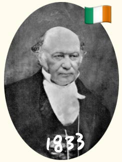
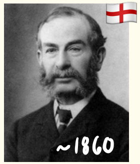
Analytical Dynamics

Integrals of Motion, Routhian Reduction, Hamilton's Equations



Hamilton



Routh

Noether

Last Time:

- Kinetic Energy: $T = \overbrace{\frac{1}{2}\dot{q}^T M(q,t)\,\dot{q}}^{T_2} + \overbrace{\beta^T(q,t)\,\dot{q}}^{T_1} + T_0(q,t)$

- Equations of Motion: $\left.\frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{q}_k}\right] - \left[\frac{\partial \mathcal{L}}{\partial q_k}\right] = Q_{k,nc}\ \right]$ n second-order ODEs

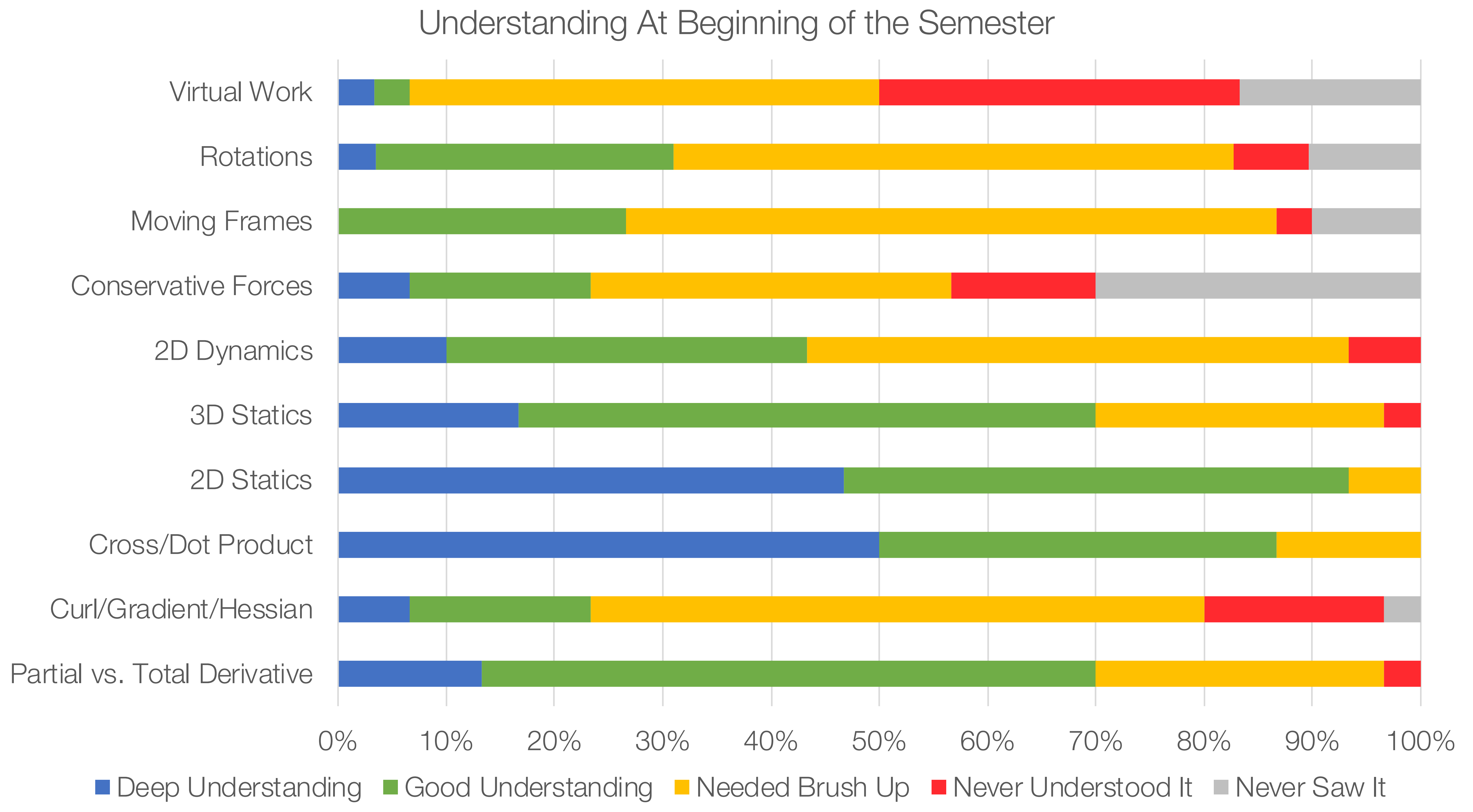
Today: ⓪ Survey results & Modal wrap up

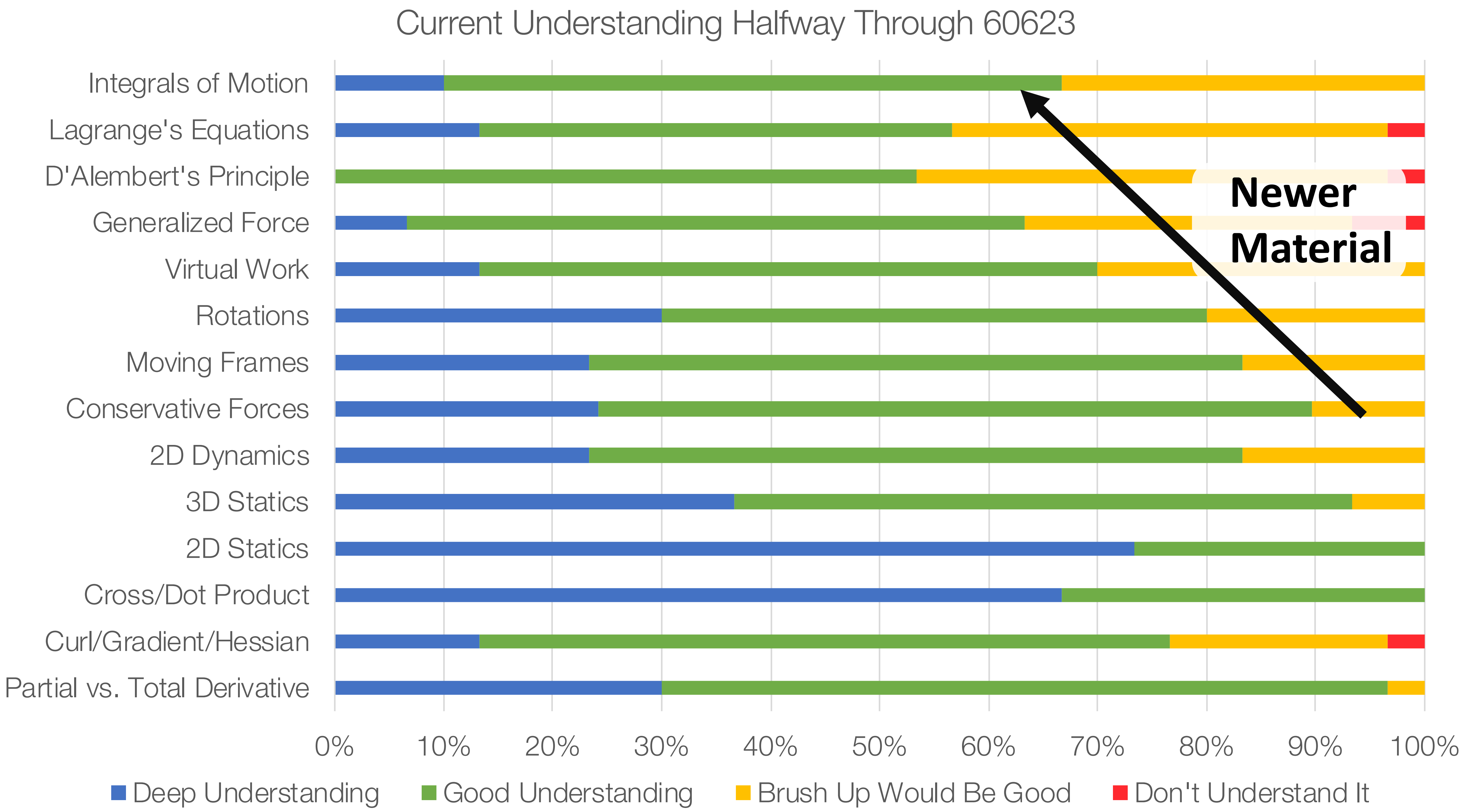
① Generalized Momentum (Analog of generalized force)

② Use integrals of motion to reduce # of ODEs we must solve (Routhian Reduction)

③ Alternate form of EOM as 2n first-order ODEs

$$\left(\underset{\dot{q}}{\text{Gen. Coords}},\ \underset{\dot{q}}{\text{Gen. vels.}}\right) \longrightarrow \left(\underset{q}{\text{Gen. coords}},\ \underset{\pi}{\text{Gen. momentum}}\right)$$


Understanding At Beginning of the Semester
Virtual Work
Rotations
Moving Frames
Conservative Forces
2D Dynamics
3D Statics
2D Statics
Cross/Dot Product
Curl/Gradient/Hessian
Partial vs. Total Derivative
0%
10%
20%
30%
40%
50%
60%
70%
80%
90%
100%
Deep Understanding
Good Understanding
Needed Brush Up
Never Understood It
Never Saw It



Current Understanding Halfway Through 60623
Integrals of Motion
Lagrange's Equations
D'Alembert's Principle
Generalized Force
Virtual Work
Rotations
Moving Frames
Conservative Forces
2D Dynamics
3D Statics
2D Statics
Cross/Dot Product
Curl/Gradient/Hessian
Partial vs. Total Derivative
Newer Material
0%
10%
20%
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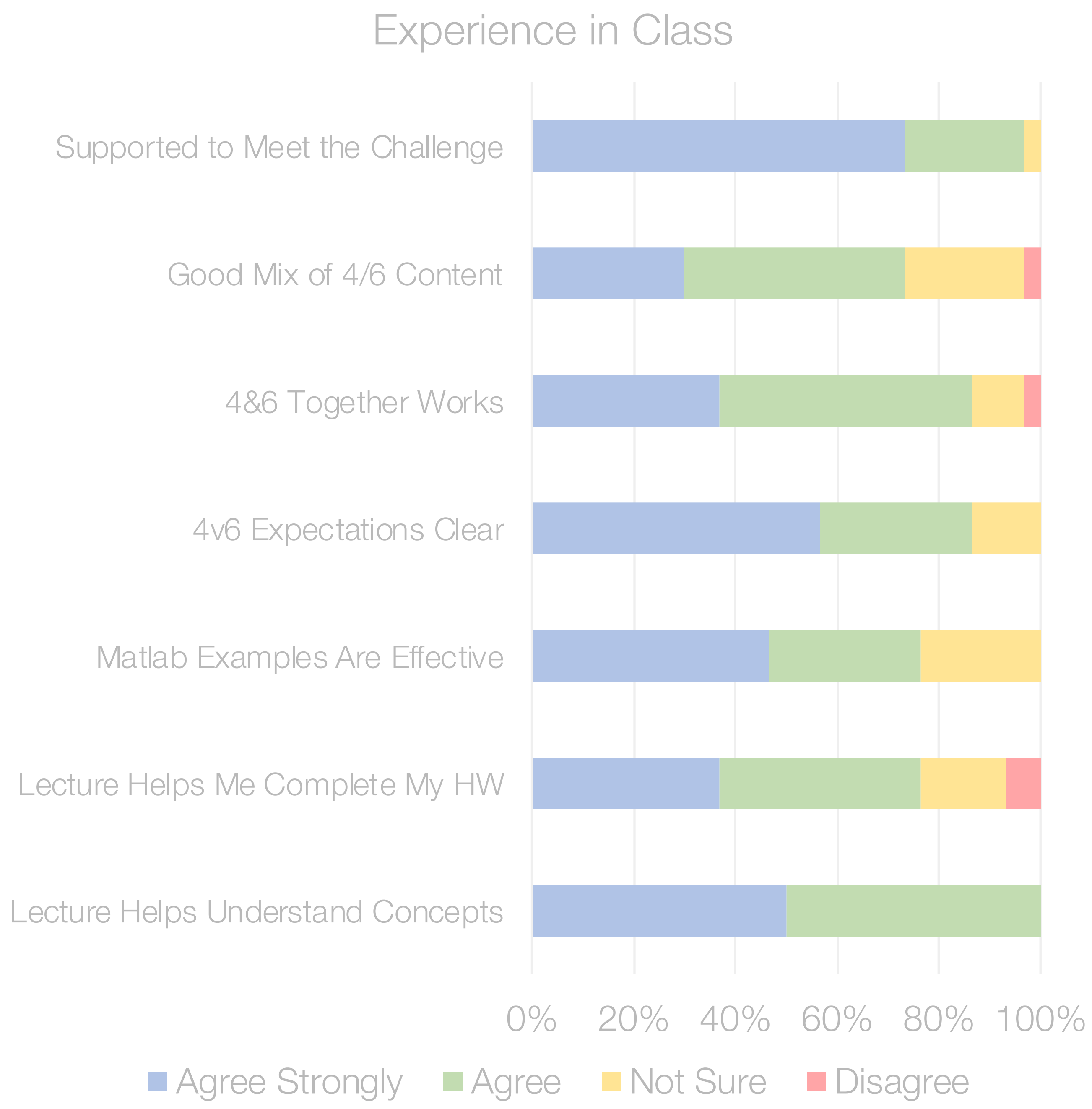
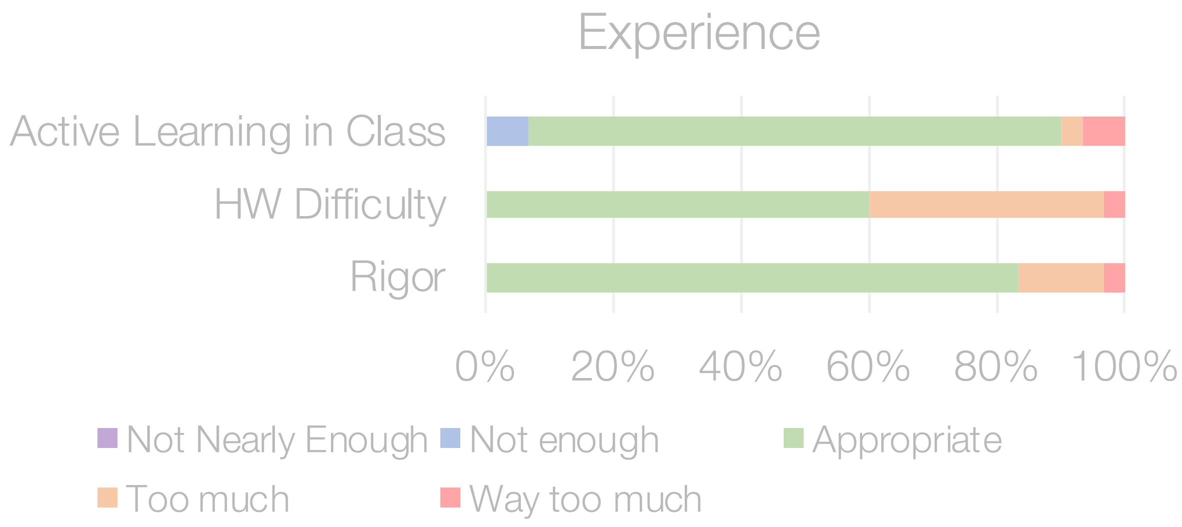
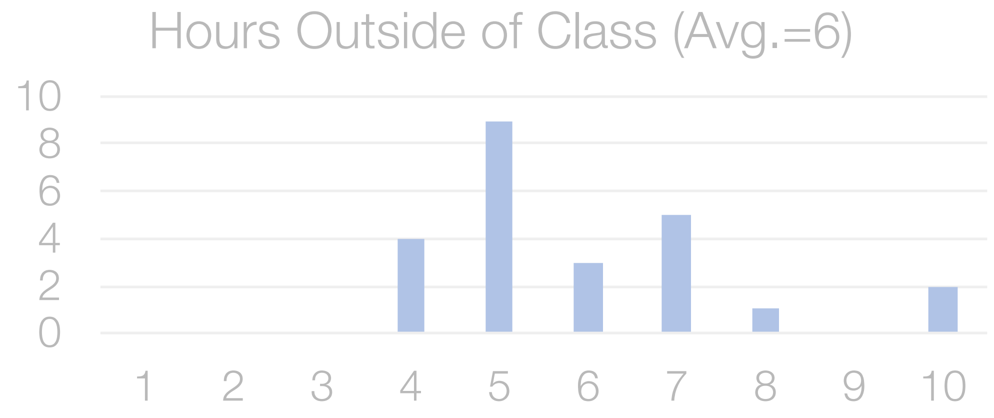
50%
60%
70%
80%
90%
100%
Deep Understanding
Good Understanding
Brush Up Would Be Good
Don't Understand It



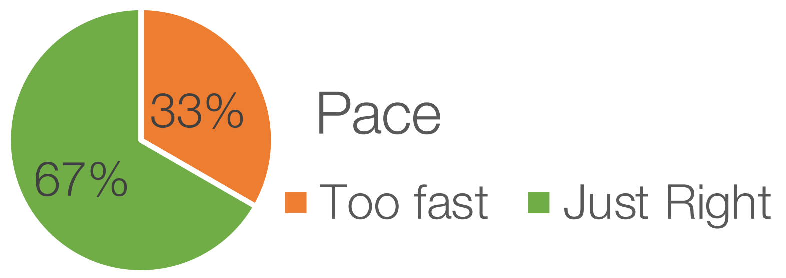
Experience in Class
Supported to Meet the Challenge
Good Mix of 4/6 Content
4&6 Together Works
4v6 Expectations Clear
Matlab Examples Are Effective
Lecture Helps Me Complete My HW
Lecture Helps Understand Concepts
0%
20%
40%
60%
80%
100%
Agree Strongly
Agree
Not Sure
Disagree



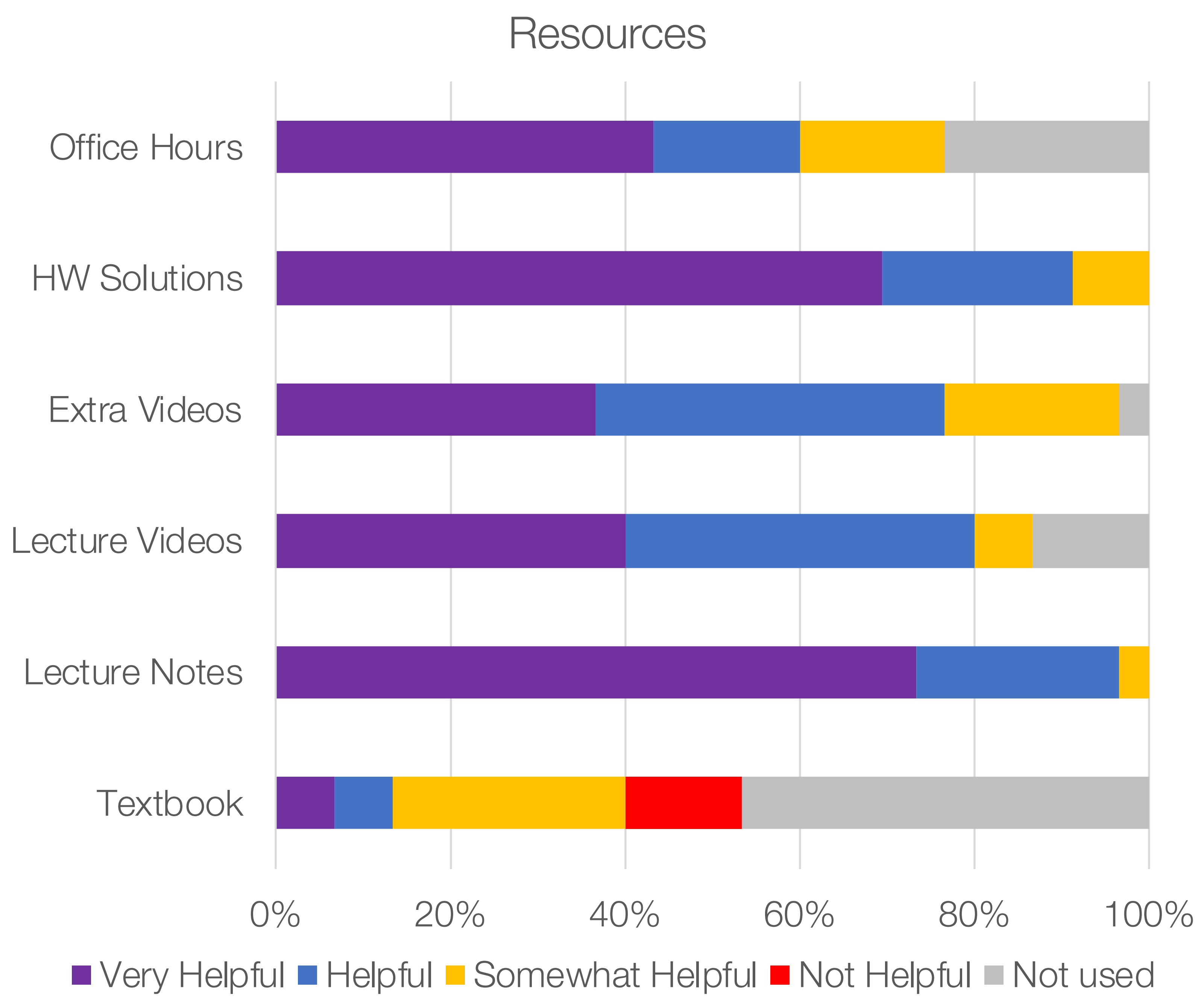
Experience
Active Learning in Class
HW Difficulty
Rigor
0%
20%
40%
60%
80%
100%
Not Nearly Enough
Not enough
Appropriate
Too much
Way too much

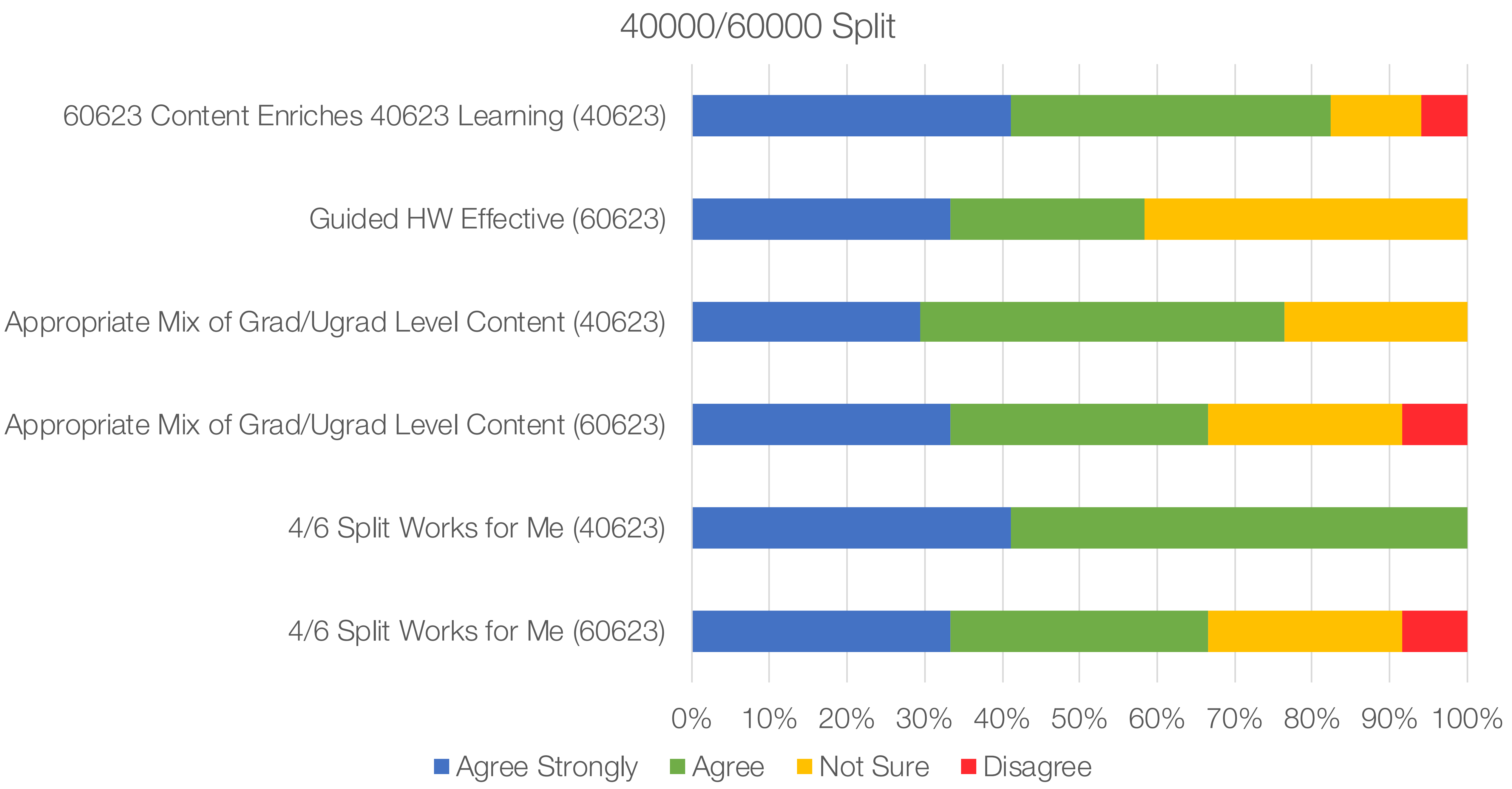


Hours Outside of Class (Avg.=6)
10
8
6
4
2
0
1
2
3
4
5
6
7
8
9
10



33%
67%
Pace
Too fast
Just Right


Resources
Office Hours
HW Solutions
Extra Videos
Lecture Videos
Lecture Notes
Textbook
0%
20%
40%
60%
80%
100%
Very Helpful
Helpful
Somewhat Helpful
Not Helpful
Not used


40000/60000 Split
60623 Content Enriches 40623 Learning (40623)
Guided HW Effective (60623)
Appropriate Mix of Grad/Ugrad Level Content (40623)
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4/6 Split Works for Me (40623)
4/6 Split Works for Me (60623)
0%
10%
20%
30%
40%
50%
60%
70%
80%
90%
100%
Agree Strongly
Agree
Not Sure
Disagree

40/60 Mix

- Good/Very Good (13)
- Clarification regarding split in lecture helps (3)
- Extra level of HW for 60K is good (3)
- Nice to see 60K but not have to master it (3)
- Great for undergrad in 60000 due to more peers
- Grading scale difference could be clarified
- 60K questions not covered in lecture -> could supplement with videos
- 60K questions not covered in lecture -> not fair to ask us to watch extra videos

Sources of Difficulty

- Keeping track of it all (6)
    - Physically grounding some of the math (2)
    - Terminology & nomenclature
    - Keeping track of indices. Scalar vs. vector
    - All the new “principles” and keeping them straight
- Generalized forces (3)
- Applying lecture concepts to HW (2)
- Leaps in derivations (2)
- Size of class and time of day (2)
- 3D rotations / Non inertia reference frames (2)
- Acceleration (linear and angular)
- Lagrange’s equation w/ constraints (60623) [See Panopto for more detail]
- MATLAB
- No way to check HW answers
- HWs at the start of the semester
- Examples in class easier than examples in HW

Helpful

- Examples in class (8)
- In-class activities / active learning (7)
- Combination of derivation + examples (4)
- Lectures (various aspects) (4)
- 1-on-1 in office hours (3)
- MATLAB Examples (2)
- Supplemental videos (2)
- Recorded lectures (2)
- Lecture notes
- Reflecting on concept after example
- Examples like the HW in class
- Lectures where AD makes dynamics easier (virtual work, Lagrange, etc.)

Suggestions

- More practice problems, worked out solutions (4)
- More complex / MATLAB examples (3)
- Less fast for most important concepts / more pauses (3)
- More examples in class (2)
- Practice exam
- Animations of motion
- Easier but more frequent homework
- MATLAB required homework problems
- Start a problem in class, let us finish it
- Go over most challenging problem from the HW
- Don’t spend time passing out HW since time is tight
- Clarify rotating vs. inertially fixed frames
- Larger font size
- Post skipped steps in lecture notes

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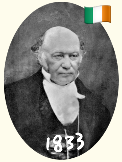
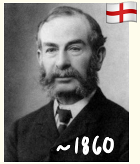


Hamilton



Routh

Noether

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- **Kinetic Energy**: $T = \underbrace{\frac{1}{2}\dot{q}^T M(q,t)\,\dot{q}}_{T_2} + \underbrace{\beta^T(q,t)\,\dot{q}}_{T_1} + T_0(q,t)$

- **Equations of Motion**: $\left.\frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{q}_k}\right] - \left[\frac{\partial \mathcal{L}}{\partial q_k}\right] = Q_{k,nc}\ \right]$ n second-order ODEs

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Summary Last Time

- Natural System (no prescribed motion for any part of the system)
  - Lagranges equations take the form

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + \nabla_q V = \underline{Q}_{nc}$$

- When $\nabla_q V\big|_{q=q^*} = 0 \Rightarrow$ small vibrations are solutions of $M(q^*)\ddot{q} + K^*(q-q^*) = 0$

- Vibration mode shapes : $\text{EigVecs}(M^{*-1}K^*)$ Frequencies: $\sqrt{\text{EigVals}(M^{*-1}K^*)}$

17/

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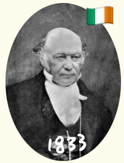
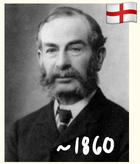


Hamilton



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2. Use integrals of motion to reduce # of ODEs we must solve (Routhian Reduction)

3. Alternate form of EOM as 2n first-order ODEs

$$(\underset{\dot{q}}{\text{Gen. Coords}},\ \underset{\dot{q}}{\text{Gen. vels.}}) \longrightarrow (\underset{q}{\text{Gen. coords}},\ \underset{\underline{\Pi}}{\text{Gen. momentum}})$$

Generalized Momentum: Motivation and Definition

$$\frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{q}_k}\right] - \underbrace{\left[\frac{\partial \mathcal{L}}{\partial q_k}\right]}_{=0} = \underbrace{Q_{k,nc}}_{=0}$$

What happens when $Q_{k,nc} = 0$ and $\frac{\partial \mathcal{L}}{\partial q_k} = 0$?

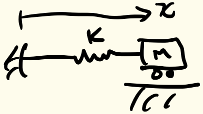
$\frac{\partial \mathcal{L}}{\partial \dot{q}_k}$ stays constant! (i.e., it is an integral of motion)

Defn: We define the generalized momentum associated w/ q_k as $\pi_k = \frac{\partial \mathcal{L}}{\partial \dot{q}_k}$

Exercise: Show that $\underline{\pi} = \begin{bmatrix} \pi_1 \\ \vdots \\ \pi_n \end{bmatrix} = \sum_i \{J_i\}^T m_i \{\dot{r}_i\}$ for a system of particles

Generalized Momentum: Simple Examples and General Case

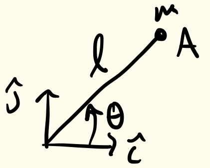
① Mass Spring system



$$\mathcal{L} = T - V = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} k x^2$$

$$\pi_x = \frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x} \quad \text{usual momentum}$$

② Particle in polar coordinates



$$\underset{\sim}{q} = [l, \theta]^T$$

π_l = Linear momentum along l

π_θ = Angular momentum about the origin

Analagous to the case of generalized forces

③ <u>General case</u>: $\underline{\pi} = M \underset{\sim}{\dot{q}} + \underset{\sim}{B} \Rightarrow \underset{\sim}{\dot{q}} = M^{-1}(\underline{\pi} - \underset{\sim}{B})$

④ <u>For a natural system</u>: $\underline{\pi} = M \underset{\sim}{\dot{q}}$

Using Conserved Quantities To Reduce # of EOM

Defn: We say a coordinate q_k is cyclic (OR ignorable) if it does not appear in the Lagrangian

$$\frac{\partial \mathcal{L}}{\partial q_k} = 0$$

Routh's question: Suppose you have ℓ cyclic coordinates, can we use the fact that we have ℓ conserved quantities to simplify EOM?

$$\boxed{\begin{array}{c}\text{EOM} \\ n \text{ 2nd order ODEs}\end{array}} \xRightarrow{\ \ell \text{ cons.}\ } \boxed{\begin{array}{c} n-\ell \text{ 2nd order} \\ \text{Diff eqs} \\ \text{for non cyclic} \\ \text{coords}\end{array}}$$

Routh's Approach: For simplicity 1 cyclic coordinate (q_n)

$$(q_1, \ldots, q_{n-1}, q_n, \dot{q}_1, \ldots, \dot{q}_{n-1}, \dot{q}_n)$$

change of variables

Doesn't appear in $\mathcal{L}$

$$(q_1, \ldots, q_{n-1}, q_n, \dot{q}_1, \ldots, \dot{q}_{n-1}, \pi_n)$$

Constant

Details: Routhian Reduction

Routhian Reduction: $\mathcal{R} = \mathcal{L}(q, \dot{q}) - \pi_n \dot{q}_n \leftarrow$ plug in $\dot{q}_n$ as a function of $q, \dot{q}_1, \ldots, \dot{q}_{n-1}, \pi_n$

$$\mathcal{R}(q, \dot{q}_1, \ldots, \dot{q}_{n-1}, \pi_n)$$ "The Routhian"

Equations of motion

$$\frac{d}{dt}\left[\frac{\partial \mathcal{R}}{\partial \dot{q}_k}\right] - \frac{\partial \mathcal{R}}{\partial q_k} = Q_{k,nc} \qquad k = 1, \ldots, n-1$$

General Procedure (Possibly more than one cyclic variable)

1. Determine cyclic coords. Partion $\dot{q}$ into $\dot{q}_{cyclic}$ and $\dot{q}_{noncyclic}$
2. Find $\dot{q}_{cyclic}$ as function of q, $\dot{q}_{noncyclic}$, Π_{cyclic}
3. Write Routhian $\mathcal{R} = \mathcal{L} - \Pi_{cyclic}^T \dot{q}_{cyclic} \leftarrow$ Use results from 2 and plug them in for $\dot{q}_{cyclic}$
4. Solve odes for $q_{noncyclic}(t)$
5. Use 2 to find $q_{cylic}(t)$ if interested.

Derivation of Previous Slide: Relating Partials of R and L

NOT IN CLASS

Notation:

$$\frac{\partial f}{\partial q} = [\nabla_q f]^T = \text{"Row vector gradient"}$$

$$\underbrace{\mathcal{R}(t, q, \dot{q}_{\text{noncyclic}}, \Pi_{\text{cyclic}})}_{①} = \underbrace{\mathcal{L} - \sum_{k,\text{cyclic}} \dot{q}_k \Pi_k}_{②}$$

Consider the total differential of $\mathcal{R}$:

$$d\mathcal{R} \overset{d①}{=} \frac{\partial \mathcal{R}}{\partial t} dt + \frac{\partial \mathcal{R}}{\partial q} dq + \sum_{k,\text{noncyclic}} \frac{\partial \mathcal{R}}{\partial \dot{q}_k} d\dot{q}_k + \sum_{k,\text{cyclic}} \frac{\partial \mathcal{R}}{\partial \Pi_k} \underbrace{d\Pi_k}$$

$=0$ along trajectories but we consider a general differential change to t, q, Π here.

$$\overset{d②}{=} \frac{\partial \mathcal{L}}{\partial t} dt + \frac{\partial \mathcal{L}}{\partial q} dq + \sum_{k,\text{noncyclic}} \frac{\partial \mathcal{L}}{\partial \dot{q}_k} d\dot{q}_k + \sum_{k,\text{cyclic}} \frac{\partial \mathcal{L}}{\partial \dot{q}_k} d\dot{q}_k - \sum_{k,\text{cyclic}} \left[d\dot{q}_k \Pi_k + \dot{q}_k d\Pi_k \right]$$

$\frac{\partial \mathcal{L}}{\partial \dot{q}_k} = \Pi_k$ so these terms cancel

Matching Terms:

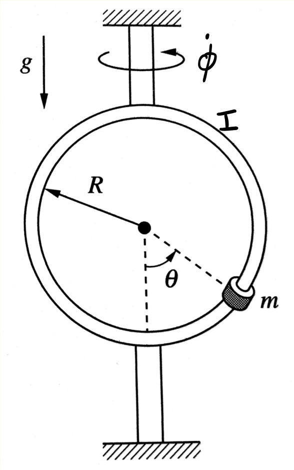
$$\frac{\partial \mathcal{R}}{\partial t} = \frac{\partial \mathcal{L}}{\partial t}$$

$$\frac{\partial \mathcal{R}}{\partial q_k} = \frac{\partial \mathcal{L}}{\partial q_k} \quad \text{For all } k!$$

$$\frac{\partial \mathcal{R}}{\partial \dot{q}_k} = \frac{\partial \mathcal{L}}{\partial \dot{q}_k} \quad \text{for all } k, \text{ non cyclic.}$$

$$\dot{q}_k = -\frac{\partial \mathcal{R}}{\partial \Pi_k} \quad \forall k, \text{cyclic}$$

Example: Ring and Bead (Ring no longer rotates with fixed rate)



Use Routhian Reduction to find an ODE for the evolution of θ

$$\mathcal{L} = T - V = \frac{1}{2} I \dot\phi^2 + \frac{1}{2} m R^2 \left(\dot\theta^2 + s_\theta^2 \dot\phi^2 \right) + m g R c_\theta$$

Gen momentum:

$$\pi_\phi = \frac{\partial \mathcal{L}}{\partial \dot\phi} = I \dot\phi + m R^2 s_\theta^2 \dot\phi = \left(I + m R^2 s_\theta^2 \right) \dot\phi$$

Total angular momentum about vertical axis

$$\dot\phi = \frac{\pi_\phi}{I + m R^2 s_\theta^2}$$

Routhian:

$$\mathcal{R} = \mathcal{L} - \pi_\phi \dot\phi = \frac{1}{2} m R^2 \dot\theta^2 - \frac{1}{2} \frac{\pi_\phi^2}{(I + m R^2 s_\theta^2)} + m g R c_\theta$$

missing in lecture

ODE for θ:

$$\frac{d}{dt}\left[\frac{\partial \mathcal{R}}{\partial \dot\theta} \right] - \frac{\partial \mathcal{R}}{\partial \theta} = m R^2 \ddot\theta - \frac{m R^2 \pi_\phi^2 c_\theta s_\theta}{(I + m R^2 s_\theta^2)^2} + m g R s_\theta = 0$$

✓

Hamiltonian Dynamics: From n second order ODEs, to 2n-First-Order ODES

Hamilton's Equations go one Step further: $(q,\dot{q}) \xrightarrow{\text{Change of Variables}} (q,\pi)$

Defn: Define the Hamiltonian $\mathcal{H} = \pi^T \dot{q} - \mathcal{L}$ ← View $\mathcal{H}$ as a function of t, q, π

Consider the total differential of $\mathcal{H}$

$$d\mathcal{H} = \frac{\partial \mathcal{H}}{\partial t}dt + \frac{\partial \mathcal{H}}{\partial q}dq + \frac{\partial \mathcal{H}}{\partial \pi}d\pi$$

$$= d\pi^T \dot{q} + \cancel{\pi^T d\dot{q}} - \frac{\partial \mathcal{L}}{\partial t}dt - \frac{\partial \mathcal{L}}{\partial q}dq - \overbrace{\cancel{\frac{\partial \mathcal{L}}{\partial \dot{q}}}}^{\pi^T}\cancel{d\dot{q}}$$

So:

$$\boxed{\begin{aligned} \dot{q}_k &= \frac{\partial \mathcal{H}}{\partial \pi_k} \\ \dot{\pi}_k &= \frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{q}_k}\right] = \frac{\partial \mathcal{L}}{\partial q_k} + Q_{k,nc} = -\frac{\partial \mathcal{H}}{\partial q_k} + Q_{k,nc} \end{aligned}}$$

Hamiltons Equations

$\boxed{\text{2n first order ODES}}$

Skipped in class. We'll return to it after the exam.

Summary:

- Generalized momentum

$$\pi_k = \frac{\partial \mathcal{L}}{\partial \dot{q}_k} \qquad \underline{\pi} = \begin{bmatrix} \pi_1 \\ \vdots \\ \pi_n \end{bmatrix} = M \dot{\underline{q}} + \overbrace{\underline{B}}^{=0 \text{ for natural system}}$$

- Routhian Reduction
    - if q_k cyclic **AND** $Q_{k,nc}=0 \Rightarrow \pi_k =$ constant along trajectories

- Routhian: $\mathcal{R}(t, q, \dot{q}_{\text{non cyclic}}, \underline{\pi}_{\text{cyclic}}) = \mathcal{L} - \sum_{k \text{ cyclic}} \pi_k \dot{q}_k$

- ODEs: $\frac{d}{dt}\left[\frac{\partial \mathcal{R}}{\partial \dot{q}_k}\right] - \frac{\partial \mathcal{R}}{\partial q_k} = Q_{k,nc}$ for k non cyclic

- Hamilton's equations:

Hamiltonian: $\mathcal{H}(t, \underline{q}, \underline{\pi}) = \underline{\pi}^T \dot{\underline{q}} - \mathcal{L}$

ODEs: $\dot{\underline{q}} = \nabla_\pi \mathcal{H} \qquad \dot{\underline{\pi}} = -\nabla_q \mathcal{H} + \underline{Q}_{nc}$