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Analytical Dynamics

Analysis of Lagrangian Dynamics

Lagrange

Admin: ① Mid semester Survey (Tues, 11:59PM) ② HW5 (Friday, 5PM) ③ Exam 2, next Weds. Debart 141

Before Break: Lagrange's Equations: $\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_k} - \frac{\partial \mathcal{L}}{\partial q_k} = Q_{k,nc} \quad k = 1, \dots, n$

Kinetic Energy: $T = \overbrace{\frac{1}{2}\dot{q}^T M(q,t)\,\dot{q}}^{T_2} + \overbrace{B^T(q,t)\,\dot{q}}^{T_1} + T_0(q,t)$

T_1, T_0: $=0$ for a natural system

Today:
- Analyze structure of Lagrange's equations
- Vibration analysis

Review: Consider a scalar function $f(q,\dot{q})$:

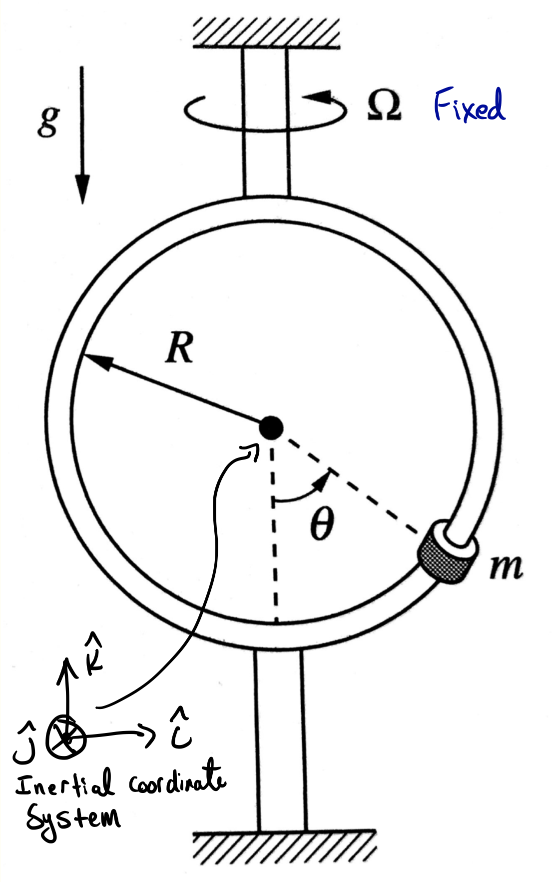
Gradient

$$\nabla_q f = \begin{bmatrix} \partial f/\partial q_1 \\ \vdots \\ \partial f/\partial q_n \end{bmatrix}$$

Hessian

$$\nabla_q^2 f = \begin{bmatrix} \partial^2 f/\partial q_1^2 & \cdots & \partial f/\partial q_1 \partial q_n \\ \vdots & \ddots & \vdots \\ \partial^2 f/\partial q_n \partial q_1 & \cdots & \partial^2 f/\partial q_n^2 \end{bmatrix}$$

Natural vs. Non-Natural Systems



$$\{r_{BEAD}(q,t)\} = \begin{bmatrix} R S\theta \, C(\Omega t) \\ R S\theta \, S(\Omega t) \\ -R C\theta \end{bmatrix}$$

$$T = \underbrace{\tfrac{1}{2}\dot{\theta}^2 (mR^2)}_{T_2} + \underbrace{0}_{T_1} + \underbrace{\tfrac{1}{2} m (R S_\theta \Omega)^2}_{T_0}$$

$M = mR^2$ (1x1 mass matrix here)

Structure of the EOM

Ignore. Uncommon in practice

Ignore for simplicity

$$\mathcal{L} = T - V = \frac{1}{2}\dot{q}^T M(q,\cancel{t})\,\dot{q} + \cancel{B(q,t)^T}\,\dot{q} - \underbrace{\left[V(q,\cancel{t}) - T_0(q,\cancel{t})\right]}_{:U = V - T_0}$$

Ignore for simplicity

$$Q_{nc} = \frac{d}{dt}\left[\nabla_{\dot{q}}\mathcal{L}\right] - \left[\nabla_{q}\mathcal{L}\right] = \frac{d}{dt}\left[M(q)\dot{q}\right] - \nabla_q\left[\tfrac{1}{2}\dot{q}^T M \dot{q}\right] + \nabla_q U$$

$$= M(q)\ddot{q} + \underbrace{\dot{M}\dot{q} - \nabla_q T_2} + \nabla_q U$$

terms here like $\dot{q}_i\dot{q}_k$ OR $\dot{q}_k^2$

This quantity referred to as coriolis & centripetal terms <u>OR</u> velocity product terms. Denote by $C\dot{q}$

MORE DETAIL NEXT PAGE

Common Form

$$\boxed{Q_{nc} = M(q)\ddot{q} + C(q,\dot{q})\dot{q} + \nabla_q U}$$

EXTRA Content: Just FYI in case of interest (important for control-focused robotics grad students)
[not on tests]

Consider the Vector: $\dot{M}\dot{q} - \nabla_q T_2$ Look at i-th component

$\dot{M}$

$$\hookrightarrow \left[\dot{M}\dot{q} - \nabla_q T_2\right]_i = \left(\left[\sum_{k=1}^{n} \frac{\partial M}{\partial q_k}\dot{q}_k\right]\dot{q}\right)_i - \frac{\partial \frac{1}{2}\dot{q}^T M(q)\dot{q}}{\partial q_i}$$

$$= \sum_{k=1}^{n}\sum_{j=1}^{n} \frac{\partial M_{ij}}{\partial q_k}\dot{q}_k \dot{q}_j - \frac{\partial}{\partial q_i}\left[\sum \tfrac{1}{2} M_{jk}\dot{q}_j \dot{q}_k\right]$$

$$= \sum_{k=1}^{n}\sum_{j=1}^{n} \underbrace{\left[\frac{\partial M_{ij}}{\partial q_k} - \frac{1}{2}\frac{\partial M_{jk}}{\partial q_i}\right]}_{\text{depends only on } q} \underbrace{\dot{q}_k \dot{q}_j}_{\text{Velocity products in every term}}$$

Since $M_{ij} = M_{ji}$ we often write this sum as:

$$= \sum_{j=1}^{n}\sum_{k=1}^{n} \frac{1}{2}\left[\frac{\partial M_{ij}}{\partial q_k} + \frac{\partial M_{ji}}{\partial q_k} - \frac{\partial M_{jk}}{\partial q_i}\right]\dot{q}_k \dot{q}_j$$

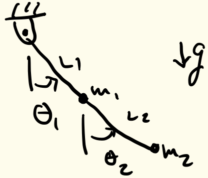
Called the Christoffel symbol of the first kind $\Gamma_{ijk}(q)$

Letting $\left[C(q,\dot{q})\right]_{ij} = \sum_{k=1}^{n} \Gamma_{ijk}(q)\,\dot{q}_k$ it follows that $\dot{M}\dot{q} - \nabla_q T_2 = C(q,\dot{q})\dot{q}$

Fun Fact: $\dot{M} = C + C^T$ using this definition. Useful in many stability proofs in control.

Example: Give $M(\underset{\sim}{q})$, $C(\underset{\sim}{q},\dot{\underset{\sim}{q}})\dot{\underset{\sim}{q}}$ and $\nabla_{\underset{\sim}{q}}V$ for the Double Pendulum

Extra Example: Not in lecture



Using the supplemental MATLAB code from Lecture 14

$$\mathcal{L} = \overbrace{\tfrac{1}{2}(m_1+m_2)L_1^2\dot{\theta}_1^2 + m_2L_2^2\dot{\theta}_2^2 + m_2L_1L_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2)}^{T} + \overbrace{m g L_1 c_1 + m_2 g(L_1c_1 + L_2c_2)}^{-V}$$

EOM from MATLAB

$$(m_1+m_2)L_1^2\ddot{\theta}_1 + m_2L_1L_2\cos(\theta_1-\theta_2)\ddot{\theta}_2 + m_2L_1L_2\sin(\theta_1-\theta_2)\dot{\theta}_2^2 + gL_1S_1(m_1+m_2) = 0$$

$$m_2L_1L_2\cos(\theta_1-\theta_2)\ddot{\theta}_1 + m_2L_2^2\ddot{\theta}_2 - m_2L_1L_2\sin(\theta_1-\theta_2)\dot{\theta}_1^2 + gm_2L_2S_2 = 0$$

Alternate Form

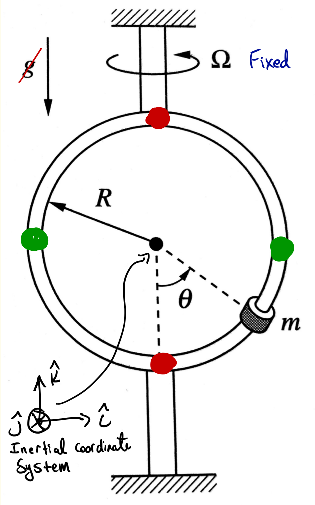
$$\underbrace{\begin{bmatrix} (m_1+m_2)L_1^2 & m_2L_1L_2\cos(\theta_1-\theta_2) \\ m_2L_1L_2\cos(\theta_1-\theta_2) & m_2L_2^2 \end{bmatrix}}_{M(\underset{\sim}{q})}\underbrace{\begin{bmatrix}\ddot{\theta}_1 \\ \ddot{\theta}_2\end{bmatrix}}_{\ddot{\underset{\sim}{q}}} + \underbrace{\begin{bmatrix} m_2L_1L_2\sin(\theta_1-\theta_2)\dot{\theta}_2^2 \\ -m_2L_1L_2\sin(\theta_1-\theta_2)\dot{\theta}_1^2 \end{bmatrix}}_{C\dot{\underset{\sim}{q}}} + \underbrace{\begin{bmatrix} gL_1S_1(m_1+m_2) \\ gL_2S_2m_2 \end{bmatrix}}_{\nabla_{\underset{\sim}{q}}V} = 0$$

Equilibrium for Non-Natural Systems

$$T = \frac{R^2 m}{2}\left[\Omega^2 s^2_\theta + \dot{\theta}^2\right] = \mathcal{L}$$

$$U = -T_0 = -\frac{1}{2} m R^2 \Omega^2 s^2_\theta$$

$$V = 0$$



$$0 = \frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right] - \frac{\partial \mathcal{L}}{\partial \theta} = mR^2\ddot{\theta} \overbrace{- mR^2\Omega^2 c_\theta s_\theta}^{\partial U/\partial \theta}$$

We consider a non natural system @ equilibrium <u>when</u> q is constant (when $\dot{q}=0$, $\ddot{q}=0$)

$$q^* \text{ an equilibrium } \Longleftrightarrow \nabla_q U\Big|_{q=q^*} = 0$$

<u>An equilibrium</u> q^* <u>is stable if</u>:

$$\left[\nabla^2_q U\Big|_{q=q^*}\right] \succ 0$$

pos definite

all positive eig vals.

Equilibria @ $\theta = k\frac{\pi}{2}$

$$\frac{\partial U}{\partial \theta} = -mR^2\Omega^2 c_\theta s_\theta$$

$$\frac{\partial^2 U}{\partial \theta^2} = -mR^2\Omega^2\left[c^2_\theta - s^2_\theta\right]$$

when $s_\theta = 0 \Rightarrow \frac{\partial^2 V}{\partial \theta^2} < 0$ (unstable)

$c_\theta = 0 \Rightarrow \partial^2 V/\partial \theta^2 > 0$ (stable)

Equilibrium and Linearization:

Consider the case when $M(q,t)=M(q)$, $T_1=0$, $U(q,t)=U(q)$, $Q_{nc}=0$

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + \nabla_q U = 0$$

Derivation on next slide

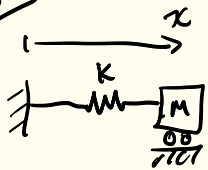
For an equilibrium q^*, locally, the dynamics follow linearization:

$$M(q^*)\ddot{q} + \nabla_q^2 U\Big|_{q=q^*}(q-q^*) = 0$$

Denote as $M^* = M(q^*) = \nabla_{\dot{q}}^2 T\Big|_{q=q^*}$

we call this the stiffness matrix ($n \times n$ matrix) Denote as K^*

Simple case



$m\ddot{x} + K(x-x_0) = 0$ oscillation @ freq $= \sqrt{K/m}$

Extra detail: Linearization around equilibrium point q^* [Detail not on tests]

$$0 = M(q)\ddot{q} + C(q,\dot{q})\dot{q} + \nabla_q U$$

We consider the above in the vicinity of $(q,\dot{q},\ddot{q}) = (q^*, \underbrace{0}_{\dot{q}^*}, \underbrace{0}_{\ddot{q}^*})$.

Look at $q(t) = q^* + \Delta q(t)$

$$0 = M(q^*+\Delta q)(\ddot{q}^* + \Delta\ddot{q}) + C(q^*+\Delta q, \dot{q}^*+\Delta\dot{q})(\dot{q}^* + \Delta\dot{q}) + \nabla_q U\Big|_{q=q^*+\Delta q}$$

Red terms are zero

$$= M(q^*)\ddot{q}^* + C(q^*,\dot{q}^*)\dot{q}^* + \nabla U\Big|_{q=q^*} + \left[\sum_{k=1}^{n} \frac{\partial M}{\partial q_k}\Delta q_k\right]\ddot{q}^* + M(q^*)\Delta\ddot{q}$$

$$+ \left[\frac{\partial C}{\partial q_k}\Delta q_k + \frac{\partial C}{\partial \dot{q}_k}\Delta\dot{q}_k\right]\dot{q}^* + C(q^*,\dot{q}^*)\Delta\dot{q}_k + \left(\nabla_q^2 U\Big|_{q=q^*}\right)\Delta q + \text{h.o.t.}$$

$$0 = M(q^*)\ddot{q} + K^*(q-q^*) + \text{"higer order terms"}$$

Modal Analysis

Consider a stable equilibrium q^* with $K^* > 0$

The solutions to $M^*\ddot{q} + K^*(q - q^*) = 0$ are described by n modes: $\underline{u}_r e^{\lambda_r t} + q^*$

$\underline{u}_r$ (nx1 vector)

- Find roots of $\det(\lambda^2 M + K) = 0$
  $\det(\lambda^2 I + M^{-1}K) = 0$ $\Rightarrow$ all roots occur in imaginary pairs
  $\lambda_r = \pm i\omega_r$ $\quad$ ω_r^2 an eig val of $M^{*-1}K^*$

- We call ω_r the r-th natural frequency

- The r-th mode shape is given by the eigenvector of $M^{*-1}K^*$ as

$$\left[M^{*-1}K^* - \omega_r^2 I\right]\underline{u}_r = 0$$

- Any solution to $M^*\ddot{q} + K^*(q - q^*) = 0$ can be written as:

$$q(t) = q^* + \sum_{r=1}^{n} \underline{u}_r A_r \cos(\omega_r t + \phi_r)$$

Amplitude $\quad$ phase

Extra Detail: Modal Analysis [Not on test]

① $\boxed{\text{Roots of } \det(\lambda^2 M + K) = 0}$

$$\det(\lambda^2 M + K) = 0 \iff \det(M^{-1})\det(\lambda^2 M + K) = 0 = \det(\lambda^2 I + M^{-1}K)$$

Since $M \succ 0$, $M^{-1} \succ 0$

② $\boxed{\text{Roots come in complex pairs:}}$

Consider $M^{-1/2}$ the symmetric square root of M^{-1}: i.e., $(M^{-1/2})^2 = M^{-1}$ $\quad (M^{-1/2} \succ 0)$

$$\det(\lambda^2 M + K) = 0 \iff 0 = \det(M^{-1/2})\det(\lambda^2 M + K)\det(M^{-1/2})$$

$$= \det(\lambda^2 I + M^{-1/2} K M^{-1/2}) \quad (*)$$

Since $K \succ 0$, $M^{-1/2} K M^{-1/2} \succ 0$, so all eig vals of $M^{-1/2} K M^{-1/2}$ are positive

i.e. roots of $\det(M^{-1/2} K M^{-1/2} - \mu I) = 0$ have $\mu > 0$

$\Rightarrow$ roots λ of $(*)$ satisfy $\lambda^2 < 0 \Rightarrow$ roots λ come in complex pairs

Extra Detail: Modal Analysis Cont [Not on test]

③ $\boxed{\text{The mode shapes are } K^* \text{ and } M^* \text{ orthogonal}}$

we say two vectors v, w are Q orthogonal ($Q = Q^T > 0$) if $v^T Q w = 0$

Let $\underline{u}_r$ and $\underline{u}_\ell$ mode shapes for ω_r and ω_ℓ with $\omega_r \neq \omega_\ell$

By Definition:

$$\omega_r^2 M^* \underline{u}_r = K^* \underline{u}_r \qquad \omega_\ell^2 M^* \underline{u}_\ell = K^* \underline{u}_\ell$$

ⓐ
$$\begin{aligned}\underline{u}_\ell^T K^* \underline{u}_r &= \omega_r^2 \underline{u}_\ell^T M^* \underline{u}_r \\ &= (\underline{u}_\ell^T K^* \underline{u}_r)^T \\ &= \underline{u}_r^T K^* \underline{u}_\ell = \omega_\ell^2 \underline{u}_r^T M^* \underline{u}_\ell\end{aligned}$$

$$\Rightarrow 0 = \underbrace{(\omega_r^2 - \omega_\ell^2)}_{\neq 0} \underline{u}_r^T M^* \underline{u}_\ell$$

$$\boxed{\therefore \underline{u}_r^T M^* \underline{u}_\ell = 0}$$

ⓑ
$$\begin{aligned}\underline{u}_\ell^T M^* \underline{u}_r &= \frac{1}{\omega_r^2} \underline{u}_\ell^T K^* \underline{u}_r \\ &= (\underline{u}_\ell^T M^* \underline{u}_r)^T \\ &= \underline{u}_r^T M^* \underline{u}_\ell = \frac{1}{\omega_\ell^2} \underline{u}_r^T K^* \underline{u}_\ell\end{aligned}$$

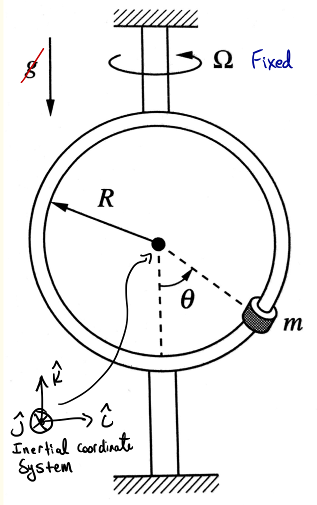
$$0 = \underbrace{\left[\frac{1}{\omega_r^2} - \frac{1}{\omega_\ell^2}\right]}_{\neq 0} \underline{u}_r^T K^* \underline{u}_\ell$$

$$\boxed{\therefore \underline{u}_r^T K^* \underline{u}_\ell = 0}$$

Equilibrium for Non-Natural Systems

$$T = \frac{R^2 m}{2}\left[\Omega^2 s_\theta^2 + \dot{\theta}^2\right] = \mathcal{L}$$

$$U = -\frac{1}{2} m R^2 \Omega^2 s_\theta^2$$



$q^* = \pi/2$: Give nat. freqs. and mode shapes for oscillations around q^*

$$M^* = M(q^*) = mR^2$$

$$K^* = \nabla_q^2 U\Big|_{q=q^*} = \frac{\partial^2 U}{\partial \theta^2} = mR^2\Omega^2$$

$$M^{*-1}K^* = \Omega^2$$

Just one oscillation mode since one DoF

$$\omega_r = \Omega$$

$$\underline{v}_r = 1$$

Equilibria @ $\theta = K\frac{\pi}{2}$

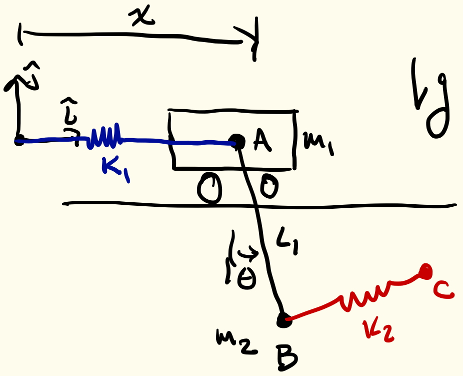
$$\frac{\partial U}{\partial \theta} = -mR^2\Omega^2 s_\theta c_\theta$$

$$\frac{\partial^2 U}{\partial \theta^2} = -mR^2\Omega^2\left[c_\theta^2 - s_\theta^2\right]$$

when $c_\theta = 0 \Rightarrow$ stable

$s_\theta = 0 \Rightarrow$ unstable

MATLAB Example: Cart Pole

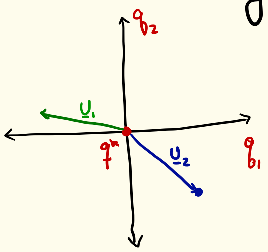


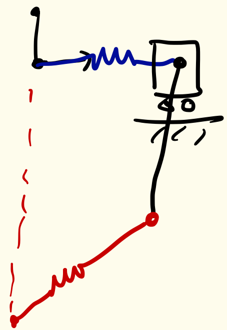
- Spring rest lengths 0m
- <u>Case:</u> $x_0=0$ $y_0=-2m$ $L_1=1m$

$$q_{eq} = \begin{bmatrix} x_{eq} \\ \theta_{eq} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \ \underline{\text{OR}} = \begin{bmatrix} 0 \\ \pi \end{bmatrix}$$

$r_c = x_0\hat{i} + y_0\hat{j}$

<u>Mode Shapes</u>: If initial condition starts along $\underline{u}_i$, it stays along $\underline{u}_i$.





Summary

- Natural System (no prescribed motion for any part of the system)
  - Lagranges equations take the form

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + \nabla_q V = Q_{nc}$$

$M(q) = \nabla^2_{\dot{q}} T$ Coriolis and Centripetal Velocity-product terms

Positive Definite Mass matrix

- When $\nabla_q V\big|_{q=q^*} = 0 \Rightarrow$ Small vibrations are solutions of $M(q^*)\ddot{q} + K^*(q-q^*) = 0$

$= \nabla^2_q V\big|_{q=q^*}$

- Vibration modes: $\mathrm{EigVecs}(M^{*-1}K^*)$ Frequencies: $\sqrt{\mathrm{EigVals}(M^{*-1}K^*)}$

- Non-natural systems (prescribed motion for some part)
  - $T = \frac{1}{2}\dot{q}^T M \dot{q} + \beta^T \dot{q} + T_0$ $\Leftarrow$ often the gyroscopic term $\beta = 0$
  - System Behaves as if potential $= V - T_0$