MATH 255 Cheat Sheet

Lecture Notes 1

Definitions

1. Cluster/limit point : Every ε-neighbourhood of x contains a point of S, i.e. every neighbourhood contains infinitely many points, i.e. there exists a sequence in S which converges to x.
2. Closed set $\iff$ contains all its cluster points
3. Interior point, i.e. $x \in S^o$ if $\exists\ \varepsilon$ such that $B(x, \varepsilon) \subseteq S$
4. Isolated point if $\exists \varepsilon$ s.t. $B(x, \varepsilon) \cap S = \{x\}$
5. Boundary point if $\forall \varepsilon$, $B(x, \varepsilon) \cap S \neq \emptyset$ and $B(x, \varepsilon) \cap S^c \neq \emptyset$
6. Closure of a set $\overline{S} = S \cup \partial S = S \cup S'$
7. **Compact** if $\{G_\alpha\}_{\alpha \in I}$ is an open cover of S, $\exists$ a finite subcover s.t. $S \subseteq G_{\alpha_1} \cup ... \cup G_{\alpha_n}$
8. Continuity:

Results

1. K_n a sequence of compact sets s.t. $K_{n-1} \subseteq K_n$, then the intersection of all K_n is compact and non-empty.
2. Perfect $\implies$ uncountable.

Lecture Notes 2 - Metric Spaces

Definitions

1. **Metric space** X:
   (a) $d(x, y) \geq 0 \forall x, y \in X$
   (b) $d(x, y) = 0 \iff x = y$
   (c) $d(x, y) = d(y, x)$
   (d) $d(x, y) \leq d(x, z) + d(z, y) \forall x, y, z \in X$
2. Open ball in X: $B(x, \varepsilon) := \{y \in X\ :\ d(x, y) < \varepsilon\}$
3. S open in X if $\forall x \in S$, $\exists \varepsilon > 0$ s.t. $\{y \in X \mid d(x, y) < \varepsilon\} \subseteq S$
4. Perfect in X if closed and every point is a cp.
5. $E \subseteq X$ is bounded if $\exists x \in X$ and $R > 0$ s.t. $\forall y \in E,\ d(x, y) < R$.
6. S is dense in X if $\overline{S} = X$, i.e. every $x \in S$ is a cp of X, i.e. $\forall x \in X, \forall \varepsilon > 0$, $\exists$ a point of S in $B(x, \varepsilon)$.
7. X is separable if it has a countable dense subset.
8. $x \in X$ is a condensation point if $\forall \varepsilon > 0$, $\exists$ uncountably many points of X in $B(x, \varepsilon)$.
9. $K \subseteq X$ is **sequentially compact** if every infinite subset E of K has a cluster point in K. That is, every sequence in K has a subsequence converging in K.
10. A set $S \subseteq X$ is **totally bounded** if $\forall \varepsilon > 0$, $\exists$ finitely many $x_n \in S$ s.t. $S \subseteq B(x_1, \varepsilon) \cup ... \cup B(x_N, \varepsilon)$.
11. A collections of subsets of E labeled as $\mathcal{F}$ has the **FIP** if whenever $F_1, ..., F_n \in \mathcal{F}$, we have
$$\cap_{i=1}^n F_i \neq \emptyset$$

Results

1. The union of arbitrary open sets is open.
2. The union of finitely many closed sets is closed.
3. The intersection of arbitrary closed sets is closed.
4. The intersection of finitely many open sets is open.
5. $E \subseteq Y \subseteq X$. Then E is open relative to $Y \iff \exists G$ open in X s.t. $E = G \cap Y$.
6. $f : E \to \mathbb{R}$ is continuous on E if the inverse image of any open set in $\mathbb{R}$ is open relative to E.
7. $K \subseteq Y \subseteq X$ Then K is compact relative to X $\iff$ it is compact relative to Y.
8. Compact $\implies$ closed & bounded (in any metric space).
9. Closed subsets of compact sets are compact.
10. F closed, K compact $\implies F \cap K$ compact.
11. **Sequentially Compact** $\iff$ **Compact**.
12. $K \subseteq X$, K is compact $\iff$ K is closed and every collection $\mathcal{F}$ of closed subsets of K which has the FIP satisfies $\cap_{F \in \mathcal{F}} F_i \neq \emptyset$
13. Totally bounded $\implies$ separable.
14. Sequentially compact $\implies$ separable.

Lecture Notes 3 - Sequences & Continuous Functions in Metric Spaces

1. a_n converges to a if $\forall \varepsilon > 0$, $\exists N$ s.t. $n \geq N \implies d(a_n, a) < \varepsilon$
2. X a compact metric space $\implies$ every sequence in X has a convergent subsequence to a point in X.
3. $p_n \in X$ cpt m.s. then the set of subsequential limits is closed.
4. **Cauchy sequence**: A sequence p_n in a metric space (X, d) is Cauchy if for every $\varepsilon > 0$, $\exists N$ s.t. $n \geq m \geq N \implies d(p_n, p_m) < \varepsilon$
5. Convergent $\implies$ Cauchy.
6. X cpt $\implies$ every Cauchy sequence converges to a point in X.
7. **Complete** m.s. if every Cauchy sequence converges.
8. $E \subseteq X,\ f : E \to Y$, $\lim_{n \to p} f(x) = q$ if $\forall \varepsilon > 0, \exists \delta$ s.t.
$$d_X(x, p) < \delta \implies d_Y(f(x), q) < \varepsilon$$
9. $\lim_{x \to p} f(x) = q \iff \forall p_n \neq p$ s.t. $\lim_{n \to \infty} p_n = p$, we have $\lim_{n \to \infty} f(p_n) = q$.
10. $f : X \to Y$ cts at p $\forall \varepsilon > 0, \exists \delta > 0$ s.t. if $d_X(x, p) < \delta \implies d_Y(f(x), f(p)) < \varepsilon$.
11. $f : X \to Y$ is cts $\iff \forall$ open set $V \in Y$, $f^{-1}(V)$ is open in X.
12. $f : X \to Y$ is cts $\iff \forall p_n \to p \in X$, we have $f(p_n) \to f(p) \in Y$.
13. Let X cpt m.s., if $f : X \to Y$ is cts, then $f(X) \subseteq Y$ is cpt.
14. $f : X \to Y$ is **Uniformly continuous** if $\forall \varepsilon$, $\exists \delta > 0$ s.t. $\forall p, q \in X$
$$d_X(p, q) < \delta \implies d_Y(f(p), f(q)) < \varepsilon$$
15. X cpt and $f : X \to Y \implies f$ uniformly cts.
16. **Contraction**: X a complete m.s. and $f : X \to X$ s.t. for $0 < \alpha < 1$,
$$d(f(x), f(y)) \leq \alpha d(x, y)\ \forall x, y \in X$$
17. The Banach Fixed Point Theorem: Let f a contraction on a complete metric space. Then there exists a unique $x \in X$ such that $f(x) = x$.

Lecture Notes 4 - Normed Vector Spaces

1. **Minkowsky inequality**: Let $x = (x_1, ..., x_N)$, $y = (y_1, ..., y_N) \in \mathbb{R}^n$ and let $1 \leq p < \infty$ then,
$\left(\sum_{n=1}^N (|x_n| + |y_n|)^p\right)^{1/p} \leq \left(\sum_{n=1}^N |x_n|^p\right)^{1/p} + \left(\sum_{n=1}^N |y_n|^p\right)^{1/p} = \|x\|_p + \|y\|_p$
2. **Holder Inequality**: let x, y be as above, $1 < p, q < \infty$ be conjugate exponents ($p^{-1} + q^{-1} = 1$) then,
$\sum_{n=1}^N |x_n||y_n| \leq \left(\sum_{n=1}^N |x_n|^p\right)^{1/p} \left(\sum_{n=1}^N |y_n|^q\right)^{1/q} = \|x\|_p \|y\|_q$
3. $\|\cdot\|_A$ and $\|\cdot\|_B$ are **equivalent norms** if $\exists c_1, c_2 > 0$ s.t. $\forall x \in X$, $\|x\|_A \leq c_1 \|x\|_B$ and $\|x\|_B \leq c_2 \|x\|_A$
4. All norms are equivalent in $\mathbb{R}^N$ and on finite v.s.
5. $T : X \to Y$ is bounded if $\exists c > 0$ s.t. $\|T(x)\|_Y \leq c\|x\|_X\ \forall x \in X$.
6. $T : X \to Y$, TFAE:
   a) T is bounded
   b) T is cts at all points in X
   c) T is cts at $x = 0$
7. X f.d. v.s. $\implies T : X \to Y$ is bounded.
8. **Operator norm**: $\|T\|_{op} = \sup_{x \neq 0} \frac{\|T(x)\|_Y}{\|x\|_X} = \sup_{\|x\|_X = 1} \|T(x)\|_Y = \inf\{c \geq 0 \mid \|T(x)\|_Y \leq c\|x\|_X \forall x\}$
9. $F : X \to Y$, $U \subseteq X$ open, $p \in U$, then F is differentiable at p if $\exists$ bounded linear operator $T : X \to Y$ s.t.
$$\lim_{h \to 0} \frac{\|F(p + h) - F(p) - T(h)\|_Y}{\|h\|_X} = 0$$
Then, T is the **derivative** of F at p.

Lecture Notes 5 - Infinite Series

1. $\sum_{k=1}^{\infty} a_k$ converges $\iff$ the sequence of partial sums s_n converges, where $s_n := a_1 + ... + a_n$.
2. Geometric series $\sum_{k=0}^{\infty} r^n$ converges to $\frac{1}{1-r}$ if $r < 1$, diverges o.w.
3. Harmonic series $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges.
4. p-series $\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges $\iff$ $1 < p < \infty$.
5. $\sum a_k$ converges $\implies$ $\lim_{k\to\infty} a_k = 0$.
6. Cauchy criterion for convergence: $\sum a_k$ converges $\iff$ $\forall\varepsilon > 0$, $\exists M \in \mathbb{N}$ s.t. $m > n \geq M \implies |s_m - s_n| = |a_{n+1} + ... + a_m| < \varepsilon$
7. Direct comparison test: $0 \leq a_k \leq b_k\ \forall k > K$, then
   a) $\sum b_k$ converges $\implies$ $\sum a_k$ converges.
   b) $\sum a_k$ diverges $\implies$ $\sum b_k$ diverges.
8. Limit comparison test 1: $0 < a_k, b_k$ s.t. $r = \lim_{k\to\infty} a_k/b_k$
   a) $r \neq 0$: $\sum a_k$ converges $\iff$ $\sum b_k$ converges.
   b) $r = 0$: $\sum b_k$ converges $\implies$ $\sum a_k$ converges.
9. Limit comparison test 2: a_k, b_k s.t. $r = \lim_{k\to\infty} |a_k|/|b_k|$
   a) $r \neq 0$: $\sum a_k$ converges abs. $\iff$ $\sum b_k$ conv. abs.
   b) $r = 0$: $\sum b_k$ converges abs. $\implies$ $\sum a_k$ converges abs.
10. $\sum a_k$ converges $\implies$ every regrouping converges.
11. Absolute convergence $\implies$ convergence
12. Ratio test: let $a_k \neq 0$, s.t. $\lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} = r$, then if
    a) $r < 1$, $\sum a_k$ converges abs.
    b) $r > 0$, $\sum a_k$ diverges.
    c) $r = 0$, inconclusive.
13. Alternating series test: $a_k \geq 0$ non-increasing converging to 0. Then $\sum(-1)^{k+1} a_k$ converges.
14. **Dirichlet's test**: a_k a decreasing sequence s.t. $\lim_{n\to\infty} a_k = 0$ and b_k s.t. the partial sums of b_k are bounded. Then $\sum a_k b_k$ converges.
15. **Abel's test**: a_k a convergent monotone sequence, $\sum b_k$ converges, then $\sum a_k b_k$ converges.
16. a_k converges absolutely $\implies$ every rearrangement of a_k converges to the same point.
17. a_k converges conditionally, α any real number. Then there exists a rearrangement of a_k which converges to α.

Lecture Notes 6 - Integration

1. Tagged partition $\dot{P}$ on $[a, b]$ is defined as $\{[x_{i-1}, x_i], t_i\}_{i=0}^n$ where t_i is the point chosen for the subinterval.
2. $\|\dot{P}\| = \max\{[x_{i-1}, x_i]\}_{i=0}^n$
3. **Riemann sum** $S(f, \dot{P}) := \sum_{i=1}^n f(t_i)(x_i - x_{i-1})$.
   $$\lim_{n\to\infty} \sum_{i=1}^n f(x_i)\Delta x = \int_a^b f(x)dx$$
   where $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$
4. $f \in R[a, b]$ if $\exists L \in \mathbb{R}$ s.t. $\forall\varepsilon > 0, \exists\delta_\varepsilon$ s.t. if $\|\dot{P}\| < \delta_\varepsilon$,
   $$|S(f, \dot{P}) - L| < \varepsilon$$
   Then, L is the *unique* integral.
5. Let $f, g \in R[a, b]$, then
   a) $\int_a^b kf = k\int_a^b f$
   b) $\int_a^b f + g = \int_a^b f + \int_a^b g$
   c) $f(x) \leq g(x) \forall x \in [a, b] \implies \int_a^b f \leq \int_a^b g$ 6
6. $f \in R[a, b] \implies f$ bounded on $[a, b]$
7. **Cauchy criterion**: $f \in R[a, b] \iff \forall\varepsilon > 0, \exists\mu_\varepsilon$ s.t. if $\|\dot{P}\|, \|\dot{Q}\| < \mu_\varepsilon$, then
   $$|S(f, \dot{P}) - S(f, \dot{Q})| \leq \varepsilon$$
8. **Squeeze**: $f \in R[a, b] \iff \forall\varepsilon > 0, \exists\alpha, \omega \in R[a, b]$ s.t.
   $$\alpha(x) \leq f(x) \leq \omega(x) \quad \forall x \in [a, b]$$
   and such that
   $$\int_a^b (\omega - \alpha) < \varepsilon$$
9. $f : [a, b] \to \mathbb{R}$ cts $\implies$ $f \in R[a, b]$.
10. $f \in R[a, b] \implies \forall c \in [a, b],\ f|_{[a,c]} \wedge f|_{[c,b]}$ are Riemann integrable and in particular
    $$\int_a^b f = \int_a^c f + \int_c^b f$$