Analytical Dynamics

D'Alembert's Principle

D'Alembert

Lagrange

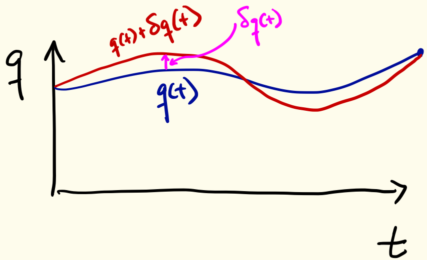
Admin:

- HW 4 Due next Wednesday

Road Map:

Virtual Work over time → (+) → Lagrangian Dynamics

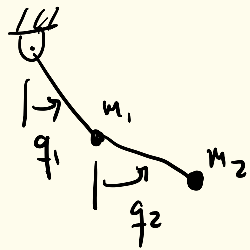
Calculus of Variations → (+)



- Revisit Generalized Forces - Conservative & Nonconservative Forces
- Introduce D'Alembert's principle (Upgrades the principle of virtual work from Statics to dynamics)
- Apply variational calculus w/ D'Alembert's principle to obtain Lagrange's equations. (next time)

Review: Setup

- System w/ N paritcles, mass m_i, position $\underline{r}_i$
- Described by n generalized coordinates q_k
- Today: No additional constraints on q
- Applied forces F_i, Constraint forces F_i'



Summary from Last Wednesday

- Constraint forces do no virtual work under virtual displacements that are consistent with the constraints.

- Virtual work of Applied Forces

$$\delta W = \sum_{i=1}^{N} \underline{F}_i \cdot \delta \underline{r}_i = \sum_{k=1}^{n} Q_k \, \delta q_k$$

- Generalized Forces

$$Q_k = \sum_{i=1}^{N} \frac{\partial \underline{r}_i}{\partial q_k} \cdot \underline{F}_i$$

Closer look @ Q_k

- At equilibrium:

$\sum Q_k \delta q_k = 0$ for all δq that satisfy constraints.

$$\delta \underline{r}_i = \sum_{k=1}^{n} \frac{\partial \underline{r}_i}{\partial q_k} \delta q_k$$

Let $J_i = \left[\frac{\partial \underline{r}_i}{\partial q_1} \cdots \frac{\partial \underline{r}_i}{\partial q_n} \right]$

$$\Rightarrow \delta \underline{r}_i = J_i \begin{bmatrix} \delta q_1 \\ \vdots \\ \delta q_n \end{bmatrix} = J_i \delta q$$

Three Methods to Compute Generalized Forces

① Write out δW and collect coefficients of variations δq_k

$$\delta W = \delta q_1 Q_1 + \dots + \delta q_n Q_n$$

Contrast between these two highlighted on next slide

② Compute $\frac{\partial \underline{r}_i}{\partial q_k}$ or $\frac{\partial \underline{v}_i}{\partial \dot{q}_k}$ (they are equal)

Derivation on 2nd next slide

$$Q_k = \sum_{i=1}^{N} \frac{\partial \underline{r}_i}{\partial q_k} \cdot \underline{F}_i \implies Q = \begin{bmatrix} Q_1 \\ \vdots \\ Q_n \end{bmatrix} = \sum_{i=1}^{N} \{J_i\}^T \{F_i\}$$

③ For conservative force field $dW = -dV \implies \delta W = -\delta V$

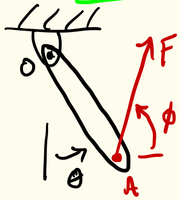
Derivation on 2nd next slide

$$\implies Q_k = -\frac{\partial V}{\partial q_k}$$

[Generalizes Force results from particles]

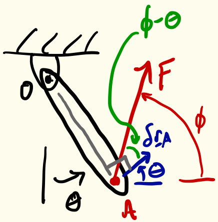
Find the generalized force Q_θ from the Applied force F shown:

EXTRA: NOT IN CLASS



$$|r_{A/O}| = \ell$$

Approach ①:



δr_A is perpendicular to the bar OA

so, it makes angle θ w/ the horizontal.

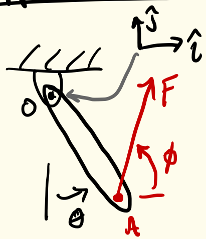
$$|\delta r_A| = \ell\,\delta\theta$$

$$\delta W = \underline{F} \cdot \delta \underline{r}_A = F\,|\delta r_A| \cos(\phi-\theta)$$

$$= F\ell\cos(\phi-\theta)\,\delta\theta \quad \Rightarrow \quad \boxed{Q_\theta = F\ell\cos(\phi-\theta)}$$

Note: We never wrote down $\underline{r}_A$ in this approach! Instead we used spatial reasoning to define the direction and magnitude of $\delta \underline{r}_A$ directly.

Approach ②:



$$\underline{r}_A = \ell(\hat{i}\,s_\theta - \hat{j}\,c\theta) \Rightarrow \frac{\partial \underline{r}_A}{\partial \theta} = \ell(c_\theta\,\hat{i} + s_\theta\,\hat{j})$$

Note: $\delta \underline{r}_A = \frac{\partial \underline{r}_A}{\partial \theta}\delta\theta$ in this case since only one GC

$$\underline{F} = F(c_\phi\,\hat{i} + s_\phi\,\hat{j})$$

$$Q_\theta = \frac{\partial \underline{r}_A}{\partial \theta} \cdot \underline{F} = F\ell\underbrace{\left[c_\theta c_\phi + s_\theta s_\phi\right]}_{\text{Trig identity } = c(\theta-\phi) = c(\phi-\theta)} = \boxed{F\ell\cos(\phi-\theta)}$$

Derivation: $Q = \sum_i \{J_i\}^T \{F_i\}$

$$Q_k = \sum_i \frac{\partial r_i}{\partial q_k} \cdot F_i$$

If we express any two vectors $\underline{a}, \underline{b}$ in a common frame then $\underline{a} \cdot \underline{b} = \{a\}^T \{b\}$. So:

$$Q_k = \sum_i \left\{\frac{\partial r_i}{\partial q_k}\right\}^T \{F_i\}$$

Stacking up all the Q_k's into a vector

$$Q = \begin{bmatrix} Q_1 \\ \vdots \\ Q_n \end{bmatrix} = \sum_i \begin{bmatrix} \{\partial r_i / \partial q_1\}^T \\ \vdots \\ \{\partial r_i / \partial q_n\}^T \end{bmatrix} \{F_i\}$$

$$= \sum_i \underbrace{\left[\left\{\frac{\partial \underline{r}_i}{\partial q_1}\right\} \cdots \left\{\frac{\partial \underline{r}_i}{\partial q_n}\right\} \right]^T}_{= \{J_i\}^T} \{F_i\}$$

$$= \sum_i \{J_i\}^T \{F_i\}$$

EXTRA: NOT IN CLASS

Derivation δW from a conservative force

Consider a potential function

$$V(r_1(q), \ldots, r_N(q))$$

Suppose $r_i = x_i \hat{\imath} + y_i \hat{\jmath} + z_i \hat{k}$

The active force on any particle i is

$$\underline{F}_i = -\frac{\partial V}{\partial x_i}\hat{\imath} - \frac{\partial V}{\partial y_i}\hat{\jmath} - \frac{\partial V}{\partial z_i}\hat{k}$$

The virtual work of these forces is

$$\delta W = \sum_{i=1}^{N} \underline{F}_i \cdot \delta \underline{r}_i$$

where

$$\delta r_i = \delta x_i \hat{\imath} + \delta y_i \hat{\jmath} + \delta z_i \hat{k}$$

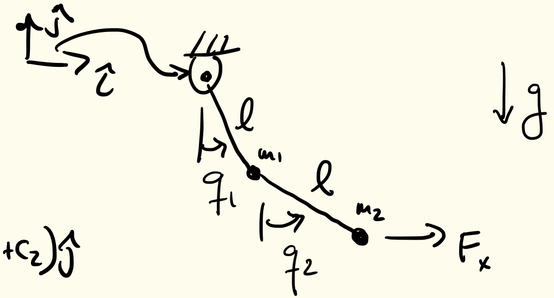
and $\delta x_i = \sum_k \frac{\partial x_i}{\partial q_k} \delta q_k$

$$= \sum_{i=1}^{N} \sum_{k=1}^{n} -\left[\frac{\partial V}{\partial x_i}\frac{\partial x_i}{\partial q_k} + \frac{\partial V}{\partial y_i}\frac{\partial y_i}{\partial q_k} + \frac{\partial V}{\partial z_i}\frac{\partial z_i}{\partial q_k}\right] \delta q_k$$

$$= -\sum_{k=1}^{n} \underbrace{\left[\sum_{i=1}^{N} \left[\frac{\partial V}{\partial x_i}\frac{\partial x_i}{\partial q_k} + \frac{\partial V}{\partial y_i}\frac{\partial y_i}{\partial q_k} + \frac{\partial V}{\partial z_i}\frac{\partial z_i}{\partial q_k}\right] \right]}_{= \frac{\partial V}{\partial q_k}!} \delta q_k$$

$$\Rightarrow \delta W = -\delta V = \sum_k -\frac{\partial V}{\partial q_k} \delta q_k$$

Example: Find the Generalized forces associated with q_1 and q_2.



① Kinematics of keypoints $\underline{r}_i(q)$

$$\underline{r}_1 = l(S_1 \hat{i} - C_1 \hat{j})$$

$$\underline{r}_2 = l(S_1 + S_2)\hat{i} - l(C_1 + C_2)\hat{j}$$

② Potential Energy $V(q)$

$$V = -m_1 g l C_1 - m_2 g l (C_1 + C_2)$$

③ Contribution of conservative forces to Q

$$Q_c = -\nabla_q V = \begin{bmatrix} -\partial V / \partial q_1 \\ -\partial V / \partial q_2 \end{bmatrix} = \begin{bmatrix} -S_1 g l (m_1 + m_2) \\ -m_2 g l S_2 \end{bmatrix}$$

④ Contribution from non conservative forces

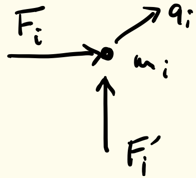
$$Q_{1,nc} = \frac{\partial \underline{r}_2}{\partial q_1} \cdot (F_x \hat{i}) = l C_1 F_x \qquad Q_{2,nc} = \frac{\partial \underline{r}_2}{\partial q_2} \cdot F_x \hat{i} = l C_2 F_x$$

⑤ $Q = Q_c + Q_{nc}$

$$Q_1 = -S_1 g l (m_1 + m_2) + l C_1 F_x$$

$$Q_2 = -S_2 g l m_2 + l C_2 F_x$$

D'Alembert's Principle



$$0 = \sum_{i=1}^{N} (F_i + F_i' - m_i a_i) \cdot \delta \underline{r}_i$$

$$= \sum_{i=1}^{N} (F_i - m_i a_i) \cdot \delta \underline{r}_i \qquad \text{if } \delta \underline{r}_i \text{ are consistent w/ constraints}$$

[D'Alembert's principle]

$$= \delta W - \sum_i m_i \ddot{\underline{r}}_i \cdot \delta \underline{r}_i$$

Using D'Alembert's principle for EoM:

we'll call this the inertial force for particle i. Just the force required for motion

$$\delta W = \sum_{k=1}^{n} Q_k \delta q_k = \sum_{k=1}^{n} \sum_{i=1}^{N} m_i \ddot{\underline{r}}_i \cdot \frac{\partial r_i}{\partial q_k} \delta q_k$$

$$\Rightarrow 0 = \sum_{k=1}^{n} \left(Q_k - \sum_{i=1}^{N} m_i \ddot{\underline{r}}_i \cdot \frac{\partial \underline{r}_i}{\partial q_k} \right) \delta q_k \qquad \forall \delta q \text{ consistent w/ constraints}$$

If no constraints

$$Q_k = \sum_{i=1}^{N} m_i \ddot{r}_i \cdot \frac{\partial r_i}{\partial q_k}$$

(Generalized applied force = Generalized Inertial force)

MATLAB Example

Use D'Alembert's principle to

Find Equations of motion.

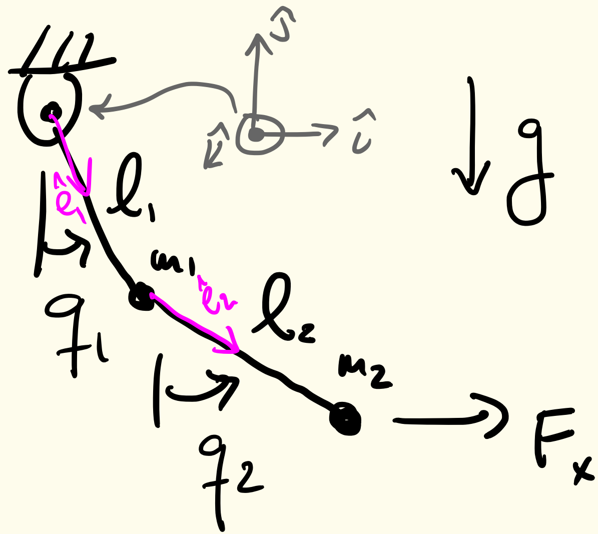
① Keypoints & Kinematics $r_i(q)$

② Generalized Active Forces $F_i = F_{i,c} + F_{i,nc}$

$$Q_{active} = -\nabla V + \sum_i \{J_i\}^T \{F_{i,nc}\}$$

③ Generalized inertial Forces

$$Q_{inertial} = \sum_{i=1}^{N} \{J_i\}^T \{m_i \ddot{r}_i\}$$



$$Q_{k,inertial} = \sum_{i=1}^{N} \frac{\partial \underline{r}_i}{\partial q_k} \cdot m_i \underline{\ddot{r}}_i$$

④ EOM $\quad Q_{active} = Q_{inertial}$

Summary

- **Generalized Forces:**

when: $\underline{F}_i = \overbrace{\underline{F}_{i,c}}^{\text{Conservative}} + \overbrace{\underline{F}_{i,nc}}^{\text{Non-Conservative}}$

	Always	when: $\underline{F}_i = \underline{F}_{i,c} + \underline{F}_{i,nc}$	
One-By-One	$Q_k = \underbrace{\sum_{i=1}^{N} \frac{\partial \underline{r}_i}{\partial q_k} \cdot \overbrace{\underline{F}_i}^{\text{Active Force}}}_{\text{Generalized Force Associated with } q_k}$	$Q_k = -\frac{\partial V}{\partial q_k} + \sum_{i=1}^{N} \frac{\partial \underline{r}_i}{\partial q_k} \cdot \underline{F}_{i,nc}$	Use this form when working by Hand
All Together	$Q = \begin{bmatrix} Q_1 \\ \vdots \\ Q_n \end{bmatrix} = \sum_{i=1}^{N} \{J_i\}^T \{\underline{F}_i\}$	$Q = -\nabla_q V + \sum_{i=1}^{N} \{J_i\}^T \{\underline{F}_{i,nc}\}$	Use this form when working w/ MATLAB

- **D'Alembert's Principle**

$$\sum_{i=1}^{N} (\underline{F}_i - m_i \underline{a}_i) \cdot \delta \underline{r}_i = 0 \quad \Rightarrow \quad \delta W = \sum_{i=1}^{N} m_i \ddot{\underline{r}}_i \cdot \delta \underline{r}_i$$
(δr_i consistent w/ constraints)

$$\sum_{k=1}^{n} Q_k \delta q_k = \sum_{k=1}^{n} \underbrace{\left[\sum_{i=1}^{N} \frac{\partial \underline{r}_i}{\partial q_k} \cdot \overbrace{(m_i \ddot{\underline{r}}_i)}^{\text{Inertial Force}} \right]}_{\text{Generalized Inertial Force Associated w/ } q_k} \delta q_k$$
($\forall \, \delta q$ consistent w/ constraints)