Lecture 12: Pontryagin Wrap Up

<u>Last time</u>

- Articulated Body Algorithm (Invented in Russia in 1974
  $\Rightarrow$ Featherstone rediscovered in
  Britan in early 1980s)

- Pontryagins Maximum principle
  - Necassary condition for open-loop optimality

HJB: Closed-loop sufficiency :: PMP: open-loop necessity

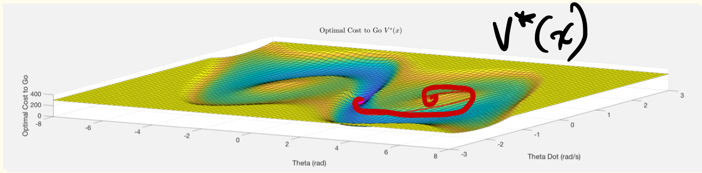
<u>Today</u>:

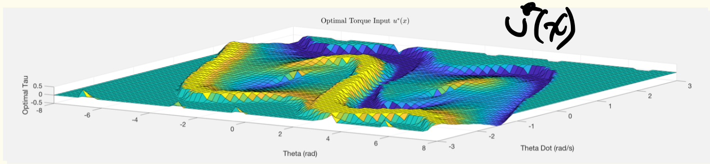
- Digging into PMP
- Relationship to Mechanics

REVIEW

Hamilton - Jacobi - Bellman

$$-\frac{\partial V^*}{\partial t} = \min_u \left[\ell(t,x,u) + \frac{\partial V^*}{\partial x} f(t,x,u) \right] \quad (*)$$





Pros:

- Gives an optimal feedback policy (i.e., solve OCP everywhere)
- If V^* satisfies $(*)$ and

$$u(t) = \arg\min \left[\ell(t,x(t),u(t)) + \frac{\partial V^*}{\partial x} f(t,x(t),u(t)) \right] \Rightarrow u(t) \text{ is optimal}$$

i.e., HJB provides a sufficient condition for optimality

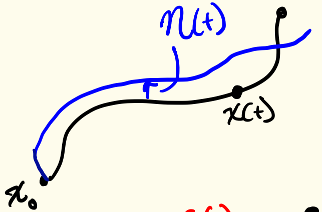
Cons:

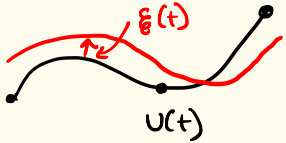
- Its hard to solve the HJB
- Usually happy if you can find an optimal $u^*(t)$ (open-loop).

REVIEW

Toward a necessary condition for $u(\cdot)$ to be extremal:

$$V(x_0, u(\cdot)) = \int_{t_0}^{t_f} \left(l(t, x, u) + \lambda^T(t)\left[f - \dot{x}\right] \right) dt + l_f(x(t_f))$$





Integration By Parts:

$$\int_{t_0}^{t_f} \lambda^T \dot{x}\, dt = \lambda^T x \Big|_{t_0}^{t_f} - \int_{t_0}^{t_f} \dot{\lambda}^T x\, dt$$

Functional:

$$V(x_0, u(\cdot)) = \int_{t_0}^{t_f} \Big(\overbrace{l + \lambda^T f}^{:= \mathcal{H}(t, x, u, \lambda)} + \dot{\lambda}^T x \Big) dt + l_f(x(t_f)) - \lambda^T x \Big|_{t_0}^{t_f}$$

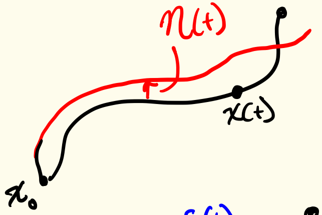
Consider a small Perturbation $\xi(t)$ to $u(t)$ w/ perturbation $\eta(t)$ to $x(t)$:

$$\frac{d}{dt}(x + \eta) = f(t, x + \eta, u + \xi) = f(t, x, u) + \frac{\partial f}{\partial x}\eta + \frac{\partial f}{\partial u}\xi + \text{h.o.t}$$

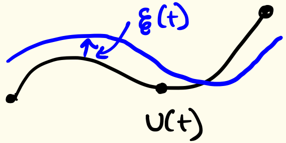
$$\dot{\eta} = \frac{\partial f}{\partial x}\eta + \frac{\partial f}{\partial u}\xi + \text{h.o.t.}$$

REVIEW

Toward a necessary condition for $u(\cdot)$ to be extremal:



$$V(x_0, u(\cdot)) = \int_{t_0}^{t_f} \left(\mathcal{H}(t, x, u, \lambda) + \dot{\lambda}^T x \right) dt + l_f(x(t_f)) - \lambda^T x \Big|_{t_0}^{t_f}$$



$$\dot{\eta} = \frac{\partial f}{\partial x}\eta + \frac{\partial f}{\partial u}\xi + h.o.t.$$

First-order Variation

$$V(x_0, u(\cdot)+\xi(\cdot)) = V(x_0, u(\cdot)) + \int_{t_0}^{t_f} \left[\cancel{\frac{\partial \mathcal{H}}{\partial x}\eta} + \frac{\partial \mathcal{H}}{\partial u}\xi + \cancel{\dot{\lambda}^T \eta} \right] dt + \cancel{\frac{\partial l_f}{\partial x}\xi(t_f) - \lambda(t_f)^T \xi(t_f)} + h.o.t.$$

Careful choice of $\lambda(t)$

$$\lambda(t_f) = \left[\frac{\partial l_f}{\partial x}\Big|_{x=x(t_f)} \right]^T \qquad \dot{\lambda} = -\left[\frac{\partial \mathcal{H}}{\partial x}\right]^T = -\nabla_x \mathcal{H} = -\nabla_x l - \left[\frac{\partial f}{\partial x}\right]^T \lambda$$

Back to variation

$$V(x_0, u(\cdot)+\xi(\cdot)) - V(x_0, u(\cdot)) = \int_{t_0}^{t_f} \frac{\partial H}{\partial u}\xi \, dt + h.o.t.$$

if $u(\cdot)$ optimal $\Rightarrow \frac{\partial H}{\partial u} = 0$

every where along the path

REVIEW

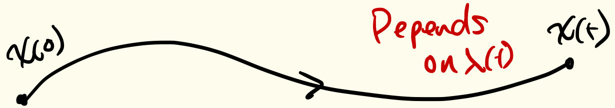
Pontryagin's Minimum Principle: $\mathcal{H} = \ell + \lambda^T f$

Suppose $x^*(t)$, $u^*(t)$ is the optimal state/control trajectory starting @ $x(0) = x_0$.
Then there exists a co-state trajectory $\lambda(t)$ with $\lambda(t_f) = \nabla_x \ell_f \big|_{x^*(t_f)}$ satisfying

1. $\dot{x}^* = \nabla_\lambda \mathcal{H}(t, x^*(t), u^*(t), \lambda(t))$

2. $-\dot{\lambda} = \nabla_x \mathcal{H}(t, x^*, u^*(t), \lambda(t))$

3. $u^*(t) = \arg\min_{\tilde{u}} \left[\mathcal{H}(t, x^*(t), \tilde{u}, \lambda(t)) \right] = \arg\min_{\tilde{u}} \left[\ell(t, x^*(t), \tilde{u}) + \lambda(t)^T f(t, x^*(t), \tilde{u}) \right]$



Note:

- Pontryagin's principle is a necessary condition for optimality. Sufficient conditions exist, but they are complex. (See text by Bryson and Ho.)

- Pontryagin's principle holds under much weaker conditions than are required for HJB to admit a solution.
  - Proof in Liberzon 4.2 (25 + pages)

- Under suitable conditions

$$\lambda(t) = \nabla_x V^* \Big|_{t,\, x = x^*(t)}$$

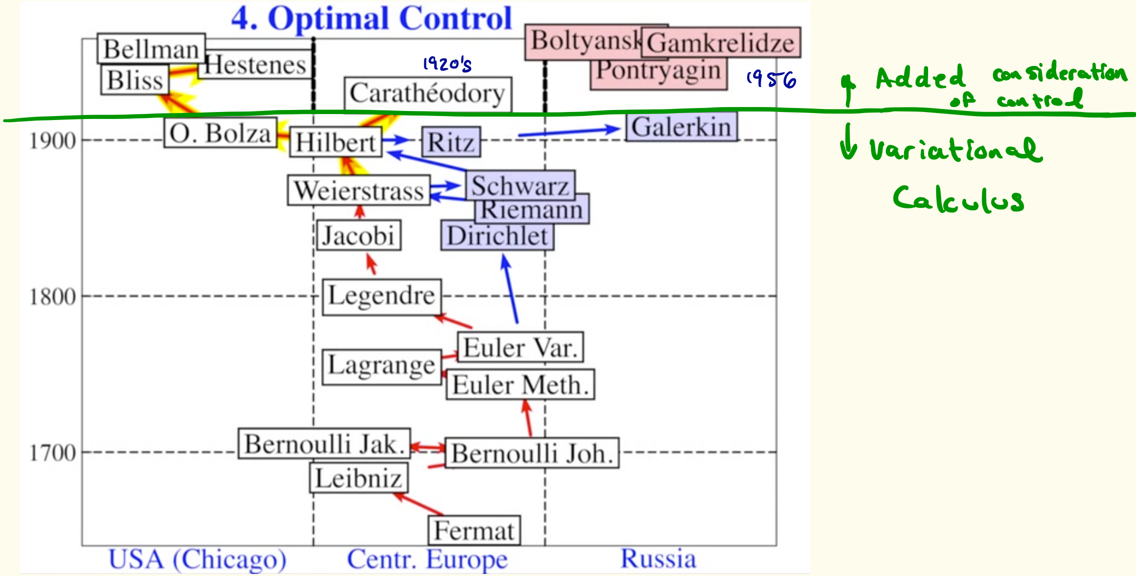
Units of $\lambda_i = \dfrac{\text{cost}}{\text{Units } x_i}$

- $\lambda(t)$ called adjoint variables, co-state, or sometimes a Lagrange multiplier

e.g., if your dynamics were f_2 for $t \in [t_1, t_2]$ then

$$\Delta V = \int_{t_1}^{t_2} \lambda(t)^T \left[f_2(t,x,u) - f(t,x,u) \right] dt$$

holds away from optimal trajectories too!



- PMP & HJB discovered independently during cold war era
- In 1926 Carathéodory had a paper w/ same eqns. as PMP, so some glory should be shared.

From: Wanner, On the discovery of Lagrange multipliers

Example: LQR

$$V(x_0, u(\cdot)) = \int_0^{t_f} \tfrac{1}{2}\left[x^T Q x + u^T R u\right] dt + \tfrac{1}{2} x(t_f)^T Q_f\, x(t_f) \qquad x(0) = x_0$$

$$\text{s.t.} \quad \dot{x} = Ax + Bu \qquad x \in \mathbb{R}^n, \; u \in \mathbb{R}^m$$

Hamiltonian:

$$\mathcal{H} = \tfrac{1}{2}\left[x^T Q x + u^T R u\right] + \lambda^T \left[Ax + Bu\right]$$

Costate:

$$-\dot{\lambda} = \nabla_x \mathcal{H} = Qx + A^T \lambda$$

$$\lambda(t_f) = \nabla_x \left(\tfrac{1}{2} x^T Q_f x\right)\Big|_{x = x(t_f)} = Q_f\, x(t_f)$$

Control:

$$\nabla_u H = 0 = Ru + B^T \lambda$$

$$u = -R^{-1} B^T \lambda$$

Closed-loop:

$$\dot{x} = Ax - BR^{-1}B^T \lambda$$

$$\Rightarrow \frac{d}{dt}\begin{bmatrix} x \\ \lambda \end{bmatrix} = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix}$$

$$x(0) = x_0$$

$$\lambda(t_f) = Q\, x(t_f)$$

Two-point Boundary Val. Problem

Compared to HJB approach:

HJB:

$$u^*(x) = -\underbrace{R^{-1}B^T P}_{K} x$$

$$V^* = \frac{1}{2} x^T P x$$

PMP:

$$u^* = -R^{-1}B^T \lambda$$

Relationship

$$\lambda = \nabla_x V^* = Px$$

Same optimal control for HJB & PMP ✓

- It can be shown that

$$\underbrace{\begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix}}_{:= \mathcal{H}_{mat}} \in \mathbb{R}^{2n \times 2n} \quad \text{is similar to} \quad \begin{bmatrix} (A-BK) & -BR^{-1}B^T \\ 0 & -(A-BK)^T \end{bmatrix} \in \mathbb{R}^{2n \times 2n}$$

- Closed-loop dynamics for $x(t)$ under optimal feedback:

$$\dot{x} = \underbrace{(A-BK)}_{\text{all negative eigen values}} x$$

- so $\mathcal{H}_{mat}$ has eig vals $+/- \mu_i$ where μ_i are eig vals of $A-BK$

Using this result to solve the CARE: Suppose $A-BK$ is diagonalizable...

- Consider $w_1, \dots, w_n$ eig vecs of $A-BK$ $\quad W = [w_1, \dots, w_n]$. Let $x(0) = w_i \quad \lambda(0) = P x(0)$

  Under the closed-loop policy

  - $\dot{x}(t) = \mu_i x(t)$ since w_i an eigenvector of $A-BK$.
  - $\dot{\lambda}(t) = P\dot{x}(t) = \mu_i P x(t) = \mu_i \lambda(t)$

$$\Rightarrow \frac{d}{dt}\begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix} = -\mu_i \begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix} = \mathcal{H}_{mat} \begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix}$$

$\Rightarrow \begin{bmatrix} w_i \\ Pw_i \end{bmatrix} \in \mathbb{R}^{2n}$ is an eig. vec. of $\mathcal{H}_{mat}$ with eig val μ_i

$\Rightarrow \begin{bmatrix} W \\ PW \end{bmatrix} \in \mathbb{R}^{2n \times n}$ is the matrix of all the eigen vecs of $\mathcal{H}_{mat}$ corresponding to negative eigenvalues

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How to solve the CARE: $0 = PA + A^TP + Q - PBR^{-1}B^TP$

① Form $\mathcal{H}_{mat} = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix}$

② Find it's eigen vectors for negative eigen values (n of them)

③ Stack eig. vecs into a $2n \times n$ matrix

$$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} \quad Y_1, Y_2 \in \mathbb{R}^{n \times n}$$

④ $P = Y_2 Y_1^{-1}$

Solves the CARE!

Relationship to Mechanics:

$V :=$ potential energy, $T :=$ Kinetic energy

Not the optimal cost to go

Hamiltonian Mechanics

$$\mathcal{H} := \frac{1}{2}\dot{q}^T H(q)\dot{q} + V(q) = T+V$$

$$p = H(q)\dot{q}$$

$$\mathcal{H} = \frac{1}{2}p^T H(q)^{-1} p + V(q)$$

Hamiltonian Dynamics: (For a conservative system)

$$\left.\begin{aligned} \dot{q} &= \nabla_p \mathcal{H} \\ \dot{p} &= -\nabla_q \mathcal{H} \end{aligned}\right]$$

Same equations as in the pontryagin principle

Backing out Nature's Cost

u plays the role of steering $q \Rightarrow \dot{q} = u$

Hamiltonian Dynamics are necessary conditions for the optimal control problem:

$$\max \left[\int_0^{\Delta t} \underbrace{\ell(q,u) + p^T(u-\dot{q})}_{\mathcal{H}}\, dt \right]$$

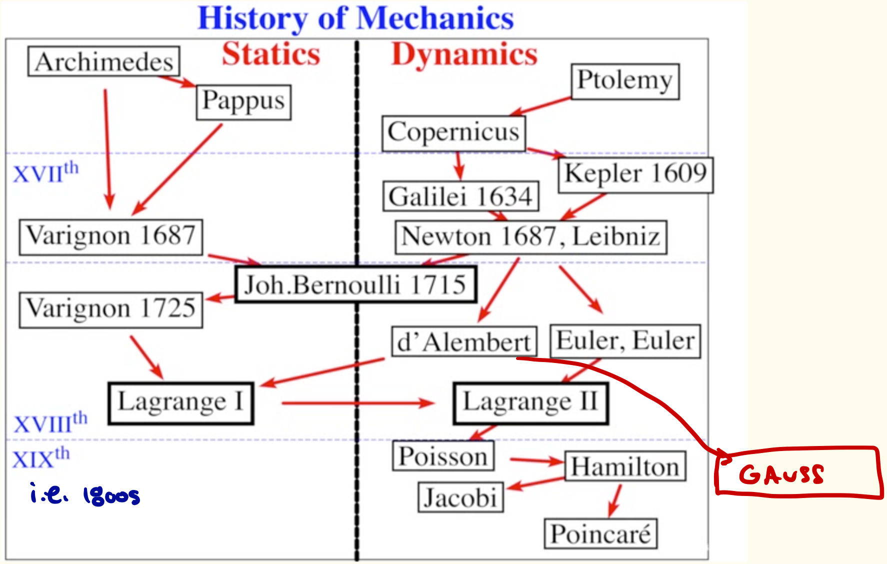
$$\mathcal{H} = T+V$$

$$= \ell(q,u) + p^T u = \ell(q,\dot{q}) + \underbrace{p^T\dot{q}}_{2T}$$

$$\Rightarrow \ell(q,\dot{q}) = V - T$$

$T-V$ is the Lagrangian from Dynamics or AME 50551

$\Rightarrow$ Nature's cost: $\min \int_0^{\Delta t} (T-V)\,dt$



- Lagrange played a role in developing variational calc. as core of analytical dynamics.
- Lagrangian dynamics preceeded Hamiltonian Dynamics (backward of my notes here)

From: Wenner, On the discovery of Lagrange multipliers