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Analytical Dynamics

Analysis of Lagrangian Dynamics

Lagrange

Admin: Mid Semester Survey over break

Last Time: Hamilton's Principle: $\int_{t_1}^{t_2} [\delta T + \delta W]\, dt = 0 \Rightarrow \int_{t_1}^{t_2} \left[\delta \mathcal{L} + \sum_k Q_{k,nc}\, \delta q_k\right] dt = 0$ (δq sat. constr.)

Principle of Least Action: $\delta \int_{t_1}^{t_2} \mathcal{L}\, dt = 0$ (δq unc.)
(no non-conservative forces)

Lagrange's Equations: $\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_k} - \frac{\partial \mathcal{L}}{\partial q_k} = Q_{k,nc}$ $\quad k = 1, \dots, n$ $\quad$ (δq unc.) [n second order Diff eqs.

Today:
- Analyze structure of Lagrange's equations
- ~~Vibration analysis~~ (next time)

Review: Consider a scalar function $f(q, \dot{q})$:

Gradient

$$\nabla_q f = \begin{bmatrix} \partial f/\partial q_1 \\ \vdots \\ \partial f/\partial q_n \end{bmatrix}$$

Hessian

$$\nabla_q^2 f = \begin{bmatrix} \partial^2 f/\partial q_1^2 & \cdots & \partial f/\partial q_1 \partial q_n \\ \vdots & \ddots & \vdots \\ \partial^2 f/\partial q_1 \partial q_n & \cdots & \partial^2 f/\partial q_n^2 \end{bmatrix}$$

Fuzziest Point - 10-14-19

(6) D'Alembert's Principle & the Generalized Inertial Force
(5) Generalized forces and their physical meaning
(4) Nonconservative forces vs. conservative forces
(4) Constraints on generalized coordinates and on virtual displacements

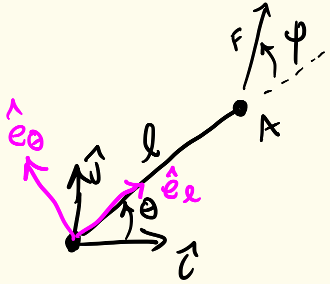
(2) Virtual Work
(2) Relationship between Euler-Lagrange equation and generalized forces
(2) Analyzing virtual displacements with partial derivatives

(1) Following notation
(1) Virtual displacements
(1) Euler Lagrange
(1) Active vs. nonconservative force
(1) Constrained Lagrangian
(1) All the different derivative types (total, partial, variation)
(1) MATLAB

Generalized Forces: A Case with a Clean Physical Interpretation

Polar Coordinates

$$q = [\ell, \theta]^T$$



Position & Velocity

$$\underline{r}_A = \ell c_\theta \hat{\imath} + \ell s_\theta \hat{\jmath}$$

$$\dot{\underline{r}}_A = \dot{\ell}\,[c_\theta \hat{\imath} + s_\theta \hat{\jmath}] + \ell\dot{\theta}\,[-s_\theta \hat{\imath} + c_\theta \hat{\jmath}] = \dot{\ell}\,\hat{e}_\ell + \ell\dot{\theta}\,\hat{e}_\theta$$

(Infinitessimal displacement over infinitessimal time)

Virtual Displacement

$$\delta \underline{r}_A = \delta\ell\,\hat{e}_\ell + \ell\,\delta\theta\,\hat{e}_\theta$$

(Infinitessimal Displacement with no change in time)

Virtual Work

$$\underline{F} = F(c_\varphi \hat{e}_\ell + s_\varphi \hat{e}_\theta)$$

Captures work done when ℓ changes by $\delta\ell$ and all other coords are fixed

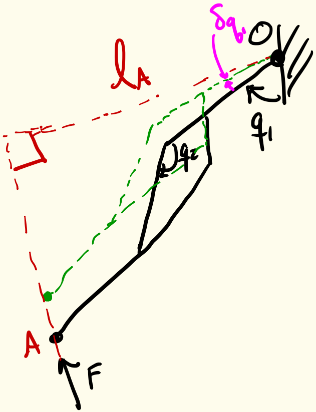
$$\delta W = \underline{F} \cdot \delta\underline{r}_A = \underbrace{F c_\varphi}_{Q_\ell}\,\delta\ell + \underbrace{F s_\varphi \ell}_{Q_\theta}\,\delta\theta$$

Generalized Forces

$Q_\ell = F c_\varphi$ (N) Force along $\hat{e}_\ell$

$Q_\theta = F \ell s_\varphi$ (Nm) Moment about the origin

Generalized Forces: Case with less clear physical interpretation



$$\delta W = Q_1 \delta q_1 + Q_2 \delta q_2$$

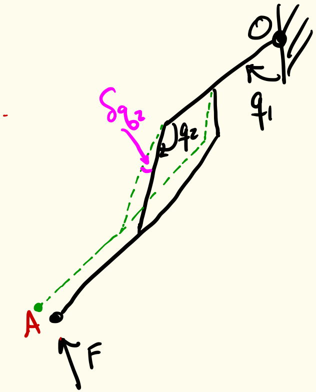
- Find Q_1 by looking at work done when q_1 changes by δq_1 and all other coords are fixed

$$\delta W = Q_1 \delta q_1 = \underbrace{l_A F}_{Nm} \delta q_1$$

moment about O (Nm)

✓ Clear meaning for Q_1

- Find Q_2 by looking at work done when q_2 changes by δq_2 and all other coords are fixed



  - $Q_2 \delta q_2$ gives δW when only q_2 is varied

  - Mathematically

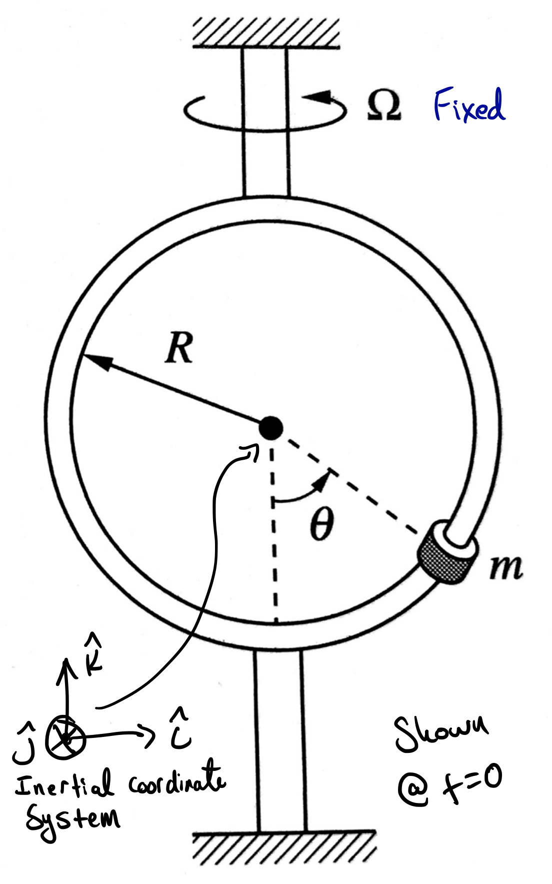
No clean form since kinematics of the four bar are complex.

$$Q_2 \delta q_2 = \delta \underline{r}_A \cdot \underline{F} = \left(\frac{\partial \underline{r}_A}{\partial q_2} \delta q_2 \right) \cdot \underline{F}$$

$$\Rightarrow Q_2 = \frac{\partial \underline{r}_A}{\partial q_2} \cdot \underline{F} \quad \text{(units, Nm)}$$

✗ No clear meaning for Q_2. It is still valid mathematically but harder to think about physically.

Natural vs. Non-Natural Systems



$$\{\underline{r}_{Bead}(\theta,t)\} = \underbrace{\begin{bmatrix} c(\Omega t) & -s(\Omega t) & 0 \\ s(\Omega t) & c(\Omega t) & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{R_3(\Omega t)} \begin{bmatrix} R s_\theta \\ 0 \\ -R c_\theta \end{bmatrix}$$

$$= \begin{bmatrix} R s_\theta \, c(\Omega t) \\ R s_\theta \, s(\Omega t) \\ -R c_\theta \end{bmatrix}$$

- When some motion is perscribed we say the system is <u>non-natural</u>
  - equivalent to $\underline{r}_i(q,t)$ depends on time for some particle in the system.

Structure of the Kinetic Energy

$$q = [q_1, \dots, q_n]^T$$

- System of particles $\underline{r}_i(q,t)$
- Velocity of each particle $\{\underline{v}_i\} = \sum_{k=1}^{n} \left\{\frac{\partial \underline{r}_i}{\partial q_k}\right\} \dot{q}_k + \left\{\frac{\partial \underline{r}_i}{\partial t}\right\}$

$$J_i = \left[\left\{\frac{\partial \underline{r}_i}{\partial q_1}\right\} \cdots \left\{\frac{\partial \underline{r}_i}{\partial q_n}\right\}\right] \quad \text{3xn matrix}$$

$$= \{J_i\}\dot{q} + \left\{\frac{\partial \underline{r}_i}{\partial t}\right\}$$

- $T = \sum_{i=1}^{N} \frac{1}{2} m_i \{\underline{v}_i\}^T\{\underline{v}_i\} = \sum_{i=1}^{N} \frac{1}{2} m_i \left[\{J_i\}\dot{q} + \left\{\frac{\partial \underline{r}_i}{\partial t}\right\}\right]^T \left[\{J_i\}\dot{q} + \left\{\frac{\partial \underline{r}_i}{\partial t}\right\}\right]$

$$= \frac{1}{2}\dot{q}^T\left[\sum_{i=1}^{N} m_i \{J_i\}^T\{J_i\}\right]\dot{q} \qquad T_2 \quad \text{quadratic in } \dot{q}$$

$$+ \left[\sum_{i=1}^{N} m_i \left\{\frac{\partial \underline{r}_i}{\partial t}\right\}^T\{J_i\}\right]\dot{q} \qquad T_1 \quad \text{Linear in } \dot{q}$$

$$+ \frac{1}{2}\sum_{i=1}^{N} m_i \left\{\frac{\partial \underline{r}_i}{\partial t}\right\}^T\left\{\frac{\partial \underline{r}_i}{\partial t}\right\} \qquad T_0 \quad \text{No gen. vels}$$

0 if a natural system

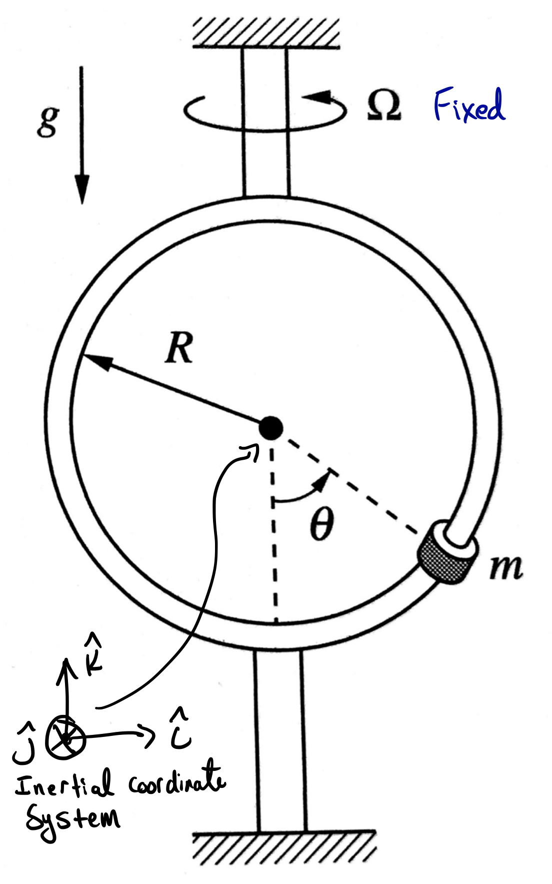
$$T = \frac{1}{2}\dot{q}^T M(q,t)\dot{q} + B(q,t)^T\dot{q} + T_0(q,t)$$

Mass matrix
nxn matrix

called the gyroscopic vector
(nx1)

Exercise
$M = \nabla^2_{\dot{q}} T$

Kinetic Energy Example: Non-Natural System



$$\{r_{BEAD}(q,t)\} = \begin{bmatrix} R s\theta \, c(\Omega t) \\ R s\theta \, s(\Omega t) \\ -R c\theta \end{bmatrix}$$

$$\{J\} = \left\{\frac{\partial r}{\partial \theta}\right\} = \begin{bmatrix} R c\theta \, c(\Omega t) \\ R c\theta \, s(\Omega t) \\ R s\theta \end{bmatrix} \qquad \left\{\frac{\partial r}{\partial t}\right\} = \begin{bmatrix} -R s\theta \, s(\Omega t)\Omega \\ R s\theta \, c(\Omega t)\Omega \\ 0 \end{bmatrix}$$

$$T_2 = \frac{1}{2}\dot{\theta}^2 m \{J\}^T\{J\} = \frac{1}{2}\dot{\theta}^2 m\left[R^2 \cancel{c^2\theta} \, \cancel{c^2\Omega t} + \cancel{R^2 c^2\theta \, s^2\Omega t} \; \cancel{+ R^2 s^2\theta}\right]$$

$$= \frac{1}{2}\dot{\theta}^2 [mR^2]$$

$$T_1 = 0$$

$$T_0 = \frac{1}{2} m \left\{\frac{\partial r}{\partial t}\right\}^T \left\{\frac{\partial r}{\partial t}\right\} = \frac{1}{2} m R^2 s_\theta^2 \, \Omega^2$$

$$= \frac{1}{2} m \left(R s_\theta \Omega\right)^2$$

$$T = \frac{1}{2}\dot{\theta}^2 (mR^2) + \frac{1}{2} m (R s_\theta \Omega)^2$$