CE394M: Isoparametric elements and Gauss integration

Krishna Kumar

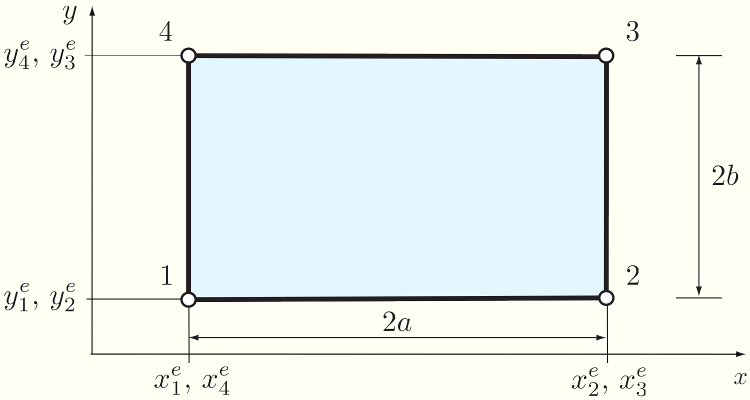
University of Texas at Austin

krishnak@utexas.edu

February 9, 2019

Overview

4-noded rectangular element



Four-node rectangular element. The nodes are by definition numbered counter-clockwise.

4-noded rectangular element

As the element has four nodes, it is necessary to start with a polynomial that has four parameters.

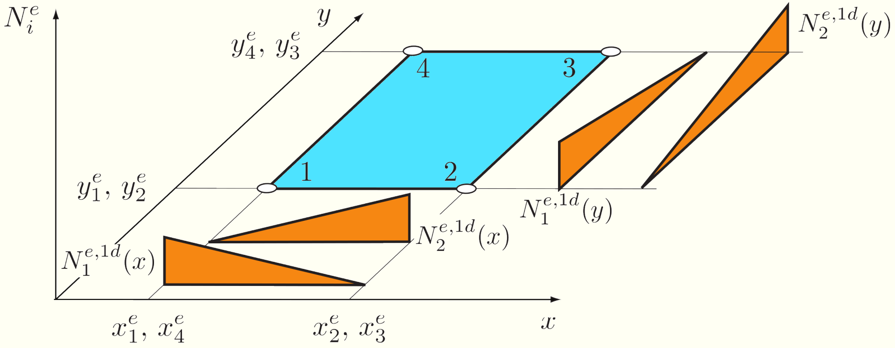
$$Te = \alpha_0^e + \alpha_1^e x + \alpha_2^e y + \alpha_3^e xy$$

It is possible to express $(\alpha_0^e, \alpha_1^e, \alpha_2^e, \alpha_3^e)$ in terms of the nodal values $(T_1^e, T_2^e, T_3^e, T_4^e)$. A derivation shape functions is tedious as it is necessary to invert a 4 x 4 matrix.

The Shape Functions should be 1 at each node, and 0 otherwise can be used to determine the 4 coefficients.

4-noded rectangular element

An alternative and more elegant approach is to construct the shape functions by the **tensor product method.** This is based on taking products of one-dimensional shape functions.



Construction of two dimensional shape functions.

4-noded rectangular element

For example, N_2^e, which has to have the value one at node 2 and zero at the other nodes, is obtained by taking the product of the one-dimensional shape functions $N_2^{e,1d}(x)$ and $N_1^{e,1d}(y)$.

$$N_2^e = N_2^{e,1d}(x) \times N_1^{e,1d}(y)$$

As visible in the figure above the product $N_2^{e,1d}(x) \times N_1^{e,1d}(y)$ has the value one at node 2 and is zero at nodes 1, 3 and 4.

$$N_2^e(x,y) = \frac{x - x_1^e}{x_2^e - x_1^e}\frac{y - y_4^e}{y_1^e - y_4^e} = -\frac{1}{A^e}(x - x_1^e)(y - y_4^e)$$

4-noded rectangular element

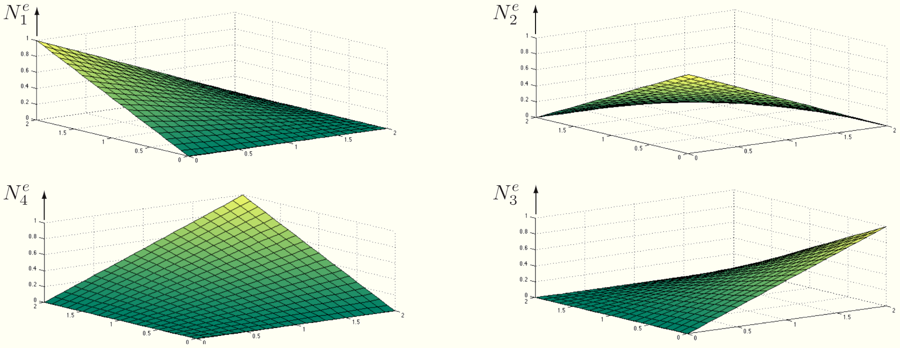
The four shape functions, also called **bilinear shape functions**, for the quadrilateral element are:

$$
\begin{aligned}
N_1^e(x,y) &= \frac{x - x_2^e}{x_1^e - x_2^e}\frac{y - y_4^e}{y_1^e - y_4^e} = -\frac{1}{A^e}(x - x_2^e)(y - y_4^e)\\
N_2^e(x,y) &= \frac{x - x_1^e}{x_2^e - x_1^e}\frac{y - y_4^e}{y_1^e - y_4^e} = -\frac{1}{A^e}(x - x_1^e)(y - y_4^e)\\
N_3^e(x,y) &= \frac{x - x_1^e}{x_2^e - x_1^e}\frac{y - y_1^e}{y_4^e - y_1^e} = -\frac{1}{A^e}(x - x_1^e)(y - y_1^e)\\
N_4^e(x,y) &= \frac{x - x_2^e}{x_1^e - x_2^e}\frac{y - y_1^e}{y_4^e - y_1^e} = -\frac{1}{A^e}(x - x_2^e)(y - y_1^e)
\end{aligned}
$$

where A^e is the area of the element.

4-noded rectangular element

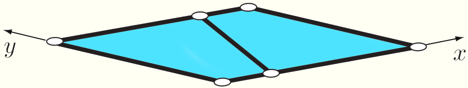
The four shape functions are plotted in the following figure:

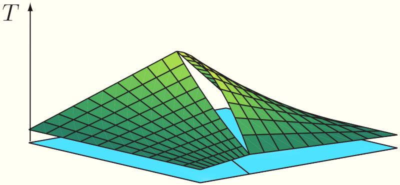


Four shape functions of the rectangular element (on $[0, 2] \times [0, 2]$).

4-noded rectangular element

If the sf equations are used for interpolating the temperature field over arbitrary quadrilaterals, the scalar field (e.g., temperature) across the element boundaries will be not continuous.

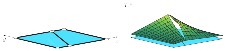




Two element mesh with rectangle shape functions. Although the two nodal values on the edge agree, the temperature distribution is still discontinuous.

4-noded rectangular element

If the sf equations are used for interpolating the temperature field over arbitrary quadrilaterals, the scalar field (e.g., temperature) across the element boundaries will be not continuous.

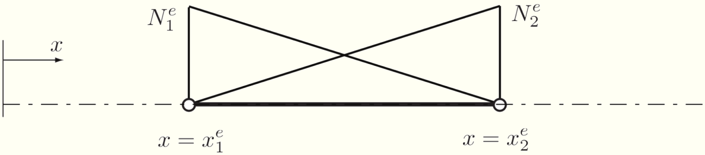


Two element mesh with rectangle shape functions. Although the two nodal values on the edge agree, the temperature distribution is still discontinuous.

The computed shape functions are suitable for rectangles and could be used with meshes consisting only of rectangles, but they are not suitable for arbitrary quadrilaterals.

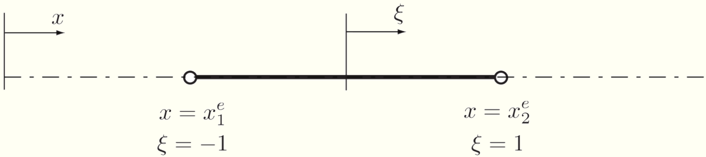
Therefore, these shape functions are of limited use for practical applications. To obtain the shape functions for arbitrary quadrilaterals we need to visit the idea of isoparametric mapping.

Isoparametric mapping in 1D

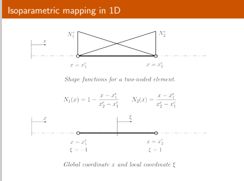


Shape functions for a two-noded element.

$$N_1(x) = 1 - \frac{x - x_1^e}{x_2^e - x_1^e} \qquad N_2(x) = \frac{x - x_1^e}{x_2^e - x_1^e}$$



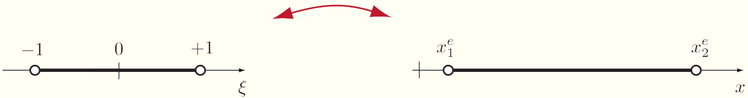
Global coordinate x and local coordinate ξ



Although isoparametric mapping is not particularly useful in one dimension, it is very helpful for understanding the general approach.

Isoparametric mapping in 1D

consider a coordinate transformation which transforms (maps) the coordinate x into a local (element specific) coordinate ξ:



Mapping of the parent element onto the physical element.

The coordinate ξ fulfills the relationships:
$x = x_1^e$ at $\xi = -1$ and $x = x_2^e$ at $\xi = 1$
"stretch transformation" of $x(\xi)$:

$$x(\xi) = x_1^e + \frac{1}{2}(x_2^e - x_1^e)(1 + \xi)$$
$$= \frac{x_1^e + x_2^e}{2} + \frac{x_2^e - x_1^e}{2}\xi$$

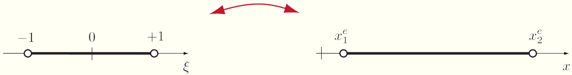
Isoparametric mapping in 1D

The shape functions $N_1^e = (1 - x/l)$ and $N_2^e = (x/l)$ can also be expressed using ξ:

$$N_1^e(\xi) = \frac{1}{2}(1 - \xi) \qquad N_2^e(\xi) = \frac{1}{2}(1 + \xi)$$

The key idea of the isoparametric concept is to use these shape functions for writing the coordinate transformation between x and ξ

$$\begin{aligned} x(\xi) &= N_1^e(\xi) x_1^e + N_2^e(\xi) x_2^e \\ &= \frac{1}{2}(1 - \xi) x_1^e + \frac{1}{2}(1 + \xi) x_2^e \\ &= \frac{x_1^e + x_2^e}{2} + \frac{x_2^e - x_1^e}{2}\xi \end{aligned}$$



Mapping of the parent element onto the physical element.

Isoparametric mapping in 1D

- The coordinate ξ is usually called the **natural coordinate** and always lies by definition between -1 and +1.
- The parent element is solely for numerical purposes.
- The finite element analysis is still performed over the physical domain.

In an isoparametric element the field variable, like displacement, is approximated with the same set of shape functions as those used for the coordinate transformation:

$$u(\xi) = N_1^e(\xi)a_1^e + N_2^e(\xi)a_2^e$$

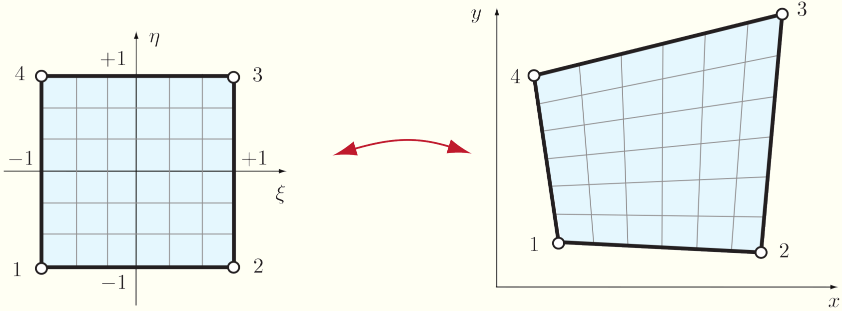
To compute the derivatives which appear in the weak form the chain rule is used:

$$\frac{du}{dx} = \frac{du}{d\xi}\frac{d\xi}{dx}$$

The derivative $d\xi/dx$ is determined from the mapping between ξ and x

Isoparametric mapping of a quadrilateral element

The idea of isoparametric mapping is used for deriving shape functions for arbitrary quadrilateral elements:



Mapping of the bi-unit parent element onto the quadrilateral element in the physical space.

Isoparametric mapping of a quadrilateral element

The bi-unit square is the parent domain and ξ and η are its natural coordinates.
To map points from the parent domain onto the quadrilateral in the physical domain the four nodal shape functions are used:

$$x(\xi, \eta) = \mathbf{N}^{4Q}(\xi, \eta) x^e \quad y(\xi, \eta) = \mathbf{N}^{4Q}(\xi, \eta) y^e$$

where $N^{4Q}(\xi, \eta)$ are the four-node element shape functions in the natural coordinates and x^e and y^e are the vectors of the element coordinates:

$$\mathbf{x}^e = \begin{bmatrix} x_1^e \\ x_2^e \\ x_3^e \\ x_4^e \end{bmatrix} \quad \mathbf{y}^e = \begin{bmatrix} y_1^e \\ y_2^e \\ y_3^e \\ y_4^e \end{bmatrix}$$

Isoparametric mapping of a quadrilateral element

As the parent element is a bi-unit square its shape functions are identical to those of the rectangular element expressed in ξ and η coordinates.

$$N_1^{4Q}(\xi,\eta) = \frac{1}{4}(1-\xi)(1-\eta)$$
$$N_2^{4Q}(\xi,\eta) = \frac{1}{4}(1+\xi)(1-\eta)$$
$$N_3^{4Q}(\xi,\eta) = \frac{1}{4}(1+\xi)(1+\eta)$$
$$N_4^{4Q}(\xi,\eta) = \frac{1}{4}(1-\xi)(1+\eta)$$

Isoparametric shape functions

The temperature will be approximated with the same shape functions:

$$T^e = \mathbf{N}^{4Q}(\xi, \eta)\mathbf{a}^e$$

The element is called *isoparametric* because the temperature approximation and the mapping of the geometry is accomplished with the same shape functions.

The displacement will be approximated as:

$$\mathbf{u}^e = \mathbf{N}^{4Q}(\xi, \eta)\mathbf{a}^e$$

$$= \begin{bmatrix} N_1^{4Q} & 0 & N_2^{4Q} & 0 & N_3^{4Q} & 0 & N_4^{4Q} & 0 \\ 0 & N_1^{4Q} & 0 & N_2^{4Q} & 0 & N_3^{4Q} & 0 & N_4^{4Q} \end{bmatrix} \begin{bmatrix} a_{1x}^e \\ a_{1y}^e \\ a_{2x}^e \\ a_{2y}^e \\ a_{3x}^e \\ a_{3y}^e \\ a_{4x}^e \\ a_{4y}^e \end{bmatrix}$$