Lecture 10: Dynamics as an optimal "control" problem

Last time - LQR: The optimal policy is linear

The optimal cost to go is quadratic

	Control law	Cost to go $x^T P x$, P satisfies
Continuous ∞ Horizon	$u^* = -R^{-1}B^T P x$	$0 = Q + A^T P + PA - PBR^{-1}B^T P$
Discrete ∞ Horizon	$u^* = -(R + B^T P B)^{-1} B^T P A x$	$P = Q + A^T[P - PB(R + B^T P B)^{-1} B^T P]A$

Today

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- Robot Dynamics as optimal control

A FEW HINTS ON HW3 ...

PROBLEM 1: Hints

From Lecture 6:

$$\overset{\times 3}{\underset{-}{\curvearrowright}} \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 8 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 0 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix} \qquad \begin{aligned} x_2 &= -1 \\ x_1 &= 3 \end{aligned}$$

Method 2 cont: Given τ Find $\ddot{q}$ by solving

$$\begin{bmatrix} H & -J_c^T \\ J_c & 0 \end{bmatrix} \begin{bmatrix} \ddot{q} \\ f_c \end{bmatrix} = \begin{bmatrix} S^T \tau_j - C\dot{q} - \tau_g \\ -\dot{J}\dot{q} \end{bmatrix}$$

Row 2 = Row 2 $- J_c H^{-1}$ (Row 1)

- You can solve for force needed to maintain the contact as:

$$f_c = \underbrace{(J_c H^{-1} J_c^T)^{-1}}_{\Lambda_c \text{ contact inertia}} \left[-\dot{J}_c \dot{q} - J_c H^{-1} (S^T \tau_j - C\dot{q} - \tau_g) \right]$$

- **Remark:** The resulting $\ddot{q}$ is the same as the solution to the least-squares problem

$$\min_{\ddot{q}} \frac{1}{2} [\ddot{q} - \ddot{q}_{uc}]^T H [\ddot{q} - \ddot{q}_{uc}]$$
$$\text{s.t.} \quad J_c \ddot{q} + \dot{J}_c \dot{q} = 0$$

where $\ddot{q}_{uc} = H^{-1}[S^T \tau_j - C\dot{q} - \tau_g]$ is the unconstrained result

Problem 3: Discounted LQR

Consider a problem: $0 \leq \gamma \leq 1$

$$V_\gamma^*(x_0) = \min_{u[\cdot]} \underbrace{\sum_{k=0}^{\infty} \gamma^k \bar{\ell}(x_k, u_k)}_{V_\gamma(x_0, u[\cdot])} = \bar{\ell}(x_0, u_0) + \gamma \ell(x_1, u_1) + \gamma^2 \ell(x_2, u_2) + \cdots$$

$$x_{n+1} = f(x_n, u_n)$$

x_0 fixed

$0 < \gamma < 1$

Bellman Eqn: $\gamma = 1$

$$V_1^*(x) = \min_u \left[\bar{\ell}(x, u) + V_1^*(f(x, u)) \right]$$

$\gamma = 0$: Tragically Myopic Version of Bellman's eqn.

$$V_0^*(x) = \min_u \left[\bar{\ell}(x, u) \right]$$

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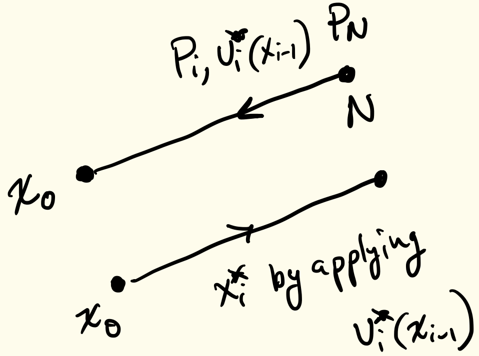
Discrete Time LQR:

$$\min_{u[\cdot],\, x[\cdot]} \left[\sum_{i=1}^{N} x_i^T Q_i x_i + u_i^T R_i u_i \right]$$

$$\text{s.t.} \quad x_i = A_i x_{i-1} + B_i u_i$$



Value Iteration: $V^*(i, x) = x^T P_i x \qquad P_N = Q_N$

$$P_{i-1} = Q_i + A_i^T \left[P_i - P_i B_i \left(B_i^T P_i B_i + R_i \right)^{-1} B_i^T P_i \right] A_i$$

$$u_i^*(x) = -\left(B_i^T P_i B_i + R_i \right)^{-1} B_i^T P_i A_i x_{i-1}$$

Infinite Horizon Time invariant Case: $V^*(x) = x^T P x$

$$P = Q + A^T \left[P - PB \left(B^T P B + R \right)^{-1} B^T P \right] A$$

Generalization 1: Cross terms in the cost

Problem Setup:

$$\min_{u[\cdot]} \left[\sum_{k=1}^{N} x_{k-1} Q_{k-1} x_{k-1} + 2 x_{k-1}^T S_k u_k + u_k^T R_k u_k \right] + x_N^T Q_N x_N$$

$$x_k = A_k x_{k-1} + B_k u_k$$

Optimal Cost to Go: $V^*(k, x) = x^T P_k x$

$$P_{k-1} = Q_{k-1} + A_k^T P_k A_k - \left(B_k^T P_k A_k + S_k^T\right)^T \left(R_k + B_k^T P_k B_k\right)^{-1} \left(B_k^T P_k A_k + S_k^T\right)$$

Optimal Control:

$$u_k^* = -\left(R_k + B_k^T P_k B_k\right) \left[B_k^T P_k A_k + S_k^T \right] x_{k-1}$$

Generalization #2: 2nd order Cost, Affine Dynamics $x_k \in \mathbb{R}^n$, $u_k \in \mathbb{R}^m$

$$\min_{u[\cdot]} \sum_{k=0}^{N} \left[x_{k-1}^T Q_{k-1} x_{k-1} + u_k^T R_k u_k + 2 x_{k-1}^T S_k u_k + 2 q_{k-1}^T x_{k-1} + 2 r_k^T u_k + q_{0,k} \right]$$

$$+ x_N^T Q_N x_n + 2 q_N^T x_N + q_{0,N}$$

$$\text{s.t.} \quad x_k = A_k x_{k-1} + B_k u_k + d_k$$

$d_k \in \mathbb{R}^n$, $q_k \in \mathbb{R}^n$, $r_k \in \mathbb{R}^m$, $q_{0,k} \in \mathbb{R}$

Transform the problem: $x' = \begin{bmatrix} x \\ 1 \end{bmatrix} \in \mathbb{R}^{n+1}$

Dynamics: $x_i' = A_i' x_{i-1}' + B_i' u_i'$

$$= \begin{bmatrix} A_i & d_i \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_{i-1} \\ 1 \end{bmatrix} + \begin{bmatrix} B_i \\ 0 \end{bmatrix} u_i$$

Running Cost: $Q_i' = \begin{bmatrix} Q_i & q_i \\ q_i^T & q_{0,i} \end{bmatrix}$ $\quad S_i' = \begin{bmatrix} S_i \\ r_i^T \end{bmatrix}$ $\quad R_i' = R_i$

$$\text{Running Cost} = x_{k-1}'^T Q_{k-1}' x_{k-1}' + 2 x_{k-1}'^T S_k' u_k + u_k^T R_k' u_k$$

Generalization #2: 2nd order Cost, Affine Dynamics

Equivalent Problem:

$$\min_{u[\cdot]} \sum_{k=0}^{N} \left[x_{k-1}'^T Q_{k-1}' x_{k-1}' + 2 x_{k-1}' S_k' u_k + u_k^T R_k' u_k \right] + x_N'^T Q_N' x_N$$

$$x_k' = A_k' x_{k-1}' + B_k' u_k$$

$$x_0' = \begin{bmatrix} x_0 \\ 1 \end{bmatrix}$$

Optimal cost to go: $V^*(k, x') = x'^T P_k' x' = \begin{bmatrix} x \\ 1 \end{bmatrix}^T \left[\begin{array}{c|c} P_2 & p_1 \\ \hline p_1^T & p_0 \end{array} \right] \begin{bmatrix} x \\ 1 \end{bmatrix}$

$P_2 \in \mathbb{R}^{n \times n}$, $p_1 \in \mathbb{R}^n$, $p_0 \in \mathbb{R}$

$$= x^T P_2 x + 2 p_1^T x + p_0$$

Optimal control law

$$u_k^*(x') = -K' x' = -\begin{bmatrix} K_2 & K_1 \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} = -K_2 x - K_1$$

$K_2 \in \mathbb{R}^{m \times n}$
$K_1 \in \mathbb{R}^m$

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Optimal Control For Rigid Body Dynamics Simulation

$$H\ddot{q} + C\dot{q} + \tau_g = \tau \qquad q \in \mathbb{R}^N$$

RNEA: Given $q, \dot{q}, \ddot{q}$ Find τ

$\Rightarrow$ RNEA is efficient $O(N)$ math ops to compute τ

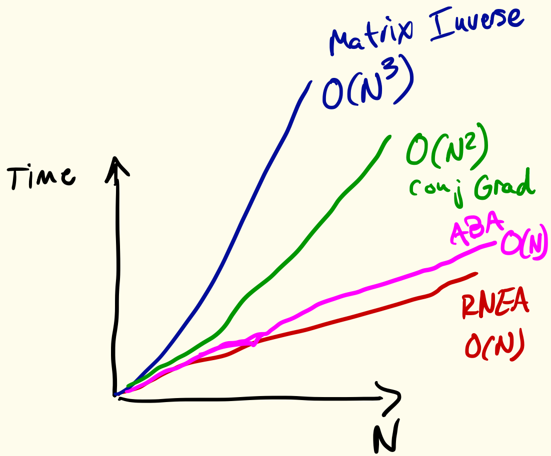
- **Simulation:** Given $q, \dot{q}, \tau$ find $\ddot{q}$

① $\ddot{q} = H^{-1}(q)\left[\tau - C\dot{q} - \tau_g\right]$ $O(N^3)$

② Solve $H(q)\ddot{q} = \tau - C\dot{q} - \tau_g$ using Conj. Grad

- N steps, $O(N)$ per step $O(N^2)$
- $Hx = \text{RNEA}(q, \dot{q}=0, \ddot{q}=x, a_g=0)$

③ Articulated Body Algorithm (ABA)



Recursive Newton Euler

Inputs: $q, \dot{q}, \ddot{q}, {}^{\circ}a_g$, (model)   Outputs: τ

$v_0 = 0 \qquad a_0 = -{}^{\circ}a_g$

for $i = 1 \ldots N$

$$v_i = {}^{i}X_{p(i)} v_{p(i)} + \phi_i \dot{q}_i$$

$$a_i = {}^{i}X_{p(i)} a_{p(i)} + \phi_i \ddot{q}_i + v_i \times \phi_i \dot{q}_i$$

$$f_i = I_i a_i + v_i \times^{*} I_i v_i$$

end

for $i = N \ldots 1$

$$\tau_i = \phi_i^T f_i$$

$$f_{p(i)} = f_{p(i)} + {}^{i}X_{p(i)}^T f_i$$

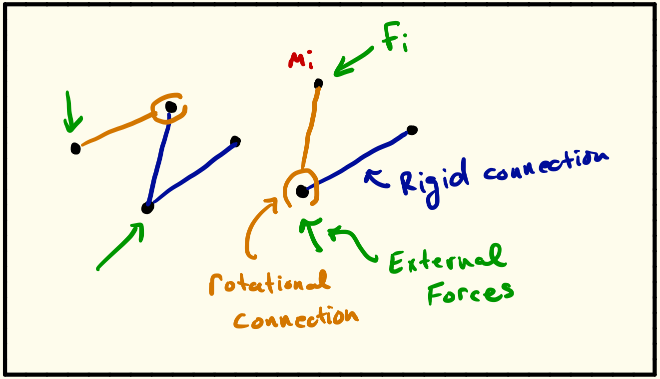
end

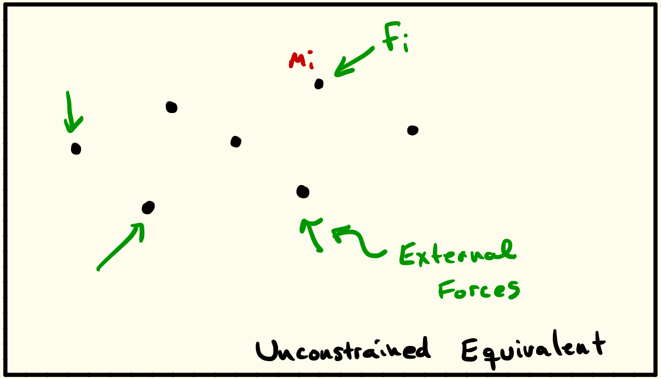
$$H(q)\ddot{q} + C(q, \dot{q})\dot{q} + \tau_g$$

$$= RNEA(q, \dot{q}, \ddot{q}, {}^{\circ}a_g)$$

Recall: Gauss principle

Consider a collection of particles w/ some constraints





Let $\ddot{x}_i^u = \frac{F_i}{m_i}$ the unconstrained accelerations

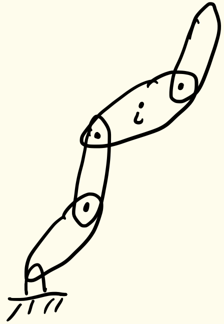
The actual accelerations $(\ddot{x}_i)$ deviate from the unconstrained ones in a least-squares sense

$$\underset{\{\ddot{x}_i\}}{\operatorname{argmin}} \sum_i m_i \| \ddot{x}_i - \ddot{x}_i^u \|_2^2$$

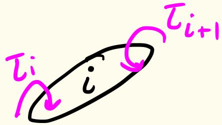
s.t. $\{\ddot{x}_i\}$ satisfy motion constraints

Gauss Principle For Rigid Body Chains

Joints impose constraints. So let's consider an unconstrained case without them...



Unconstrained case:



$$f_i^{net} = \phi_i \tau_i - {}^{i}X^*_{i+1} \phi_{i+1} \tau_{i+1}$$

$$= \phi_i \tau_i - {}^{i+1}X_i^T \phi_{i+1} \tau_{i+1}$$

$$= I_i a_i^{uc} + v_i \times^* I_i v$$

$$\Rightarrow a_i^{uc} = I_i^{-1}\left[f_i^{net} - v_i \times^* I_i v_i \right]$$

Gauss Principle:

$$\min_{\ddot{q},\, a_i} \sum_{i=1}^{N} (a_i^{uc} - a_i)^T I_i (a_i^{uc} - a_i)$$

$$\text{s.t.} \quad a_i = {}^{i}X_{i-1} a_{i-1} + \phi_i \ddot{q}_i + v_i \times \phi_i \dot{q}_i$$

$$a_0 = \begin{bmatrix} 0 \\ -\vec{g} \end{bmatrix}$$

Gauss Principle For Rigid Body Chains

$$\min_{\ddot{q},a} \sum_{i=1}^{N} \tfrac{1}{2}(a_i^u - a_i)^T I_i (a_i^u - a_i)$$

$$\text{s.t. } a_i = {}^iX_{i-1}\, a_{i-1} + \phi_i \ddot{q}_i + [v_i \times]\phi_i \dot{q}_i$$

$$a_0 = -\begin{bmatrix} 0_{3\times1} \\ \vec{g} \end{bmatrix}$$

Mapping

$a_i \Longleftrightarrow x_i$

$\ddot{q}_i \Longleftrightarrow u_i$

$I_i \Longleftrightarrow Q_i$

$-I_i a_i^u \Longleftrightarrow c_i$

${}^iX_{i-1} \Longleftrightarrow A_i$

$\phi_i \Longleftrightarrow B_i$

$v_i \times \phi_i \dot{q}_i \Longleftrightarrow d_i$

$a_i^{uT} I_i a_i^u \Longleftrightarrow c_{0,i}$

LQ optimal control

$$\min_{\{x_i, u_i\}} \sum \tfrac{1}{2} x_i^T Q_i x_i + c_i^T x_i + c_{0,i}$$

$$\text{S.t. } x_i = A_i x_{i-1} + B_i u_i + d_i$$

x_0 fixed

$$Q_i' = \begin{bmatrix} Q_i & c_i \\ c_i^T & c_{0,i} \end{bmatrix} \quad A_i' = \begin{bmatrix} A_i & d_i \\ 0 & 1 \end{bmatrix}$$

$$B_i' = \begin{bmatrix} B_i \\ 0 \end{bmatrix}$$

- $\ddot{q}$ can be found with

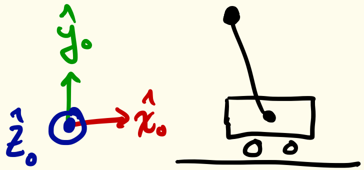
  1. Forward sweep base to tips to compute v_i (and a_i^{uc})

  2. Backward sweep to compute optimal cost to go

     $$V^*(i, a_i) = \begin{bmatrix} a_i \\ 1 \end{bmatrix}^T P_i' \begin{bmatrix} a_i \\ 1 \end{bmatrix}, \quad \ddot{q}_i(a_{i-1})$$

  3. A subsequent sweep to compute $\ddot{q}_i$, a_i for the optimal solution.

Example: Cart Pole



NEXT TIME

Sample Code available online

Summary:

- LQR: Can be used to solve
  - Problems with state/control cross costs $x^T S u$
  - Problems with general 2nd order cost & affine dynamics

- Rigid Body Dynamics Simulation
  - Most efficient simulators apply Value iteration on the gauss cost to go! $O(N)$