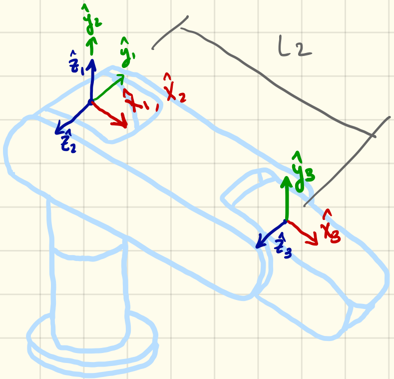



$$^1\omega_1 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix}$$

$$^2\omega_2 = \underbrace{{}^2R_1}_{\theta_2}\ \underbrace{{}^1\omega_1}_{\uparrow \dot{\theta}_1} + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_2 \end{bmatrix}$$

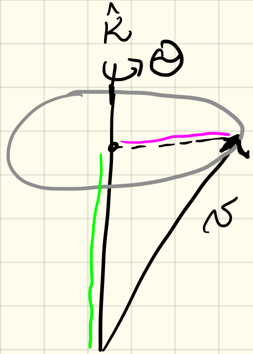
$$= \begin{bmatrix} s_2 \dot{\theta}_1 \\ c_2 \dot{\theta}_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_2 \end{bmatrix}$$

# Trick for 3D rotations: Turn it into a 2D rotation

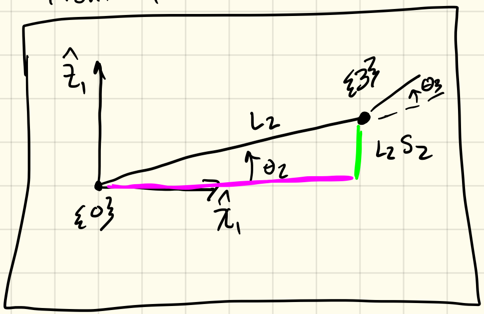



$$
{}^0p_3(\theta_1, \theta_2) = {}^0R_1 \, {}^1p_3(\theta_2)
$$

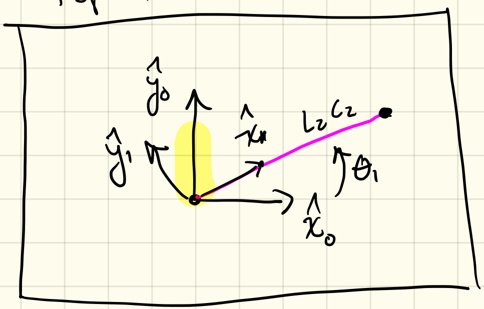
$$
= \begin{bmatrix} c_1 L_2 c_2 \\ s_1 L_2 c_2 \\ L_2 s_2 \end{bmatrix}
$$

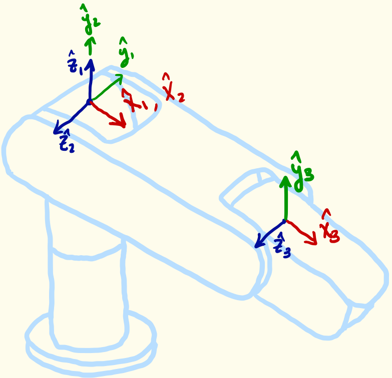



Front View



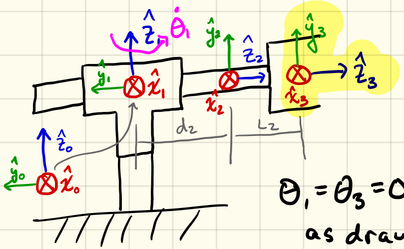

Top View





$\hat{y}_2$
$\hat{y}_1$
$\hat{z}_1$
$\hat{x}_1$
$\hat{x}_2$
$\hat{z}_2$
$\hat{y}_3$
$\hat{x}_3$
$\hat{z}_3$

# Jacobians




$\theta_1 = \theta_3 = 0°$
as drawn

**Find:** $\quad {}^3J_3 = \left[\begin{array}{c|c|c} S_3 & 0 & 0 \\ c_3 & 0 & 0 \\ 0 & 0 & 1 \\ \hline c_3(d_2+L_2) & 0 & 0 \\ -S_3(d_2+L_2) & 0 & 0 \\ 0 & 1 & 0 \end{array}\right]$

Last Column: ${}^3\omega_3$, ${}^3v_3$ when $\dot\theta_3 = 1$

$${}^3\omega_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad {}^3v_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

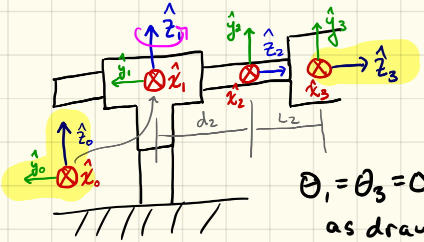
2nd Column: ${}^3\omega_3$, ${}^3v_3$ when $\dot d_2 = 1$

$${}^3\omega_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad {}^3v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

1st Column: ${}^3\omega_3$ and ${}^3v_3$ when $\dot\theta_1 = 1$

$$\uparrow \dot\theta_1 \quad \Rightarrow \quad {}^3\omega_3 = \begin{bmatrix} S_3 \\ C_3 \\ 0 \end{bmatrix} = {}^3R_1 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\uparrow \dot\theta_1 \times \xrightarrow[d_2+L_2]{{}^1p_3} = \otimes_{d_2+L_2} \quad \Rightarrow \quad {}^3v_3 = \begin{bmatrix} C_3(d_2+L_2) \\ -S_3(d_2+L_2) \\ 0 \end{bmatrix}$$



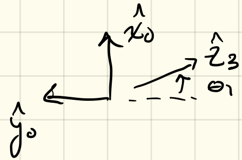

$\theta_1 = \theta_3 = 0°$
as drawn

**Find:**

$$
{}^0J_3 = \left[\begin{array}{c|c|c} 0 & 0 & s_1 \\ 0 & 0 & -c_1 \\ 1 & 0 & 0 \\ \hline c_1(d_2+l_2) & s_1 & 0 \\ s_1(d_2+l_2) & -c_1 & 0 \\ 0 & 0 & 0 \end{array}\right]
$$

**Last column:** ${}^0\omega_3$, ${}^0v_3$, $\dot{\theta}_3 = 1$

$$
{}^0\omega_3 = \xrightarrow[\dot{\theta}_3]{} = \begin{bmatrix} s_1 \\ -c_1 \\ 0 \end{bmatrix}
$$

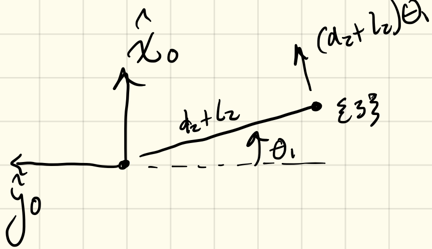



**2nd Column:** ${}^0\omega_3$, ${}^0v_3$ $\dot{d}_2 = 1$

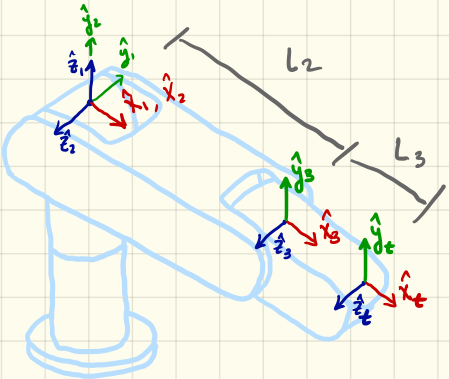
$$
{}^0\omega_3 = 0 \qquad {}^0v_3 = \xrightarrow[\dot{d}_2]{}
$$

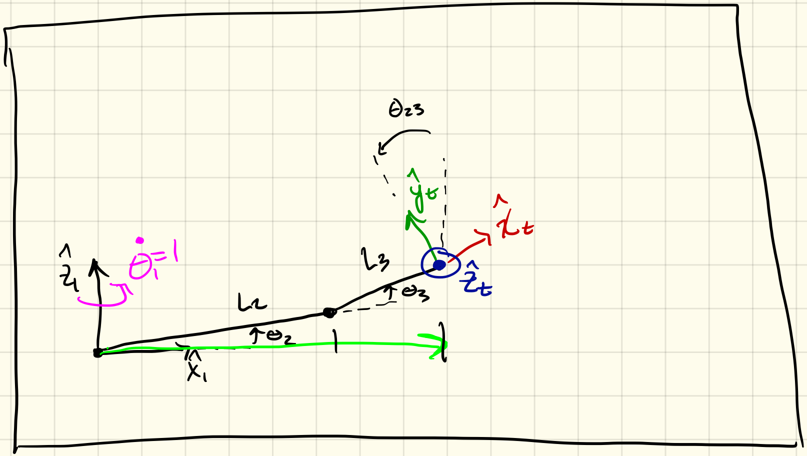
**1st column:** ${}^0\omega_3$, ${}^0v_3$ $\dot{\theta}_1 = 1$

$$
{}^0\omega_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad {}^0v_3 = \begin{bmatrix} c_1(d_2+l_2) \\ s_1(d_2+l_2) \\ 0 \end{bmatrix}
$$



# Reasoning w/ Rotations about Parallel Axes:




$$
{}^tJ_t = \left[\begin{array}{c|c|c} S_{23} & 0 & 0 \\ C_{23} & 0 & 0 \\ 0 & 1 & 1 \\ \hline 0 & L_2 S_3 & 0 \\ 0 & L_2 C_3 + L_3 & L_3 \\ L_2C_2 + L_3C_{23} & 0 & 0 \end{array}\right] \begin{array}{c} {}^t\omega_t \\ \\ {}^tv_t \end{array}
$$

$$\dot\theta_1 = 1 \qquad \dot\theta_2 = 1 \qquad \dot\theta_3 = 1$$

$$\dot\theta_1 = 1$$

$$
{}^t\omega_t = \uparrow_{\dot\theta_1} = \begin{bmatrix} S_{23} \\ C_{23} \\ 0 \end{bmatrix}
$$

$$
{}^tv_t = \uparrow_{\dot\theta_1} \times \underset{L_2C_2+L_3C_{23}}{\longrightarrow} = \underset{L_2C_2+L_3C_{23}}{\otimes}
$$

$$
= \begin{bmatrix} 0 \\ 0 \\ L_2C_2 + L_3C_{23} \end{bmatrix}
$$

# Reasoning w/ Rotations about Parallel Axes: