CE394M: An introduction to constitutive modeling

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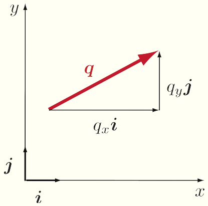
Overview

The objective of constitutive modelling is the determination of stiffness tensor **C**, a relation between stress and strain tensors.

Vector calculus

A vector is expressed in terms of its components and the unit vectors in the $x-$ and $y-$ directions.



$$\mathbf{q} = q_x \mathbf{i} + q_y \mathbf{j}$$

where q_x is the $x-$component and q_y is the $y-$component and i and j are basis vectors (are unit length).

Scalar product:

$$\mathbf{q} \cdot \mathbf{r} = \mathbf{q}^T \mathbf{r} = \begin{bmatrix} q_x & q_y \end{bmatrix} \begin{bmatrix} r_x \\ r_y \end{bmatrix} = q_x r_x + q_y r_y$$

Vector calculus

Grad: If the del operator acts on a scalar field, say temperature $T(x, y)$, it produces a vector that points in the direction of the steepest slope.

$$\nabla \mathbf{T} = \begin{bmatrix} \frac{\partial T}{\partial x} \\ \frac{\partial T}{\partial y} \end{bmatrix}$$

Divergence: The scalar product of the del operator with a vector field q gives the divergence

$$div\mathbf{q} = \nabla \cdot \mathbf{q} = \frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y}$$

Notice the divergence of a vector field is a scalar.

Divergence theorem

$$\int_\Omega div\mathbf{q} d\Omega = \oint_\Gamma \mathbf{q} \cdot \mathbf{n} d\Gamma$$

Vector calculus

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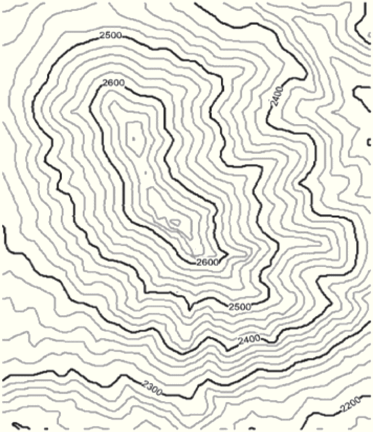
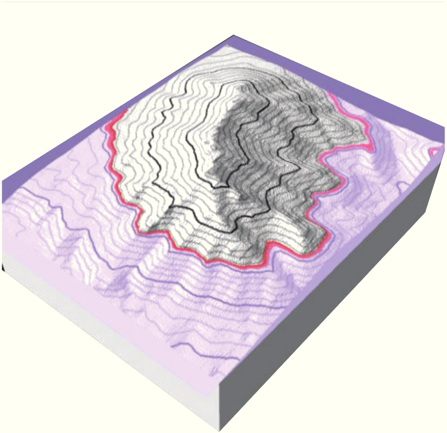
Gauss's theorem or Divergence theorem is a result that relates the flow (that is, flux) of a vector field through a surface to the behavior of the tensor field inside the surface.

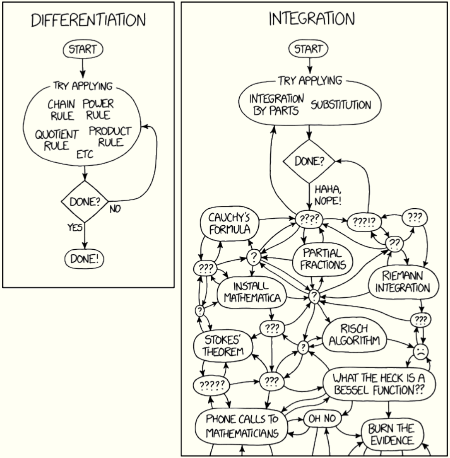
Vector calculus

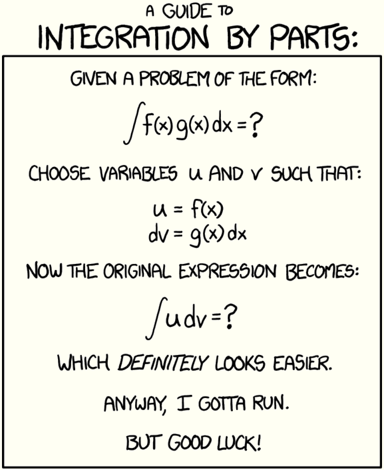




Contour map for a terrain (left) and the associated three-dimensional model (right). If T is interpreted as the height, the vector ΔT points in the direction of the steepest slope.

Differention and Integration





XKCD - Randall Munroe

The summation convention

Suppose **x** and **y** are vectors, and **A** and **B** are matrices. Write a few common combinations in terms of their components:

- **Dot product:**

$$\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^{n=3} x_i y_i$$

- **Matrix-vector product:**

$$[\mathbf{A}\mathbf{x}]_i = \sum_{j=1}^{n} A_{ij} x_j$$

- **Matrix-vector product:**

$$[\mathbf{A}\mathbf{B}]_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj}$$

Every sum goes with an index which is repeated twice. Non-repeated indices are not summed.

We can use a simplified notation by adopting the summation convention (due to Einstein), Do not write the summation symbol $\sum$. A repeated index implies summation. (An index may not appear more than twice on one side of an equality.) Using the summation convention

- **Dot product:**

$$\mathbf{x} \cdot \mathbf{y} = x_i y_i$$

- **Matrix-vector product:**

$$[\mathbf{Ax}]_i = A_{ij} x_j$$

- **Matrix-vector product:**

$$[\mathbf{AB}]_{ij} = A_{ik} B_{kj}$$

This may seem a very peculiar trick, with no obvious benefit. However, it will turn out to be surprisingly powerful, and make many calculations involving vector identities and vector differential identities much simpler.

The Kronecker delta δ_{ij}

The identity matrix:

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We define the '*Kronecker delta*' as:

$$\delta_{ij} = \begin{cases} 1 & i = j, \\ 0 & i \neq j \end{cases}$$

We know that: $\mathbf{Iy} = \mathbf{y}$

$$\delta_{ij} y_j = y_i$$

In other words 'if one index of δ_{ij} is summed, the effect is to swap this to the other index'.

The permutation symbol ϵ_{ijk}

Cross product of two vectors:

$$\mathbf{x} \times \mathbf{y} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = \begin{bmatrix} x_2 y_3 - x_3 y_2, & x_3 y_1 - x_1 y_3, & x_1 y_2 - x_2 y_1 \end{bmatrix}$$

where e_i are the basis for the vectors. We have assumed that e_i are the unit vectors for Cartesian coordinates (you may have seen the basis vectors written as i, j and k). To express the cross product in index notation, we will use the permutation symbol ϵ_{ijk}. The permutation symbol ϵ_{ijk} is defined as:

$$\epsilon_{ijk} = \begin{cases} 1 & \text{if } (ijk) \text{ is an even permutation of } (1, 2, 3), \\ -1 & \text{if } (ijk) \text{ is an odd permutation of } (1, 2, 3), \\ 0 & \text{otherwise} \end{cases}$$

The permutation symbol ϵ_{ijk}

For example:

$$\epsilon_{112} = 0$$
$$\epsilon_{312} = 1$$
$$\epsilon_{132} = -1$$

The permutation symbol is also known as the 'alternating symbol' or the 'Levi-Civita symbol'. Using the permutation symbol, we can write the cross product of two vectors as:

$$[\mathbf{x} \times \mathbf{y}] = \epsilon_{ijk} x_j y_k$$

To prove this, for each i sum over j and k. The permutation symbol possesses a number of 'symmetries',

$\epsilon_{ijk} = \epsilon_{kij} = \epsilon_{jki}$ (cyclic permutation)

$= -\epsilon_{jik} = -\epsilon_{ikj} = -\epsilon_{kji}$ (switch pair ij, switch pair ki, switch pair jk)

Vector derivatives

The real power of index notation is revealed when we look at vector differential identities. The vector derivatives known as the gradient, the divergence and the curl can all be written in terms of the operator ∇

$$\nabla = \left[\frac{\partial}{\partial x_1}, \quad \frac{\partial}{\partial x_2}, \quad \frac{\partial}{\partial x_3}\right]$$

where $[x_1, x_2, x_3]$ are the components of the position vector $\mathbf{x}$.

- **Gradient:**

$$grad\phi : [\nabla\phi]_i = \frac{\partial\phi}{\partial x_i} = \left[\frac{\partial\phi}{\partial x_1}, \quad \frac{\partial\phi}{\partial x_2}, \quad \frac{\partial\phi}{\partial x_3}\right]$$

- **Divergence:**

$$div\mathbf{u} : \nabla \cdot \mathbf{u} = \frac{\partial u}{\partial x_i} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}$$

- **Curl:** $[\mathbf{a} \times \mathbf{b}]_i = \epsilon_{ijk} a_j b_k$

$$curl\mathbf{u} : [\nabla \times \mathbf{u}]_i = \epsilon_{ijk} \frac{\partial u_k}{\partial x_j}$$

What is a Tensor?

- A tensor is a geometric object that maps in a multi-linear manner geometric vectors, scalars, and other tensors to a resulting tensor. Vectors and scalars which are often used in elementary physics and engineering applications, are considered as the simplest tensors.
- Tensors, defined mathematically, are simply arrays of numbers, or functions, that transform according to certain rules under a change of coordinates. Tensors characterize the properties of a physical system.
- An elementary example of mapping, describable as a tensor, is the dot product, which maps two vectors to a scalar.
- A tensor may consist of a single number, in which case it is referred to as a tensor of order zero, or simply a scalar. An example of a scalar field would be the density of a fluid as a function of position.
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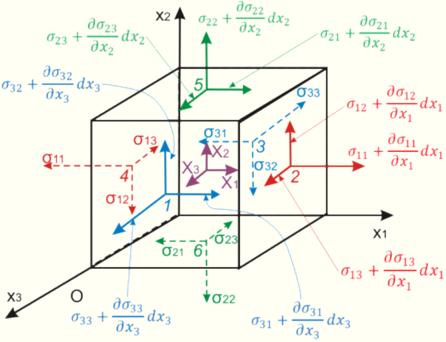
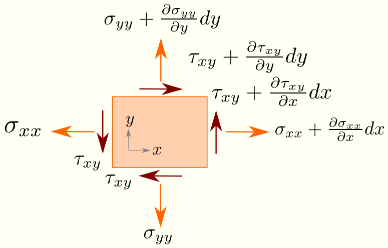
Assuming a basis of a real vector space, e.g., a coordinate frame in the ambient space, a tensor can be represented as an organized multidimensional array of numerical values with respect to this specific basis. Changing the basis transforms the values in the array in a characteristic way that allows to define tensors as objects adhering to this transformational behavior. For example, there are invariants of tensors that must be preserved under any change of the basis, thereby making only certain multidimensional arrays of numbers a tensor. Compare this to the array representing ε_{ijk} not being a tensor, for the sign change under transformations changing the orientation.

Stresses





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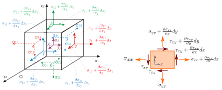
Stresses

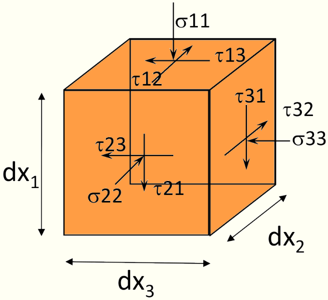


- Stress state at a point is defined by σ_{ij} in a frame of reference.
- Equilibrium of moments require $sigma_{xz} = \sigma_{zx}, \ldots$ etc.,

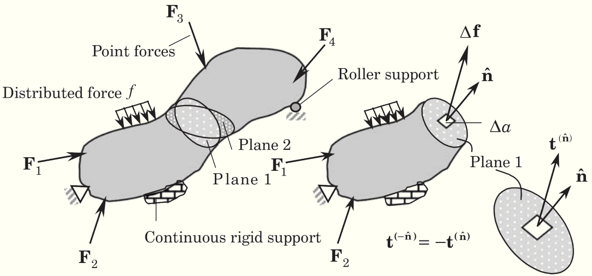
- σ_{xz} stress acting on plane perpendicular to axis x and in the direction of z
- σ_{xx} stress acting on plane perpendicular to axis x and in the direction of x

Stresses

- 9 components of the stress tensor.
- 6 stresses: $\sigma_{11}, \sigma_{22}, \sigma_{33}, \tau_{12}, \tau_{23}, \tau_{31}$.
- $\tau_{21} = -\tau_{12}, \tau_{32} = -\tau_{23}, \tau_{13} = -\tau_{31}$
- Compression is positive
- Shear stress, anti-clockwise is positive
- In order to write the components in a more concise way we can use the indices notation: σ_{ij} (use $i = 1, 2, 3$ and $j = 1, 2, 3$)
- Correspondence from x, y, z to $1, 2, 3$ (e.g., $\sigma_{11} = \sigma_{xx}, \sigma_{12} = \sigma_{xy}$)



Stress vector on a plane



Stress vector on a plane normal to $\hat{\mathbf{n}}$ (Reddy., 2008)

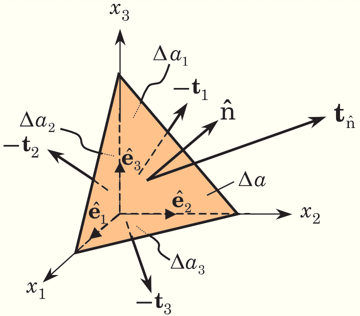
If we denote by $\Delta(\mathbf{f}\hat{\mathbf{n}})$ the force on a small area $\hat{\mathbf{n}}$ located at the position x, the stress vector can be defined:

$$\mathbf{t}(\hat{\mathbf{n}}) = \lim_{\Delta a \to 0} \frac{\Delta \mathbf{f}(\hat{\mathbf{n}})}{\Delta a}$$

Cauchy stress is the true stress, that is, stress in the deformed configuration.

Cauchy stress tensor

To establish the relationship between $\mathbf{t}$ and $\hat{\mathbf{n}}$ we now set up an infinitesimal tetrahedron in Cartesian coordinates:



If $-\mathbf{t}_1, -\mathbf{t}_2, -\mathbf{t}_3$ and $\mathbf{t}$ denote the stress vectors in the outward directions on the faces of the infinitesimal tetrahedron whose areas are $\Delta a_1, \Delta a_2, \Delta a_3$, and Δa, respectively. Δv is the volume of the tetrahedron, ρ the density, f the body force per unit mass, and $\mathbf{a}$ the acceleration.

Cauchy stress tensor

we have by Newton's second law for the mass inside the tetrahedron:

$$\mathbf{t}\Delta a - \mathbf{t}_1\Delta a_1 - \mathbf{t}_1\Delta a_1 - \mathbf{t}_1\Delta a_1 + \rho\Delta v\mathbf{f} = \rho\Delta v\mathbf{a}$$

Since the total vector area of a closed surface is zero (gradient theorem):

$$\Delta a\hat{\mathbf{n}} - \Delta a_1\hat{\mathbf{e}}_1 - \Delta a_2\hat{\mathbf{e}}_2 - \Delta a_3\hat{\mathbf{e}}_3 = \mathbf{0}$$

$$\Delta a_1 = (\hat{\mathbf{n}}\cdot\hat{\mathbf{e}}_1)\Delta a, \quad \Delta a_2 = (\hat{\mathbf{n}}\cdot\hat{\mathbf{e}}_2)\Delta a, \quad \Delta a_3 = (\hat{\mathbf{n}}\cdot\hat{\mathbf{e}}_3)\Delta a.$$

The volume Δv can be expressed as: $\Delta v = (\Delta h/3)\Delta a$
where Δh is the perpendicular distance from the origin to the slant face.

$$\mathbf{t} = (\hat{\mathbf{n}}\cdot\hat{\mathbf{e}}_1)\mathbf{t}_1 + (\hat{\mathbf{n}}\cdot\hat{\mathbf{e}}_2)\mathbf{t}_2 + (\hat{\mathbf{n}}\cdot\hat{\mathbf{e}}_3)\mathbf{t}_3 + \rho\frac{\Delta h}{3}(\mathbf{a} - \mathbf{f})$$

Cauchy stress tensor

In the limit when the tetrahedron shrinks to a point $\Delta h \to 0$:

$$\mathbf{t} = (\hat{\mathbf{n}} \cdot \hat{\mathbf{e}}_1)\mathbf{t}_1 + (\hat{\mathbf{n}} \cdot \hat{\mathbf{e}}_2)\mathbf{t}_2 + (\hat{\mathbf{n}} \cdot \hat{\mathbf{e}}_3)\mathbf{t}_3 = (\hat{\mathbf{n}} \cdot \hat{\mathbf{e}}_i)\mathbf{t}_i$$

where the summation convention is used.

$$\mathbf{t} = \hat{\mathbf{n}} \cdot (\hat{\mathbf{e}}_1\mathbf{t}_1 + \cdot\hat{\mathbf{e}}_2\mathbf{t}_2 + \hat{\mathbf{e}}_3\mathbf{t}_3).$$

The terms in the parenthesis is the **stress tensor** σ:

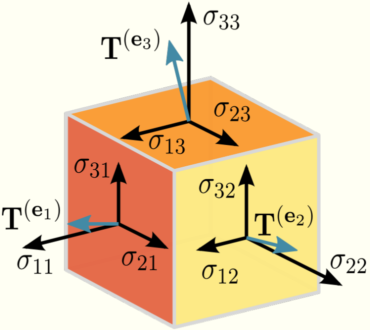
$$\sigma \equiv \hat{\mathbf{e}}_1\mathbf{t}_1 + \cdot\hat{\mathbf{e}}_2\mathbf{t}_2 + \hat{\mathbf{e}}_3\mathbf{t}_3$$

The stress tensor is a property of the medium that is independent of the $\hat{\mathbf{n}}$

$$\mathbf{t}(\hat{\mathbf{n}}) = \hat{\mathbf{n}}\sigma = \sigma^T\hat{\mathbf{n}}.$$

The stress vector $\mathbf{t}$ represents the vectorial stress on a plane whose normal is $\hat{\mathbf{n}}$. σ is the *Cauchy stress tensor* defined to be the *current force per unit deformed area*. In Cartesian component, the Cauchy formula is: $t_i = n_j\sigma_{ji}$.

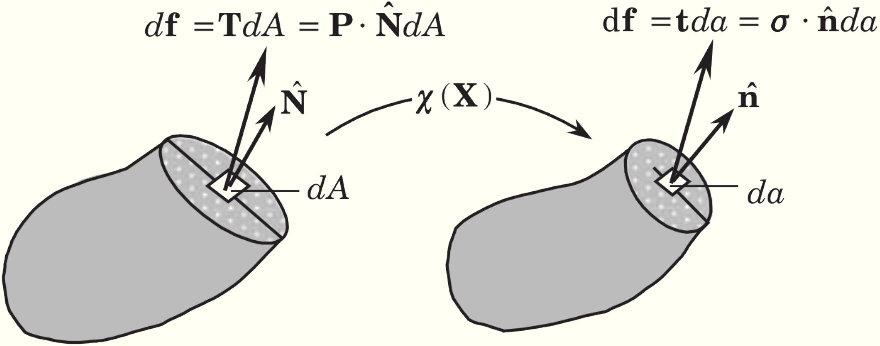
Cauchy stress tensor



Wikipedia

The Cauchy stress tensor σ, which takes a directional unit vector e as input and maps it to the stress vector $T(e)$, which is the force (per unit area) exerted by material on the negative side of the plane orthogonal to e against the material on the positive side of the plane, thus expressing a relationship between these two vectors

Cauchy stress vs Piola-Kirchoff stress

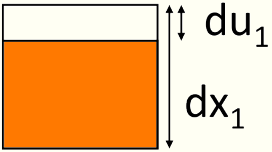


An introduction to continuum mechanics - J. N. Reddy (2008)

- The first Piola–Kirchhoff stress tensor, also referred to as the *nominal stress tensor*, or *Lagrangian stress tensor*, gives the current force per unit undeformed area.

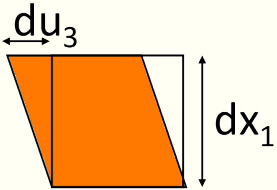
Strains

Normal strain



$$\varepsilon_{11} = du_1/dx_1$$

Shear strain



$$\gamma_{13} = du_3/dx_1$$

- 6 strains: $\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}, \gamma_{12}, \gamma_{23}, \gamma_{31}$.
- $\gamma_{21} = -\gamma_{12}, \gamma_{32} = -\gamma_{23}, \gamma_{13} = -\gamma_{31}$
- Compression is positive
- anti-clockwise is positive

Green's strain tensor

If u_1 is displacement in x_1 direction. Then:

$$\varepsilon_{11} = \frac{\partial u_1}{\partial x_1} + \frac{1}{2}\left[\left(\frac{\partial u_1}{\partial x_1}\right)^2 + \left(\frac{\partial u_2}{\partial x_2}\right)^2 + \left(\frac{\partial u_3}{\partial x_3}\right)^2\right]$$

$$\varepsilon_{12} = \frac{1}{2}\left\{\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} + \frac{1}{2}\left[\left(\frac{\partial u_1}{\partial x_1}\frac{\partial u_1}{\partial x_2}\right) + \left(\frac{\partial u_2}{\partial x_1}\frac{\partial u_2}{\partial x_2}\right) + \left(\frac{\partial u_3}{\partial x_1}\frac{\partial u_3}{\partial x_2}\right)\right]\right\}$$

Engineering shear strain: $\gamma_{12} = 2\varepsilon 12$

.

Typically, we assume small displacements + small strains. Therefore the quadratic terms (higher order) can be ignored. Pile driving or progression of slope failure cannot be modeled as a small strain problem!

$$\varepsilon_{11} = \frac{\partial u_1}{\partial x_1} \quad \varepsilon_{22} = \frac{\partial u_2}{\partial x_2} \quad \varepsilon_{33} = \frac{\partial u_3}{\partial x_3}$$

$$\varepsilon_{12} = \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}$$

Linearization of strain tensor

Ignoring higher order terms is called as the linearization of the strain tensor. This assumption allows for two simplifications:

1. Calculate strains based on undeformed/original geometry
2. Calculate matrix **B** based on originl / undeformed element geomtry. (**B** is the strain-displacement matrix)

Alternative, use natural strain approach:

$$\varepsilon_n = \int_{l_{initial}}^{l_{final}} \frac{dl}{l} = -\ln\left(\frac{l_{final}}{l_{initial}}\right) = -\ln\left(1 - \frac{\Delta l}{l_0}\right) = -\ln\left(1 - \varepsilon_{engg}\right)$$

For a 1D deformation of a bar

- $l_0 = 5, l_f = 4.9$ $\varepsilon = 0.1/5.0 = 2\%$ $\quad \varepsilon_n = \ln(5/4.9) = 2.02\%$
- $l_0 = 5, l_f = 4$ $\varepsilon = 1/5.0 = 20\%$ $\quad \varepsilon_n = \ln(5/4) = 22.3\%$

As long as strains are small, small strain formulation is very close - OK! (If you have large strains - a correction / alternatives is to update the **B** matrix every iteration to adhust for finite strains).