CE394M: Introduction to the Finite Element Method

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Overview

1. Galerkin methods
2. Strong form
3. Weak form

Finite Element Analysis

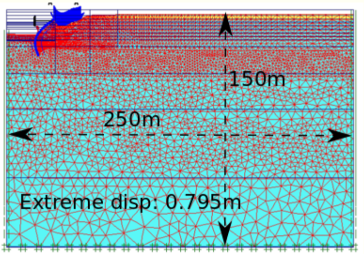

Fig. FE Mesh

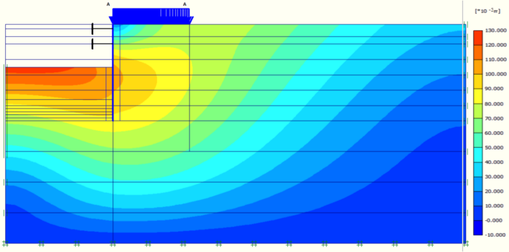

Fig. Displacement profile

Singapore Nicoll highway excavation FE analysis

Galerkin:Ritz method

1. Define the functional u for which you wish to find stationary points.
2. Choose a combination of linearly independent functions that will be used to approximate the solution. These will be called 'basis functions'. The amplitudes of these functions will be the unknowns that you will determine. The basis functions must satisfy the Dirichlet ('fixed') boundary conditions.
3. Insert the approximate solution into the functional that is now denoted by u_h.
4. Take the directional derivative of u_h with respect to the unknown amplitudes of the basis functions.
5. Determine the amplitudes of the basis functions which yield a stationary point of u_h.

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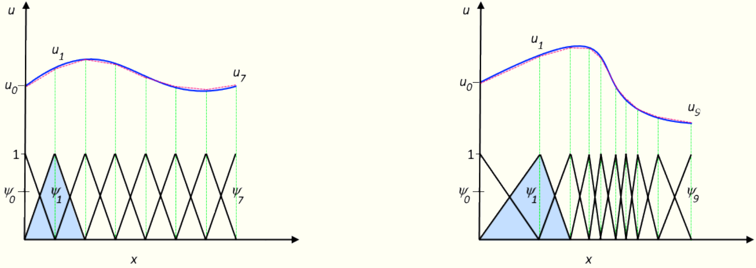
Numerical methods for partial differential equations are tools for finding approximate solutions and are normally used with the aid of a computer. A number of numerical methods are closely linked to a variational form of the differential equation. A group of such methods are known as Galerkin methods. The Ritz method is an example of a Galerkin method, and the finite element method is another. If a numerical method is properly formulated (and the equation is stable), the more effort (read computer time) that is expended, the closer one gets to the exact solution.

Finite Element Approximations

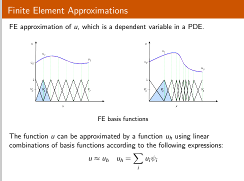
FE approximation of u, which is a dependent variable in a PDE.



FE basis functions

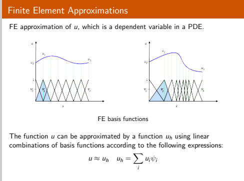
The function u can be approximated by a function u_h using linear combinations of basis functions according to the following expressions:

$$u \approx u_h \quad u_h = \sum_i u_i \psi_i$$



The reality of partial differential equations is that in most cases it is not possible to find an analytical solution. This is particularly so for equations on complicated geometries (as is common in engineering), nonlinear equations and equations with complicated source terms and boundary conditions.

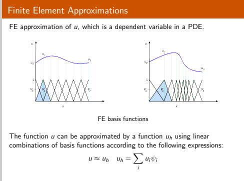
Instead, an approximation of the equations needs to be constructed, typically based upon different types of discretizations. These discretization methods approximate the PDEs with numerical model equations, which can be solved using numerical methods. The solution to the numerical model equations are, in turn, an approximation of the real solution to the PDEs. The finite element method (FEM) is used to compute such approximations.



Take, for example, a function u that may be the dependent variable in a PDE (i.e., temperature, electric potential, pressure, etc.) The function u can be approximated by a function uh using linear combinations of basis functions according to the following expressions:

$$u \approx u_h \quad u_h = \sum_i u_i \psi_i$$

Here, ψ_i denotes the basis functions and u_i denotes the coefficients of the functions that approximate u with uh. The figure below illustrates this principle for a 1D problem. u could, for instance, represent the temperature along the length (x) of a rod that is nonuniformly heated. Here, the linear basis functions have a value of 1 at their respective nodes and 0 at other nodes. In this case, there are seven elements along the portion of the x-axis, where the function u is defined (i.e., the length of the rod).

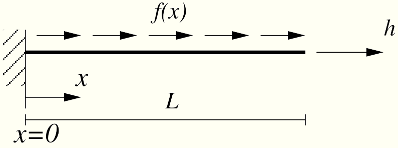


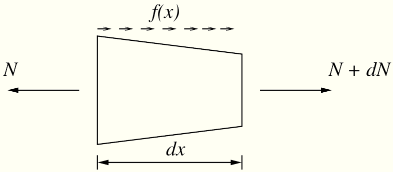
One of the benefits of using the finite element method is that it offers great freedom in the selection of discretization, both in the elements that may be used to discretize space and the basis functions. In the figure above, for example, the elements are uniformly distributed over the x-axis, although this does not have to be the case. Smaller elements in a region where the gradient of u is large could also have been applied.

Both of these figures show that the selected linear basis functions include very limited support (nonzero only over a narrow interval) and overlap along the x-axis. Depending on the problem at hand, other functions may be chosen instead of linear functions.

`https://www.comsol.com/multiphysics/finite-element-method`

Strong form of the equilibrium equation for a 1-D bar





where f is a distributed force and h as a force applied at the end of the bar

The equilibrium equation can be derived by considering an infinitesimal bar:

$$-\frac{dN}{dx} = f$$

where N is the normal force in the bar and f is the distributed force along the bar.

Boundary value problem of a 1-D bar

For linear elasticity

$$N = A\sigma = EA\frac{du}{dx} = EA\varepsilon$$

where $A(x)$ is the area of the bar, $E(x)$ is Young's modulus u is the displacement and $\varepsilon = du/dx$ is the strain.

$$-\frac{d}{dx}\left(EA\frac{du}{dx}\right) = f$$

which is a second-order differential equation. BCs:

1. $u = 0$ at $x = 0$ (displacement or 'Dirichlet' boundary condition),
2. $EA\varepsilon = h$ at $x = L$ (force or 'Neumann' boundary condition).

We now have a well-defined boundary value problem that can be solved.

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To formulate a boundary value problem, we need to assume a constitutive model which defines the relationship between stress and deformation, and we need to supply boundary conditions.

Weak form of the equilibrium equations of a 1D bar

The general derivation of the weak form of any equation from the strong form follows a standard procedure:

1. Multiply the strong equation by a weight function v which is equal to zero where Dirichlet (displacement) boundary conditions are applied, but is otherwise arbitrary (Another condition is that it must be sufficiently continuous. The degree of continuity required depends on the properties of the equation being considered.)
2. Use integration by parts to 'shift' derivatives to the weight function
3. Insert the Neumann (force) boundary conditions

We then want to find a solution u to the weak form that holds for all v . The weight function is also known as the '*test*' function.

Weak form of the equilibrium equations of a 1D bar

Multiplying equilibrium equation by an arbitrary weight function v and integrating along the bar:

$$-\int_0^L v\frac{dN}{dx}\,\mathrm{d}x. = \int_0^L vf\,\mathrm{d}x.$$

we require that $v(0) = 0$ because of the displacement boundary condition at $x = 0$.

$$-\int_0^L \frac{dv}{dx}N\,\mathrm{d}x. = \int_0^L vf\,\mathrm{d}x + vN|_{x=0}^{x=L}.$$

Since $v(0) = 0$, inserting the constitutive relationship and taking into account the force boundary condition at $x = L$.

$$-\int_0^L \frac{dv}{dx}EA\frac{du}{dx}\,\mathrm{d}x. = \int_0^L vf\,\mathrm{d}x + v(L)h.$$

The task is to find u with $u(0) = 0$ such that this equation is satisfied for all v.

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Strong form is the conventional PDE. The strong form imposes continuity and differentiability requirements on the potential solutions to the equation. Weak form is an alternate representation of the differential equation. The weak form relaxes these requirements on solutions to a certain extent. This means that a larger set of functions are solutions of the weak form. By construction all solutions of the strong form satisfy the weak form but not vice-versa.

Weak formulations are often referred to as 'variational formulations'. In fact, the weak form is more general than the strong form. The weak form of an equation does not generally make an equation easier to solve analytically (it may make it harder), but is usually a more suitable form for mathematical analysis (allowing us to say things about the properties of the equation without knowing the solution) and for numerical solution methods.

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The task is to find u with $u(0) = 0$ such that this equation is satisfied for all v.

An important observation is that the weak form for the bar contains at most first-order derivatives, whereas the strong form for this problem contained second-order derivatives. We have transferred one derivative from the displacement field u to weight function v using integration by parts. This is a key feature of the weak form and is crucial for finite element methods. If the symbol v in the weak form was replaced by the symbol $\delta\varepsilon$, the weak form in the case of the bar resembles the virtual work equation. It is in fact equivalent. However, the process that we have followed is general and does not require any knowledge of virtual work, and can be applied to any differential equation.