Assignment 2, COMP 252, Winter 2024, Jan 21, 2024. Due Jan 30, 2024, 11 pm.

Exercise 1. ALGORITHM DESIGN. You are given n vectors $x_1, \ldots, x_n$ in Z^n, where Z is the set of all integers. Design an efficient algorithm in the RAM model for computing for each x_i one of its nearest neighbors among the other points, using the standard Euclidean metric to measure distances. You can't use real numbers, and operations like square root are not available. Nevertheless, show how this can be done in $o(n^3)$ worst-case time.

Exercise 2. DYNAMIC PROGRAMMING: COMPUTING THE OPTIMAL STAR. We are given a matrix of pairwise distances $D[i, j]$ between cities i and j, where $1 \leq i, j \leq n$. These must be connected in a star, i.e., there is a central city (the core of the star), which, when removed, leaves one or more chains. The objective is to design a dynamic programming algorithm to find the star of the smallest total length. Observe that we restrict the star to one tentacle (chain), then this is the shortest traveling salesman path problem of the January 23, 2024 lecture. Please make sure that your solution does not use more than exponential time (in n).

Exercise 3. INDUCTION. We are given the recurrence

$$T_n = 2T_{n/a} + 7T_{n/a^2} + 1, n \geq a^2,$$

where $a \geq 2$ is a given integer, and n is restricted to be a power of a. We also know that $T_1 = T_a = 1$.

(i) Show by induction that $T_n = \Omega(n^c)$ for some positive constant c, and find (implicitly or explicitly) the largest such c. (Thus, c depends upon a in an intricate manner.)

(ii) Show by induction that $T_n = O(n^c)$ for the same maximal c found in part (i).

Exercise 4. SORTING WITH DUPLICATES. We are concerned here with sorting n numbers with possible duplicates: the total number of different numbers in the input is k, an unknown number between one and n. This can be done in time $O(n \log_2(k + 1))$ with a ternary comparison oracle.

(i) Give a divide-and-conquer algorithm that achieves this.

(ii) Prove the complexity claim.