Checking if $f_G = \left(\dot{h}_G - \begin{bmatrix}0\\mg\end{bmatrix}\right) \in CWC$

Solve the Feasibility Problem

$$\min \quad 0$$

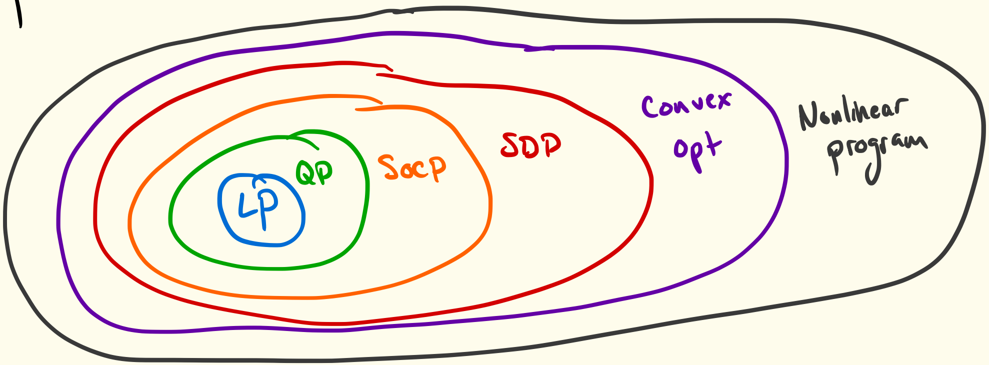
$$\text{s.t.} \quad f_G = \sum^{G} X^*_{c_j} \begin{bmatrix}0\\f_{c_j}\end{bmatrix}$$

each $f_{c_j} \in$

$\text{norm}([f_x, f_y]) \leq \mu f_z$

Implementation in CVX

- This problem is a second-order cone problem (SOCP).

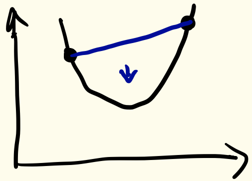


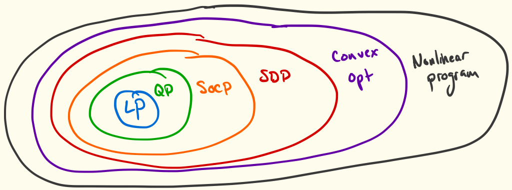
Class	Cost	Constraints
LP	$c^T x$	$Ax \le b$
QP	$Q \succeq 0$, $\frac{1}{2} x^T Q x + c^T x$	$Ax \le b$
SOCP	" "	$Ax \le b$, $\lVert Cx \rVert \le d^T x$
SDP Semi-Definite Prog.	" "	$Ax \le b$, $\lVert Cx \rVert \le d^T x$, $F_0 + \sum_i F_i x_i \succeq 0$
Convex	$f(x)$ convex func.	$x \in C$ C a convex set

<u>$f(x)$ is convex if</u>

$$f(\lambda x + (1-\lambda) y) \le \lambda f(x) + (1-\lambda) f(y)$$

$$\forall x, y \quad \forall \lambda \in [0, 1]$$





Lecture 21 - Model Predictive Control and ZMP Planning

- On flat ground (under a few assumptions) ZMP is sufficient for checking feasibility of motion
  - ⇒ Use ZMP to check if we can catch ourselves from falling
  - ⇒ Use ZMP to plan walking motions
- We will pursue the second through model predictive control (MPC)
  - ⇒ MPC = Re-solving many trajectory optimization problems online
  - ⇒ We can do this if we formulate the right problem ...

"Stable" locomotion is not falling down

- Let $\mathcal{X}$ is the state space, $\mathcal{F}$ the fallen states

- We say a state x_0 is Viable if there exists a controlled trajectory of the system such that $x(t; x_0) \notin \mathcal{F} \ \forall t > 0$ — Hard to verify

- The set of all viable states is called the viability kernel

- We will say that a state is captured if its in static equilibrium.

    - 0-step capturable if it can be captured w/o stepping

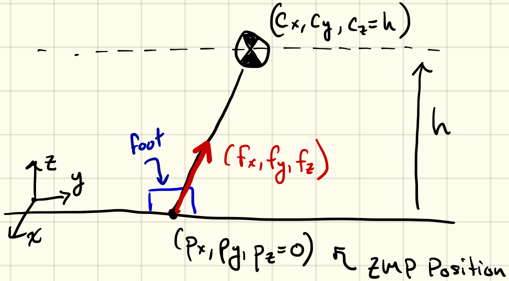
    - 1-Step " if it can be captured in 1 step

- evidence (and analysis) suggests that for bipeds, 2 steps is enough

"Capturing" the LIP

Assumptions: ① The CoM is @ a constant height h
② The angular momentum about the CoM is zero

Linear Inverted Pendulum



CoM Dynamics: $\ddot{c}_x = \omega^2 (c_x - p_x) \quad \omega = \sqrt{g/h}$

$$\frac{d}{dt} \underbrace{\begin{bmatrix} c_x \\ \dot{c}_x \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 0 & 1 \\ \omega^2 & 0 \end{bmatrix}}_{A} \begin{bmatrix} c_x \\ \dot{c}_x \end{bmatrix} + \underbrace{\begin{bmatrix} 1 \\ -\omega^2 \end{bmatrix}}_{B} p_x$$

Remarks

- Controllable canonical form $\Rightarrow$ CoM position & velocity are controllable via ZMP

- $\det(\lambda I - A) = \lambda^2 - \omega^2 \Rightarrow \lambda = +/- \omega$

Understanding the LIP Dynamics:

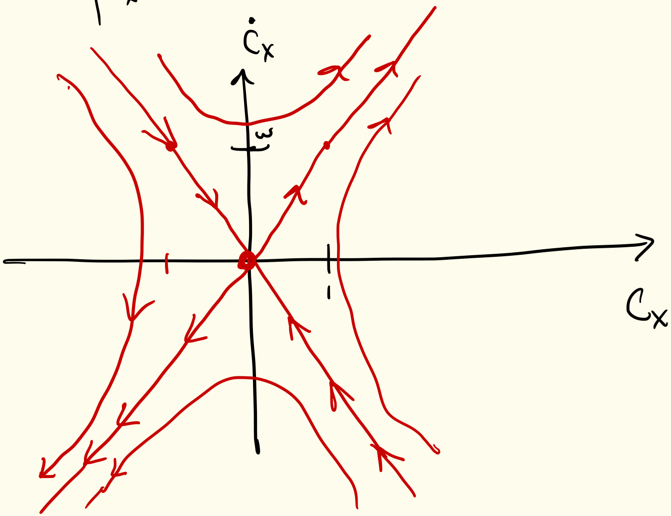
$$\frac{d}{dt}\begin{bmatrix} c_x \\ \dot{c}_x \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \omega^2 & 0 \end{bmatrix} \begin{bmatrix} c_x \\ \dot{c}_x \end{bmatrix} + \begin{bmatrix} 0 \\ -\omega^2 \end{bmatrix} p_x$$

Eig vals: $+/-\ \omega$

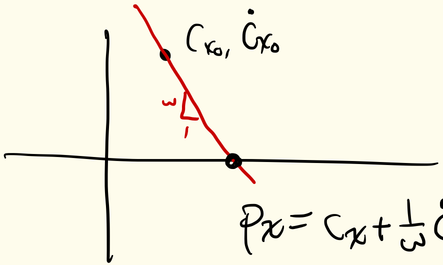
Eig vecs: $\lambda = \omega \quad \begin{bmatrix} 1 \\ \omega \end{bmatrix}$

$\lambda = -\omega \quad \begin{bmatrix} -1 \\ \omega \end{bmatrix}$

$p_x = 0$



Given $c_{x_0}, \dot{c}_{x_0}$ where should you place p_x s.t. $c_x \to p_x$, $\dot{c}_x \to 0$?



$$p_x = c_x + \frac{1}{\omega}\dot{c}_x$$

0-step capturable if $c_x + \frac{1}{\omega}\dot{c}_x \in$ Support Poly

1-Step capturable if " " a step places it in Support Poly

Understanding the LIP: The Capture Point

- Consider the change of variables $(c_x, \dot c_x) \Rightarrow (c_x, \underbrace{c_x + \tfrac{1}{\omega}\dot c_x}_{\xi_x \text{ : capture point}})$

- Dynamics

$$\frac{d}{dt}\begin{bmatrix} c_x \\ \xi_x \end{bmatrix} = \begin{bmatrix} -\omega(c_x - \xi_x) \\ +\omega(\xi_x - p_x) \end{bmatrix} = \begin{bmatrix} -\omega & \omega \\ 0 & \omega \end{bmatrix}\begin{bmatrix} c_x \\ \xi_x \end{bmatrix} + \begin{bmatrix} 0 \\ -\omega \end{bmatrix} p_x$$

$$p_x \longrightarrow \boxed{\dot\xi_x = \omega(\xi_x - p_x)} \xrightarrow{\ \xi_x\ } \boxed{\dot c_x = -\omega(c_x - \xi_x)} \xrightarrow{\ c_x\ }$$

① ξ_x diverges from p_x

C_x converges toward ξ_x

- If I have c_{x_0}, ξ_{x_0} with $p_x(t) = \xi_{x_0}$ then

  ① $\xi_x(t) = \xi_{x_0}$

  ② $C_x(t) \to \xi_{x_0}$ as $t \to \infty$ (exp. w/ rate ω)

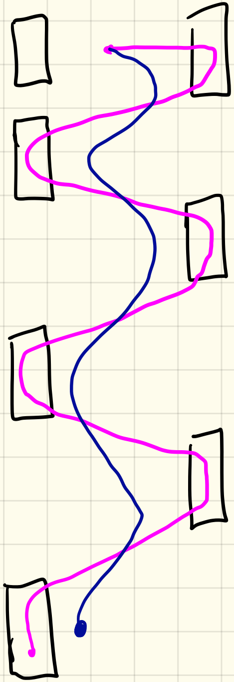
Also called extrapolated com (XCom)

OR the divergent component of motion (DCM)

ZMP Planning For Walking: Take 1, Unconstrained

Suppose I have some ZMP reference: $p_{zmp}^{d}(t) = \begin{bmatrix} p_x \\ p_y \end{bmatrix}$

Consider the LIP w $p_{com} = \begin{bmatrix} c_x \\ c_y \end{bmatrix}$



$$p_{zmp} = p_{com} - \frac{1}{\omega^2}\ddot{p}_{com}$$

Linear System $\quad x = \begin{bmatrix} p_{com} \\ \dot{p}_{com} \end{bmatrix} \quad u = \ddot{p}_{com} \quad y = p_{zmp}$

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

$$\min_{u(\cdot)} \int_0^{t_f} \left[\| Cx + Du - y^d(t) \|_Q^2 + u^T R u \right] dt$$

s.to $\quad \dot{x} = Ax + Bu$

x_0 given $\qquad x(t_f)$ fixed

Time varying LQR problem. Solve with RDE.

ZMP Planning for Walking: Take 2: Constrained

Discretize Dynamics:

$$\begin{bmatrix} p_{com} \\ \dot{p}_{com} \end{bmatrix}_{k+1} = \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p_{com} \\ \dot{p}_{com} \end{bmatrix}_{k} + \begin{bmatrix} \frac{1}{2}\Delta t^2 \\ \Delta t \end{bmatrix} \ddot{p}_{com_k}$$

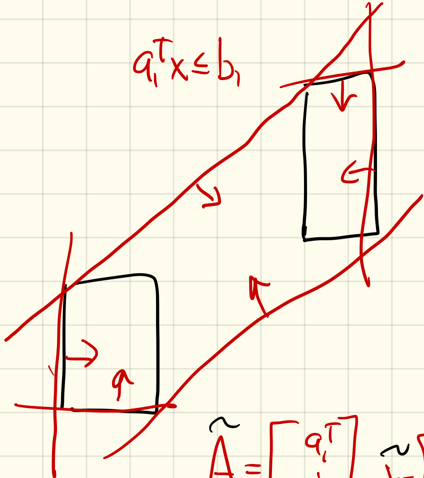
$$\min_{\{x_k, u_k\}} \sum \|Cx_k + Du_k - y_k^d\|_Q^2 + \|u_k\|_R^2$$

s.t. $x_{k+1} = Ax_k + Bu_k$

$Cx_k + Du_k \in \text{Support Polygon}_k$

$\tilde{A}_k (Cx_k + D_k) \le \tilde{b}_k$

$\epsilon_N \in \text{Support Polygon}_N$



$$\tilde{A} = \begin{bmatrix} a_1^T \\ \vdots \\ a_6^T \end{bmatrix} \quad \tilde{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_6 \end{bmatrix}$$

$$\tilde{A} x \le \tilde{b}$$