Lecture 23 - Model Predictive Control

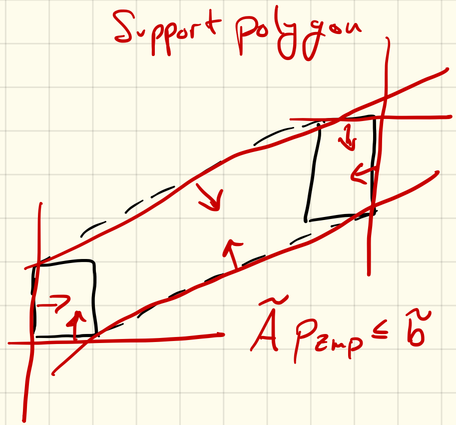
- Model Predictive Control
  - Online trajectory optimization to generate controllers
  - Constraints includes
  - Stability / Feasibility hard to prove

- Today
  - Specify MPC precisely
  - Show examples for how it fails
  - Describe theory that allows us to prevent failure.

ZMP Planning for Walking: Take 2: Constrained

Discretize Dynamics:

$$\begin{bmatrix} p_{com} \\ \dot{p}_{com} \end{bmatrix}_{k+1} = \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p_{com} \\ \dot{p}_{com} \end{bmatrix}_{k} + \begin{bmatrix} \frac{1}{2}\Delta t^2 \\ \Delta t \end{bmatrix} \ddot{p}_{com_k}$$



$$\min_{x_k, u_k} \sum_{k=0}^{N-1} \| Cx_k + Du_k - y_k^d \|_Q^2 + \| u_k \|_R^2$$

$$\text{s.t.} \quad \underbrace{Cx_k + Du_k}_{ZMP_k} \in \text{Support Polygon}_k$$

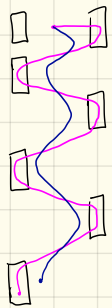
Can be written as:

$$= \tilde{A}_k (Cx_k + Dx_k) \le \tilde{b}_k$$

$$x_{k+1} = A x_k + B u_k$$

$$\varepsilon_N \in \text{Support Polygon}_N$$

This problem is a QP



Model Predictive Control:

- Solving a single traj. opt problem is limiting
  - all models are wrong (planned trajectories never tracked pefectly)
  - Disturbances may futther challenge application
- Conventional controls strategies ignore constraints
- Main idea of MPC: Consider an OCP

$$U^*[\cdot, x_{init}] = \underset{U[\cdot]}{\text{argmin}} \sum_{k=0}^{N-1} \ell(x_k, u_k) + \ell_f(x_N)$$

$$\text{s.t. } x_{k+1} = f(x_k, u_k) \qquad u_k \in \mathcal{U}$$

$$x_0 = x_{init} \qquad x_k \in \mathcal{X}$$

MPC $U^*[0, x]$ as control policy

- Issues: Feasible? Timing for solution? Stability?

Slides From:

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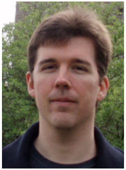

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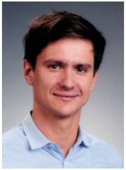

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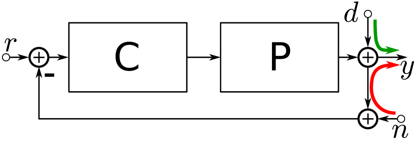
Two Different Perspectives

Classical design: design C

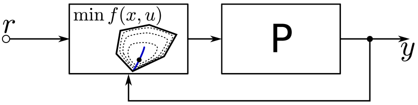


Dominant issues addressed

- Disturbance rejection $(d \to y)$
- Noise insensitivity $(n \to y)$
- Model uncertainty

(usually in *frequency domain*)

MPC: real-time, repeated optimization to choose $u(t)$

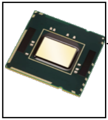


Dominant issues addressed

- Control constraints (limits)
- Process constraints (safety)

(usually in *time domain*)

MPC: Applications



Computer control — ns

μs — Power systems

Traction control — ms

Walking Robots

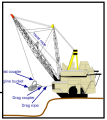
Seconds — Buildings

Refineries — Minutes

Hours — Nurse rostering

Train scheduling — Days

Weeks — Production planning



Infinite Time Constrained Optimal Control (what we would like to solve)

$$
\begin{aligned}
J_0^*(x(0)) = \min \ & \sum_{k=0}^{\infty} q(x_k, u_k) \\
\text{s.t. } & x_{k+1} = Ax_k + Bu_k, k = 0, \ldots, N-1 \\
& x_k \in \mathcal{X}, u_k \in \mathcal{U}, k = 0, \ldots, N-1 \\
& x_0 = x(0)
\end{aligned}
$$

- We can't compute a control law from this formulation since it would require ∞ of opt. variables

Receding Horizon Control (what we can sometimes solve)

Same as our $l_f(x_{t+N})$

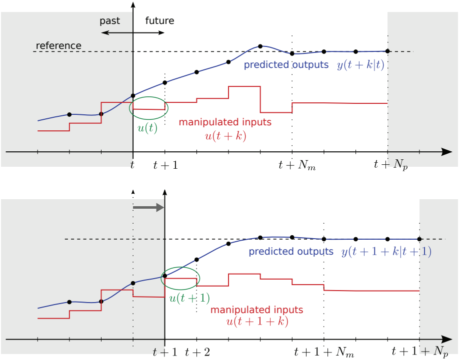
$$
\begin{aligned}
J_t^*(x(t)) = \quad \min_{U_t} \quad & p(x_{t+N}) + \sum_{k=0}^{N-1} q(x_{t+k}, u_{t+k}) \\
\text{subj. to} \quad & x_{t+k+1} = Ax_{t+k} + Bu_{t+k},\ k = 0, \ldots, N-1 \\
& x_{t+k} \in \mathcal{X},\ u_{t+k} \in \mathcal{U},\ k = 0, \ldots, N-1 \\
& x_{t+N} \in \mathcal{X}_f \\
& x_t = x(t)
\end{aligned}
\tag{1}
$$

where $\mathcal{U}_t = \{u_t, \ldots, u_{t+N-1}\}$.

$p(x_{t+N})$ approximates tail of the cost

$\mathcal{X}_f$: approximates the tail of the constraints

On-line Receding Horizon Control



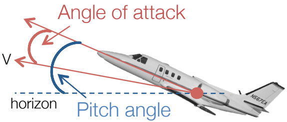
1. At each sampling time, solve a Constrained Finite-Time Opt. Control (CFTOC) problem
2. Apply the optimal input **only during** $[t, t+1]$
3. At $t+1$ solve a CFTOC over a **shifted horizon** based on new state measurements
4. The resultant controller is referred to as **Receding Horizon Controller (RHC)** or **Model Predictive Controller (MPC)**.

Example: Cessna Citation Aircraft

Linearized continuous-time model:
(at altitude of 5000m and a speed of 128.2 m/sec)

$$\dot{x} = \begin{bmatrix} -1.2822 & 0 & 0.98 & 0 \\ 0 & 0 & 1 & 0 \\ -5.4293 & 0 & -1.8366 & 0 \\ -128.2 & 128.2 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} -0.3 \\ 0 \\ -17 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} x$$



- Input: elevator angle
- States: x_1: angle of attack, x_2: pitch angle, x_3: pitch rate, x_4: altitude
- Outputs: pitch angle and altitude
- Constraints: elevator angle ± 0.262rad ($\pm 15°$), elevator rate ± 0.524rad ($\pm 60°$), pitch angle ± 0.349 ($\pm 39°$)

Open-loop response is unstable (open-loop poles: $0, 0, -1.5594 \pm 2.29i$)

LQR and Linear MPC with Quadratic Cost

- Quadratic cost
- Linear system dynamics
- Linear constraints on inputs and states

LQR

$$J_\infty(x(t)) = \min \sum_{k=0}^{\infty} x_t^T Q x_t + u_k^T R u_k$$
$$\text{s.t. } x_{k+1} = Ax_k + Bu_k$$
$$x_0 = x(t)$$

MPC

$$J_0^*(x(t)) = \min_{U_0} \sum_{k=0}^{N-1} {x_k}^T Q x_k + {u_k}^T R u_k$$
$$\text{s.t. } x_{k+1} = Ax_k + Bu_k$$
$$x_k \in \mathcal{X},\ u_k \in \mathcal{U}$$
$$x_0 = x(t)$$

Assume: $Q = Q^T \succeq 0,\ R = R^T \succ 0$

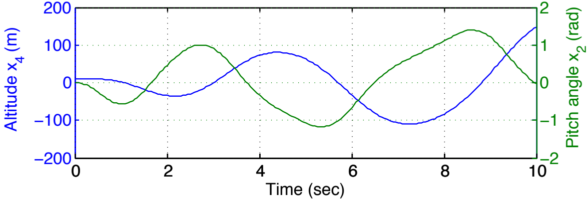
Example: LQR with saturation

Linear quadratic regulator with saturated inputs.

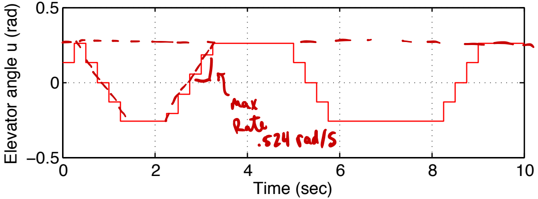
At time $t = 0$ the plane is flying with a deviation of $10m$ of the desired altitude, i.e. $x_0 = [0; 0; 0; 10]$

Problem parameters:

Sampling time 0.25sec, $Q = I$, $R = 10$



- Closed loop system is unstable
- Applying LQR and saturating can lead to instability



Saturated @.262rad

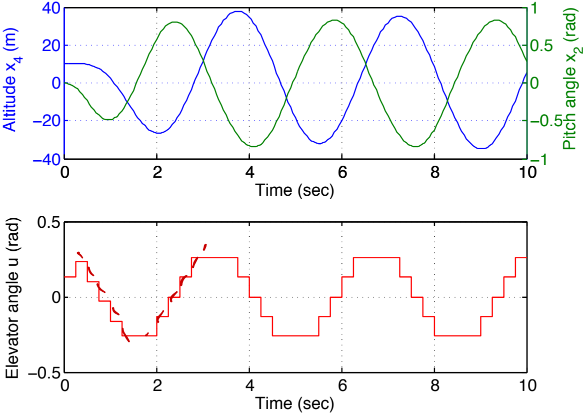
Example: MPC with Bound Constraints on Inputs

MPC controller with input constraints $|u_i| \leq 0.262$

Rate constraints not modeled w/in MPC but enforce during execution

Problem parameters:

Sampling time 0.25sec,
$Q = I,\ R = 10,\ N = 10$



The MPC uses knowledge that the elevator will saturate, but doesn't consider rate constraints

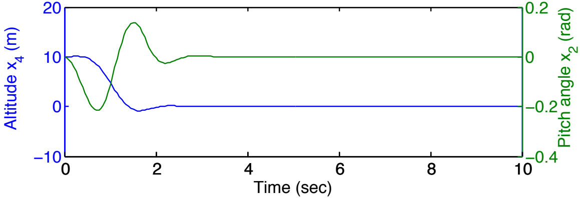
$\Rightarrow$ system converges to a limit cycle

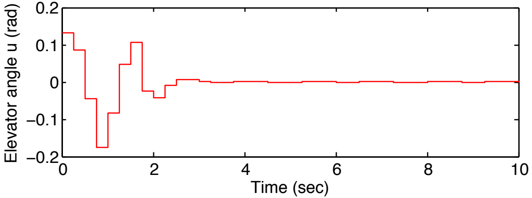
Example: MPC with all Input Constraints

MPC controller with input constraints $|u_i| \leq 0.262$ and rate constraints $|\dot{u}_i| \leq 0.349$ approximated by $|u_k - u_{k-1}| \leq 0.349\, T_s$

Problem parameters:

$\boxed{\begin{array}{l}\text{Sampling time } 0.25\text{sec},\\ Q = I,\ R = 10,\ N = 10\end{array}}$





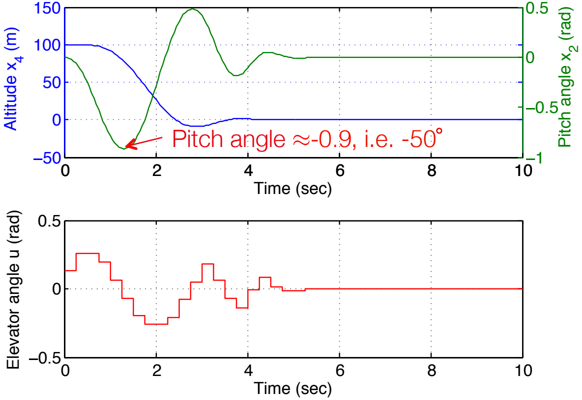
- All constraints considered in MPC
- In this case the equilibrium @ 0 is stable.

Example: Inclusion of state constraints

MPC controller with input constraints $|u_i| \leq 0.262$ and rate constraints $|\dot{u}_i| \leq 0.349$ approximated by $|u_k - u_{k-1}| \leq 0.349\, T_s$

Problem parameters:

$\boxed{\text{Sampling time } 0.25\text{sec}, \; Q = I, \; R = 10, \; N = 10}$

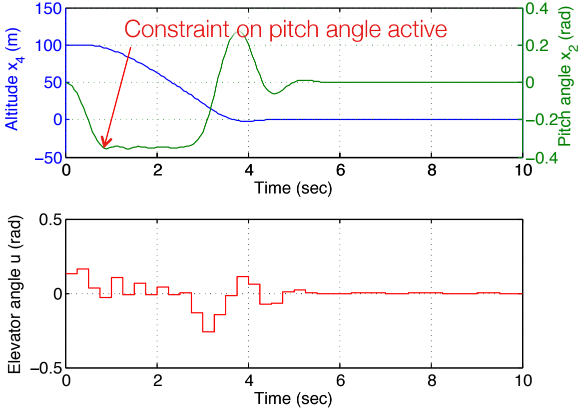


Example: Inclusion of state constraints

MPC controller with input constraints $|u_i| \leq 0.262$ and rate constraints $|\dot{u}_i| \leq 0.349$ approximated by $|u_k - u_{k-1}| \leq 0.349\, T_s$

Problem parameters:

Sampling time 0.25sec, $Q = I,\ R = 10,\ N = 10$



Add state constraints for passenger comfort:

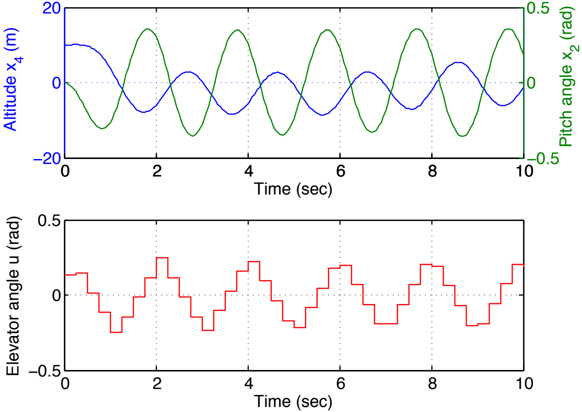
$$|x_2| \leq 0.349$$

Example: Short horizon

MPC controller with input constraints $|u_i| \leq 0.262$ and rate constraints $|\dot{u}_i| \leq 0.349$ approximated by $|u_k - u_{k-1}| \leq 0.349\,T_s$

Problem parameters:

Sampling time 0.25sec,
$Q = I$, $R = 10$, $N = 4$



Decrease in prediction Horizon causes loss of Stability properties

Model Predictive Control: Consider a simple case when $f(0,0)=0$ and

$$\beta_1 \|x\| \leq \ell(x,u) \qquad \forall x, \forall u \in \mathcal{U}$$

If: $N=\infty$ finite cost implies (roughly)

- Closed-loop Stability
- Persistent feasibility along the closed-loop trajectory

When $\underline{\ell_f(x)=0}$: Main contributor to loss of stability is a short prediction horizon.

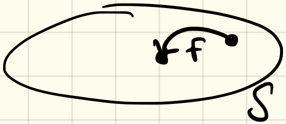
Feasibility and stability in MPC - Solution

Main idea: Introduce terminal cost and constraints to explicitly ensure feasibility and stability:

$$
\begin{aligned}
J_0^*(x_0) = \quad \min_{U_0} \quad & p(x_N) + \sum_{k=0}^{N-1} q(x_k, u_k) && \text{Terminal Cost} \\
\text{subj. to} \quad & \\
& x_{k+1} = Ax_k + Bu_k,\ k = 0, \ldots, N-1 \\
& x_k \in \mathcal{X},\ u_k \in \mathcal{U},\ k = 0, \ldots, N-1 \\
& x_N \in \mathcal{X}_f && \text{Terminal Constraint} \\
& x_0 = x(t)
\end{aligned}
$$

$p(\cdot)$ and $\mathcal{X}_f$ are chosen to mimic an infinite horizon.

A few Definitions: $x \in \mathbb{R}^n$ $\quad D \subseteq \mathbb{R}^n$



$$x_{n+1} = f(x_n) \quad , \quad f(0)=0 \quad (*)$$

- A set S is forward invariant if $\forall x \in S \quad f(x) \in S$

  (If you start in S, you stay there.)

- A function $V: D \to \mathbb{R}$ is called an exponential lyapunov function if $\exists\, K_1, K_2 > 0$, $K_3 > 0$

$$K_1 \|x\| \leq V(x) \leq K_2 \|x\|$$

AND

$$\underline{V(x) - V(f(x)) \geq K_3 \|x\|} \quad \forall x \in D$$

$V(x_k)$ decreases along trajectories...

If $(*)$ admits a ELF V then

- the origin is exponentially stable $\left[\|x_k\| \leq c\,\gamma^k \|x_0\| \quad \text{for some } c>0,\ \gamma \in (0,1) \right]$
- Any level set $\{x \mid V(x) \leq c\} \subseteq D$ is forward invariant

A Few More: $x_{n+1} = f(x_n, u_n)$ $\quad x \in \mathbb{R}^n \quad u \in \mathcal{U} \subseteq \mathbb{R}^m$

- A set S is called control invariant if $\forall x \in S \ \exists u \in \mathcal{U}$ s.t. $f(x, u) \in S$ (if you start in S, it is possible to stay there)

- Example: Let $\mathcal{X}_{Bad} \subseteq \mathbb{R}^n$ be a set of states to avoid. The set of viable states S is controlled invariant (it's actually the largest controlled invariant set that does not intersect $\mathcal{X}_{Bad}$.)

- A function $V(x)$ is called an exponential control lyapunov function if

$$k_1 \|x\| \leq V(x) \leq k_2 \|x\|$$

$$\forall x \in D \ \exists u \in \mathcal{U} \quad V(x) - V(f(x, u)) \geq k_3 \|x\|$$

Takeaway: If a system admits an ECLF then the origin can be exponentially stabilized in closed loop.

Model Predictive Control: Consider a simple case when $f(0,0)=0$ and

- Consider a terminal state constraint $x_N \in \mathcal{X}_N = \mathcal{X}_f$
- Define backwards Reachable sets

$$\mathcal{X}_{i-1} = \{ x_{i-1} \in \mathcal{X} \mid \exists u \in \mathcal{U} \text{ s.t. } f(x_{i-1}, u) \in \mathcal{X}_i \}$$

$\mathcal{X}_0$ is the set of initial states where the MPC problem is feasible.

- Proposition 1: The MPC problem will be persistently feasible from any initial state $x_0 \in \mathcal{X}_0$ if $\mathcal{X}_1$ is controlled invariant

  ← incorrect in lecture. Please fix notes.

  ← fix notes

- Proposition 2: $\mathcal{X}_1$ is controlled invariant if $\mathcal{X}_f$ is controlled invariant.

Takeaway: If $\mathcal{X}_f$ choosen to be control invariant then the MPC is persistently feasible.

Proof of Propositions:

① Suppose $\mathcal{X}_1$ is control invariant. Then:

Justify as a simple exercise.

$$\mathcal{X}_1 \subseteq \{x \in \mathcal{X} \mid u \in \mathcal{U}, f(x,u) \in \mathcal{X}_1\} = \mathcal{X}_0$$

So. Given $x_0 \in \mathcal{X}_0$ if we follow the MPC control law u_0^*

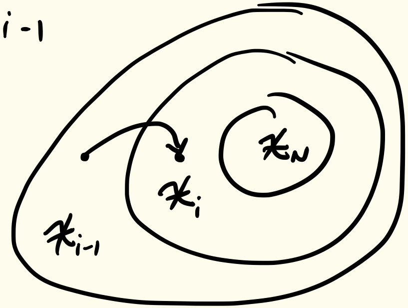
$$f(x_0, u_0^*) \in \mathcal{X}_1 \subseteq \mathcal{X}_0 \Rightarrow \text{next state is feasible for MPC!}$$

② Suppose $\mathcal{X}_f$ is control invariant. We show $\mathcal{X}_{i-1}$ is control invariant by induction. Suppose $\mathcal{X}_i$ is CI then

$$\mathcal{X}_i \subseteq \{x \in \mathcal{X} \mid u \in \mathcal{U}, f(x,u) \in \mathcal{X}_i\} = \mathcal{X}_{i-1}$$

So, let $x \in \mathcal{X}_{i-1}$. By its definition $\exists\, u$ s.t. $f(x,u) \in \mathcal{X}_i$. Yet, since $\mathcal{X}_i \subseteq \mathcal{X}_{i-1}$

$\Rightarrow f(x,u) \in \mathcal{X}_{i-1}$. Thus $\mathcal{X}_{i-1}$ is CI.



What about Stability? 4 options Guaranee it

Suppose: $\ell(x,u) \geq \beta\|x\| \quad \forall x \in \mathcal{X}, u \in \mathcal{U} \qquad (\beta > 0)$ AND $\ell(0,0) = 0$

options:

① Pick $\mathcal{X}_f = \{0\}$ (Very small $\mathcal{X}_0$ as a result ☹)

② Pick $\mathcal{X}_f$ and $\ell_f(x)$ such that

- $\ell_f(x)$ is an ECLF on $\mathcal{X}_f$ and
- $\mathcal{X}_f$ forward invariant.

(Larger $\mathcal{X}_0$ as a result)

③ Pick N "large enough" and let $\ell_f(x) = 0 \quad \mathcal{X}_f = \mathcal{X}$

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$\mathcal{X}_0$ gives basin of attraction (but may be hard to compute for nonlinear systems...)

Option 4 for Stability:

Consider an OCP w/o constraints and find $V_N^*(x)$ the OCTG for a horizon length N, such that V_N^* an ECLF for the unconstrained problem.

Let $u_N^*(x)$ the corresponding optimal policy

- V_N^* often much easier to find when ignoring constraints

Then for the constrained MPC (any horizon length you please)

- Set $l_f(x) = V_N^*(x)$ (If you are close enough to $x=0$ the constraints don't affect the tail of the cost.)

- Set $\mathcal{X}_f$ any set such that ⓐ $\mathcal{X}_f$ invariant under $u_N^*(x)$ and ⓑ $u_N^*(x) \subset \mathcal{U} \quad \forall x \in \mathcal{X}_f$

sufficient condition for forward invariance of $\mathcal{X}_f$

Choice of Terminal Set and Cost: Summary

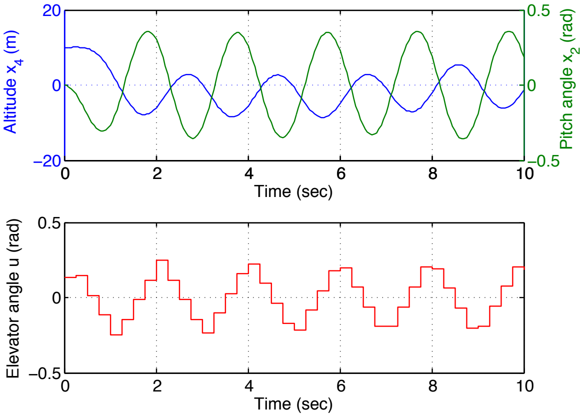
- Terminal constraint $\mathcal{X}_f = \{0\}$ provides a sufficient condition for stability
- Region of attraction without terminal constraint may be larger than for MPC with terminal constraint but characterization of region of attraction extremely difficult
- $\mathcal{X}_f = 0$ simplest choice but small region of attaction for small N
- Solution for linear systems with quadratic cost
- In practice: Enlarge horizon and check stability by sampling
- With larger horizon length N, region of attraction approaches maximum control invariant set

Example: Short horizon

MPC controller with input constraints $|u_i| \leq 0.262$ and rate constraints $|\dot{u}_i| \leq 0.349$ approximated by $|u_k - u_{k-1}| \leq 0.349\, T_s$

Problem parameters:

Sampling time 0.25sec, $Q = I$, $R = 10$, $N = 4$



Decrease in the prediction horizon causes loss of the stability properties

Example: Short horizon

MPC controller with input constraints $|u_i| \leq 0.262$ and rate constraints $|\dot{u}_i| \leq 0.349$ approximated by $|u_k - u_{k-1}| \leq 0.349\,T_s$

Problem parameters:

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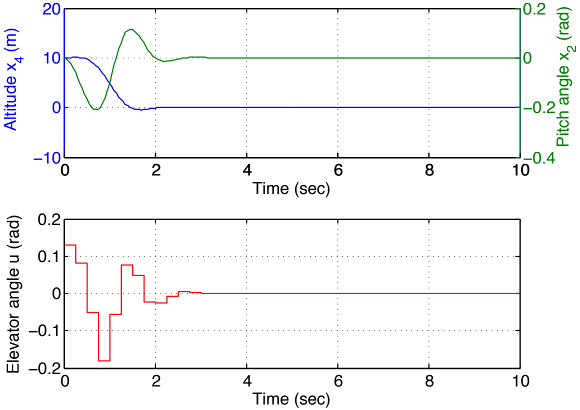

Stability of the origin can be guaranteed even with a short prediction horizon if appropriate $\mathcal{X}_f$ and l_f imposed...

Summary

Finite-horizon MPC may not be stable!

Finite-horizon MPC may not satisfy constraints for all time!

- An infinite-horizon provides stability and invariance.
- We 'fake' infinite-horizon by forcing the final state to be in an invariant set for which there exists an invariance-inducing controller, whose infinite-horizon stabilizability is captured in a control lyapunov function that serves as a final cost.
- These ideas extend to non-linear systems, but the sets are difficult to compute.