Lecture 22 - Dynamics of a rigid body

Announcements:

1. HW6 Due Fri @ 4PM, Fitz 365
2. Midterm instructor feedback Quiz on Sakai
   (Bonus points on Exam 2 for completion)

Last Time: The inertia tensor

$$^{i}I = \begin{bmatrix} ^{i}I_{xx} & -^{i}I_{xy} & -^{i}I_{xz} \\ -^{i}I_{xy} & ^{i}I_{yy} & -^{i}I_{yz} \\ -^{i}I_{xz} & -^{i}I_{yz} & ^{i}I_{zz} \end{bmatrix}$$

Parallel Axis Thm $\Rightarrow$

$$^{i}I_{xx} = {}^{c_i}I_{xx} + m_i\left(y_c^2 + z_c^2\right)$$

$$^{i}I_{xy} = {}^{c_i}I_{xy} + m_i\left(x_c y_c\right)$$

Today:

- Rotational analog of $F = ma = \frac{d}{dt}\left[mv\right]$
- Dynamics of two bodies

Dynamics of a Rigid Body

$${}^{i}\dot{v}_{C_i} = {}^{i}R_0 \left[\frac{d}{dt}\, {}^{0}v_{C_i}\right]$$

Linear:

Linear momentum

$${}^{0}F_i = \frac{d}{dt}\left[m_i\, {}^{0}v_{C_i}\right] = m_i\, {}^{0}\dot{v}_{C_i}$$

Net force on body i

$$\boxed{{}^{i}F_i = m_i\, {}^{i}\dot{v}_{C_i} \qquad \text{Newton's Equation}}$$

Angular

Angular momentum of body i about C_i

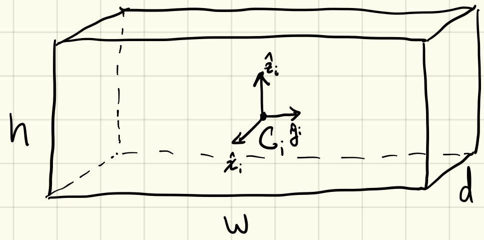
$${}^{0}N_i = \frac{d}{dt}\left[{}^{0}R_i\, {}^{C_i}I\, {}^{i}\omega_i\right]$$

Net moment on Body i about C_i

$$= {}^{0}\omega_i \times {}^{0}R_i\, {}^{C_i}I\, {}^{i}\omega_i + {}^{0}R_i\, {}^{C_i}I\, {}^{i}\dot{\omega}_i$$

$$\boxed{{}^{i}N_i = {}^{C_i}I\, {}^{i}\dot{\omega}_i + {}^{i}\omega_i \times \left[{}^{C_i}I\, {}^{i}\omega_i\right] \qquad \text{Euler's Equation}}$$

Example: Newtons Equations For Rect. Prism



$${}^{C_i}I_{xx} = \frac{m}{12}\left(w^2 + h^2\right)$$

$${}^{C_i}I_{yy} = \frac{m}{12}\left(h^2 + d^2\right)$$

$${}^{C_i}I_{zz} = \frac{m}{12}\left(d^2 + w^2\right)$$

$${}^{i}\omega_i = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \qquad {}^{i}\dot{\omega}_i = \begin{bmatrix} \dot{\omega}_x \\ \dot{\omega}_y \\ \dot{\omega}_z \end{bmatrix}$$

$${}^{C_i}I = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix}$$

$${}^{i}N_i = \begin{bmatrix} N_x \\ N_y \\ N_z \end{bmatrix} = {}^{C_i}I\,{}^{i}\dot{\omega}_i + {}^{i}\omega_i \times {}^{C_i}I\,{}^{i}\omega_i$$

$$= \begin{bmatrix} I_{xx}\dot{\omega}_x \\ I_{yy}\dot{\omega}_y \\ I_{zz}\dot{\omega}_z \end{bmatrix} + \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \times \begin{bmatrix} I_{xx}\omega_x \\ I_{yy}\omega_y \\ I_{zz}\omega_z \end{bmatrix}$$

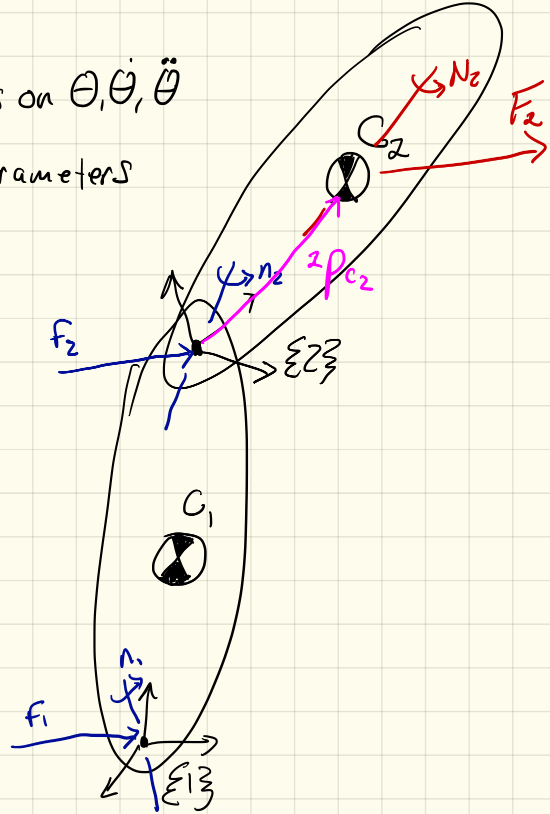
$$= \begin{bmatrix} I_{xx}\dot{\omega}_x + (I_{zz} - I_{yy})\,\omega_y\omega_z \\ I_{yy}\dot{\omega}_y + (I_{xx} - I_{zz})\,\omega_x\omega_z \\ I_{zz}\dot{\omega}_z + (I_{yy} - I_{xx})\,\omega_x\omega_y \end{bmatrix}$$

Simplified form of Euler's equations when axes $\{i\}$ match principal axes

Dynamics Analysis of Two Bodies:

Given: ${}^1\dot{v}_{c_1}, {}^2\dot{v}_{c_2}, {}^1\omega_1, {}^2\omega_2, {}^1\dot{\omega}_1, {}^2\dot{\omega}_2$ depends on $\theta, \dot{\theta}, \ddot{\theta}$

$m_1, m_2, {}^1P_{c_1}, {}^2P_{c_2}, {}^{c_1}I, {}^{c_2}I$ fixed parameters



Find: ${}^1f_1, {}^1n_1, {}^2f_2, {}^2n_2$

$$\boxed{{}^2F_2 = m_2\, {}^2\dot{v}_{c_2} = {}^2f_2}$$

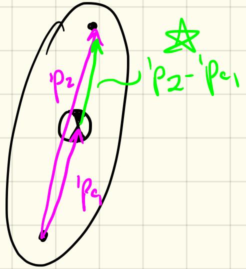
$$\begin{aligned} {}^2N_2 &= {}^{c_2}I\, {}^2\dot{\omega}_2 + {}^2\omega_2 \times {}^{c_2}I\, {}^2\omega_2 \\ &= {}^2n_2 - {}^2P_{c_2} \times {}^2f_2 \end{aligned}$$

$$\boxed{{}^2n_2 = {}^2N_2 + {}^2P_{c_2} \times {}^2F_2}$$

Analysis of Body 1 : ${}^1F_1 = m_1\,{}^1\dot{v}_{c_1}$ $\quad {}^1N_1 = {}^{c_1}I\,{}^1\dot{\omega}_1 + {}^1\omega_1 \times {}^{c_1}I\,{}^1\omega_1$

Net Force: ${}^1F_1 = {}^1f_1 - {}^1R_2\,{}^2f_2$

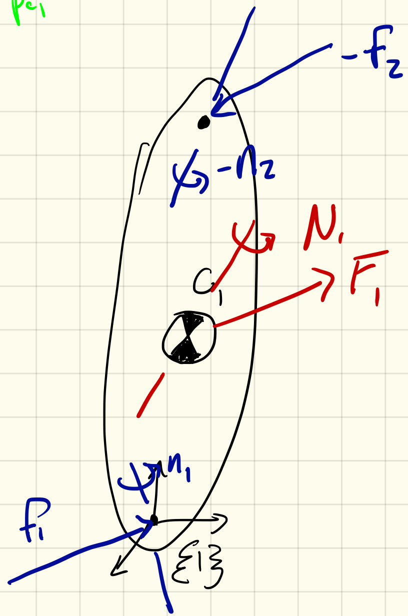
$$ {}^1f_1 = \underbrace{{}^1F_1}_{\text{Force for motion of Body 1}} + \underbrace{{}^1R_2\,{}^2f_2}_{\text{Force for motion of Body 2}} $$



Net Moment about C_1:

$$ {}^1N_1 = {}^1n_1 - {}^1R_2\,{}^2n_2 - {}^1p_{c_1} \times {}^1f_1 - ({}^1p_2 - {}^1p_{c_1}) \times {}^1R_2\,{}^2f_2 $$

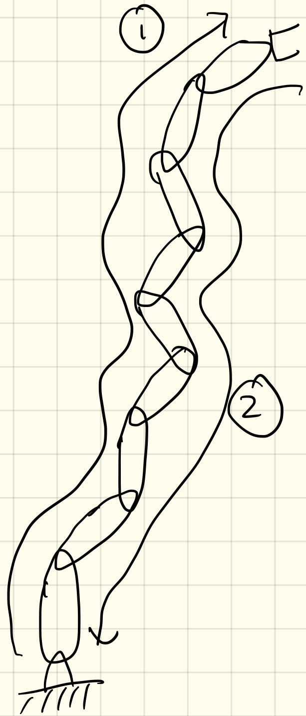
$$ {}^1n_1 = \underbrace{{}^1N_1 + {}^1p_{c_1} \times {}^1F_1}_{\text{Moment for motion of Body 1}} + \underbrace{{}^1R_2\,{}^2n_2 + {}^1p_2 \times {}^1R_2\,{}^2f_2}_{\text{Moment for motion of Body 2}} $$



We did this analysis for two bodies but you can do it for any number of them!

Inverse Dynamics Outline: Where we are headed after break

- Given $\theta, \dot{\theta}, \ddot{\theta}$ find torques to impart the motion



1. Kinematics analysis to determine
   ${}^i\dot{v}_{c_i}$, ${}^i\omega_i$, ${}^i\dot{\omega}_i$

   Newton & Euler to find ${}^iF_i$, ${}^iN_i$

2. Static force/moment propagation to find
   ${}^if_i$ and ${}^in_i$