10/

Exam 2 Review

- Exam Weds.
  - in 141 DeBartolo
  - Open notes/book/MATLAB
  - No internet use except to access Sakai
- Exam prep resources on Sakai
  - Topic List
  - Example Exam from Dr. Plecnik (~ Similar style)
- Today
  - Exam Layout ✓
  - Concept overview ✓
  - Practice Test

Terminology:

Constraint Force: Exists to fix some part of motion that is constrained

Active/Applied Force: Forces onto or between particles in a system that aren't constraint forces. (e.g., an externally applied push, internal actuator force, gravity force, spring force.)

Property of Forces on a system

Conservative: Force can be given as -gradient of some potential

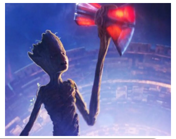
nonconservative: Everything Else

Property of a system & its motion

natural system: No fixed motions

non natural system: Some portion of system motion is fixed

Concept Map (assume all constraints captured by GC)



Generalized Coordinates

$$\mathbf{q} = [q_1, \ldots, q_n]^T$$

$$\mathbf{r}_i = \mathbf{r}_i(\mathbf{q}, t) \qquad i = 1, \ldots, N$$

Virtual Displacements

$\delta\mathbf{r}_1, \ldots, \delta\mathbf{r}_N$

occur in zero time

Newton's Laws

$$\mathbf{F}_i + \mathbf{F}'_i = m_i \ddot{\mathbf{r}}_i$$

Active Constr.

Constraint forces do no virtual work under virtual displacements consistent with the constraints

Virtual Work

$$\delta W = \sum_i \mathbf{F}_i \cdot \delta\mathbf{r}_i$$

$$\delta W = \mathbf{Q}^T \delta\mathbf{q} = \sum_k Q_k \delta q_k$$

Generalized Forces

$$Q_k = \sum_i \frac{\partial \mathbf{r}_i}{\partial q_k} \cdot \mathbf{F}_i$$

$$\mathbf{Q} = \sum_i \{\mathbf{J}_i\}^T \{\mathbf{F}_i\}$$

D'Alembert's Principle

$$\underbrace{\sum_i \mathbf{F}_i \cdot \delta\mathbf{r}_i}_{\sum_k Q_k \delta q_k} = \sum_i m_i \ddot{\mathbf{r}}_i \cdot \delta\mathbf{r}_i$$

EOM Via D'Alembert

$$Q_{k,\mathrm{inertial}} = \sum_i \frac{\partial \mathbf{r}_i}{\partial q_k} \cdot (m_i \ddot{\mathbf{r}}_i)$$

EOM: $Q_k = Q_{k,\mathrm{inertial}}$

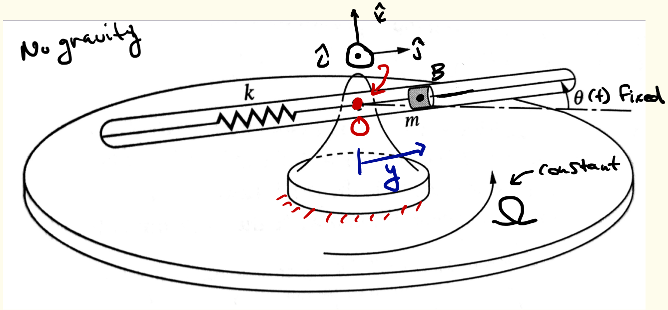
Calc. of Variations

EOM Via Lagrange

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathscr{L}}{\partial \dot{q}_k} - \frac{\partial \mathscr{L}}{\partial q_k} = Q_{k,\mathrm{nc}}$$

D'Alembert Example

Frame Fixed to rotating Tube $\quad q = [y, \cancel{\theta}]$



D'Alembert:

$$Q_y = m\underline{a}_B \cdot \overbrace{\frac{\partial r_B}{\partial y}}^{\hat{J}}$$

Kinematics:

$$\underline{r}_B = y\hat{J} \qquad \frac{\partial r_B}{\partial y} = \hat{J}$$

$$\underline{v}_B = \dot{y}\hat{J} + y\dot{\hat{J}}$$

$$\underline{a}_B = \ddot{y}\hat{J} + 2\dot{y}\dot{\hat{J}} + y\ddot{\hat{J}}$$

No component along $\hat{J}$

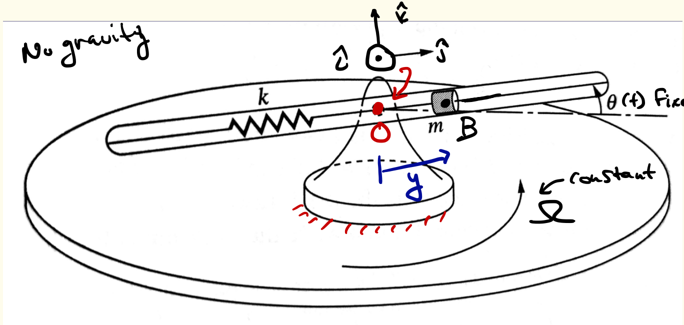
Angular Velocity of frame

$$\underline{\omega} = \dot{\theta}\hat{\imath} + \Omega\left(c_\theta \hat{K} + s_\theta \hat{J}\right)$$

$$\dot{\hat{J}} = \underline{\omega} \times \hat{J} = \left[\dot{\theta}\hat{\imath} + \Omega\left(c_\theta \hat{K} + s_\theta \hat{J}\right)\right] \times \hat{J}$$

$$= \dot{\theta}\hat{K} - \Omega c_\theta \hat{\imath}$$

D'Alembert Example



$$Q_y = m\,\underline{a}_B \cdot \overbrace{\frac{\partial \underline{r}_B}{\partial y}}^{\hat{\jmath}}$$

Kinematics Cont

$$\underline{a}_B = \ddot{y}\,\hat{\jmath} + 2\dot{y}\,\dot{\hat{\jmath}} + y\,\ddot{\hat{\jmath}}$$

We don't care about this term in the end

$$\dot{\hat{\jmath}} = \dot{\theta}\,\hat{k} - \Omega c_\theta\,\hat{\imath}$$

$$\dot{\theta}\,\underline{\omega}\times\hat{k} - \Omega c_\theta\,\underline{\omega}\times\hat{\imath} = \underline{\omega}\times\overbrace{(\dot{\theta}\,\hat{k} - \Omega c_\theta\,\hat{\imath})}^{\dot{\hat{\jmath}}}$$

$$\ddot{\hat{\jmath}} = \ddot{\theta}\,\hat{k} + \Omega s_\theta \dot{\theta}\,\hat{\imath} + \dot{\theta}\,\dot{\hat{k}} - \Omega c_\theta\,\dot{\hat{\imath}} = \ddot{\theta}\,\hat{k} + \Omega s_\theta \dot{\theta}\,\hat{\imath} + \underline{\omega}\times\dot{\hat{\jmath}}$$

$$= \ddot{\theta}\,\hat{k} + \Omega s_\theta \dot{\theta}\,\hat{\imath} + \left[\dot{\theta}\,\hat{\imath} + \Omega(c_\theta\,\hat{k} + s_\theta\,\hat{\jmath})\right] \times \left[\dot{\theta}\,\hat{k} - \Omega c_\theta\,\hat{\imath}\right]$$

$$= \bullet\,\hat{\imath} + \left[-\dot{\theta}^2 - \Omega^2 c_\theta^2\right]\hat{\jmath} + \bullet\,\hat{k}$$

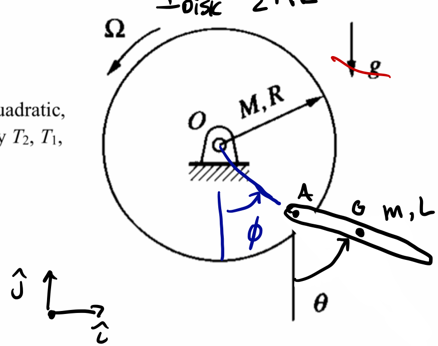
$$\underline{a}_B = \bullet\,\hat{\imath} + \left[\ddot{y} - y\dot{\theta}^2 - y\Omega^2 c_\theta^2\right]\hat{\jmath} + \bullet\,\hat{k}$$

Gen. Force: $V = \frac{1}{2}ky^2 \Rightarrow Q_y = -ky$

EOM: $-ky = m(\ddot{y} - y\dot{\theta}^2 - y\Omega^2 c_\theta^2)$

1. (20 points) A link of mass m and Length L is pinned to the edge of a disk of mass M and radius R, as shown in the figure. A servomotor keeps the angular velocity of the disk constant at Ω. Gravity acts on the system.

$I_{Disk} = \frac{1}{2}MR^2$



Provide expressions for:

(a) The kinetic energy of the system.

(b) Identify degrees of freedom and the quadratic, linear, and constant parts of kinetic energy T_2, T_1, and T_0

$q = \theta$

Kinematics of Key points

$$\underline{r}_{G,rod} = R(-c_\phi \hat{j} + s_\phi \hat{i}) + \frac{L}{2}(-c_\theta \hat{j} + s_\theta \hat{i})$$

$$\underline{v}_{G,rod} = R\dot{\phi}(s_\phi \hat{j} + c_\phi \hat{i}) + \frac{L}{2}\dot{\theta}(s_\theta \hat{j} + c_\theta \hat{i})$$

$$\|v_{G,rod}\|^2 = (R\dot{\phi}c_\phi + \tfrac{L}{2}\dot{\theta}c_\theta)^2 + (R\dot{\phi}s_\phi + \tfrac{L}{2}\dot{\theta}s_\theta)^2$$

$$= R^2\dot{\phi}^2 + \frac{L^2}{4}\dot{\theta}^2 + RL\dot{\phi}\dot{\theta}[c_\theta c_\phi + s_\phi s_\theta]$$

Kinetic Energy

$$T = T_{DISK} + T_{ROD}$$

This approach always gives the correct kinetic energy. Other approaches only work in special cases.

$$= \frac{1}{2} I_{G,Disk}\dot{\phi}^2 + \frac{1}{2} I_{G,rod}\dot{\theta}^2 + \frac{1}{2} m \|v_{G,rod}\|^2$$

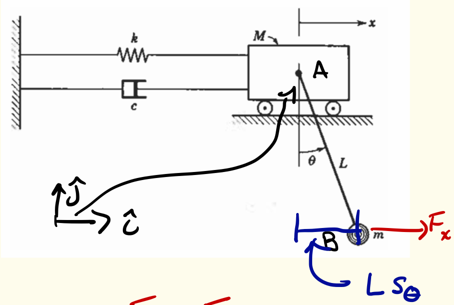
Inertia about CoM

Velocity of CoM

$$= \frac{1}{2}\left(\frac{1}{2}MR^2\right)\Omega^2 + \frac{1}{2}\left(\frac{1}{12}mL^2\right)\dot{\theta}^2 + \frac{1}{2}mR^2\Omega^2 + \frac{1}{2}m\frac{L^2}{4}\dot{\theta}^2 + \frac{1}{2}mRL\dot{\theta}\Omega\cos(\theta-\phi)$$

$$T_2 = \frac{1}{6}mL^2\dot{\theta}^2 \qquad T_1 = \frac{1}{2}mRL\Omega\dot{\theta}\cos(\theta-\phi) \qquad T_0 = \left(\frac{1}{4}M + \frac{1}{2}m\right)R^2\Omega^2$$

2. (40 points) The pendulum cart system has a linear spring (stiffness k, rest position $x = 0$) and linear damper (coefficient c) attached to it. The cart has mass M, and pendulum of length L has a point mass of m. A force of $F\sin\omega t$ is applied to the point mass in the x-direction.



Provide expressions for:
(a) The non-conservative generalized forces associated with x and θ coordinates
(b) The system's kinetic energy.
(c) The system's potential energy.
(d) The system's equations of motion using Lagrange's formula

Similar to HW so we'll skip it here. See Soln to HW4.

$F_x = F\sin(\omega t)$

(a) $\underline{r}_A = x\hat{i}$ $\qquad$ $\underline{F}_A = -c\dot{x}\hat{i}$

$\underline{r}_B = (x + LS_\theta)\hat{i} - LC_\theta\hat{j}$ $\qquad$ $\underline{F}_B = F\sin(\omega t)\hat{i}$

$$Q_x = \frac{\partial \underline{r}_A}{\partial x}\cdot\underline{F}_A + \frac{\partial \underline{r}_B}{\partial x}\cdot\underline{F}_B$$

$$= [\hat{i}]\cdot[-c\dot{x}\hat{i}] + [\hat{i}]\cdot[F\sin(\omega t)\hat{i}] = \boxed{-c\dot{x} + F\sin\omega t = Q_x}$$

Alternate Approach

SEE VIDEO for Physical Explanation.

$\delta W = Q_x\delta x$ when $\delta x \neq 0$ $\delta\theta = 0$: $\delta W = -c\dot{x}\delta x + F\sin(\omega t)\delta x = Q_x\delta x$

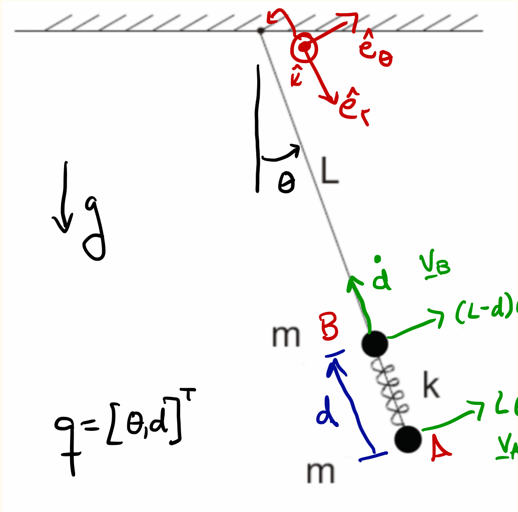
$\delta W = Q_\theta\delta\theta$ when $\delta x = 0$ $\delta\theta \neq 0$: $\delta W = LC_\theta\delta\theta F_x = LC_\theta F\sin(\omega t)\delta\theta = Q_\theta\delta\theta$

$$\Rightarrow \boxed{Q_\theta = LC_\theta F\sin(\omega t)}$$

Check w/ Standard way: $Q_\theta = \dfrac{\partial \underline{r}_A}{\partial\theta}\cdot\underline{F}_A + \dfrac{\partial \underline{r}_B}{\partial\theta}\cdot\underline{F}_B$

3. (40 points) Two point masses (m each) are connected by a linear spring (stiffness k), one rigidly attached to the end of a massless pendulum of length L and the other sliding without friction along the pendulum. The distance between masses is measured by d. The linear spring exerts no force when d=0. The angle of the pendulum is θ measured from the downward vertical.

Provide expressions for:
(a) The system's Lagrangian.
(b) The system's stiffness matrix.
(c) Two of the system's equilibrium points (there are 4 all together).



Potential Energy

$$V = -mgLc_\theta - mg(L-d)c_\theta + \tfrac{1}{2}kd^2$$

$$= mgc_\theta[d-2L] + \tfrac{1}{2}kd^2$$

$$\underline{\omega} = \dot{\theta}\hat{k}$$

$$\dot{\hat{e}}_r = \underline{\omega}\times\hat{e}_r = \dot{\theta}\hat{e}_\theta$$

$$\dot{\hat{e}}_\theta = \underline{\omega}\times\hat{e}_\theta = -\dot{\theta}\hat{e}_r$$

Kinematics & Kinetic Energy

$$\underline{r}_A = L\hat{e}_r \qquad \underline{v}_A = L\dot{\hat{e}}_r = L\theta\hat{e}_\theta$$

$$\underline{r}_B = (L-d)\hat{e}_r \qquad \underline{v}_B = -\dot{d}\hat{e}_r + (L-d)\dot{\hat{e}}_r = -\dot{d}\hat{e}_r + (L-d)\dot{\theta}\hat{e}_\theta$$

$$T = T_A + T_B = \tfrac{1}{2}m(L\dot{\theta})^2 + \tfrac{1}{2}m[(L-d)\dot{\theta}]^2 + \tfrac{1}{2}m\dot{d}^2$$

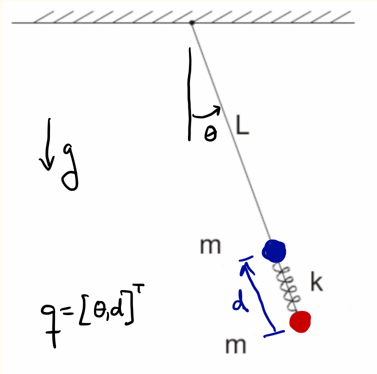
Partials of V

$$\frac{\partial V}{\partial \theta} = -mgs_\theta[d-2L] \qquad \frac{\partial V}{\partial d} = mgc_\theta + kd$$

$$[K] = \begin{bmatrix} \partial^2 V/\partial\theta^2 & \partial^2 V/\partial\theta\partial d \\ \partial^2 V/\partial\theta\partial d & \dfrac{\partial^2 V}{\partial d^2} \end{bmatrix} = \boxed{\begin{bmatrix} mgc_\theta[2L-d] & -mgs_\theta \\ -mgs_\theta & k \end{bmatrix}} \quad (b)$$

(a)
$$\boxed{\begin{aligned} \mathcal{L} &= T - V \\ &= \tfrac{1}{2}mL^2\dot{\theta}^2 + \tfrac{1}{2}m(L-d)^2\dot{\theta}^2 + \tfrac{1}{2}m\dot{d}^2 \\ &\quad - mgc_\theta[d-2L] - \tfrac{1}{2}kd^2 \end{aligned}}$$

(c) Two of the system's equilibrium points (there are 4 all together). ✓
(d) Assess stability at two of the system's equilibrium points. [60623 only]
(e) [Extra Credit +5 points] Assess stability at the other two equilibrium points. [60623 only]



$$\frac{\partial V}{\partial \theta} = -mg S_\theta \left[d-2L\right] = 0 \qquad \frac{\partial V}{\partial d} = mg C_\theta + Kd = 0$$

$$[K] = \begin{bmatrix} mg C_\theta [2L-d] & -mg S_\theta \\ -mg S_\theta & K \end{bmatrix}$$

Equilibrium

$\frac{\partial V}{\partial \theta} = 0$ ⓐ $\theta = 0$ ⓑ $\theta = \pi$ ⓒ $d = 2L$

Cases:

ⓐ $\frac{\partial V}{\partial d} = 0 \Rightarrow \boxed{\begin{matrix}\theta = 0 \\ d = -\frac{mg}{K}\end{matrix}}$

$$[K] = \begin{bmatrix} \overbrace{mg\left[2L + \frac{mg}{K}\right]}^{>0} & 0 \\ 0 & K \end{bmatrix} > 0$$

Stable

ⓑ $\frac{\partial V}{\partial d} = 0 \Rightarrow \boxed{\begin{matrix}\theta = \pi \\ d = \frac{mg}{K}\end{matrix}}$

$$[K] = \begin{bmatrix} mg\left[\frac{mg}{K} - 2L\right] & 0 \\ 0 & K \end{bmatrix}$$

$mg - 2LK > 0$

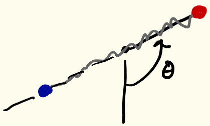
↳ Stable when $mg > 2LK$

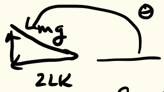
ⓒ $\frac{\partial V}{\partial d} = 0 \Rightarrow mg C_\theta + 2LK = 0$

$C_\theta = \frac{-2LK}{mg}$

exists when: $2LK \le mg$

$$\boxed{\begin{matrix}\theta = \pm \operatorname{acos}\left(\frac{-2LK}{mg}\right) \\ d = 2L\end{matrix}}$$





$$S_\theta = \frac{\sqrt{(mg)^2 - (2LK)^2}}{(mg)^2}$$

Missing in video

Any 2x2 matrix A has all pos. eig vals. iff $tr(A) > 0$ $det(A) > 0$

$$[K] = \begin{bmatrix} 0 & -\sqrt{(mg)^2 - (2LK)^2} \\ * & K \end{bmatrix}$$

$$det(K) = -\left[(mg)^2 - (2LK)^2\right] < 0$$

Unstable