

# CE394M: Linear Elasticity

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## 1 Linear Elasticity

# Isotropic linear elastic stress-strain relations

The linear relationship between the stress and strain tensor is a linear one. The stress component is a linear combination of the strain tensor:

$$\begin{aligned}\sigma_{ij} = & C_{ij11}\varepsilon_{11} + C_{ij12}\varepsilon_{12} + C_{ij13}\varepsilon_{13} + \\ & C_{ij21}\varepsilon_{21} + C_{ij22}\varepsilon_{22} + C_{ij23}\varepsilon_{23} + \\ & C_{ij31}\varepsilon_{31} + C_{ij32}\varepsilon_{32} + C_{ij33}\varepsilon_{33}\end{aligned}$$

The most general form for *linear* stress-strain relations for a *Cauchy elastic* material is given by:

$$\sigma_{ij} = B_{ij} + C_{ijkl}\varepsilon_{kl}$$

Where  $B_{ij}$  is the components of initial stress tensor corresponding to the initial strain free (when all strain components  $\varepsilon_{kl} = 0$ ).  $C_{ijkl}$  is the tensor of material *elastic constants*.

If it is assumed that the initial strain free state corresponds to an *initial stress free state*, that is  $B_{ij} = 0$ , the equations reduces to:

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl}$$

# Observation on linear elasticity

- 1  $\sigma_{ij} = C_{ijkl}\varepsilon_{kl}$  is a general expression relating stress to strains for a linear solid.
- 2  $C_{ijkl}$  is a 4th order tensor containing 81 terms (we trick using symmetry and reduce order).
- 3  $C_{ijkl}$  material response functions having dimensions  $F/L^2$ .
- 4 Homogeneous:  $C_{ijkl}$  independent of position
- 5 Isotropic:  $C_{ijkl}$  independent of frame of reference.
- 6 Because the stress is symmetric:  $\sigma_{ij} = \sigma_{ji}$ ,  $C_{ijkl} = C_{jikl}$ . Strain is symmetric  $\varepsilon_{kl} = \varepsilon_{lk}$  and  $C_{ijkl} = C_{ijlk}$ . Hence the number of independent variables drop from 81 to 36.
- 7 Both the stress and the strain tensor have only 6 independent values, therefor write them as vecotrs, then the stiffness tensor can be written as a matrix (compromise I can not rotate tensor).

# Stress-strain relationship

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{23} \\ \varepsilon_{31} \end{bmatrix}$$

$C_{ijkl}$  is a tensor of material *elastic constants*. However, the above  $[\mathbf{C}]$  is not a tensor anymore. So we can not rotate the matrix to another frame of reference. This relationship is useful for isotropic materials, where  $\mathbf{C}$  is independent of the frame of reference.

$$\{\sigma\} = [\mathbf{C}] \{\varepsilon\}$$

The inverse of the relationship (Compliance matrix):

$$\{\varepsilon\} = [\mathbf{D}] \{\sigma\} \quad [\mathbf{D}] = [\mathbf{C}]^{-1}$$

# Isotropic Linear Elastic Stress-strain relationship

The *isotropic tensor*  $C_{ijkl}$ :

$$C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) + \alpha (\delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk})$$

Where  $\lambda$ ,  $\mu$ , and  $\alpha$  are scalar constants. Since  $C_{ijkl}$  must satisfy symmetry,  $\alpha = 0$ .

$$C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk})$$

So the stress:

$$\sigma_{ij} = \lambda \delta_{ij} \delta_{kl} \varepsilon_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) \varepsilon_{kl}$$

$$\sigma_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij}$$

Hence for an isotropic linear elastic material, there are only two independent material constants,  $\lambda$  and  $\mu$ , which are called *Lame's constants*.

# Hooke's law

Empirical observation:

$$\Delta \varepsilon_a = \Delta \sigma_{axial} \cdot \frac{1}{E} \rightarrow \Delta \varepsilon_{11} = \frac{\Delta \sigma_{11}}{E}$$

Where  $E$  is defined as the *Young's modulus*.

The lateral strains are defined as:

$$\Delta \varepsilon_{22} = -\mu \Delta \varepsilon_{11}$$

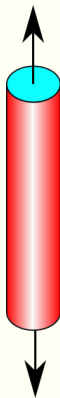
$$\Delta \varepsilon_{33} = -\mu \Delta \varepsilon_{11}$$

Using superposition for principal stresses:

$$\varepsilon_{11} = (1/E) [\sigma_{11} - \mu \sigma_{22} - \mu \sigma_{33}]$$

$$\varepsilon_{22} = (1/E) [-\mu \sigma_{11} + \sigma_{22} - \mu \sigma_{33}]$$

$$\varepsilon_{33} = (1/E) [-\mu \sigma_{11} - \mu \sigma_{22} + \sigma_{33}]$$



# Hooke's law

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \end{Bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\mu & -\mu \\ -\mu & 1 & -\mu \\ -\mu & -\mu & 1 \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \end{Bmatrix}$$

It is possible to invert the matrix to obtain the generalized Hooke's law  
 $[\sigma] = [\mathbf{C}][\varepsilon]$