"/

# Analytical Dynamics

## Generalized Coordinates, Constraints, & Virtual Work

Admin:

- Exam 1 corrections Due next Monday 10/7
- HW 3 Due next Wednesday 10/9 ⟵ 2D Rigid Body Review & Example Problem on Panopto

Goals for Today:

Fundamentals of Analytical Mechanics

- Generalized Coordinates
- Constraints on Particles / Systems of Particles
- Virtual Dispacements & Fundamental Axiom of Analytical Mechanics

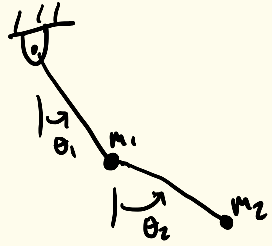
- Virtual Work

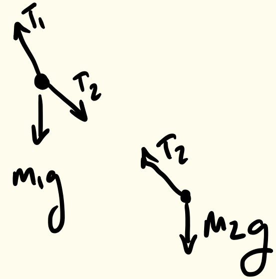
# Contrast Between Newtonian and Analytical Approaches

## Newtonian

- Centered on vectorial quantities (Momentum, velocity, etc.)

- Requires significant creativity to obtain EOM
    - Combining Newton's equation for multiple particles / bodies
    - Sampling the result along carefully chosen directions to avoid constraint forces




## Analytical

- Centered on scalar quantities (work, energy, etc)

- Less creativity to obtain EOM, but more caluclus

- Automatic way to eliminate constraint forces from analysis

## Generalized Coordinates:

- Generalized coordinates: A set of parameters that uniquely defines the configuration of a mechanism.
    - Denoted by $q$
    - GC often chosen so that $\#GC = \#DoFs$
    - Many possible definitions are valid for a given system.

      e.g., Double pendulum

      $$q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} \text{(Absolute Angles)} \quad \text{OR} \quad q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} \theta_1 \\ \theta_2 - \theta_1 \end{bmatrix} \text{(Relative Angles)}$$

      Valid choice / Also Valid

    - Any particle $i$ has position given by some function $\underline{r}_i(q, t)$
- Generalized Velocity: Time derivative of GC $\Rightarrow \dot{q}$
- State Space: $\begin{bmatrix} q \\ \dot{q} \end{bmatrix}$

## Constraints:

Sometimes we use more GC than needed, and then we add constraints

- Constraint Eqn: $f(q,t)=0$ or $g(q,\dot{q},t)=0$

  today      After Exam 2

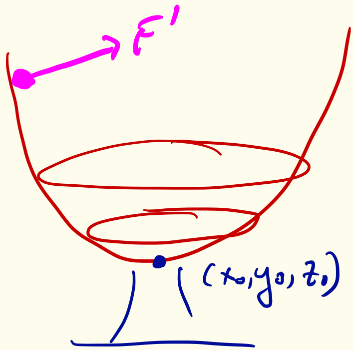
  Describe geometry and/or kinematics of contact between bodies or particles

Example: Particle in a bowl

$$\mathbf{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k} \qquad q = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Constraint: $z - z_0 = (x-x_0)^2 + (y-y_0)^2$

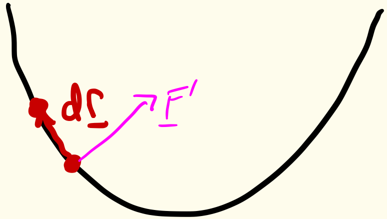
Any particle on a surface can be described by $f(x,y,z,t)=0$



# Constraints (cont.):

Particle on a Surface: Pfaffian form:

$$0 = df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial t}dt$$

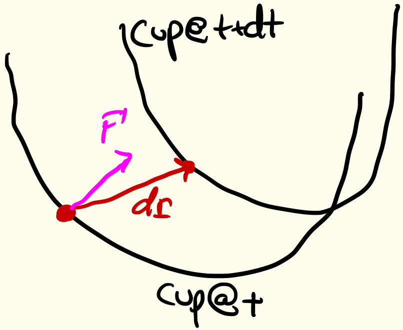



(A) Surface is Stationary $\left(\frac{\partial f}{\partial t} = 0\right)$

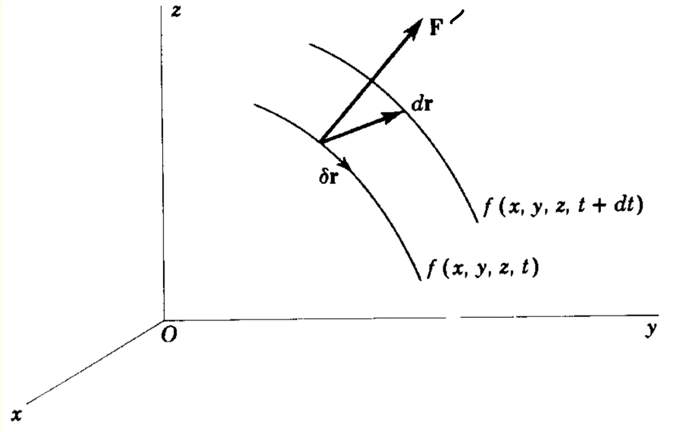
$dW = \underline{F}' \cdot d\underline{r} = 0$ workless Constraint

(B) Surface is moving $\left(\frac{\partial f}{\partial t} \neq 0\right)$

$\Rightarrow$ constraint may do work!



# Virtual Displacement:

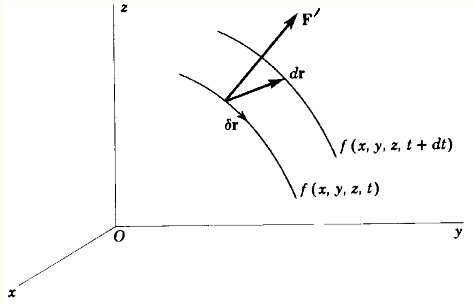



- Features
  - Infinitessimal
  - Occur instantaneously (time fixed)
- Notation
  - $\delta \underline{r}$ for a particle
  - $\delta \underline{q}$ for a system

- Variation operator (just like d but w/ time fixed)
  - $\delta \underline{r} = \delta x \hat{\imath} + \delta y \hat{\jmath} + \delta z \hat{k}$ (if $\hat{\imath}, \hat{\jmath}, \hat{k}$ inertial)
  - $\delta f = \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial z}\delta z$ (if $\delta f = 0$ we say that $\delta \underline{r}$ are consistent w/ constraints)

# Virtual Work:

$$\delta W = \underline{F}' \cdot \delta \underline{r}$$




- Particle on a surface
  - $\underline{F}'$ is normal to surface
  - $\delta \underline{r}$ is tangent to the surface if consistent w/ constraint

$\Rightarrow F'$ do no virtual work under virtual displacements consistent w/ the constraint!

For a system of particles $\underline{r}_1, \ldots, \underline{r}_N$

Virtual work from a set of forces $\underline{F}_1, \ldots, \underline{F}_N$

$$\delta W = \sum_{i=1}^{N} \underline{F}_i \cdot \delta \underline{r}_i$$

Recall: $\underline{r}_i(\underline{q}, t)$

$$\underline{q} = \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix}$$

$$\delta \underline{r}_i = \sum_{k=1}^{n} \frac{\partial \underline{r}_i}{\partial q_k} \delta q_k$$

$$= \begin{bmatrix} \frac{\partial \underline{r}_i}{\partial q_1} & \cdots & \frac{\partial \underline{r}_i}{\partial q_n} \end{bmatrix} \begin{bmatrix} \delta q_1 \\ \vdots \\ \delta q_n \end{bmatrix}$$

$$= J_i \, \delta \underline{q}$$

Jacobian of $\underline{r}_i$ w.r.t. $\underline{q}$

# Lions, Tigers, and Bears of calculus Notation:

$d$: (Pronounced "Dee") a total differential (e.g., $dm$ or $d\underline{r}$)

- $d\underline{r}$ is an infinitessimal dispacement that occurs over an infinitessimal amount of time $dt$

$\delta$: (Pronounced "delta") a variation (e.g., $\delta\underline{r}$ of $\delta q$)

- Rougly, same as $d$ but ignoring time changes (see Literature folder on Sakai for other explanations)
- $\delta\underline{r}$ is a virtual displacement. It occurs instantaneously without any change in time.

$\partial$: (Pronounced "partial" or sometimes "del") Used to indicate partial derivatives

- e.g. $\frac{\partial F}{\partial q_i}$
- $\partial q_i$ by itself is nonsense.

# Fundamental Axiom of Analytical Mechanics

I call it an axiom because I am not aware of a proof of it from any other first principles.

Consider a system of N particles with a set of M Pfaffian constraints:

$$(*) \quad a_{j1}\delta q_1 + \cdots + a_{jn}\delta q_n = 0 \qquad j=1,\dots,M$$

And net constraint forces of each particle $F_i'$

The total virtual work of the constraint forces is zero under any virtual displacements that are consistent with the constraints.

$$0 = \delta W = \sum_{i=1}^{N} \underline{F}_i' \cdot \delta \underline{r}_i = \sum_{i=1}^{N} \sum_{k=1}^{n} F_i' \cdot \frac{\partial r_i}{\partial q_k} \delta q_k$$

For any $\delta q$ that satisfies $(*)$

# Virtual Work and Equilibrium:

$$R_i = F_i + F_i'$$

Resultant, Applied Forces, Constraint Forces

**Virtual Work** $\delta W = \sum R_i \cdot \delta \underline{r}_i$

**At Equilibrium:** $R_i = 0 \Rightarrow \delta W = 0 = \sum (F_i + F_i') \cdot \delta \underline{r}_i$

$$= \sum_i F_i \cdot \delta r_i \quad \text{if } \delta r_i \text{ consistent w/ constraints}$$

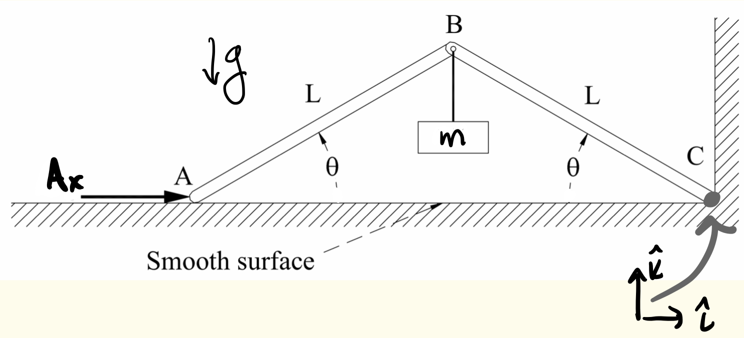
$$0 = \sum_K \underbrace{\sum_i F_i \cdot \frac{\partial r_i}{\partial q_K}}_{:= Q_K} \delta q_K = \sum_K Q_K \delta q_K$$

$:= Q_K$ generalized force associated w/ $q_K$

If no constraints on $q$ then all $Q_K = 0$ at equilibrium!

# Example: Give the force Ax necessary to hold the system at equilibrium

(Bars massless)




$q = z_B$

$\boxed{q = \theta}$

$$\delta W = Q_\theta \, \delta\theta$$

$$= \left[\frac{\partial r_B}{\partial \theta} \cdot F_B + \frac{\partial r_A}{\partial \theta} \cdot F_A\right] \delta\theta$$

Key Points

$$\underline{r}_B = -Lc_\theta \hat{i} + Ls_\theta \hat{k}$$

$$\underline{r}_A = -2Lc_\theta \hat{i}$$

Virtual work

$$\delta W = \Big[ [Ls_\theta \hat{i} + \underline{Lc_\theta \hat{k}}] \cdot [\underline{-mg\hat{k}}] + [2Ls_\theta \hat{i}] \cdot [A_x \hat{i}] \Big] \delta\theta$$

$$= [-mgLc_\theta + 2Lc_\theta A_x] \delta\theta$$

$$\boxed{A_x = \frac{mgc_\theta}{2s_\theta} = \frac{mg}{2}\cot(\theta)}$$

# Summary

- Constraint forces do no virtual work under virtual displacements that are consistent with the constraints.

- Virtual work of Applied Forces

$$\delta W = \sum_{i=1}^{N} \underline{F}_i \cdot \delta \underline{r}_i = \sum_{k=1}^{n} Q_k \, \delta q_k$$

- Generalized Forces

$$Q_k = \sum_{i=1}^{N} \frac{\partial \underline{r}_i}{\partial q_k} \cdot \underline{F}_i$$

conservative & non-conservative parts:

$$\boxed{Q_k = -\frac{\partial V}{\partial q_k} + Q_{k,nc}}$$

We'll see this next week...

- At equilibrium:

$$\sum Q_k \, \delta q_k = 0$$ for all $\delta q$ that satisfy constraints.