CE394M: An introduction to constitutive modeling

Krishna Kumar

University of Texas at Austin

krishnak@utexas.edu

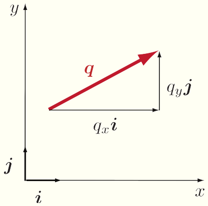
March 10, 2019

Overview

1. Review of vector calculus
2. Definition of stress and strain tensors

The objective of constitutive modelling is the determination of stiffness tensor **C**, a relation between stress and strain tensors.

A vector is expressed in terms of its components and the unit vectors in the $x-$ and $y-$ directions.



$$\mathbf{q} = q_x\mathbf{i} + q_y\mathbf{j}$$

where q_x is the $x-$component and q_y is the $y-$component and i and j are basis vectors (are unit length).

Scalar product:

$$\mathbf{q}\cdot\mathbf{r} = \mathbf{q}^T\mathbf{r} = \begin{bmatrix} q_x & q_y \end{bmatrix}\begin{bmatrix} r_x \\ r_y \end{bmatrix} = q_x r_x + q_y r_y$$

Vector calculus

Grad: If the del operator acts on a scalar field, say temperature $T(x, y)$, it produces a vector that points in the direction of the steepest slope.

$$\nabla \mathbf{T} = \begin{bmatrix} \frac{\partial T}{\partial x} \\ \frac{\partial T}{\partial y} \end{bmatrix}$$

Divergence: The scalar product of the del operator with a vector field q gives the divergence

$$div\mathbf{q} = \nabla \cdot \mathbf{q} = \frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y}$$

Notice the divergence of a vector field is a scalar.
Divergence theorem

$$\int_\Omega div\mathbf{q} d\Omega = \oint_\Gamma \mathbf{q} \cdot \mathbf{n} d\Gamma$$

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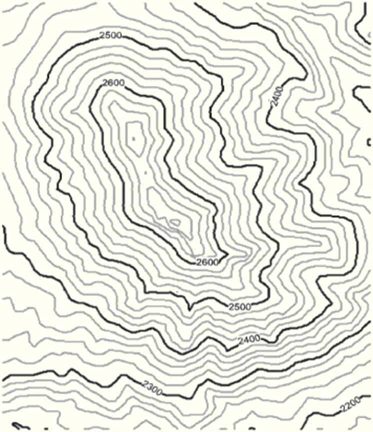
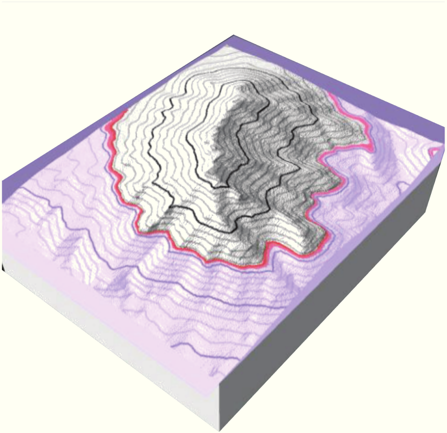
Gauss's theorem or Divergence theorem is a result that relates the flow (that is, flux) of a vector field through a surface to the behavior of the tensor field inside the surface.

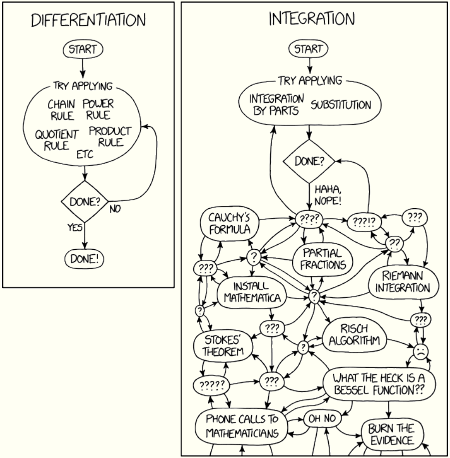
Vector calculus

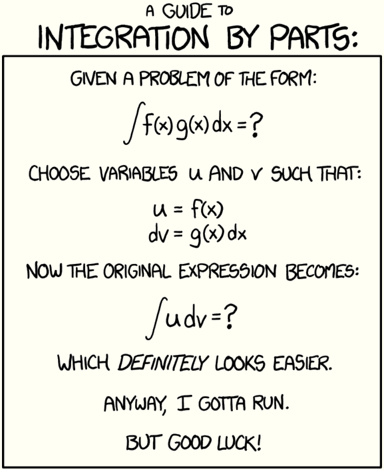




Contour map for a terrain (left) and the associated three-dimensional model (right). If T is interpreted as the height, the vector ΔT points in the direction of the steepest slope.

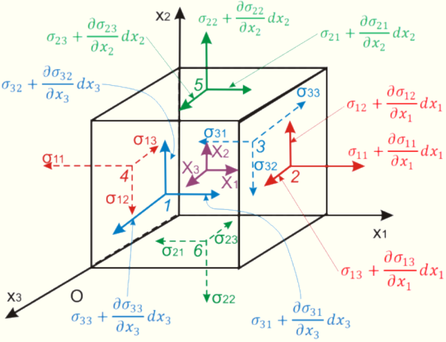
Differention and Integration

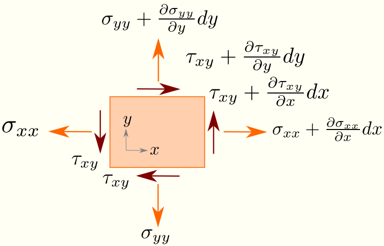




XKCD - Randall Munroe

Stresses





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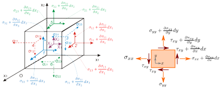
- Equilibrium of moments require $sigma_{xz} = \sigma_{zx}, \ldots$ etc.,

Stresses



- Stress state at a point is defined by σ_{ij} in a frame of reference.
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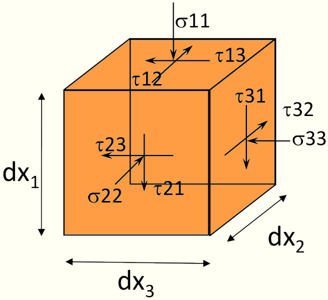
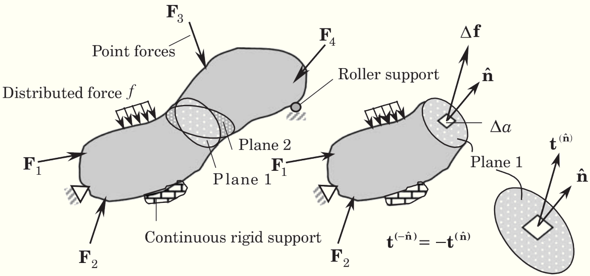
- σ_{xz} stress acting on plane perpendicular to axis x and in the direction of z
- σ_{xx} stress acting on plane perpendicular to axis x and in the direction of x

Stresses

- 9 components of the stress tensor.
- 6 stresses: $\sigma_{11}, \sigma_{22}, \sigma_{33}, \tau_{12}, \tau_{23}, \tau_{31}$.
- $\tau_{21} = -\tau_{12}, \tau_{32} = -\tau_{23}, \tau_{13} = -\tau_{31}$
- Compression is positive
- Shear stress, anti-clockwise is positive
- In order to write the components in a more concise way we can use the indices notation: σ_{ij} (use $i = 1, 2, 3$ and $j = 1, 2, 3$)
- Correspondence from x, y, z to $1, 2, 3$ (e.g., $\sigma_{11} = \sigma_{xx}, \sigma_{12} = \sigma_{xy}$)



Stress vector on a plane



Stress vector on a plane normal to $\hat{\mathbf{n}}$ (Reddy., 2008)

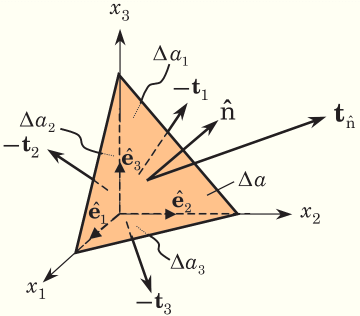
If we denote by $\Delta(\mathbf{f}\hat{\mathbf{n}})$ the force on a small area $\hat{\mathbf{n}}$ located at the position x, the stress vector can be defined:

$$\mathbf{t}(\hat{\mathbf{n}}) = \lim_{\Delta a \to 0} \frac{\Delta \mathbf{f}(\hat{\mathbf{n}})}{\Delta a}$$

Cauchy stress is the true stress, that is, stress in the deformed configuration.

Cauchy stress tensor

To establish the relationship between $\mathbf{t}$ and $\hat{\mathbf{n}}$ we now set up an infinitesimal tetrahedron in Cartesian coordinates:



If $-\mathbf{t}_1, -\mathbf{t}_2, -\mathbf{t}_3$ and $\mathbf{t}$ denote the stress vectors in the outward directions on the faces of the infinitesimal tetrahedron whose areas are $\Delta a_1, \Delta a_2, \Delta a_3$, and Δa, respectively. Δv is the volume of the tetrahedron, ρ the density, f the body force per unit mass, and $\mathbf{a}$ the acceleration.

Cauchy stress tensor

we have by Newton's second law for the mass inside the tetrahedron:

$$\mathbf{t}\Delta a - \mathbf{t}_1\Delta a_1 - \mathbf{t}_1\Delta a_1 - \mathbf{t}_1\Delta a_1 + \rho\Delta v\mathbf{f} = \rho\Delta v\mathbf{a}$$

Since the total vector area of a closed surface is zero (gradient theorem):

$$\Delta a\hat{\mathbf{n}} - \Delta a_1\hat{\mathbf{e}}_1 - \Delta a_2\hat{\mathbf{e}}_2 - \Delta a_3\hat{\mathbf{e}}_3 = \mathbf{0}$$

$$\Delta a_1 = (\hat{\mathbf{n}}\cdot\hat{\mathbf{e}}_1)\Delta a, \quad \Delta a_2 = (\hat{\mathbf{n}}\cdot\hat{\mathbf{e}}_2)\Delta a, \quad \Delta a_3 = (\hat{\mathbf{n}}\cdot\hat{\mathbf{e}}_3)\Delta a.$$

The volume Δv can be expressed as: $\Delta v = (\Delta h/3)\Delta a$
where Δh is the perpendicular distance from the origin to the slant face.

$$\mathbf{t} = (\hat{\mathbf{n}}\cdot\hat{\mathbf{e}}_1)\mathbf{t}_1 + (\hat{\mathbf{n}}\cdot\hat{\mathbf{e}}_2)\mathbf{t}_2 + (\hat{\mathbf{n}}\cdot\hat{\mathbf{e}}_3)\mathbf{t}_3 + \rho\frac{\Delta h}{3}(\mathbf{a} - \mathbf{f})$$

Cauchy stress tensor

In the limit when the tetrahedron shrinks to a point $\Delta h \to 0$:

$$\mathbf{t} = (\hat{\mathbf{n}} \cdot \hat{\mathbf{e}}_1)\mathbf{t}_1 + (\hat{\mathbf{n}} \cdot \hat{\mathbf{e}}_2)\mathbf{t}_2 + (\hat{\mathbf{n}} \cdot \hat{\mathbf{e}}_3)\mathbf{t}_3 = (\hat{\mathbf{n}} \cdot \hat{\mathbf{e}}_i)\mathbf{t}_i$$

where the summation convention is used.

$$\mathbf{t} = \hat{\mathbf{n}} \cdot (\hat{\mathbf{e}}_1\mathbf{t}_1 + \cdot\hat{\mathbf{e}}_2\mathbf{t}_2 + \hat{\mathbf{e}}_3\mathbf{t}_3).$$

The terms in the parenthesis is the **stress tensor** σ:

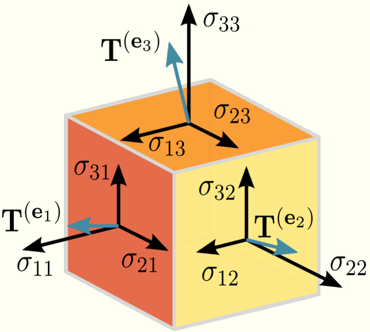
$$\sigma \equiv \hat{\mathbf{e}}_1\mathbf{t}_1 + \cdot\hat{\mathbf{e}}_2\mathbf{t}_2 + \hat{\mathbf{e}}_3\mathbf{t}_3$$

The stress tensor is a property of the medium that is independent of the $\hat{\mathbf{n}}$

$$\mathbf{t}(\hat{\mathbf{n}}) = \hat{\mathbf{n}}\sigma = \sigma^T\hat{\mathbf{n}}.$$

The stress vector $\mathbf{t}$ represents the vectorial stress on a plane whose normal is $\hat{\mathbf{n}}$. σ is the *Cauchy stress tensor* defined to be the *current force per unit deformed area*. In Cartesian component, the Cauchy formula is: $t_i = n_j\sigma_{ji}$.

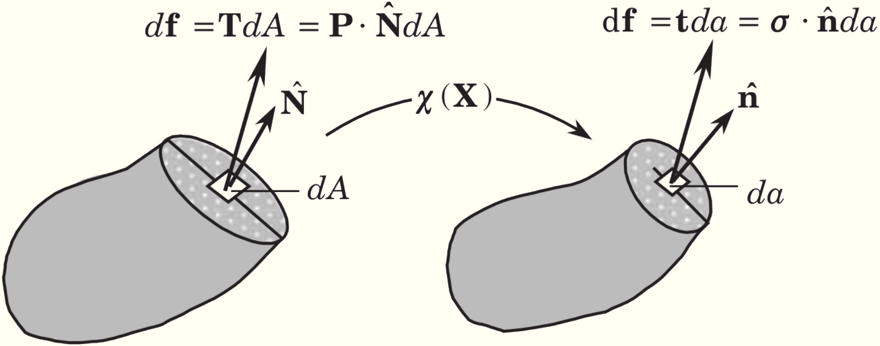
Cauchy stress tensor



Wikipedia

The Cauchy stress tensor σ, which takes a directional unit vector e as input and maps it to the stress vector $T(e)$, which is the force (per unit area) exerted by material on the negative side of the plane orthogonal to e against the material on the positive side of the plane, thus expressing a relationship between these two vectors

Cauchy stress vs Piola-Kirchoff stress



An introduction to continuum mechanics - J. N. Reddy (2008)

- The first Piola–Kirchhoff stress tensor, also referred to as the *nominal stress tensor*, or *Lagrangian stress tensor*, gives the current force per unit undeformed area.