CE394M: Isoparametric elements and Gauss integration

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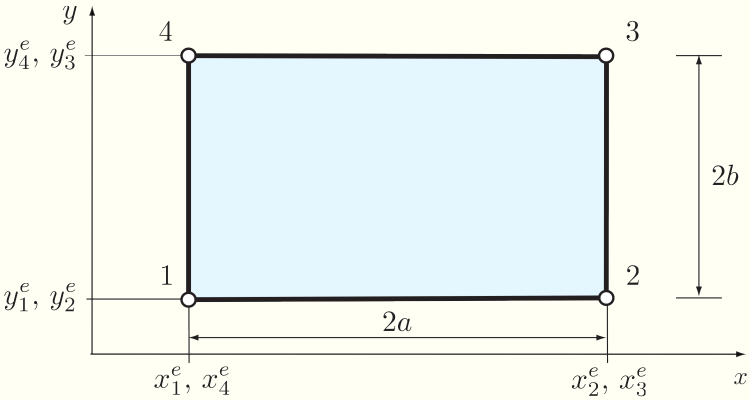
Overview

4-noded rectangular element



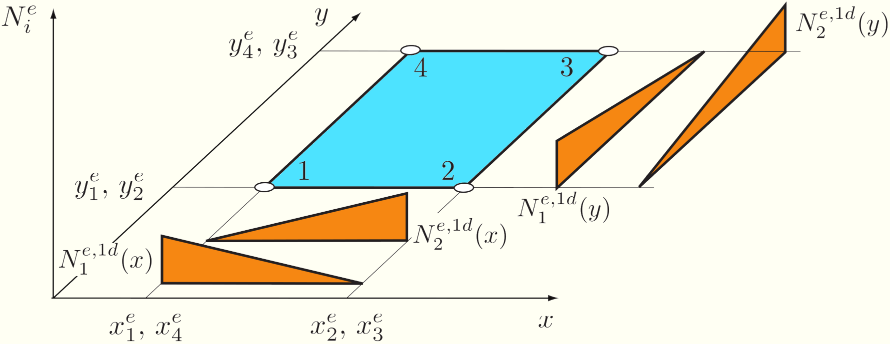
Four-node rectangular element. The nodes are by definition numbered counter-clockwise.

4-noded rectangular element

As the element has four nodes, it is necessary to start with a polynomial that has four parameters.

4-noded rectangular element

An alternative and more elegant approach is to construct the shape functions by the **tensor product method.** This is based on taking products of one-dimensional shape functions.



Construction of two dimensional shape functions.

4-noded rectangular element

For example, N_2^e, which has to have the value one at node 2 and zero at the other nodes, is obtained by taking the product of the one-dimensional shape functions $N_2^{e,1d}(x)$ and $N_1^{e,1d}(y)$.

4-noded rectangular element

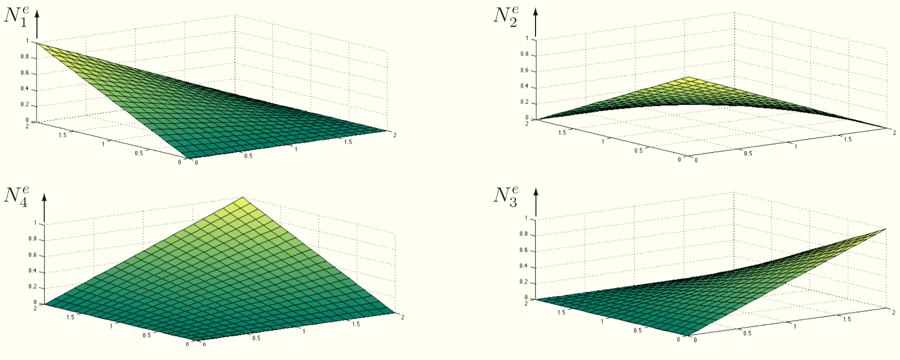
The four shape functions, also called **bilinear shape functions**, for the quadrilateral element are:

$$
\begin{aligned}
N_1^e(x,y) &= \frac{x - x_2^e}{x_1^e - x_2^e}\frac{y - y_4^e}{y_1^e - y_4^e} = \frac{1}{A^e}(x - x_2^e)(y - y_4^e)\\
N_2^e(x,y) &= \frac{x - x_1^e}{x_2^e - x_1^e}\frac{y - y_4^e}{y_1^e - y_4^e} = -\frac{1}{A^e}(x - x_1^e)(y - y_4^e)\\
N_3^e(x,y) &= \frac{x - x_1^e}{x_2^e - x_1^e}\frac{y - y_1^e}{y_4^e - y_1^e} = \frac{1}{A^e}(x - x_1^e)(y - y_1^e)\\
N_4^e(x,y) &= \frac{x - x_2^e}{x_1^e - x_2^e}\frac{y - y_1^e}{y_4^e - y_1^e} = -\frac{1}{A^e}(x - x_2^e)(y - y_1^e)
\end{aligned}
$$

where A^e is the area of the element.

4-noded rectangular element

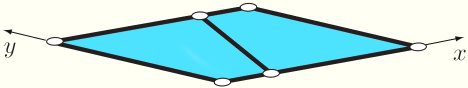
The four shape functions are plotted in the following figure:

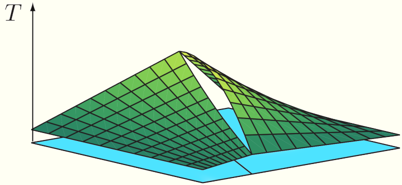


Four shape functions of the rectangular element (on $[0, 2] \times [0, 2]$).

4-noded rectangular element

If the sf equations are used for interpolating the temperature field over arbitrary quadrilaterals, the scalar field (e.g., temperature) across the element boundaries will be not continuous.



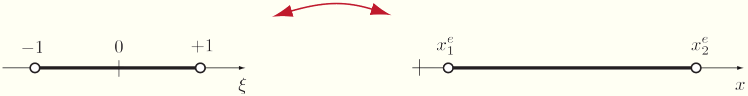


Two element mesh with rectangle shape functions. Although the two nodal values on the edge agree, the temperature distribution is still discontinuous.

Isoparametric mapping in 1D

Isoparametric mapping in 1D

consider a coordinate transformation which transforms (maps) the coordinate x into a local (element specific) coordinate ξ:



Mapping of the parent element onto the physical element.

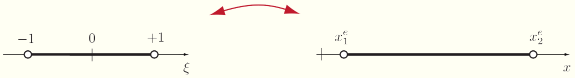
The coordinate ξ fulfills the relationships:

"stretch transformation" of $x(\xi)$:

Isoparametric mapping in 1D

The shape functions $N_1^e = (1 - x/l)$ and $N_2^e = (x/l)$ can also be expressed using ξ:

The key idea of the isoparametric concept is to use these shape functions for writing the coordinate transformation between x and ξ



Mapping of the parent element onto the physical element.

Isoparametric mapping in 1D

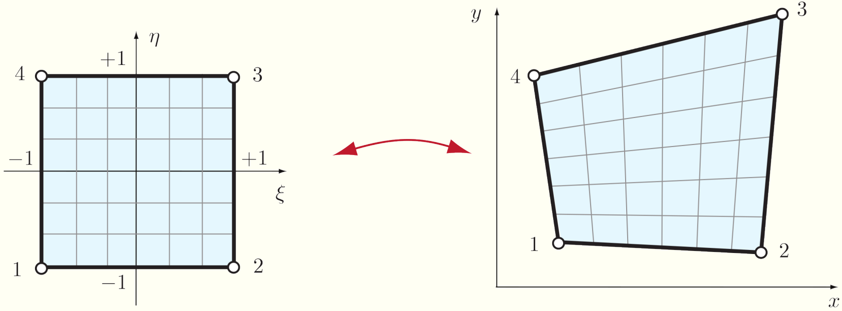
- The coordinate ξ is usually called the **natural coordinate** and always lies by definition between -1 and $+1$.
- The parent element is solely for numerical purposes.
- The finite element analysis is still performed over the physical domain.

In an isoparametric element the field variable, like displacement, is approximated with the same set of shape functions as those used for the coordinate transformation:

To compute the derivatives which appear in the weak form the chain rule is used:

Isoparametric mapping of a quadrilateral element

The idea of isoparametric mapping is used for deriving shape functions for arbitrary quadrilateral elements:



Mapping of the bi-unit parent element onto the quadrilateral element in the physical space.

Isoparametric mapping of a quadrilateral element

The bi-unit square is the parent domain and ξ and η are its natural coordinates.

To map points from the parent domain onto the quadrilateral in the physical domain the four nodal shape functions are used:

Isoparametric mapping of a quadrilateral element

As the parent element is a bi-unit square its shape functions are identical to those of the rectangular element expressed in ξ and η coordinates.

$$
\begin{aligned}
N_1^{4Q}(\xi,\eta) &= \frac{1}{4}(1-\xi)(1-\eta)\\
N_2^{4Q}(\xi,\eta) &= \frac{1}{4}(1+\xi)(1-\eta)\\
N_3^{4Q}(\xi,\eta) &= \frac{1}{4}(1+\xi)(1+\eta)\\
N_4^{4Q}(\xi,\eta) &= \frac{1}{4}(1-\xi)(1+\eta)
\end{aligned}
$$

Isoparametric shape functions

The temperature will be approximated with the same shape functions:

The element is called *isoparametric* because the temperature approximation and the mapping of the geometry is accomplished with the same shape functions.
The displacement will be approximated as:

Derivatives isoparametric shape functions

The gradient of the temperature for the four-node (isoparametric) quadrilateral element is:

with

Derivatives isoparametric shape functions

To compute shape function derivatives the chain rule will be used:

written as matrices and vectors this becomes:
$\mathbf{J}^e$ is the Jacobian which contains the derivatives of the physical coordinates with respect to the natural coordinates.

Derivatives isoparametric shape functions

The derivatives required for the weak form are computed by inverting the above expression:

The inverse of $\mathbf{J}^e$ is:

Where $|\mathbf{J}^e|$ is the determinant of the Jacobian, which represents the ratio of an area element in the physical domain to the corresponding area element in the parent domain.

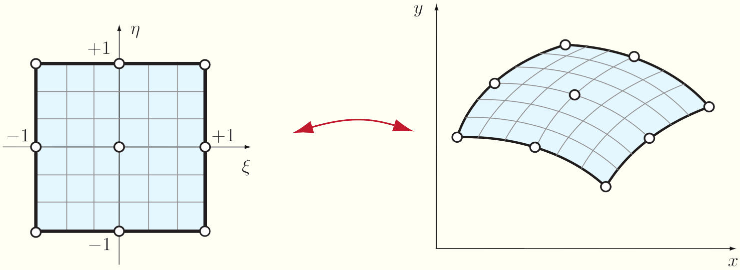
Derivatives isoparametric shape functions

The isoparametric mapping is used to compute the Jacobian:

which leads to:

$$\mathbf{J}^e = \begin{bmatrix} \frac{\partial N_1^{4Q}}{\partial \xi} & \frac{\partial N_2^{4Q}}{\partial \xi} & \frac{\partial N_3^{4Q}}{\partial \xi} & \frac{\partial N_4^{4Q}}{\partial \xi} \\ \frac{\partial N_1^{4Q}}{\partial \eta} & \frac{\partial N_2^{4Q}}{\partial \eta} & \frac{\partial N_3^{4Q}}{\partial \eta} & \frac{\partial N_4^{4Q}}{\partial \eta} \end{bmatrix} \begin{bmatrix} x_1^e & y_1^e \\ x_2^e & y_2^e \\ x_3^e & y_3^e \\ x_4^e & y_4^e \end{bmatrix}$$

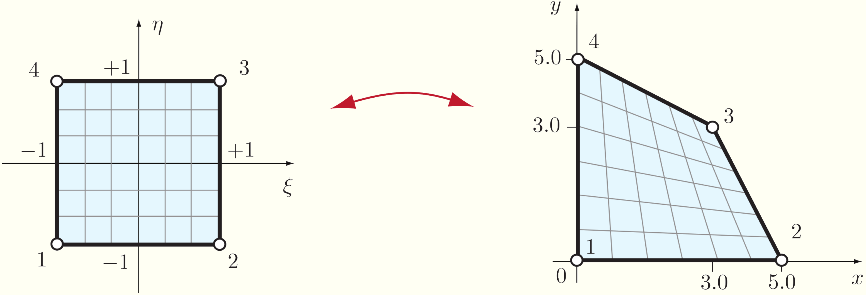
Higher order quadrilateral element



Nine-node ispoarametric quadrilateral in parameter (left) and physical space (right).

Effect of element shape on isoparametric mapping

Consider the following isoparametric mapping for a four-node quadrilateral element:



Effect of element shape on isoparametric mapping

The Jacobian of this mapping:

$$\mathbf{J}^e = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix}$$

is computed from

Effect of element shape on isoparametric mapping

After some algebra we obtain:

$$\mathbf{J}^e = \begin{bmatrix} 2 - \frac{\eta}{2} & -\frac{1}{2} - \frac{1}{2}\eta \\ -\frac{1}{2} - \frac{1}{2}\xi & 2 - \frac{1}{2}\xi \end{bmatrix}$$

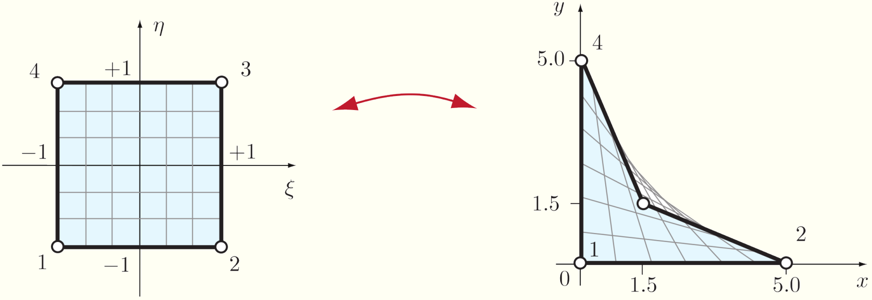
The Jacobian has to be invertible for computing the derivatives of the shape functions with respect to the physical coordinates.
For the mapping to be invertible, the determinant of the Jacobian has to be larger than zero over the entire element:

$$det\mathbf{J}^e = \frac{5}{4}(3 - \xi - eta)$$

which is the case for this mapping.

Effect of element shape on isoparametric mapping

In contrast to the previous mapping, it can be for the following mapping shown:



that the determinant of the Jacobian is zero or negative close to the non-convex corner.

Numerical integration (Quadrature)

- During the finite element solution procedure, it is necessary to integrate various quantities, for instance the stiffness matrix or force vectors.
- For isoparametric elements these quantities must in general be integrated numerically due to the presence of the Jacobian.
- Although there are many different numerical integration techniques, in finite elements Gauss integration is preferred.

Review of Basic Integration Rules

Consider the following integral which is to be integrated numerically:

$$\int_a^b f(x)dx$$

The integration domain $[a, b]$ is first split into N equidistant subintervals:

with:

There are various schemes for approximating the integral using the function values at N positions

Rectangle rule (constant approximation)

Integration of $f(x)$ with the rectangle rule between $x = a$ and $x = b$.

Trapezoidal rule (linear approximation)

Integration of $f(x)$ with the trapezoidal rule between $x = a$ and $x = b$.

Integration rule

Simpson's rule (quadratic approximation)

$$\int_a^b f(x)dx \approx \frac{1}{3}h(f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+\cdots+4f(x_{N-1})+f(x_N))$$

Notice that all these integration rules can be written in the following form:

Gauss integration in 1D

Gauss integration formulas are always given over the parent domain $[-1, 1]$:

Two-point Gauss integration in 1D

Integration of $f(\xi)$ between $\xi = -1$ and $\xi = 1$ using two Gauss points:

Gauss integration weights and position in 1D

N	location ξ_I	weight w_I
1	0	2
2	$-1/\sqrt{3}$	1
	$1/\sqrt{3}$	1
3	$-\sqrt{0.6}$	$5/9$
	0	$8/9$
	$\sqrt{0.6}$	$5/9$

With N Gauss points the integration is exact up to polynomial order $(2N - 1)$.

Gauss integration

$$\int_{-1}^{1} (3\xi^2 + \xi) d\xi$$

The quadrature approximation is:

$$\int_{-1}^{1} f(\xi) d\xi \approx \sum_{I=1}^{N} w_I f(\xi_I)$$

where GP is the number of Gauss points.

one-point Gaussian integration:

Gauss integration

The quadrature approximation is:

$$\int_{-1}^{1} f(\xi) d\xi \approx \sum_{I=1}^{N} w_I f(\xi_I)$$

Two-point Gaussian integration:

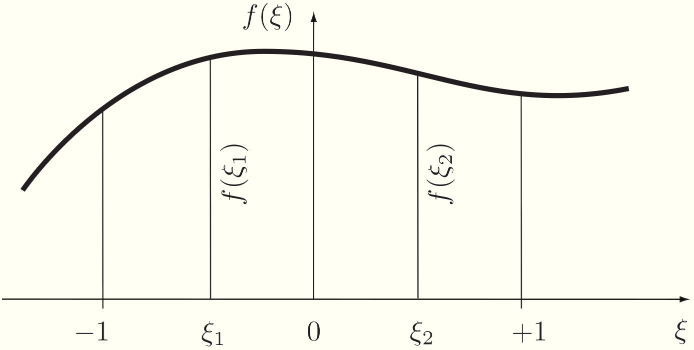
Three-point Gaussian integration:

Gauss integration

Assume that we want to integrate an arbitrary cubic polynomial over the parent domain $[-1, 1]$:

$$f(\xi) = \alpha_1 + \alpha_2\xi + \alpha_3\xi^2 + \alpha_4\xi^3$$

where $\alpha_1, \alpha_2, \alpha_3$ and α_4 are constants. The integral will be approximated with two integration points:



Gauss integration

Integration of $f(\xi)$ between $\xi = -1$ and $\xi = 1$ using two Gauss points:

$$\begin{aligned}\int_{-1}^{+1} f(\xi) d\xi &= w_1 f(\xi_1) + w_2 f(\xi_2) \\ &= w_1(\alpha_1 + \alpha_2\xi + \alpha_3\xi^2 + \alpha_4\xi^3) \\ &+ w_2(\alpha_1 + \alpha_2\xi + \alpha_3\xi^2 + \alpha_4\xi^3)\end{aligned}$$

Alternatively, the integration can be performed analytically:

$$\begin{aligned}\int_{-1}^{+1} f(\xi) d\xi &= \int_{-1}^{+1} (\alpha_1 + \alpha_2\xi + \alpha_3\xi^2 + \alpha_4\xi^3) dx \\ &= \left[\alpha_1\xi + \alpha_2\frac{\xi^2}{2} + \alpha_3\frac{\xi^3}{3} + \alpha_4\frac{\xi^4}{4}\right]_{-1}^{+1} \\ &= 2\alpha_1 + 0 + \frac{2}{3}\alpha_3 + 0\end{aligned}$$

Gauss integration

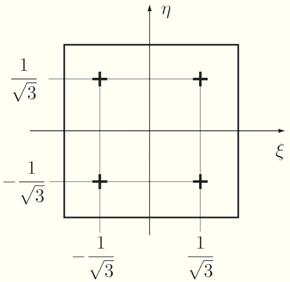
Comparing the coefficients of the constants in equations:

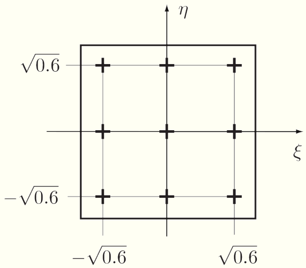
This is a set of four nonlinear equations for determining the values of w_1, w_2, ξ_1 and ξ_2. Symmetry considerations require that

Integration over quadrilateral elements

For the integration over the bi-unit parent element we have:

$$\int_{-1}^{+1}\int_{-1}^{+1} f(\xi,\eta)d\xi d\eta = \int_{-1}^{+1}\left(\int_{-1}^{+1} f(\xi,\eta)d\xi\right)d\eta$$
$$= \sum_{J=1}^{M}\left(\sum_{I=1}^{N} f(\xi_I,\eta_J)w_I\right)w_K$$





Four-point (left) and nine-point (right) integration over bi-unit square.

Evaluation of FE integrals

The integration domain for finite element matrices and vectors is the physical element domain. Therefore, we need to consider the isoparametric mapping from the the parent domain ($[-1,+1] \times [-1,+1]$) to the physical element domain Ω_e:

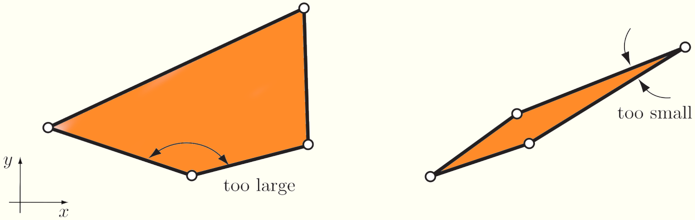
$$\begin{aligned}
\mathbf{K}^e &= \int_{\Omega_e} \mathbf{B}^{e^T}(x,y)\mathbf{D}\mathbf{B}^e(x,y)d\Omega \\
&= \int_{\Omega_e} \mathbf{B}^{e^T}(x,y)\mathbf{D}\mathbf{B}^e(x,y)dxdy \\
&= \int_{-1}^{+1}\int_{-1}^{+1} \mathbf{B}^{e^T}(\xi,\eta)\mathbf{D}\mathbf{B}^e(\xi,\eta)\left|\mathbf{J}^e(\xi,\eta)\right| d\xi d\eta
\end{aligned}$$

where $|\mathbf{J}^e|$ is the Jacobian of the isoparametric mapping and takes care of the mapping of infinitesimal area elements from parent to the physical domain.

$$dxdy = |\mathbf{J}^e|\, d\xi d\eta$$

Number of integration points to use

Modeling considerations: Element geometries



Quadrilateral element geometries to be avoided.

Modeling considerations: Higher-order element

Notice for higher order elements, like the nine-noded one, the position of the mid-nodes contribute to the element distortion. Therefore, they must lie at a certain distance from the corner nodes.