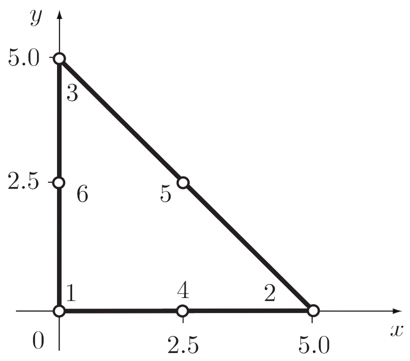
Assignment 4: FEM: Isoparametric elements, Gauss integration, solvers and errors (20 points)
Assigned: 25th February 2019
Due: 8th March 2019

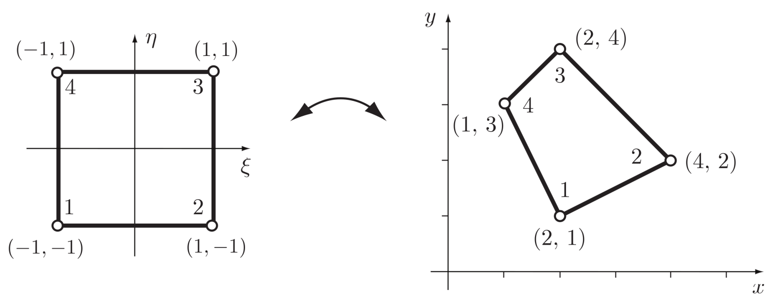
1. For the six noded-triangle shown. The nodal temperatures are $T^e = \begin{bmatrix}300 & 0 & 0 & 0 & 340 & 0\end{bmatrix}^T$. Compute the temperature and its gradient at the point P with the coordinates $x = 1.5$ and $y = 2.0$.



The shape functions for a 6-noded triangle element are:

$$N_1 = 2(1-\xi-\eta)^2 - (1-\xi-\eta) \qquad N_2 = 2\xi^2 - \xi$$
$$N_3 = 2\eta^2 - \eta \qquad N_4 = 4\xi(1-\xi-\eta)$$
$$N_5 = 4\xi\eta \qquad N_6 = 4\eta(1-\xi-\eta)$$

2. For the isoparametric mapping shown below
   (a) Compute the x and y coordinates of the point $\xi = 0.5$, $\eta = 0.5$ in the physical domain.
   (b) Compute $\frac{\partial N_1}{\partial x}$ for the same point.



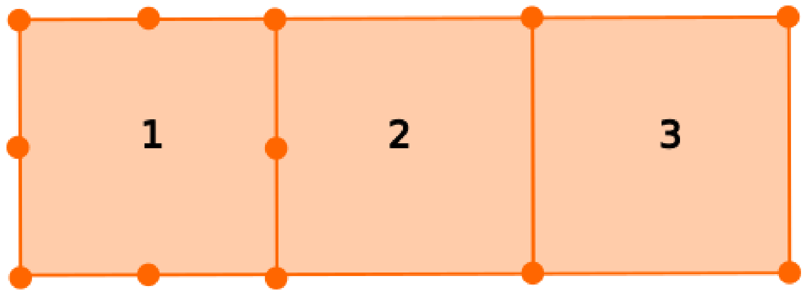
3. Evaluate the following three integrals using one, two and three-point Gauss integration. Compare your results with the results of analytic integration.

(a) $\int_{-1}^{+1}(3\xi^2 + 2\xi)d\xi$

(b) $\int_{-1}^{+1} \cos \xi d\xi$

(c) $\int_0^3 (3x^2 + x)dx$

4. Consider the following quadrilateral mesh, sketch (1D is fine) the temperature distribution across the mesh assuming a hear source on the left boundary of the mesh and comment on the suitability of the mesh for finite element computation and the need for the transition element (element #2)?



5. Develop a Python script for the Newton Raphson method to solve non-linear force-displacement relationships. Test and comment on the accuracy and efficiency of the NR to solve: $f = -2u^2 + 2u$, where u denotes the displacement and f refers to the internal force. For the analytical solution plot the displacement u between 0 and 1.

6. Two FE simulations of a 3D foundation settlement are performed using a structured mesh of tri-linear (8-noded) hexahedral elements. Assume the geometry of the mesh is cubic and all the elements are hexahedron and of the same size. Case A uses n elements and Case B uses $2n$ elements, where n is large.

   (a) How many non-zero entries would you expect in most rows of the global stiffness matrix for Case A and Case B?

   (b) Compute the approximate increase in memory required to build the global stiffness matrix for Case B compared to Case A?

   (c) Cases A and B are solved using an LU solver. Based on the theoretical complexity of the LU solver estimate the increase in solver time in going from Case A to Case B.

   (d) If the error in L^2 norm of the solution is proportional to Ch^2, where h is the length of an element edge and C is a problem constant that does not depend on h, estimate the reduction in the error for Case B compared to Case A.