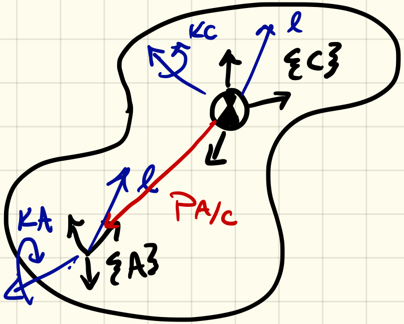
Lecture 4: Multibody Dynamics

- **Last Time:**
  - Spatial accel w/moving frames $\quad {}^{A}a_{Body} = \left[{}^{A}v_{A}\times\right]{}^{A}v_{Body} + \frac{d}{dt}\left[{}^{A}v_{Body}\right]$
  - Spatial cross product $\quad {}^{0}\dot{X}_{A} = \left[{}^{0}v_{A}\times\right]{}^{0}X_{A}$
  - Spatial cross product in any frame $\quad \left[{}^{0}v_{A}\times\right]{}^{0}v_{Body} = {}^{0}X_{A}\left[\left[{}^{A}v_{A}\times\right]{}^{A}v_{Body}\right]$
  - Spatial force $\quad {}^{A}f = \begin{bmatrix} {}^{A}n_{A} \\ {}^{A}f \end{bmatrix} = {}^{A}X_{B}^{*}\begin{bmatrix} {}^{B}n_{B} \\ {}^{B}f \end{bmatrix} \qquad {}^{A}X_{B}^{*} = {}^{B}X_{A}^{T}$

Today:
- $f = m\,\ddot{p}_{com}$ spatial edition
- Multibody conventions
- Multibody Dynamics

Spatial Momentum:



$${}^C h = \begin{bmatrix} {}^C k_C \\ {}^C \ell \end{bmatrix} \begin{matrix} \leftarrow \text{Angular momentum about C} \\ \leftarrow \text{linear momentum} \end{matrix}$$

$$= \underbrace{\begin{bmatrix} {}^C \bar{I} & \\ & m \end{bmatrix}}_{{}^C I} \underbrace{\begin{bmatrix} {}^C\omega \\ {}^C v_C \end{bmatrix}}_{{}^C V}$$

$\bar{I} \in \mathbb{R}^{3\times3}$ inertia tensor about CoM

$${}^A h = \begin{bmatrix} {}^A k_A \\ {}^A \ell \end{bmatrix} = \begin{bmatrix} {}^A R_C \, {}^C k_C - {}^A R_C ({}^C p_{A/C} \times) {}^C \ell \\ {}^A R_C \, {}^C \ell \end{bmatrix} = {}^A X_C^* \; {}^C h$$

Spatial momenta transform same as spatial forces! (i.e., they are both elements of the 6D VS of spatial momenta/forces)

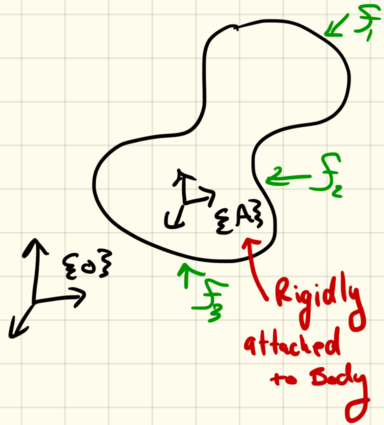
$${}^A h = {}^A X_C^* \, {}^C h = {}^A X_C^* \, {}^C I \, {}^C V = \underbrace{{}^A X_C^* \, {}^C I \, {}^C X_A}_{{}^A I} \, {}^A V$$

Assume ${}^A R_C = I_{3\times3}$

$${}^A I = \begin{bmatrix} {}^C\bar{I} + m S({}^A p_{com}) S({}^A p_{com})^T & m S({}^A p_{com}) \\ m S({}^A p_{com})^T & m I_{3\times3} \end{bmatrix}$$

rotational inertia of body about {A} (Parallel axis thm. Steiner's thm.)

Spatial Dynamics of a Rigid Body:



$$^0h = \begin{bmatrix} {}^0k_0 \\ {}^0\ell \end{bmatrix} = {}^0X_A^* \, {}^AI \, {}^AX_0 \, {}^0v_A$$

$$\frac{d}{dt}\,{}^0h = \underbrace{\begin{bmatrix} {}^0n_0 \\ {}^0f \end{bmatrix}}_{\text{net spatial force on body}} = {}^0f = {}^A\dot{X}_0^T \, {}^AI \, {}^AX_0 \, {}^0v_A + {}^AX_0^T \, {}^AI \, {}^A\dot{X}_0 \, {}^0v_A + {}^AX_0^T \, {}^AI \, {}^AX_0 \, {}^0a_A$$

$$= [{}^0v_A\times^*] \underbrace{{}^0X_A^* \, {}^AI \, {}^AX_0}_{{}^0I} \, {}^0v_A - {}^0X_A^* \, {}^AI \, {}^AX_0 \, \cancelto{0}{{}^0v_A \times {}^0v_A} + \underbrace{{}^0X_A^* \, {}^AI \, {}^AX_0}_{{}^0I} \, {}^0a_A$$

$$= [{}^0v_A\times^*] \, {}^0I \, {}^0v_A + {}^0I \, {}^0a_A$$

Aside

$$[{}^A\dot{X}_0^T] = [{}^A\dot{X}_0]^T = [{}^0\dot{X}_A^{-1}]^T = [-{}^AX_0 \, [{}^0v_A\times] \, \cancel{{}^0X_A \, {}^0X_A^{-1}}]^T = -\overbrace{[{}^0v_A\times]^T \, {}^0X_A^*}^{[{}^0v_A\times^*]}$$

$[{}^0v_A\times^*]$ ← Spatial cross product applied to momenta/forces

Spatial Equation of motion for a rigid body:

$$\boxed{{}^{A}f = {}^{A}I\,{}^{A}a + {}^{A}v\times^{*}\,{}^{A}I\,{}^{A}v}$$

Generalizes Eulers Eqn:

$${}^{c}n_{c} = {}^{c}I\,{}^{c}\dot{\omega} + {}^{c}\omega\times{}^{c}I\,{}^{c}\omega$$

works in any frame

(even when A is away from the com of the body)

$${}^{A}f = {}^{A}\left[\frac{d}{dt}h\right] = {}^{A}\left[\frac{d}{dt}\left[Iv\right]\right]$$

$$f = \underbrace{Ia}_{\text{accounts for the fact that } v \text{ is changing}} + \underbrace{v\times^{*}Iv}_{\text{Accounts for the fact that the inertia is moving}}$$

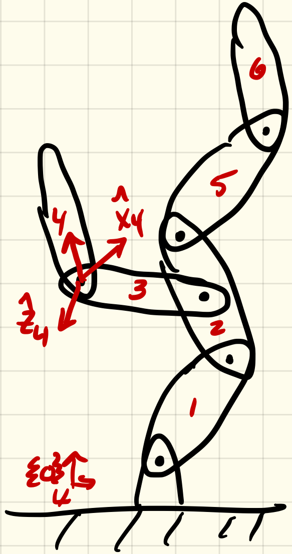
Multibody Systems: Conventions

(1) Number bodies 1 to N

- predecessor (parent) toward root as $p(i)$ e.g., $p(5)=2$
- $p(i)<i$ for all i in valid numberings
- $c(i)$ children (e.g. $c(2)=\{3,5\}$)

(2) Joint i connects $p(i)$ to i

$q_i \in \mathbb{R}$ its joint angle

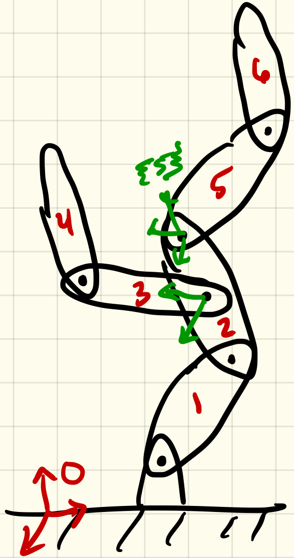


(3) Frame $\{i\}$ attached to body i immediately after joint i

${}^i\phi_i \in \mathbb{R}^6$ spatial velocity from $\dot{q}_i = 1$

$${}^4\phi_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ \hdashline 0 \\ 0 \\ 0 \end{bmatrix}$$

Multibody Systems: Velocities & Accelerations



$${}^5V_5 = {}^5X_1\, {}^1\Phi_1\, \dot q_1 + {}^5X_2\, {}^2\Phi_2\, \dot q_2 + {}^5\Phi_5\, \dot q_5$$

$$= {}^5X_2 \underbrace{\left[{}^2X_1\, {}^1\Phi_1\, \dot q_1 + {}^2\Phi_2\, \dot q_2\right]}_{{}^2V_2} + {}^5\Phi_5\, \dot q_5$$

Velocity Prop

$$\boxed{{}^iV_i = {}^iX_{p(i)}\, {}^{p(i)}V_{p(i)} + {}^i\Phi_i\, \dot q_i}$$

Accel Prop

$$\boxed{{}^ia_i = {}^iX_{p(i)}\, {}^{p(i)}a_{p(i)} + {}^iV_i \times {}^i\Phi_i\, \dot q_i + {}^i\Phi_i\, \ddot q_i}$$

Derivative in a coordinate-free sense

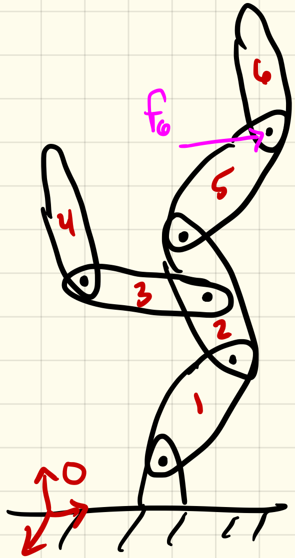
$${}^i\left[\frac{d}{dt}\Phi_i\, \dot q_i\right] = \frac{d}{dt}\left[{}^i\Phi_i\, \dot q_i\right] + \underbrace{{}^iV_i \times {}^i\Phi_i\, \dot q_i}$$

Derivative taken in moving coordinates

Extra term to account for moving basis vectors

Multibody Systems: Forces

Let f_i denote the force onto body i from body $p(i)$



Body 6

$$f_6 = I_6 a_6 + v_6 \times^* I_6 v_6$$

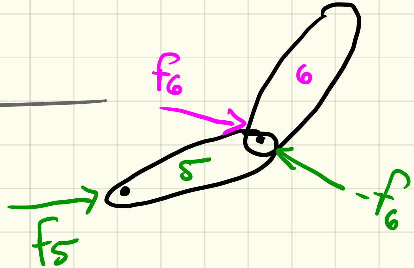
Body 5

$$f_5 - f_6 = I_5 a_5 + v_5 \times^* I_5 v_5$$

$$\Rightarrow \quad f_5 = f_6 + I_5 a_5 + v_5 \times I_5 v_5$$

Body 2

$$f_2 = f_3 + f_5 + I_2 a_2 + v_2 \times^* I_2 v_2$$



In general

$$f_i = I_i a_i + v_i \times^* I_i v_i + \sum_{j \in c(i)} f_j$$

$$\tau_i = \phi_i^T f_i$$

Recursive Newton Euler $\tau(q, \dot{q}, \ddot{q})$

$v_0 = 0 \qquad a_0 = -{}^0a_g$

for $i = 1$ to N

$$v_i = {}^iX_{p(i)}\, v_{p(i)} + \phi_i \dot{q}_i$$

$$a_i = {}^iX_{p(i)}\, a_{p(i)} + \phi_i \ddot{q} + v_i \times \phi_i \dot{q}_i$$

$$f_i = I_i a_i + v_i \times^* I_i v_i \quad (**)$$

end

for $i = N$ to 1

$$\tau_i = \phi_i^T f_i$$

$$f_{p(i)} = f_{p(i)} + {}^iX_{p(i)}^T f_i$$

end

- when algo finishes f_i is correct

${}^0a_g$ = gravity acceleration.

- This "trick" causes τ_i to include gravity effects
- However a_i is now "off" by a_g in comparisson to its true spatial acceleration.
  - e.g., with this trick

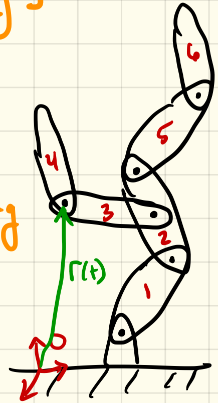
$$\underbrace{a_4}_{\text{in the algo}} = a_{4,true} - {}^4X_0\, {}^0a_g$$

$$= \begin{bmatrix} {}^4\dot{\omega}_4 \\ {}^4\ddot{r} - {}^4\omega_4 \times {}^4\dot{r} - {}^4R_0 \vec{g} \end{bmatrix}$$

- If one wants to avoid this issue, revise (**) to

$$f_i = I_i a_i + v_i \times^* I_i v_i - I_i\, {}^iX_0\, {}^0a_g$$

and initialize $a_0 = 0$.



Summary

Compare to 3D vector version from AME 50551

$${}^0\omega_0 = {}^0\dot{\omega}_0 = {}^0\dot{v}_0 = 0$$

$i = 0:5$

$${}^{i+1}\omega_{i+1} = {}^{i+1}R_i\, {}^i\omega_i + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_{i+1} \end{bmatrix}$$

$${}^{i+1}\dot{\omega}_{i+1} = {}^{i+1}R_i\, {}^i\dot{\omega}_i + {}^{i+1}\omega_{i+1} \times \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_{i+1} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \ddot{\theta}_{i+1} \end{bmatrix}$$

$${}^{i+1}\dot{v}_{i+1} = {}^{i+1}R_i\left({}^i\dot{v}_i + {}^i\dot{\omega}_i \times {}^iP_{i+1} + {}^i\omega_i \times {}^i\omega_i \times {}^iP_{i+1}\right) + 2\, {}^{i+1}\omega_{i+1} \times \begin{bmatrix} 0 \\ 0 \\ \dot{d}_{i+1} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \ddot{d}_{i+1} \end{bmatrix}$$

$${}^{i+1}\dot{v}_{C_{i+1}} = {}^{i+1}\dot{v}_{i+1} + {}^{i+1}\dot{\omega}_{i+1} \times {}^{i+1}P_{C_{i+1}} + {}^{i+1}\omega_{i+1} \times {}^{i+1}\omega_{i+1} \times {}^{i+1}P_{C_{i+1}}$$

$${}^{i+1}F_{i+1} = m_{i+1}\, {}^{i+1}\dot{v}_{C_{i+1}}$$

$${}^{i+1}N_{i+1} = {}^{C_{i+1}}I\, {}^{i+1}\dot{\omega}_{i+1} + {}^{i+1}\omega_{i+1} \times {}^{C_{i+1}}I\, {}^{i+1}\omega_{i+1}$$

end

$i = 6:1$

$${}^if_i = {}^iR_{i+1}\, {}^{i+1}f_{i+1} + {}^iF_i$$

$${}^in_i = {}^iR_{i+1}\, {}^{i+1}n_{i+1} + {}^iN_i + {}^iP_{i+1} \times {}^iR_{i+1}\, {}^{i+1}f_{i+1} + {}^iP_{C_i} \times {}^iF_i$$

$\tau_i = [0\ 0\ 1]\, {}^in_i$ or $\tau_i = [0\ 0\ 1]\, {}^if_i$

end

Page 188

Summary:

- Dynamics of a rigid Body

$$f = I a + v \times^* I v$$

Valid w/ any choice of frame

- Recursive Newton Euler
  - chains velocities, accelerations, & forces
  - Computes $\tau(q, \dot{q}, \ddot{q})$ with a number of math ops. that scales linearly w/ N