Corollaries to Fubini and Tonelli's theorems

Corollary 1

If $A \subseteq \mathbb{R}^d$ is measurable, then for a.e. $y \in \mathbb{R}^{d_2}$, A^y is measurable and

$$\int_{\mathbb{R}^{d_2}} m(A^y)\, dy = m(A)$$

Proof: By Tonelli's Theorem, applied to χ_A, we obtain that for a.e. $y \in \mathbb{R}^{d_2}$, $\chi_A^y := \chi_{A^y}$ is measurable, hence $A^y = \chi_{A^y}^{-1}(\{1\})$ is measurable. Moreover, Tonelli's also gives that

$$y \longmapsto \int_{\mathbb{R}^{d_1}} \chi_{A_y} = m(A_y)$$

is measurable, and $\displaystyle\int_{\mathbb{R}^{d_2}} m(A^y)\, dy = \int_{\mathbb{R}^d} \chi_A = m(A)$ ■

Corollary 2: Tonelli's Thm for measurable subsets

Let $A \subseteq \mathbb{R}^d$ be measurable and $f : A \mapsto [0,\infty]$ measurable.

(i) For a.e. $y \in \mathbb{R}^{d_2}$, f^y is measurable on A^y (which is measurable by cor. 1)

(ii) $y \longmapsto \int_{A^y} f^y$ is measurable on $\mathbb{R}^{d_2}$

(iii) $\displaystyle\int_{\mathbb{R}^{d_2}} \left(\int_{A^y} f^y \right) dy = \int_A f$

Proof: Apply Tonelli's to $\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases}$ ■

Corollary 3: Fubini's Theorem for measurable sets

Let $A \subseteq \mathbb{R}^d$ be measurable and $f: A \to \overline{\mathbb{R}}$ integrable over A.

(i) For a.e. $y \in \mathbb{R}^{d_2}$, f^y is integrable over A^y

(ii) $y \mapsto \int_{A^y} f^y$ is integrable over $\mathbb{R}^{d_2}$

(iii) $\int_{\mathbb{R}^{d_2}} \left(\int_{A^y} f^y \right) dy = \int_A f$

Proof: Apply Fubini's to $\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases}$ ■

Proposition

For every measurable set $A_1 \subseteq \mathbb{R}^{d_1}$, $A_2 \subseteq \mathbb{R}^{d_2}$, $A_1 \times A_2$ is measurable, and $m(A_1 \times A_2) = m(A_1) m(A_2)$ with the convention that $0 \cdot \infty = 0$.

Lemma $\forall A_1 \subseteq \mathbb{R}^{d_1}, A_2 \subseteq \mathbb{R}^{d_2}$, $m_*(A_1 \times A_2) \leq m_*(A_1) m_*(A_2)$

Proof of lemma:

- Case $m_*(A_1) < \infty$, $m_*(A_2) < \infty$.

By definition, $\exists$ $(Q_{k,1})_{k\in\mathbb{N}}$, $(Q_{k,2})_{k\in\mathbb{N}}$ open cubes such that $A_1 \subseteq \bigcup_{k=1}^{\infty} Q_{k,1}$ and $A_2 \subseteq \bigcup_{k=1}^{\infty} Q_{k,2}$, with $\sum_{k=1}^{\infty} \mathrm{Vol}(Q_{k,1}) \leq m_*(A_1) + \varepsilon$, $\sum_{k=1}^{\infty} \mathrm{Vol}(Q_{k,2}) \leq m_*(A_2) + \varepsilon$, then $A_1 \times A_2 \subseteq \left(\bigcup_{k=1}^{\infty} Q_{k,1}\right) \times \left(\bigcup_{k'=1}^{\infty} Q_{k',2}\right) = \bigcup_{k,k'=1}^{\infty} Q_{k,1} \times Q_{k',2}$.

Since the sets $Q_{k,1} \times Q_{k',2}$ are rectangles, it follows that

$$m_*(A_1 \times A_2) \leq \sum_{k,k'=1}^{\infty} Vol(Q_{k,1} \times Q_{k',2}) = \sum_{k,k'=1}^{\infty} Vol(Q_{k,1}) Vol(Q_{k',2})$$

$$= \left(\sum_{k=1}^{\infty} Vol(Q_{k,1})\right) \times \left(\sum_{k'=1}^{\infty} Vol(Q_{k',2})\right)$$

$$\leq (m_*(A_1) + \varepsilon) \times (m_*(A_2) + \varepsilon)$$

Passing to the limit as $\varepsilon \to 0$ gives $m_*(A_1 \times A_2) \leq m_*(A_1) m(A_2)$

- Case $m_*(A_1) = \infty$, $m_*(A_2) > 0$: nothing to prove since $m_*(A_1) m_*(A_2) = \infty$.

- Case $m_*(A_1) = \infty$, $m_*(A_2) = 0$:

Let $A_{1,k} = A_1 \cap B(0,k)$ so that $A_1 = \bigcup_{k=1}^{\infty} A_{1,k}$ and $m_*(A_{1,k}) < \infty$.

Then, $A_1 \times A_2 = \bigcup_{k=1}^{\infty} A_{1,k} \times A_2$, hence by countable subadditivity,

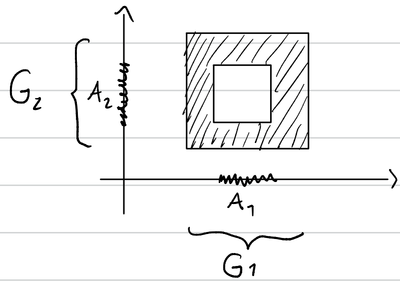
$$m_*(A_1 \times A_2) \leq \sum_{k=1}^{\infty} m_*(A_{1,k} \times A_2) = \sum_{k=1}^{\infty} m_*(A_{1,k}) \underbrace{m_*(A_2)}_{=0} = 0 = m_*(A_1) m_*(A_2)$$

$\blacksquare$

Proof of the Proposition:

We can write $A_1 = G_1 \setminus N_1$ and $A_2 = G_2 \setminus N_2$ where $G_1 = \bigcap_{k=1}^{\infty} O_{k,1}$ and $G_2 = \bigcap_{k=1}^{\infty} O_{k,2}$ are G_δ-sets and $N_1 \subseteq G_1$, $N_2 \subseteq G_2$ of measure 0. Then, $G_1 \times G_2 = \left(\bigcap_{k=1}^{\infty} O_{k,1}\right) \times \left(\bigcap_{k'=1}^{\infty} O_{k',2}\right)$. Since the sets $O_{k,1}, O_{k',2}$ are open, we obtain that the sets $O_{k,1} \times O_{k',2}$ are open, hence $G_1 \times G_2$ is a G_δ-set.

$$(G_1 \times G_2) \setminus (A_1 \times A_2) = (G_1 \times N_2) \cup (N_1 \times G_2)$$



▨ : $(G_1 \times G_2) \setminus (A_1 \times A_2)$

Since $m(N_1) = m(N_2) = 0$, we obtain that $m(G_1 \times N_2) = m(N_1 \times G_2) = 0$,

and so $m((G_1 \times G_2) \setminus (A_1 \times A_2)) = 0$. Therefore,

$$A_1 \times A_2 = \underbrace{(G_1 \times G_2)}_{G_\delta\text{-set}} \setminus \underbrace{((G_1 \times G_2) \setminus (A_1 \times A_2))}_{\text{of measure } 0} \quad \text{is measurable}$$

Since $A_1 \times A_2$ is measurable, we can apply $\overbrace{\text{Tonelli's Theorem}}^{\text{In fact, cor 1 of Tonelli's}}$ to $\chi_{A_1 \times A_2}$

and obtain $$m(A_1) m(A_2) = \int_{\mathbb{R}^{d_2}} \underbrace{m((A_1 \times A_2)^y)}_{= m(A_1) \chi_{A_2}(y)} = m(A)$$

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Corollary 4:

Let $f : A_1 \to \overline{\mathbb{R}}$ measurable, $A_2 \subseteq \mathbb{R}^{d_2}$ measurable. Then,

$$\tilde{f} : \begin{array}{l} A_1 \times A_2 \longrightarrow \overline{\mathbb{R}} \\ (x, y) \longmapsto f(x) \end{array} \quad \text{is measurable}$$

Proof: $\tilde{f}^{-1}([-\infty, c)) = f^{-1}([-\infty, c)) \times A_2$ is measurable since $f^{-1}([-\infty, c))$ and A_2 are measurable, use proposition above.

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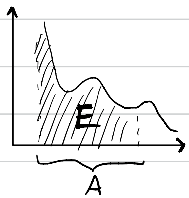
Proposition

Let $d_1 = d-1$ and $d_2 = 1$, $A \subseteq \mathbb{R}^{d_1}$ measurable and $f : A \to [0,\infty]$. Then, f is measurable $\iff E = \{(x,y) \in A \times \mathbb{R} : 0 \le y \le f(x)\}$ is measurable. Furthermore, if f is measurable, then $m(E) = \int_A f(x)\,dx$.



Proof $(\Rightarrow)$ Assume f is measurable.

Observe that $E = \tilde{f}^{-1}([0,\infty])$ where $\tilde{f} : A \times [0,\infty) \to \overline{\mathbb{R}}$
$$(x,y) \mapsto f(x) - y$$
which is measurable since $x \mapsto f(x)$ and $y \mapsto -y$ are measurable, hence their sum is measurable. This shows E is measurable.

$(\Leftarrow)$ Assume E is measurable.

Observe that $E_x = \begin{cases} [0, f(x)] & \text{if } x \in A \\ \emptyset & \text{if } x \notin A \end{cases}$

We then obtain that $x \longmapsto m(E_x) = \begin{cases} f(x) & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$ is measurable,

i.e. f is measurable. Moreover, we have $m(E) = \int_{\mathbb{R}^{d-1}} m(E_x)\,dx = \int_A f(x)\,dx$

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