Lecture 21 - Dynamics of a rigid body

Announcements:

- See Sakai Media Gallery for a 10 minute wrap up on numeric IK

$$\Delta\Theta = {}^{0}J_t^{-1} \begin{bmatrix} {}^{0}\hat{k}\phi \\ \hline {}^{0}p_t^d - {}^{0}p_t \end{bmatrix} \qquad {}^{0}R_t^d = e^{S({}^{0}\hat{k})\phi}\, {}^{0}R_t$$

- No class Friday: Hand in HW in Fitz 365 by 4PM
  - ⇒ Video lecture to be posted, watch over break

Today:

- Dynamics Intro
- Dynamics of a rigid body
  - Describing the mass Distribution

Kinematics: Study of motion w/o regard for the forces that cause it

Dynamics: Study of the relationship between motion and force

We often study this relationship in one of two ways

1. $\tau \rightarrow \ddot{\theta}$ Forward Dynamics (Simulation)

   Given $\theta, \dot{\theta}, \tau$ what is $\ddot{\theta}$?

2. $\ddot{\theta} \rightarrow \tau$ Inverse Dynamics (control)

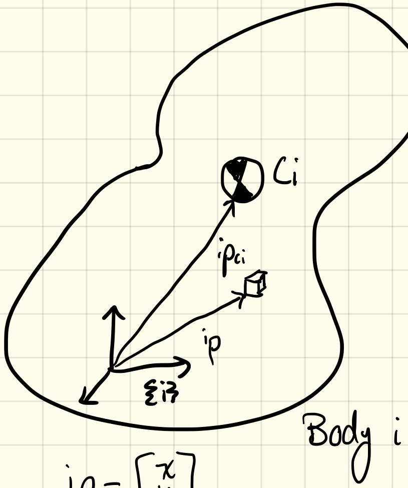
   Given $\theta, \dot{\theta}, \ddot{\theta}$ what τ is required for motion?

What is the link between forces and motion? Mass!

$$F = ma$$

Mass Distribution of a rigid Body

$\rho({}^ip)$: Density at point ${}^ip$



$${}^ip = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Mass

$$m_i = \iiint_{V_i} \rho({}^ip)\,dV$$

V_i ← Volume of body i

Center of mass

$${}^ip_{c_i} = \frac{1}{m_i}\iiint_{V_i} {}^ip\,\rho({}^ip)\,dV$$

Newton's law

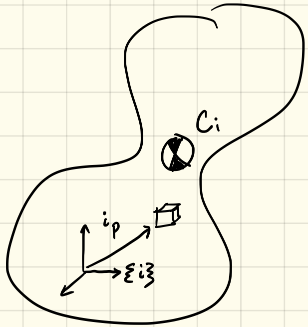
Linear momentum

$$\frac{d}{dt}\left[m_i\, v_{c_i} \right] = F_i$$

← Net Force on Body i

The Inertia Tensor

$$^{i}\underline{I} = \iiint_{V_i} S(^{i}p)\, S(^{i}p)^T \rho(^{i}p)\, dV$$



$^{i}p = [x, y, z]$

- If you have a pure rotation for $\{i\}$ $\Rightarrow$ angular momentum of the body is given by $^{i}\underline{I}\, ^{i}\omega$:

$$^{i}\underline{I} = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix}$$

Mass products of inertia

Mass moments of inertia

$$I_{xx} = \iiint (y^2 + z^2)\, \rho(^{i}p)\, dV$$

(always positive)

$$I_{xy} = \iiint (xy)\, \rho(^{i}p)\, dV$$

(could be positive, negative, or zero)

Poll Everywhere: Rectangular Prism Uniform Density

C_i is located @ origin of $\{i\}$

$w > h > d$

Which of the Following Are True:

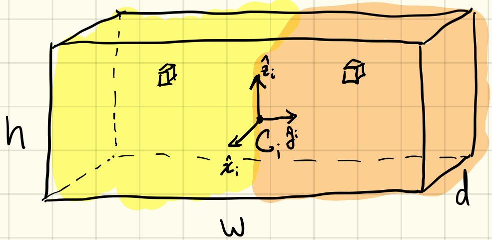
Ⓐ $I_{xy} = I_{yz} = I_{xz} = 0$ ✓

Ⓑ $I_{xx} > I_{zz} > I_{yy}$ ✓

Ⓒ $I_{xx} < I_{zz} < I_{yy}$

Ⓓ A & B

Ⓔ A & C



If a body symmetric x-z plane

$$I_{xy} = I_{zy} = 0$$

If a body " x-y plane

$$I_{xz} = I_{yz} = 0$$

Mass Distribution : Inertia Tensor about CoM

$${}^{c_i}I = \iiint_V S({}^ip - {}^ip_{c_i}) S({}^ip - {}^ip_{c_i})^T \rho({}^ip) dV$$

Generalization of the parallel AxB Theorem

$${}^iI = {}^{c_i}I + m_i S({}^ip_{c_i}) S({}^ip_{c_i})^T$$

$$= {}^{c_i}I + m_i \left[\|{}^ip_{c_i}\|^2 I - {}^ip_{c_i} \, {}^ip_{c_i}^T \right]$$

$$\begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix}$$

$${}^iI_{xx} = {}^{c_i}I_{xx} + m_i(y_c^2 + z_c^2) \qquad {}^iI_{xy} = {}^{c_i}I_{xy} + m_i x_c y_c$$

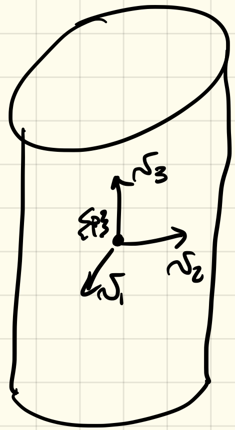
Properties of Inertia Tensors

- Inertia tensor is symmetric. All positive eigenvalues.

- **Eigenvectors**: v_1, v_2, v_3 pricipal axes

  Eigenvalues: Pricipal moments of inertia $\lambda_1, \lambda_2, \lambda_3$

$$
{}^{P}\underline{I} = \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{bmatrix}
$$



Principal moments of inertia:

$$
\lambda_1 + \lambda_2 \geq \lambda_3
$$

"=" when distribution degenerate.