CE394M: Critical State and Cam-Clay

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April 28, 2019

Overview

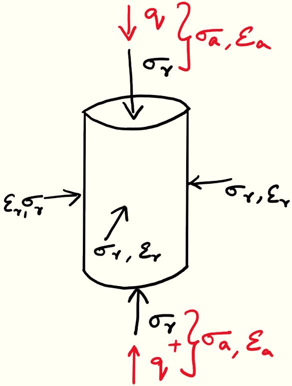
Critical State Soil Mechanics

Roscoe et al., (1958), Schofield & Worth (1968), Wood (1990):

- Provides a conceptual framework in which to interpret stress-strain-strength-volumetric strain response of soil.
- Started as a qualitative, rather than a mathematical model
- A unified framework of known or observed soil responses: drained / undrained / etc

Critical state variables

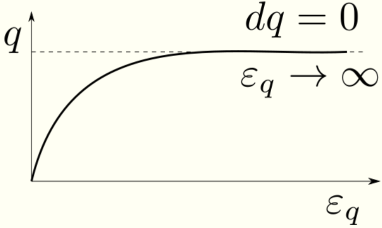
- Mean stress: $p' = \frac{\sigma'_a + 2\sigma'_r}{3} = p - u$.
- Deviatoric stress: $q = \sigma'_a - \sigma'_r = \sigma_a - \sigma_r$
- Specific volume: $v = \frac{V_T}{V_s} = \frac{V_s + V_v}{V_s} = 1 + e$.

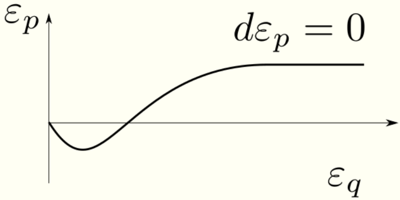


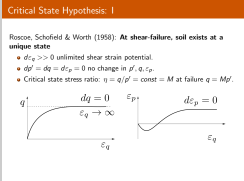
Critical State Hypothesis: I

Roscoe, Schofield & Worth (1958): **At shear-failure, soil exists at a unique state**

- $d\varepsilon_q >> 0$ unlimited shear strain potential.
- $dp' = dq = d\varepsilon_p = 0$ no change in p', q, ε_p.
- Critical state stress ratio: $\eta = q/p' = const = M$ at failure $q = Mp'$.



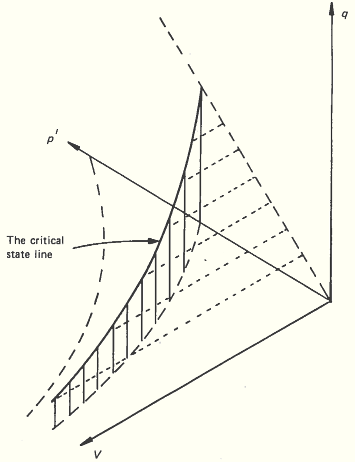




Soil is sheared to a point where stresses are stationary ($dq = dp' = 0$) with no futher change in volume ($d\varepsilon_p = 0$), unlimited shear strains ($d\varepsilon_q >> 0$) and q/p' has a fixed value: **critical state**.
M can be related to phi': $M = \frac{6 \sin \phi'}{3 - \sin \phi'}$.

Critical State Hypothesis: II

Critical state is a function of q, p', v.



The CSL (p', v, q) space is given by the intersection of two planes: $q = Mp'$ and a cruved vertical plane $v = \Gamma - \lambda \ln p'$

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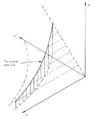

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Critical state curve connecting critical state points:

- Crticial state line
- Defined in 3D but we'll look at projections into $q - p'$ and $v - p'$ space

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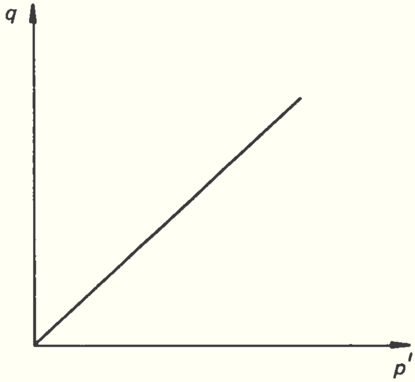

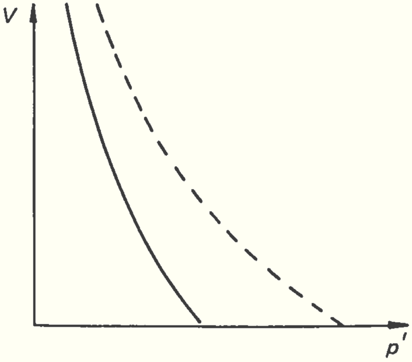

The CSL in (a) (p', q) plot and (b) (p', v) plot (isotropic normal compression line is shown in dashed)

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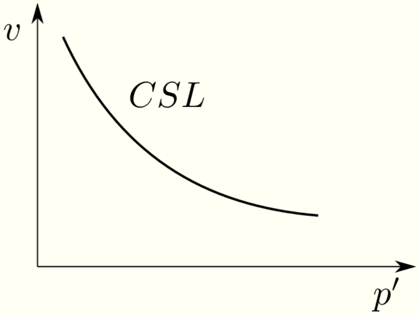

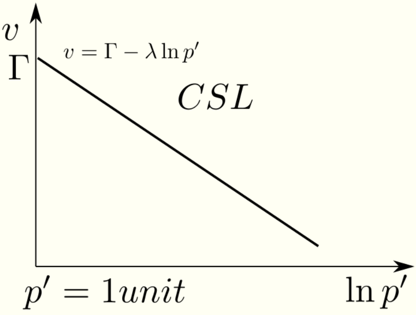

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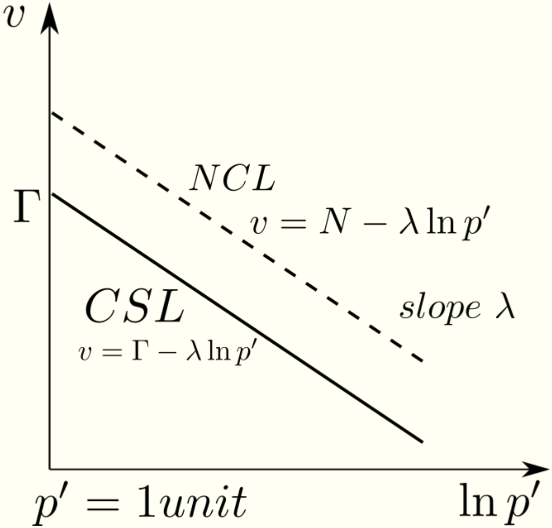
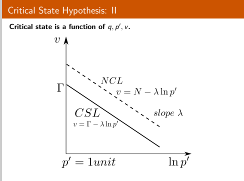
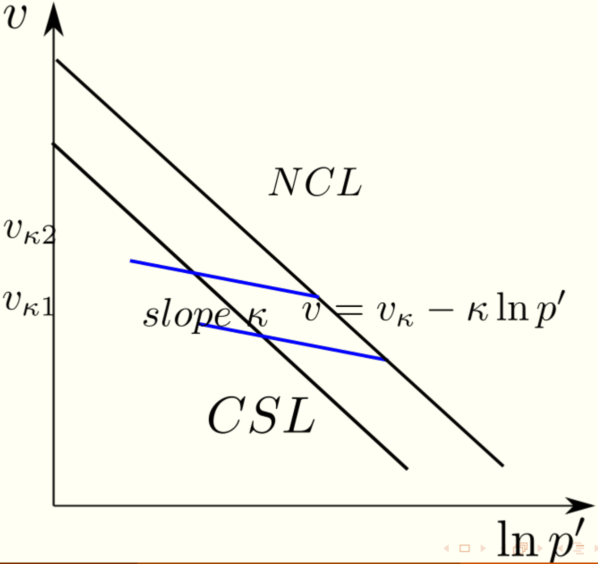
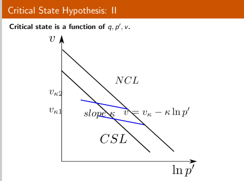




Isotropic virgin compression line (VCL) $\eta = 0$. NCL is parallel to CSL. VCL is $\eta = 0$, while CSL $\eta = M$. Oedometer falls between VCL and CSL at a constant η: $0 < \eta < M$.

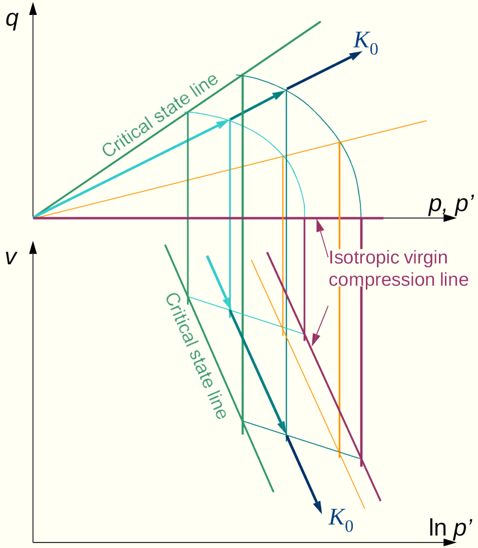
Critical state is a function of q, p', v.



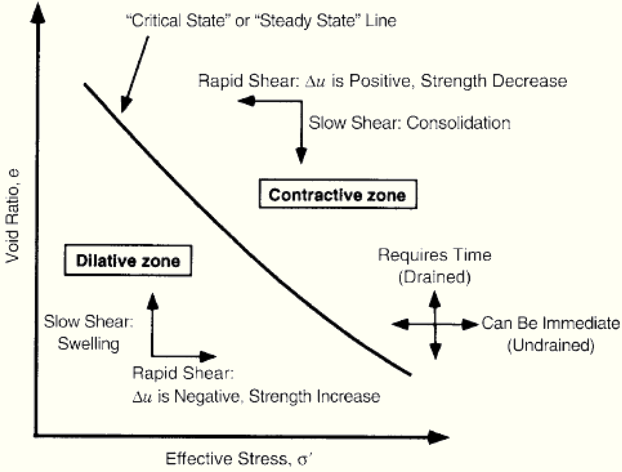


v_κ depends on which κ line you are on. $\kappa \neq c_r$ and $\lambda \neq C_c$

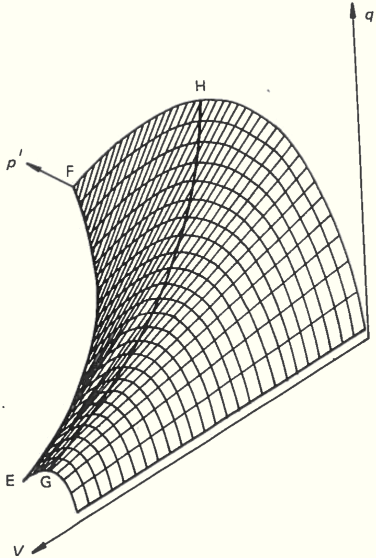
Stress paths $\sigma_3'/\sigma_1' = K_c = const$



Clay behavior



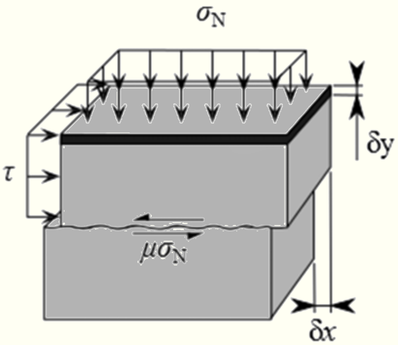
Critical state boundary surface



Summary of critical state behavior

- Can only traverse NCL in one direction
- Can traverse RCL (κ-line) in both directions
- To move from one κ-line to another must move along NCL. Hence, plastic volumetric strains must occur.
- Critical state line is **NOT** a yield surface. It's where it's going but a lot of plastic straining is needed to get there. (if $CSL = F = 0$) then with associative flow rule $d\varepsilon_p^p \neq 0$ at critical state. Real F is horizontal at critical state.

Stress - dilatancy theory (Taylor, 1948)



Work in friction and dilation:

$$\tau dx - \sigma'_n dy = \mu \sigma'_n dx$$

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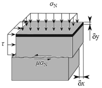

Work in friction and dilation:

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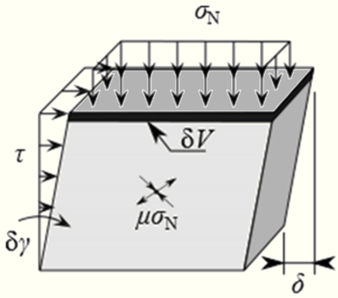
Taylor (1948) proposed a stress-dilatancy theory based on the work balance equation: The external work corresponds to the product of the measured displacements and forces (assuming that the elastic deformation is negligible). The internal work corresponds to the frictional force.

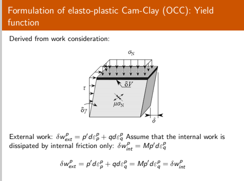
Formulation of elasto-plastic Cam-Clay (OCC): Yield function

Derived from work consideration:



External work: $\delta w^p_{ext} = p' d\varepsilon^p_p + q d\varepsilon^p_q$ Assume that the internal work is dissipated by internal friction only: $\delta w^p_{int} = Mp' d\varepsilon^p_q$

$$\delta w^p_{ext} = p' d\varepsilon^p_p + q d\varepsilon^p_q = Mp' d\varepsilon^p_q = \delta w^p_{int}$$



This dissipation function can be regarded simply as generalisation of Taylor's equation. It should be noted that both Taylor's equation and Cam-Clay dissipation function equation assume that when there is some combination of volumechange (dy or $\partial\varepsilon_v$) and of shear distortion (dx or $\partial\varepsilon_s$) it is the shear strain that determines the dissipation rate. The dilation or volume change is a geometrical consequence of interlocking, and does not appear explicitly in the dissipation function.

Cam-Clay (OCC): Stress dilatancy relation

$$p' d\varepsilon_p^p + q d\varepsilon_q^p = M p' d\varepsilon_q^p$$

Rearranging the terms (divide by $p' d\varepsilon_q^p$):

$$\frac{d\varepsilon_p^p}{d\varepsilon_q^p} = M - \frac{q}{p'} = M - \eta$$

Where $\eta = q/p'$ is defined as the stress-ratio. This equation is known as the dilatancy expression and expresses the ratio in plastic volumetric and deviatoric components.

$$q/p < M: \quad \frac{d\varepsilon_p^p}{d\varepsilon_q^p} > 0 \rightarrow \quad d\varepsilon_p^p > 0 \quad \text{Contractive response}$$

$$q/p > M: \quad \frac{d\varepsilon_p^p}{d\varepsilon_q^p} > 0 \rightarrow \quad d\varepsilon_p^p < 0 \quad \text{Dilative response}$$

$$q/p = M: \quad d\varepsilon_p^p = 0 \quad \text{No volume change}$$

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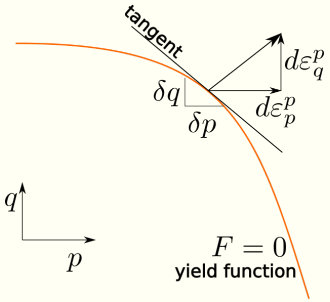
The critical state is defined by an absence of volume change or, in other words, a nil dilatancy conditions. Therefore, at critical state, the stress-dilatancy rule yields to the critical state condition $\eta = M$.

Cam-Clay (OCC): flow-rule

The original idea was very simple. The yield locus must be such that each associated flow rule $(\delta\varepsilon_p, \delta\varepsilon_q)$ would be orthogonal to the tangent to the yield locus.

$$\frac{\delta\varepsilon_p}{\delta\varepsilon_q} = -\frac{\delta\varepsilon_q}{\delta\varepsilon'_p}$$



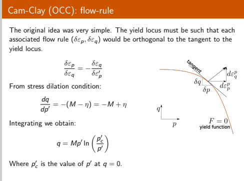
From stress dilation condition:

$$\frac{dq}{dp'} = -(M - \eta) = -M + \eta$$

Integrating we obtain:

$$q = Mp' \ln\left(\frac{p'_c}{p'}\right)$$

Where p'_c is the value of p' at $q = 0$.



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Original Cam-Clay integration

$$\eta = q/p \quad \rightarrow d\eta = \frac{\partial\eta}{\partial q}dq + \frac{\partial\eta}{\partial p'}dp'$$

Which gives:

$$d\eta = \frac{dq}{p} - \frac{q}{p^2}dp \rightarrow \quad dq = pd\eta + \eta dp$$

We know from flow rule and orthogonality: $dq = dp(-M + \eta)$
Equating the above 2 equations:

$$dp = pd\eta + \eta dp = dp(-M + \eta)$$

$$pd\eta = -Mdp \rightarrow \quad deta = -M\frac{dp}{p}$$

Integrating this expression we obtain:

$$\eta = -M \ln p + C \tag{1}$$

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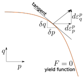

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Where p'_c is the value of p' at $q = 0$.

Original Cam-Clay integration

$$\eta = -M \ln p + C \qquad (2)$$

To find the constants, for $\eta = 0$, we get $p = p_c$:

$$0 = -M \ln p_c + C \quad C = M \ln p_c$$

Which gives:

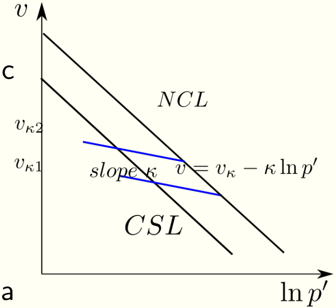
$$\eta = M \ln p_c - M \ln p$$
$$q/p = M \ln\left(p_c/p\right)$$

Yield function:

$$F = q - Mp' \ln(p'_c/p') = 0$$

Cam-Clay (OCC): Elastic properties

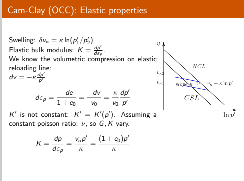
Swelling: $\delta v_\kappa = \kappa \ln(p'_1/p'_2)$
Elastic bulk modulus: $K = \frac{dp'}{d\varepsilon_p}$.
We know the volumetric compression on elastic reloading line:
$dv = -\kappa \frac{dp'}{p'}$



$$d\varepsilon_p = \frac{-de}{1+e_0} = \frac{-dv}{v_0} = \frac{\kappa}{v_0}\frac{dp'}{p'}$$

K' is not constant: $K' = K'(p')$. Assuming a constant poisson ratio: ν, so G, K vary.

$$K = \frac{dp}{d\varepsilon_p} = \frac{v_o p'}{\kappa} = \frac{(1+e_0)p'}{\kappa}$$



Observation:

- Stiffness K increases with p': correct.
- Stiffness increases with void ratio (not right)!

Note: The original derivation assumed that there were no recoverable (elastic) shear strains so $G = \infty$. We can find the stress-strain relationships for a single element in this case, but for a finite element forumulation we need to have a finite G^e. So there are two options:

- Define $G = f(e, p')$.
- Use a constant "elastic" Poisson ratio. Ratio between the shear and bulk modulus is constant. $2G/K = const$.

The first alternative has the shortcoming that depending on the choice of G we may have unreasonable values of the "elastic" Poisson's ratio. I prefer the second choice.

Cam-Clay (OCC): Hardening law

We need to define how the yield surface hardens as plastic work is being performed. Only "*memory*" parameter in our yield surface is the size: p'_c. From the isotropic NCL:

$$d\varepsilon_p = \frac{-dv}{v} = \frac{-de}{1+e} = \frac{+\lambda}{v}\frac{dp'_c}{p'_c}$$

But the increment in elastic volumetric strain is:

$$d\varepsilon_p^e = \left(\frac{-dv}{v}\right)^{elastic} = +\frac{\kappa}{v}\left(\frac{dp'_c}{p'_c}\right)$$

Therefore the increment of p_c can be related to the increment of plastic volumetric strain:

$$d\varepsilon_p^p = d\varepsilon_p - d\varepsilon_p^e = (\lambda - \kappa)\left(\frac{dp'_c}{p'_c}\right) \rightarrow \quad dp'_c = \left(\frac{v \cdot p'_c}{(\lambda - \kappa)}\right) \cdot d\varepsilon_p^p$$

Cam-Clay (OCC): Hardening law

We have seen that the hardening law:

$$H = -\left(\frac{\partial F}{\partial Wp}\right)\left(\frac{\partial Wp}{\partial \varepsilon^p}\right)^T \cdot \frac{\partial G}{\partial \sigma}$$

W_p is the vector of memory parameters. In our case, the CC model has only one parameter: p'_c and it's variation is only a function of the plastic volumetric strain. So:

$$H = -\left(\frac{\partial F}{\partial p'_c}\right)\left(\frac{\partial p'_c}{\partial \varepsilon^p}\right)^T \cdot \frac{\partial G}{\partial \sigma}$$

Cam-Clay (OCC): Hardening law

We know:

$$\frac{\partial F}{\partial p'_c} = -Mp'/p'_c$$

$$\frac{\partial p'_c}{\partial \varepsilon^p} = \frac{v}{(\lambda - \kappa)} p'_c$$

$$\frac{\partial G}{\partial \sigma} = P_p = Q_p = M - \eta$$

$$\boxed{H = -\left(\frac{\partial F}{\partial p'_c}\right)\left(\frac{\partial p'_c}{\partial \varepsilon^p}\right)^T \cdot \frac{\partial G}{\partial \sigma} = M\frac{(M-\eta)}{(\lambda - \kappa)} \cdot (1+e) \cdot p'}$$