CE394M: An introduction to plasticity

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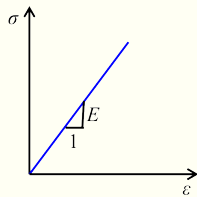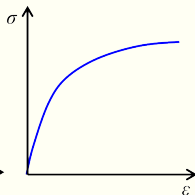
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April 9, 2019

Overview

1 Constitutive modeling

2 Classical plasticity
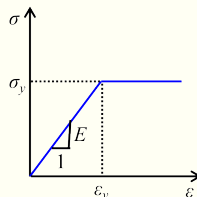   - Equations of plasticity

Constitutive law

Constitutive law is the stress-strain relationship: $\sigma = f(\varepsilon) \rightarrow \sigma = \mathbf{D} \cdot \varepsilon$
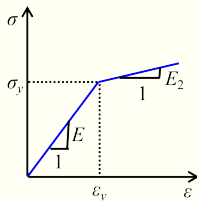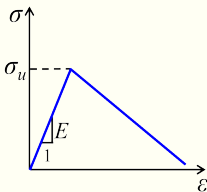

Linearly Elastic

Nonlinear
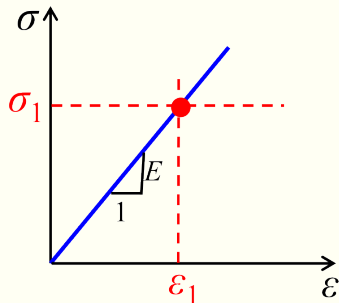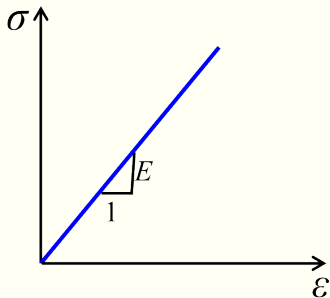
Elastic – perfectly Plastic

Elastic – Linear Hardening

Elastic – Linear Softening

Isotropic linear elasticity

$\sigma = \mathbf{D}^{\mathbf{el}} \cdot \varepsilon$ where $\mathbf{D}^{\mathbf{el}}$ is the elastic stiffness matrix.

The knowledge of strain alone allows us to obtain the stress value.

Hyperbolic model (Duncan and Chang., 1970)

$$(\sigma_1 - \sigma_3) = \frac{\varepsilon}{a + b\varepsilon}$$

$$a = \frac{1}{E_i} \quad b = \frac{1}{(\sigma_1 - \sigma_3)_{ult}}$$
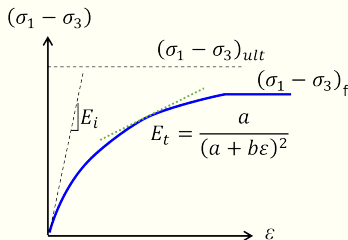
Hyperbolic stress-strain relationship
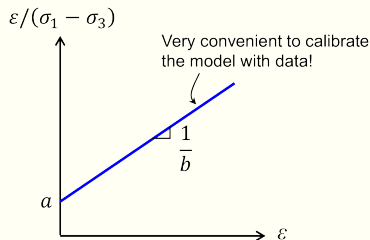
Transformation of hyperbolic relations

There is no physical meaning to $(\sigma_1 - \sigma_3)_{ult}$. The $(\sigma_1 - \sigma_3)_f$ is determined from the strength criteria $\tau = c + \sigma' \tan \phi'$ (drained) or $\tau = s_u$ (total stress undrained).

Classical plasticity
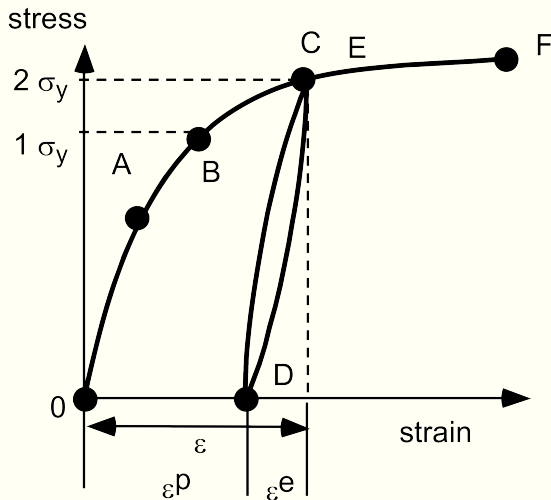
Specimen 1 at point 0 same as specimen 2 at point D, but yield stress of specimen 2 is greater than the yield stress of specimen 1 due to **plastic hardening**.

Classical plasticity
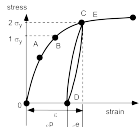
Specimen 1 at point 0 same as specimen 2 at point D, but yield stress of specimen 2 is greater than the yield stress of specimen 1 due to **plastic hardening**.

Uniaxial tension test on metal bar:

O→A	Linear elastic, reversible
A→B	Nonlinear elastic, reversible
B	Starts to yield
B→C	Nonlinear elasto-plastic, irreversible
C→D	Elastic with hysteresis
C→F	Nonlinear elasto-plastic, irreversible
F	Peak stress at failure

$\varepsilon = \varepsilon^e + \varepsilon^p$, where e is elastic and p is plastic.

Classical plasticity

- **Plastic behavior**: The direction of plastic strains is governed by the current stress state σ. $d\sigma = f(d\varepsilon) \rightarrow d\sigma = \mathbf{D}^e \cdot d\varepsilon$.
- **Elastic behavior**: The direction of elastic strains is governed by the stress state increment $\delta\sigma$ direction.
- **Plastic models**
  - Rigid - perfectly plastic model - used in static limit equilibrium analysis (no elastic strain and no strain hardening / softening)
  - Linear elastic - perfectly plastic model (Drucker-Prager and Mohr Coulomb models)
  - Hybrid model (nonlinear elastic with perfectly plastic - Duncan and Chang)
  - Work (or strain) hardening plasticity model (Cam-Clay model)
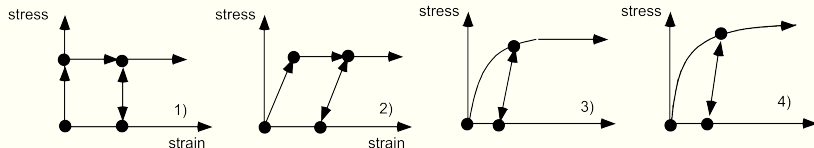
Elasto-plastic materials

Main distinctive feature of elasto-plastic materials: "*irreversibility*" $\rightarrow$
Plastic deformation ε^p
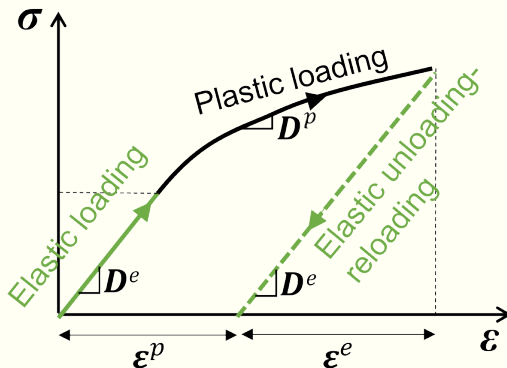
$$\varepsilon = \varepsilon^e + \varepsilon^p \quad d\varepsilon = d\varepsilon^e + d\varepsilon^p$$

Basic concepts of classical plasticity

To formulate an elasto-plastic constitutive model we need:

1. **Elastic stress-strain relationship:** $\sigma = \mathbf{D}^e \varepsilon^e = \mathbf{D}^e(\varepsilon - \varepsilon^p)$.
   Describe elastic response.

2. **Yield function**: defines the condition for the onset of plastic strain. Depends on the stress state σ and state parameters (e.g., in MC they are cohesion and friction angle). $F(\sigma', Wp) = 0$.

3. **Plastic potential ($G(\sigma, Wp) = 0$)** defines the direction of plastic strains. Depends on stress state σ and state parameter (for e.g., is dilatancy in MC). Note that the direction of $d\varepsilon^p$ doesn't depend on $d\sigma$ but on the actual stress state σ. **Flow rule** $\varepsilon^p = \lambda(dG/d\sigma)$.

4. **Hardening rule / Hardening law (h)** defines how F changes with plastic strains. Yield function $F = f(\text{stress state}, W_p)$, where W_p is a function of plastic strains. Describes the evolution of state parameters depending on the plastic strain ε^p.

Yield functions

defines when plastic strains occur. If the material is isotropic, we can use the principal stresses to define the stress state.

Wikipedia

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the principal stresses to define the stress state.
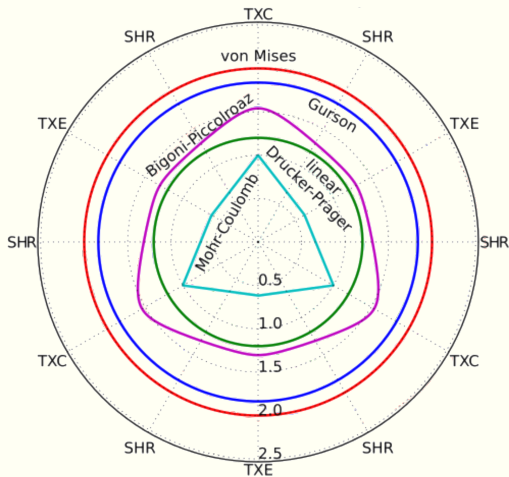
Wikipedia
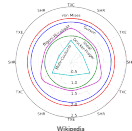
Yield function $F(\sigma, Wp) = 0$.

- if $F = 0$ under loading: yielding and plastic strains and in unloading: elastic strains.

- if $F < 0$ elastic domain.

- if $F > 0$ impossible.
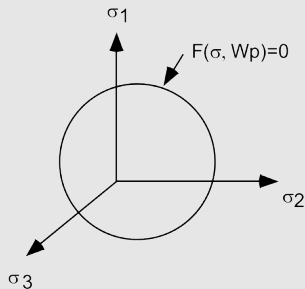
How the threshold of yielding changes with plastic strain or how the yield function changes with plastic strain.

σ_1

F(σ, Wp1)=0

F(σ, Wp2)=0

Wp1<Wp2

σ_2

σ_3

stress

Wp1

Wp2

strain

$$Wp = f(\varepsilon^p)$$
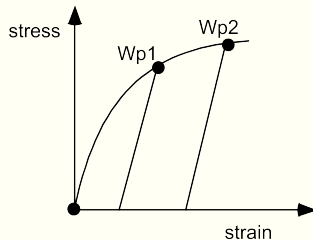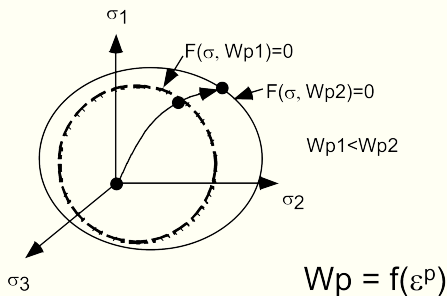
Describes the incremental stress-strain relationship.

$$d\sigma = D^e \cdot d\varepsilon^e = D^e \cdot (d\varepsilon - d\varepsilon^p)$$

Where D^e is the elastic stiffness matrix. ^{e} denotes the elastic part.
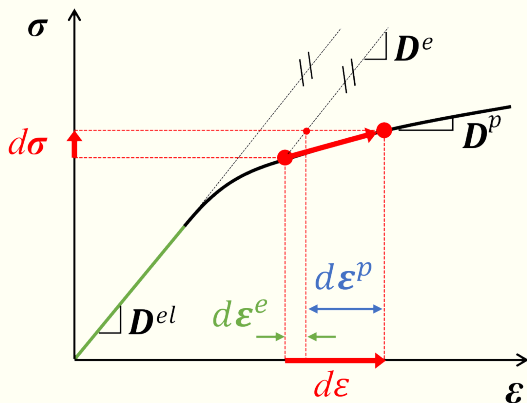
Equations of elasto-plasticity: 2. Flow rule

- Specifies the direction of plastic strain increments at every yield stress state. It is very important because it controls the ratio of the volumetric and deviatoric components (e.g., dilatancy of the material).

- States that the plastic strain increments ($d\varepsilon^p$) are normal to the plastic potential surface (G).

$$d\varepsilon^p = d\lambda \cdot \vec{P}$$

Note: typically $\vec{P}$ is **not** a unit vector, so $\vec{P}$ also controls the magnitude of $d\vec{\varepsilon_p}$ (in addition to the direction).

- In many cases, $\vec{P}$ is chosen as the gradient of a function g (if it exists) such that:

$$\vec{P} = \frac{\partial G}{\partial \sigma}$$

Drucker's postulate

Drucker (1952) established that for a stable inelastic material in a closed stress cycle, a positive work must be done.

$$dW^{plastic} > 0 \rightarrow d\sigma \cdot d\varepsilon^p > 0$$

This requirement is satisfied if the normality condition is assumed (but is not the only solution). The gradient of the yield surface (i.e., normal to the surface)

$$\vec{Q} = \frac{\partial F}{\partial \sigma} = \vec{P}$$

It also imposes a constraint that the yield surface must be convex.

2019-04-09

CE394M: intro to plasticity
└─ Classical plasticity
   └─ Equations of plasticity
      └─ Drucker's postulate
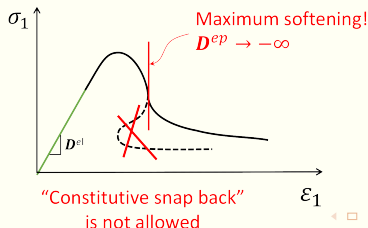
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$$\vec{Q} = \frac{\partial F}{\partial \sigma} = \vec{F}$$

It also imposes a constraint that the yield surface must be convex.

$$Q = \frac{\partial F}{\partial \sigma}$$

$$P = \frac{\partial G}{\partial \sigma}$$

Associated and non-associated plasticity

- **Associated flow rule** - Major principal plastic strain increment ($d\varepsilon_1^p$) is in the direction of the major principal stress (σ_1), and the axes of the principal strain increment and principal stress coincide, i.e., the yield and plastic functions coincide. This results in a symmetric constitutive matrix D^p.

- **Nonassociated flow rule** - The axes of the principal strain increment and principal stress do not coincide.

Associated flow rule

Non-associated flow rule
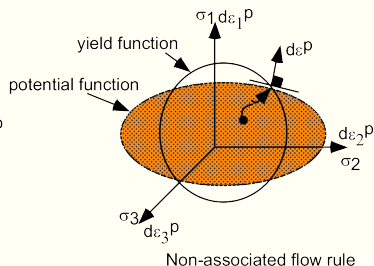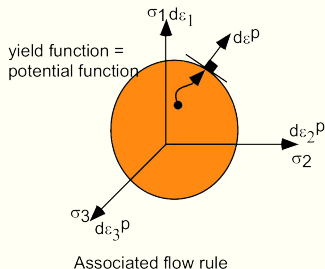
States that the elastic limit is defined by the yield surface, enforcing points in plastic condition to **remain** on the yield surface.

$$dF = \left(\frac{\partial F}{\partial \sigma}\right)^T \cdot d\sigma + \frac{\partial F}{\partial Wp} \cdot dWp = 0$$

Where D^e is the elastic stiffness matrix. ^{e} denotes the elastic part. E.g., strain hardening in tension for steel.
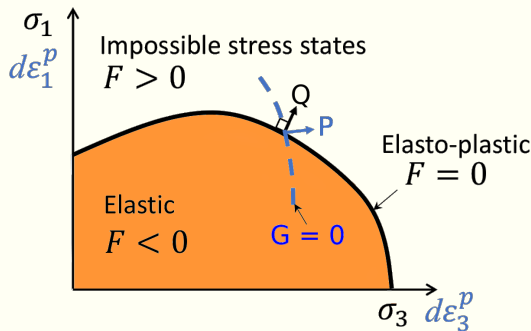
The consistency condition can be written as:

$$dF = \left(\frac{\partial F}{\partial \sigma}\right)^T \cdot d\sigma + \frac{\partial F}{\partial Wp} \cdot dWp \qquad = 0$$

$$= \left(\frac{\partial F}{\partial \sigma}\right)^T \cdot d\sigma + \left(\frac{\partial F}{\partial Wp}\right) \cdot \left(\frac{\partial Wp}{\partial \varepsilon^p}\right)^T \cdot d\varepsilon^p \qquad = 0$$

$$= \left(\frac{\partial F}{\partial \sigma}\right)^T \cdot d\sigma + \left(\frac{\partial F}{\partial Wp}\right) \left(\frac{\partial Wp}{\partial \varepsilon^p}\right)^T \cdot \frac{\partial G}{\partial \sigma} d\lambda \qquad = 0$$

$$dF = \left(\frac{\partial F}{\partial \sigma}\right)^T \cdot d\sigma - H d\lambda \qquad = 0.$$

if $H > 0$: Hardening, if $H = 0$: perfect plasticity, if $H < 0$: softening.

Linear elastic – hardening plastic material $H > 0$
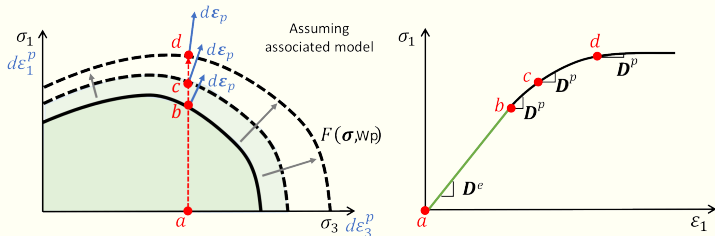
Linear elastic – softening plastic material $H < 0$
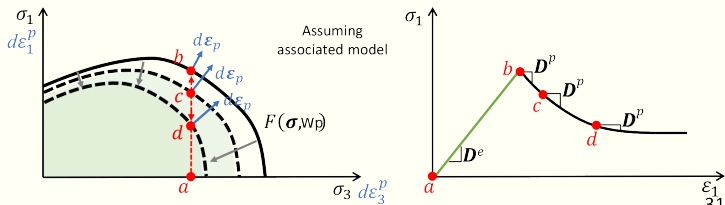
Isotropic v kinematic hardening

Two primary types of hardening laws: density and kinematic hardening.
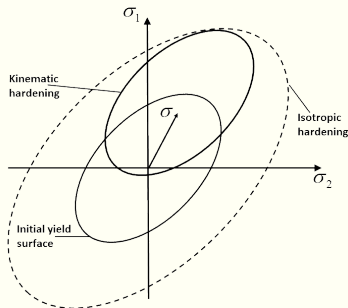
1. **Density hardening**: also referred to as isotropic hardening.
   $(\sigma_1 - \sigma_3) - 2c(h) = 0$
2. **Kinematic hardening**: typically describing fabric anisotropy. No change in size of yield surface but translating in stress space.
   $F = (\sigma_1 - \sigma_3) - \alpha(h) - 2c = 0$.

CE394M: intro to plasticity
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Isotropic v kinematic hardening

Two primary types of hardening laws: density and kinematic hardening.

- **Density hardening**: also referred to as isotropic hardening.
  $(\sigma_1 - \sigma_3) - 2c(h) = 0$
- **Kinematic hardening**: typically describing fabric anisotropy. No change in size of yield surface but translating in stress space.
  $F = (\sigma_1 - \sigma_3) - \alpha(h) - 2c = 0$.

e.g. $(\sigma_1 - \sigma_3) - 2c(h) = 0$

Isotropic hardening
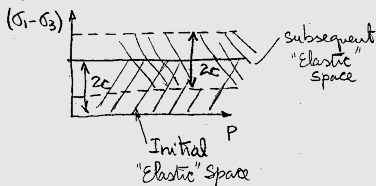
Kinematic hardening
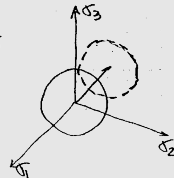
Basic concepts of elasto-plasticity

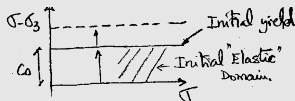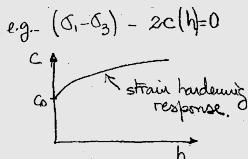The response of the material is:

- **elastic**: as long as the stress state remains **within** the yield surface.

$$F(\sigma, Wp) < 0 \rightarrow d\sigma = D^e \cdot d\varepsilon.$$

- **elasto-plastic**: if the stress state is **on** the yield surface

$$F(\sigma, Wp) = 0 \rightarrow d\sigma = D^p \cdot d\varepsilon.$$

Note: When the response is elastic, we have no problem! Knowing the strain, we directly can calculate the stress ($d\sigma = D^e \cdot d\varepsilon$).
The difficulty arises when the material response is elasto-plastic, we need to determine D^p!

Equations of elasto-plasticity

For an elasto-plastic material:

- Input material: $D^e, \frac{\partial F}{\partial \sigma}, \frac{\partial G}{\partial \sigma}, H$
- Results from FE analysis: $d\varepsilon$
- unknowns: $d\varepsilon^p, d\lambda, d\sigma$
- What we are interested: $d\sigma$

1. Stress-strain: $d\sigma = D^e \cdot (d\varepsilon - d\varepsilon^p)$
2. Consistency-condition: $dF = \frac{\partial F}{\partial \sigma}^T \cdot d\sigma - H d\lambda$
3. Flow rule: $d\varepsilon^p = \frac{\partial G}{\partial \sigma} d\lambda$.

Solution procedure:

1. Eqs 1 and 3 in eq 2 $\rightarrow d\lambda$
2. $d\lambda$ in Eq 3 $\rightarrow d\varepsilon^p$
3. $d\varepsilon^p$ in Eq 1 $\rightarrow d\sigma$

Equations of elasto-plasticity

Eq 1 and 3 (stress-strain & flow rule) in Eq 2 (consistency condition) for $d\lambda$

$$dF = \left(\frac{\partial F}{\partial \sigma}\right)^T \cdot D^e \cdot \left(d\varepsilon - \frac{\partial G}{\partial \sigma}d\lambda\right) - Hd\lambda = 0$$

$$d\lambda = \frac{\left(\frac{\partial F}{\partial \sigma}\right)^T \cdot D^e \cdot d\varepsilon}{\left(\frac{\partial F}{\partial \sigma}\right)^T \cdot D^e \left(\frac{\partial G}{\partial \sigma}\right) + H}$$

$$\left(\frac{\partial F}{\partial \sigma}\right)^T \cdot D^e \frac{\partial G}{\partial \sigma} + H > 0$$

$d\lambda$ in Eq 3 $\rightarrow d\varepsilon^p$

$$d\varepsilon^p = \frac{\partial G}{\partial \sigma}d\lambda = \frac{\partial G}{\partial \sigma}\frac{\left(\frac{\partial F}{\partial \sigma}\right)^T \cdot D^e \cdot d\varepsilon}{\left(\frac{\partial F}{\partial \sigma}\right)^T \cdot D^e \left(\frac{\partial G}{\partial \sigma}\right) + H}$$

Equations of elasto-plasticity

$d\varepsilon^p$ in Eq 1 $\rightarrow$ $d\sigma$

$$d\sigma = D^e \cdot (d\varepsilon - d\varepsilon^p)$$

$$= D^e \cdot \left(d\varepsilon - \frac{\partial G}{\partial \sigma} \frac{\left(\frac{\partial F}{\partial \sigma}\right)^T \cdot D^e \cdot d\varepsilon}{\left(\frac{\partial F}{\partial \sigma}\right)^T \cdot D^e \left(\frac{\partial G}{\partial \sigma}\right) + H} \right)$$

$$d\sigma' = \left[D^e - \frac{D^e \left(\frac{\partial G}{\partial \sigma'}\right) \left(\frac{\partial F}{\partial \sigma'}\right)^T D^e}{-\left(\frac{\partial F}{\partial W_p}\right) \left(\frac{\partial W_p}{\partial \varepsilon^p}\right)^T \left(\frac{\partial G}{\partial \sigma'}\right) + \left(\frac{\partial F}{\partial \sigma'}\right)^T D^e \left(\frac{\partial G}{\partial \sigma'}\right)} \right] d\varepsilon$$

Plastic loading v elastic unloading

Depending on α, we might have three different scenarios:
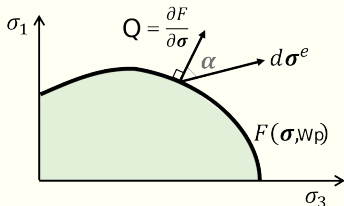
If $\alpha < 90°$ $\Rightarrow \left(\frac{\partial F}{\partial \boldsymbol{\sigma}}\right)^T \cdot \boldsymbol{D}^e \cdot d\boldsymbol{\varepsilon} > 0$

$\Rightarrow d\lambda > 0 \Rightarrow d\boldsymbol{\varepsilon}^p > 0$

$\Rightarrow$ **Elastoplastic loading**

$\Rightarrow dF = 0$

If $\alpha = 90°$ $\Rightarrow \left(\frac{\partial F}{\partial \boldsymbol{\sigma}}\right)^T \cdot \boldsymbol{D}^e \cdot d\boldsymbol{\varepsilon} = 0$

$\Rightarrow d\lambda = 0 \Rightarrow d\boldsymbol{\varepsilon}^p = 0$

$\Rightarrow$ **Neutral loading**

$\Rightarrow dF = 0$

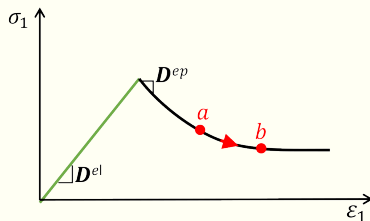If $\alpha > 90°$ $\Rightarrow \left(\frac{\partial F}{\partial \boldsymbol{\sigma}}\right)^T \cdot \boldsymbol{D}^e \cdot d\boldsymbol{\varepsilon} < 0$

$\Rightarrow d\lambda < 0 \Rightarrow d\boldsymbol{\varepsilon}^p = 0$

$\Rightarrow$ **Elastic unloading**
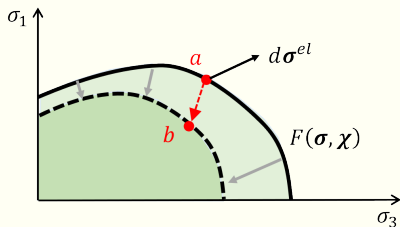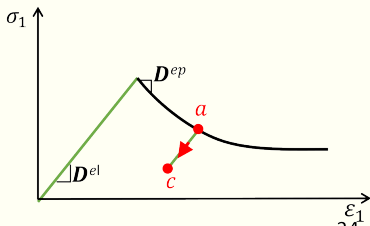
$\Rightarrow dF < 0$

Plastic loading v elastic unloading

Don't confuse **elastoplastic loading with softenting**...

...with **elastic unloading**
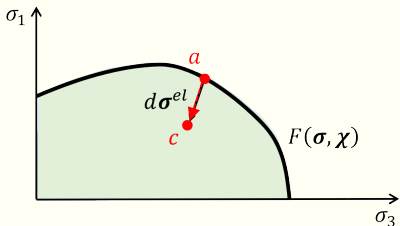
Problems associated to softening

1. The modeling of **strain softening** (i.e. strength decreases from pick to residual conditions) leads to the localization phenomenon.

2. Localization shows up when all deformation is absorbed by one element, while the neighboring elements remain under elastic conditions.

3. The solution depends on the element size, hence the localization is a **mesh dependent numerical problem**.

4. **Solution:** regularization techniques: non-local integration, strain softening models dependent on the element size,...

5. Be careful: Most of the commercial software do not include regularization techniques.

Don't confuse **elastoplastic loading with softenting**...

5 elements → **1 fails**
10 elements → **1 fails**
1,000,000 elements → **1 fails**

Localization =mesh dependent numerical problem

FE Mesh initialization

Input material properties

Input initial stresses $\boldsymbol{\sigma}$

Input load increment $\Delta \boldsymbol{f}$

Calc. element stiffness $K_{ij} \approx \sum_{p}^{n_p} B_i^p D^p B_j^p W_p$

Assembling and solve equations $\Delta \boldsymbol{u} = \boldsymbol{K}^{-1} \cdot \Delta \boldsymbol{f}$ **Equilibrium eq.**

Calc. strain and stress increment $\Delta \boldsymbol{\varepsilon} = \boldsymbol{B} \cdot \Delta \boldsymbol{u}$ **Compatibility eq.**
$\Delta \boldsymbol{\sigma} = \boldsymbol{D} \cdot \Delta \boldsymbol{\varepsilon}$ **Constitutive eq.**

Accumulate stress, etc. $\boldsymbol{u} = \boldsymbol{u} + \Delta \boldsymbol{u}$
$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon} + \Delta \boldsymbol{\varepsilon}$
$\boldsymbol{\sigma} = \boldsymbol{\sigma} + \Delta \boldsymbol{\sigma}$

Load Increment loop

Iteration loop

END