CE394M: An introduction to plasticity

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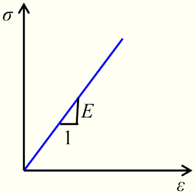
Overview

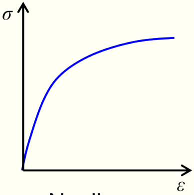
1. Constitutive modeling

2. Plasticity
   - Equations of plasticity

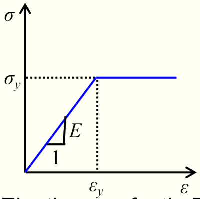
Constitutive law is the stress-strain relationship: $\sigma = f(\varepsilon) \rightarrow \sigma = \mathbf{D} \cdot \varepsilon$



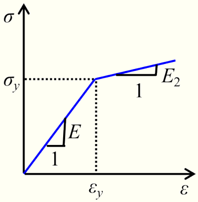
Linearly Elastic



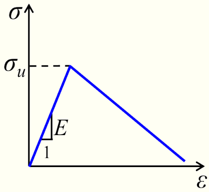
Nonlinear



Elastic – perfectly Plastic

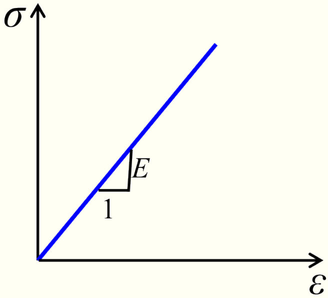


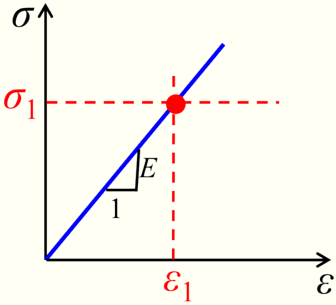
Elastic – Linear Hardening



Elastic – Linear Softening

$\sigma = \mathbf{D}^{\mathbf{el}} \cdot \varepsilon$ where $\mathbf{D}^{\mathbf{el}}$ is the elastic stiffness matrix.



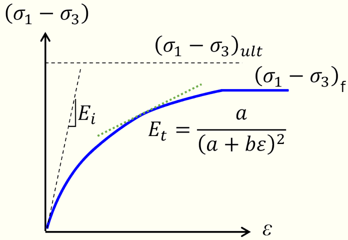


The knowledge of strain alone allows us to obtain the stress value.

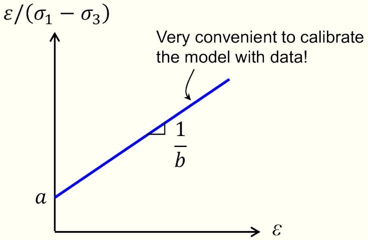
Hyperbolic model (Duncan and Chang., 1970)

$$(\sigma_1 - \sigma_3) = \frac{\varepsilon}{a + b\varepsilon}$$

$$a = \frac{1}{E_i} \quad b = \frac{1}{(\sigma_1 - \sigma_3)_{ult}}$$



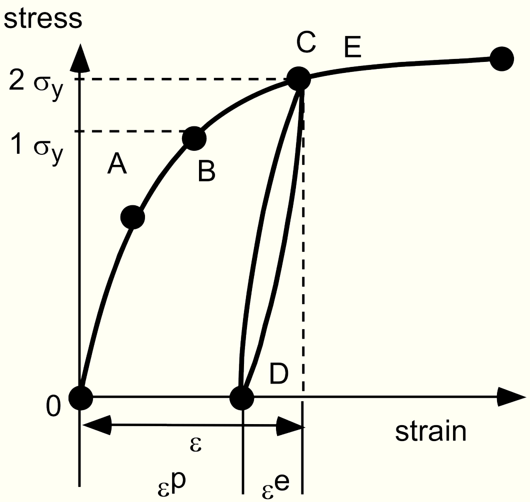
Hyperbolic stress-strain relationship



Transformation of hyperbolic relations

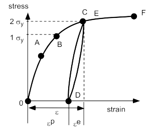
There is no physical meaning to $(\sigma_1 - \sigma_3)_{ult}$. The $(\sigma_1 - \sigma_3)_f$ is determined from the strength criteria $\tau = c + \sigma' \tan \phi'$ (drained) or $\tau = s_u$ (total stress undrained).

Classical plasticity



Specimen 1 at point 0 same as specimen 2 at point D, but yield stress of specimen 2 is greater than the yield stress of specimen 1 due to **plastic hardening**.

Classical plasticity



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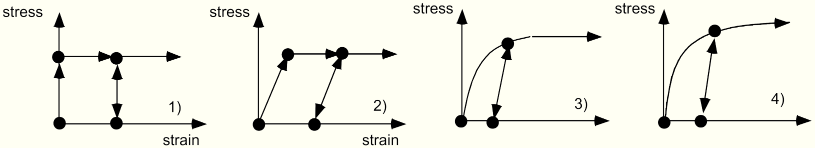
Uniaxial tension test on metal bar:

O→A	Linear elastic, reversible
A→B	Nonlinear elastic, reversible
B	Starts to yield
B→C	Nonlinear elasto-plastic, irreversible
C→D	Elastic with hysteresis
C→F	Nonlinear elasto-plastic, irreversible
F	Peak stress at failure

$\varepsilon = \varepsilon^e + \varepsilon^p$, where e is elastic and p is plastic.

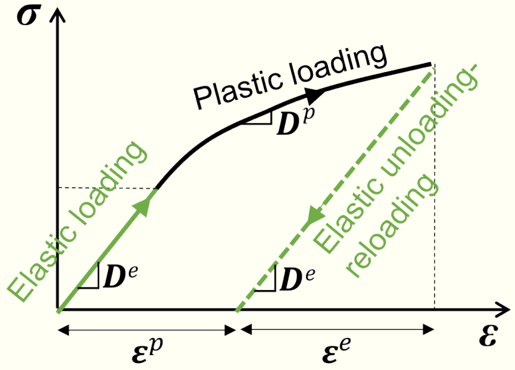
Classical plasticity

- **Plastic behavior**: The direction of plastic strains is governed by the current stress state σ. $d\sigma = f(d\varepsilon) \rightarrow d\sigma = \mathbf{D}^e \cdot d\varepsilon$.
- **Elastic behavior**: The direction of elastic strains is governed by the stress state increment $\delta\sigma$ direction.
- **Plastic models**
  - Rigid - perfectly plastic model - used in static limit equilibrium analysis (no elastic strain and no strain hardening / softening)
  - Linear elastic - perfectly plastic model (Drucker-Prager and Mohr Coulomb models)
  - Hybrid model (nonlinear elastic with perfectly plastic - Duncan and Chang)
  - Work (or strain) hardening plasticity model (Cam-Clay model)



Elasto-plastic materials

Main distinctive feature of elasto-plastic materials: "*irreversibility*" $\rightarrow$ **Plastic deformation** ε^p



$$\varepsilon = \varepsilon^e + \varepsilon^p \quad d\varepsilon = d\varepsilon^e + d\varepsilon^p$$

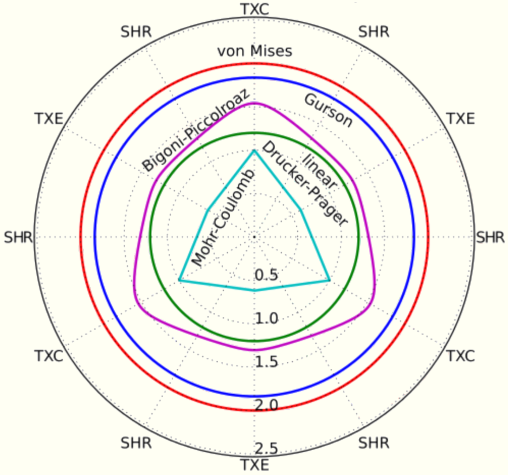
Basic concepts of classical plasticity

To formulate an elasto-plastic constitutive model we need:

1. **Elastic stress-strain relationship:** $\sigma = \mathbf{D}^e \varepsilon^e = \mathbf{D}^e(\varepsilon - \varepsilon^p)$. Describe elastic response.
2. **Yield function**: defines the condition for the onset of plastic strain. Depends on the stress state σ and state parameters (e.g., in MC they are cohesion and friction angle). $F(\sigma', Wp) = 0$.
3. **Plastic potential ($G(\sigma, Wp) = 0$)** defines the direction of plastic strains. Depends on stress state σ and state parameter (for e.g., is dilatancy in MC). Note that the direction of $d\varepsilon^p$ doesn't depend on $d\sigma$ but on the actual stress state σ. **Flow rule** $\varepsilon^p = \lambda(dG/d\sigma)$.
4. **Hardening rule / Hardening law (h)** defines how F changes with plastic strains. Yield function $F = f(\text{stress state}, W_p)$, where W_p is a function of plastic strains. Describes the evolution of state parameters depending on the plastic strain ε^p.

Yield functions

defines when plastic strains occur. If the material is isotropic, we can use the principal stresses to define the stress state.



Wikipedia

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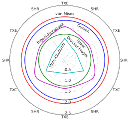
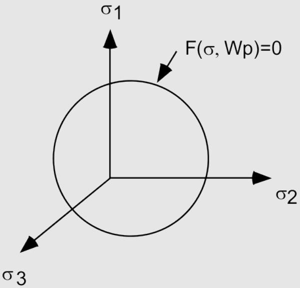


Wikipedia

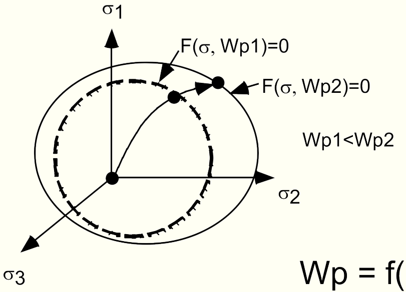
Yield function $F(\sigma, Wp) = 0$.

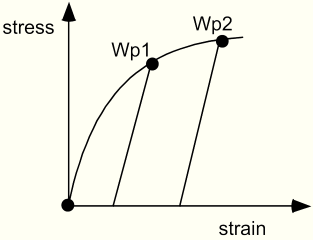
- if $F = 0$ under loading: yielding and plastic strains and in unloading: elastic strains.
- if $F < 0$ elastic domain.
- if $F > 0$ impossible.



Hardening law

How the threshold of yielding changes with plastic strain or how the yield function changes with plastic strain.





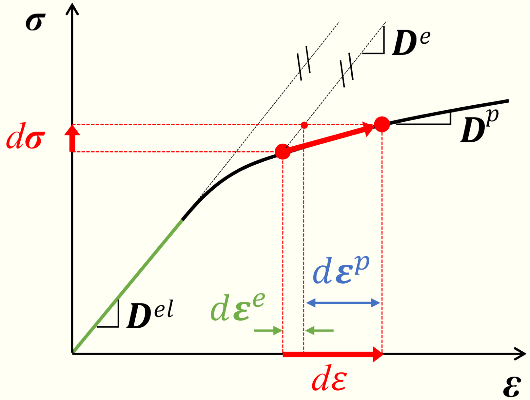
$$Wp = f(\varepsilon^p)$$

Equations of elasto-plasticity: 1. Stress-strain relation

Describes the incremental stress-strain relationship.

$$d\sigma = D^e \cdot d\varepsilon^e = D^e \cdot (d\varepsilon - d\varepsilon^p)$$

Where D^e is the elastic stiffness matrix. ^{e} denotes the elastic part.



Equations of elasto-plasticity: 2. Flow rule

- Specifies the direction of plastic strain increments at every yield stress state. It is very important because it controls the ratio of the volumetric and deviatoric components (e.g., dilatancy of the material).
- States that the plastic strain increments ($d\varepsilon^p$) are normal to the plastic potential surface (G).

$$d\varepsilon^p = d\lambda \cdot \vec{P}$$

  Note: typically $\vec{P}$ is **not** a unit vector, so $\vec{P}$ also controls the magnitude of $d\vec{\varepsilon_p}$ (in addition to the direction).
- In many cases, $\vec{P}$ is chosen as the gradient of a function g (if it exists) such that:

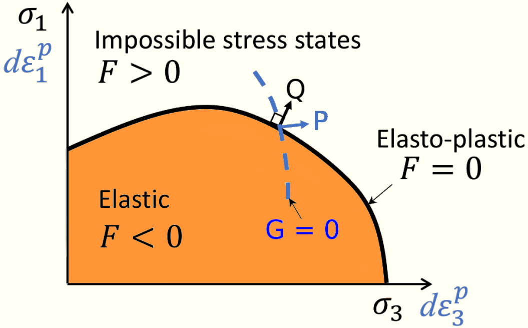
$$\vec{P} = \frac{\partial G}{\partial \sigma}$$

Equations of elasto-plasticity: 3. Consistency condition

States that the elastic limit is defined by the yield surface, enforcing points in plastic condition to **remain** on the yield surface.

$$dF = \left(\frac{\partial F}{\partial \sigma}\right)^T \cdot d\sigma + \frac{\partial F}{\partial Wp} \cdot dWp = 0$$

Where D^e is the elastic stiffness matrix. ^{e} denotes the elastic part.
E.g., strain hardening in tension for steel.



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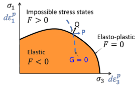


$$n = \frac{\partial F}{\partial \sigma}$$
$$m = \frac{\partial G}{\partial \sigma}$$

Drucker's postulate

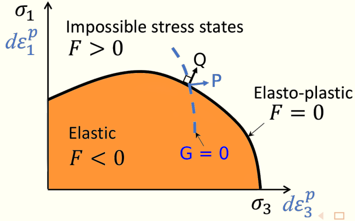
Drucker (1952) established that for a stable inelastic material in a closed stress cycle, a positive work must be done.

$$dW^p > 0 \rightarrow d\sigma \cdot d\varepsilon^p > 0$$

This requirement is satisfied if the normality condition is assumed (but is not the only solution). The gradient of the yield surface (i.e., normal to the surface)

$$\vec{Q} = \frac{\partial F}{\partial \sigma} = \vec{P}$$

It also imposes a constraint that the yield surface must be convex.



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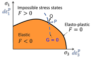


$$Q = \frac{\partial F}{\partial \sigma}$$
$$P = \frac{\partial G}{\partial \sigma}$$

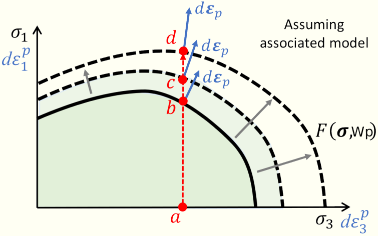
The consistency condition can be written as:

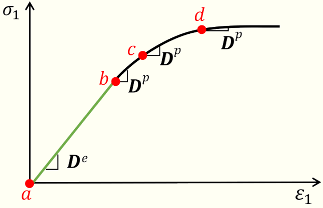
$$
\begin{aligned}
dF &= \left(\frac{\partial F}{\partial \sigma}\right)^T \cdot d\sigma + \frac{\partial F}{\partial Wp} \cdot dWp && = 0 \\
&= \left(\frac{\partial F}{\partial \sigma}\right)^T \cdot d\sigma + \frac{\partial F}{\partial Wp} \cdot \frac{\partial Wp}{\partial \varepsilon^p} \cdot d\varepsilon^p && = 0 \\
&= \left(\frac{\partial F}{\partial \sigma}\right)^T \cdot d\sigma + \frac{\partial F}{\partial Wp} \frac{\partial Wp}{\partial \varepsilon^p} \cdot \frac{\partial G}{\partial \sigma} d\lambda && = 0 \\
dF &= \left(\frac{\partial F}{\partial \sigma}\right)^T \cdot d\sigma - H d\lambda && = 0.
\end{aligned}
$$

if $H > 0$: Hardening, if $H = 0$: perfect plasticity, if $H < 0$: softening.

Hardening v Softening

Linear elastic – hardening plastic material $H > 0$





Linear elastic – softening plastic material $H < 0$