Lecture 5: Dynamics - Structure & Identification

- **Last Time:**
  - Spatial Equation of Motion: $f = \underline{I} a + v \times^* \underline{I} v$
  - Velocity propagation: $v_i = {}^{i}X_{p(i)} v_{p(i)} + \phi_i \dot{q}_i$
  - Acceleration propagation: $a_i = {}^{i}X_{p(i)} a_{p(i)} + \phi_i \ddot{q}_i + v_i \times {}^{i}\phi_i \dot{q}_i$
  - Recursive Newton Euler Algorithm: $\tau = RNEA(q, \dot{q}, \ddot{q})$

Today:

- Breaking down the RNEA: $\tau = H(q)\ddot{q} + C(q, \dot{q})\dot{q} + \tau_g$
- Optimization for system identification.

Recursive Newton Euler

Inputs: $q, \dot{q}, \ddot{q}$ (model) Outputs: τ

$v_0 = 0 \qquad a_0 = -{}^0a_g$

for $i = 1 \ldots N$

$v_i = {}^iX_{p(i)} v_{p(i)} + \phi_i \dot{q}_i$ ← Linear in $\dot{q}$

$a_i = {}^iX_{p(i)} a_{p(i)} + \phi_i \ddot{q}_i + v_i \times \phi_i \dot{q}_i$ ← $\ddot{q}$ appear linearly, $\dot{q}$ terms appear bilinearly

$f_i = I_i a_i + v_i \times^* I_i v_i$ ← Linear in $\ddot{q}$, bilinear in $\dot{q}$

end

for $i = N \ldots 1$

$\tau_i = \phi_i^T f_i$ ← τ linear in $\ddot{q}$, bilinear in $\dot{q}$

$f_{p(i)} = f_{p(i)} + {}^iX_{p(i)}^T f_i$

end

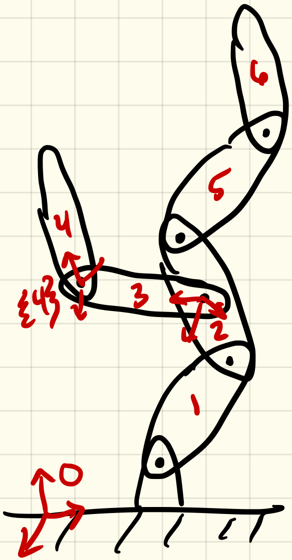
${}^iv_i = J_i \dot{q}$ ← Jacobian

${}^ia_i = J_i \ddot{q} + \dot{J}_i \dot{q} + v_i \times J_i \dot{q} - {}^iX_0 {}^0a_g$

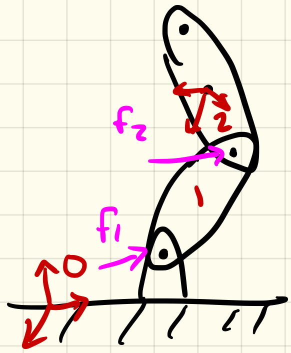
Structure in the equations

$$ {}^5V_5 = {}^5X_1\,{}^1\phi_1\,\dot q_1 + {}^5X_2\,{}^2\phi_2\,\dot q_2 + {}^5\phi_5\,\dot q_5 $$

$$ = \underbrace{\begin{bmatrix} {}^5X_1\,{}^1\phi_1 & {}^5X_2\,{}^2\phi_2 & 0_{6\times1} & 0_{6\times1} & {}^5\phi_5 & 0_{6\times1} \end{bmatrix}}_{\text{Jacobian for Body 5}} \begin{bmatrix} \dot q_1 \\ \vdots \\ \dot q_6 \end{bmatrix} $$



Torque analysis Simplified Case: Two Bodies



$$f_2 = I_2 a_2 + v_2 \times^* I_2 v_2$$

$$f_1 = \left[I_1 a_1 + v_1 \times^* I_1 v_1 \right] + {}^2X_1^T f_2$$

Torque @ Joint 1:

$$\tau_1 = \phi_1^T f_1$$

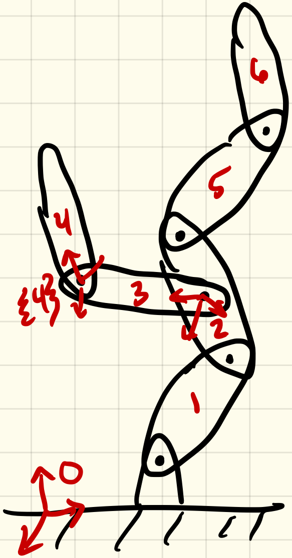
$$= \phi_1^T \left[I_1 a_1 + v_1 \times^* I_1 v_1 + {}^2X_1^T \left(I_2 a_2 + v_2 \times^* I_2 v_2 \right) \right]$$

$$= \underbrace{\phi_1^T \left[I_1 a_1 + v_1 \times^* I_1 v_1 \right]}_{\text{torque @ joint 1 due to motion of Body 1}} + \underbrace{\phi_1^T \, {}^2X_1^T \left(I_2 a_2 + v_2 \times^* I_2 v_2 \right)}_{\text{torque @ Joint 1 due to motion of Body 2}}$$

Structure in the equations

$${}^5V_5 = {}^5X_1\,{}^1\phi_1\,\dot q_1 + {}^5X_2\,{}^2\phi_2\,\dot q_2 + {}^5\phi_5\,\dot q_5$$

$$= \underbrace{\begin{bmatrix} {}^5X_1\,{}^1\phi_1 & {}^5X_2\,{}^2\phi_2 & 0_{6\times1} & 0_{6\times1} & {}^5\phi_5 & 0_{6\times1} \end{bmatrix}}_{\text{Jacobian for Body 5}} \begin{bmatrix} \dot q_1 \\ \vdots \\ \dot q_6 \end{bmatrix}$$



Torque @ Joint 1 required to move body 5:

$${}^1\phi_1^T\,{}^5X_1^T\left(I_5 a_5 + v_5 \times^* I_5 v_5\right) = \tau_1$$

More generally

$$\begin{bmatrix} \tau_1 \\ \vdots \\ \tau_6 \end{bmatrix} = J_5^T \left[I_5 a_5 + v_5 \times^* I_5 v_5 \right]$$

Putting it together

$$\tau = \sum_{i=1}^{N} J_i^T \left[I_i a_i + v_i \times^* I_i v_i \right]$$

$$= \sum_{i=1}^{N} J_i^T I_i \left[J_i \ddot{q} + \dot{J}_i \dot{q} + v_i \times J_i \dot{q} - {}^{i}X_0 \, {}^{0}a_g \right] + J_i^T v_i \times^* I_i J_i \dot{q}$$

$$= \underbrace{\left[\sum_{i=1}^{N} J_i^T I_i J_i \right]}_{\substack{\text{Mass Matrix} \\ H(q) \in \mathbb{R}^{N \times N}}} \ddot{q} + \underbrace{\left[\sum_{i=1}^{N} J_i^T \left[I_i \dot{J}_i + I_i v_i \times J_i + v_i \times^* I_i J_i \right] \right]}_{\substack{C(q,\dot{q}) \in \mathbb{R}^{n \times n} \text{ (linear in } \dot{q}) \\ \text{Coriolis Matrix}}} \dot{q} + \underbrace{\left[\sum_{i=1}^{N} J_i^T I_i \, {}^{i}X_0 \left[-{}^{0}a_g \right] \right]}_{\substack{\tau_g \\ \text{Generalized gravitational} \\ \text{Force}}}$$

$$\boxed{\tau = H(q)\ddot{q} + C(q,\dot{q})\dot{q} + \tau_g}$$

Exercise: T, V Kinetic & Potential energy

$\Rightarrow T = \frac{1}{2} \dot{q}^T H(q) \dot{q}$

$\Rightarrow \nabla_q V = \tau_g$

Properties:

① $H(q) = H(q)^T$ is positive definite if $I_i \succ 0 \quad \forall i$

② For the matrix $C(q, \dot{q})$ on the prev. page $[\dot{H} - 2C]$ is skew symmetric (HW 2)

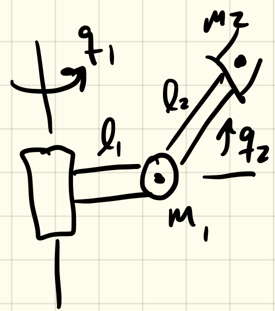
③ Many other definitions of $C(q, \dot{q})$ are valid [i.e., Give the same product $C(q,\dot{q})\dot{q}$]

$$C(q,\dot{q})\dot{q} = \begin{bmatrix} -2\overbrace{m_2 l_2 (l_1 + l_2 \cos(q_2)) \sin(q_2)}^{\beta} \dot{q}_1 \dot{q}_2 \\ m_2 l_2 (l_1 + l_2 \cos(q_2)) \sin(q_2)\, \dot{q}_1^2 \end{bmatrix}$$

$$= \underbrace{\beta \begin{bmatrix} -2\dot{q}_2 & 0 \\ \dot{q}_1 & 0 \end{bmatrix}} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \underbrace{\beta \begin{bmatrix} 0 & -2\dot{q}_1 \\ \dot{q}_1 & 0 \end{bmatrix}} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}$$

Both valid choices of C

2 DoF Example For ③



Properties:

④ Another option

$$[C]_{ij} = \sum_k \underbrace{\frac{1}{2}\left[\frac{\partial H_{ij}}{\partial q_k} + \frac{\partial H_{ik}}{\partial q_j} - \frac{\partial H_{jk}}{\partial q_i}\right]}_{} \dot{q}_k$$

Γ_{ijk}: Christoffel symbols of the first kind

⑤ All valid C satisfy

$$\dot{q}^T\left[\dot{H}(q,\dot{q}) - 2C(q,\dot{q})\right]\dot{q} = 0$$

Why: Equivalent to conservation of energy when $\tau = 0$.

⑥ Only special choices of C satisfy ② $\alpha^T\left[\dot{H} - 2C\right]\alpha = 0 \quad \forall \alpha \in \mathbb{R}^n$

Model Identification: Considerations

- Kinematic Model ($^{i}X_{p(i)}$, $^{i}\phi_i$)

    - Link lengths, structural angles
    - Sometimes accurate via CAD

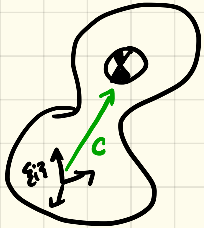
        (Machinists accurate down to 10s of ums)

    - Else kinematic calibration may be required

- Dynamic Models
    - Kinematic Model + Inertia model
    - Often difficult to model exactly in CAD
        - Due to screws, wires, sensors, etc. that may be tedious to model
    - Alternative: Build model from data

Inertial Parameters of A Rigid Body

Notation clarification. $c \in \mathbb{R}^3$, vector to CoM. Not the Coriolis Matrix here.



Inertia Matrix:

$$\mathcal{I}_i = \begin{bmatrix} {}^i\bar{I} & m\,S(c) \\ m\,S(c)^T & m\,\mathbf{1}_{3\times3} \end{bmatrix} \qquad {}^i\bar{I} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{xy} & I_{yy} & I_{yz} \\ I_{xz} & I_{yz} & I_{zz} \end{bmatrix}$$

Parameters:

$$\pi_i = [m,\ mc_x,\ mc_y,\ mc_z,\ I_{xx},\ I_{yy},\ I_{zz},\ I_{yz},\ I_{xz},\ I_{xy}]^T \in \mathbb{R}^{10}$$

Key: $\mathcal{I}_i$ is linear in these parameters

$$\mathcal{I}_i = \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & \mathbf{1}_{3\times3} \end{bmatrix}}_{:=K_1} \underbrace{m}_{\pi_{i,1}} + \underbrace{\begin{bmatrix} 0 & S\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) \\ S\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)^T & 0 \end{bmatrix}}_{:=K_2} \underbrace{mc_x}_{\pi_{i,2}} + \cdots$$

$$\underbrace{\begin{bmatrix} \begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix} & 0 \\ 0 & 0 \end{bmatrix}}_{:=K_5} \underbrace{I_{xx}}_{\pi_{i,5}} + \cdots + \underbrace{\begin{bmatrix} \begin{bmatrix}0&1&0\\1&0&0\\0&0&0\end{bmatrix} & 0 \\ 0 & 0 \end{bmatrix}}_{:=K_{10}} \underbrace{I_{xy}}_{\pi_{i,10}}$$

$\Rightarrow \mathcal{I}_i = \sum_{j=1}^{10} K_j\, \pi_{i,j}$ $\Rightarrow$ Since RNEA is linear in $\mathcal{I}_i$, it is linear in π_i!

The Regressor Matrix:

- Consider Body i, parameter k:
  - Run RNEA $I_i = K_k$ (all other I's are zero)
  - Let the resultant τ be saved as $y_{ik} \in \mathbb{R}^{N_J}$ ← # Joint DoFs
- Collect results:

Rigid Bodies

$$Y = \left[y_{1,1} \cdots y_{1,10} \cdots\cdots y_{N_B,1} \cdots y_{N_B,10} \right]$$

$$\pi = \left[\pi_{1,1} \cdots \pi_{1,10} \cdots\cdots \pi_{N_B,1} \cdots \pi_{N_B,10} \right]^T$$

- $\tau = \sum_{i=1}^{N_B} \sum_{k=1}^{10} y_{ik} \pi_{ik} = Y(q, \dot{q}, \ddot{q})\, \pi$

  - Holds only if you have perfect sensing for $q, \dot{q}, \ddot{q}, \tau$
  - Holds only if no friction or other modelling errors
    (e.g., non-rigid bodies that flex)

Least-Squares Identification: Formulation

$$\min_{\pi} \sum_{s \in \text{Samples}} \| Y(q^{(s)}, \dot{q}^{(s)}, \ddot{q}^{(s)})\pi - \tau^{(s)} \|^2$$

Notes:

1. Can be written in QP form $\Rightarrow$ $\min_{\pi} \frac{1}{2}\pi^T Q \pi + c^T \pi + \text{constant}$

2. Q is PSD: Follows since

$$\mathcal{N}_Y = \{ \pi \in \mathbb{R}^{10 N_B} \mid Y(q, \dot{q}, \ddot{q})\pi = 0 \quad \forall q, \dot{q}, \ddot{q} \}$$ is non empty

3. Implication: You can have a good dynamic model even if parameters are not a good match to ground truth!

Summary:

1. $\tau = H(q)\ddot{q} + C(q,\dot{q})\dot{q} + \tau_g(q)$

2. Reressor Form

$$\tau = Y(q,\dot{q},\ddot{q})\,\pi$$

"Linearity of inverse dynamics in the inertial params"

3. Dynamic model identification = Least squares