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Analytical Dynamics

Holonomic vs. Nonholonomic Constraints

Admin: HW 8 Due Wednesday

Last Time: • D'Alembert's Equations w/ 3D Rigid Bodies

Today:
- Review Problem 2 from HW 7
- A Brief recap of Key Ideas for D'Alemberts principle applied to 3D bodies

REVIEW

- Holonomic vs. Nonholonomic Constraints

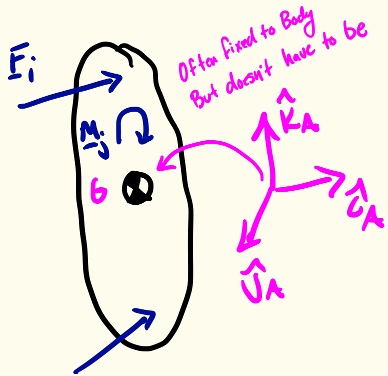
NEW

Pre-reqs: Null space of a matrix $\mathcal{N}(A) = \{ x \mid Ax = 0 \}$

Span of a set of vectors: $\text{Span}(\{w_1, \ldots, w_m\}) = \left\{ \sum_{i=1}^{m} \alpha_i w_i \mid \alpha_1, \ldots, \alpha_m \in \mathbb{R} \right\}$

Review: Newton's and Euler's Equations

For Any Rigid Body



$$m\,\underline{a}_G = \sum \underline{F}_i = \underline{F}_{net}$$

$$\dot{\underline{H}}_G = \sum_j \underline{M}_j + \sum_i (\underline{r}_i - \underline{r}_G) \times \underline{F}_i = \underline{M}_{G,net}$$

$${}^A\{\underline{H}_G\} = {}^A\underline{\underline{I}}_G \; {}^A\{\underline{\omega}\}$$

without a subscript denotes angular velocity of the Body

$${}^A\{\dot{\underline{H}}_G\} = \frac{d}{dt}\,{}^A\{\underline{H}_G\} + {}^A\{\underline{\omega}_A\} \times {}^A\{\underline{H}_G\}$$

Angular velocity of frame A (Same as $\underline{\omega}$ if A Body fixed)

$${}^A\underline{\underline{I}}_G = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix}$$

if A aligned w/ principal Axes

Euler's Equation: In any frame A (Derivation on next Slide)

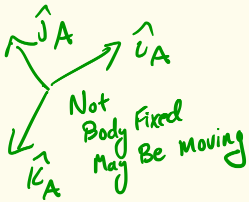
(whether A is body fixed or not, whether inertial or not)

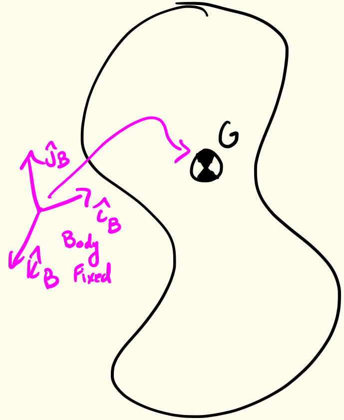
$${}^A\{\underline{M}_{G,net}\} = {}^A\underline{\underline{I}}_G \; {}^A\{\dot{\underline{\omega}}\} + {}^A\{\underline{\omega}\} \times {}^A\underline{\underline{I}}_G \; {}^A\{\underline{\omega}\}$$

Always angular velocity of the Body. I was wrong regarding using $\underline{\omega}_A$ here in class.

Euler's Equation about CoM, but expressed in a non-body-fixed Frame

Extra, not in class just FYI





Rotational Inertia:

$${}^A\underline{\underline{I}}_G = {}^A R_B \, \underbrace{{}^B\underline{\underline{I}}_G}_{\text{Constant since B Body Fixed}} \, {}^B R_A$$

Angular Momentum About G, Expressed in A

$${}^A\{\underline{H}_G\} = {}^A\underline{\underline{I}}_G \, {}^A\{\underline{\omega}\} = {}^A R_B \, {}^B\underline{\underline{I}}_G \, {}^B R_A \, {}^A\{\underline{\omega}\}$$

$$= {}^A R_B \, {}^B\underline{\underline{I}}_G \, {}^B\{\underline{\omega}\}$$

Rate of change in Angular Momentum about G, Expressed in A

$${}^A\{\underline{\dot{H}}_G\} = \frac{d}{dt}\, {}^A\{\underline{H}_G\} + {}^A\{\underline{\omega}_A\} \times {}^A\{\underline{H}_G\}$$

$$= \underbrace{{}^A\dot{R}_B}_{=\left[{}^A\widetilde{\omega_{B/A}}\right]{}^A R_B} \, {}^B\underline{\underline{I}}_G \, {}^B\{\underline{\omega}\} + {}^A R_B \, \cancel{{}^B\dot{\underline{\underline{I}}}_G}^{\,0} \, {}^B\{\underline{\omega}\} + {}^A R_B \, {}^B\underline{\underline{I}}_G \, \underbrace{{}^B\{\underline{\dot{\omega}}\}}_{= {}^B R_A \, {}^A\{\dot{\omega}\}} + {}^A\{\underline{\omega}_A\} \times {}^A\{\underline{H}_G\}$$

This is the term I missed in my off the coff answer in class

$$= {}^A\{\underline{\omega}_{B/A}\} \times \underbrace{{}^A R_B \, {}^B\underline{\underline{I}}_G \, {}^B\{\underline{\omega}\}}_{{}^A\{\underline{H}_G\}} + {}^A\underline{\underline{I}}_G \, {}^A\{\underline{\dot{\omega}}\} + {}^A\{\underline{\omega}_A\} \times {}^A\{\underline{H}_G\}$$

$$= {}^A\{\underbrace{\underline{\omega}_{B/A} + \underline{\omega}_A}_{=\underline{\omega}_B = \underline{\omega}}\} \times {}^A\{\underline{H}_G\} + {}^A\underline{\underline{I}}_G \, {}^A\{\underline{\dot{\omega}}\} = \boxed{{}^A\underline{\underline{I}}_G \, {}^A\{\underline{\dot{\omega}}\} + {}^A\{\underline{\omega}\} \times {}^A\underline{\underline{I}}_G \, {}^A\{\underline{\omega}\}}$$

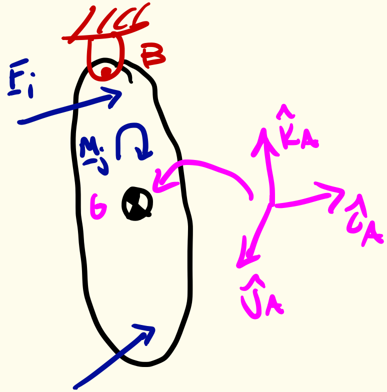
Review: Newton's and Euler's Equations

When rotating purely about B

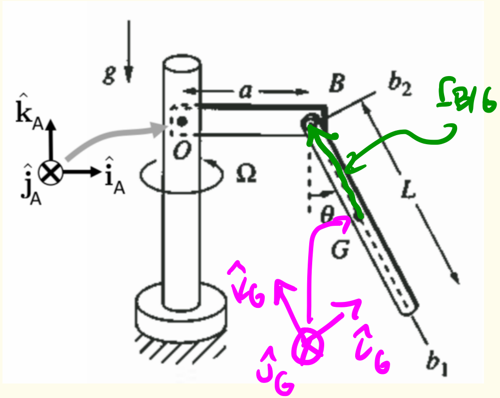
$$ {}^{A}\{\underline{H}_B\} = \underbrace{{}^{A}\underline{I}_B}_{\text{Rotational Inertia about B, expressed in A}} {}^{A}\{\underline{\omega}\} $$

$$ \dot{\underline{H}}_B = \underline{M}_{B,net} $$

$$ {}^{A}\{\dot{\underline{H}}_B\} = {}^{A}\underline{I}_B\, {}^{A}\{\dot{\underline{\omega}}\} + {}^{A}\{\underline{\omega}\} \times {}^{A}\underline{I}_B\, {}^{A}\{\underline{\omega}\} $$



Problem 2 (Euler's Equations): Consider the spinning pendulum consisting of a slender bar of mass m and length L attached to the arm (length a) of a shaft rotating with constant speed Ω. Using Newton's and Euler's equations, give the equation of motion and the reaction force vector $\mathbf{F}$ onto the rod at B. Express $\mathbf{F}$ using $\hat{\mathbf{i}}_A$, $\hat{\mathbf{j}}_A$, and $\hat{\mathbf{k}}_A$, which are assumed attached to the rotating vertical bar. Give each answer in terms of system parameters (e.g., m, L, g, Ω) and the generalized coordinate and its derivative θ, $\dot{\theta}$, $\ddot{\theta}$.



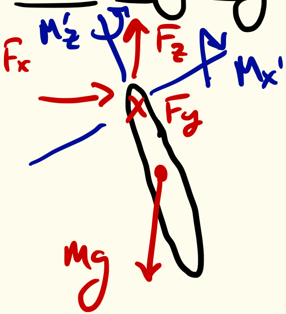
Strategy

1. Obtain $\underline{a}_G$ from acceleration analysis similar to HW2

2. $$\underline{F}_{net} = \underbrace{F_x\hat{i}_A + F_y\hat{j}_A + F_z\hat{k}_A}_{\underline{F}} - mg\,\hat{k}_A = m\,\underline{a}_G$$

$$\underline{F} = m\,\underline{a}_G + mg\,\hat{k}_A$$

Free Body Diag.

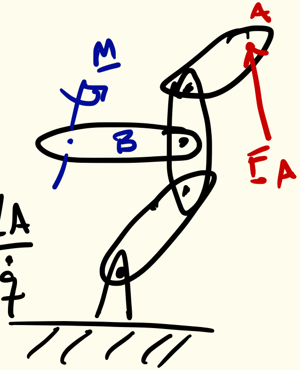


3. $${}^G I_G = \frac{1}{12} mL^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\boxed{\text{Pull off second component for EOM}}$$

$${}^G I_G\,{}^G\{\dot{\underline{\omega}}_{rod}\} + {}^G\{\underline{\omega}_{rod}\} \times {}^G I_G\,{}^G\{\underline{\omega}_{rod}\} = {}^G\{\underline{r}_{B/G} \times \underline{F}\} + \begin{bmatrix} M_x' \\ 0 \\ M_z' \end{bmatrix}$$

Generalized Forces: Addressing 3D Forces and Moments



- If a force acts at a point A on a body it contributes generalized Force

$$\frac{\partial \underline{r}_A}{\partial \underline{q}} = \frac{\partial \underline{v}_A}{\partial \underline{\dot{q}}}$$

$$Q_k = \frac{\partial \underline{r}_A}{\partial q_k} \cdot \underline{F}_A \qquad \underline{Q} = \left\{\frac{\partial \underline{r}_A}{\partial \underline{q}}\right\}^T \{\underline{F}_A\} = \left\{\frac{\partial \underline{v}_A}{\partial \underline{\dot{q}}}\right\}^T \{\underline{F}_A\}$$

$$\underline{Q} = F2Q({}^0\{\underline{F}_A\}, {}^0\{r_A\})$$ in MATLAB

- If a moment acts on a Body $\underline{B}$ it contributes GF

$$Q_k = \frac{\partial \underline{\omega}_B}{\partial \dot{q}_k} \cdot \underline{M} \qquad \underline{Q} = \left\{\frac{\partial \underline{\omega}_B}{\partial \underline{\dot{q}}}\right\}^T \{\underline{M}\}$$

$$\underline{Q} = M2Q(\{M\}, \{\underline{\omega}_B\})$$

- The total generalized force is the sum of the contributions from all forces & moments.

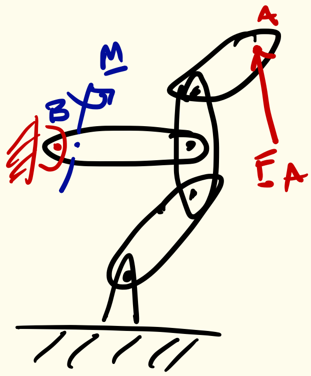
Generalized Inertial Force for 3D Rigid Bodies

- The generalized inertial force from a body B is given by:

$$Q_k = \frac{\partial \underline{V}_G}{\partial \dot{q}_k} \cdot (m \underline{a}_G) + \frac{\partial \underline{\omega}_B}{\partial \dot{q}_k} \cdot \underline{\dot{H}}_G$$

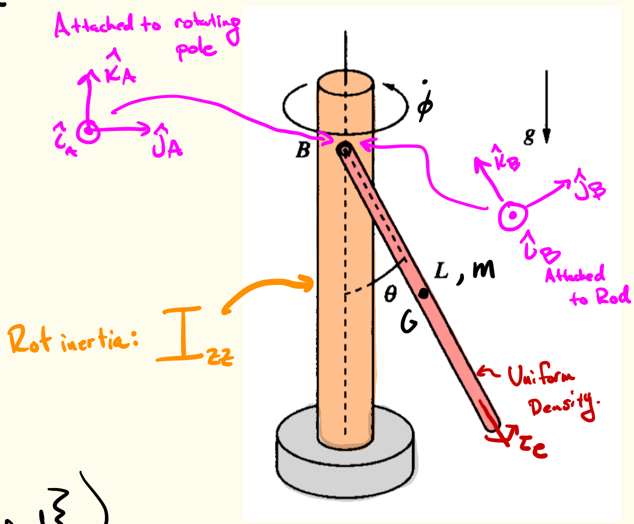
$$\underline{Q} = \left\{ \frac{\partial \underline{v}_G}{\partial \dot{\underline{q}}} \right\}^T \left\{ m \underline{a}_G \right\} + \left\{ \frac{\partial \underline{\omega}_B}{\partial \dot{\underline{q}}} \right\}^T \left\{ \underline{\dot{H}}_G \right\}$$

$$= \left\{ \frac{\partial \underline{\omega}_B}{\partial \dot{\underline{q}}} \right\}^T \left\{ \underline{\dot{H}}_B \right\} \quad \text{if purely rotating about point } B$$



- The total generalized force is the sum over all bodies.

Example: Same as last time, but Pole has Rot. Inertia
Give EOM Via D'Alembert



Strategy:

2 Bodies, Both purely rotate about B

$$\underline{Q}_{inertial} = M2Q\left({}^{A}\{\dot{\underline{H}}_{pole,B}\}, {}^{A}\{\underline{\omega}_{pole}\}\right)$$

$$+ M2Q\left({}^{B}\{\dot{\underline{H}}_{rod,B}\}, {}^{B}\{\underline{\omega}_{rod}\}\right)$$

$$\underline{Q} = \overbrace{F2Q\left({}^{0}\{m\underline{g}\}, {}^{0}\{\underline{r}_G\}\right)}^{-\nabla_q V} + M2Q\left({}^{B}\{\tau_e \hat{k}_B\}, {}^{B}\{\underline{\omega}_{rod}\}\right)$$