Analytical Dynamics

The End

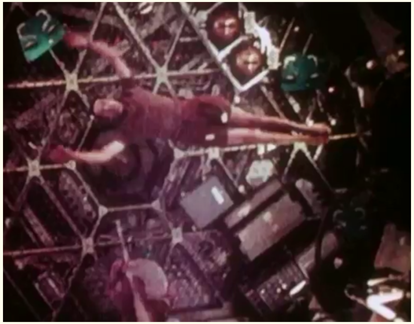
Admin:
- HW 9 Solution online tonight
- Final: Tuesday Dec 17th 4:15-6:15 PM (140 DeBartolo)
- Final review list available on Sakai
- Please complete your CIFs! Last Day 12/15 (Sunday)

Today:
- Discuss Final Format
- Review Concepts & Problems
- Discuss my long-term goals for learning outcomes

Other Exploits of Thomas Kane

- Conservation of angular momentum is a nonholonomic constraint
- Cats (& Astronauts) can flip over while having zero angular momentum

https://m.youtube.com/watch?v=5TgtW0wDg9E

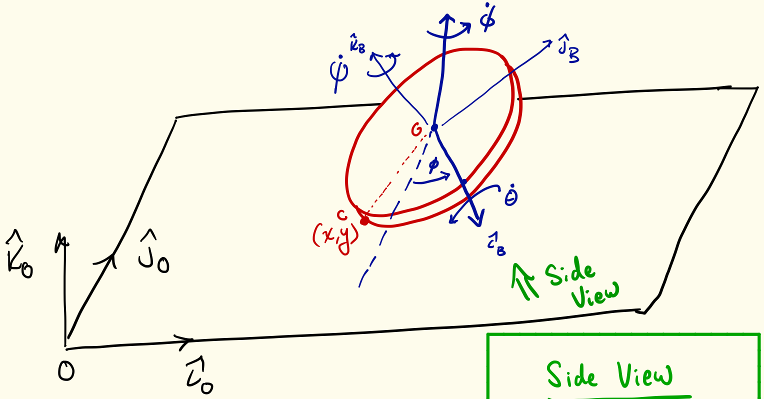


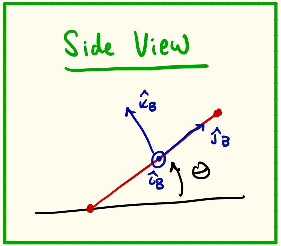
Dropped upside down, the cat shown on this page lands neatly on its feet. The spaceman (in this case a trampolinist in a spacesuit) carefully mimics the cat's motions, a maneuver that may come in handy when he finds himself wafting weightless Out There. The cat's knack is instinctive; to copy it the less gifted spaceman must rely on a series of mathematical equations developed by a professor of applied mechanics.

SPACE / A Copycat Astronaut

Example: Thin Coin (MATLAB online)

Additional Kane's example





$$^{B}\underline{I}_G = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_1 & 0 \\ 0 & 0 & I_3 \end{bmatrix}$$

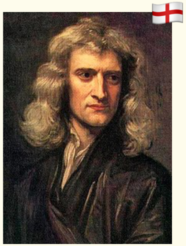
Notes on Sakai w/ MATLAB

with Lecture 27

AME 40623/60623 Analytical Dynamics

Newtonian Dynamics

Analytical Dynamics



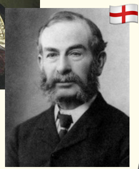
Newton 1687

D'Alembert 1742

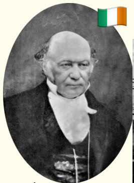
Euler 1765

Lagrange 1778

Gauss 1829



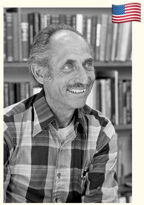
Routh 1860



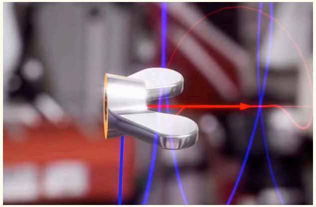
Hamilton 1833

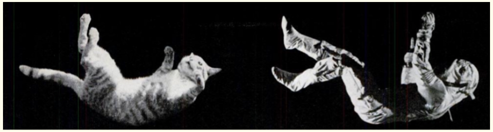
Jourdain 1909

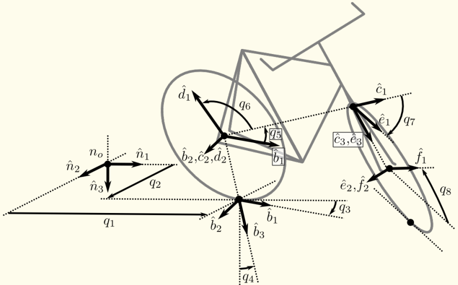
Noether 1915



Kane 1961







Final Details:

- When : Tues. Dec. 17th 4:15-6:15PM
- Where: 140 DeBartolo
- Rules: • Open note / laptop / MATLAB
  - 120 minutes

What to Expect on Final

- 5 problems (at least one from extra probs on Sakai), 20 pts each
  1. 40623 vs. 60623 Grab bag
  2. Rigid Body Dynamics (3D)
  3. D'Alembert's Principle (2D)
  4. Lagrangian Dynamics (3D)
  5. Kane's method (3D-ish)

Part 1: Newtonian Fundamentals	Part 2: Analytical Fundamentals	Part 3: Analytical Dyn. w/ 3D rigid bodies
• Impulse & Momentum $\dot{\underline{p}} = \underline{F}_{net}$ $\dot{\underline{H}}_G = \underline{M}_{G,net}$ • Equations of motion by sampling $\underline{F}=m\underline{a}$ along directions to avoid constraint forces • Reference Frames • Angular velocity • Moving Frames — Example next • Equilibria & stability • Integrals of Motion • Conservative Forces $\underline{F}_i = -\nabla V(\underline{r}_i)$	• Virtual Work • Generalized Forces $\underline{Q}_c = -\nabla_q V$ $\underline{Q}_{nc} = \sum_i \left\{\frac{\partial \underline{r}_i}{\partial q}\right\}^T \{\underline{F}_i\}$ • D'Alembert's principle $\underline{Q}_c + \underline{Q}_{nc} = \underline{Q}_{inertial}$ $= \sum_i \left\{\frac{\partial \underline{r}_i}{\partial q}\right\}^T \{m_i \underline{a}_i\}$ • Lagrangian Dynamics $\frac{d}{dt}\left[\nabla_{\dot{q}} \mathcal{L}\right] - \nabla_q \mathcal{L} = \underline{Q}_{nc}$ $\mathcal{L} = T - V$ Example next • Natural & non-natural systems • Hamiltonian Dynamics & Routhian Reduction • Modal Analysis	• Angular Momentum & Rot. Inertia ${}^A\{\underline{H}_G\} = {}^A\underline{I}_G\, {}^A\{\omega\}$ Example next • Euler's Equation ${}^A\underline{I}_G\, {}^A\{\dot{\underline{\omega}}\} + {}^A\{\underline{\omega}\} \times {}^A\underline{I}_G\, {}^A\{\underline{\omega}\}$ $= {}^A\{\dot{\underline{H}}_G\} = {}^A\{\underline{M}_{G,net}\}$ Example next • Kinetic Energy $T = \frac{1}{2} m \|\underline{v}_G\|^2 + \frac{1}{2}\, {}^A\{\underline{\omega}\}^T\, {}^A\underline{I}_G\, {}^A\{\underline{\omega}\}$ • Generalized Forces $\underline{Q} = \sum_i \left\{\frac{\partial \underline{v}_i}{\partial \dot{q}}\right\}^T \{\underline{F}_i\} + \sum_j \left\{\frac{\partial \underline{\omega}_j}{\partial \dot{q}}\right\}^T \{\underline{M}_j\}$ $\underline{Q}_{inertial} = \sum_i \left\{\frac{\partial \underline{\omega}_i}{\partial \dot{q}}\right\}^T \{\dot{\underline{H}}_{G_i}\} + \left\{\frac{\partial \underline{v}_{G_i}}{\partial \dot{q}}\right\}^T \{m_i \underline{a}_i\}$ Example next • Kane's method

Concept Map (assume all constraints captured by GC)

EXERCISE: CREATE A VERSION OF THIS MAP FOR THE COURSE OVERALL

Generalized Coordinates

$$\mathbf{q} = [q_1, \ldots, q_n]^T$$

$$\mathbf{r}_i = \mathbf{r}_i(\mathbf{q}, t)$$
$$i = 1, \ldots, N$$

Virtual Displacements

$\delta \mathbf{r}_1, \ldots, \delta \mathbf{r}_N$
occur in zero time

Newton's Laws

$$\mathbf{F}_i + \mathbf{F}'_i = m_i \ddot{\mathbf{r}}_i$$

Active Constr.

Constraint forces do no virtual work under virtual displacements consistent with the constraints

Virtual Work

$$\delta W = \sum_i \mathbf{F}_i \cdot \delta \mathbf{r}_i$$

D'Alembert's Principle

$$\underbrace{\sum_i \mathbf{F}_i \cdot \delta \mathbf{r}_i}_{\sum_k Q_k \delta q_k} = \sum_i m_i \ddot{\mathbf{r}}_i \cdot \delta \mathbf{r}_i$$

Generalized Forces

$$Q_k = \sum_i \frac{\partial \mathbf{r}_i}{\partial q_k} \cdot \mathbf{F}_i$$

$$\mathbf{Q} = \sum_i \{\mathbf{J}_i\}^T \{\mathbf{F}_i\}$$

$$\delta W = \mathbf{Q}^T \delta \mathbf{q}$$
$$= \sum_k Q_k \delta q_k$$

EOM Via D'Alembert

$$Q_{k,\text{inertial}} = \sum_i \frac{\partial \mathbf{r}_i}{\partial q_k} \cdot (m_i \ddot{\mathbf{r}}_i)$$

EOM: $Q_k = Q_{k,\text{inertial}}$

Calc. of Variations

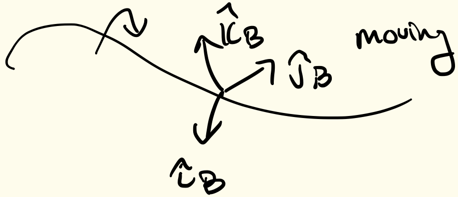
EOM Via Lagrange

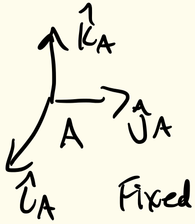
$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial \mathscr{L}}{\partial \dot{q}_k} - \frac{\partial \mathscr{L}}{\partial q_k} = Q_{k,\text{nc}}$$

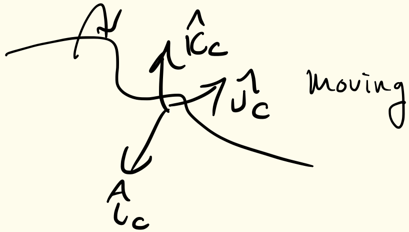
Moving Frames

$$\underline{\omega}_C = \alpha \hat{\imath}_A + \beta \hat{\jmath}_A + \gamma \hat{k}_A$$

$$= \omega_x \hat{\imath}_B + \omega_y \hat{\jmath}_B + \omega_z \hat{k}_B$$







Angular acceleration

$$\dot{\underline{\omega}}_C = \dot{\alpha} \hat{\imath}_A + \dot{\beta} \hat{\jmath}_A + \dot{\gamma} \hat{k}_A$$

$$= \underbrace{\dot{\omega}_x \hat{\imath}_B + \dot{\omega}_y \hat{\jmath}_B + \dot{\omega}_z \hat{k}_B}_{\text{Accounts for change of coordinates}} + \underbrace{\underline{\omega}_B \times \underline{\omega}_C}_{\text{Accounts for change of coordinate vectors.}} \quad (*)$$

Equivalent

In coordinates:

$${}^{B}\{\underline{\omega}_C\} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

$${}^{B}\{\dot{\underline{\omega}}_C\} = \frac{d}{dt}\,{}^{B}\{\underline{\omega}_C\} + {}^{B}\{\underline{\omega}_B\} \times {}^{B}\{\underline{\omega}_C\} \quad (*)$$

$${}^{C}\{\dot{\underline{\omega}}_C\} = \frac{d}{dt}\,{}^{C}\{\underline{\omega}_C\} + \cancel{{}^{C}\{\underline{\omega}_C\} \times {}^{C}\{\underline{\omega}_C\}}^{\,0}$$

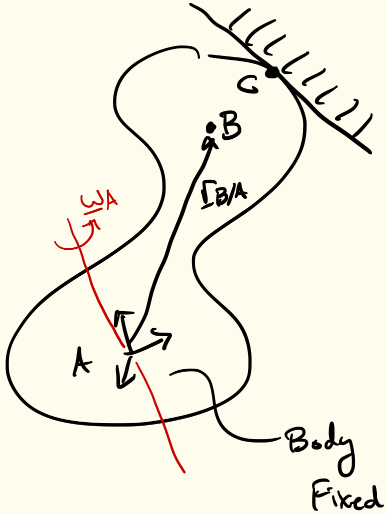
$\boxed{\text{Special case}}$

Application of Transport Thm: Rigid Body

$$\underline{r}_{B/A} = x\,\hat{i}_A + y\,\hat{j}_A + z\,\hat{k}_A$$

$$\dot{\underline{r}}_{B/A} = \underline{\omega}_A \times \underline{r}_{B/A}$$

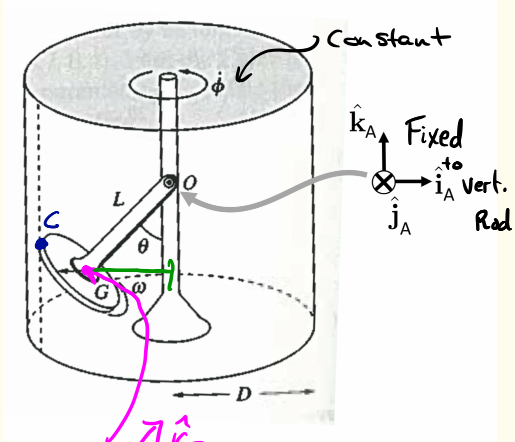
$$\underline{V}_B = \dot{\underline{r}}_A + \dot{\underline{r}}_{B/A} = \underline{V}_A + \underline{\omega}_A \times \underline{r}_{B/A}$$

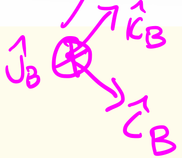


If c in contact:

$$\underline{V}_c = \underline{V}_A + \underline{\omega}_A \times \underline{r}_{c/A} = 0$$

Example: HW 7 problem 3: Find ω and $\dot{\underline{\omega}}_{Disk}$





B fixed To rod

$\theta = 30°$ Disk radius R, mass m

C fixed to Disk coincides w/ contact @ this instant.

$$\underline{\omega}_A = \dot\phi \hat{k}_A = \dot\phi(\hat{k}_B C_\theta - \hat{i}_B S_\theta) = \omega_B$$

$$\underline{\omega}_{Disk} = \dot\phi \hat{k}_A + \omega \hat{k}_B \qquad \hat{k}_A = \hat{k}_B C_\theta - \hat{i}_B S_\theta$$

$$= (\dot\phi C_\theta + \omega)\hat{k}_B - \dot\phi S_\theta \hat{i}_B$$

Using contact condition $\underline{V}_C = 0$ $\quad \underline{V}_G = -L S_\theta \dot\phi \hat{j}_B$

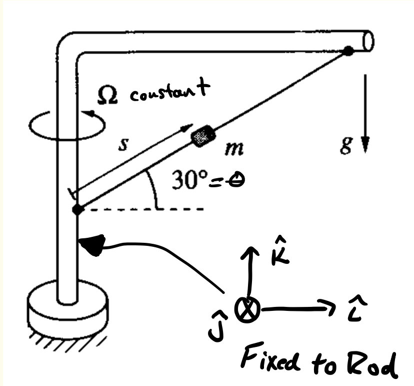
$$\underline{V}_C = \underline{V}_G + \underline{\omega}_{Disk} \times (-R\hat{i}_B)$$

$$= -L S_\theta \dot\phi \hat{j}_B - (\dot\phi C_\theta + \omega) R \hat{j}_B = 0 \qquad \omega = -\dot\phi \left[\frac{L S_\theta + R C_\theta}{R}\right]$$

$$\underline{\omega}_{Disk} = -\frac{L}{R} S_\theta \dot\phi \hat{k}_B - \dot\phi S_\theta \hat{i}_B$$

$$\dot{\underline{\omega}}_{Disk} = \underline{\omega}_B \times \underline{\omega}_{Disk} = -\dot\phi^2 S_\theta \left[\frac{L}{R} S_\theta + C_\theta\right] \hat{j}_B$$

LAGRANGIAN EXAMPLE



Setup: Find T_2, T_1, T_0, M and s @ equilibrium

$$\underline{v} = \dot{s}\left[\hat{\imath}\frac{\sqrt{3}}{2} + \hat{k}\frac{1}{2}\right] + \frac{\sqrt{3}}{2} s\Omega\hat{\jmath}$$

$$T = \frac{1}{2}m\left(\dot{s}^2 + \frac{3}{4}s^2\Omega^2\right) \qquad T_2 = \frac{1}{2}m\dot{s}^2$$

$$V = m\frac{g}{2}s \qquad T_1 = 0$$

$$T_0 = \frac{3}{8}m s^2\Omega^2$$

$$M = m$$

@Equilibrium:

$$\frac{\partial(V-T_0)}{\partial s} = 0 = \frac{\cancel{m}g}{2} - \frac{3}{4}\cancel{m}s\Omega^2 \qquad \boxed{s = \frac{2g}{3\Omega^2}}$$

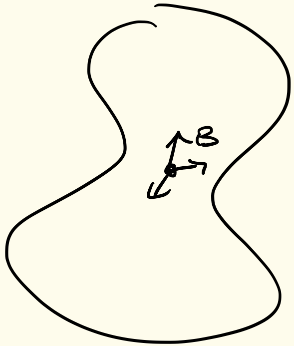
$$\frac{\partial^2(V-T_0)}{\partial s^2} = -\frac{3}{4}m\Omega^2 \leftarrow \text{not stable!}$$

Be prepared for modal analysis, Routhian reduction (G0623)

Quick Trick: Angular momentum

Suppose B body Fixed at COM and

$${}^{B}\underline{I}_G = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix}$$



We know

$${}^{B}\{\underline{H}_G\} = {}^{B}\underline{I}_G \; {}^{B}\{\underline{\omega}\}$$

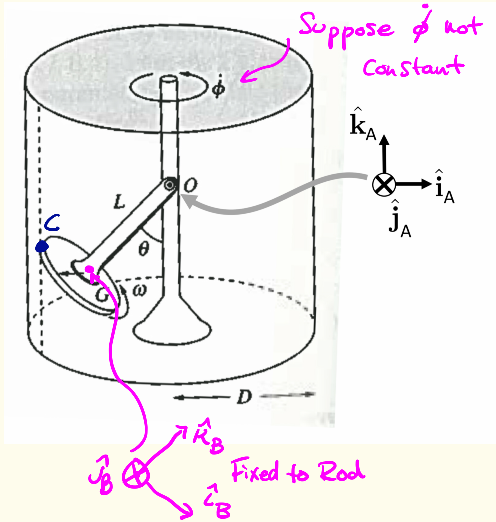
If: $\underline{\omega} = \omega_x \hat{i}_B + \omega_y \hat{j}_B + \omega_z \hat{k}_B$

$$\Rightarrow \quad \underline{H}_G = I_{xx}\omega_x \hat{i}_B + I_{yy}\omega_y \hat{j}_B + I_{zz}\omega_z \hat{k}_B$$

Since ${}^{B}\{\underline{H}_G\} = \begin{bmatrix} I_{xx}\omega_x \\ I_{yy}\omega_y \\ I_{zz}\omega_z \end{bmatrix}$

Give $\underline{H}_G$ and $\dot{\underline{H}}_G$:

Disk mass m, radius R



Previous Results:

$$\underline{\omega}_{Disk} = -\frac{L}{R} S_\theta \dot{\phi}\, \hat{k}_B - \dot{\phi} S_\theta \hat{i}_B = -\dot{\phi}\left[\frac{L}{R} S_\theta \hat{k}_B + S_\theta \hat{i}_B\right]$$

$$\dot{\underline{\omega}}_{Disk} = -\dot{\phi}^2 S_\theta \left[\frac{L}{R} S_\theta + C_\theta\right] \hat{j}_B$$

Using Previous Trick:

$$\underline{H}_{G,Disk} = \underbrace{\frac{1}{2} m R^2}_{I_{zz}} \left(-\frac{L}{R} S_\theta \dot{\phi}\, \hat{k}_B\right) + \underbrace{\frac{1}{4} m R^2}_{I_{xx}} \left(-\dot{\phi} S_\theta \hat{i}_B\right)$$

$$= -\frac{1}{2} m R L S_\theta \dot{\phi} \hat{k}_B - \frac{1}{4} m R^2 \dot{\phi} S_\theta \hat{i}_B$$

$$\dot{\underline{H}}_{G,Disk} = \left[-\frac{1}{2} m R L S_\theta \hat{k}_B - \frac{1}{4} m R^2 S_\theta \hat{i}_B\right] \ddot{\phi} + \overbrace{\left[-\frac{1}{2} m R L S_\theta^2 \dot{\phi}^2 - \frac{1}{4} m R^2 \dot{\phi}^2 S_\theta C_\theta\right] \hat{j}_B}^{\underline{\omega}_B \times \underline{H}_{G,Disk}}$$

Give inertial Force $Q_{\phi,inertial}$

$$\underline{V}_G = -L S_\theta \dot{\phi}\, \hat{j}_B \qquad \underline{a}_G = -L S_\theta \ddot{\phi}\, \hat{j}_B - L S_\theta \dot{\phi} \overbrace{\left[-C_\theta \hat{i}_B - S_\theta \hat{k}_B\right]}^{\hat{i}_A} \dot{\phi}$$

$$Q_{\phi,inertial} = \frac{\partial \underline{v}_G}{\partial \dot{\phi}} \cdot m \underline{a}_G + \frac{\partial \underline{\omega}_{Disk}}{\partial \dot{\phi}} \cdot \dot{\underline{H}}_G = + L^2 S_\theta^2 m \ddot{\phi} + \frac{1}{2} m L^2 S_\theta^2 \ddot{\phi} + \frac{1}{4} m R^2 S_\theta^2 \ddot{\phi}$$

$$= \left[\frac{3}{2} m L^2 + \frac{1}{4} m R^2\right] S_\theta^2 \ddot{\phi}$$

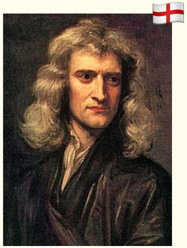
What I hope you'll take from the class

- Highest level:
  1. Fundamental skills to go from a mechanical system to a mathematical Model of it
  2. Ability to use (or confidence that you could use) state-of-the-art Symbolic tools to scale these fundamentals to complex systems

Tools to help manage complexity

- Integrals of motion ← constant quantities simplify analysis
- Routhian Reduction ← Fewer EOM
- Hamiltonian ← 1st order ODEs for EOM
- Kane's equations ← Additional freedom w/ generalized Speeds means more options to simplify EOM

What good do I get out of knowing all these different Strategies for Dynamics

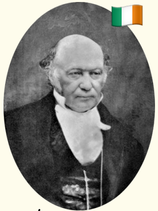


Newton
1687

D'Alembert
1742

Euler
1765

Lagrange
1778



Hamilton
1833

Noether
1915

- In industry: Use two different methods to check your work!
- In research: Being conversant in all approaches will make it easier to read the literature (Many labs adopt one formalism...)
- In both: Different formalisms expose structure to help develop controllers.

Overall:

- I hope you enjoyed the course!
- Thanks for all the feedback throughout (Especially w.r.t. the 40/60 split setup)
- Please complete your CIFs @ cif.nd.edu
  - What was most/least interesting part of the course?
  - Where would you have liked to spend more/less time?
  - More MATLAB? Forced conversion?

Any and all feedback will be valued

Analytical Dynamics

The End

Admin:

- HW 9 Solution online tonight
- Final: Tuesday Dec 17th 4:15 - 6:15 PM (140 DeBartolo)
- Final review topic list available on Sakai
- Please complete your CIFs! Last Day 12/15 (Sunday)

Good Luck On

All Your Finals!