Lecture 3: Spatial Dynamics

- Announcements
  - HW 1: Prob 7c: Now for glory & Bonus (+5/100)
  - HW 1: Due Thursday

- Last Time
  - Spatial velocity / acceleration of a rigid body
  - Change of basis (X matrix)

- Today
  - Moving frames
  - Spatial forces & momentum
  - Dynamics of a rigid body
  - Recursive Newton Euler (probably not today)

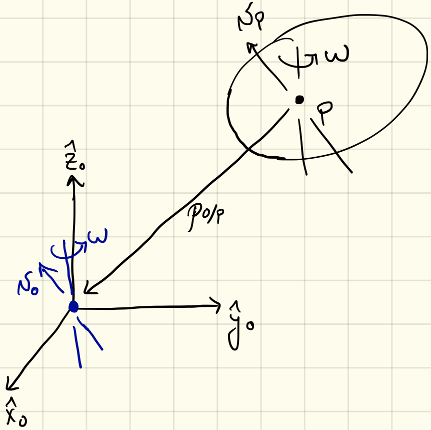
- On Deck: Multibody Equations of Motion

Spatial Velocity of a Rigid Body:

$${}^{0}v = \begin{bmatrix} {}^{0}\omega \\ {}^{0}v_0 \end{bmatrix}$$

← angular velocity of body

← Velocity of the body-fixed particle coincident w/ origin of $\{0\}$



Spatial Acceleration of a Rigid Body

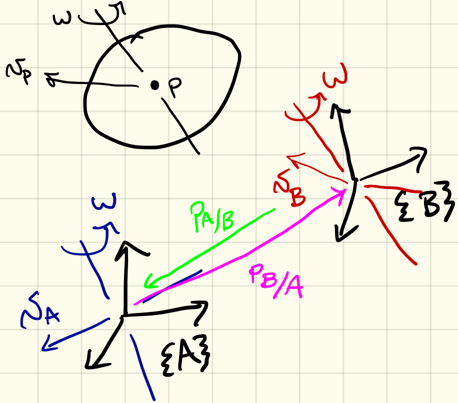
$${}^{0}a = {}^{0}[\dot{v}] = \begin{bmatrix} {}^{0}\dot{\omega} \\ {}^{0}\dot{v}_0 \end{bmatrix}$$

← rate of change in stream velocity of particles passing through origin of $\{0\}$

> The space of all spatial velocities is a 6-D vector space. Any frame sets a basis for this space via
>
> ① Pure translation along $\hat{x}, \hat{y}, \hat{z}$
>
> ② Pure rotations about $\hat{x}, \hat{y}, \hat{z}$ with axes passing through the frame origin

Spatial accelerations are elements of the same Vec. Sp.

Transforming Spatial Vectors:



$${}^A V = \begin{bmatrix} {}^A\omega \\ {}^A v_A \end{bmatrix} \qquad {}^B V = \begin{bmatrix} {}^B\omega \\ {}^B v_B \end{bmatrix}$$

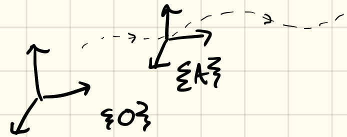
$S({}^B p_{A/B})\,{}^B R_A$

$S(a)b = a \times b$

$${}^B V = \begin{bmatrix} {}^B\omega \\ {}^B v_B \end{bmatrix} = \underbrace{\begin{bmatrix} {}^B R_A & 0 \\ {}^B p_{A/B} \times {}^B R_A & {}^B R_A \end{bmatrix}}_{{}^B X_A} \begin{bmatrix} {}^A\omega \\ {}^A v_A \end{bmatrix} = {}^B X_A \, {}^A V$$

Change of basis from Plucker coordinates w.r.t. {A} to Plücker coords w.r.t. {B}

Spatial acceleration **More Generally:** Consider a fixed frame $\{0\}$ a moving frame $\{A\}$. Consider a rigid body with Spatial velocity $^0V_{Body}(t)$.



$$^0a_{Body} = \lim_{\Delta t \to 0} \frac{^0V_{Body}(t+\Delta t) - {^0V_{Body}}(t)}{\Delta t}$$

$$^Aa_{Body} = {^AX_0}\,{^0a_{Body}} = {^A\left[\dot{V}_{Body}\right]} \neq \lim_{\Delta t \to 0} \frac{^AV_{Body}(t+\Delta t) - {^AV_{Body}}(t)}{\Delta t} = \frac{d}{dt}\left[{^AV_{Body}}\right]$$

- First 3 components of $^Aa_{Body}$ give angular acceleration
- Last 3 give the rate of change in stream velocity of Body-fixed particles passing through $\{A\}$ if $\{A\}$ stayed fixed @ its current location.

Spatial Cross Product: Working w/ moving frames

HW2

$${}^0\dot{X}_A = ({}^0V_A \times)\,{}^0X_A$$

$${}^0V_{Body} = \begin{bmatrix} {}^0\omega_{Body} \\ {}^0v_{O,Body} \end{bmatrix} \qquad {}^0V_A = \begin{bmatrix} {}^0\omega_A \\ {}^0v_{O,A} \end{bmatrix}$$

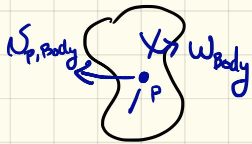
$${}^0a_{Body} = \frac{d}{dt}\left[{}^0V_{Body}\right] = \frac{d}{dt}\left[{}^0X_A\,{}^AV_{Body}\right] = {}^0\dot{X}_A\,{}^AV_{Body} + {}^0X_A\left[\frac{d}{dt}\,{}^AV_{Body}\right]$$

$$= \left[\underbrace{({}^0V_A \times)}\,{}^0X_A\right]{}^AV_{Body} + {}^0X_A\left[\frac{d}{dt}\,{}^AV_{Body}\right]$$

$$:= \begin{bmatrix} S({}^0\omega_A) & 0 \\ S({}^0v_{O,A}) & S({}^0\omega_A) \end{bmatrix}$$

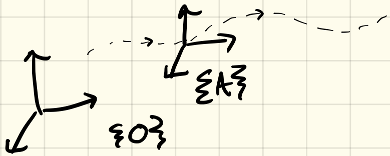
Expressed in $\{0\}$

$${}^0a_{Body} = ({}^0V_A \times)\,{}^0X_A\,{}^AV_{Body} + {}^0X_A\left[\frac{d}{dt}\,{}^AV_{Body}\right]$$

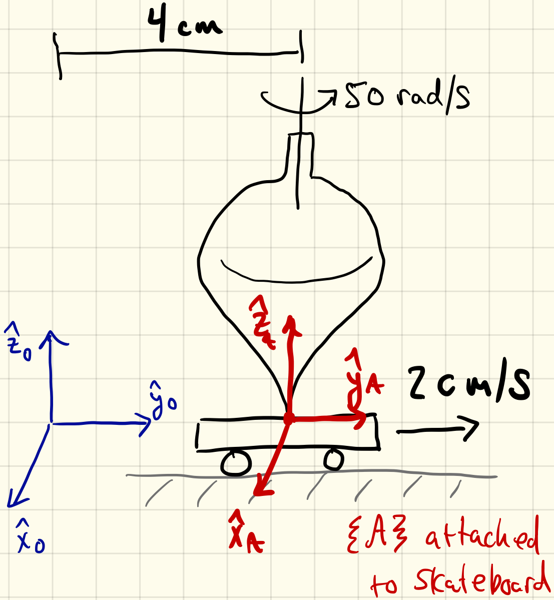


Expressed in $\{A\}$

$${}^Aa_{Body} = \underbrace{{}^AV_A \times {}^AV_{Body}}_{\text{accounts for moving coord sys}} + \underbrace{\left[\frac{d}{dt}\,{}^AV_{Body}\right]}_{\text{Derivative in coordinates}} = {}^AX_0\,{}^0a_{Body}$$



Example: The top returns (on a skateboard)



Give ${}^{0}V_{top}$, ${}^{0}a_{top}$

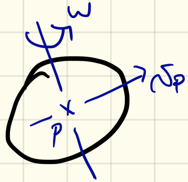
$$ {}^{0}V_{top} = \begin{bmatrix} 0 \\ 0 \\ 50\ \mathrm{rad/s} \\ \hdashline 2\ \mathrm{m/s} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \hdashline 0 \\ 2\ \mathrm{cm/s} \\ 0 \end{bmatrix} \qquad {}^{A}V_{top} = \begin{bmatrix} 0 \\ 0 \\ 50\ \mathrm{rad/s} \\ \hdashline 0 \\ 2\ \mathrm{cm/s} \\ 0 \end{bmatrix} $$

$$ = \begin{bmatrix} 0 \\ 0 \\ 50\ \mathrm{rad/s} \\ \hdashline 2\ \mathrm{m/s} + 1\ \mathrm{m/s^2}\, t \\ 2\ \mathrm{cm/s} \\ 0 \end{bmatrix} $$

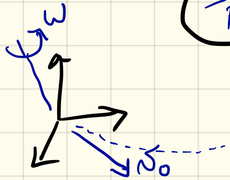
$$ {}^{0}a_{top} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \hdashline 1\ \mathrm{m/s^2} \\ 0 \\ 0 \end{bmatrix} $$

$$ {}^{A}a_{top} = \left({}^{A}V_{A}\times\right){}^{A}V_{top} + \cancel{\left[\frac{d}{dt}\,{}^{A}V_{top}\right]}^{\,0} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \hdashline 0 \\ 2\ \mathrm{cm/s} \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ 50\ \mathrm{rad/s} \\ \hdashline 0 \\ 2\ \mathrm{cm/s} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \hdashline 1\ \mathrm{m/s^2} \\ 0 \\ 0 \end{bmatrix} $$

Relationship Between Spatial & Conventional Accel

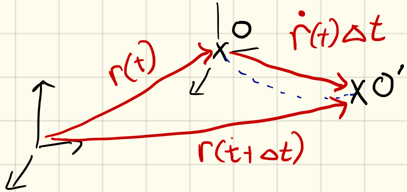


Again $\dot{v}_o$ is not the acceleration of any point on the Body!



$\dot{r}(t) = v_o(t)$ ~ $r(t)$ gives position of the particle that was at O when $t=0$.

$$\dot{r}(t+\Delta t) = v_o(t+\Delta t) + \omega(t+\Delta t)\times \dot{r}(t)\Delta t$$



$$\dot{v}_o = \lim_{\Delta t\to 0} \frac{v_o(t+\Delta t) - v_o(t)}{\Delta t}$$

$$= \lim_{\Delta t\to 0} \frac{\dot{r}(t+\Delta t) - \omega(t+\Delta t)\times \dot{r}(t)\Delta t - \dot{r}(t)}{\Delta t}$$

$$= \ddot{r}(t) - \omega(t)\times \dot{r}(t)$$

Conventional acceleration of one particle

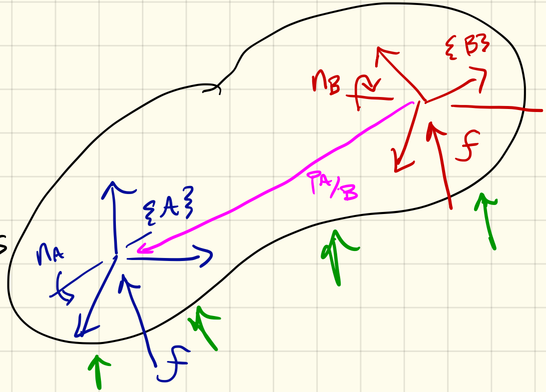
$${}^{o}a = \begin{bmatrix} {}^{o}\dot{\omega} \\ \hline \ddot{r}(t) - {}^{o}\omega(t)\times {}^{o}\dot{r}(t) \end{bmatrix}$$

Rate of change in the flow velocity of particles passing through O

Take away

We'll work w/ spatial accels. but if you need to convert to a true acceleration of a point => conversion is easy.

Spatial Forces (wrenches):



- Consider a body with many forces on it and a frame {B}.

  The net effect of these forces can be expressed as
  1. A force f passing through {B}
  2. A moment n_B about origin of {B}

$$ {}^B f = \begin{bmatrix} {}^B n_B \\ {}^B f \end{bmatrix} $$

- Similarly, you can do the same using a different frame, {A}

$$ {}^A f = \begin{bmatrix} {}^A n_A \\ {}^A f \end{bmatrix} = \begin{bmatrix} {}^A R_B \, {}^B n_B - {}^A R_B \, {}^B p_{A/B} \times {}^B f \\ {}^A R_B \, {}^B f \end{bmatrix} = \underbrace{\begin{bmatrix} {}^A R_B & -{}^A R_B \, S({}^B p_{A/B}) \\ 0 & {}^A R_B \end{bmatrix}}_{:= {}^A X^*_B} \underbrace{\begin{bmatrix} {}^B n_B \\ {}^B f \end{bmatrix}}_{{}^B f} $$

Spatial force transform

Relationship Between Transforms

Recall: $v \times w = S(v)w$ where $S(v) = \begin{bmatrix} 0 & -v_z & v_y \\ v_z & 0 & -v_x \\ -v_y & v_x & 0 \end{bmatrix}$ Skew symmetric! $S(v)^T = -S(v)$

From Past Notes

$${}^{B}X_{A} = \begin{bmatrix} {}^{B}R_{A} & 0 \\ S({}^{B}p_{A/B})\,{}^{B}R_{A} & {}^{B}R_{A} \end{bmatrix} \qquad {}^{A}X^{*}_{B} = \begin{bmatrix} {}^{A}R_{B} & -{}^{A}R_{B}\,S({}^{B}p_{A/B}) \\ 0 & {}^{A}R_{B} \end{bmatrix}$$

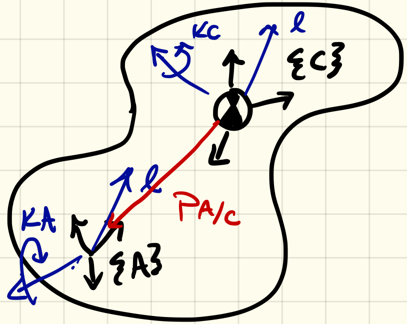
Taking a transpose...

$${}^{B}X_{A}^{T} = \begin{bmatrix} {}^{A}R_{B} & \left(S({}^{B}p_{A/B})\,{}^{B}R_{A}\right)^{T} \\ 0 & {}^{A}R_{B} \end{bmatrix} = \begin{bmatrix} {}^{A}R_{B} & -{}^{A}R_{B}\,S({}^{B}p_{A/B}) \\ 0 & {}^{A}R_{B} \end{bmatrix}$$

$$\therefore\ {}^{B}X_{A}^{T} = {}^{A}X^{*}_{B}$$

When taking the transpose of a spatial transform you swap the sub/super script AND switch the type (velocity or force) of the transform.

Spatial Momentum:



$$^C h = \begin{bmatrix} ^C\kappa_C \\ ^C\ell \end{bmatrix} \begin{matrix} \leftarrow \text{Angular momentum about } C \\ \leftarrow \text{linear momentum} \end{matrix}$$

$$= \underbrace{\begin{bmatrix} ^C\bar{I} & \\ & m \end{bmatrix}}_{^C I} \underbrace{\begin{bmatrix} ^C\omega \\ ^C v_C \end{bmatrix}}_{^C v}$$

$\bar{I} \in \mathbb{R}^{3\times 3}$ inertia tensor about CoM

$$^A h = \begin{bmatrix} ^A\kappa_A \\ ^A\ell \end{bmatrix} = \begin{bmatrix} ^A R_C\, ^C\kappa_C - {}^A R_C \left(^C p_{A/C} \times \right) {}^C\ell \\ ^A R_C\, ^C\ell \end{bmatrix} = {}^A X_C^*\ {}^C h$$

Spatial momenta transform same as spatial forces! (i.e., they are both elements of the 6D VS of spatial momenta/forces)