

MATH 204 Cheat Sheet

Simple Linear Regression

$$Y_i = \beta_0 + \beta_1 x_i + \epsilon_i$$

where

$$\hat{\beta}_1 = \frac{\sum_{i=1}^n (y_i - \bar{y})(x_i - \bar{x})}{\sum_{i=1}^n (x_i - \bar{x})^2} = \frac{S_{xy}}{S_{xx}}$$

$$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}$$

β_1 is the change in the mean of Y_i for a 1 unit increase in x_i ,

β_0 is the mean when $x_i = 0$

$$S_{XX} = \sum (x_i - \bar{x})^2, S_{YY} = \sum (y_i - \bar{y})^2,$$

$$S_{XY} = \sum (x_i - \bar{x})(y_i - \bar{y})$$

Estimating σ^2

- Standard deviation of $\hat{\beta}_1$: $\sigma_{\hat{\beta}_1} = \sqrt{\text{var}(\hat{\beta}_1)} = \sigma / \sqrt{S_{XX}}$
- Variance of residuals: $\hat{\sigma}^2 = \frac{1}{n-2} \sum_{i=1}^n (y_i - \hat{y}_i)^2 = \frac{SSE}{n-2}$
- $SSE = S_{YY} - \hat{\beta}_1 S_{XY}$
- $\hat{\sigma}_{\hat{\beta}_1} = \hat{\sigma} / \sqrt{S_{XX}}$

Inference about β_1

- When the error terms are normal, $\hat{\beta}_1 \sim \mathcal{N}(\beta_1, \sigma^2 / S_{XX})$
- $T = \frac{\hat{\beta}_1 - \beta_1}{\hat{\sigma} / \sqrt{S_{XX}}} \sim t_{n-2}$

$$\mathcal{H}_0 : \beta_1 = 0 \quad \text{vs} \quad \mathcal{H}_a : \beta_1 \neq 0$$

$$T_{obs} = \frac{\hat{\beta}_1}{\hat{\sigma}_{\hat{\beta}_1}} = \frac{\hat{\beta}_1}{\hat{\sigma} / \sqrt{S_{XX}}}$$

Compare T_{obs} with the student distribution $t_{n-2, \alpha/2}$ to get RR.

- Could get same conclusion from p-value, which illustrates the probability that our results occurred under \mathcal{H}_0 .
- Confidence interval for β_1 : $\hat{\beta}_1 \pm t_{n-2, \alpha/2} \frac{\hat{\sigma}}{\sqrt{S_{XX}}}$.

ANOVA

- $SS_{reg} = S_{YY} - SSE = \sum (y_i - \bar{y})^2 - \sum (y_i - \hat{y}_i)^2 = \sum (\hat{y}_i - \bar{y})^2$
- $T \sim t_v$, $T^2 \sim \mathcal{F}(1, v)$, where the latter is the Fisher-Snedecor dis.
- ANOVA table guide:
 - (X, Sum Sq) = SS_{reg}
 - (Residuals, Sum Sq) = SSE
 - (Residuals, Df) = $n - 2$

4. lm summary table

- t-value (slope): $T_{obs} = \frac{\hat{\beta}_1}{\hat{\sigma}_{\hat{\beta}_1}}$
- F-statistic : T_{obs}^2
- Residual std error: $\hat{\sigma}$

Correlation

- $\text{corr}(X, Y) = \text{corr}(Y, X)$
- $r = S_{XY} / \sqrt{S_{XX} S_{YY}}$ is an estimator for ρ (the true pop. correlation).
- $(1 - \alpha)100\%$ confidence interval for ρ : transform r to $z = 0.5 \ln\left(\frac{1+r}{1-r}\right)$. Build an interval: $z \pm \frac{z_{\alpha/2}}{\sqrt{n-3}} = (c_l, c_u)$, where $z_{\alpha/2}$ is from the standard Normal table. Then, the interval is $\left(\frac{e^{2c_l} - 1}{e^{2c_l} + 1}, \frac{e^{2c_u} - 1}{e^{2c_u} + 1}\right)$
- Coefficient of determination: $R^2 = 1 - SSE / S_{YY}$

Estimating response

- Mean response confidence interval:
 $\hat{y}_0 \pm t_{n-2, \alpha/2} \hat{\sigma} \sqrt{1/n + (x_0 - \bar{x})^2 / S_{XX}}$
- Individual value Y_0 confidence interval:
 $\hat{y}_0 \pm t_{n-2, \alpha/2} \hat{\sigma} \sqrt{1 + 1/n + (x_0 - \bar{x})^2 / S_{XX}}$

Residual Analysis

- Assumptions: ϵ_i are independent, $E(\epsilon_i) = 0$, $\text{var}(\epsilon_i) = \sigma^2$, $\epsilon_i \sim \mathcal{N}(0, \sigma^2)$
- Check Normality with QQ plot and histogram of the studentized residuals, which have mean 0, all residuals should lie within 3 std deviations.
- Check $E(\epsilon_i) = 0$ by plotting studentized residuals against fitted values. Points should have equal variance and zero mean, i.e. evenly distributed.

Polynomial Regression

$Y_i = \beta_0 + \beta_1 x_i + \beta_2 x_i^2 + \dots + \beta_p x_i^p \epsilon_i$, not all intermediate powers need be present.

Higher-order terms are specified using the $I(\cdot)$ function in R.

- Test that the quadratic term is zero: $H_0 : \beta_2 = 0$.
- If rejected, use linear and quadratic terms in model.
- If not rejected, there is no evidence that the quadratic model gives significant improvement over the linear model.

Multiple Regression (2+ covariates)

$$Y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \dots + \beta_k x_{ik} + \epsilon_i$$

The model is linear in the parameters (β_i), not necessarily in the covariates (x_i). Same assumptions are made about the residuals.

β_j is the change in the mean of Y_i for a 1 unit increase of x_{ij} when holding all other variables constant.

- $\hat{\sigma}^2 = (n - (K + 1))^{-1} \sum (y_i - \hat{y}_i)^2 = SSE / (n - (K + 1))$ where $(K + 1)$ is the number of coefficients β_i in the model.
- Can test each coefficient individually with same hypothesis as in simple regression. In which case, we test for e.g. β_j after adjusting for all other variables.
- Confidence interval for β_j : $\hat{\beta}_j \pm t_{n-(K+1), \alpha/2} \cdot \hat{\sigma}_{\hat{\beta}_j}$
- Global Fit

$$R_a^2 = 1 - \frac{n-1}{n-(K+1)} \left(\frac{SSE}{S_{YY}} \right) = 1 - \frac{n-1}{n-K-1} (1 - R^2)$$

e.g. if $R_a^2 = 0.80$, then we say that the model explains 80% of the variance in Y.

Overall hypothesis:

$$\mathcal{H}_0 : \beta_1 = \beta_2 = \dots = 0 \quad \mathcal{H}_a : \text{at least one } \beta_j \neq 0$$

$$F_{statistic} = \frac{(S_{YY} - SSE) / K}{SSE / (n - (K + 1))} = \frac{R^2 / K}{(1 - R^2) / (n - (K + 1))}$$

\mathcal{H}_0 is rejected for $F > \mathcal{F}_{\alpha, K, n-(K+1)}$.

Interaction

if an interaction is suspected between X_1 and X_2 , we incorporate the interaction by setting

$$\begin{aligned} Y_i &= \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \beta_3 x_{i1} x_{i2} + \epsilon_i \\ &= \beta_0 + \beta_1 x_{i1} + (\beta_2 + \beta_3 x_{i1}) x_{i2} + \epsilon_i \\ &= \beta_0 + (\beta_1 + \beta_3 x_{i2}) x_{i1} + \beta_2 x_{i2} + \epsilon_i \end{aligned}$$

In the above model, a 1-unit increase in x_2 for a fixed x_1 corresponds to an estimated $\hat{\beta}_2 + \hat{\beta}_3 x_1$ increase in Y_i .
 1. Fit the model including the two covariates and the interaction.
 2) Conduct a global F-test with $\mathcal{H}_0 : \beta_1 = \beta_2 = \beta_3 = 0$
 3) If rejected, test for an interaction by using a Student t-test to test $\mathcal{H}_0 : \beta_3 = 0$. If rejected, stop. Otherwise, re-fit the model without the interaction.

Qualitative

Set $Z_i = 0 \forall i$ for reference group and $Z_i = \begin{cases} 1 & \text{if condition i} \\ 0 & \text{otherwise} \end{cases}$

$$Y_i = \beta_0 + \beta_1 z_i + \beta_2 z_2 + \epsilon_i$$

Where $\hat{\beta}_0 = \mu_0$, $\hat{\beta}_1 = \mu_1 - \mu_0$, and $\hat{\beta}_2 = \mu_2 - \mu_0$, and $\mathcal{H}_0 : \beta_1 = \beta_2 = 0 \iff \mathcal{H}_0 : \mu_2 = \mu_1 = \mu_0$, $\mathcal{H}_a : \text{at least 1 } \beta_i \neq 0$