Lecture 11 – 9/13/2018 – IK Solution Methods

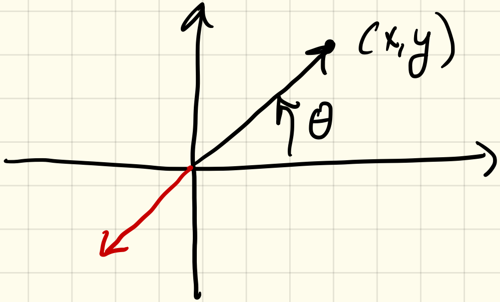
- No one algorithm solves symbolic IK problems

- This lecture provides a few common approaches
  [More available on Sakai under video resources.]

A quick comment on atan2:

$$\theta = \text{atan2}(y, x)$$

$$= \text{atan2}(\alpha y, \alpha x) \qquad \alpha > 0$$



$$\theta \neq \text{atan2}(\alpha y, \alpha x) \qquad \alpha < 0$$

off by 180°

$\underline{l}s_1 = p_y$

$\underline{l}c_1 = p_x$

$$\theta_1 = \text{atan2}(p_y, p_x)$$

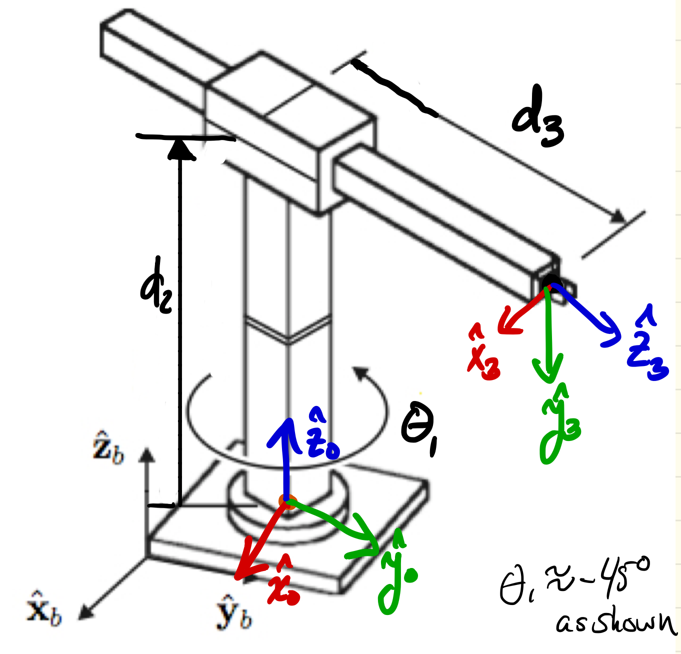
Simple Example: RPP

Find ${}^0T_3$: (Forward Kin)

$$ {}^0T_3 = \begin{bmatrix} c_1 & 0 & -s_1 & -d_3 s_1 \\ s_1 & 0 & c_1 & d_3 c_1 \\ 0 & -1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

Solve for $\underbrace{\theta_1, d_2, d_3}_{\text{unknowns}}$ (Inv. Kin)

$$ \begin{bmatrix} c_1 & 0 & -s_1 & -d_3 s_1 \\ s_1 & 0 & c_1 & d_3 c_1 \\ 0 & -1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\text{known}} $$



$$ \boxed{d_2 = p_z} \checkmark $$

$$ \boxed{\theta_1 = \operatorname{atan2}(r_{21}, r_{11})} \checkmark $$

$$ d_3 = \begin{cases} \dfrac{p_y}{c_1} & \text{if } \theta_1 \neq \pm 90° \\ -p_x/s_1 & \text{o/w} \end{cases} $$

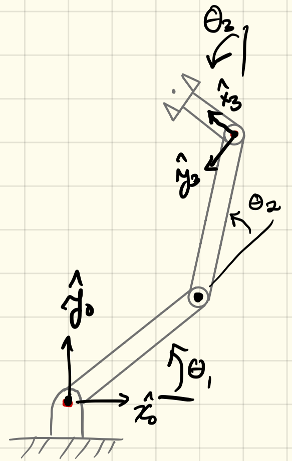
$$ -p_x s_1 + p_y c_1 = s_1^2 d_3 + c_1^2 d_3 = d_3 $$

$$ \boxed{d_3 = -p_x s_1 + p_y c_1} $$

Example 2: Full IK of the 3R Manip

$$\underbrace{\begin{bmatrix} c_{123} & -s_{123} & 0 & l_1c_1 + l_2c_{12} \\ s_{123} & c_{123} & 0 & l_1s_1 + l_2s_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{{}^0T_3} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Solve for $\theta_1, \theta_2, \theta_3$



Strategy: Solve for θ_1 and θ_2 using wrist position
Then Solve for θ_3 using wrist orientation

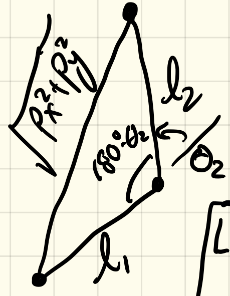
Square & Add:

$$p_x^2 + p_y^2 = (l_1c_1 + l_2c_{12})^2 + (l_1s_1 + l_2s_{12})^2$$

$$\vdots$$

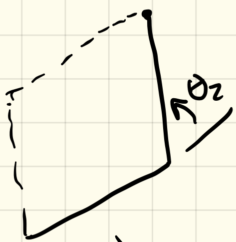
$$= l_1^2 + l_2^2 - 2l_1l_2\cos(180° - \theta_2)$$

$$= l_1^2 + l_2^2 + 2l_1l_2c_2$$



Law of Cosines

$$c_2 = \frac{p_x^2 + p_y^2 - l_1^2 - l_2^2}{2l_1l_2}$$



$$s_2 = \pm\sqrt{1 - c_2^2}$$

$$\theta_2 = \operatorname{atan2}(s_2, c_2)$$

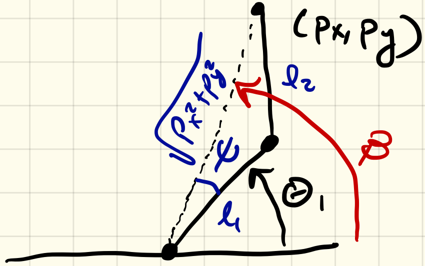
2 solutions for θ_2

Finding θ_1:

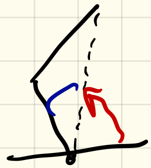
$$\beta = \operatorname{atan2}(p_y, p_x)$$

$$\ell_2^2 = \ell_1^2 + p_x^2 + p_y^2 - 2\ell_1\sqrt{p_x^2+p_y^2}\cos\psi$$

(law of cosines



$$\psi = \arccos\left(\frac{\ell_1^2 + p_x^2 + p_y^2 - \ell_2^2}{2\ell_1\sqrt{p_x^2+p_y^2}}\right)$$



Solution 1

$$\theta_1 = \beta - \psi$$

$$\theta_2 = \operatorname{atan2}\left(\sqrt{1-c_2^2}, c_2\right)$$

Solution 2

$$\theta_1 = \beta + \psi$$

$$\theta_2 = \operatorname{atan2}\left(-\sqrt{1-c_2^2}, c_2\right)$$

Finding θ_3:

$$\begin{bmatrix} c_{123} & -s_{123} & 0 & l_1 c_1 + l_2 c_{12} \\ s_{123} & c_{123} & 0 & l_1 s_1 + l_2 s_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\theta_1 + \theta_2 + \theta_3 = \operatorname{atan2}(r_{21}, r_{11})$$

$$\boxed{\theta_3 = \operatorname{atan2}(r_{21}, r_{11}) - \theta_1 - \theta_2}$$

1 solution for each θ_1 and θ_2

Summary

$$\begin{bmatrix} c_{123} & -s_{123} & 0 & l_1 c_1 + l_2 c_{12} \\ s_{123} & c_{123} & 0 & l_1 s_1 + l_2 s_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Solve for $\theta_1, \theta_2, \theta_3$

Intermediate Quantities:

$$c_2 = \frac{p_x^2 + p_y^2 - l_1^2 - l_2^2}{2 l_1 l_2}$$

$$\beta = \text{atan2}(p_y, p_x)$$

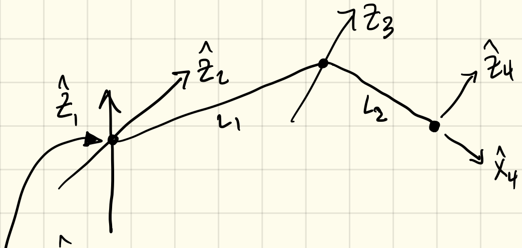
$$\psi = \text{acos}\left(\frac{l_1^2 + p_x^2 + p_y^2 - l_2^2}{2\sqrt{p_x^2 + p_y^2}\, l_1}\right)$$

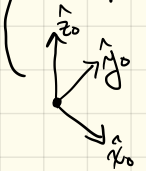
Solution:

$$\boxed{\begin{aligned} \theta_2 &= \text{atan2}\left(\pm\sqrt{1 - c_2^2},\ c_2\right) \\ \theta_1 &= \beta \mp \psi \\ \theta_3 &= \text{atan2}(r_{21}, r_{11}) - \theta_1 - \theta_2 \end{aligned}}$$

2 solutions

Example 3: Simplified Puma: Position IK





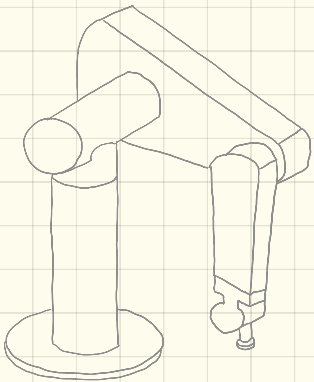
i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	θ_1
2	$-\pi/2$	0	0	θ_2
3	0	L_1	0	θ_3
4	0	L_2	0	0

$$
{}^0T_4 = \begin{bmatrix} c_1 c_{23} & -c_1 s_{23} & -s_1 & c_1(c_{23}L_2 + c_2 L_1) \\ s_1 c_{23} & -s_1 s_{23} & c_1 & s_1(c_{23}L_2 + c_2 L_1) \\ -s_{23} & -c_{23} & 0 & -s_{23}L_2 - s_2 L_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

Position IK:

$$
\begin{bmatrix} c_1(c_{23}L_2 + c_2 L_1) \\ s_1(c_{23}L_2 + c_2 L_1) \\ -s_{23}L_2 - s_2 L_1 \end{bmatrix} = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}
$$

Solve for $\theta_1, \theta_2, \theta_3$



How Many Solutions? $\boxed{4}$

- Elbow up vs. down
- Facing fwd. vs. back