4/

Reference Frames

3D Rotations & Angular Velocity

- Admin
  - HW 1: Due Weds. Group office hours tonight 5-6:30 Dbrt 232
  - HW 2: Coming Soon! Due next weds

- Last time

  Finishing up whirlwind review of Newtonian Particle Dynamics
  - Work & Energy
  - Equilibrium
  - Integrals of Motion (Started)

  Will come back up multiple times throughout the semester in new contexts

- Today
  - Wrap up Integrals of Motion
  - 3D Kinematics analysis (today @ velocity level, Weds at accel level)

Example: $m\ddot{x} = \underbrace{-Kx}_{F}$ ① Find an integral of motion $F = -\frac{d}{dx}\left(\frac{1}{2}Kx^2\right)$

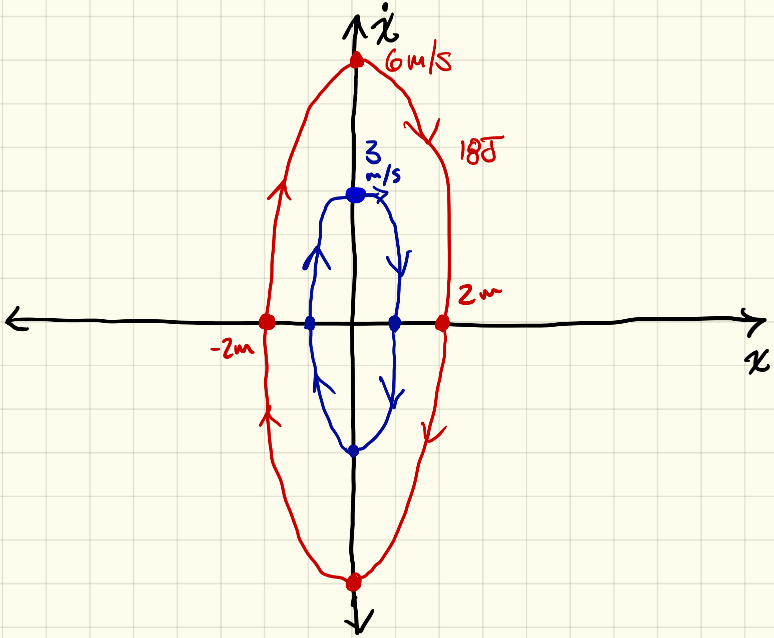
② Sketch the behavior of the system from initial conditions below.

Suppose $K = 9$ N/m, $m = 1$ kg.

① $-Kx$ conservative
⇒ total energy is an integral of motion

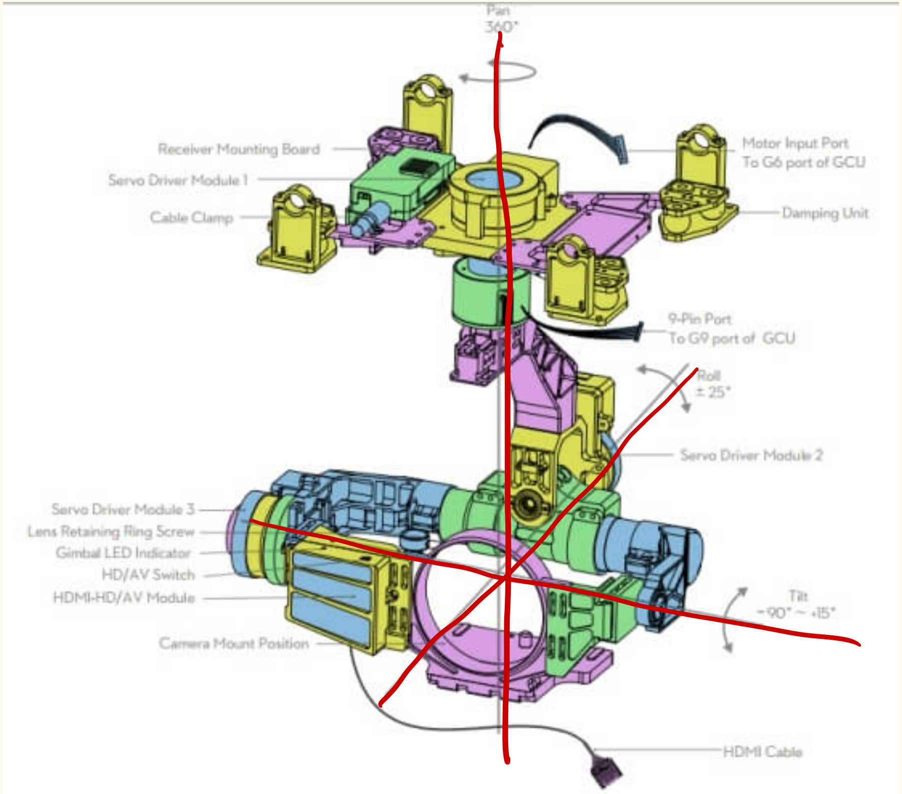
$$\boxed{C(x,\dot{x}) = \frac{1}{2}Kx^2 + \frac{1}{2}m\dot{x}^2}$$

$$C_0 = \frac{1}{2}Kx_0^2 + \frac{1}{2}m\dot{x}_0^2$$
$$= \frac{1}{2}K\,x(t)^2 + \frac{1}{2}m\,\dot{x}(t)^2$$

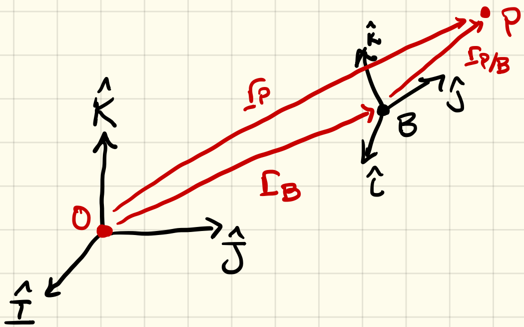


Reference Frames in action: Camera Stabilization

https://m.youtube.com/watch?v=XdlmoLAbbiQ



Multiple Reference Frames



Suppose $\underline{r}_B = 0$ (i.e, only motion of B is rotation)

$$\underline{r} = \underline{r}_P = X\hat{\underline{I}} + Y\hat{J} + Z\hat{K}$$

$$= x\hat{i} + y\hat{j} + z\hat{k}$$

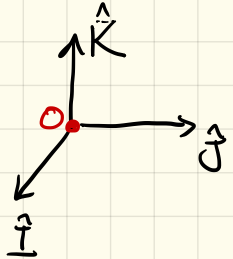
$$\underline{V}_P = \dot{X}\hat{\underline{I}} + \dot{Y}\hat{J} + \dot{Z}\hat{K}$$

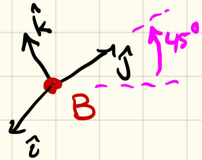
$$= \underbrace{\dot{x}\hat{i} + \dot{y}\hat{j} + \dot{z}\hat{k}}_{\text{Velocity observed in moving frame}} + \underbrace{x\dot{\hat{i}} + y\dot{\hat{j}} + z\dot{\hat{k}}}_{\text{velocity due to frame motion}}$$

$$\underline{a}_P = \underbrace{\ddot{x}\hat{i} + \ddot{y}\hat{j} + \ddot{z}\hat{k}}_{\text{aceleration observed in the moving frame}} + \underbrace{x\ddot{\hat{i}} + y\ddot{\hat{j}} + z\ddot{\hat{k}}}_{\text{acceleration of moving frame}} + \underbrace{2\left[\dot{x}\dot{\hat{i}} + \dot{y}\dot{\hat{j}} + \dot{z}\dot{\hat{k}}\right]}_{\text{Coriolis terms}}$$

Representing orientation: Rotation Matrix

- How to specify $\hat{\imath}, \hat{\jmath}, \hat{k}$? Express in $\hat{I}, \hat{J}, \hat{K}$





- Example for $\hat{\imath}$:

$$\hat{\imath} = r_{11}\hat{I} + r_{21}\hat{J} + r_{31}\hat{K}$$

where $r_{11} = \underbrace{\hat{\imath}\cdot\hat{I}}_{\text{projection}}$ $\quad r_{21} = \hat{\imath}\cdot\hat{J} \quad r_{31} = \hat{\imath}\cdot\hat{K}$

projection of $\hat{\imath}$ onto $\hat{I}$ = cos("angle between $\hat{\imath}$ and $\hat{I}$")

- We collect this process for $\hat{\imath}, \hat{\jmath}, \hat{k}$ into a matrix

$$^{O}R_{B} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} \hat{I}\cdot\hat{\imath} & \hat{I}\cdot\hat{\jmath} & \hat{I}\cdot\hat{k} \\ \hat{J}\cdot\hat{\imath} & \hat{J}\cdot\hat{\jmath} & \hat{J}\cdot\hat{k} \\ \hat{K}\cdot\hat{\imath} & \hat{K}\cdot\hat{\jmath} & \hat{K}\cdot\hat{k} \end{bmatrix}$$

Each row and each column must have unit magnitude.

- Example if $\hat{I} = \hat{\imath}$ and $\hat{\imath}, \hat{\jmath}, \hat{k}$ rotated as above

$$^{O}R_{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 45^\circ & \cos 135^\circ \\ 0 & \cos 45^\circ & \cos 45^\circ \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 45^\circ & -\sin 45^\circ \\ 0 & \sin 45^\circ & \cos 45^\circ \end{bmatrix} := R_1(45^\circ)$$

Working with vectors vs. vectors of numbers:

$\underline{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ is an arrow in 3D space

${}^B\{\underline{r}\} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is a 3x1 column vector of numbers

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Dot Product

$$\underline{r}_1 \cdot \underline{r}_2 = \{r_1\}^T\{r_2\} = x_1 x_2 + y_1 y_2 + z_1 z_2$$

Same result no matter what frame is used

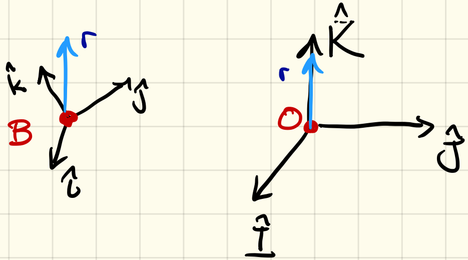
Cross product

$$\underline{r}_1 \times \underline{r}_2 = (y_1 z_2 - z_1 y_2)\hat{\imath} + (z_1 x_2 - x_1 z_2)\hat{\jmath} + (x_1 y_2 - y_1 x_2)\hat{k}$$

$${}^B\{r_1 \times r_2\} = \begin{bmatrix} y_1 z_2 - z_1 y_2 \\ \vdots \\ \vdots \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & -z_1 & y_1 \\ z_1 & 0 & -x_1 \\ -y_1 & x_1 & 0 \end{bmatrix}}_{{}^B[\tilde{r}_1]} \underbrace{\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}}_{{}^B\{r_2\}}$$

Multiple Frames: $^O\{r\} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \quad ^B\{r\} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

Properties of Rotation Matrices



① $^O R_B = \left[{^O\{\hat{\imath}\}} \;\; {^O\{\hat{\jmath}\}} \;\; {^O\{\hat{k}\}} \right]$

$^B R_O = \left[{^B\{\hat{I}\}} \;\; {^B\{\hat{J}\}} \;\; {^B\{\hat{K}\}} \right]$

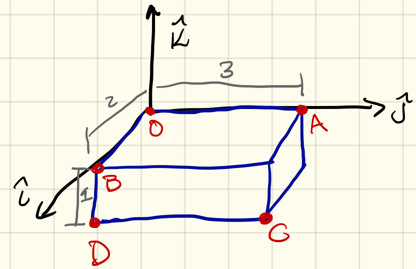
② $^O\{\underline{r}\} = {^O R_B} \, {^B\{\underline{r}\}}$

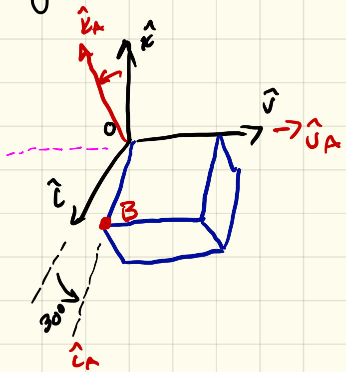
③ $^O R_B^T \, {^O R_B} = \begin{bmatrix} ^O\{\hat{\imath}\}^T \\ ^O\{\hat{\jmath}\}^T \\ ^O\{\hat{k}\}^T \end{bmatrix} \left[{^O\{\hat{\imath}\}} \;\; {^O\{\hat{\jmath}\}} \;\; \{\hat{k}\} \right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow {^O R_B^T} = {^O R_B^{-1}}$

④ $^O R_B^T = {^B R_O}$

⑤ $^O R_B = {^O R_A} \, {^A R_B}$ (Rotations can be chained)

Example: Rotate the box about $\vec{OA}$ by $30°$, then about $\vec{OB}$ by $90°$ What are the final coordinates of D w.r.t. $\hat{i}, \hat{j}, \hat{k}$?



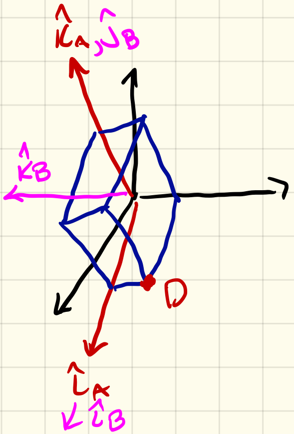


$$ {}^0R_A = \begin{bmatrix} {}^0\{\hat{i}_A\} & {}^0\{\hat{j}_A\} & {}^0\{\hat{k}_A\} \end{bmatrix} $$

$$ = \begin{bmatrix} C30° & 0 & S30° \\ 0 & 1 & 0 \\ -S30° & 0 & C30° \end{bmatrix} := R_z(30°) $$

$$ {}^AR_B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} $$

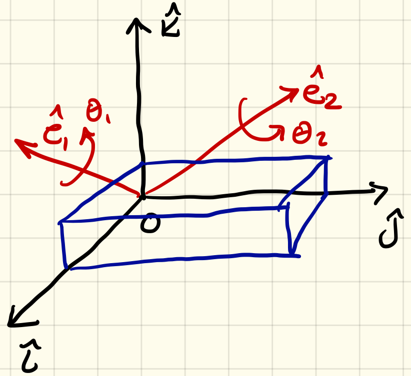
$$ {}^B\{r''_D\} = {}^0\{r_D\} $$



Twice rotated r_D

$$ {}^0\{r''\} = {}^0R_A \, {}^AR_B \, \underbrace{{}^B\{r''_D\}}_{= \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}} = \begin{bmatrix} \sqrt{3} \\ 1 \\ -1 \end{bmatrix} $$

Concept Question: Consider two unit vectors $\hat{e}_1$ and $\hat{e}_2$ ($\hat{e}_1 \neq \hat{e}_2$). Consider the following cases.



① The block rotates by angle θ_1 about $\hat{e}_1$ then by angle θ_2 about $\hat{e}_2$

② The block rotates by angle θ_2 about $\hat{e}_2$ then by angle θ_1 about $\hat{e}_1$.

Assume $0 < \theta_1 < 180°$, $0 < \theta_2 < 180°$

True or False: The block ends up in the same position for both cases

Ⓐ True (Always)

Ⓑ True if $e_1 \perp e_2$

Ⓒ False (always)

Ⓓ none of the above.

Next time...

Summary

- Orientation of one frame relative to another can be described with a rotation matrix: ${}^0R_B$
  - Columns of ${}^0R_B$ are $\left[{}^0\{\hat{i}\} \;\; {}^0\{\hat{j}\} \;\; {}^0\{\hat{k}\} \right]$
  - For any vector $\underline{r}$ $\quad {}^0\{\underline{r}\} = {}^0R_B \, {}^B\{\underline{r}\}$
  - Composition of rotations ${}^0R_B = {}^0R_A \, {}^AR_B$