Lecture 31 - Lagrangian Dynamics

- Examples
- Structure of The Eqns.

Alternate Method: Lagrangian Equations of Motion

Potential Energy: $U(\theta)$

Kinetic Energy: $K(\theta, \dot{\theta})$

Lagrangian: $\mathcal{L} = K - U$ (Analytical Dynamics)

$$\boxed{\tau = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} - \frac{\partial \mathcal{L}}{\partial \theta}}$$

$$\frac{\partial \mathcal{L}}{\partial \theta} = \begin{bmatrix} \partial \mathcal{L}/\partial \theta_1 \\ \vdots \\ \partial \mathcal{L}/\partial \theta_n \end{bmatrix}$$

Lagrangian: General Case $\quad (m_i , {}^{c_i}I)$

- Velocity Propagation (Kinetic Energy)

$$ {}^{i}\omega_i = {}^{i}R_{i-1} \, {}^{i-1}\omega_{i-1} + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_i \end{bmatrix} $$

$$ {}^{i}v_i = {}^{i}R_{i-1} \left[{}^{i-1}v_{i-1} + {}^{i-1}\omega_{i-1} \times {}^{i-1}p_i \right] + \begin{bmatrix} 0 \\ 0 \\ \dot{d}_i \end{bmatrix} $$

$$ {}^{i}v_{c_i} = {}^{i}v_i + {}^{i}\omega_i \times {}^{i}p_{c_i} $$

$$ K_i = \frac{1}{2} m_i \, \| {}^{i}v_{c_i} \|^2 + \frac{1}{2} \, {}^{i}\omega_i^T \, {}^{c_i}I \, {}^{i}\omega_i $$

$$ K = \sum_i K_i $$

- Kinematic Propagation (Potential Energy)

  - $$ {}^{0}T_i = {}^{0}T_{i-1} \, {}^{i-1}T_i = \begin{bmatrix} {}^{0}R_i & {}^{0}p_i \\ 0 & 1 \end{bmatrix} $$

  - $$ {}^{0}p_{c_i} = {}^{0}p_i + {}^{0}R_i \, {}^{i}p_{c_i} $$

  - $$ U_i = - m_i \, {}^{0}a_g^T \, {}^{0}p_{c_i} \qquad U = \sum_i U_i \qquad \downarrow g \quad {}^{0}a_g = \begin{bmatrix} 0 \\ 0 \\ -g \end{bmatrix} $$

Comparisson

RNEA

- Forward Prop
  - ${}^i\omega_i$, ${}^i\dot{\omega}_i$, ${}^i\dot{v}_i$, ${}^i\dot{v}_{c_i}$
- Newton/Euler Equs
  - ${}^iF_i$, ${}^iN_i$
- Backward pass
  - ${}^if_i$, ${}^in_i$
  - $\hookrightarrow \tau_i$

(Turn the crank Matrix Multiplication)

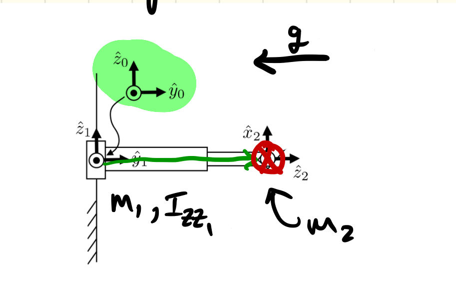
Lagrange

- Forward Prop
  - ${}^iv_i$, ${}^i\omega_i$, ${}^iv_{c_i}$, ${}^0T_i$, ${}^0p_{c_i}$
- Energy for each Body
  - K_i, U_i
- Lagranges equs

$$\tau = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} - \frac{\partial \mathcal{L}}{\partial \theta}$$

(Turn the crank taking partial derivatives)

Example



$$^0p_{c_1}, \; ^0p_{c_2}, \; ^1v_{c_1}, \; ^2v_{c_2}, \; ^1\omega_1, \; ^2\omega_2$$

$$U_1 = 0$$

$$^1v_{c_1} = 0 \qquad ^1\omega_1 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix}$$

$$K_1 = 0 + \underbrace{\frac{1}{2} I_{zz_1} \dot{\theta}_1^2}_{\frac{1}{2} \, ^1\omega_1 \, ^{c_1}I \, ^1\omega_1}$$

__Potential Body 2__

$$U_2 = m_2 c_1 g d_2$$

__Kinetic Body 2__

$$^2v_{c_2} = \begin{bmatrix} 0 \\ -d_2\dot{\theta}_1 \\ \dot{d}_2 \end{bmatrix} \qquad ^2\omega_2 = \begin{bmatrix} \dot{\theta}_1 \\ 0 \\ 0 \end{bmatrix}$$

$$K_2 = \frac{1}{2} m_2 \left(d_2^2 \dot{\theta}_1^2 + \dot{d}_2^2\right) + \frac{1}{2} \, ^2\omega_2^T \, ^{c_2}\cancel{I} \, ^2\omega_2$$

Example cont:

$$K = \frac{1}{2} I_{zz}\dot{\theta}_1^2 + \frac{1}{2} m_2(\dot{d}_2^2 + d_2^2\dot{\theta}_1^2)$$

$$U = g d_2 c_1 m_2$$

$$\Theta = \begin{bmatrix} \theta_1 \\ d_2 \end{bmatrix}$$

$$\mathcal{L} = K - U = \frac{1}{2} I_{zz}\dot{\theta}_1^2 + \frac{1}{2} m_2(\dot{d}_2^2 + d_2^2\dot{\theta}_1^2) - g d_2 c_1 m_2$$

$$\tau = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\Theta}} - \frac{\partial \mathcal{L}}{\partial \Theta}$$

$$\tau_1 = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}_1} - \frac{\partial \mathcal{L}}{\partial \theta_1} = \frac{d}{dt}\left[I_{zz}\dot{\theta}_1 + m_2 d_2^2\dot{\theta}_1\right] - \underbrace{\left[g d_2 s_1 m_2\right]}_{\partial \mathcal{L}/\partial \theta_1}$$

Terms from $M(\Theta)\ddot{\Theta}$

Coriolis

Gravity Terms

$$\tau_1 = I_{zz}\ddot{\theta}_1 + 2 m_2 d_2 \dot{d}_2 \dot{\theta}_1 + m_2 d_2^2 \ddot{\theta}_1 - g d_2 s_1 m_2$$

$$\tau_2 = \frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{d}_2}\right] - \left[\frac{\partial \mathcal{L}}{\partial d_2}\right] = \frac{d}{dt}\left[m_2 \dot{d}_2\right] - \left[m_2 d_2 \dot{\theta}_1^2 - g c_1 m_2\right]$$

centripetal

$$\tau_2 = m_2 \ddot{d}_2 - m_2 d_2 \dot{\theta}_1^2 + g c_1 m_2$$

Structure of Dynamic Eqns:

$M(\theta)\ddot{\theta} + V(\theta,\dot{\theta}) = \tau - \nabla_\theta U$

$$M(\theta)\ddot{\theta} + V(\theta,\dot{\theta}) + G(\theta) = \tau = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} - \frac{\partial \mathcal{L}}{\partial \theta}$$

$$\tau = \frac{d}{dt}\left[\frac{\partial K}{\partial \dot{\theta}}\right] - \frac{\partial K}{\partial \theta} + \frac{\partial U}{\partial \theta} \qquad \mathcal{L} = K - U$$

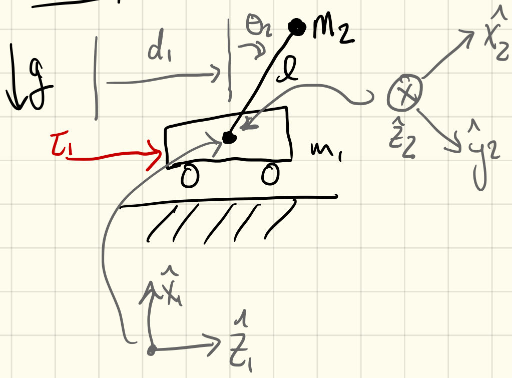
$$\tau_i = \frac{d}{dt}\left[\frac{\partial K}{\partial \dot{\theta}_i}\right] - \frac{\partial K}{\partial \theta_i} + \frac{\partial U}{\partial \theta_i} = \left[\sum_{j=1}^{n} \underbrace{\frac{\partial}{\partial \theta_j}\left[\frac{\partial K}{\partial \dot{\theta}_i}\right]}_{\theta,\dot{\theta}} \dot{\theta}_j + \underbrace{\frac{\partial}{\partial \dot{\theta}_j}\left[\frac{\partial K}{\partial \dot{\theta}_i}\right]}_{\theta} \ddot{\theta}_j\right] - \underbrace{\frac{\partial K}{\partial \theta_i}}_{\theta,\dot{\theta}} + \underbrace{\frac{\partial U}{\partial \theta_i}}_{\theta}$$

$$M(\theta) = \begin{bmatrix} \partial^2 K/(\partial\dot{\theta}_1)^2 & \partial^2 K/\partial\dot{\theta}_1\partial\dot{\theta}_2 & \cdots & \partial^2 K/\partial\dot{\theta}_1\partial\dot{\theta}_n \\ & \ddots & & \vdots \\ & & \ddots & \\ & & & \partial^2 K/\partial\dot{\theta}_n\partial\dot{\theta}_n \end{bmatrix} \qquad G(\theta) = \begin{bmatrix} \partial U/\partial\theta_1 \\ \vdots \\ \partial U/\partial\theta_n \end{bmatrix}$$

Hessian of Kinetic Energy!

$V(\theta,\dot{\theta}) = \ \ldots$

Example 2: Cart Pole (PR)



① Kinetic & Potential Energies

$$U_1 = 0 \qquad K_1 = \frac{1}{2} m_1 \| {}^1 v_{c_1} \|^2 + \frac{1}{2} {}^1\omega_1^T \, {}^{c_1} I \, {}^1\omega_1$$

$$= \frac{1}{2} m_1 \dot{d}_1^2$$

$$U_2 = m_2 \ell c_2 g \qquad K_2 = \frac{1}{2} m_2 \| {}^2 v_2 \|^2$$

$$= \frac{1}{2} m_2 \left(\dot{d}_1^2 + \ell^2 \dot{\theta}^2 + 2 \ell c_2 \dot{d}_1 \dot{\theta}_2 \right)$$

$${}^2 v_2 = \xrightarrow{\dot{d}_1} + \dot{\theta}_2 \otimes \times \overset{\ell}{\nearrow} = \xrightarrow{\dot{d}_1} + \overset{\ell \dot{\theta}_2}{\searrow}$$

$$= \begin{bmatrix} \dot{d}_1 s_2 \\ \dot{d}_1 c_2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \ell \dot{\theta}_2 \\ 0 \end{bmatrix} = \begin{bmatrix} \dot{d}_1 s_2 \\ \dot{d}_1 c_2 + \ell \dot{\theta}_2 \\ 0 \end{bmatrix}$$

② Lagrangian

$$\mathcal{L} = K - U = \frac{1}{2} (m_1 + m_2) \dot{d}_1^2 + \frac{1}{2} m_2 \ell^2 \dot{\theta}_2^2 + m_2 \ell c_2 \dot{d}_1 \dot{\theta}_2 - m_2 \ell c_2 g$$

Cart Pole Continued

$$\mathcal{L} = K - U = \tfrac{1}{2}(m_1+m_2)\dot{d}_1^2 + \tfrac{1}{2} m_2 \ell^2 \dot{\theta}_2^2 + m_2 \ell c_2 \dot{d}_1 \dot{\theta}_2 - m_2 \ell c_2 g$$

$$\tau_1 = \frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{d}_1}\right] - \frac{\partial \mathcal{L}}{\partial d_1} = \frac{d}{dt}\left[(m_1+m_2)\dot{d}_1 + m_2 \ell c_2 \dot{\theta}_2\right] - 0$$

$$= (m_1+m_2)\ddot{d}_1 + m_2 \ell c_2 \ddot{\theta}_2 - m_2 \ell s_2 \dot{\theta}_2^2$$

$$0 = \tau_2 = \frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{\theta}_2}\right] - \frac{\partial \mathcal{L}}{\partial \theta_2} = \frac{d}{dt}\left[m_2 \ell^2 \dot{\theta}_2 + m_2 \ell c_2 \dot{d}_1\right] - \left[-m_2 \ell s_2 \dot{d}_1 \dot{\theta}_2 + m_2 g \ell s_2\right]$$

$$= m_2 \ell^2 \ddot{\theta}_2 + m_2 \ell c_2 \ddot{d}_1 \cancel{- m_2 \ell s_2 \dot{d}_1 \dot{\theta}_2} \cancel{+ m_2 \ell s_2 \dot{d}_1 \dot{\theta}_2} - m_2 g \ell s_2$$

Equations of Motion

$$\begin{bmatrix} \tau_1 \\ 0 \end{bmatrix} = \underbrace{\begin{bmatrix} m_1+m_2 & m_2 \ell c_2 \\ m_2 \ell c_2 & m_2 \ell^2 \end{bmatrix}}_{M(\theta)} \underbrace{\begin{bmatrix} \ddot{d}_1 \\ \ddot{\theta}_2 \end{bmatrix}}_{\ddot{\theta}} + \underbrace{\begin{bmatrix} -m_2 \ell s_2 \dot{\theta}_2^2 \\ 0 \end{bmatrix}}_{V(\theta, \dot{\theta})} + \underbrace{\begin{bmatrix} 0 \\ -m_2 g \ell s_2 \end{bmatrix}}_{G(\theta)}$$