Compito 6 ①

1) a) $V = \left\langle \begin{pmatrix}1\\3\\0\\2\\1\end{pmatrix}, \begin{pmatrix}1\\2\\-1\\1\\0\end{pmatrix}, \begin{pmatrix}-1\\0\\3\\1\\2\end{pmatrix}, \begin{pmatrix}1\\1\\-1\\-1\\-1\end{pmatrix}, \begin{pmatrix}0\\-1\\-2\\0\\-1\end{pmatrix} \right\rangle$ trovare una base

$$\begin{pmatrix}1&3&0&2&1\\1&2&-1&1&0\\-1&0&3&1&2\\1&1&-1&-1&-1\\0&-1&-2&0&-1\end{pmatrix} \to \begin{pmatrix}1&3&0&2&1\\0&-1&-1&-1&-1\\0&3&3&3&3\\0&-2&-1&-3&-2\\0&-1&-2&0&-1\end{pmatrix} \to \begin{pmatrix}1&3&0&2&1\\0&1&1&1&1\\0&3&3&3&3\\0&-2&-1&-3&-2\\0&-1&-2&0&-1\end{pmatrix} \to \begin{pmatrix}1&3&0&2&1\\0&1&1&1&1\\0&0&0&0&0\\0&0&1&-1&0\\0&0&-1&1&0\end{pmatrix} \to \begin{pmatrix}1&3&0&2&1\\0&1&1&1&1\\0&0&1&-1&0\\0&0&-1&1&0\\0&0&0&0&0\end{pmatrix} \to$$

$$\begin{pmatrix}1&3&0&2&1\\0&1&1&1&1\\0&0&1&-1&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix} \quad \text{una base è} \quad \left(\begin{pmatrix}1\\3\\0\\2\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\\1\\1\end{pmatrix}, \begin{pmatrix}0\\0\\1\\-1\\0\end{pmatrix} \right)$$

$$W = \left\{ \begin{pmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{pmatrix} \in \mathbb{R}^5 \;\middle|\; 3x_1 - x_2 - x_4 + 2x_5 = 0,\ 3x_1 - x_2 + x_3 = 0 \right\} =$$

$$\left\{ \begin{pmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{pmatrix} \in \mathbb{R}^5 \;\middle|\; \begin{pmatrix}3&-1&0&-1&2\\3&-1&1&0&0\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \right\}$$

$$\left(\begin{array}{ccccc|c}3&-1&0&-1&2&0\\3&-1&1&0&0&0\end{array}\right) \to \left(\begin{array}{ccccc|c}3&-1&0&-1&2&0\\0&0&1&1&-2&0\end{array}\right) \to \left(\begin{array}{ccccc|c}1&-1/3&0&-1/3&2/3&0\\0&0&1&1&-2&0\end{array}\right) \to \begin{cases} x_1 - 1/3\,x_2 - 1/3\,x_4 + 2/3\,x_5 = 0 \\ x_3 + x_4 - 2x_5 = 0 \end{cases}$$

$$\to \begin{cases} x_1 = 1/3\,x_2 + 1/3\,x_4 - 2/3\,x_5 \\ x_3 = -x_4 + 2x_5 \end{cases} \quad \text{quindi}$$

$$W = \left\{ \begin{pmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{pmatrix} \in \mathbb{R}^5 \;\middle|\; x_1 = 1/3\,x_2 + 1/3\,x_4 - 2/3\,x_5,\ x_3 = -x_4 + 2x_5 \right\} = \left\{ \begin{pmatrix}1/3\,x_2 + 1/3\,x_4 - 2/3\,x_5\\x_2\\-x_4 + 2x_5\\x_4\\x_5\end{pmatrix} \;\middle|\; x_2, x_4, x_5 \in \mathbb{R} \right\}$$

$$= \left\{ x_2\begin{pmatrix}1/3\\1\\0\\0\\0\end{pmatrix} + x_4\begin{pmatrix}1/3\\0\\-1\\1\\0\end{pmatrix} + x_5\begin{pmatrix}-2/3\\0\\2\\0\\1\end{pmatrix} \;\middle|\; x_2, x_4, x_5 \in \mathbb{R} \right\} = \left\langle \begin{pmatrix}1/3\\1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}1/3\\0\\-1\\1\\0\end{pmatrix}, \begin{pmatrix}-2/3\\0\\2\\0\\1\end{pmatrix} \right\rangle \text{ e}$$

una base di W è $\left(\begin{pmatrix}1/3\\1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}1/3\\0\\-1\\1\\0\end{pmatrix}, \begin{pmatrix}-2/3\\0\\2\\0\\1\end{pmatrix} \right)$ perché la matrice era fortemente ridotta.

Alternativa per calcolare una base di W:
si vede subito che $x_4 = 3x_1 - x_2 + 2x_5$ e $x_3 = -3x_1 + x_2$. Quindi

$$W = \left\{ \begin{pmatrix}x_1\\x_2\\-3x_1+x_2\\3x_1-x_2+x_5\\x_5\end{pmatrix} \;\middle|\; x_1, x_2, x_5 \in \mathbb{R} \right\} = \left\{ x_1\begin{pmatrix}1\\0\\-3\\3\\0\end{pmatrix} + x_2\begin{pmatrix}0\\1\\1\\-1\\0\end{pmatrix} + x_5\begin{pmatrix}0\\0\\0\\2\\1\end{pmatrix} \;\middle|\; x_1, x_2, x_5 \in \mathbb{R} \right\} = \left\langle \begin{pmatrix}1\\0\\-3\\3\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\\-1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\2\\1\end{pmatrix} \right\rangle$$

i vettori $\begin{pmatrix}1\\0\\-3\\3\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\\-1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\2\\1\end{pmatrix}$ sono linearmente indipendenti (perché??) e generano W.

Quindi una base di W è $\left(\begin{pmatrix}1\\0\\-3\\3\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\\-1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\2\\1\end{pmatrix} \right)$.

b) $U = \left\langle \begin{pmatrix}1\\3\\0\\2\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\\1\\1\end{pmatrix}, \begin{pmatrix}0\\0\\1\\-1\\0\end{pmatrix} \right\rangle \quad c_1\begin{pmatrix}1\\3\\0\\2\\1\end{pmatrix} + c_2\begin{pmatrix}0\\1\\1\\1\\1\end{pmatrix} + c_3\begin{pmatrix}0\\0\\1\\-1\\0\end{pmatrix} = \begin{pmatrix}1\\1\\2\\1\\1\end{pmatrix}$?

$$\left(\begin{array}{ccc|c}1&0&0&1\\3&1&0&1\\0&1&1&2\\2&1&-1&1\\1&1&0&1\end{array}\right) \to \left(\begin{array}{ccc|c}1&0&0&1\\0&1&0&-2\\0&1&1&2\\0&1&-1&-1\\0&1&0&0\end{array}\right) \to \left(\begin{array}{ccc|c}1&0&0&1\\0&1&0&-2\\0&0&1&4\\0&0&-1&1\\0&0&0&2\end{array}\right)$$

il sistema non ha soluzioni, quindi $\vec{u} \notin U$.

$\vec{u} \notin W$ perché non soddisfa le due equazioni

Alternativa: usare la base trovata di W e ragionare come sopra.

2) a) La dimensione di $\mathbb{R}^3$ è 3, quindi ogni base contiene 3 vettori. Perciò non è una base

b) $c_1\begin{pmatrix}1\\1\\-1\end{pmatrix}+c_2\begin{pmatrix}4\\1\\-1\end{pmatrix}+c_3\begin{pmatrix}0\\1\\-1\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$ $\left(\begin{array}{ccc|c}1&4&0&0\\1&1&1&0\\-1&-1&-1&0\end{array}\right)\to\left(\begin{array}{ccc|c}1&4&0&0\\0&-3&1&0\\0&3&-1&0\end{array}\right)\to\left(\begin{array}{ccc|c}1&4&0&0\\0&1&-1/3&0\\0&0&0&0\end{array}\right)$ ci sono infinite soluzioni

I vettori non sono linearmente indipendenti, perciò non formano una base di $\mathbb{R}^3$.

c) $c_1\begin{pmatrix}3\\2\\2\end{pmatrix}+c_2\begin{pmatrix}-1\\2\\1\end{pmatrix}+c_3\begin{pmatrix}0\\1\\0\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$ $\left(\begin{array}{ccc|c}3&-1&0&0\\2&2&1&0\\2&1&0&0\end{array}\right)\to\left(\begin{array}{ccc|c}1&-1/3&0&0\\2&2&1&0\\2&1&0&0\end{array}\right)\to\left(\begin{array}{ccc|c}1&-1/3&0&0\\0&8/3&1&0\\0&5/3&0&0\end{array}\right)\to\left(\begin{array}{ccc|c}1&-1/3&0&0\\0&1&3/8&0\\0&5/3&0&0\end{array}\right)\to$

$\left(\begin{array}{ccc|c}1&-1/3&0&0\\0&1&3/8&0\\0&0&-5/8&0\end{array}\right)\to\left(\begin{array}{ccc|c}1&-1/3&0&0\\0&1&3/8&0\\0&0&1&0\end{array}\right)\to\left(\begin{array}{ccc|c}1&-1/3&0&0\\0&1&0&0\\0&0&1&0\end{array}\right)\to\left(\begin{array}{ccc|c}1&0&0&0\\0&1&0&0\\0&0&1&0\end{array}\right)\to\begin{cases}c_1=0\\c_2=0\\c_3=0\end{cases}$

I vettori sono linearmente indipendenti. La dimensione di $\mathbb{R}^3$ è 3. Tre vettori linearmente indipendenti in uno spazio di dimensione 3 generano lo spazio. Quindi i vettori formano una base di $\mathbb{R}^3$.

3) La matrice di F è $A=\begin{pmatrix}1&0&2&-1\\2&-1&0&2\\-1&1&2&-3\\3&-1&2&1\end{pmatrix}$

a) $\mathrm{Ker}(F)=\{\vec{x}\in\mathbb{R}^4 \mid F(\vec{x})=\vec{0}\}=\{x\in\mathbb{R}^4 \mid A\vec{x}=\vec{0}\}$.

$\left(\begin{array}{cccc|c}1&0&2&-1&0\\2&-1&0&2&0\\-1&1&2&-3&0\\3&-1&2&1&0\end{array}\right)\to\left(\begin{array}{cccc|c}1&0&2&-1&0\\0&-1&-4&4&0\\0&1&4&-4&0\\0&-1&-4&4&0\end{array}\right)\to\left(\begin{array}{cccc|c}1&0&2&-1&0\\0&1&4&-4&0\\0&0&0&0&0\\0&0&0&0&0\end{array}\right)\to\begin{cases}x_1+2x_3-x_4=0\\x_2+4x_3-4x_4=0\end{cases}\to\begin{cases}x_1=-2x_3+x_4\\x_2=-4x_3+4x_4\end{cases}$

$\mathrm{Ker}(F)=\left\{\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}\in\mathbb{R}^4 \,\middle|\, x_1=-2x_3+x_4,\ x_2=-4x_3+4x_4\right\}=\left\{\begin{pmatrix}-2x_3+x_4\\-4x_3+4x_4\\x_3\\x_4\end{pmatrix} \,\middle|\, x_3,x_4\in\mathbb{R}\right\}=$

$\left\{x_3\begin{pmatrix}-2\\-4\\1\\0\end{pmatrix}+x_4\begin{pmatrix}1\\4\\0\\1\end{pmatrix} \,\middle|\, x_3,x_4\in\mathbb{R}\right\}=\left\langle\begin{pmatrix}-2\\4\\1\\0\end{pmatrix},\begin{pmatrix}1\\4\\0\\1\end{pmatrix}\right\rangle$ (controllare che i vettori sono in ker F !!)

Una base di Ker(F) è $\left\langle\begin{pmatrix}-2\\4\\1\\0\end{pmatrix},\begin{pmatrix}1\\4\\0\\1\end{pmatrix}\right\rangle$.

$\mathrm{Im}(F)=\left\langle\begin{pmatrix}1\\2\\-1\\3\end{pmatrix},\begin{pmatrix}0\\-1\\1\\-1\end{pmatrix},\begin{pmatrix}2\\0\\2\\2\end{pmatrix},\begin{pmatrix}-1\\2\\-3\\1\end{pmatrix}\right\rangle$

$\begin{pmatrix}1&2&-1&3\\0&-1&1&-1\\2&0&2&2\\-1&2&-3&1\end{pmatrix}\to\begin{pmatrix}1&2&-1&3\\0&-1&1&-1\\0&-4&4&-4\\0&4&-4&4\end{pmatrix}\to\begin{pmatrix}1&2&-1&3\\0&1&-1&1\\0&0&0&0\\0&0&0&0\end{pmatrix}$ $\mathrm{Im}(F)=\left\langle\begin{pmatrix}1\\2\\-1\\3\end{pmatrix},\begin{pmatrix}0\\1\\-1\\1\end{pmatrix}\right\rangle$

una base di Im(F) è $\left\langle\begin{pmatrix}1\\2\\-1\\3\end{pmatrix},\begin{pmatrix}0\\1\\-1\\1\end{pmatrix}\right\rangle$ si osserva che $2+2=4$.

b) $\begin{pmatrix}1\\0\\0\\0\end{pmatrix}\in\mathrm{Im}(F)$ se e solo se $\begin{pmatrix}1\\0\\0\\0\end{pmatrix}\in\left\langle\begin{pmatrix}1\\-2\\-1\\3\end{pmatrix},\begin{pmatrix}0\\1\\-1\\1\end{pmatrix}\right\rangle$ se e solo se esistono $c_1,c_2\in\mathbb{R}$ con

$c_1\begin{pmatrix}1\\-2\\-1\\3\end{pmatrix}+c_2\begin{pmatrix}0\\1\\-1\\1\end{pmatrix}=\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$ $\left(\begin{array}{cc|c}1&0&1\\-2&1&0\\-1&-1&0\\3&1&0\end{array}\right)\to\left(\begin{array}{cc|c}1&0&1\\0&1&2\\0&-1&1\\0&1&-3\end{array}\right)\to\left(\begin{array}{cc|c}1&0&1\\0&1&2\\0&0&3\\0&0&-1\end{array}\right)$ il sistema non ha soluzioni quindi $\begin{pmatrix}1\\0\\0\\0\end{pmatrix}\notin\mathrm{Im}(F)$.

4) a) $c_1\begin{pmatrix}2\\5\end{pmatrix}+c_2\begin{pmatrix}5\\12\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$ $\left(\begin{array}{cc|c}2&5&0\\5&12&0\end{array}\right)\to\left(\begin{array}{cc|c}1&5/2&0\\5&12&0\end{array}\right)\to\left(\begin{array}{cc|c}1&5/2&0\\0&-1/2&0\end{array}\right)\to\left(\begin{array}{cc|c}1&5/2&0\\0&1&0\end{array}\right)\to\left(\begin{array}{cc|c}1&0&0\\0&1&0\end{array}\right)\to\begin{cases}c_1=0\\c_2=0\end{cases}$

i vettori sono linearmente indipendenti. Due vettori linearmente indipendenti in uno spazio di dimensione 2 generano lo spazio, quindi formano una base

$c_1\begin{pmatrix}2\\5\end{pmatrix}+c_2\begin{pmatrix}5\\12\end{pmatrix}=\begin{pmatrix}7\\16\end{pmatrix}$ $\left(\begin{array}{cc|c}2&5&7\\5&12&16\end{array}\right)\to\left(\begin{array}{cc|c}1&5/2&7/2\\5&12&16\end{array}\right)\to\left(\begin{array}{cc|c}1&5/2&7/2\\0&-1/2&-3/2\end{array}\right)\to\left(\begin{array}{cc|c}1&5/2&7/2\\0&1&3\end{array}\right)\to\left(\begin{array}{cc|c}1&0&-4\\0&1&3\end{array}\right)$

$\to\begin{cases}c_1=-4\\c_2=3\end{cases}$ $\left[\begin{pmatrix}7\\16\end{pmatrix}\right]_b=\begin{pmatrix}-4\\3\end{pmatrix}$ (controllo $-4\begin{pmatrix}2\\5\end{pmatrix}+3\begin{pmatrix}5\\12\end{pmatrix}=\begin{pmatrix}7\\16\end{pmatrix}$ ok!!).

<u>Alternativa</u>: $[I]^b_e=\begin{pmatrix}2&5\\5&12\end{pmatrix}$ $[I]^e_b=\left([I]^b_e\right)^{-1}=\begin{pmatrix}-12&5\\5&-2\end{pmatrix}$

$\left[\begin{pmatrix}7\\6\end{pmatrix}\right]_b=[I]^e_b\left[\begin{pmatrix}7\\6\end{pmatrix}\right]_e=\begin{pmatrix}-12&5\\5&-2\end{pmatrix}\begin{pmatrix}7\\6\end{pmatrix}=\begin{pmatrix}-4\\3\end{pmatrix}$.

b) $c_1\begin{pmatrix}1\\2\end{pmatrix}+c_2\begin{pmatrix}5\\6\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$ $\left(\begin{array}{cc|c}1&5&0\\2&6&0\end{array}\right)\to\left(\begin{array}{cc|c}1&5&0\\0&-4&0\end{array}\right)\to\left(\begin{array}{cc|c}1&5&0\\0&1&0\end{array}\right)\to\left(\begin{array}{cc|c}1&0&0\\0&1&0\end{array}\right)\to\begin{cases}c_1=0\\c_2=0\end{cases}$

i vettori sono linearmente indipendenti. Due vettori linearmente indipendenti in uno spazio di dimensione 2 generano lo spazio, quindi formano una base.

$c_1\begin{pmatrix}1\\2\end{pmatrix}+c_2\begin{pmatrix}5\\6\end{pmatrix}=\begin{pmatrix}-4\\4\end{pmatrix}$ $\left(\begin{array}{cc|c}1&5&-4\\2&6&4\end{array}\right)\to\left(\begin{array}{cc|c}1&5&-4\\0&-4&12\end{array}\right)\to\left(\begin{array}{cc|c}1&5&-4\\0&1&-3\end{array}\right)\to\left(\begin{array}{cc|c}1&0&11\\0&1&-3\end{array}\right)\to\begin{cases}c_1=11\\c_2=-3.\end{cases}$

$\left[\begin{pmatrix}-4\\4\end{pmatrix}\right]_b=\begin{pmatrix}11\\-3\end{pmatrix}$ (controllo $11\begin{pmatrix}1\\2\end{pmatrix}-3\begin{pmatrix}5\\6\end{pmatrix}=\begin{pmatrix}-4\\4\end{pmatrix}$ ok!!).

<u>Alternativa</u> $[I]_e^b=\begin{pmatrix}1&5\\2&6\end{pmatrix}$ $[I]_b^e=\left([I]_e^b\right)^{-1}=\begin{pmatrix}-6/4&5/4\\2/4&-1/4\end{pmatrix}.$

$\left[\begin{pmatrix}-4\\4\end{pmatrix}\right]_b=[I]_b^e\left[\begin{pmatrix}-4\\4\end{pmatrix}\right]_e=\begin{pmatrix}-6/4&5/4\\2/4&-1/4\end{pmatrix}\begin{pmatrix}-4\\4\end{pmatrix}=\begin{pmatrix}11\\-3\end{pmatrix}.$

5

a) $c_1\begin{pmatrix}1\\2\\3\end{pmatrix}+c_2\begin{pmatrix}0\\1\\2\end{pmatrix}+c_3\begin{pmatrix}1\\0\\1\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$ $\left(\begin{array}{ccc|c}1&0&1&0\\2&1&0&0\\3&2&1&0\end{array}\right)\to\left(\begin{array}{ccc|c}1&0&1&0\\0&1&-2&0\\0&2&-2&0\end{array}\right)\to\left(\begin{array}{ccc|c}1&0&1&0\\0&1&-2&0\\0&0&2&0\end{array}\right)\to\left(\begin{array}{ccc|c}1&0&1&0\\0&1&-2&0\\0&0&1&0\end{array}\right)\to\left(\begin{array}{ccc|c}1&0&0&0\\0&1&0&0\\0&0&1&0\end{array}\right)\to\begin{cases}c_1=0\\c_2=0\\c_3=0\end{cases}$

I vettori sono linearmente indipendenti. Tre vettori linearmente indipendenti in uno spazio di dimensione 3 generano lo spazio, quindi formano una base.

$c_1\begin{pmatrix}1\\2\\3\end{pmatrix}+c_2\begin{pmatrix}0\\1\\2\end{pmatrix}+c_3\begin{pmatrix}1\\0\\1\end{pmatrix}=\begin{pmatrix}1\\1\\1\end{pmatrix}$ $\left(\begin{array}{ccc|c}1&0&1&1\\2&1&0&1\\3&2&1&1\end{array}\right)\to\left(\begin{array}{ccc|c}1&0&1&1\\0&1&-2&-1\\0&2&-2&-2\end{array}\right)\to\left(\begin{array}{ccc|c}1&0&1&1\\0&1&-2&-1\\0&0&2&0\end{array}\right)\to\left(\begin{array}{ccc|c}1&0&1&1\\0&1&-2&-1\\0&0&1&0\end{array}\right)\to\left(\begin{array}{ccc|c}1&0&0&1\\0&1&0&-1\\0&0&1&0\end{array}\right)\to\begin{cases}c_1=1\\c_2=-1\\c_3=0\end{cases}$

$\left[\begin{pmatrix}1\\1\\1\end{pmatrix}\right]_b=\begin{pmatrix}1\\-1\\0\end{pmatrix}$ (controllo $\begin{pmatrix}1\\1\\1\end{pmatrix}=\begin{pmatrix}1\\2\\3\end{pmatrix}-\begin{pmatrix}0\\1\\2\end{pmatrix}$ ok!!) vedi b) per un modo alternativa.

b) $c_1\begin{pmatrix}1\\1\\1\end{pmatrix}+c_2\begin{pmatrix}0\\1\\1\end{pmatrix}+c_3\begin{pmatrix}0\\0\\1\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$ $\left(\begin{array}{ccc|c}1&0&0&0\\1&1&0&0\\1&1&1&0\end{array}\right)\to\left(\begin{array}{ccc|c}1&0&0&0\\0&1&0&0\\0&1&1&0\end{array}\right)\to\left(\begin{array}{ccc|c}1&0&0&0\\0&1&0&0\\0&0&1&0\end{array}\right)\to\begin{cases}c_1=0\\c_2=0\\c_3=0\end{cases}$

i vettori sono linearmente indipendenti. Tre vettori linearmente indipendenti in uno spazio di dimensione 3 generano lo spazio, quindi formano una base.

$[I]_e^b=\begin{pmatrix}1&0&0\\1&1&0\\1&1&1\end{pmatrix}$ $\left(\begin{array}{ccc|ccc}1&0&0&1&0&0\\1&1&0&0&1&0\\1&1&1&0&0&1\end{array}\right)\to\left(\begin{array}{ccc|ccc}1&0&0&1&0&0\\0&1&0&-1&1&0\\0&1&1&-1&0&1\end{array}\right)\to\left(\begin{array}{ccc|ccc}1&0&0&1&0&0\\0&1&0&-1&1&0\\0&0&1&0&-1&1\end{array}\right)$ $[I]_b^e=\left([I]_e^b\right)^{-1}=\begin{pmatrix}1&0&0\\-1&1&0\\0&-1&1\end{pmatrix}$

$[\vec{x}]_b=[I]_b^e[\vec{x}]_e$ quindi $\left[\begin{pmatrix}3\\7\\13\end{pmatrix}\right]_b=\begin{pmatrix}1&0&0\\-1&1&0\\0&-1&1\end{pmatrix}\begin{pmatrix}3\\7\\13\end{pmatrix}=\begin{pmatrix}3\\4\\6\end{pmatrix}$ (controllo: $3\begin{pmatrix}1\\1\\1\end{pmatrix}+4\begin{pmatrix}0\\1\\1\end{pmatrix}+6\begin{pmatrix}0\\0\\1\end{pmatrix}=\begin{pmatrix}3\\7\\13\end{pmatrix}$ ok!!)

vedi a) per un modo alternativo.

6 $[\vec{x}]_b=\begin{pmatrix}2\\-1\end{pmatrix}$ cioè $\vec{x}=2\begin{pmatrix}8\\4\\-1\end{pmatrix}+(-1)\begin{pmatrix}5\\2\\-1\end{pmatrix}=\begin{pmatrix}11\\6\\-1\end{pmatrix}$

7 a) $[T]_e^e=\begin{pmatrix}1&2\\3&4\end{pmatrix}$ $[I]_e^b=\begin{pmatrix}1&1\\1&2\end{pmatrix}$ $\left(\begin{array}{cc|cc}1&1&1&0\\1&2&0&1\end{array}\right)\to\left(\begin{array}{cc|cc}1&1&1&0\\0&1&-1&1\end{array}\right)\to\left(\begin{array}{cc|cc}1&0&2&-1\\0&1&-1&1\end{array}\right)$ $[I]_b^e=\left([I]_e^b\right)^{-1}=\begin{pmatrix}2&-1\\-1&1\end{pmatrix}$

$[T]_b^b=[I]_b^e[T]_e^e[I]_e^b=\begin{pmatrix}2&-1\\-1&1\end{pmatrix}\begin{pmatrix}1&2\\3&4\end{pmatrix}\begin{pmatrix}1&1\\1&2\end{pmatrix}=\begin{pmatrix}-1&-1\\4&6\end{pmatrix}$

b) $[T]_e^e=\begin{pmatrix}7&-1\\-6&8\end{pmatrix}$ $[I]_e^b=\begin{pmatrix}1&-1\\2&3\end{pmatrix}$ $\left(\begin{array}{cc|cc}1&-1&1&0\\2&3&0&1\end{array}\right)\to\left(\begin{array}{cc|cc}1&-1&1&0\\0&5&-2&1\end{array}\right)\to\left(\begin{array}{cc|cc}1&-1&1&0\\0&1&-2/5&1/5\end{array}\right)\to\left(\begin{array}{cc|cc}1&0&3/5&1/5\\0&1&-2/5&1/5\end{array}\right)$

$[I]_b^e=\left([I]_e^b\right)^{-1}=\begin{pmatrix}3/5&1/5\\-2/5&1/5\end{pmatrix}$

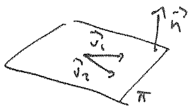
$[T]_b^b=[I]_b^e[T]_e^e[I]_e^b=\begin{pmatrix}3/5&1/5\\-2/5&1/5\end{pmatrix}\begin{pmatrix}7&-1\\-6&8\end{pmatrix}\begin{pmatrix}1&-1\\2&3\end{pmatrix}=\begin{pmatrix}5&0\\0&10\end{pmatrix}$

si osserva che $T:\begin{pmatrix}1\\2\end{pmatrix}\to5\begin{pmatrix}1\\2\end{pmatrix}$ e $\begin{pmatrix}-1\\3\end{pmatrix}\mapsto10\begin{pmatrix}-1\\3\end{pmatrix}$ come risulta dal calcolo!

(8) $[T]^b_b = \begin{pmatrix}1&9\\9&7\end{pmatrix}$ $[I]^b_e = \begin{pmatrix}3&5\\5&8\end{pmatrix}$ $\left(\begin{array}{cc|cc}3&5&1&0\\5&8&0&1\end{array}\right) \to \left(\begin{array}{cc|cc}1&5/3&1/3&0\\5&8&0&1\end{array}\right) \to \left(\begin{array}{cc|cc}1&5/3&1/3&0\\0&-1/3&-5/3&1\end{array}\right) \to \left(\begin{array}{cc|cc}1&5/3&1/3&0\\0&1&5&-3\end{array}\right) \to$

$\left(\begin{array}{cc|cc}1&0&-8&5\\0&1&5&-3\end{array}\right)$ $[I]^e_b = ([I]^b_e)^{-1} = \begin{pmatrix}-8&5\\5&-3\end{pmatrix}$

$[T]^e_e = [I]^b_e [T]^b_b [I]^e_b = \begin{pmatrix}3&5\\5&8\end{pmatrix}\begin{pmatrix}1&9\\9&7\end{pmatrix}\begin{pmatrix}-8&5\\5&-3\end{pmatrix} = \begin{pmatrix}-74&54\\-111&82\end{pmatrix}.$

(9) 
sia $R:\mathbb{R}^3 \to \mathbb{R}^3$ la riflessione rispetto al piano $x+2y+3z=0$

$\vec n = \begin{pmatrix}1\\2\\3\end{pmatrix}$ $\vec v_1 = \begin{pmatrix}1\\1\\-1\end{pmatrix}$ $\vec v_2 = \begin{pmatrix}-1\\2\\-1\end{pmatrix}$ allora $\vec v_1$ e $\vec v_2$ sono perpendicolari ad $\vec n$.

$(\vec v_1, \vec v_2, \vec n)$ è una base di $\mathbb{R}^3$:

$c_1\begin{pmatrix}1\\1\\-1\end{pmatrix} + c_2\begin{pmatrix}-1\\2\\-1\end{pmatrix} + c_3\begin{pmatrix}1\\2\\3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$ $\left(\begin{array}{ccc|c}1&-1&1&0\\1&2&2&0\\-1&-1&3&0\end{array}\right) \to \left(\begin{array}{ccc|c}1&-1&1&0\\0&3&1&0\\0&-2&4&0\end{array}\right) \to \left(\begin{array}{ccc|c}1&-1&1&0\\0&1&1/3&0\\0&-2&4&0\end{array}\right) \to \left(\begin{array}{ccc|c}1&-1&1&0\\0&1&1/3&0\\0&0&14/3&0\end{array}\right) \to \left(\begin{array}{ccc|c}1&-1&1&0\\0&1&1/3&0\\0&0&1&0\end{array}\right) \to$

$\left(\begin{array}{ccc|c}1&-1&0&0\\0&1&0&0\\0&0&1&0\end{array}\right) \to \left(\begin{array}{ccc|c}1&0&0&0\\0&1&0&0\\0&0&1&0\end{array}\right) \to \begin{cases}c_1=0\\c_2=0\\c_3=0\end{cases}$ i vettori sono linearmente indipendenti

Tre vettori linearmente indipendenti in un spazio di dimensione 3 generano lo spazio quindi formano una base.

sia $b = (\vec v_1, \vec v_2, \vec v_3)$ allora $R: \vec v_1 \to \vec v_1$ quindi $[R(\vec v_1)]_b = \begin{pmatrix}1\\0\\0\end{pmatrix}$, $R: \vec v_2 \to \vec v_2$ quindi $[R(\vec v_2)]_b = \begin{pmatrix}0\\1\\0\end{pmatrix}$

$R: \vec n \mapsto -\vec n$ quindi $[R(\vec n)]_b = \begin{pmatrix}0\\0\\-1\end{pmatrix}$. $[R]^b_b = \begin{pmatrix}1&0&0\\0&1&0\\0&0&-1\end{pmatrix}.$

$[I]^b_e = \begin{pmatrix}1&-1&1\\1&2&2\\-1&-1&3\end{pmatrix}$ $\left(\begin{array}{ccc|ccc}1&-1&1&1&0&0\\1&2&2&0&1&0\\-1&-1&3&0&0&1\end{array}\right) \to \left(\begin{array}{ccc|ccc}1&-1&1&1&0&0\\0&3&1&-1&1&0\\0&-2&4&1&0&1\end{array}\right) \to \left(\begin{array}{ccc|ccc}1&-1&1&1&0&0\\0&1&1/3&-1/3&1/3&0\\0&-2&4&1&0&1\end{array}\right) \to \left(\begin{array}{ccc|ccc}1&-1&1&1&0&0\\0&1&1/3&-1/3&1/3&0\\0&0&14/3&1/3&2/3&1\end{array}\right) \to$

$\left(\begin{array}{ccc|ccc}1&-1&1&1&0&0\\0&1&1/3&-1/3&1/3&0\\0&0&1&1/14&2/14&3/14\end{array}\right) \to \left(\begin{array}{ccc|ccc}1&-1&0&13/14&-2/14&3/14\\0&1&0&-5/14&4/14&-1/14\\0&0&1&1/14&2/14&3/14\end{array}\right) \to \left(\begin{array}{ccc|ccc}1&0&0&8/14&2/14&-4/14\\0&1&0&-5/14&4/14&-1/14\\0&0&1&1/14&2/14&3/14\end{array}\right)$

$[I]^e_b = \begin{pmatrix}8/14&2/14&-4/14\\-5/14&4/14&-1/14\\1/14&2/14&3/14\end{pmatrix}.$ $[R]^e_e = [I]^b_e [R]^b_b [I]^e_b = \begin{pmatrix}1&-1&1\\1&2&2\\-1&-1&3\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\0&0&-1\end{pmatrix}\begin{pmatrix}8/14&2/14&-4/14\\-5/14&4/14&-1/14\\1/14&2/14&3/14\end{pmatrix} =$

$\begin{pmatrix}12/14&-4/14&-6/14\\-4/14&6/14&-12/14\\-6/14&-12/14&-4/14\end{pmatrix}.$ (Fai il controllo che $\begin{pmatrix}1\\1\\-1\end{pmatrix} \mapsto \begin{pmatrix}1\\1\\-1\end{pmatrix}$, $\begin{pmatrix}-1\\2\\-1\end{pmatrix} \mapsto \begin{pmatrix}-1\\2\\-1\end{pmatrix}$ e $\begin{pmatrix}1\\2\\3\end{pmatrix} \mapsto -\begin{pmatrix}1\\2\\3\end{pmatrix}$)

(10) sia $\vec x = \begin{pmatrix}x_1\\x_2\end{pmatrix}$ e $\vec y = \begin{pmatrix}y_1\\y_2\end{pmatrix}$ i vettori della base che cerchiamo. sia $b = (\vec x, \vec y)$, allora

$\begin{pmatrix}1\\2\end{pmatrix} = 3\vec x + 5\vec y$ e $\begin{pmatrix}3\\4\end{pmatrix} = 2\vec x + 3\vec y$. Cioè $\begin{cases}3x_1+5y_1=1\\3x_2+5y_2=2\end{cases}$ e $\begin{cases}2x_1+3y_1=3\\2x_2+3y_2=4\end{cases}$

Risolvendo il sistema di 4 equazioni trovi $x_1=12$, $x_2=14$ $y_1=-7$, $y_2=-8$

cioè $\vec x = \begin{pmatrix}12\\14\end{pmatrix}$ $\vec y = \begin{pmatrix}-7\\-8\end{pmatrix}.$

(11) $c_1\begin{pmatrix}1\\0\end{pmatrix} + c_2\begin{pmatrix}a\\c\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$ $\left(\begin{array}{cc|c}1&a&0\\0&c&0\end{array}\right) \to \left(\begin{array}{cc|c}1&a&0\\0&1&0\end{array}\right) \to \left(\begin{array}{cc|c}1&0&0\\0&1&0\end{array}\right)$ i vettori sono linearmente indipendenti. Due vettori linearmente indipendenti in un spazio di dimensione 2 generano lo spazio. Quindi i vettori formano una base di $\mathbb{R}^2$.

$[I]^b_e = \begin{pmatrix}1&a\\0&c\end{pmatrix}$ $[I]^e_b = ([I]^b_e)^{-1} = \begin{pmatrix}1&-a/c\\0&1/c\end{pmatrix}$

$[T]^b_b = [I]^e_b [T]^e_e [I]^b_e = \begin{pmatrix}1&-a/c\\0&1/c\end{pmatrix}\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}1&a\\0&c\end{pmatrix} = \begin{pmatrix}0&bc-da\\1&a+d\end{pmatrix}$

controllo:

$\vec v_1 \mapsto \begin{pmatrix}a\\c\end{pmatrix}$ $\left[\begin{pmatrix}a\\c\end{pmatrix}\right]_b = \begin{pmatrix}0\\1\end{pmatrix}$

$\vec v_2 \mapsto \begin{pmatrix}a^2+bc\\ac+dc\end{pmatrix} = (bc-da)\begin{pmatrix}1\\0\end{pmatrix} + (a+d)\begin{pmatrix}a\\c\end{pmatrix}$ quindi $\left[\begin{pmatrix}a^2+bc\\ac+dc\end{pmatrix}\right]_b = \begin{pmatrix}bc-da\\a+d\end{pmatrix}$ ok!