Lecture 24 – Operational-Space Control

High-level:

Given some low-dimensional template, how do we get our robot to follow it?

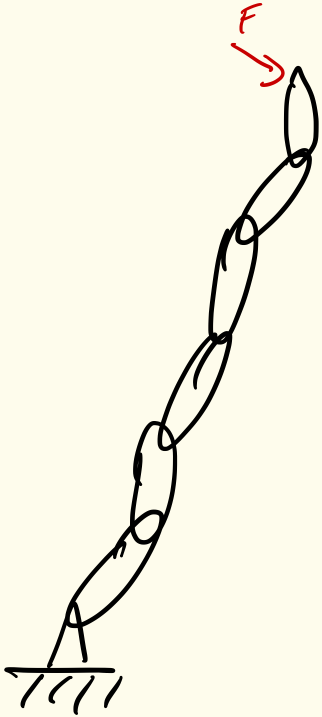
Today:

- Operational-Space Control of Manipulators
- Extensions to Legged Robots
- Maybe: Formulation as a QP

Motivation:

Beyond template tracking we often can characterize desired behaviors in task-specic operational-spaces

- e.g. tool position for a welding robot
- CoM position for a walking robot
- Gaze direction for a humanoid



Option 1:

Task-space goal trajectory

↓ (Inverse Kinematics, Resolved acceleration)

Joint space goal trajectory

↓ Joint space control

Joint Torques

Option 2:

Task space goal trajectory

↓ Task-space control

Task Forces

↓ $J^T F$

Joint Torques

Task-Space Dynamics: v denotes task velocities

$$(*)\quad H\ddot{q} + C\dot{q} + \tau_G = \tau + J^T F \qquad v = J\dot{q} \qquad \dot{v} = J\ddot{q} + \dot{J}\dot{q} \quad (**)$$

"Project" Dynamics to task space

$$JH^{-1}(*) \Rightarrow J\ddot{q} + JH^{-1}(C\dot{q} + \tau_g) = JH^{-1}\tau + JH^{-1}J^T F$$

$$(**) \Rightarrow \dot{v} - \dot{J}\dot{q} + JH^{-1}(C\dot{q} + \tau_G) = JH^{-1}\tau + \underbrace{JH^{-1}J^T}_{\Lambda^{-1}} F$$

Λ operational-space inertia

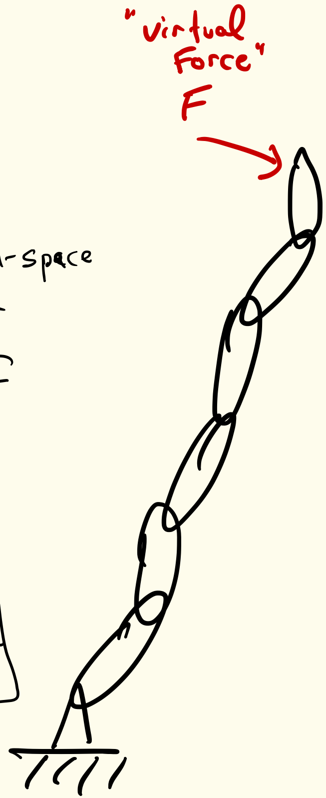
Mult by Λ $\Rightarrow$

$$\Lambda\dot{v} + (\Lambda J H^{-1} C - \Lambda\dot{J})\dot{q} + \Lambda J H^{-1}\tau_G = \Lambda J H^{-1}\tau + f$$

$$\begin{tikzcd}
\ddot{q} \arrow[r, bend left, "H"] \arrow[d, bend right, "J"'] & \tau \arrow[l, bend left, "H^{-1}"] \\
\dot{v} \arrow[r, bend left, "\Lambda"] & F \arrow[l, bend left, "\Lambda^{-1}"] \arrow[u, bend right, "J^T"']
\end{tikzcd}$$

Task-Space Dynamics

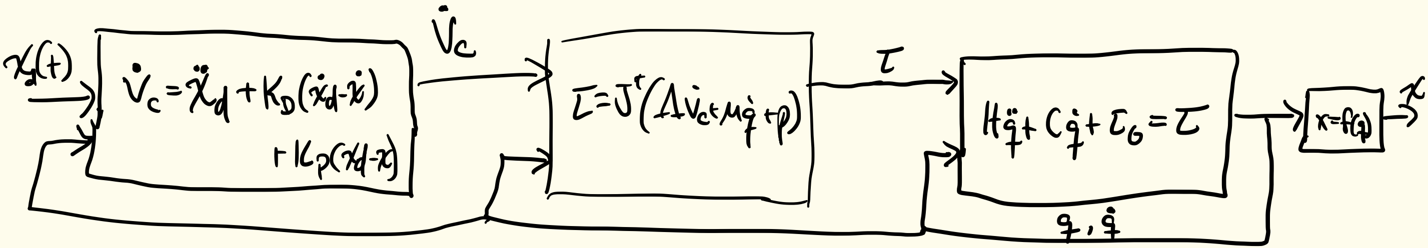
$$\boxed{\Lambda(q)\dot{v} + \mu(q,\dot{q})\dot{q} + p(q) = F + \Lambda J H^{-1}\tau}$$



Task Space Control $\tau = J^T(\Lambda \dot{v}_c + \mu \dot{q} + \rho)$

- Consider the case where $x = f(q) \in \mathbb{R}^3$ is the end-effector position
  $v = \dot{x}$

- Mapping from $\dot{v}_c \to x$ is the same as $\dot{v}_c \to \boxed{\int} \to \boxed{\int} \to x$
  we partial feedback linearized the system
  feedback linearized the task space dynamics



- Suppose you have a desired trajectory $x_d(t)$. If $K_p \succ 0$, and $K_D \succ 0$ then the $e(t) = x_d(t) - x(t) \to 0$ as $t \to \infty$

Task Space Control

$$\Lambda(q)\dot{v} + \mu(q,\dot{q})\dot{q} + p_G(q) = F + \Lambda J H^{-1}\tau$$

Let: $F = \Lambda \dot{v}_c + \mu\dot{q} + p_G$, $\tau = 0$ $\qquad \dot{v}_c \in \mathbb{R}^3$

Then: $\dot{v} = \dot{v}_c$. The dynamics of the robot are the same as if the end effector were constrained to accelerate at $\dot{v}_c$. (can be shown via gauss principle)

Note: $\Lambda J H^{-1}$ is a left inverse of J^T

$$(\Lambda J H^{-1}) J^T = (\Lambda)\overbrace{(J H^{-1} J^T)}^{\Lambda^{-1}} = I$$

Control: Set $\tau = J^T(\Lambda \dot{v}_c + \mu\dot{q} + p)$, $F = 0$ Results in the same dynamics!

Redundancy: Suppose $q \in \mathbb{R}^n$, $x \in \mathbb{R}^m$ $m < n$

- More degrees of freedom than task variables.

- Let $\bar{J} = H^{-1} J^T \Lambda \in \mathbb{R}^{n \times m}$ $\bar{J}^T \in \mathbb{R}^{m \times n} \in$

  $\bar{J}^T$ has a null space

$J\bar{J} = I$

$\bar{J}^T J^T = I$

From Before:

$$\Lambda \dot{v} + \mu \dot{q} + p = \bar{J}^T \tau + F$$

- So. If $\tau \in \mathcal{N}(\bar{J}^T)$ then it does not affect the task

- Prop: $(I - J^T \bar{J}^T)\tau_0 \in \mathcal{N}(\bar{J}^T)$ $\forall \tau_0 \in \mathbb{R}^n$

- Proof: $\bar{J}^T \underbrace{(I - J^T \bar{J}^T)}_{N^T} \tau_0 = (\bar{J}^T - \cancelto{I}{\bar{J}^T J^T} \bar{J}^T)\tau_0 = 0$ $JN = 0$

Implication: $\tau = J^T F + N^T \tau_0$ for any τ_0 provides same task acceleration.

what about in contact?

J_c = Contact Jacobian $\qquad$ J_t = Task Jacobian

$$H\ddot{q} + C\dot{q} + \tau_G = S^T\tau_j + J_c^T f_c \qquad J_c\dot{q} = 0$$

"Projected" Equations:

$$H\ddot{q} + N_c^T C\dot{q} + J_c^T \Lambda \dot{J}_c \dot{q} + N_c^T \tau_g = N_c^T S^T \tau_j \quad (*)$$

Same as before

$(*)\ J_t H^{-1}(\cdot) =$

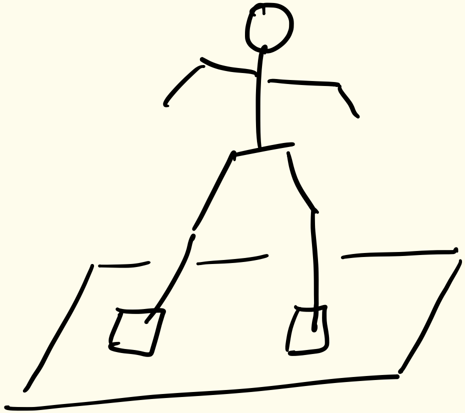
Shorthand: $J_{t|c} = J_t N_c$

$$\Lambda_{t|c} = \left(J_{t|c} H^{-1} J_{t|c}^T\right)^{-1}$$

$$\bar{J}_{t|c} = H^{-1} J_{t|c}^T \Lambda_{t|c}$$

$$\mu_{t|c} = \bar{J}_{t|c}^T C + \Lambda_{t|c} J_t H^{-1} J_c^T \Lambda_c \dot{J}_c - \Lambda_{t|c}\dot{J}_t$$

$$p_{t|c} = \bar{J}_{t|c}^T \tau_G$$



Constraint-consistent task space dynamics

$$\Lambda_{t|c}\dot{v}_t + \mu_{t|c}\dot{q} + p_{t|c} = \bar{J}_{t|c}^T S^T \tau$$

Same form as before!

Derivation of Previous Page

Property: $N_c^2 = N_c$ (projecting twice is the same as projecting once.)

Property: $N_c H^{-1} = H^{-1} N_c = N_c H^{-1} N_c^T$

$$N_c H^{-1} = (I - \bar{J}_c J_c) H^{-1} = (I - H^{-1} J_c^T \Lambda_c J_c) H^{-1} = H^{-1}(I - J_c^T \underbrace{\Lambda_c J_c H^{-1}}_{\bar{J}_c^T}) = H^{-1} \underbrace{(I - J_c^T \bar{J}_c^T)}_{N_c^T}$$

Multiply (*) by $J_t H^{-1}$:

$$H\ddot{q} + N_c^T C\dot{q} + J_c^T \Lambda_c \dot{J}_c \dot{q} + N_c^T \tau_g = N_c^T S^T \tau_j \quad (*)$$

$$\Rightarrow \underbrace{J_t \ddot{q}}_{\dot{v}_t - \dot{J}_t \dot{q}} + \underbrace{J_t N_c}_{J_{t|c}} H^{-1} C\dot{q} + J_t H^{-1} J_c^T \Lambda_c \dot{J}_c \dot{q} + \underbrace{J_t N_c}_{J_{t|c}} \tau_g = \underbrace{J_t N_c}_{J_{t|c}} H^{-1} S^T \tau$$

Multiply both sides by $\Lambda_{t|c}$

$$\Lambda_{t|c} \dot{v}_t + \bar{J}_{t|c}^T C\dot{q} + \Lambda_{t|c} J_t H^{-1} J_c^T \Lambda_c \dot{J}_c \dot{q} - \Lambda_{t|c} \dot{J}_t \dot{q} + \bar{J}_{t|c}^T \tau_G =$$