CE394M: FEM solvers and errors

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Overview

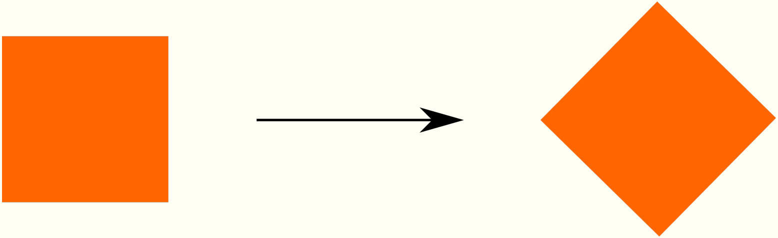
1. Solving non-linear problems
   - Iterative solvers
   - Tangent stiffness
   - Newton Raphson

Linear and non-linear problems

- **Linear problems**
  - The response can only be approximated as linear if its deformations/motions are small.
  - In linear analyses, the response to individual load cases can be scaled and added to the results from other linear analyses, which is the principle of superposition.
- **Non-linear problems**
  - Superposition is invalid.
  - The solution is an incremental/iterative process.
  - An iteration is the solution of a system of equations linearised about the current state of the nonlinear physical problem.

Non-linearity in geotechnical engineering

- **Material non-linearity**
  - plasticity
- **Contact**
  - discontinuous source of non-linearity
- **Large deformations and motions of a geotechnical structures**
  - rotation, rigid body motion
  - often ignored, still in the research area.



No strains if linear strain-displacement relations are used.

Linear solvers

To solve a system of linear equations of the form $\mathbf{Ka} = \mathbf{b}$, there are two families of methods of solvers that can be used: direct and iterative.

- Direct solvers solve a system of linear equations in a predefined number of steps.
- Methods are based on Gauss elimination, with the most common method being LU decomposition.
- The time required to solve a linear system increases with the number of computer operations performed.
- For a direct linear solver applied to a dense system of size $n \times n$: CPU time $= Cn^3$.
- If the number of degrees of freedom in a finite element simulation is doubled, the time required to solve the system will increase by a factor of 8!

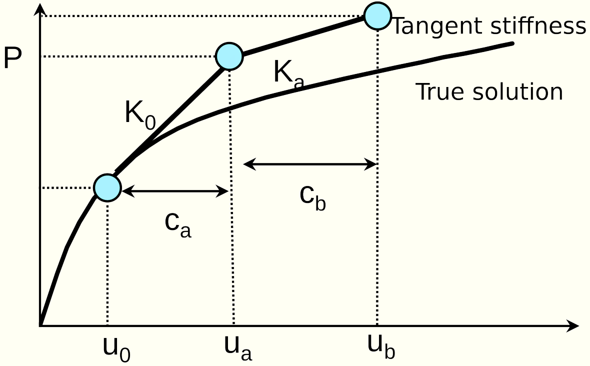
Iterative solvers

The system of equations that we derived from the finite element approximation of the BVP:

$$F_{\mathrm{int}}(u) - F_{\mathrm{ext}} = 0$$

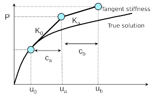
At this point we remind ourselves that in the case of finite deformations, both F_{int} and F_{ext} are in general very nonlinear terms.
The integration has to be carried over the current volume and surface (of the finite element under consideration) that may in general depend on u in a highly non-linear fashion.

Tangent stiffness method



- Many small increments are need to obtain accurate solution
- Need to perform a parametric study to find the optimum incremental
- Defining increments is very important
- Program - SAGE CRISP

Tangent stiffness method



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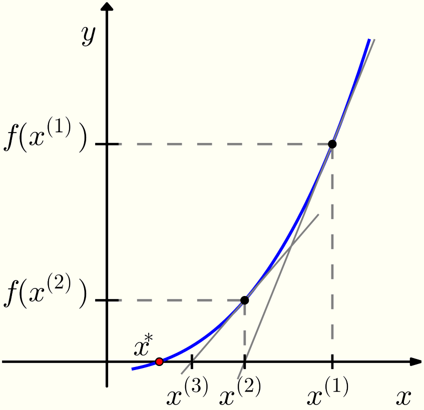
In the incremental solution we divide the load into increments $\Delta P = \lambda P$, where λ is also known as a load factor and apply a repeated solution of $\Delta u = K^{-1} \Delta P$.

Basically we divide the load into substeps, and treat each as linear - but that is usually not accurate enough and inefficient.

Newton Raphson method



$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

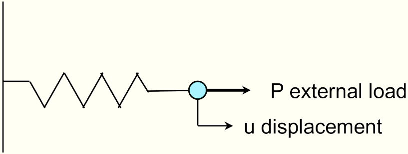
The incremental loading is expressed as follows. The external load vector F_{ext} is gradually increased from 0 in order to reach a desired value $\mathbf{F}^*$. Assuming that $\mathbf{F}^*$ itself remains constant during the analysis in terms of its 'direction' and only its magnitude is changing, we can write $F_{\text{ext}} = q$ known just to simplify our expression for the system of equations. Then we can control how the external load vector increases or decreases by introducing a scalar quantity λ and express the system as follows:

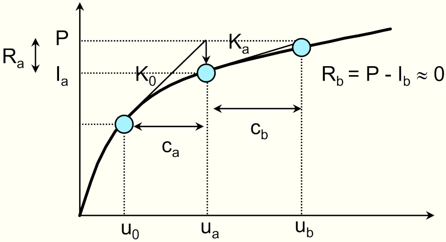
$$R(u) = F_{\text{int}} - F_{\text{ext}} \rightarrow R(u) = F_{\text{int}} - \lambda F_{\text{ext}} = 0$$

Thus by increasing or decreasing λ we can control our load. We are interested in u and λ. At every increment, we change slightly the value of λ and try to determine u satisfying $R(u) = 0$.

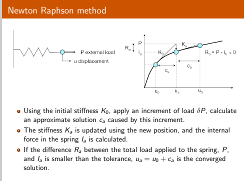
$$\Delta u = [K_T]_{u0}^{-1} - (\Delta \lambda q)$$

Newton Raphson method





- Using the initial stiffness K_0, apply an increment of load δP, calculate an approximate solution c_a caused by this increment.
- The stiffness K_a is updated using the new position, and the internal force in the spring I_a is calculated.
- If the difference R_a between the total load applied to the spring, P, and I_a is smaller than the tolerance, $u_a = u_0 + c_a$ is the converged solution.

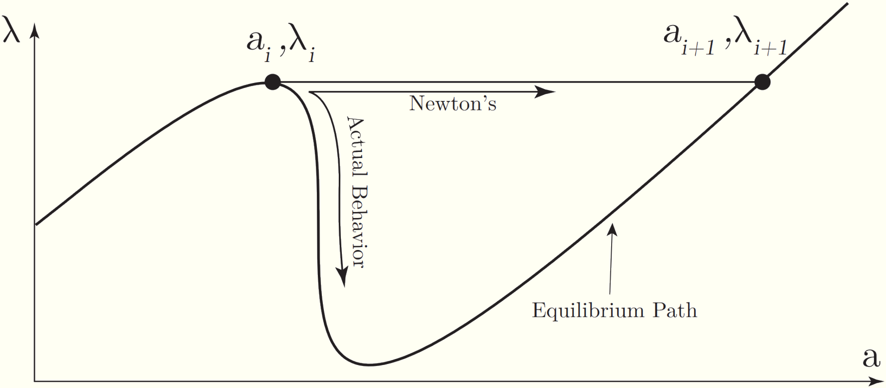


- If R_a is not small, a new displacement correction c_b is calculated by solving $c_b = R_a/K_a$
- The new displacement u_b is updated, and the internal force I_b in the updated configuration is calculated.
- The new force residual R_b is obtained. If $R_b < \mathit{tolerance}$, the solution is converged. If not, continue the iteration.

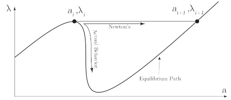
Newton Raphson method: Tolerance and Convergence

- Program - ABAQUS
- Newton-Raphson is the most standard method to solve nonlinear problems in FE.
- A large error in the initial estimate can contribute to non-convergence of the algorithm.
- **Tolerance**
  - must be small enough to ensure that the approximate solution is close to the exact mathematical solution.
  - must be large enough so that reasonable number of iterations are performed.
- **Quadratic convergence**
  - If the tangent stiffness is calculated correctly, R should reduce quadratically from one iteration to the next.

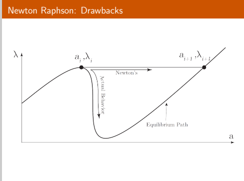
Newton Raphson: Drawbacks



Newton Raphson: Drawbacks

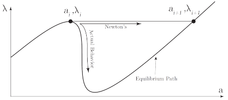


Newton's method fails to accurately follow the 'equilibrium' path once the tangent stiffness reaches zero. That happens due to the formulation of Newton's method, and in particular that it restricts the parameter λ to change monotonically every increment 3 . The definition of a limit point then (saddle points excluded), suggests that in order to remain on the equilibrium path you need to change your loading pattern depending on whether the limit point is a local maximum or maximum in the $u-\lambda$ space. This problem can be better conceptualized in Figure 2.2.



In terms of mechanical systems then, this method is able to solve any non-linear system of equations very efficiently but only up to the critical point (if any). In the case shown in Figure 2.2,Newton's method fails in load–control. Now in many cases, one way to circumvent problems like these is to use displacement control, where you can continuously increase the displacements u and still remain on the equilibrium curve. In general however, apart from Snap–Through behaviors under load control, a problem may exhibit Snap– Back behaviors under displacement control or even both as shown in Figure 2.3. The main problem is, that in most actual applications, the structural response, and therefore the equilibrium path, for the structure under consideration is unknown, and therefore one does not know what type of behavior to expect.

Newton Raphson: Drawbacks



As a general rule, if the problem under consideration requires information after its critical/failure points then Newton's method is not a good choice. Buckling analysis and non-linear materials that exhibit work softening are just two example problems that cannot be solved using Newton's method. Furthermore, very often, strong nonlinearities that arise in finite deformation problems may eventually lead to such behaviors and thus it is necessary to introduce a numerical technique to solve such problem with strongly nonlinear behaviors.