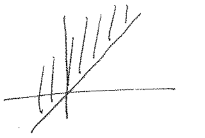
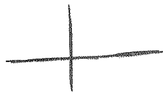
(1a) $X = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; x_1 \leq x_2 \right\}$



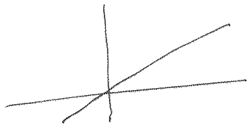
non è un sottospazio di $\mathbb{R}^2$ perché $\begin{pmatrix} 0 \\ 1 \end{pmatrix} \in X$ ma $(-1)\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \notin X$

$Y = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; x_1 x_2 = 0 \right\}$



non è un sottospazio di $\mathbb{R}^2$ perché $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \in Y$, $\begin{pmatrix} 0 \\ 1 \end{pmatrix} \in Y$ ma $\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \notin Y$

$Z = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; x_1 = 3x_2 \right\}$



è un sottospazio di $\mathbb{R}^2$ perché:

1) $\begin{pmatrix} 0 \\ 0 \end{pmatrix} \in Z$ perché $0 = 3 \cdot 0$.

2) se $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \in Z$ allora $a_1 = 3a_2$ e $b_1 = 3b_2$ quindi $a_1 + b_1 = 3a_2 + 3b_2 = 3(a_2 + b_2)$

cioè $\begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \end{pmatrix} \in Z$. Perciò $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \in Z$

3) se $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \in Z$ e $k \in \mathbb{R}$ allora $a_1 = 3a_2$ quindi $ka_1 = k3a_2 = 3(ka_2)$

cioè $\begin{pmatrix} ka_1 \\ ka_2 \end{pmatrix} \in Z$. Perciò $k\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \in Z$

b) $P = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; x_1 = 0, \; x_2 - x_3 = 0 \right\}$ è un sottospazio di $\mathbb{R}^3$ perché

1) $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \in P$ perché $0 = 0$ e $0 - 0 = 0$

2) se $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \in P$ e $\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \in P$, allora $a_1 = 0$ $a_2 - a_3 = 0$, $b_1 = 0$, $b_2 - b_3 = 0$

$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \end{pmatrix}$ allora $a_1 + b_1 = 0 + 0 = 0$ $(a_2 + b_2) - (a_3 + b_3) = a_2 - a_3 + b_2 - b_3 = 0 + 0 = 0$

quindi $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \in P$.

3) se $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \in P$ e $k \in \mathbb{R}$ allora $a_1 = 0$ $a_2 - a_3 = 0$.

$k\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} ka_1 \\ ka_2 \\ ka_3 \end{pmatrix}$, allora $ka_1 = k0 = 0$ $ka_2 - ka_3 = k(a_2 - a_3) = k.0 = 0$

quindi $k\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \in P$.

$Q = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; x_1 = x_2 \right\}$ è un sottospazio di $\mathbb{R}^3$ perché.

1) $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \in Q$ perché $0 = 0$

2) se $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ e $\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \in Q$ allora $a_1 = a_2$ e $b_1 = b_2$

$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \end{pmatrix}$ e $a_1 + b_1 = a_2 + b_2$ quindi $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \in Q$

3) se $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \in Q$ e $k \in \mathbb{R}$ allora $a_1 = a_2$

$k\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} ka_1 \\ ka_2 \\ ka_3 \end{pmatrix}$ e $ka_1 = ka_2$ quindi $k\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \in Q$.

$R = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; x_1 + x_2 + x_3 = 3 \right\}$ non è un sottospazio di $\mathbb{R}^3$ perché $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \notin R$.

$S = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; x_1^2 = x_2 \right\}$ non è un sottospazio di $\mathbb{R}^3$ perché $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \in R$ ma $2\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} \notin R$ perché $4 \neq 2$.

(2) X, Y sottospazi di $\mathbb{R}^3$

$X \cap Y$ è un sottospazio di $\mathbb{R}^3$ perchè

1) $\vec{0} \in X$ e $\vec{0} \in Y$ quindi $\vec{0} \in X \cap Y$

2) se $\vec{a}, \vec{b} \in X \cap Y$ allora $\vec{a}, \vec{b} \in X$ quindi $\vec{a}+\vec{b} \in X$, e $\vec{a}, \vec{b} \in Y$ quindi $\vec{a}+\vec{b} \in Y$.
Perciò $\vec{a}+\vec{b} \in X \cap Y$

3) se $\vec{a} \in X \cap Y$ e $k \in \mathbb{R}$ allora $\vec{a} \in X$ quindi $k\vec{a} \in X$, $\vec{a} \in Y$ quindi $k\vec{a} \in Y$ Perciò $k\vec{a} \in X \cap Y$.

$X \cup Y$ prima si osserva:

1) $\vec{0} \in X$ e $\vec{0} \in Y$ quindi $\vec{0} \in X \cup Y$

3) se $\vec{a} \in X \cup Y$ e $k \in \mathbb{R}$
se $\vec{a} \in X$ allora $k\vec{a} \in X$ quindi $k\vec{a} \in X \cup Y$
se $\vec{a} \in Y$ allora $k\vec{a} \in Y$ quindi $k\vec{a} \in X \cup Y$. Perciò $k\vec{a} \in X \cup Y$.

2) se $\vec{a}, \vec{b} \in X \cup Y$ allora $\vec{a} \in X$ o $\vec{a} \in Y$ e $\vec{b} \in X$ o $\vec{b} \in Y$
se $\vec{a}, \vec{b} \in X$ allora $\vec{a}+\vec{b} \in X$ quindi $\vec{a}+\vec{b} \in X \cup Y$.
se $\vec{a}, \vec{b} \in Y$ allora $\vec{a}+\vec{b} \in Y$ quindi $\vec{a}+\vec{b} \in X \cup Y$.
se $\vec{a} \in X \setminus Y$ e $\vec{b} \in Y \setminus X$ allora $\vec{a}+\vec{b} \notin X \cup Y$ perchè:
$\vec{a}+\vec{b} \notin X$ perchè se $\vec{a}+\vec{b} \in X$ allora $(\vec{a}+\vec{b})-\vec{a} \in X$ cioè $\vec{b} \in X$. contraddizione.
$\vec{a}+\vec{b} \notin Y$ perchè se $\vec{a}+\vec{b} \in Y$ allora $(\vec{a}+\vec{b})-\vec{b} \in Y$ cioè $\vec{a} \in Y$. contraddizione.

Quindi $X \cup Y$ è un sottospazio di $\mathbb{R}^3$ se e solo se $X \subseteq Y$ o $Y \subseteq X$.

$X \setminus Y$ non è un sottospazio di $\mathbb{R}^3$ perchè $\vec{0} \notin X \setminus Y$.

(3) a) $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ $\operatorname{Im}(T) = \left\langle \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \end{pmatrix} \right\rangle$, si osserva che A è invertibile e $\operatorname{Im}(T) = \mathbb{R}^2$

Infatti $A^{-1} = -\frac{1}{2}\begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 3/2 & -1/2 \end{pmatrix}$ quindi $\vec{e}_1, \vec{e}_2 \in \operatorname{Im}(T)$

Infatti $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \to \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 3/2 & -1/2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 3/2 \end{pmatrix}$ quindi $\begin{pmatrix} 1 \\ 0 \end{pmatrix} = -2\begin{pmatrix} 1 \\ 3 \end{pmatrix} + \frac{3}{2}\begin{pmatrix} 2 \\ 4 \end{pmatrix}$

$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \to \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 3/2 & -1/2 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1/2 \end{pmatrix}$ quindi $\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 2 \\ 4 \end{pmatrix}$.

segue che $\vec{e}_1, \vec{e}_2 \in \left\langle \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \end{pmatrix} \right\rangle$ quindi $\langle \vec{e}_1, \vec{e}_2 \rangle \subseteq \left\langle \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \end{pmatrix} \right\rangle$

ma $\langle \vec{e}_1, \vec{e}_2 \rangle = \mathbb{R}^2$ e $\left\langle \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \end{pmatrix} \right\rangle \subseteq \mathbb{R}^2$ quindi $\left\langle \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \end{pmatrix} \right\rangle = \mathbb{R}^2$.

b) $A = \begin{pmatrix} 1 & 4 \\ 3 & 12 \end{pmatrix}$ $\operatorname{Im}(T) = \left\langle \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 12 \end{pmatrix} \right\rangle$ $\begin{pmatrix} 4 \\ 12 \end{pmatrix} = 4\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ quindi $\left\langle \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 12 \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} 1 \\ 3 \end{pmatrix} \right\rangle$

$\operatorname{Im}(T) = \left\{ t\begin{pmatrix} 1 \\ 3 \end{pmatrix} \mid t \in \mathbb{R} \right\}$ è la retta $\begin{cases} x = t \\ y = 3t \end{cases}$ $t \in \mathbb{R}$.

c) $A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ -2 & -4 & -6 & -8 \end{pmatrix}$ $\operatorname{Im}(T) = \left\langle \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 2 \\ -4 \end{pmatrix}, \begin{pmatrix} 3 \\ -6 \end{pmatrix}, \begin{pmatrix} 4 \\ -8 \end{pmatrix} \right\rangle$

$\operatorname{Im}(T) = \left\{ \begin{pmatrix} 1 & 2 & 3 & 4 \\ -2 & -4 & -6 & -8 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \;\middle|\; \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \mathbb{R}^4 \right\} = \left\{ x_1\begin{pmatrix} 1 \\ -2 \end{pmatrix} + x_2\begin{pmatrix} 2 \\ -4 \end{pmatrix} + x_3\begin{pmatrix} 3 \\ -6 \end{pmatrix} + x_4\begin{pmatrix} 4 \\ -8 \end{pmatrix} \;\middle|\; x_1, x_2, x_3, x_4 \in \mathbb{R} \right\}$

$= \left\{ (x_1 + 2x_2 + 3x_3 + 4x_4)\begin{pmatrix} 1 \\ -2 \end{pmatrix} \;\middle|\; x_1, x_2, x_3, x_4 \in \mathbb{R} \right\} = \left\{ t\begin{pmatrix} 1 \\ -2 \end{pmatrix} \;\middle|\; t \in \mathbb{R} \right\}$.

è la retta $\begin{cases} x = t \\ y = -2t \end{cases}$ $t \in \mathbb{R}$.

d) $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ $\operatorname{Im}(T) = \left\langle \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\rangle$ è la retta $\begin{cases} x = t \\ y = t \\ z = t \end{cases}$ $t \in \mathbb{R}$.

e) $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $\operatorname{Im}(T) = \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle = \mathbb{R}^3$

f) $A=\begin{pmatrix}2&1&3\\3&4&2\\6&5&7\end{pmatrix}$ $\quad \mathrm{Im}(T)=\left\langle \begin{pmatrix}2\\3\\6\end{pmatrix},\begin{pmatrix}1\\4\\5\end{pmatrix},\begin{pmatrix}3\\2\\7\end{pmatrix}\right\rangle$ $\quad \begin{pmatrix}1\\4\\5\end{pmatrix}$ non è un multiplo di $\begin{pmatrix}2\\3\\6\end{pmatrix}$

quindi $\left\langle \begin{pmatrix}2\\3\\6\end{pmatrix},\begin{pmatrix}1\\4\\5\end{pmatrix}\right\rangle$ è un piano. $\begin{pmatrix}3\\2\\7\end{pmatrix}\in\left\langle \begin{pmatrix}2\\3\\6\end{pmatrix},\begin{pmatrix}1\\4\\5\end{pmatrix}\right\rangle$? cioè $\begin{pmatrix}3\\2\\7\end{pmatrix}=x_1\begin{pmatrix}2\\3\\6\end{pmatrix}+x_2\begin{pmatrix}1\\4\\5\end{pmatrix}$ per certo $x_1,x_2\in\mathbb{R}$?

$$\left(\begin{array}{cc|c}2&1&3\\3&4&2\\6&5&7\end{array}\right)\to\left(\begin{array}{cc|c}1&1/2&3/2\\3&4&2\\6&5&7\end{array}\right)\to\left(\begin{array}{cc|c}1&1/2&3/2\\0&5/2&-5/2\\0&2&-2\end{array}\right)\to\left(\begin{array}{cc|c}1&1/2&3/2\\0&1&-1\\0&2&-2\end{array}\right)\to\left(\begin{array}{cc|c}1&1/2&3/2\\0&1&-1\\0&0&0\end{array}\right)\to\left(\begin{array}{cc|c}1&0&2\\0&1&-1\\0&0&0\end{array}\right)$$

Quindi $\begin{pmatrix}3\\2\\7\end{pmatrix}=2\begin{pmatrix}2\\3\\6\end{pmatrix}-\begin{pmatrix}1\\4\\5\end{pmatrix}$ e quindi $\begin{pmatrix}3\\2\\7\end{pmatrix}\in\left\langle \begin{pmatrix}2\\3\\6\end{pmatrix},\begin{pmatrix}1\\4\\5\end{pmatrix}\right\rangle$

perciò $\mathrm{Im}\,T=\left\langle \begin{pmatrix}2\\3\\6\end{pmatrix},\begin{pmatrix}1\\4\\5\end{pmatrix}\right\rangle$

$\begin{vmatrix}\vec{i}&\vec{j}&\vec{k}\\2&3&6\\1&4&5\end{vmatrix}=-9\vec{i}-4\vec{j}+5\vec{k}=\begin{pmatrix}-9\\-4\\5\end{pmatrix}$ $\quad \mathrm{Im}(T)$ è il piano $-9x-4y+5z=0$.

4 a) $c_1\begin{pmatrix}1\\2\end{pmatrix}+c_2\begin{pmatrix}3\\6\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$ $\quad\left(\begin{array}{cc|c}1&3&0\\2&6&0\end{array}\right)\to\left(\begin{array}{cc|c}1&3&0\\0&0&0\end{array}\right)\to\{c_1+3c_2=0\to\{c_1=-3c_2$

sono linearmente dipendenti e.g. $-3\begin{pmatrix}1\\2\end{pmatrix}+\begin{pmatrix}3\\6\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$.

b) $c_1\begin{pmatrix}7\\11\end{pmatrix}+c_2\begin{pmatrix}11\\7\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$ $\quad\left(\begin{array}{cc|c}7&11&0\\11&7&0\end{array}\right)\to\left(\begin{array}{cc|c}1&11/7&0\\11&7&0\end{array}\right)\to\left(\begin{array}{cc|c}1&11/7&0\\0&-72/7&0\end{array}\right)\to\left(\begin{array}{cc|c}1&11/7&0\\0&1&0\end{array}\right)\to\left(\begin{array}{cc|c}1&0&0\\0&1&0\end{array}\right)\to\begin{cases}c_1=0\\c_2=0\end{cases}$

sono linearmente indipendenti

c) sono linearmente dipendenti perché $0\cdot\begin{pmatrix}7\\11\end{pmatrix}+3\begin{pmatrix}0\\0\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$.

d) $c_1\begin{pmatrix}1\\2\\0\end{pmatrix}+c_2\begin{pmatrix}1\\2\\3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$ $\quad\left(\begin{array}{cc|c}1&1&0\\2&2&0\\0&3&0\end{array}\right)\to\left(\begin{array}{cc|c}1&1&0\\0&0&0\\0&3&0\end{array}\right)\to\left(\begin{array}{cc|c}1&1&0\\0&3&0\\0&0&0\end{array}\right)\to\left(\begin{array}{cc|c}1&1&0\\0&1&0\\0&0&0\end{array}\right)\to\left(\begin{array}{cc|c}1&0&0\\0&1&0\\0&0&0\end{array}\right)\to\begin{cases}c_1=0\\c_2=0\end{cases}$

sono linearmente indipendenti

e) $c_1\begin{pmatrix}1\\1\\1\end{pmatrix}+c_2\begin{pmatrix}1\\2\\3\end{pmatrix}+c_3\begin{pmatrix}1\\3\\6\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$ $\quad\left(\begin{array}{ccc|c}1&1&1&0\\1&2&3&0\\1&3&6&0\end{array}\right)\to\left(\begin{array}{ccc|c}1&1&1&0\\0&1&2&0\\0&2&5&0\end{array}\right)\to\left(\begin{array}{ccc|c}1&1&1&0\\0&1&2&0\\0&0&1&0\end{array}\right)\to\left(\begin{array}{ccc|c}1&1&0&0\\0&1&0&0\\0&0&1&0\end{array}\right)\to\left(\begin{array}{ccc|c}1&0&0&0\\0&1&0&0\\0&0&1&0\end{array}\right)\to\begin{cases}c_1=0\\c_2=0\\c_3=0\end{cases}$

sono linearmente indipendenti

f) $c_1\begin{pmatrix}1\\2\\3\end{pmatrix}+c_2\begin{pmatrix}4\\5\\6\end{pmatrix}+c_3\begin{pmatrix}7\\8\\9\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$ $\quad\left(\begin{array}{ccc|c}1&4&7&0\\2&5&8&0\\3&6&9&0\end{array}\right)\to\left(\begin{array}{ccc|c}1&4&7&0\\0&-3&-6&0\\0&-6&-12&0\end{array}\right)\to\left(\begin{array}{ccc|c}1&4&7&0\\0&1&2&0\\0&-6&-12&0\end{array}\right)\to\left(\begin{array}{ccc|c}1&4&7&0\\0&1&2&0\\0&0&0&0\end{array}\right)\to\left(\begin{array}{ccc|c}1&0&-1&0\\0&1&2&0\\0&0&0&0\end{array}\right)$

$\to\begin{cases}c_1-c_3=0\\c_2+2c_3=0\end{cases}\to\begin{cases}c_1=c_3\\c_2=-2c_3\end{cases}$ sono linearmente dipendenti e.g. $1\cdot\begin{pmatrix}1\\2\\3\end{pmatrix}-2\begin{pmatrix}4\\5\\6\end{pmatrix}+\begin{pmatrix}7\\8\\9\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$.

5 a) $c_1\begin{pmatrix}1\\3\end{pmatrix}+c_2\begin{pmatrix}1\\-1\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$ $\quad\left(\begin{array}{cc|c}1&1&0\\3&-1&0\end{array}\right)\to\left(\begin{array}{cc|c}1&1&0\\0&-4&0\end{array}\right)\to\left(\begin{array}{cc|c}1&1&0\\0&1&0\end{array}\right)\to\left(\begin{array}{cc|c}1&0&0\\0&1&0\end{array}\right)\to\begin{cases}c_1=0\\c_2=0\end{cases}$

i vettori sono linearmente indipendenti

sia $\vec{x}\in\mathbb{R}^2$ $\quad\vec{x}=\begin{pmatrix}x_1\\x_2\end{pmatrix}$ $\quad c_1\begin{pmatrix}1\\3\end{pmatrix}+c_2\begin{pmatrix}1\\-1\end{pmatrix}=\begin{pmatrix}x_1\\x_2\end{pmatrix}$

$\left(\begin{array}{cc|c}1&1&x_1\\3&-1&x_2\end{array}\right)\to\left(\begin{array}{cc|c}1&1&x_1\\0&-4&x_2-3x_1\end{array}\right)\to\left(\begin{array}{cc|c}1&1&x_1\\0&1&-1/4(x_2-3x_1)\end{array}\right)$ il sistema ha sempre soluzioni

quindi $\vec{x}$ è una combinazione lineare di $\begin{pmatrix}1\\3\end{pmatrix}$ e $\begin{pmatrix}1\\-1\end{pmatrix}$. Quindi i vettori generano $\mathbb{R}^2$

segue che $\left\{\begin{pmatrix}1\\3\end{pmatrix},\begin{pmatrix}1\\-1\end{pmatrix}\right\}$ è una base di $\mathbb{R}^2$

b) $c_1\begin{pmatrix}1\\2\end{pmatrix}+c_2\begin{pmatrix}2\\-3\end{pmatrix}+c_3\begin{pmatrix}3\\2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$ $\quad\left(\begin{array}{ccc|c}1&2&3&0\\2&-3&2&0\end{array}\right)\to\left(\begin{array}{ccc|c}1&2&3&0\\0&-7&-4&0\end{array}\right)\to\left(\begin{array}{ccc|c}1&2&3&0\\0&1&4/7&0\end{array}\right)$

il sistema ha infinite soluzioni quindi i vettori sono linearmente dipendenti

quindi non formano una base di $\mathbb{R}^2$.

c) $c_1\begin{pmatrix}1\\3\end{pmatrix}+c_2\begin{pmatrix}-2\\-6\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$ $\quad -2\begin{pmatrix}1\\3\end{pmatrix}=\begin{pmatrix}-2\\-6\end{pmatrix}$ quindi i vettori sono linearmente dipendenti

Quindi non formano una base di $\mathbb{R}^2$.

6 La matrice che rappresenta T è $\begin{pmatrix}1&-1&1&0\\1&-1&0&1\end{pmatrix}$.

a) $\operatorname{Ker}(T) = \{ \vec{x} \in \mathbb{R}^4 \mid T(\vec{x}) = \vec{0} \} = \{ \vec{x} \in \mathbb{R}^4 \mid A\vec{x} = \vec{0} \}$

$\begin{pmatrix} 1 & -1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \\ 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ quindi $\begin{pmatrix} 3 \\ 2 \\ 3 \\ -2 \end{pmatrix} \notin \operatorname{Ker}(T)$.

b) $\operatorname{Im}(T) = \left\langle \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle \qquad \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

quindi $\begin{pmatrix} 1 \\ 2 \end{pmatrix} \in \operatorname{Im}(T)$.

7 a) $\left(\begin{array}{cc|c} 1 & 2 & 0 \\ 3 & 4 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & -2 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 1 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right) \qquad \operatorname{Ker}(T) = \{ \vec{x} \in \mathbb{R}^2 \mid T(\vec{x}) = \vec{0} \} =$

$\{ \vec{x} \in \mathbb{R}^2 \mid A\vec{x} = \vec{0} \} = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 \,\middle|\, \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\} = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 \,\middle|\, x_1 = 0 \ x_2 = 0 \right\} = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$

non ha una base.

b) $\left(\begin{array}{cc|c} 2 & 3 & 0 \\ 6 & 9 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 3/2 & 0 \\ 6 & 9 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 3/2 & 0 \\ 0 & 0 & 0 \end{array}\right) \to \{ x_1 + 3/2\, x_2 = 0 \to \{ x_1 = -3/2\, x_2$

$\operatorname{Ker}(T) = \{ \vec{x} \in \mathbb{R}^2 \mid T(\vec{x}) = \vec{0} \} = \{ \vec{x} \in \mathbb{R}^2 \mid A\vec{x} = \vec{0} \} = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 \,\middle|\, x_1 = -3/2\, x_2 \right\} =$

$\left\{ \begin{pmatrix} -3/2\, x_2 \\ x_2 \end{pmatrix} \,\middle|\, x_2 \in \mathbb{R} \right\} = \left\{ x_2 \begin{pmatrix} -3/2 \\ 1 \end{pmatrix} \,\middle|\, x_2 \in \mathbb{R} \right\} = \left\langle \begin{pmatrix} -3/2 \\ 1 \end{pmatrix} \right\rangle$ una base è $\left(\begin{pmatrix} -3/2 \\ 1 \end{pmatrix} \right)$

c) $\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 2 & 3 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right) \to \begin{cases} x_1 - x_3 = 0 \\ x_2 + 2x_3 = 0 \end{cases} \to \begin{cases} x_1 = x_3 \\ x_2 = -2x_3 \end{cases}$

$\operatorname{Ker}(T) = \{ \vec{x} \in \mathbb{R}^3 \mid T(\vec{x}) = \vec{0} \} = \{ \vec{x} \in \mathbb{R}^3 \mid A\vec{x} = \vec{0} \} = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 \,\middle|\, x_1 = x_3,\ x_2 = -2x_3 \right\} =$

$\left\{ \begin{pmatrix} x_3 \\ -2x_3 \\ x_3 \end{pmatrix} \,\middle|\, x_3 \in \mathbb{R} \right\} = \left\{ x_3 \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \,\middle|\, x_3 \in \mathbb{R} \right\} = \left\langle \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \right\rangle$ una base è $\left(\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \right)$

d) $(1\ 2\ 3 \mid 0) \to \{ x_1 + 2x_2 + 3x_3 = 0 \to \{ x_1 = -2x_2 - 3x_3$

$\operatorname{Ker}(T) = \{ \vec{x} \in \mathbb{R}^3 \mid T(\vec{x}) = \vec{0} \} = \{ \vec{x} \in \mathbb{R}^3 \mid A\vec{x} = \vec{0} \} = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 \,\middle|\, x_1 = -2x_2 - 3x_3 \right\} =$

$\left\{ \begin{pmatrix} -2x_2 - 3x_3 \\ x_2 \\ x_3 \end{pmatrix} \,\middle|\, x_2, x_3 \in \mathbb{R} \right\} = \left\{ \begin{pmatrix} -2x_2 \\ x_2 \\ 0 \end{pmatrix} + \begin{pmatrix} -3x_3 \\ 0 \\ x_3 \end{pmatrix} \,\middle|\, x_2, x_3 \in \mathbb{R} \right\} = \left\{ x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} \,\middle|\, x_2, x_3 \in \mathbb{R} \right\}$

$= \left\langle \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} \right\rangle$ una base è $\left(\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} \right)$

e) $\left(\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 3 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \to \begin{cases} x_1 = 0 \\ x_2 = 0 \end{cases} \qquad \operatorname{Ker}(T) = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$ non ha una base.

f) $\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \to \{ x_1 + x_2 + x_3 = 0 \to \{ x_1 = -x_2 - x_3$

$\operatorname{Ker}(T) = \{ \vec{x} \in \mathbb{R}^3 \mid T(\vec{x}) = \vec{0} \} = \{ \vec{x} \in \mathbb{R}^3 \mid A\vec{x} = \vec{0} \} = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 \,\middle|\, x_1 = -x_2 - x_3 \right\} =$

$\left\{ \begin{pmatrix} -x_2 - x_3 \\ x_2 \\ x_3 \end{pmatrix} \,\middle|\, x_2, x_3 \in \mathbb{R} \right\} = \left\{ \begin{pmatrix} -x_2 \\ x_2 \\ 0 \end{pmatrix} + \begin{pmatrix} -x_3 \\ 0 \\ x_3 \end{pmatrix} \,\middle|\, x_2, x_3 \in \mathbb{R} \right\} = \left\{ x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \,\middle|\, x_2, x_3 \in \mathbb{R} \right\}$

$= \left\langle \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\rangle$ una base è $\left(\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right)$

8 a) $\operatorname{Im}(T) = \left\langle \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right\rangle$

$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}$ una base di $\operatorname{Im}(T)$ è $\left(\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \right)$

b) $\operatorname{Im}(T) = \left\langle \begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}1\\5\\7\end{pmatrix} \right\rangle$ $\begin{pmatrix}1&1&1\\1&2&3\\1&5&7\end{pmatrix} \to \begin{pmatrix}1&1&1\\0&1&2\\0&4&6\end{pmatrix} \to \begin{pmatrix}1&1&1\\0&1&2\\0&0&-2\end{pmatrix} \to \begin{pmatrix}1&1&1\\0&1&2\\0&0&1\end{pmatrix}$

una base di $\operatorname{Im}(T)$ è $\left(\begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}0\\1\\2\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix} \right)$

c) $\operatorname{Im}(T) = \left\langle \begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix} \right\rangle$ $\begin{pmatrix}0&0&0\\1&0&0\\0&1&1\end{pmatrix} \to \begin{pmatrix}1&0&0\\0&1&1\\0&0&0\end{pmatrix}$

una base di $\operatorname{Im}(T)$ è $\left(\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix} \right)$

d) $\operatorname{Im}(T) = \left\langle \begin{pmatrix}1\\2\\3\\4\end{pmatrix}, \begin{pmatrix}5\\6\\7\\8\end{pmatrix}, \begin{pmatrix}3\\2\\1\\0\end{pmatrix} \right\rangle$ $\begin{pmatrix}1&2&3&4\\5&6&7&8\\3&2&1&0\end{pmatrix} \to \begin{pmatrix}1&2&3&4\\0&-4&-8&-12\\0&-4&-8&-12\end{pmatrix} \to \begin{pmatrix}1&2&3&4\\0&1&2&3\\0&-4&-8&-12\end{pmatrix}$

$\to \begin{pmatrix}1&2&3&4\\0&1&2&3\\0&0&0&0\end{pmatrix}$ una base di $\operatorname{Im}(T)$ è $\left(\begin{pmatrix}1\\2\\3\\4\end{pmatrix}, \begin{pmatrix}0\\1\\2\\3\end{pmatrix} \right)$

e) $\operatorname{Im}(T) = \left\langle \begin{pmatrix}1\\2\\3\\4\end{pmatrix}, \begin{pmatrix}2\\4\\6\\8\end{pmatrix}, \begin{pmatrix}3\\6\\9\\12\end{pmatrix} \right\rangle$ $\begin{pmatrix}1&2&3&4\\2&4&6&8\\3&6&9&12\end{pmatrix} \to \begin{pmatrix}1&2&3&4\\0&0&0&0\\0&0&0&0\end{pmatrix}$

una base di $\operatorname{Im}(T)$ è $\left(\begin{pmatrix}1\\2\\3\\4\end{pmatrix} \right)$

f) $\operatorname{Im}(T) = \left\langle \begin{pmatrix}0\\0\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}2\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}3\\4\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \right\rangle.$

$$\begin{pmatrix}0&0&0&0\\1&0&0&0\\2&0&0&0\\0&1&0&0\\3&4&0&0\\0&0&1&0\end{pmatrix} \to \begin{pmatrix}1&0&0&0\\0&0&0&0\\2&0&0&0\\0&1&0&0\\3&4&0&0\\0&0&1&0\end{pmatrix} \to \begin{pmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\0&1&0&0\\0&4&0&0\\0&0&1&0\end{pmatrix} \to \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\\0&4&0&0\\0&0&1&0\end{pmatrix} \to \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&1&0\end{pmatrix} \to \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}$$

una base di $\operatorname{Im}(T)$ è $\left(\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \right).$