CE394M: Introduction to the Finite Element Method

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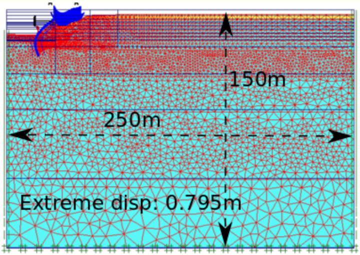
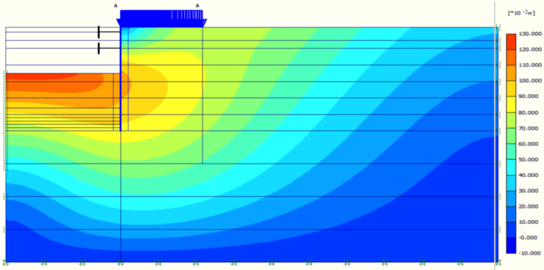
Overview

1. Galerkin methods
2. Strong form
3. Weak form
4. Finite Element formulation
5. Shape functions

Finite Element Analysis



(a) FE Mesh



(b) Displacement profile

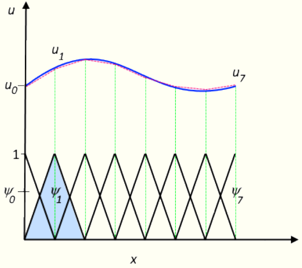
Singapore Nicoll highway excavation FE analysis

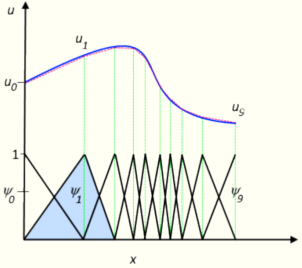
Galerkin:Ritz method

1. Define the functional u for which you wish to find stationary points.
2. Choose a combination of linearly independent functions that will be used to approximate the solution. These will be called 'basis functions'. The amplitudes of these functions will be the unknowns that you will determine. The basis functions must satisfy the Dirichlet ('fixed') boundary conditions.
3. Insert the approximate solution into the functional that is now denoted by u_h.
4. Take the directional derivative of u_h with respect to the unknown amplitudes of the basis functions.
5. Determine the amplitudes of the basis functions which yield a stationary point of u_h.

Finite Element Approximations

FE approximation of u, which is a dependent variable in a PDE.



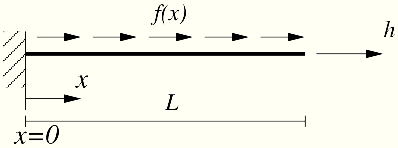


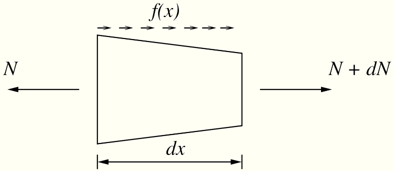
FE basis functions

The function u can be approximated by a function u_h using linear combinations of basis functions according to the following expressions:

$$u \approx u_h \quad u_h = \sum_i u_i \psi_i$$

Strong form of the equilibrium equation for a 1-D bar





where f is a distributed force and h as a force applied at the end of the bar

The equilibrium equation can be derived by considering an infinitesimal bar:

$$-\frac{dN}{dx} = f$$

where N is the normal force in the bar and f is the distributed force along the bar.

Boundary value problem of a 1-D bar

For linear elasticity

$$N = A\sigma = EA\frac{du}{dx} = EA\varepsilon$$

where $A(x)$ is the area of the bar, $E(x)$ is Young's modulus u is the displacement and $\varepsilon = du/dx$ is the strain.

$$-\frac{d}{dx}\left(EA\frac{du}{dx}\right) = f$$

which is a second-order differential equation. BCs:

1. $u = 0$ at $x = 0$ (displacement or 'Dirichlet' boundary condition),
2. $EA\varepsilon = h$ at $x = L$ (force or 'Neumann' boundary condition).

We now have a well-defined boundary value problem that can be solved.

Weak form of the equilibrium equations of a 1D bar

The general derivation of the weak form of any equation from the strong form follows a standard procedure:

1. Multiply the strong equation by a weight function v which is equal to zero where Dirichlet (displacement) boundary conditions are applied, but is otherwise arbitrary (Another condition is that it must be sufficiently continuous. The degree of continuity required depends on the properties of the equation being considered.)
2. Use integration by parts to 'shift' derivatives to the weight function
3. Insert the Neumann (force) boundary conditions

We then want to find a solution u to the weak form that holds for all v . The weight function is also known as the '*test*' function.

Weak form of the equilibrium equations of a 1D bar

Multiplying equilibrium equation by an arbitrary weight function v and integrating along the bar:

$$-\int_0^L v \frac{dN}{dx} \, \mathrm{d}x. = \int_0^L vf \, \mathrm{d}x.$$

we require that $v(0) = 0$ because of the displacement boundary condition at $x = 0$.

$$\int_0^L \frac{dv}{dx} N \, \mathrm{d}x. = \int_0^L vf \, \mathrm{d}x + vN|_{x=0}^{x=L}.$$

Since $v(0) = 0$, inserting the constitutive relationship and taking into account the force boundary condition at $x = L$.

$$\int_0^L \frac{dv}{dx} EA \frac{du}{dx} \, \mathrm{d}x. = \int_0^L vf \, \mathrm{d}x + v(L)h.$$

The task is to find u with $u(0) = 0$ such that this equation is satisfied for all v.

FE formulation of a 1-D bar

Approximates the PDE by replacing the unknown function (the displacement u of the elastic bar) and the weight function (v) by approximate fields u_h and v_h. Inserting these fields into the weak equilibrium equation for a bar:

$$\int_0^L \frac{dv_h}{dx} EA \frac{du_h}{dx} \, \mathrm{d}x. = \int_0^L v_h f \, \mathrm{d}x + v_h(L) h.$$

We now allow only a limited number of possibilities for v_h and u_h. Therefore, it is now unlikely that u_h will be equal to the exact solution.

Basis functions

The approximate displacement field u_h is represented by a set of '*basis functions*' $N_i(x)$:

$$u_h(x) = \sum_i^n N_i(x) a_i$$

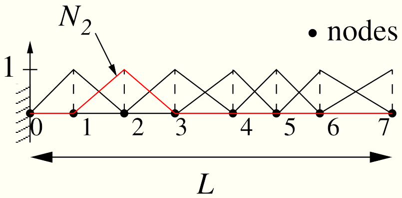
x_i: discrete number of points (known as nodes)
a_i: degrees of freedom
n: number of nodes
The approximate strain field:

$$\varepsilon_h = \frac{du_h}{dx} = \sum_i^n \frac{dN_i(x)}{dx} a_i$$

The task of the FE formulation will be to find the coefficients a_i, which will the approximate solution u_h and ε_h.

FE shape functions

The simplest finite element basis functions in 1D hat-like continuous, piece-wise linear polynomials.

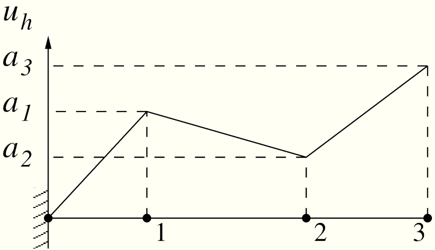


Each node i has its own shape function and its own degree of freedom. A shape function is equal to one at its own node, and zero at all others.

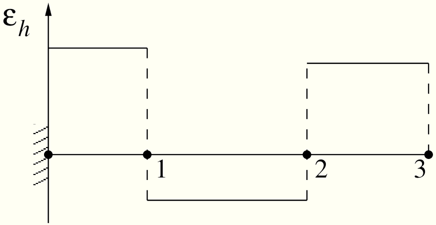
$$N_i(x) = \begin{cases} \frac{x - x_{i-1}}{x_i - x_{i-1}} & x_{i-1} < x \leq x_i \,, \\ \frac{x_{i+1} - x}{x_{i+1} - x_i} & x_i < x < x_{i+1} \,, \\ 0 & \text{otherwise.} \end{cases}$$

FE shape functions

For a bar divided into three elements, the displacement and strain fields could have the form



(a) displacement field



(b) strain field

FE formulation of a 1-D bar

Weak form:

$$\int_0^L EA \frac{dv_h}{dx}\frac{du_h}{dx}\,\mathrm{d}x. = \int_0^L v_h f\,\mathrm{d}x + v_h(L)h.$$

Using basis functions for u_h and v_h:

$$\int_0^L EA\left(\sum_i^n \frac{dN_i}{dx}a_i^*\right)\left(\sum_j^n \frac{dN_j}{dx}a_j\right)\mathrm{d}x = \int_0^L \left(\sum_i^n N_i a_i^*\right) f\,\mathrm{d}x + \left(\sum_i^n N_i(L)a_i^*\right) h$$

since a_i^* and a_j are not a function of x, we take it out.

$$\sum_i^n a_i^*\left(\sum_j^n a_j \int_0^L EA\frac{dN_i}{dx}\frac{dN_j}{dx}\,\mathrm{d}x\right) = \sum_i^n a_i^* \int_0^L N_i\, f\,\mathrm{d}x + \sum_i^n N_i(L)a_i^*\, h$$

FE formulation of a 1-D bar

Since a_i^* is arbitrary for each i we set $a_{k=i}^* = 1$ and $a_{k \neq i}^* = 1$. Then for each i we have an equation with n unknowns (the values of a_j):

$$i = 1: \quad \sum_j^n a_j \int_0^L EA \frac{dN_1}{dx} \frac{dN_j}{dx} dx = \int_0^L N_1 f dx + N_i(L) h,$$

$$i = 2: \quad \sum_j^n a_j \int_0^L EA \frac{dN_2}{dx} \frac{dN_j}{dx} dx = \int_0^L N_2 f dx + N_i(L) h,$$

$$\vdots$$

$$i = n: \quad \sum_j^n a_j \int_0^L EA \frac{dN_n}{dx} \frac{dN_j}{dx} dx = \int_0^L N_n f dx + N_i(L) h,$$

FE formulation of a 1-D bar

A system of linear equations is most conveniently expressed as a matrix:

$$\mathbf{K}\mathbf{a} = \mathbf{b}$$

Stiffness matrix:

$$K_{ij} = \int_0^L EA \frac{dN_i}{dx} \frac{dN_j}{dx} dx$$

right-hand side vector:

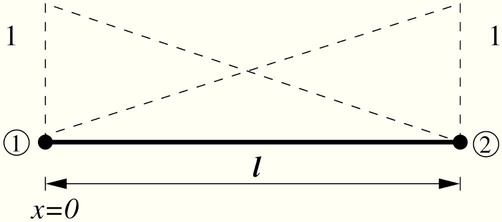
$$b_i = \int_0^L N_i f dx + N_i(L) h.$$

Finite Element Method: Formulation

$$
\begin{aligned}
[\mathbf{K}]\,\mathbf{u} &= F \\
\mathbf{u} &= [\mathbf{K}]^{-1}\,F
\end{aligned}
$$

	Property [K]	**Behavior $\{\mathbf{u}\}$**	**Action $\{\mathbf{F}\}$**
Elastic	stiffness	displacement	force
Thermal	conductivity	temperature	heat source
Fluid	viscosity	velocity	body force

Linear shape functions



A polynomial shape function is equal to one at its own node, and zero at all other nodes of the element

The displacement field inside the element is given by

$$
\begin{aligned}
u_h(x) &= N_1(x)a_1 + N_2(x)a_2 \\
&= \left(-\frac{x}{l} + 1\right) a_1 + \left(-\frac{x}{l}\right) a_2
\end{aligned}
$$

Linear shape functions: displacements

Writing displacement field using matrices and vectors:

$$u_h = \mathbf{N}\mathbf{a_e}$$

where the matrix $\mathbf{N}$ has the shape functions:

$$\mathbf{N} = \begin{bmatrix} N_1(x) & N_2(x) \end{bmatrix}$$

Matrix $\mathbf{a_e}$ contains the degrees of freedom for an element:

$$\mathbf{a_e} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

Linear shape functions: strains

The strain field is written as:

$$
\begin{aligned}
\varepsilon_h(x) &= \frac{dN_1(x)}{dx} a_1 + \frac{dN_2(x)}{dx} a_2 \\
&= \left(-\frac{1}{l}\right) a_1 + \left(-\frac{1}{l}\right) a_2
\end{aligned}
$$

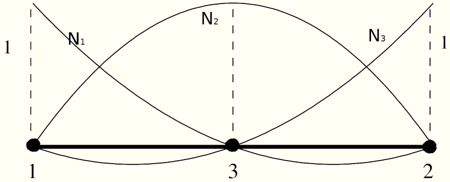
Strain inside an element:

$$\varepsilon_h = \mathbf{B}\mathbf{a_e}$$

where the matrix **B** is the derivatives of the shape functions:

$$\mathbf{B} = \begin{bmatrix} \frac{dN_1(x)}{dx} & \frac{dN_2(x)}{dx} \end{bmatrix}$$

Quadratic element: shape functions



The shape functions:

$$N_1 = a_1 + b_1 x + c_1 x^2\,,$$
$$N_2 = a_2 + b_2 x + c_2 x^2\,,$$
$$N_3 = a_3 + b_3 x + c_3 x^2$$

$x_1 = -1$, $x_2 = 1$ and $x_3 = 0$:

$$N_1 = \frac{x^2}{2} - \frac{x}{2}\,,$$
$$N_2 = \frac{x^2}{2} + \frac{x}{2}\,,$$
$$N_3 = -x^2 + 1$$