Lecture 2: Classification of Optimization Problems & Dynamic Modeling Intro

- Last Time
  - Multivariable Calc Review
  - Quadratic Forms
  - Optimality Criteria (Unconstrained case)

- Today
  - Optimality Criteria (constrained case)
  - Problem classification
  - Dynamic Modeling Intro

Optimality Criteria: $f(x): \mathbb{R}^n \to \mathbb{R}$

- Suppose x^* is a local minimizer of f. Then

  ⓐ $\left.\frac{df}{dx}\right|_{x=x^*} = 0$ MULTI DIM ANALOG $\Rightarrow$ $\nabla_x f = 0$

  ⓑ $\frac{d^2 f}{(dx)^2} \geq 0$ $\quad$ $\nabla_{xx} f \succeq 0$

  Necessary conditions

  $x^2, x^3, x^4, -x^4$ @ $x=0$

- Let $x_c \in \mathbb{R}^n$ if

  ⓐ $\nabla_x f|_{x=x_c} = 0$

  ⓑ $\nabla_{xx} f|_{x=x_c} \succ 0$

  Then x_c is a local optimizer of f

  Sufficient conditions

  x^2 @ $x=0$

Conditions For Global Optimality

Let $x_c \in \mathbb{R}^n$ if

x^2, x^4 @ $x=0$

ⓐ $\nabla_x f|_{x=x_c} = 0$

ⓑ $\nabla_{xx} f \succeq 0 \quad \forall x \in \mathbb{R}^n$

$\Rightarrow$ x_c is a global minimizer of f

What about constraints?

Example: A farmer has purchased 100m of fence and would like to maximize the area of a pen for their cattle. The farmer is stubborn and will only consider rectangular arrangements. How should the farmer arrange their pen?

$$2\ell + 2w = 100m$$

① $\max_{\ell, w} \ \ell w$ $\quad\Big|\quad f(x)$

s.t. $2\ell + 2w - 100 = 0$ $\quad\Big|\quad g(x) = 0$

② Introduce a Lagrange multiplier $\lambda \in \mathbb{R}$

$$L(\ell, w, \lambda) = \ell w + \lambda(2\ell + 2w - 100)$$
$$= f(x) + \lambda g(x)$$

③ First-order necessary conditions

$$\left.\begin{aligned} \frac{\partial L}{\partial \ell} &= w + 2\lambda = 0 \\ \frac{\partial L}{\partial w} &= \ell + 2\lambda = 0 \end{aligned}\right] \quad w = \ell = 25m, \quad \lambda = -12.5 \frac{m^2}{m} \quad \Big] \ \nabla_x L = 0$$

$$\frac{\partial L}{\partial \lambda} = 2\ell + 2w - 100 = 0 \quad \Big] \ \frac{\partial L}{\partial \lambda} = 0$$

Intro to Problem Classes:

Linear Programs ← Can be solved with $n \geq 10^6$, $m \geq 10^3$

$$\min_{x} \; c^T x$$
$$\text{s.t.} \; Ax \leq b$$
$$\underline{x} \leq x \leq \bar{x}$$

$c \in \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $\underline{x}, \bar{x} \in \mathbb{R}^n$ Problem data

$x \in \mathbb{R}^n$ variables for optimization

Example: $\min_{x_1, x_2} \; 3x_1 + 2x_2$

s.t. $x_1 + x_2 \geq 1$

$2x_1 - x_2 \leq 4$

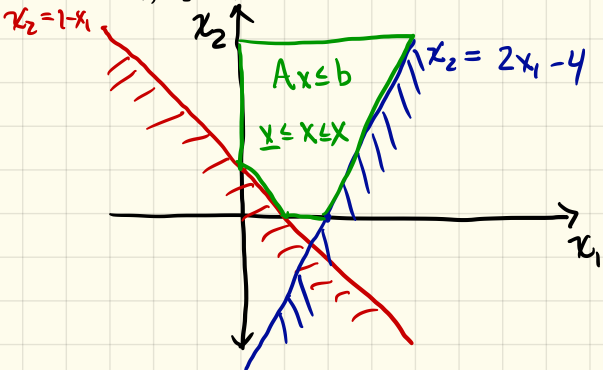
$0 \leq x_1 \leq 4$

$0 \leq x_2 \leq 4$

$0 \leq x_1, x_2 \leq 4$

$\underline{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\bar{x} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$ $c = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$A = \begin{bmatrix} -1 & -1 \\ 2 & -1 \end{bmatrix}$ $b = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$



In general the set

$$\{x \mid Ax \leq b, \; \underline{x} \leq x \leq \bar{x}\}$$

is the intersection of half space and is a polytope in $\mathbb{R}^n$

Quadratic Programs (QPs)

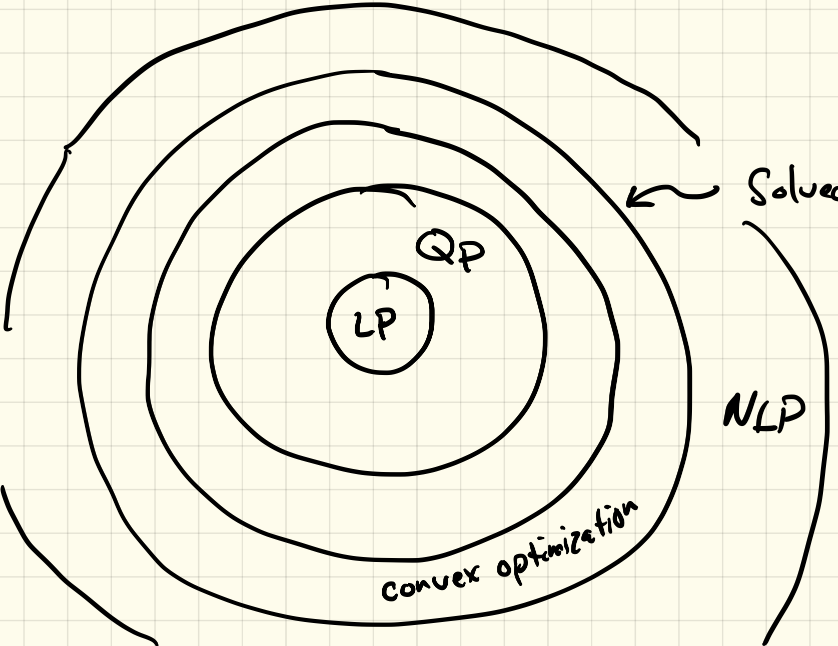
$x \in \mathbb{R}^n \qquad n \geq 10^4$

$$\min_x \tfrac{1}{2} x^T Q x + c^T x \qquad Q \succeq 0$$

$$\text{s.t. } Ax \leq b$$

$$\underline{x} \leq x \leq \overline{x}$$

- Can can be solved w/ tens of thousands of variables.
  (maybe more if Q low rank)



General NLPs

$$\min_x \; f(x)$$
$$\text{s.t.} \; g(x) \le 0$$

$f(x): \mathbb{R}^n \to \mathbb{R}$

$g(x): \mathbb{R}^n \to \mathbb{R}^m$

Can be solved to local optimality (sometimes) but few guaranzees exist...

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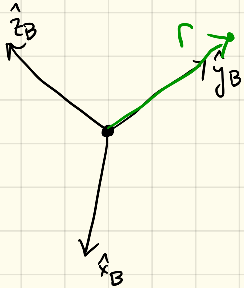
  - Problem classification ✓
  - Dynamic Modeling Intro
    - Frames
    - Velocity of a Rigid Body

Frames:

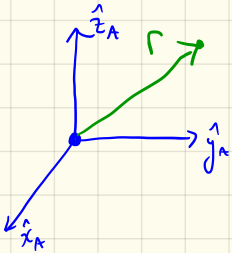
- **Frame:** 3 coordinate vectors $\hat{x}, \hat{y}, \hat{z}$ and an origin.
  - $\hat{x}, \hat{y}, \hat{z}$ mutually orthogonal
  - $\hat{x} \times \hat{y} = \hat{z}$ (right handed coordinate system)

- **Rotation Matrix:** Specifies orientation of one frame w.r.t. another



$$^{A}R_B = \left[{}^{A}\hat{x}_B \;\; {}^{A}\hat{y}_B \;\; {}^{A}\hat{z}_B \right] \in \mathbb{R}^{3\times3}$$

$\hat{x}_B$ w.r.t. $\hat{x}_A, \hat{y}_A, \hat{z}_A$



(a) Valid rotations

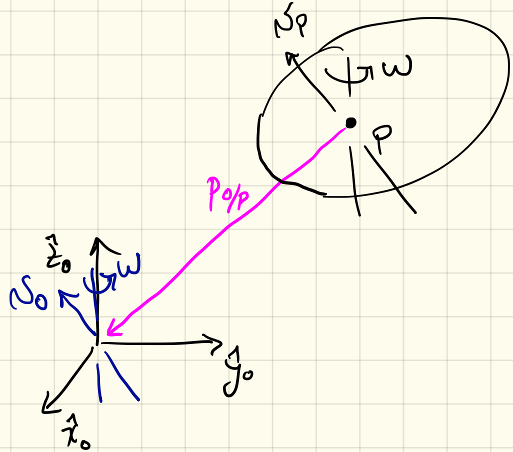
① $R^T R = I$ ② $\det(R) = 1$

(b) Use of rotations: change of basis

$$^{A}r = {}^{A}R_B \, {}^{B}r$$

Velocity of a Rigid Body:

- Rotating with angular velocity ω
- Point P moving w/ linear velocity v_p



- Introduce frame {0}

  v_0 is the linear velocity of a body-fixed point @ 0 at the current instant

  $$v_0 = v_p + \omega \times p_{o/p}$$

- Body translating at a linear velocity v_0 while rotating w/ angular velocity ω about an axis passing through 0. The following 6 elements are a <u>basis</u> for all possible rigid body velocities
  - Translation along $\hat{x}_0, \hat{y}_0, \hat{z}_0$
  - Rotation about $\hat{x}_0, \hat{y}_0, \hat{z}_0$

$${}^0V = \begin{bmatrix} {}^0\omega \\ \hline {}^0v_0 \end{bmatrix}$$

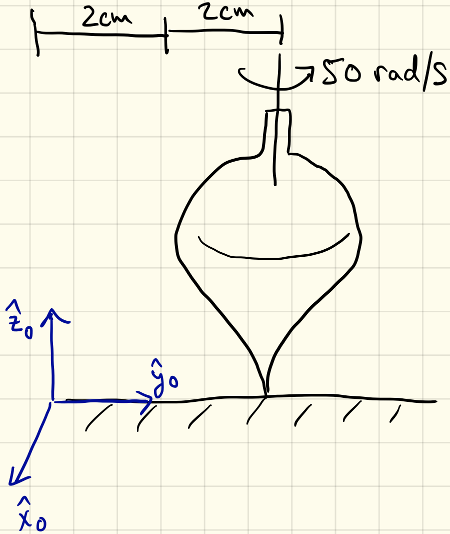
Spatial Velocity

Plücker coordinates of V in frame {0}

Example: Spinning Top

What is ${}^0V_{top}$?

$${}^0V_{top} = \begin{bmatrix} {}^0\omega \\ \hline {}^0v_o \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 50\text{ rad/s} \\ \hline 2\text{ m/s} \\ 0 \\ 0 \end{bmatrix}$$



Acceleration of a Rigid Body

$${}^0a_{top} = \lim_{\Delta t \to 0} \frac{{}^0V_{top}(t+\Delta t) - {}^0V_{top}(t)}{\Delta t}$$

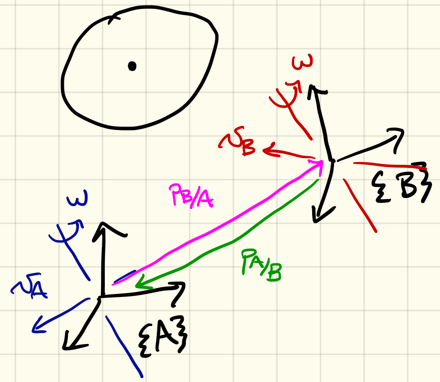
In this case:

- ${}^0a_{top}(t) = 0$

More Generally

- First 3 components of spat. accel. angular acceleration. ${}^0\dot{\omega}$
- ${}^0\dot{v}_o = ?$ Rate of change in velocity flow of Body-Fixed particles passing through O

Transforming Spatial Vectors:



$${}^A V = \begin{bmatrix} {}^A\omega \\ {}^A v_A \end{bmatrix} \qquad {}^B V = \begin{bmatrix} {}^B\omega \\ {}^B v_B \end{bmatrix}$$

$$v_B = v_A + \omega \times p_{B/A}$$

In coordinates

$${}^B\omega = {}^B R_A \, {}^A\omega$$

$${}^B v_B = {}^B R_A \, {}^A v_A - {}^B\omega \times {}^B p_{A/B} = {}^B R_A \, {}^A v_A + {}^B p_{A/B} \times {}^B R_A \, {}^A\omega$$

Collect terms

$${}^B V = \begin{bmatrix} {}^B\omega \\ {}^B v_B \end{bmatrix} = \underbrace{\begin{bmatrix} {}^B R_A & 0 \\ {}^B p_{A/B} \times {}^B R_A & {}^B R_A \end{bmatrix}}_{{}^B X_A} \begin{bmatrix} {}^A\omega \\ {}^A v_A \end{bmatrix} = {}^A V$$

change of basis from Plücker coordinates w.r.t. {A} to w.r.t. {B}