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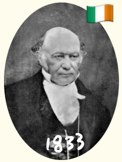
Analytical Dynamics

Exam Wrap Up & Hamiltonian Dynamics



Hamilton

Noether

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(1) HW 6 Due Wednesday

Gameplan: • Go Over Exam 2

• Finish last piece of analytical fundamentals:

• Equations of Motion: $\frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{q}_k}\right] - \left[\frac{\partial \mathcal{L}}{\partial q_k}\right] = Q_{k,nc}$] n Second-order ODEs

Generalized Momentum Π_k

• Today: Alternate form of EOM as 2n First-order ODEs

(Gen. Coords , Gen. vels.) $\longrightarrow$ (Gen. coords , Gen. momentum)

$\dot{q}$ $\quad$ $\dot{q}$ $\quad$ q $\quad$ Π

Generalized Momentum: Motivation and Definition

REVIEW

$$\frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{q}_k}\right] - \left[\frac{\partial \mathcal{L}}{\partial q_k}\right] = Q_{k,nc}$$

What happens when $Q_{k,nc} = 0$ and $\frac{\partial \mathcal{L}}{\partial q_k} = 0$?

$\frac{\partial \mathcal{L}}{\partial \dot{q}_k}$ stays constant over time! (i.e., it is an integral of motion)

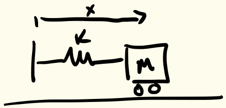
Defn: We define the generalized momentum associated with q_k as $\pi_k = \frac{\partial \mathcal{L}}{\partial \dot{q}_k} = \frac{\partial T}{\partial \dot{q}_k}$

$$\mathcal{L}(t, q, \dot{q}) = T(t, q, \dot{q}) - V(t, q)$$

Generalized Momentum: Simple Examples and General Case

REVIEW

① Harmonic oscillator

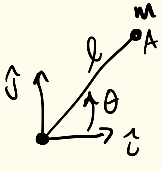


$$\mathcal{L} = T - V = \tfrac{1}{2} m \dot{x}^2 - \tfrac{1}{2} K x^2$$

$$\pi_x = \frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x}$$

② Particle in polar coordinates

$$q = [l, \theta]^T$$



π_l = Linear momentum along l

π_θ = Angular momentum about origin

③ General Case: $\underline{\pi} = M \dot{\underline{q}} + \underline{\beta} \Rightarrow \dot{\underline{q}} = M^{-1}(\underline{\pi} - \underline{\beta})$

④ For natural system: $\underline{\pi} = M \dot{\underline{q}}$

Hamiltonian Dynamics: From n second order ODEs, to 2n-First-Order ODES

Key Step: $(q, \dot{q}) \xrightarrow{\text{change of variables}} (q, \pi)$

$\mathcal{L}(t, q, \dot{q})$

Defn: Define the Hamiltonian $\mathcal{H} = \underline{\pi}^T \dot{q} - \mathcal{L}$ ← View as a function of $t, q, \underline{\pi}$

$\mathcal{H}(t, q, \underline{\pi})$

Consider the total differential of $\mathcal{H}$

$$d\mathcal{H} = \frac{\partial \mathcal{H}}{\partial t} dt + \frac{\partial \mathcal{H}}{\partial q} dq + \frac{\partial \mathcal{H}}{\partial \pi} d\underline{\pi}$$

$$= d\underline{\pi}^T \dot{q} + \cancel{\pi^T d\dot{q}} - \frac{\partial \mathcal{L}}{\partial t} dt - \frac{\partial \mathcal{L}}{\partial q} dq - \cancel{\overbrace{\frac{\partial \mathcal{L}}{\partial \dot{q}}}^{\underline{\pi}^T} d\dot{q}}$$

$$\boxed{\dot{q}_k = \frac{\partial \mathcal{H}}{\partial \pi_k}}$$

$$\dot{\pi}_k = \frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{q}_k}\right] = \frac{\partial \mathcal{L}}{\partial q_k} + Q_{k,nc} = \boxed{-\frac{\partial \mathcal{H}}{\partial q_k} + Q_{k,nc}}$$

2n-first-order ODES

$$\frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{q}_k}\right] - \frac{\partial \mathcal{L}}{\partial q_k} = Q_{k,nc}$$

Hamiltonian Dynamics: Properties (60623)

EXTRA: NOT IN CLASS
JUST FYI

① $\mathcal{H} = T_2 + U$

$= T + V$ for natural systems (i.e., total Energy)

② $\frac{d}{dt}\mathcal{H} = -\frac{\partial \mathcal{L}}{\partial t} + \dot{q}^T \underline{Q}_{nc}$

$= \dot{q}^T \underline{Q}_{nc}$ (for natural systems)

$= 0$ (for natural systems w/ only conservative forces)

Hamiltonian Dynamics: Use Case -> Impacts

Continuous:

$$\dot{q}_k = \frac{\partial \mathcal{H}}{\partial \pi_k}$$

$$\dot{\pi}_k = -\frac{\partial \mathcal{H}}{\partial q_k} + Q_{k,nc}$$

Integrate across impact:

$$q_k^+ = q_k^-$$

$$\pi_k^+ = \pi_k^- + \underbrace{\hat{Q}_{k,nc}}$$

Impulsive Generalized Force

$$\int_{t^-}^{t^+} Q_{k,nc} \, dt$$

Result of Impact:

$$q^+ = q^-$$

Form 1: $\Delta \underline{\pi} = \underline{\hat{Q}}_{nc}$

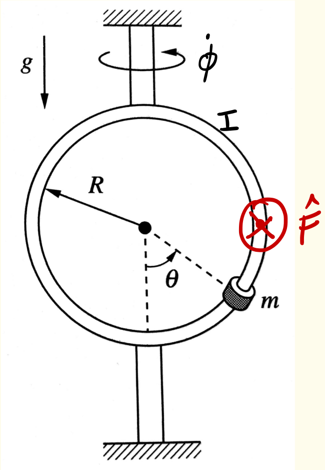
Change in generalized momentum
= generalized impulsive force

[only works for instantaneous impulse, not over time (due to $-\frac{\partial \mathcal{H}}{\partial q_k}$ in $\dot{\underline{\pi}}_k$)]

Form 2: $M \Delta \dot{\underline{q}} = \underline{\hat{Q}}_{nc}$

Impact Example: Ring and Bead

$$\mathcal{L} = T - V = \frac{1}{2} I \dot{\phi}^2 + \frac{1}{2} m R^2 \left(\dot{\theta}^2 + s_\theta^2 \dot{\phi}^2\right) + m g R c_\theta$$



Which variables change during impact?

ⓐ neither $\dot{\theta}$ nor $\dot{\phi}$

ⓑ only $\dot{\theta}$

ⓒ only $\dot{\phi}$

ⓓ both $\dot{\phi}$ and $\dot{\theta}$

ⓔ It depends on whether or not $\sin(\theta) = 0$

Give $\dot{\phi}^+$ and $\dot{\theta}^+$ after impact as a function of $\dot{\theta}^-$ and $\dot{\phi}^-$ before it

$$\pi_\theta = m R^2 \dot{\theta} \qquad Q_\theta = 0$$

$$\pi_\phi = \left[m R^2 s_\theta^2 + I\right] \dot{\phi} \qquad Q_\phi = F R$$

$$\Rightarrow \quad \pi_\theta^+ = \pi_\theta^-$$

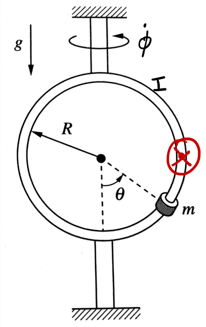
$$\pi_\phi^+ = \pi_\phi^- + \hat{F} R$$

$$\boxed{\begin{aligned} \dot{\theta}^+ &= \dot{\theta}^- \\ \dot{\phi}^+ &= \dot{\phi}^- + \frac{\hat{F} R}{m R^2 s_\theta^2 + I} \end{aligned}}$$

Hamiltonian Example: Ring and Bead (at home)

NOT IN CLASS

MATLAB on Sakai in Resources/Lecture MATLAB



F always perp. to Ring

$$\mathcal{L} = T - V = \tfrac{1}{2}I\dot\phi^2 + \tfrac{1}{2}mR^2\left(\dot\theta^2 + s_\theta^2\dot\phi^2\right) + mgRc_\theta$$

$$= \tfrac{1}{2}\dot\phi^2\left(I + mR^2 s_\theta^2\right) + \tfrac{1}{2}mR^2\dot\theta^2 + mgRc_\theta$$

① Give Hamiltonian $\mathcal{H}$

$$\pi_\theta = mR^2\dot\theta \qquad \Rightarrow \qquad \dot\theta = \frac{\pi_\theta}{mR^2}$$

$$\pi_\phi = \left[mR^2 s_\theta^2 + I\right]\dot\phi \qquad \dot\phi = \frac{\pi_\phi}{I + mR^2 s_\theta^2}$$

We use these expressions when evaluating $\mathcal{H}$

$$\mathcal{H} = \pi_\theta\dot\theta + \pi_\phi\dot\phi - \mathcal{L}(\theta,\phi,\dot\theta,\dot\phi) = \frac{\pi_\phi^2}{(I+mR^2s_\theta^2)} + \frac{\pi_\theta^2}{mR^2} - \overbrace{\left[\frac{1}{2}\frac{\pi_\theta^2}{I+mR^2s_\theta^2} + \frac{1}{2}\frac{\pi_\phi^2}{mR^2} + mgRc_\theta\right]}^{\mathcal{L}}$$

$$= \boxed{\frac{\pi_\phi^2}{2(I+mR^2s_\theta^2)} + \frac{\pi_\theta^2}{2mR^2} + mgRc_\theta}$$

② Hamilton's Equations

$$\dot\theta = \frac{\partial\mathcal{H}}{\partial\pi_\theta} = \frac{\pi_\theta}{mR^2} \qquad \dot\pi_\theta = -\frac{\partial\mathcal{H}}{\partial\theta} + \cancel{Q_{\theta,nc}}^{\,0} = \frac{\pi_\phi^2\, mR^2 s_\theta c_\theta}{(I+mR^2s_\theta^2)^2} + mgRs_\theta$$

$$\dot\phi = \frac{\partial\mathcal{H}}{\partial\pi_\phi} = \frac{\pi_\phi}{I+mR^2s_\theta^2} \qquad \dot\pi_\phi = -\cancel{\frac{\partial\mathcal{H}}{\partial\phi}}^{\,0} + Q_{\phi,nc} = FR$$

Summary:

- Generalized momentum

$$\pi_k = \frac{\partial \mathcal{L}}{\partial \dot{q}_k} \qquad \underline{\pi} = \begin{bmatrix} \pi_1 \\ \vdots \\ \pi_n \end{bmatrix} = M \dot{\underline{q}} + \overbrace{\underline{B}}^{=0 \text{ for natural system}}$$

- Hamilton's equations:

must be expressed in terms of $\underline{q}$ and $\underline{\pi}$

Hamiltonian: $\mathcal{H}(t, \underline{q}, \underline{\pi}) = \underline{\pi}^T \dot{\underline{q}} - \mathcal{L}$

$$= \underbrace{\tfrac{1}{2} \underline{\pi}^T M^{-1}(\underline{q}) \underline{\pi} + V}$$ for natural systems

Fun Fact
This means $\mathcal{H}$ = Total Energy!

ODEs: $\dot{\underline{q}} = \nabla_{\pi} \mathcal{H} \qquad \dot{\underline{\pi}} = -\nabla_{\underline{q}} \mathcal{H} + \underline{Q}_{nc}$

Impacts: $\underline{\pi}^+ = \underline{\pi}^- + \hat{Q}_{nc}$

$$M(\underline{q}^+)\dot{\underline{q}}^+ = M(\underline{q}^-)\dot{\underline{q}}^- + \hat{Q}_{nc} \qquad \underline{q}^+ = \underline{q}^-$$