ECE5463: Introduction to Robotics

Lecture Note 5: Velocity of a Rigid Body

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Outline

- Introduction
- Rotational Velocity
- Change of Reference Frame for Twist (Adjoint Map)
- Rigid Body Velocity

Introduction

- For a moving particle with coordinate $p(t) \in \mathbb{R}^3$ at time t, its (linear) velocity is simply $\dot{p}(t)$

- A moving rigid body consists of infinitely many particles, all of which may have different velocities. What is the velocity of the rigid body?

- Let $T(t)$ represent the configuration of a moving rigid body at time t. A point p on the rigid body with (homogeneous) coordinate $\tilde{p}_b(t)$ and $\tilde{p}_s(t)$ in body and space frames:

$$\tilde{p}_b(t) \equiv \tilde{p}_b, \quad \tilde{p}_s(t) = T(t)\tilde{p}_b$$

Introduction

- Velocity of p is $\frac{d}{dt}\tilde{p}_s(t) = \dot{T}(t)p_b$

- $\dot{T}(t)$ is not a good representation of the velocity of rigid body
  - There can be 12 nonzero entries for $\dot{T}$.
  - May change over time even when the body is under a constant velocity motion (constant rotation + constant linear motion)

- Our goal is to find effective ways to represent the rigid body velocity.

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Illustrating Example

- **Question:** Given the orientation $R(t)$ of a rotating frame as a function of time t, what is the the angular velocity?

- We start with an example for which we know the answer, then we generalize to obtain a formal answer

- **Example:** Suppose {b} starts with an initial orientation $R(0)$ and rotates about $\hat{\mathrm{x}}$ at unit constant speed (i.e. we know the angular velocity at time $t > 0$ is $\omega = (1, 0, 0)^T$), where

$$R(0) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad Rot(\hat{\mathrm{x}}; \theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\theta & -s_\theta \\ 0 & s_\theta & c_\theta \end{bmatrix}$$

  Consider a point p rigidly attached to frame {b} with coordinates $p_s(t)$ and $p_b(t)$ in {s} and {b} frames.

Illustrating Example (Continued)

- $p_s(t) = R(t)p_b \Rightarrow \dot{p}_s(t) = \dot{R}(t)p_b$

- Since we know the motion in this example, we must have $\dot{p}_s(t) = \omega \times p_s(t)$, where $\omega = (1, 0, 0)$

- Conclusion:

Properties of Rotation Matrices

- **Property:** For any $\omega \in \mathbb{R}^3$ and $R \in SO(3)$, we have

$$R[\omega]R^T = [Rw]$$

- **Property:** Let $R(t) \in SO(3)$ be differentiable in t, then $\dot{R}(t)R^{-1}(t)$ and $R^{-1}(t)\dot{R}(t)$ are both skew symmetric, i.e. they are in $so(3)$.

Rotational Velocity Representation

- **Rotational Velocity in space frame:** Let $R_{sb}(t)$ be the orientation of a rotating frame {b} at time t. Then the (instantaneous) angular velocity vector w of frame {b} is given by

$$[\omega_s] = \dot{R}_{sb} R_{sb}^{-1}$$

where ω_s is the {s}-frame coordinate of w.

- Note the angular velocity w is a free vector, which can be represented in different frames.
- Its coordinates ω_c and ω_d in frames {c} and {d} satisfy

$$\omega_c = R_{cd}\omega_d$$

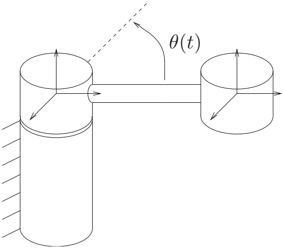
Rotational Velocity in Body Frame

- **Rotational velocity in body frame:** Consider the same set up as the previous slide where $R_{sb}(t)$ is the orientation of the rotating frame {b}.
  - ω_b denotes the body-frame representation of w, i.e. $\omega_b = R_{bs}(t)\omega_s = R_{sb}^{-1}(t)\omega_s$

$$\Rightarrow [\omega_b] = R_{sb}^{-1}\dot{R}_{sb}$$

  - Note: ω_b is NOT the angular velocity relative to a moving frame. It is rather the velocity relative to the *stationary* frame that is instantaneously coincident with the rotating body frame.

Example of Rotational Velocity



$$R(t) = \begin{bmatrix} \cos\theta(t) & -\sin\theta(t) & 0 \\ \sin\theta(t) & \cos\theta(t) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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Change of Reference Frame for Twist

- Given two frames {c} and {d} with $T = (R, p)$ representing the configuration of {d} relative to {c}. The same rigid body motion can be represented in {c} or in {d} using the twist $\mathcal{V}_c = (\omega_c, v_c)$ or $\mathcal{V}_d = (\omega_d, v_d)$, respectively. How do these two twists relate to each other?

Change of Reference Frame for Twist (Continued)

- $\Rightarrow \begin{bmatrix} \omega_c \\ v_c \end{bmatrix} = \begin{bmatrix} R & 0 \\ [p]R & R \end{bmatrix} \begin{bmatrix} \omega_d \\ v_d \end{bmatrix}$

Adjoint Map

- Given $T = (R, p) \in SE(3)$, its **adjoint representation (adjoint map)** $[\mathrm{Ad}_T]$ is

$$[\mathrm{Ad}_T] = \begin{bmatrix} R & 0 \\ [p]R & R \end{bmatrix}$$

- Adjoint map changes reference frames for twist vector. If T is configuration of {d} relative to {c}, then the twists $\mathcal{V}_c$ and V_d in two frames are related by

$$V_c = [\mathrm{Ad}_T]V_d \quad \text{or equilvalently} \quad [\mathcal{V}_c] = T[V_d]T^{-1}$$

- **Properties of Adjoint:**
  - Given $T_1, T_2 \in SE(3)$ and $\mathcal{V} = (\omega, v)$, we have

$$[\mathrm{Ad}_{T_1}][\mathrm{Ad}_{T_2}]\mathcal{V} = [\mathrm{Ad}_{T_1T_2}]\mathcal{V}$$

  - For any $T \in SE(3)$,

$$[\mathrm{Ad}_T]^{-1} = [\mathrm{Ad}_{T^{-1}}]$$

Example: Change reference frame for twist

Two frames {a} and {b} and configuration of {b} relative to {a} is $T = (R, p_0)$ with

$$R = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad p_0 = (0, 2, 0)$$

Example: Change reference frame for Twist (Continued)

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Derivation of Spatial Velocity of a Rigid Body

- **Question:** Given configuration $T_{sb}(t) = (R_{sb}(t), p_{sb}(t))$ of a moving rigid body, how to represent/find the velocity of the rigid body?

- Similar to the rotational velocity, we consider a point q attached to the body and derive its differential equation in {s} frame.

$$q_s(t) = R_{sb}(t) q_b + p_{sb}(t) \quad \Rightarrow \quad \dot{q}_s(t) = \omega_s \times q_s(t) + v_s$$

Spatial Twist and Body Twist

- Given $T_{sb}(t) = (R(t), p(t))$. Spatial velocity in space frame (called **spatial twist**) is given by

$$\mathcal{V}_s = (\omega_s, v_s), \text{ with } [\omega_s] = \dot{R}R^T, v_s = \dot{p} + \omega_s \times (-p)$$

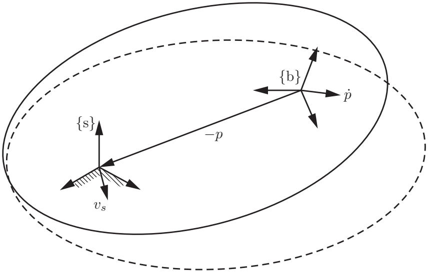
- Change reference frame to body frame will lead to **body twist**:

$$\mathcal{V}_b = (\omega_b, v_b) = [\mathrm{Ad}_{T_{bs}}]\mathcal{V}_s, \text{ where } [\omega_b] = R^T\dot{R}, v_b = R^T\dot{p}$$

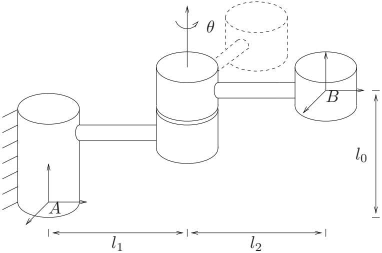
Spatial Twist and Body Twist: Interpretations

- ω_b and ω_s is the angular velocity expressed in $\{b\}$ and $\{s\}$, respectively.

- v_b is the linear velocity of the origin of $\{b\}$ expressed in $\{b\}$; v_s is the linear velocity of the origin of $\{s\}$ expressed in $\{s\}$

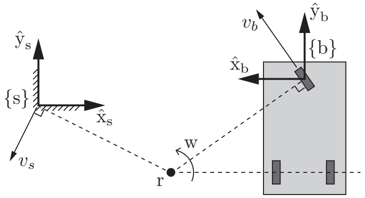


Example of Spatial/Body Twist I



$$T(t) = \begin{bmatrix} \cos\theta(t) & -\sin\theta(t) & 0 & -l_2\sin\theta(t) \\ \sin\theta(t) & \cos\theta(t) & 0 & l_1 + l_2\cos\theta(t) \\ 0 & 0 & 1 & l_0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example of Spatial/Body Twist II



$r_s = (2, -1, 0)$, $r_b = (2, -1.4, 0)$, w=2 rad/s

$$T_{sb} = \begin{bmatrix} -1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0.4 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

More Discussions