Lecture 11: Articulated Body Algorithm

Last time

- LQR Generalizations
  - cost: $x^T S u$
  - cost: $x^T Q x + u^T R u + q^T x + q_0 + r^T u + x^T S u$
    dynamics: $x_{i+1} = A x_i + B u_{i+1} + d_{i+1}$

- Dynamics as an LQR problem

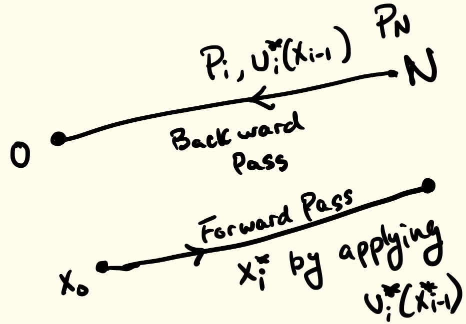
Today

- Wrap up last lecture with an example
- Intro to the Hamiltonian perspective on optimal control
  - Pontryagins Maximum Principle

Discrete Time LQR:

$$\min_{u[\cdot],\,x[\cdot]} \left[\sum_{i=1}^{N} x_i^T Q_i x_i + u_i^T R_i u_i\right]$$

$$\text{s.t.} \quad x_i = A_i x_{i-1} + B_i u_i$$



Value Iteration: $V^*(i,x) = x^T P_i x \qquad P_N = Q_N$

$$P_{i-1} = Q_i + A_i^T\left[P_i - P_i B_i \left(B_i^T P_i B_i + R_i\right)^{-1} B_i^T P_i\right] A_i$$

$$u_i^*(x) = -\left(B_i^T P_i B_i + R_i\right)^{-1} B_i^T P_i A_i x_{i-1}$$

Infinite Horizon Time invariant Case: $V^*(x) = x^T P x$

$$P = Q + A^T\left[P - PB\left(B^T P B + R\right)^{-1} B^T P\right] A$$

"Life can only be understood going backwards,

but it must be lived going forwards"

—Søren Kierkegaard

Optimal Control For Rigid Body Dynamics Simulation

$$H\ddot{q} + C\dot{q} + \tau_g = \tau \qquad q \in \mathbb{R}^N$$

- RNEA: given $q, \dot{q}, \ddot{q}$ find τ

  $\Rightarrow$ RNEA is efficient! It takes $O(N)$ math operations to compute

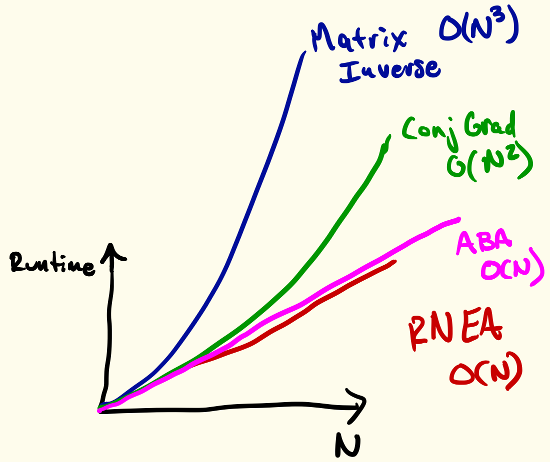
- Simulation: given $q, \dot{q}, \tau$ Find $\ddot{q}$

  ① $\ddot{q} = H^{-1}(\tau - C\dot{q} - \tau_g)$ $\quad O(N^3)$

  ② Solve $H\ddot{q} = \tau - C\dot{q} - \tau_g$ using conjugate gradient:

  - N steps, $O(N^2)$ total
  - $Hx = \text{RNEA}(q, \dot{q}=0, \ddot{q}=x, a_g=0)$ $O(N)$ per step

  ③ Articulated Body Algorithm $O(N)$



Gauss Principle For Rigid Body Chains

$$\min_{\ddot{q}, a} \sum_{i=1}^{N} \frac{1}{2}(a_i^u - a_i)^T I_i (a_i^u - a_i)$$

s.t. $a_i = {}^iX_{i-1}\, a_{i-1} + \phi_i \ddot{q}_i + [v_i \times] \phi_i \dot{q}_i$

$$a_0 = -\begin{bmatrix} 0_{3\times 1} \\ \vec{g} \end{bmatrix}$$

Mapping

$a_i \Leftrightarrow x_i$

$\ddot{q}_i \Leftrightarrow u_i$

$I_i \Leftrightarrow Q_i$

$-I_i a_i^u \Leftrightarrow c_i$

${}^iX_{i-1} \Leftrightarrow A_i$

$\phi_i \Leftrightarrow B_i$

$v_i \times \phi_i \dot{q}_i \Leftrightarrow d_i$

$a_i^{u\,T} I_i a_i^u \Leftrightarrow c_{0,i}$

LQ Optimal Control

$$\min_{\{x_i, u_i\}_{i=1}^{N}} \sum_{i=1}^{N} \frac{1}{2} x_i^T Q_i x_i + c_i^T x_i + c_{0,i}$$

s.t. $x_i = A_i x_{i-1} + B_i u_i + d_i$

x_0 fixed

$$Q_i' = \begin{bmatrix} Q_i & c_i \\ c_i^T & c_{0,i} \end{bmatrix} \quad A_i' = \begin{bmatrix} A_i & d_i \\ 0 & 1 \end{bmatrix}$$

$$B_i' = \begin{bmatrix} B_i \\ 0 \end{bmatrix}$$

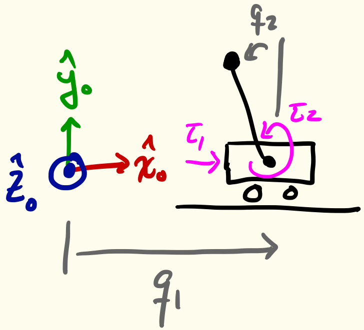
- $\ddot{q}$ can be found with
  1. A forward sweep to determine v_i (and a_i^u)
  2. A backward sweep to compute a quadratic value function

$$V^*(i, a_i) = \begin{bmatrix} a_i \\ 1 \end{bmatrix}^T P_i' \begin{bmatrix} a_i \\ 1 \end{bmatrix}$$ quadratic optimal gauss cost-to-go

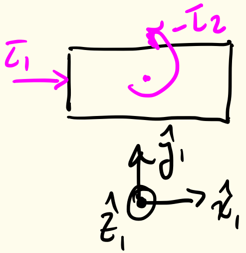
$$\ddot{q}_i = -(B_i'^T P_i' B_i')^{-1} B_i'^T P_i' \begin{bmatrix} a_{i-1} \\ 1 \end{bmatrix}$$

  3. A subsequent forward sweep to compute $\ddot{q}_i$, a_i for the optimal solution

Example: Cart Pole

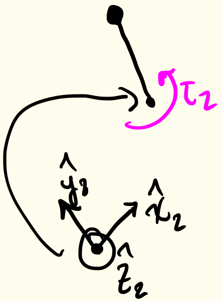


Unconstrained:



$$ {}^1 f_1^{net} = \begin{bmatrix} 0 \\ 0 \\ -\tau_2 \\ \hdashline \tau_1 \\ 0 \\ 0 \end{bmatrix} $$

$$ f_i^{net} = I_i a_i^{uc} + v_i \times^* I_i v_i $$



$$ {}^2 f_2^{net} = \begin{bmatrix} 0 \\ 0 \\ \tau_2 \\ \hdashline 0 \\ 0 \\ 0 \end{bmatrix} $$

Articulated Body Algorithm

History

- 1974 Vereshchagin
  Developed in Russia
- 1983 Featherstone
  Developed independently

Pass #1: Base to Tip (Kinematics)

$v_0 = 0$

for $i = 1$ to N

$$v_i = {}^iX_{i-1} v_{i-1} + \phi_i \dot{q}_i$$

end

Pass #2: Tips to Base (Value iteration on Optimal Gauss Cost to go)

$I_N^A = I_N \qquad p_N^A = v_N \times^* I_N v_N$

for $i = N$ to 1

$$\beta_i = \tau_i - \phi_i^T p_i^A$$

Articulated body inertia

$$I_{i-1}^A = I_{i-1} + {}^{i-1}X_i^* \overbrace{\left[I_i^A - I_i^A \phi_i \left(\phi_i^T I_i^A \phi_i \right)^{-1} \phi_i^T I_i^A \right]}^{I_i^a} {}^iX_{i-1}$$

$$p_{i-1}^A = v_{i-1} \times^* I_{i-1} v_{i-1} + {}^{i-1}X_i^* \left[p_i^A + I_i^a (v_i \times \phi_i \dot{q}_i) + I_i^A \phi_i \left(\phi_i^T I_i^A \phi_i \right)^{-1} \beta_i \right]$$

end

$$P_i' = \begin{bmatrix} I_i^A & p_i^A - \phi_i \tau_i \\ (\cdot)^T & \ast \end{bmatrix}$$

Pass #3: Base to tips

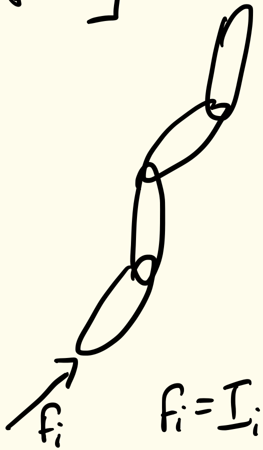
$a_0 = -a_g$

for $i = 1$ to N

$$\ddot{q}_i = \left(\phi_i^T I_i^A \phi_i \right)^{-1} \left[\beta_i - \phi_i^T I_i^A \left[{}^iX_{i-1} a_{i-1} + v_i \times \phi_i \dot{q}_i \right] \right]$$

$$a_i = {}^iX_{i-1} a_{i-1} + \phi_i \ddot{q}_i + v_i \times \phi_i \dot{q}_i$$

end

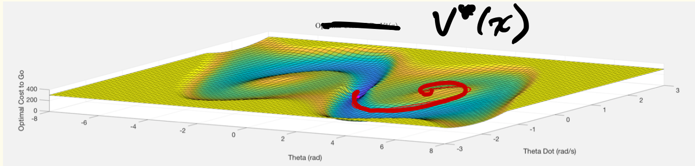


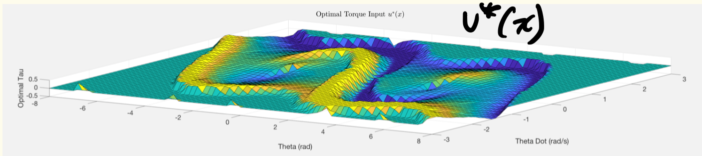
$$f_i = I_i^A a_i + p_i^A$$

I_i^A inertia felt with joints free to move.

So Far: Hamilton-Jacobi-Bellman

$$-\frac{\partial V^*}{\partial t} = \min_{u}\left[\ell(t,x,u) + \frac{\partial V^*}{\partial x} f(t,x,u) \right] \quad (*)$$





Pros:

- Gives optimal feedback policy (i.e., you solve the OCP everywhere)
- If V^* satisfies the HJB and

$$u(t) = \arg\min\left[\ell + \frac{\partial V^*}{\partial x} f \right] \quad \forall t \quad \Rightarrow \quad u(t) \text{ is optimal}$$

i.e., HJB provides a sufficient condition for optimality

Cons:

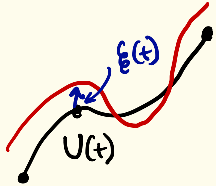
- It's hard to solve the HJB
- Usually we are happy to find an optimal open loop $u^*(t)$ (We may wasting effort to solve the HJB).

Toward a necessary condition for $u(\cdot)$ to be extremal:

$$V(x_0, u(\cdot)) = \int_{t_0}^{t_f} \left[\ell(t, x(t), u(t)) + \lambda(t)^T \left(f(t,x,u) - \dot{x}\right) \right] dt + \ell_f(x(t_f))$$

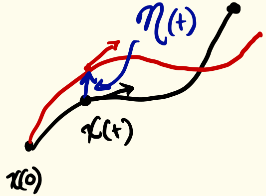
Integration by parts:

$$\int_{t_0}^{t_f} \lambda(t)^T \dot{x}(t)\, dt = \lambda(t)\, x(t) \Big|_{t_0}^{t_f} - \int_{t_0}^{t_f} \dot{\lambda}(t)^T x(t)\, dt$$



Functional:

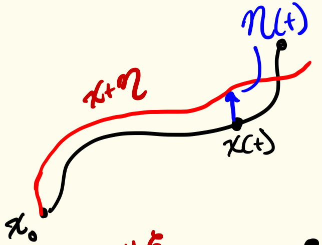
$$V = \int_{t_0}^{t_f} \Big(\overbrace{\ell + \lambda^T f}^{:= \mathcal{H}(t,x,u,\lambda)} + \dot{\lambda}^T x \Big) dt + \ell_f(x(t_f)) - \lambda^T x \Big|_{t_0}^{t_f}$$



Consider a small perturbation $\varepsilon(t)$ to $u(t)$ and $\eta(t)$ to $x(t)$

$$\frac{d}{dt}\left(x(t) + \eta(t)\right) = f\left(t, x(t) + \eta(t), u(t) + \varepsilon(t)\right)$$

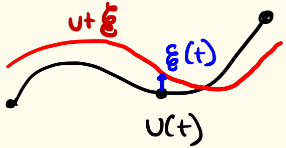
$$\cancel{\dot{x}(t)} + \dot{\eta}(t) = \cancel{f(t, x(t), u(t))} + \frac{\partial f}{\partial x}\eta(t) + \frac{\partial f}{\partial u}\varepsilon(t) + h.o.t.$$



Toward a necessary condition for $u(\cdot)$ to be extremal:

$$V(x_0, u(\cdot)) = \int_{t_0}^{t_f} \left(\mathcal{H}(t, x, u, \lambda) + \dot{\lambda}^T x \right) dt + l_f(x(t_f)) - \lambda^T x \Big|_{t_0}^{t_f}$$

$$\dot{\eta} = \frac{\partial f}{\partial x}\eta + \frac{\partial f}{\partial u}\xi + h.o.t.$$



$$V(x_0, u(\cdot)+\varepsilon(\cdot)) = V(x_0, u(\cdot)) + \int_{t_0}^{t_f} \left[\cancel{\frac{\partial \mathcal{H}}{\partial x}\eta(t)} + \frac{\partial \mathcal{H}}{\partial u}\varepsilon(t) + \cancel{\dot{\lambda}^T \eta(t)} \right]$$

$$\cancel{+ \frac{\partial l_f}{\partial x}\eta(t_f) - \lambda(t_f)^T \eta(t_f) + h.o.t}$$

$$\lambda(t_f) = \nabla_x l_f$$

$$-\dot{\lambda} = \nabla_x \mathcal{H}$$

Variation

$$V(x_0, u(\cdot)+\varepsilon(\cdot)) - V(x_0, u(\cdot)) = \int_{t_0}^{t_f} \frac{\partial \mathcal{H}}{\partial u}\varepsilon(t)\,dt + h.o.t.$$

if $u(\cdot)$ is optimal $\Rightarrow \dfrac{\partial \mathcal{H}}{\partial u}\left(t, x(t), u(t), \lambda(t)\right) = 0 \quad \forall t$

Pontryagin's Minimum Principle:

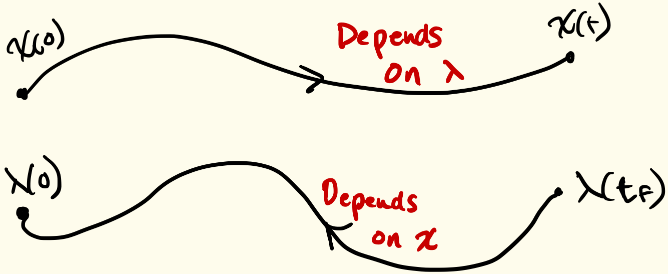
Suppose $x^*(t)$, $u^*(t)$ is the optimal state/control trajectory starting @ $x(0)=x_0$.

Then there exists a co-state trajectory $\lambda(t)$ with $\lambda(t_f) = \nabla_x l_f \big|_{x^*(t_f)}$ satisfying

① $\dot{x}^* = \nabla_\lambda \mathcal{H}(t, x^*(t), u^*(t), \lambda(t))$

② $-\dot{\lambda} = \nabla_x \mathcal{H}(t, x^*, u^*(t), \lambda(t))$

③ $u^*(t) = \arg\min_{\tilde{u}} \left[\mathcal{H}(t, x^*(t), \tilde{u}, \lambda(t)) \right] = \arg\min_{\tilde{u}} \left[l(t, x^*(t), \tilde{u}) + \lambda(t)^T f(t, x^*(t), u^*(t)) \right]$



Summary:

- Rigid Body Dynamics Simulation
  - Most efficient simulators apply value iteration on the gauss cost to go! $O(N)$
  - Quadratic portion of the optimal cost to go is the articulated-Body inertia.

- Pontryagin's Principle
  - Necessary condition for optimality
  - Requires $u^*(t)$ to minimize the Hamiltonian at each point along a traj.
  - By contrast HJB requires $u^*(t,x)$ to minimize the same quantity everywhere in space

Pontryagin: Open-loop optimality :: Bellman: closed-loop optimality