MATH 255 Cheat Sheet

Lecture Notes 1

Definitions

1. Cluster/limit point : Every ε-neighbourhood of x contains a point of S, i.e. every neighbourhood contains infinitely many points, i.e. there exists a sequence in S which converges to x.
2. Closed set $\iff$ contains all its cluster points
3. Interior point, i.e. $x \in S^o$ if $\exists\ \varepsilon$ such that $B(x, \varepsilon) \subseteq S$
4. Isolated point if $\exists \varepsilon$ s.t. $B(x, \varepsilon) \cap S = \{x\}$
5. Boundary point if $\forall \varepsilon$, $B(x, \varepsilon) \cap S \neq \emptyset$ and $B(x, \varepsilon) \cap S^c \neq \emptyset$
6. Closure of a set $\overline{S} = S \cup \partial S = S \cup S'$
7. **Compact** if $\{G_\alpha\}_{\alpha \in I}$ is an open cover of S, $\exists$ a finite subcover s.t. $S \subseteq G_{\alpha_1} \cup ... \cup G_{\alpha_n}$
8. Continuity:

Results

1. K_n a sequence of compact sets s.t. $K_{n-1} \subseteq K_n$, then the intersection of all K_n is compact and non-empty.
2. Perfect $\implies$ uncountable.

Lecture Notes 2 - Metric Spaces

Definitions

1. **Metric space** X:
   (a) $d(x, y) \geq 0 \forall x, y \in X$
   (b) $d(x, y) = 0 \iff x = y$
   (c) $d(x, y) = d(y, x)$
   (d) $d(x, y) \leq d(x, z) + d(z, y) \forall x, y, z \in X$
2. Open ball in X: $B(x, \varepsilon) := \{y \in X\ :\ d(x, y) < \varepsilon\}$
3. S open in X if $\forall x \in S$, $\exists \varepsilon > 0$ s.t. $\{y \in X \mid d(x, y) < \varepsilon\} \subseteq S$
4. Perfect in X if closed and every point is a cp.
5. $E \subseteq X$ is bounded if $\exists x \in X$ and $R > 0$ s.t. $\forall y \in E,\ d(x, y) < R$.
6. S is dense in X if $\overline{S} = X$, i.e. every $x \in S$ is a cp of X, i.e. $\forall x \in X, \forall \varepsilon > 0$, $\exists$ a point of S in $B(x, \varepsilon)$.
7. X is separable if it has a countable dense subset.
8. $x \in X$ is a condensation point if $\forall \varepsilon > 0$, $\exists$ uncountably many points of X in $B(x, \varepsilon)$.
9. $K \subseteq X$ is **sequentially compact** if every infinite subset E of K has a cluster point in K. That is, every sequence in K has a subsequence converging in K.
10. A set $S \subseteq X$ is **totally bounded** if $\forall \varepsilon > 0$, $\exists$ finitely many $x_n \in S$ s.t. $S \subseteq B(x_1, \varepsilon) \cup ... \cup B(x_N, \varepsilon)$.
11. A collections of subsets of E labeled as $\mathcal{F}$ has the **FIP** if whenever $F_1, ..., F_n \in \mathcal{F}$, we have
$$\cap_{i=1}^n F_i \neq \emptyset$$

Results

1. The union of arbitrary open sets is open.
2. The union of finitely many closed sets is closed.
3. The intersection of arbitrary closed sets is closed.
4. The intersection of finitely many open sets is open.
5. $E \subseteq Y \subseteq X$. Then E is open relative to Y $\iff \exists G$ open in X s.t. $E = G \cap Y$.
6. $f : E \to \mathbb{R}$ is continuous on E if the inverse image of any open set in $\mathbb{R}$ is open relative to E.
7. $K \subseteq Y \subseteq X$ Then K is compact relative to X $\iff$ it is compact relative to Y.
8. Compact $\implies$ closed & bounded (in any metric space).
9. Closed subsets of compact sets are compact.
10. F closed, K compact $\implies$ $F \cap K$ compact.
11. **Sequentially Compact** $\iff$ **Compact**.
12. $K \subseteq X$, K is compact $\iff$ K is closed and every collection $\mathcal{F}$ of closed subsets of K which has the FIP satisfies $\cap_{F \in \mathcal{F}} F_i \neq \emptyset$
13. Totally bounded $\implies$ separable.
14. Sequentially compact $\implies$ separable.
15.

Lecture Notes 3 - Sequences & Continuous Functions in Metric Spaces

Lecture Notes 4 - Normed Vector Spaces

1. **Minkowsky inequality**: Let $x = (x_1, ..., x_N)$, $y = (y_1, ..., y_N) \in \mathbb{R}^n$ and let $1 \leq p < \infty$ then,
$\left(\sum_{n=1}^N (|x_n| + |y_n|)^p\right)^{1/p} \leq$
$\left(\sum_{n=1}^N |x_n|^p\right)^{1/p} + \left(\sum_{n=1}^N |y_n|^p\right)^{1/p} = \|x\|_p + \|y\|_p$
2. **Holder Inequality**: let x, y be as above, $1 < p, q < \infty$ be conjugate exponents ($p^{-1} + q^{-1} = 1$) then,
$\sum_{n=1}^N |x_n||y_n| \leq \left(\sum_{n=1}^N |x_n|^p\right)^{1/p} \left(\sum_{n=1}^N |y_n|^q\right)^{1/q} = \|x\|_p \|y\|_q$
3. $\|\cdot\|_A$ and $\|\cdot\|_B$ are **equivalent norms** if $\exists c_1, c_2 > 0$ s.t. $\forall x \in X$, $\|x\|_A \leq c_1 \|x\|_B$ and $\|x\|_B \leq c_2 \|x\|_A$
4. All norms are equivalent in $\mathbb{R}^N$.
5. $T : X \to Y$ is bounded if $\exists c > 0$ s.t. $\|T(x)\|_Y \leq c\|x\|_X\ \forall x \in X$

Lecture Notes 5 - Infinite Series

1. $\sum_{k=1}^\infty a_k$ converges $\iff$ the sequence of partial sums s_n converges, where $s_n := a_1 + ... + a_n$.
2. Geometric series $\sum_{k=0}^\infty r^n$ converges to $\frac{1}{1-r}$ if $r < 1$, diverges o.w.
3. Harmonic series $\sum_{k=1}^\infty \frac{1}{k}$ diverges.
4. p-series $\sum_{k=1}^\infty \frac{1}{k^p}$ converges $\iff$ $1 < p < \infty$.
5. $\sum a_k$ converges $\implies$ $lim_{k \to \infty} a_k = 0$.
6. Cauchy criterion for convergence: $\sum a_k$ converges $\iff \forall \varepsilon > 0$, $\exists M \in \mathbb{N}$ s.t. $m > n \geq M \implies |s_m - s_n| = |a_{n+1} + ... + a_m| < \varepsilon$
7. Direct comparison test: $0 \leq a_k \leq b_k\ \forall k > K$, then
   a) $\sum b_k$ converges $\implies$ $\sum a_k$ converges.
   b) $\sum a_k$ diverges $\implies$ $\sum b_k$ diverges.
8. Limit comparison test 1: $0 < a_k, b_k$ s.t. $r = \lim_{k \to \infty} a_k / b_k$
   a) $r \neq 0$: $\sum a_k$ converges $\iff$ $\sum b_k$ converges.
   b) $r = 0$: $\sum b_k$ converges $\implies$ $\sum a_k$ converges.
9. Limit comparison test 2: a_k, b_k s.t. $r = \lim_{k \to \infty} |a_k| / |b_k|$
   a) $r \neq 0$: $\sum a_k$ converges abs. $\iff$ $\sum b_k$ conv. abs.
   b) $r = 0$: $\sum b_k$ converges abs. $\implies$ $\sum a_k$ converges abs.
10. $\sum a_k$ converges $\implies$ every regrouping converges.
11. Absolute convergence $\implies$ convergence

Lecture Notes 6 - Integration