Lecture 17: Floating Base Systems

Today:

- DDP wrap up
  - ⇒ Code Example
  - ⇒ DDP vs. Iterative LQR
- Floating Base Systems
  - ⇒ Configuration Space & Quaternions
  - ⇒ Dynamics of Floating Base Systems
    - Features of the Dynamics

<u>DDP</u> <u>algorithm</u>: Input guess controller $\bar{u}(k,x)$

① Simulate $\bar{x}_{1:N}$ via $\bar{u}$, store nominal cost $\bar{V} = \sum \ell + \ell_f(\bar{x}_N)$

② Backward Pass

Seed $V_x^*[N] = \nabla \ell_f \big|_{\bar{x}_N}$, $V_{xx}^*[N] = \nabla_{xx}^2 \ell_f \big|_{\bar{x}_n}$

for $k = N-1:0$

Form $Q_x, Q_u, Q_{xx}, Q_{ux}, Q_{uu}, Q_{ux}$ from $\bar{x}_k, \bar{u}_k, V_x[k+1]$ $V_{xx}[k+1]$

$\delta u_k^* = -\varepsilon Q_{uu}^{-1} Q_u - Q_{uu}^{-1} Q_{ux} \delta x_k$ (Store control law)

Update $V_x[k]$, $V_{xx}[k]$ via previous slide

③ Forward Pass [Algorithm parameters: $0<\gamma<1$, $0<\beta<1$ (You pick)]

ⓐ Set $\varepsilon = 1$

ⓑ Simulate a $x_{1:N}$ via the policy $u_k = \bar{u}_k - \varepsilon Q_{uu}^{-1} Q_u - Q_{uu}^{-1} Q_{ux} \underbrace{(x_k - \bar{x}_k)}_{\delta x_k}$

ⓒ Record $V = \sum \ell(x_k, u_k) + \ell_f(x_N)$, $\Delta V = \sum_{i=1}^{N} -\varepsilon\left(1-\frac{\varepsilon}{2}\right) Q_u^T Q_{uu}^{-1} Q_u$

ⓓ If $V < \bar{V} + \gamma \Delta V$ <u>then</u> go to ⓔ <u>else</u> $\varepsilon = \beta\varepsilon$ and go to ⓑ

ⓔ $\bar{x}_{1:N} \Leftarrow x_{1:N}$, $\bar{u}_{1:N} \Leftarrow u_{1:N}$, $\bar{V} \Leftarrow V$

Until convergence

Remarks:

- Invented by DQ Mayne in 1960s, Popularized by Todorov in 2010s

- Some additional care must be taken s.t. Backward pass $Q_{uu} \succ 0$ & K (Todorov 2012)

- If you change from $Q_{xx} = H_{xx} + f_x^T V_{xx} f_x$ $\quad H = \ell(x,u) + \lambda^T f(x,u)$

  To $Q_{xx} = \ell_{xx} + f_x^T V_{xx} f_x$

  (i.e. ignore 2nd derivatives of f)

  Then you get Iterative LQR (iLQR)

  - iLQR has superlinear convergence but BW pass provably has $Q_{uu} \succ 0$ if

$$\begin{bmatrix} \ell_{xx} & \ell_{xu} \\ \ell_{ux} & \ell_{uu} \end{bmatrix} \succ 0$$

(i.e., if running cost concave up)

  - Also called Sequential Linear Quadratic (SLQ) in community

SKIPPED IN CLASS BUT FYI:

Continuous-Time DDP: Backward Pass

$$① \quad -\dot{V}_{xx} = H_{xx} + f_x^T V_{xx} + V_{xx} f_x - \left[H_{ux} + f_u^T V_{xx}\right]^T H_{uu}^{-1} \left[H_{ux} + f_u^T V_{xx}\right]$$

$$② \quad -\dot{V}_x = H_x - \left[H_{ux} + f_u^T V_{xx}\right]^T H_{uu}^{-1} H_u$$

$$③ \quad -\Delta \dot{V} = -\varepsilon\left(1 - \frac{\varepsilon}{2}\right) H_u^T H_{uu}^{-1} H_u$$

In iLQR case ① is equivalent to: Ricatti Differential Equation

$$-\dot{P} = Q + A^T P + PA - \left[S + B^T P\right]^T R^{-1} \left[S + B^T P\right]$$

Derivation of Previous Slide

$$V^*(\underbrace{\bar{x}(t)+\delta x}_{x}, t) = \bar{V}(t) + \Delta V(t) + V_x(t)^T \delta x + \tfrac{1}{2}\delta x^T V_{xx}(t)\,\delta x$$

$$= \bar{V}(t) + \Delta V(t) + V_x(t)^T[x - \bar{x}(t)] + \tfrac{1}{2}[x-\bar{x}(t)]^T V_{xx}(t)[x - \bar{x}(t)]$$

HJB:

$$\left[\frac{\partial V^*}{\partial x}\right]^T = V_x(t) + V_{xx}(t)\,\delta x$$

$$0 = \min_{\delta u}\left[\ell(\bar{x}+\delta x, \bar{u}+\delta u) + \frac{\partial}{\partial t}V^*(x,t) + \boxed{\frac{\partial V^*}{\partial x}}\, f(\bar{x}+\delta x, \bar{u}+\delta u)\right]$$

$$= \min_{\delta u}\left[H(\bar{x}+\delta x, \bar{u}+\delta u, V_x + V_{xx}\delta x) + \frac{\partial}{\partial t}V^*\right] \quad (*)$$

Find δu^*

$$\delta u^* = \underset{\delta u}{\operatorname{argmin}}\left[\delta u^T H_u + \tfrac{1}{2}\delta u^T H_{uu}\delta u + \delta u^T H_{ux}\delta x + \delta u^T \overset{f_u^T}{H_{u\lambda}} V_{xx}\delta x\right]$$

$$= -H_{uu}^{-1}\left[\varepsilon H_u + \overset{:=L}{\left[H_{ux} + f_u^T V_{xx}\right]}\delta x\right]$$

Plug δu^* into $(*)$:

$$0 = H(\bar{x},\bar{u},V_x) + \delta x^T H_x + \delta u^T H_u + \delta x^T V_{xx} H_\lambda + \tfrac{1}{2}\delta x^T H_{xx}\delta x + \tfrac{1}{2}\delta u^T H_{uu}\delta u$$
$$+ \delta x^T H_{xu}\delta u + \delta x^T H_{x\lambda} V_{xx}\delta x + \delta u^T H_{u\lambda} V_{xx}\delta x + \frac{\partial}{\partial t}V^*$$

$$= \ell(\bar{x},\bar{u}) + V_x^T f(\bar{x},\bar{u}) + \delta x^T\left[H_x - L^T H_{uu}^{-1} H_u\right] - \varepsilon H_u^T H_{uu}^{-1} H_u + \delta x^T V_{xx} f(\bar{x},\bar{u}) + \tfrac{1}{2}\delta x^T H_{xx}\delta x$$
$$+ \tfrac{1}{2}[\varepsilon H_u + L\delta x]^T H_{uu}^{-1}[\varepsilon H_u + L\delta x] - [H_{ux}\delta x]^T H_{uu}^{-1}[\varepsilon H_u + L\delta x] + \delta x^T f_x^T V_{xx}\delta x - [f_u^T V_{xx}\delta x]^T H_{uu}^{-1}[\varepsilon H_u + L\delta x] + \frac{\partial}{\partial t}V^*$$

Collecting terms...

$$= \ell(\bar{x},\bar{u}) + [V_x + V_{xx}\delta x]^T f(\bar{x},\bar{u}) - \varepsilon\left(1-\tfrac{\varepsilon}{2}\right)H_u^T H_{uu}^{-1}H_u + \delta x^T\left[H_x - L^T H_{uu}^{-1}H_u\right] + \tfrac{1}{2}\delta x^T\left[H_{xx} - L^T H_{uu}^{-1} L_u + F_x^T V_{xx} + V_{xx} f_x\right]\delta x + \frac{\partial}{\partial t}V^*$$

Notation

Scalar funtion: Subsript gives gradient

$$H_u := \nabla_u H\Big|_{\bar{x},\bar{u},V_x}$$

Vector funtion: Subscript gives partial

$$f_x := \frac{\partial}{\partial x} f\Big|_{\bar{x}}$$

Partials go from left to right

$$H_{ux} := \left[\frac{\partial}{\partial x} H_u\right]\Bigg|_{\bar{x},\bar{u},V_x}$$

Other Examples:

$$H_\lambda = f\Big|_{\bar{x},\bar{u}}$$

$$H_{\lambda x} = f_x\Big|_{\bar{x},\bar{u}}$$

$$H_{x\lambda} = (H_{\lambda x})^T = f_x^T$$

From Previous Slide:

$$V^*(x,t) = \bar V(t) + \Delta V(t) + V_x(t)^T\left[x - \bar x(t)\right] + \tfrac{1}{2}\left[x - \bar x(t)\right]^T V_{xx}(t)\left[x - \bar x(t)\right]$$

$$0 = \ell(\bar x,\bar u) + \left[V_x + V_{xx}\delta x\right]^T f(\bar x,\bar u) - \varepsilon\left(1-\tfrac{\varepsilon}{2}\right) H_u^T H_{uu}^{-1} H_u + \delta x^T\left[H_x - L^T H_{uu}^{-1} H_u\right] + \tfrac{1}{2}\delta x^T\left[H_{xx} - L^T H_{uu}^{-1} L_u + f_x^T V_{xx} + V_{xx} f_x\right]\delta x + \frac{\partial}{\partial t} V^*$$

Using the form of V^*:

$$0 = \cancel{\ell(\bar x,\bar u)} + \cancel{\left[V_x + V_{xx}\delta x\right]^T f(\bar x,\bar u)} - \varepsilon\left(1-\tfrac{\varepsilon}{2}\right) H_u^T H_{uu}^{-1} H_u + \delta x^T\left[H_x - L^T H_{uu}^{-1} H_u\right] + \tfrac{1}{2}\delta x^T\left[H_{xx} - L^T H_{uu}^{-1} L_u + f_x^T V_{xx} + V_{xx} f_x\right]\delta x$$

$$+ \cancel{\dot{\bar V}} + \Delta\dot V + \dot V_x^T \delta x - \cancel{V_x^T f(\bar x,\bar u)} + \tfrac{1}{2}\delta x^T \dot V_{xx}\delta x - \cancel{f(\bar x,\bar u)^T V_{xx}\delta x}$$

rate of change in cost to go along $\bar x$ is $-\ell(\bar x,\bar u)$

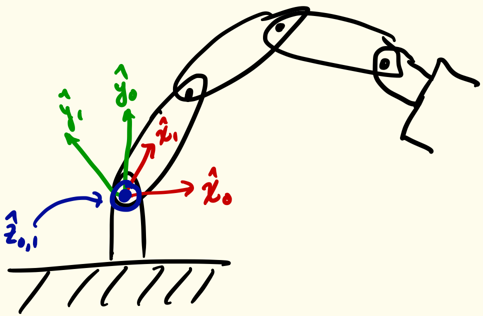
Matching Terms:

$$-\frac{d}{dt}\Delta V(t) = -\varepsilon\left(1-\tfrac{\varepsilon}{2}\right) H_u^T H_{uu}^{-1} H_u$$

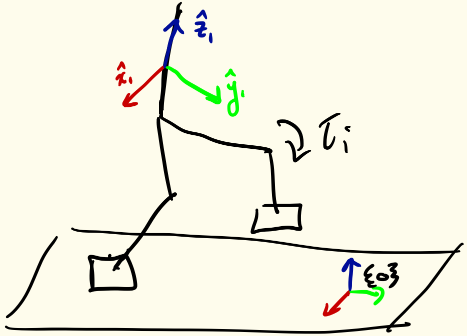
$$-\frac{d}{dt} V_x(t) = H_x - \left[H_{xu} + V_{xx} f_u\right] H_{uu}^{-1} H_u$$

$$-\frac{d}{dt} V_{xx}(t) = H_{xx} + f_x^T V_{xx} + V_{xx} f_x - \left[H_{xu} + V_{xx} f_u\right] H_{uu}^{-1}\left[H_{ux} + f_u^T V_{xx}\right]$$

Back To Modeling: Floating Base Systems



Assume
n joints
1 DoF each.



Fixed Base System

- $q \in \mathbb{R}^n$
- Connection between $\{1\}$ and $\{0\}$ is 1 DoF

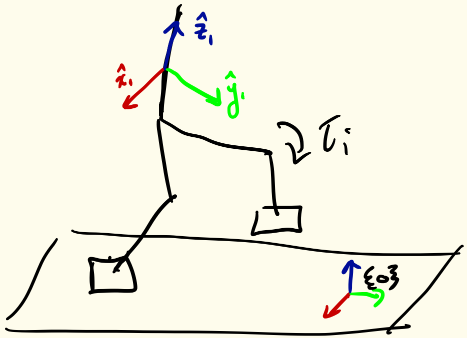
Floating-Base System

- Attach Frame $\{1\}$ to trunk
- Connection between $\{0\}$ and $\{1\}$ 6 DoF
- $q =$ ("overall" position/orientation, Body shape)

$= (q_B, q_J)$ $\quad q_J \in \mathbb{R}^n$ $\quad q_B \in ?$

Configuration Manifold for Torso

$q_B = (\text{position } \{1\}, \text{ orientation } \{1\})$



Special-Orthogonal Group:

$$SO(n) = \{ X \in \mathbb{R}^{n \times n} \mid X^T X = I, \det(X) = 1 \}$$

$SO(3)$ is the set of all rotation matrices

Special Euclidean Group:

$$SE(n) = \left\{ \begin{bmatrix} X & y \\ 0 & 1 \end{bmatrix} \,\middle|\, X \in SO(n), y \in \mathbb{R}^n \right\}$$

$SE(3)$ is the set of all Homogeneous Transforms

$$\boxed{q_B \in SE(3)}$$

Issue for Simulation:

- Euler angles (and many other) parameterizations of SO(3)) have singularities

- Consider $(\alpha,\beta,\gamma) \in \mathbb{R}^3$ $\quad {}^0R_1(\alpha,\beta,\gamma)$

$$ {}^0\dot{R}_1 = \frac{\partial R}{\partial \alpha}\dot{\alpha} + \frac{\partial R}{\partial \beta}\dot{\beta} + \frac{\partial R}{\partial \gamma}\dot{\gamma} $$

$$ = {}^0R_1 S({}^1\omega_1) $$

$(*)$ $\quad {}^0R_1^T \, {}^0\dot{R}_1 = S({}^1\omega_1)$ $\quad$ given $\alpha,\beta,\gamma,\dot{\alpha},\dot{\beta},\dot{\gamma} \Rightarrow$ you can find ${}^1\omega_1$

However: Sometimes given ${}^1\omega_1$ there are no $\dot{\alpha},\dot{\beta},\dot{\gamma}$ that satisfy $(*)$

- Known as gymbal lock

Remedy: Quaterions

Recall: Angle axis $\;{}^0R_1 = e^{S(\hat{k})\theta} \qquad \|\hat{k}\|=1$

Rodrigues Formula: $\;{}^0R_1 = I + \sin\theta\, S(\hat{k}) + (1-\cos\theta)\, S(\hat{k})^2$

Unit quaternions: $\varepsilon = [\varepsilon_0 \,;\, \varepsilon_{1:3}] \in \mathbb{R}^4 \qquad \|\varepsilon\| = 1$

$$= \left[\cos\left(\tfrac{\theta}{2}\right) ;\; \hat{k}\sin\left(\tfrac{\theta}{2}\right)\right]$$

Via Double angle

$$\hat{k}\sin(\theta) = 2\sin\left(\tfrac{\theta}{2}\right)\cos\left(\tfrac{\theta}{2}\right)\hat{k} \qquad \Rightarrow \quad 2\varepsilon_0 S(\varepsilon_{1:3}) = \sin(\theta) S(\hat{k})$$

$$\boxed{{}^0R_1 = I + 2\varepsilon_0 S(\varepsilon_{1:3}) + 2S(\varepsilon_{1:3})^2}$$

Quaterion Properties:

Defn: Quaternion Multiplication

$$\left[\varepsilon \otimes\right] = \varepsilon_0 I + \begin{bmatrix} 0 & -\varepsilon_{1:3}^T \\ \varepsilon_{1:3} & S(\varepsilon_{1:3}) \end{bmatrix}$$

$\left[{}^1\varepsilon \otimes\right] {}^2\varepsilon$ written as ${}^1\varepsilon \otimes {}^2\varepsilon$

(1) Quaternion Mult is associative

$$\left[{}^1\varepsilon \otimes\right]\left[{}^2\varepsilon \otimes\right]{}^3\varepsilon = \left[{}^1\varepsilon \otimes {}^2\varepsilon\right] \otimes {}^3\varepsilon = {}^1\varepsilon \otimes \left[{}^2\varepsilon \otimes {}^3\varepsilon\right]$$

Defn: Quaternion Conjugate

$${}^A\varepsilon_B = \left[\varepsilon_0 \; ; \; \varepsilon_{1:3}\right]$$

$${}^A\varepsilon_B^* := \left[\varepsilon_0 \; ; \; -\varepsilon_{1:3}\right]$$

(2) Conjugate is inverse

$${}^B\varepsilon_A = {}^A\varepsilon_B^*$$

(3) Rotating Vectors

$${}^A R_B \, {}^B r = {}^A r$$

$$\Rightarrow \begin{bmatrix} 0 \\ {}^A r \end{bmatrix} = {}^A\varepsilon_B \otimes \begin{bmatrix} 0 \\ {}^B r \end{bmatrix} \otimes {}^A\varepsilon_B^*$$

(4) Chaining Rotations

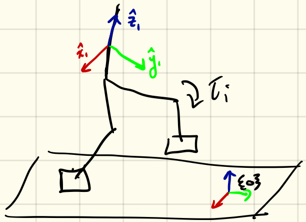
$${}^A\varepsilon_C = {}^A\varepsilon_B \otimes {}^B\varepsilon_C$$

(5) Rate of change in quat

$${}^0\dot{\varepsilon}_1 = \frac{1}{2}\begin{bmatrix} 0 \\ {}^0\omega_1 \end{bmatrix} \otimes {}^0\varepsilon_1$$

$$= \frac{1}{2} {}^0\varepsilon_1 \otimes \begin{bmatrix} 0 \\ {}^1\omega_1 \end{bmatrix}$$

Dynamics of Legged Systems:



- Attach a frame $\{1\}$ to the body and let $q_B \in SE(3)$ denote the configuration of $\{1\}$ relative to $\{0\}$

Stored on Computer with ${}^0p_1$, ${}^0\epsilon_1$

$$q = [q_B, q_J]$$

- With a slight liberty of notation

$$\dot{q} = [{}^1v_1^T, \dot{q}_J]$$

Dynamics

Top 6 rows indirectly actuated through contacts

$$H(q)\ddot{q} + C\dot{q} + G(q) = \begin{bmatrix} 0 \\ \tau_J \end{bmatrix} + \sum_c J_c^T f_c$$

Bottom n rows directly actuated

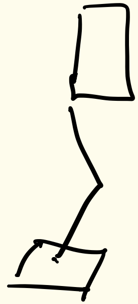
$$\begin{bmatrix} H_{bb} & H_{bj} \\ H_{jb} & H_{jj} \end{bmatrix} \begin{bmatrix} {}^1\dot{v}_1 \\ \ddot{q}_J \end{bmatrix} + \begin{bmatrix} C_b + G_b \\ C_J + G_J \end{bmatrix} =$$

Understanding The First 6 Rows

Contact Jacobian:

$$ {}^{foot}V_{foot} = {}^{foot}J_{foot}\,\dot{q} = \begin{bmatrix} J_b & J_J \end{bmatrix} \begin{bmatrix} {}^{1}V_1 \\ \dot{q}_J \end{bmatrix} $$

$$ = \underbrace{J_b}_{{}^{foot}X_1}\, {}^{1}V_1 + J_J\,\dot{q}_J $$



TO BE CONTINUED