CE394M: FEM solvers and errors

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February 10, 2019

Overview

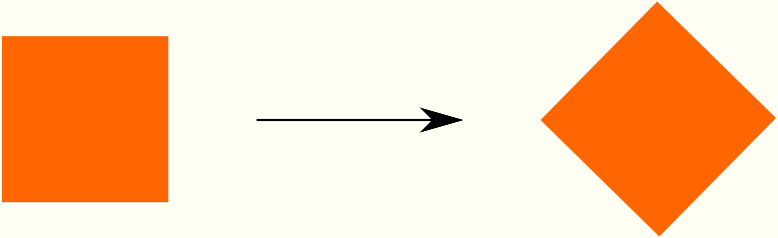
1. Solving non-linear problems
   - Iterative solvers
   - Tangent stiffness
   - Newton Raphson

Linear and non-linear problems

- **Linear problems**
  - The response can only be approximated as linear if its deformations/motions are small.
  - In linear analyses, the response to individual load cases can be scaled and added to the results from other linear analyses, which is the principle of superposition.
- **Non-linear problems**
  - Superposition is invalid.
  - The solution is an incremental/iterative process.
  - An iteration is the solution of a system of equations linearised about the current state of the nonlinear physical problem.

Non-linearity in geotechnical engineering

- **Material non-linearity**
  - plasticity
- **Contact**
  - discontinuous source of non-linearity
- **Large deformations and motions of a geotechnical structures**
  - rotation, rigid body motion
  - often ignored, still in the research area.



No strains if linear strain-displacement relations are used.

Linear solvers

To solve a system of linear equations of the form $\mathbf{Ka} = \mathbf{b}$, there are two families of methods of solvers that can be used: direct and iterative.

- Direct solvers solve a system of linear equations in a predefined number of steps.
- Methods are based on Gauss elimination, with the most common method being LU decomposition.
- The time required to solve a linear system increases with the number of computer operations performed.
- For a direct linear solver applied to a dense system of size $n \times n$: CPU time $= Cn^3$.
- If the number of degrees of freedom in a finite element simulation is doubled, the time required to solve the system will increase by a factor of 8!

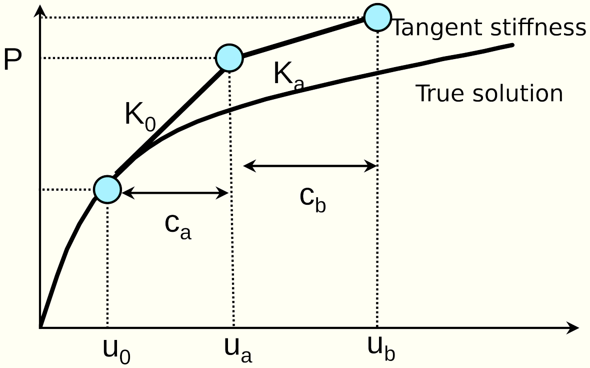
Iterative solvers

The system of equations that we derived from the finite element approximation of the BVP:

$$F_{\mathrm{int}}(u) - F_{\mathrm{ext}} = 0$$

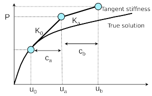
At this point we remind ourselves that in the case of finite deformations, both F_{int} and F_{ext} are in general very nonlinear terms.
The integration has to be carried over the current volume and surface (of the finite element under consideration) that may in general depend on u in a highly non-linear fashion.

Tangent stiffness method



- Many small increments are need to obtain accurate solution
- Need to perform a parametric study to find the optimum incremental
- Defining increments is very important
- Program - SAGE CRISP

Tangent stiffness method



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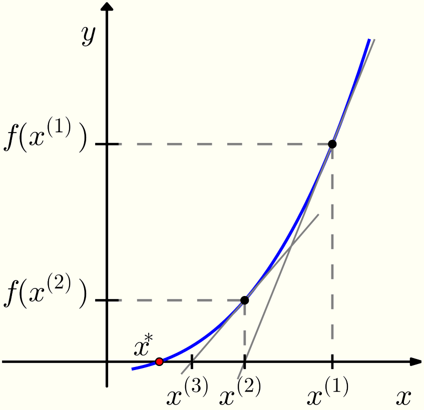
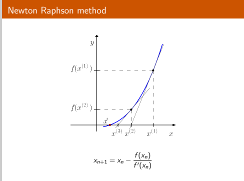
In the incremental solution we divide the load into increments $\Delta P = \lambda P$, where λ is also known as a load factor and apply a repeated solution of $\Delta u = K^{-1}\Delta P$.

Basically we divide the load into substeps, and treat each as linear - but that is usually not accurate enough and inefficient.

Newton Raphson method



$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$



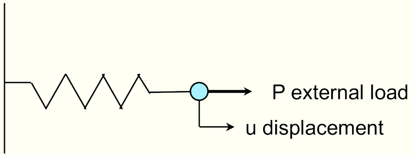
Newton's Method In Newton's method, the incremental loading is expressed as follows. The external load vector F_{ext} is gradually increased from 0 in order to reach a desired value $\mathbf{F}^*$. Assuming that $\mathbf{F}^*$ itself remains constant during the analysis in terms of its 'direction' and only its magnitude is changing, we can write $F_{\text{ext}} = q$ known just to simplify our expression for the system of equations. Then we can control how the external

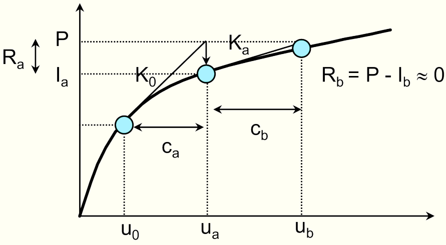
load vector increases or decreases by introducing a scalar quantity λ and express the system as follows:

$$R(u) = F_{\text{int}} - F_{\text{ext}} - > R(u) = F_{\text{int}} - \lambda F_{\text{ext}} == 0$$

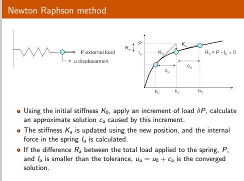
Thus by increasing or decreasing λ we can control our load incrementation.

Newton Raphson method





- Using the initial stiffness K_0, apply an increment of load δP, calculate an approximate solution c_a caused by this increment.
- The stiffness K_a is updated using the new position, and the internal force in the spring I_a is calculated.
- If the difference R_a between the total load applied to the spring, P, and I_a is smaller than the tolerance, $u_a = u_0 + c_a$ is the converged solution.



- If R_a is not small, a new displacement correction c_b is calculated by solving $c_b = R_a/K_a$
- The new displacement u_b is updated, and the internal force I_b in the updated configuration is calculated.
- The new force residual R_b is obtained. If $R_b < tolerance$, the solution is converged. If not, continue the iteration.

Newton Raphson method: Tolerance and Convergence

- Program - ABAQUS
- Newton-Raphson is the most standard method to solve nonlinear problems in FE.
- A large error in the initial estimate can contribute to non-convergence of the algorithm.
- **Tolerance**
  - must be small enough to ensure that the approximate solution is close to the exact mathematical solution.
  - must be large enough so that reasonable number of iterations are performed.
- **Quadratic convergence**
  - If the tangent stiffness is calculated correctly, R should reduce quadratically from one iteration to the next.