Lecture 17 - The Jacobian

Last Time: Velocity analysis frame by frame

$${}^{i}\omega_i = {}^{i}R_{i-1}\,{}^{i-1}\omega_{i-1} + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_i \end{bmatrix}$$

$${}^{i}v_i = {}^{i}R_{i-1}\left[{}^{i-1}v_{i-1} + {}^{i-1}\omega_{i-1} \times {}^{i-1}p_i\right] + \begin{bmatrix} 0 \\ 0 \\ \dot{d}_i \end{bmatrix} \qquad i = 1, \dots, n$$

Today: End-effector velocity & Jacobian

Θ : n-dim vector of joint variables

$$\begin{bmatrix} {}^{t}\omega_t \\ {}^{t}v_t \end{bmatrix} = {}^{t}J_t(\theta)\dot{\Theta} = \begin{bmatrix} {}^{t}J_t^{\omega} \\ \hdashline {}^{t}J_t^{v} \end{bmatrix}\dot{\Theta}$$

${}^{t}J_t$ $6 \times n$ matrix

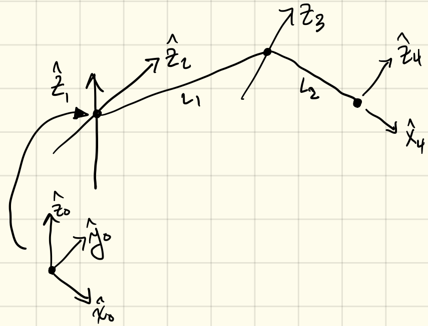
${}^{t}J_t^{\omega}$ $3 \times n$

Methods to compute the Jacobian

- Analytical $\longleftarrow$
- Geometric

① Analytical Computation of the Jacobian

$$
{}^0p_4 = \begin{bmatrix} c_1(\overset{A}{c_{23}l_2} + \overset{B}{c_2 l_1}) \\ s_1(c_{23}l_2 + c_2 l_1) \\ \underset{C}{-s_{23}L_2} \; \underset{D}{-s_2 L_1} \end{bmatrix} \qquad {}^0v_4 = {}^0J_4^v \, \dot{\Theta} = \frac{d}{dt}\,{}^0p_4
$$



$$
{}^0\dot{p}_4 = \left[\frac{\partial\, {}^0p_4}{\partial \theta_1} \cdot \dot{\theta}_1 + \frac{\partial\, {}^0p_4}{\partial \theta_2} \dot{\theta}_2 + \frac{\partial\, {}^0p_4}{\partial \theta_3} \dot{\theta}_3 \right]
$$

$$
= \underbrace{\begin{bmatrix} \frac{\partial\, {}^0p_4}{\partial \theta_1} & \frac{\partial\, {}^0p_4}{\partial \theta_2} & \frac{\partial\, {}^0p_4}{\partial \theta_3} \end{bmatrix}}_{{}^0J_4^v} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix}
$$

Note: ${}^0J_4^w$ by taking partials of ${}^0R_4$ (HW5)

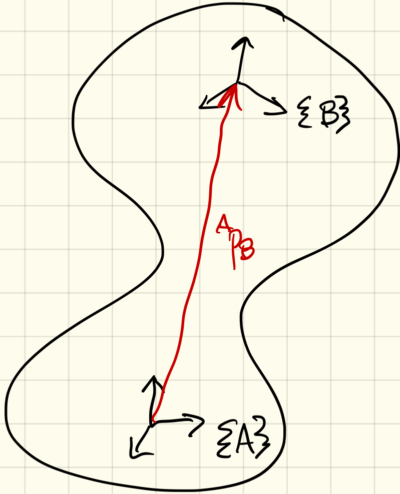
$$
{}^0J_4^v = \begin{bmatrix} -s_1(A+B) & c_1(C+D) & c_1 C \\ c_1(A+B) & s_1(C+D) & s_1 C \\ 0 & -(A+B) & -A \end{bmatrix}
$$

② Geometric Computation of Jacobian

$$\begin{bmatrix} {}^B\omega_B \\ {}^B v_B \end{bmatrix} = \begin{bmatrix} {}^B R_A \, {}^A\omega_A \\ {}^B R_A \left[{}^A v_A - \underbrace{{}^A p_B \times {}^A\omega_A}_{S({}^A p_B)\,{}^A\omega_A} \right] \end{bmatrix}$$

$$= \underbrace{\left[\begin{array}{c|c} {}^B R_A & 0 \\ \hline -{}^B R_A S({}^A p_B) & {}^B R_A \end{array}\right]}_{{}^B X_A} \begin{bmatrix} {}^A\omega_A \\ {}^A v_A \end{bmatrix}$$

${}^B X_A$ (Easily computed from ${}^A T_B$)



{A}, {B} attached to the same body

Aside: The cross product matrix

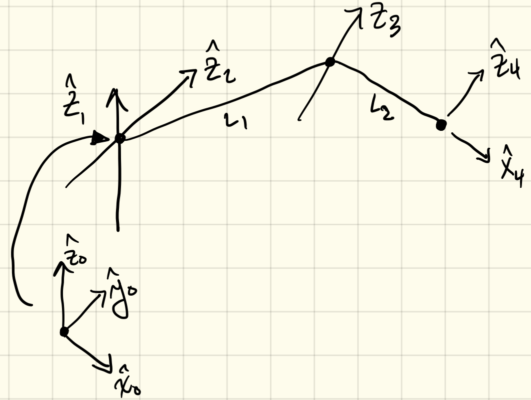
- Consider the cross product of two vectors:

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \times \begin{bmatrix} d \\ e \\ f \end{bmatrix} = \begin{bmatrix} bf - ce \\ cd - af \\ ae - bd \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{bmatrix}}_{\text{define as } S\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right)} \begin{bmatrix} d \\ e \\ f \end{bmatrix}$$

- Given two vectors ${}^{A}v_1$, ${}^{A}v_2$ (expressed in the same frame) the cross product matrix $S({}^{A}v_2)$ as above is a 3x3 matrix such that

$${}^{A}v_1 \times {}^{A}v_2 = S({}^{A}v_1)\,{}^{A}v_2$$

Computing the Jacobian Geometrically



$$\begin{bmatrix} {}^4\omega_4 \\ {}^4v_4 \end{bmatrix} = {}^4J_4\,\dot{\theta} = \begin{bmatrix} {}^4v_{4,1} & {}^4v_{4,2} & {}^4v_{4,3} \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix}$$

${}^4v_{4,1}$ is equal to $\begin{bmatrix} {}^4\omega_4 \\ {}^4v_4 \end{bmatrix}$ when $\dot{\theta}_1 = 1$

$\dot{\theta}_2, \dot{\theta}_3 = 0$

$$\begin{bmatrix} {}^4\omega_4 \\ {}^4v_4 \end{bmatrix} = {}^4X_1 \begin{bmatrix} {}^1\omega_1 \\ {}^1v_1 \end{bmatrix} = {}^4X_1 \left[\begin{array}{c} 0 \\ 0 \\ 1 \\ \hline 0 \\ 0 \\ 0 \end{array}\right]$$

Revolute Joint

$${}^tv_{t,i} = {}^tX_i \left[\begin{array}{c} 0 \\ 0 \\ 1 \\ \hline 0 \\ 0 \\ 0 \end{array}\right] = \left[\begin{array}{c} {}^iR_t^T \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\ \hline {}^iR_t^T \left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times {}^ip_t \right) \end{array}\right]$$

Prismatic

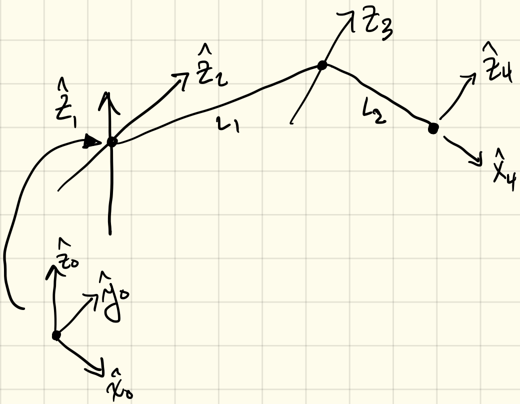
$${}^tv_{t,i} = {}^tX_i \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ \hline 0 \\ 0 \\ 1 \end{array}\right] = \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ \hline {}^iR_t^T \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{array}\right]$$

$$ {}^{B}X_{A} = \begin{bmatrix} {}^{B}R_{A} & \\ -{}^{B}R_{A}\, S({}^{A}P_{B}) & {}^{B}R_{A} \end{bmatrix} $$

$$ \begin{bmatrix} {}^{4}\omega_{4} \\ {}^{4}v_{4} \end{bmatrix} = {}^{4}J_{4}\,\dot{\Theta} = \begin{bmatrix} {}^{4}v_{4,1} & {}^{4}v_{4,2} & {}^{4}v_{4,3} \end{bmatrix} \dot{\Theta} $$

$$ {}^{1}T_{4} = \begin{bmatrix} c_{23} & -s_{23} & 0 & L_2 c_{23} + L_1 c_2 \\ 0 & 0 & 1 & 0 \\ -s_{23} & -c_{23} & 0 & -L_2 s_{23} - L_1 s_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad {}^{2}T_{4} = \begin{bmatrix} c_3 & -s_3 & 0 & L_1 + L_2 c_3 \\ s_3 & c_3 & 0 & L_2 s_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad {}^{3}T_{4} = \begin{bmatrix} 1 & 0 & 0 & L_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

Compute the Second Column of the Jacobian ${}^{4}J_{4}$: ${}^{4}v_{4,2}$



$$ {}^{4}v_{4,2} = \begin{bmatrix} {}^{2}R_{4}^{T} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\ \hdashline {}^{2}R_{4}^{T} \left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times {}^{2}P_{4} \right) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ \hdashline L_1 s_1 \\ L_1 c_1 + L_2 \\ 0 \end{bmatrix} $$