CE394M: Linear Elasticity

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Overview

Isotropic linear elastic stress-strain relations

The linear relationship between the stress and strain tensor is a linear one. The stress component is a linear combination of the strain tensor:

$$\begin{aligned}\sigma_{ij} = C_{ij11}\varepsilon_{11} + C_{ij12}\varepsilon_{12} + C_{ij13}\varepsilon_{13} + \\ C_{ij21}\varepsilon_{21} + C_{ij22}\varepsilon_{22} + C_{ij23}\varepsilon_{23} + \\ C_{ij31}\varepsilon_{31} + C_{ij32}\varepsilon_{32} + C_{ij33}\varepsilon_{33}\end{aligned}$$

The most general form for *linear* stress-strain relations for a *Cauchy elastic* material is given by:

$$\sigma_{ij} = B_{ij} + C_{ijkl}\varepsilon_{kl}$$

Where B_{ij} is the components of initial stress tensor corresponding to the initial strain free (when all strain components $\varepsilon_{kl} = 0$). C_{ijkl} is the tensor of material *elastic constants*.

If it is assumed that the initial strain free state corresponds to an *initial stress free state*, that is $B_{ij} = 0$, the equations reduces to:

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl}$$

Observation on linear elasticity

1. $\sigma_{ij} = C_{ijkl}\varepsilon_{kl}$ is a general expression relating stress to strains for a linear solid.
2. C_{ijkl} is a 4th order tensor containing 81 terms (we trick using symmetry and reduce order).
3. C_{ijkl} material response functions having dimensions F/L^2.
4. Homogeneous: C_{ijkl} independent of position
5. Isotropic: C_{ijkl} independent of frame of reference.
6. Because the stress is symmetric: $\sigma_{ij} = \sigma_{ji}$, $C_{ijkl} = C_{jikl}$. Strain is symmetric $\varepsilon_{kl} = \varepsilon_{lk}$ and $C_{ijkl} = C_{ijlk}$. Hence the number of independent variables drop from 81 to 36.
7. Both the stress and the strain tensor have only 6 independent values, therefor write them as vecotrs, then the stiffness tensor can be written as a matrix (compromise I can not rotate tensor).

Stress-strain relationship

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{23} \\ \varepsilon_{31} \end{bmatrix}$$

C_{ijkl} is a tensor of material *elastic constants*. However, the above $[\mathbf{C}]$ is not a tensor anymore. So we can not rotate the matrix to another frame of reference. This relationship is useful for isotropic materials, where **C** is independent of the frame of reference.

$$\{\sigma\} = [\mathbf{C}]\{\varepsilon\}$$

The inverse of the relationship (Compliance matrix):

$$\{\varepsilon\} = [\mathbf{D}]\{\sigma\} \qquad [\mathbf{D}] = [\mathbf{C}]^{-1}$$

Isotropic Linear Elastic Stress-strain relationship

The *isotropic tensor* C_{ijkl}:

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \alpha(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk})$$

Where $\lambda, \mu,$ and $, \alpha$ are scalar constants. Since C_{ijkl} must satisfy symmetry, $\alpha = 0$.

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

So the stress:

$$\begin{aligned}
\sigma_{ij} &= \lambda\delta_{ij}\delta_{kl}\varepsilon_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\varepsilon_{kl} \\
\sigma_{ij} &= \lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij}
\end{aligned}$$

Hence for an isotropic linear elastic material, there are only two independent material constants, λ and μ, which are called *Lame's constants*.

Hooke's law

Empirical observation:

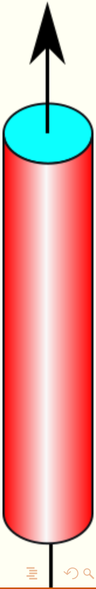
$$\Delta\varepsilon_a = \Delta\sigma_{axial} \cdot \frac{1}{E} \to \Delta\varepsilon_{11} = \frac{\Delta\sigma_{11}}{E}$$

Where E is defined as the *Young's modulus*.
The lateral strains are defined as:

$$\Delta\varepsilon_{22} = -\nu\Delta\varepsilon_{11}$$
$$\Delta\varepsilon_{33} = -\nu\Delta\varepsilon_{11}$$

Using superposition for principal stresses:

$$\varepsilon_{11} = (1/E)\left[\sigma_{11} - \nu\sigma_{22} - \nu\sigma_{33}\right]$$
$$\varepsilon_{22} = (1/E)\left[-\nu\sigma_{11} + \sigma_{22} - \nu\sigma_{33}\right]$$
$$\varepsilon_{33} = (1/E)\left[-\nu\sigma_{11} - \nu\sigma_{22} + \sigma_{33}\right]$$



Hooke's law

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \end{Bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu \\ -\nu & 1 & -\nu \\ -\nu & -\nu & 1 \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \end{Bmatrix}$$

It is possible to invert the matrix to obtain the generalized Hooke's law: $[\sigma] = [\mathbf{C}][\varepsilon]$.

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{11} \\ \sigma_{11} \\ \sigma_{11} \\ \sigma_{11} \\ \sigma_{11} \end{Bmatrix} = \alpha \begin{bmatrix} (1-\mu) & \nu & \nu & 0 & 0 & 0 \\ & (1-\nu) & \nu & 0 & 0 & 0 \\ & & (1-\nu) & 0 & 0 & 0 \\ & & & \frac{(1-2\nu)}{2} & 0 & 0 \\ & & & & \frac{(1-2\nu)}{2} & 0 \\ & & & & & \frac{(1-2\nu)}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{12} \\ 2\varepsilon_{13} \\ 2\varepsilon_{23} \end{Bmatrix}$$

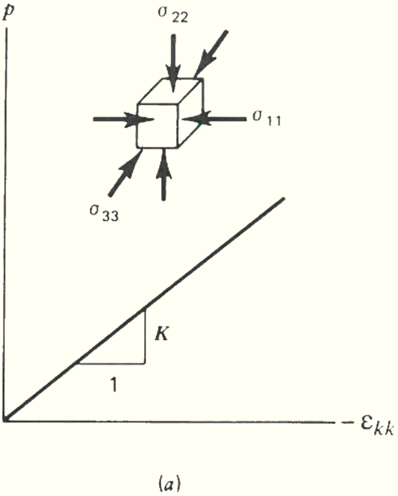
Where $\alpha = E/((1+\mu)(1-2\mu))$. Similarly, we can obtain the inverse matrix.

Hooke's law

The matrices [**C**] and [**D**] contains two indepdent variables E and μ, where $E > 0$ and $-1 \leq \mu \leq 0.5$. The matrix can also be defined in terms of Lame's constants.

$$\begin{cases} \lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} & \text{Lame's modulus (wave propagation)} \\ \mu = G = \frac{E}{2(1+\nu)} & \text{Shear modulus (shear behavior)} \\ K = \frac{E}{3(1-2\nu)} & \text{Bulk modulus (volumetric behavior)} \end{cases}$$

Isotropic linear elastic



Behavior of isotropic linear elastic material in simple tests: (a) hydrostatic compression test ($\sigma_{11} = \sigma_{22} = \sigma_{33} = p$) and (b) simple tension test (Chen 1994)

Isotropic linear elastic

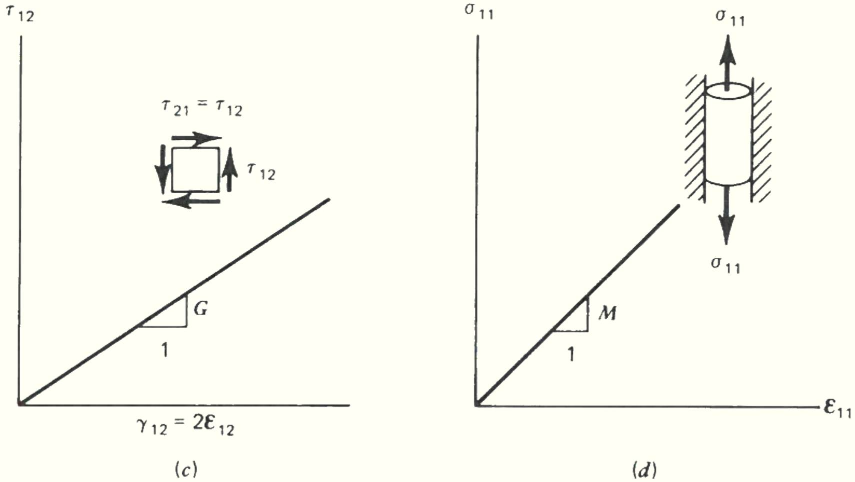
Hydrostatic compression test The non-zero components of stress: $\sigma_{11} = \sigma_{22} = \sigma_{33} = -p = \sigma_{kk}/3$.
The *Bulk modulus, K,* is defined as the ratio between the *hydrostatic pressure p* nd the corresponding volume change $\delta\varepsilon_v = \varepsilon_{kk}$.

$$K = -\frac{p}{\varepsilon_{kk}} = \lambda + \frac{2}{3}\mu$$

Simple tension test The only non-zero components of stress: $\sigma_{11} = \sigma$
The *Young's modulus, K,* is defined as the ratio between the *hydrostatic pressure p* nd the corresponding volume change $\Delta\varepsilon_v = \varepsilon_{kk}$.

$$K = -\frac{p}{\varepsilon_{kk}} = \lambda + \frac{2}{3}\mu$$

Isotropic linear elastic



Behavior of isotropic linear elastic material in simple tests: (c) pure shear test, and (d) uniaxial strain test (Chen 1994)

Isotropic linear elastic

Simple shear test

The non-zero components of stress: $\sigma_{12} = \sigma_{21} = \tau_{12} = \tau_{21} = \tau$.
The *Shear modulus, G or* μ, is defined as:

$$G = \mu = \frac{\sigma_{12}}{\gamma_{12}} = \frac{\tau}{2\varepsilon_{12}}$$

Uniaxial strain test The test is carried out by applying a uniaxial stress component σ_{11} in the axial direction of a cylindrical sample, whose lateral surface is *restrained* against lateral movement (Oedometer test). Axial strain $\varepsilon_1 1$ is the only nonvanishing component. The *constrained modulus* M or as PLAXIS calls it E_{oed} is defined as the ratio between σ_{11} and $\varepsilon_1 1$.

$$\sigma_{11} = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_{11} + \nu\cancelto{0}{\varepsilon_{22}} + \nu\cancelto{0}{\varepsilon_{33}}\right] = \frac{E(1-\nu)\varepsilon_{11}}{(1+\nu)(1-2\nu)}$$

$$M = E_{oed} = \frac{\sigma_{11}}{\varepsilon_{11}} = \frac{(1-\nu)E}{(1+\nu)(1-2\nu)} = (\lambda + 2\mu)$$