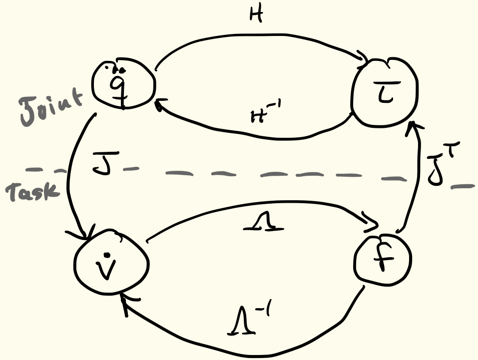
Lecture 7 - Contact Model Wrap Up & Optimal Control Intro

Today:

1. Recap of Last Lecture
2. Wrap Up on Contact Models (Linear Complementary Problems)
3. Intro to Optimal Control
   - Setup & Definitions
   - Principle of Optimality

Making Sense out of the Equations: $H\ddot{q} + C\dot{q} + \tau_g = S^T\tau + J^T f$

- J_c, J_c^T, H, H^{-1}, Λ_c, Λ_c^{-1}, N_c^T, ... AH!



Mappings:

H: "Joint accelerations" → "joint torques"

H^{-1}: "Joint torques" → "joint accels"

J: "Joint vels" → "task velocities"

"joint accels → "task accels"

J^T: "task force" → "joint torques"

$\Lambda_c^{-1} = (J H^{-1} J^T)$: "task Forces" → "task accels

Λ : "task accels" → "task Forces"

task ΔV → "task impulses"

when impacting

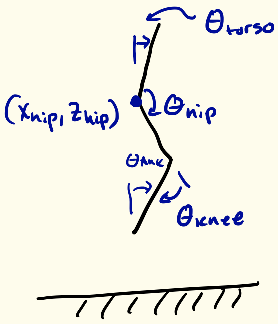
$$\hat{f} = \Lambda \Delta V$$

$$\Delta \dot{q} = H^{-1} J^T \underbrace{\Lambda \Delta V}_{\hat{f}}$$

Recap of Last Lecture: $H\ddot{q} + C\dot{q} + \tau_g = S^T\tau + J_c^T f_c$

$q_{Fl} = [x_{hip}, z_{hip}, \theta_{hip}, \theta_{kn}, \theta_{torso}]$

$q_{St} = [\theta_{ank}, \theta_{kn}, \theta_{hip}]$



Model System as 5 DoF, contact f_c follows some constitutive law

$f_c = Law(q, \dot{q})$

$\ddot{q} = H^{-1}[S^T\tau + J_c^T f_c - C\dot{q} - \tau_g]$

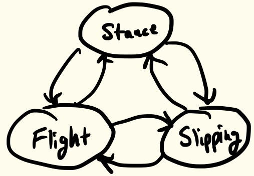
- Easy simulation Approach
- Small time steps 10μs-50μs
- No Impact impulse

Compliant — Ground Model — Rigid Ground

Do I have lots of contact mode

Yes: Treat each mode as 5 DoFs chan[g]e # of constraints (Excess coordinates)

No: Treat model as having different # of DoFs in each mode (minimal coordinates)



In each mode

$$\begin{bmatrix} H & -J_c^T \\ J_c & 0 \end{bmatrix} \begin{bmatrix} \ddot{q} \\ f_c \end{bmatrix} = \begin{bmatrix} S^T\tau - C\dot{q} - \tau_G \\ -\dot{J}_c \dot{q} \end{bmatrix}$$

- J_c changes as constraints change

In each mode

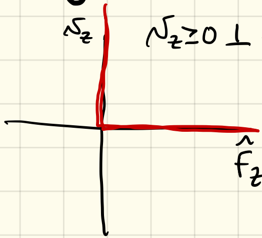
$$H'(q')\ddot{q}' + C(q', \dot{q}')\dot{q}' + \tau_G' = \tau'$$

- Constraints baked into choice of coordinates

Making contacts is impulsive

Determining when to break contact: Linear Complementary Problems

- General Idea: One of two cases
  - $v_z = 0$, $f_z \geq 0$ (the contact stays active)
  - $v_z > 0$, $f_z = 0$ (the contact releases)



$v_z \geq 0 \perp f_z \geq 0$

- Consider a timestep T

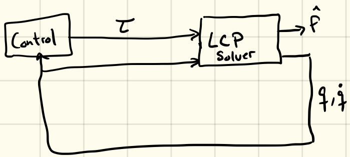
$$\ddot{q}_{t+T} \approx \frac{\dot{q}_{t+T} - \dot{q}_t}{T}$$

$$H\ddot{q} + C\dot{q} + \tau_G = S^T\tau + J^T f$$

$$\Rightarrow H_t \dot{q}_{t+T} = \underbrace{H_t \dot{q}_t - T(C_t \dot{q}_t + \tau_{Gt}) + T(S^T\tau}_{\gamma} + J^T f)$$

$$\Rightarrow J_t \dot{q}_{t+T} = V_{t+h}$$

- Allows to determine when breaking contact
- Timesteps $\approx$ 1ms - 15ms



- LCP Solver both models contacts and simulates

Linear Complementary Problem

Find $\dot{q}_{t+T}$, V_{t+T}, $\hat{f}$

$$H_t \dot{q}_{t+T} = \gamma + J_t^T \hat{f}$$

$$J_t \dot{q}_{t+T} = V_{t+T}$$

$$v_z \geq 0 , \quad \hat{f}_z \geq 0$$

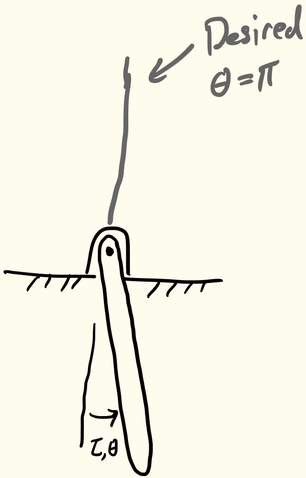
$$v_z \hat{f}_z = 0$$

Complementary condition

Introduction To Optimal Control

Conventional Control objectives:

- Rise time
- overshoot
- Settling time
- Metric notion of how good a response is.
  (All kind of arbitrary)



Control Design As Optimization

- System: $\dot{x} = f(t, x, u)$ $(*)$ $\quad x \in \mathbb{R}^n$ (state) $\quad u \in \mathbb{R}^m$ (control)

  $x(t_0) = x_0$ $\quad f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ (dynamics)

  $x_0 \in \mathbb{R}^n$ initial condition

- Cost: Functional

$$V(u(\cdot)) = \int_{t_0}^{t_f} \underbrace{\ell}_{\text{running cost}}(t, x(t), u(t))\, dt + \underbrace{\ell_f}_{\text{final, terminal cost}}(x(t_f))$$

  $x(t)$ the solution to $(*)$ starting from $x(t_0) = x_0$ and applying $u(t)$ an an input.

- Optimal control problem

$$\min_{u(\cdot)} V(u(\cdot))$$

$$\text{s.t.} \quad u(t) \in \mathcal{U}_t$$

$$x(t) \in \mathcal{X}_t$$

$$x(t_0) = x_0$$

  $\mathcal{U}_t \subseteq \mathbb{R}^m$ admissible controls

  $\mathcal{X}_t \subseteq \mathbb{R}^n$ admissible states

Pendulum Case: Example Formulation

System: $I\ddot{\theta} = \tau - mgl_1\sin\theta$

$$x = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix} \qquad \dot{x} = \underbrace{\begin{bmatrix} \dot{\theta} \\ \frac{1}{I}\left(\tau - mgl_1\sin\theta\right) \end{bmatrix}}_{F(x,u)}$$

$$u = \tau$$

Cost Functional:

$$\int_{t_0}^{\infty} \left[\left(\theta(t) - \pi\right)^2 + \tau^2(t) \right] dt$$

OCP:

$$\min_{u(\cdot)} \int_{t_0}^{\infty} \left[\left(\theta(t) - \pi\right)^2 + \tau^2(t) \right] dt$$

$$\text{s.t.} \quad |\tau(t)| \le \tau_{max}$$

$$\theta_{min} \le \theta(t) \le \theta_{max}$$

$$\dot{x} = F(x,u)$$