# Lecture 6 - Modeling Contacts

- Last time:
  - $H\ddot{q} + C\dot{q} + \tau_g = \tau$ for manipulators
  - No interaction models
- Today:
  - Modeling systems in contact
  - Two Methods
    - Minimal Coordinates
    - Excess Coordinates
  - Simulation Algorithms

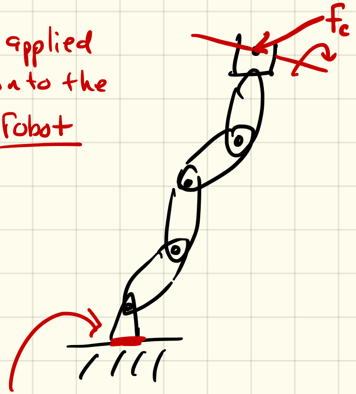
# Dynamics: Industrial manipulator w/ n joints

Configuration: $q \in \mathbb{R}^n$

Equations of motion:

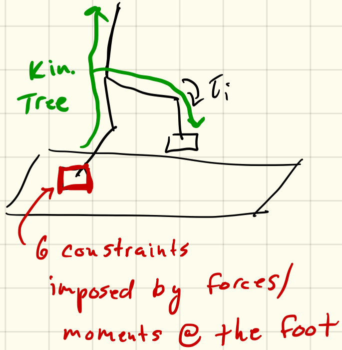
$$\underbrace{H(q)}_{\substack{n\times n \\ \text{Mass Matrix}}}\ddot{q} + \underbrace{C(q,\dot{q})\dot{q}}_{\substack{\text{Coriolis} \\ \text{\& Centripetal} \\ \text{Forces}}} + \underbrace{\tau_g(q)}_{\substack{\text{Graviational} \\ \text{Effects}}} = \underbrace{\tau}_{\text{Joint torques}} + J_c^T(q)\, f_c$$

$f_c$ applied onto the robot




interface between manipulator & world imposes 6 hard constraints.

# Method 1: Minimal coordinates




## Choose minimal coordinates:

For humanoid in single support the joint angles $q_J \in \mathbb{R}^n$ form minimal coordinates

## Dynamics:

$$H(q_J)\ddot{q}_j + C(q_J, \dot{q}_J)\dot{q}_J + \tau_G = \tau_j$$

## Disadvantage:

- If humanoid puts its foot on ground it loses 6 DoF and we would need new generalized coordinates $q' \in \mathbb{R}^{n-6}$
- This approach masks the fact that the foot is not rigidy pinned to the ground.

**Note From Last time:** Let $q' \in \mathbb{R}^m$ minimal coordinates

$$\underbrace{\tau}_{\substack{\text{generalized}\\ \text{force}}} = \underbrace{\left[\sum_{i=1}^{N} J_i^T I_i J_i\right]}_{\substack{\text{Mass Matrix}\\ H(q') \in \mathbb{R}^{m\times m}}} \ddot{q}' + \underbrace{\left[\sum_{i=1}^{N} J_i^T \left[ I_i \dot{J}_i + I_i v_i \times J_i + v_i \times^* I_i J_i \right]\right]}_{\substack{C(q',\dot{q}') \in \mathbb{R}^{m\times m} \text{ (linear in } \dot{q}')\\ \text{Coriolis Matrix}}} \dot{q}' + \underbrace{\left[\sum_{i=1}^{N} J_i^T I_i {}^iX_0 \left[-{}^0a_g\right]\right]}_{\substack{\tau_g \in \mathbb{R}^m\\ \text{Generalized gravitational}\\ \text{Force}}}$$

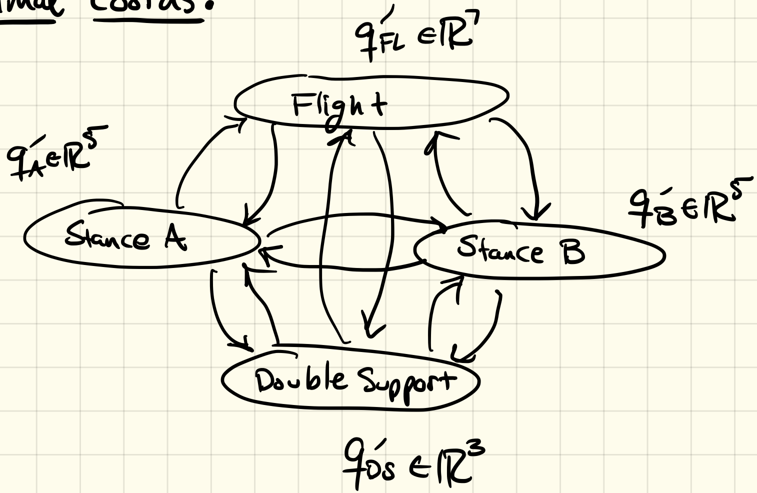
- Equations from last time still apply for systems with kinematic loops (e.g., if you have a four bar linkage in the structure)

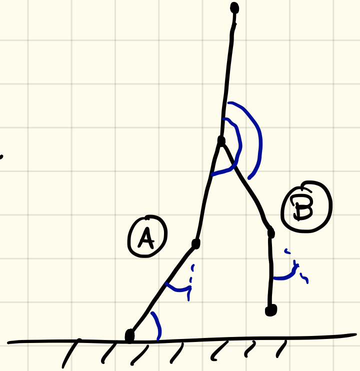
- All $J_i$'s must be redefined s.t. $v_i = J_i \dot{q}'$

- Consequence of Jourdain's Principle of Virtual Power (a.k.a., Kane's eqns.)

# Modeling w/ Minimal Coords:

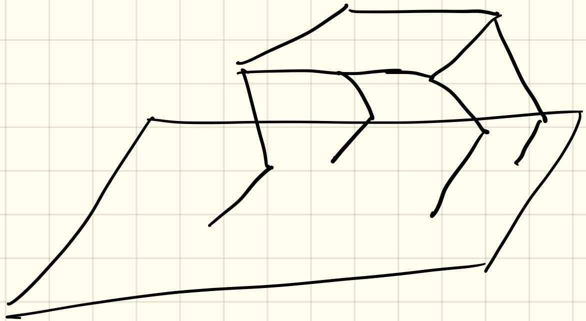
- **Bided:**
  - 4 modes
    $\{ST, FL\}^2$
  - 12 transitions
  - Complex but Doable

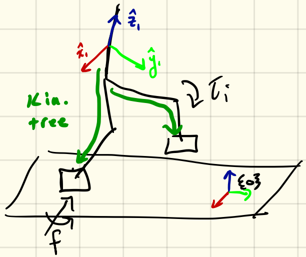



## Quadruped:

- 16 modes
- 16 x 15 transitions max
- Unmanageable quickly






# Method 2: Excess coordinates.

- Attach a frame $\{1\}$ to the body and let $q_B = ({}^0R_1, {}^0p_1)$ denote the configuration of $\{1\}$ relative to $\{0\}$

- Consider the configuration variable

$$q = [q_B, q_J] \qquad \text{w/ spatial velocity of Body} \quad {}^1V_1$$

- With a slight liberty of notation

$$\dot{q} = [{}^1V_1^T, \dot{q}_J^T]^T$$

## Dynamics:

$$\begin{bmatrix} 0 \\ I_{n\times n} \end{bmatrix} \tau_j = S^T \tau_j$$

$$H(q)\ddot{q} + C(q,\dot{q})\dot{q} + \tau_g(q)$$

$$= \begin{bmatrix} H_{bb} & H_{bj} \\ H_{jb} & H_{jj} \end{bmatrix} \begin{bmatrix} {}^1a_1 \\ \ddot{q}_j \end{bmatrix} + \begin{bmatrix} C_b \\ C_j \end{bmatrix} \dot{q} + \begin{bmatrix} \tau_{GB} \\ \tau_{Gj} \end{bmatrix} = \begin{bmatrix} 0 \\ \tau_j \end{bmatrix} + J_c^T f_c$$

No direct actuation on torso.

$$J_c \dot{q} = 0 \Rightarrow J_c \ddot{q} + \dot{J}_c \dot{q} = 0$$

Method 2 cont: Given $\tau$ Find $\ddot{q}$ by solving

$$\begin{bmatrix} H & -J_c^T \\ J_c & 0 \end{bmatrix} \begin{bmatrix} \ddot{q} \\ f_c \end{bmatrix} = \begin{bmatrix} S^T \tau_j - C\dot{q} - \tau_g \\ -\dot{J}_c \dot{q} \end{bmatrix}$$

This is now correct. It was wrong in Lecture. (The location of the transpose was swapped.)

- Contact force

$$f_c = \underbrace{(J_c H^{-1} J_c^T)^{-1}}_{\Lambda_c := \text{ contact inertia}} \left[ -\dot{J}_c \dot{q} - J_c H^{-1} (S^T \tau_j - C\dot{q} - \tau_g) \right]$$

- Plug back into eqns: "Projected Eqns of Motion"

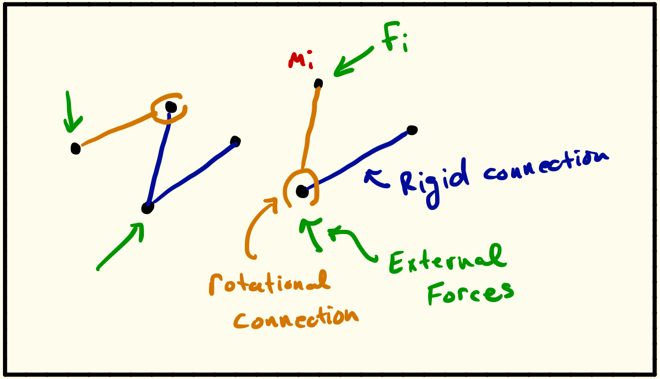
$$H\ddot{q} + N_c^T C\dot{q} + J_c^T \Lambda_c \dot{J}_c \dot{q} + N_c^T \tau_g = N_c^T S^T \tau_j$$

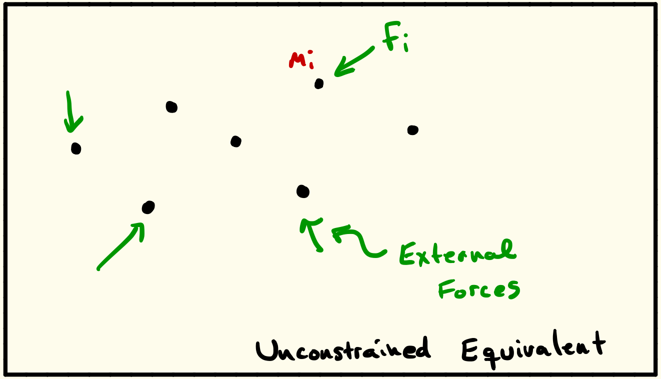
$N_c^T$ is a "Null space Projector"

$$N_c^T = \underbrace{I}_{(n+6)\times(n+6)} - J_c^T \Lambda_c J_c H^{-1} \qquad J_c H^{-1} N_c^T = 0$$

## Alternate View: Gauss principle of least constraint (Force)

Consider a collection of particles w/ some constraints




Let $\ddot{x}_i^u = \frac{f_i}{m_i}$ the unconstrained accelerations

The actual accelerations ($\ddot{x}_i$) deviate from the unconstrained ones in a least-squares sense:

$$\min_{\{\ddot{x}_i\}} \sum_i m_i \|\ddot{x}_i - \ddot{x}_i^u\|_2^2 = \sum_i m_i \left\|\frac{F_i^c}{m_i}\right\|_2^2 = \sum_i \frac{1}{m_i}\|f_i^c\|^2$$

s.t. $\{\ddot{x}_i\}$ satisfy constraints

Method 2 cont: Given $\tau$ Find $\ddot{q}$ by solving

$$\begin{bmatrix} H & -J_c^T \\ J_c & 0 \end{bmatrix} \begin{bmatrix} \ddot{q} \\ f_c \end{bmatrix} = \begin{bmatrix} S^T\tau - C\dot{q} - \tau_g \\ -\dot{J}_c\dot{q} \end{bmatrix}$$

$\Leftarrow$ From derivative of $J_c\dot{q} = 0$

- You can solve for force needed to maintain the contact as:

$$f_c = \underbrace{(J_c H^{-1} J_c^T)^{-1}}_{\Lambda_c \text{ contact inertia}} \left[ -\dot{J}_c\dot{q} - J_c H^{-1}(S^T\tau - C\dot{q} - \tau_g) \right]$$

- The resulting $\ddot{q}$ are the same as the solution to

$$\min_{\ddot{q}} \frac{1}{2} [\ddot{q} - \ddot{q}_{uc}]^T H(q) [\ddot{q} - \ddot{q}_{uc}]$$

$$\text{s.t. } J_c\ddot{q} + \dot{J}_c\dot{q} = 0$$

$$\ddot{q}_{uc} = H^{-1}[S^T\tau_j - C\dot{q} - \tau_G]$$

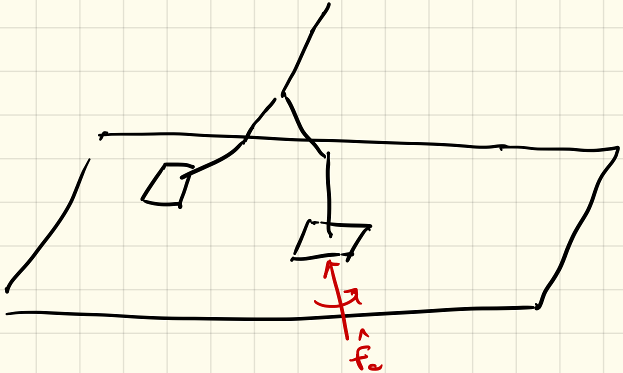
$\boxed{\text{HW3}}$

**Method 2:** Gives dynamics during contact, but what about when making contact?

- $t^-$ : time just before impact
- $t^+$ : time just after impact

$$\int_{t^-}^{t^+}\left[H\ddot{q} + \cancel{C\dot{q}} + \cancel{\tau_g}\right]dt = \int_{t^-}^{t^+}\left[\cancelto{0}{S^T\tau} + J_c^T f_c\right]dt$$

$$H(q)\left[\dot{q}^+ - \dot{q}^-\right] = J_c^T \hat{f}_c$$




Collecting Terms

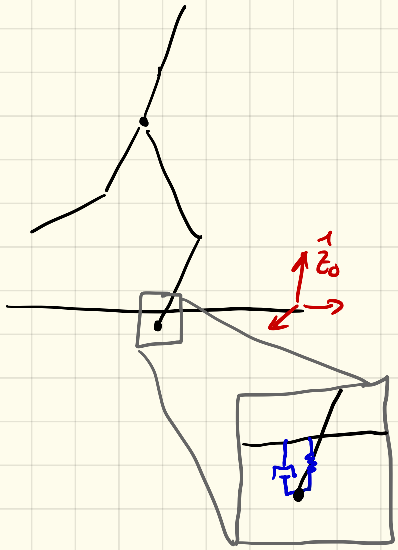
$$\begin{bmatrix} H & -J_c^T \\ J_c & 0 \end{bmatrix}\begin{bmatrix} \dot{q}^+ \\ \hat{f}_c \end{bmatrix} = \begin{bmatrix} H\dot{q}^- \\ 0 \end{bmatrix}$$ ← Assuming inelastic collision $J_c\dot{q}^+ = v_{foot}^+ = 0$

This is now correct. The transpose was in the wrong spot in lecture.

Solve for $\hat{f}_c = -(J_c H^{-1} J_c^T)^{-1} J_c \dot{q}^- = -\Lambda_c v_{foot}^-$ $\qquad \Delta T = \frac{1}{2} v_{foot}^{-T} \Lambda_c v_{foot}^-$

# Simulating w/ Excess Coordinates: Soft Contacts

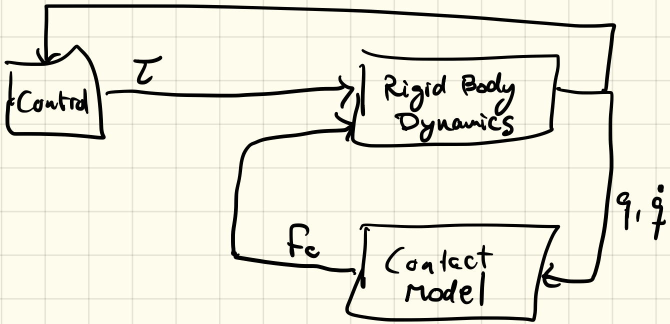
The world is not infinitely rigid
- So model force as a function of penetrating the ground




**Common:** $f_z = -K_{p_z} \Delta p_z - K_{D_z} v_z$

$$f_x = -K_{p_x} \Delta p_x - K_{D_x} v_x$$

**Better:** Adust forces for friction (maybe slide too)




**Disadvantage:**

- Small integration time steps required

Summary:

Method 1: Minimal coordinates

- Requires different coordinates in each contact mode
- Requires treatment of transitions between modes
- Suitable for system with simple contact Scenarios

Method 2: Excess Coordinates

- Allows use of one coordinate convention (simplifies code)
- Soft Contacts
  - Force results from contact deflection
  - Small time steps but easy to compute
- Hard contacts with LCP timestepping
  - Next time