Topology

- if $y \in B_r(x)$, then $\exists B_\varepsilon(y) \subseteq B_r(x)$
- $A \subseteq \mathbb{R}$ is open $\Leftrightarrow \forall x \in A, \exists \varepsilon > 0$ s.t. $B_\varepsilon(x) \subseteq A$
- Finite intersections of open sets are open
- Arbitrary unions of open sets are open
- Every open set is the union of countably many open intervals
- $\lim x_n = x \Leftrightarrow \forall \varepsilon > 0, \forall^\infty n, x_n \in B_\varepsilon(x)$
  $\Leftrightarrow \forall$ open set $U \ni x$ $\forall^\infty n$ $x_n \in U$.

- A set $F \subseteq \mathbb{R}$ is closed if its complement is open
- Finite unions of closed sets are closed
- Arbitrary intersections of closed sets are closed
- $\bar{S}$ is the closure of a set S, i.e. the smallest closed set containing S

→ For $S \subseteq \mathbb{R}, x \in \mathbb{R}$, $x \in \bar{S} \Leftrightarrow \forall \varepsilon > 0, B_\varepsilon(x) \cap S \neq \emptyset$
$\Leftrightarrow \exists$ sequence $(x_n) \subseteq S$ st. $x_n \to x$

- A set S is closed $\Leftrightarrow S = \bar{S}$
  $\Leftrightarrow \forall$ converging sequences $x_n \subseteq S$, $\lim x_n \in S$

- The boundary ∂S of a set $S \subseteq \mathbb{R}$ is the set of all $x \in \mathbb{R}$ st. $\forall \varepsilon > 0$, $B_\varepsilon(x) \cap S \neq \emptyset$ AND $B_\varepsilon(x) \cap S^c \neq \emptyset$
- $\bar{S} = S \cup \partial S$, for $S \subseteq \mathbb{R}$
- A set $S \subseteq \mathbb{R}$ is closed $\Leftrightarrow \partial S \subseteq S$
- $S \subseteq \mathbb{R}, x \in \mathbb{R}$, $x \in \partial S \Leftrightarrow \exists x_n$ & y_n st. $\lim x_n = x = \lim y_n$
- connected set $\Leftrightarrow$ convex set

- A subset $A \subseteq \mathbb{R}$ is compact $\Leftrightarrow$ it is closed and bounded. ex:
  (i) closed, bdd intervals are compact
  (ii) any finite set is compact
  (iii) any finite union of closed, bdd intervals is compact
- Any nested sequence (K_n) of nonempty compact sets has a nonempty intersection, i.e. $\bigcap_{n \in \mathbb{N}} K_n \neq \emptyset$

Cantor set C

(i) C is closed because it is the intersection of infinitely many closed intervals
(ii) C is nonempty
(iii) C has 2^N pieces of length $(\frac{1}{3})^N \Rightarrow$ its length is $2^N (\frac{1}{3})^N = (\frac{2}{3})^N \to 0$ as $N \to \infty$
(iv) C is uncountable, $C \equiv 2^{\mathbb{N}}$:= the set of all binary sequences
(v) The interior of C is empty ie., $\nexists$ nonempty open sets $\in C$
$\Leftrightarrow \forall c \in C$, c is a limit point, ie. C^c is dense in $\mathbb{R}$

- A cluster point of a set $A \subseteq \mathbb{R}$ is a point $x \in \mathbb{R}$ st. $x \in \overline{(A \setminus \{x\})}$. TFAE:
  (1) x is a cluster point of A
  (2) Any open ball $B_\varepsilon(x)$ intersects $A \setminus \{x\}$
  (3) $\exists (x_n) \subseteq A \setminus \{x\}$ st. $\lim x_n = x$.

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limits of functions let $f: D \to \mathbb{R}$, c a cluster point

- $\lim_{x \to c} f(x) = L \Leftrightarrow \forall \varepsilon$-Ball about L, $\exists \delta$-ball about c st. $f(B_\delta(c) \cap D \setminus \{c\}) \subseteq B_\varepsilon(L)$
  $\Leftrightarrow \forall \varepsilon > 0, \exists \delta_{(\varepsilon)} > 0$ st. $\forall x \in D \setminus \{c\}, |x - c| < \delta_{(\varepsilon)} \Rightarrow |f(x) - L| < \varepsilon$
- $\lim_{x \to c} f(x) = L \Leftrightarrow \forall (x_n) \subseteq D \setminus \{c\}$ converging to c, we have $f(x_n) \to L$ as $n \to \infty$
- let $f, g : D \to \mathbb{R}$ and c a cluster point of D. Suppose $\lim_{x \to c} f(x)$ & $\lim_{x \to c} g(x)$ exist, then:
  a) $\lim_{x \to c} \lambda f(x) = \lambda \cdot \lim_{x \to c} f(x)$ (b) $\lim_{x \to c} (f \pm g)(x) = \lim_{x \to c} f(x) \pm \lim_{x \to c} g(x)$
  (c) $\lim_{x \to c} (f \cdot g)(x) = \lim_{x \to c} f(x) \cdot \lim_{x \to c} g(x)$ (d) $\lim_{x \to c} f/g(x) = \lim_{x \to c} f(x) / \lim_{x \to c} g(x)$ $\neq 0$

* These follow from the sequential definition of the limit of a function

- let $A \leq f(x) \leq B$ $\forall x \in D \setminus \{c\}$, and $\lim_{x \to c} f(x)$ exists, then $A \leq \lim_{x \to c} f(x) \leq B$
- Squeeze theorem is true for limits of functions
- If $\lim_{x \to c} f(x)$ exists and is $\lessgtr A$, then $\exists B_\delta(c)$ st. $\forall x \in B_\delta(c) \cap D \setminus \{c\}$, $f(x) \lessgtr A$
- If $\lim_{x \to c} f(x)$ exists, then f is bdd around c, ie. $\exists B_\delta(c)$ st. $f(B_\delta(c) \cap D)$ is a bdd set, i.e. $\exists M > 0$ st. $|f(x)| \leq M$ $\forall x \in B_\delta(c) \cap D$
- $f(x)$ tends to $\pm\infty$ as $x \to c$ if $\forall$ neighborhood $(r, +\infty)$ (or $(-\infty, r)$), $\exists \delta(r) > 0$ st. $f(B_\delta(c) \cap D \setminus \{c\}) \subseteq (r, +\infty)$ (or $(-\infty, r)$)
  ie. $\forall r \in \mathbb{R}, \exists \delta(r) > 0$ st. $\forall x \in D$ with $0 < |x - c| < \delta$, then $f(x) \gtrless r$
- $\lim_{x \to +\infty} f(x) = L \in \mathbb{R}$ if $\forall B_\varepsilon(L), \exists (r, +\infty), r > 0$ st. $f((r, +\infty)) \subseteq B_\varepsilon(L)$
- Divergence criteria: f doesn't have limit L at $c \Leftrightarrow \exists (x_n) \subseteq D$ with $x_n \neq c$ $\forall n \in \mathbb{N}$ st. $x_n \to c$ but $(f(x_n)) \not\to L$ (diverges or converges to $p \neq L$)
- Or, two sequences x_n, y_n st. $\lim x_n = \lim y_n = c$, but $\lim f(x_n) \neq \lim f(y_n)$

Continuity

- let $f: A \to \mathbb{R}$, f is continuous at $a \in A$
  $\Leftrightarrow \forall \varepsilon > 0, \exists \delta > 0$ st. $\forall x \in A$, $|x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$
  $\Leftrightarrow$ ε-Ball def. $\forall \varepsilon > 0, \exists \delta > 0$ st. $f(B_\delta(a) \cap A) \subseteq B_\varepsilon(f(a))$
  $\Leftrightarrow B_\delta(a) \cap A \subseteq f^{-1}(B_\varepsilon(f(a)))$
  $\Leftrightarrow$ limit definition $\lim_{x \to c} f(x) = f(c)$ if c is a cluster point.
  $\Leftrightarrow$ seq. def. $\forall$ seq. $(x_n) \subseteq D$ st. $x_n \to c$, we have $f(x_n) \to f(c)$
- A function $f: D \to \mathbb{R}$ is continuous on D if it is continuous at every $x \in D$. ie. f is cts on D if $\forall x \in D$, $\forall \varepsilon > 0, \exists \delta > 0$, $|y - x| < \delta \Rightarrow |f(y) - f(x)| < \varepsilon$
- let D an open set and $f: D \to \mathbb{R}$. Then f is continuous iff the f-preimage of any open set is open
- If f is continuous, then the f-preimage of all closed sets is closed.
- let $f, g : D \to \mathbb{R}$ cts at $c \in D$, then: (d) $\frac{f}{g}$ cts at c, $g \neq 0$
  (a) $\lambda \cdot f$ cts at c $\forall \lambda \in \mathbb{R}$ (b) $f \pm g$ cts at c (c) $f \cdot g$ cts at c
- All polynomial functions are continuous
- If $f: D \to \mathbb{R}$ is cts at $c \in D$ and $f(c) \lessgtr a$, then $\exists B_\delta(c)$ st. $\forall x \in B_\delta(c) \cap D$, $f(x) \lessgtr a$
- rational function: given by the ratio of 2 polynom. ie. $f(x) = \frac{p(x)}{q(x)}$ its domain is the set of points at which $q(x) \neq 0$. They are continuous wherever they are defined
- sin & cos are cts and lipschitz with constant 1
- Continuous functions map compact sets to compact sets
- Max-min thm: let D compact and $f: D \to \mathbb{R}$ cts. Then $f(D)$ achieves its sup & inf, i.e. $\exists x_1, x_2 \in D$ st. $f(x_1) = \inf f(D)$ and $f(x_2) = \sup f(D)$
- Intermediate value Thm: if $f: [a, b] \to \mathbb{R}$ is cts, then $\forall$ value C between $f(a)$ and $f(b)$, $\exists c \in [a, b]$ st. $f(c) = C$. i.e. $f([a, b])$ contains the whole interval between $f(a)$ & $f(b)$.
- Continuous functions map intervals (connected sets) to intervals (connected sets) and closed, bdd intervals to closed, bdd intervals.
- The composition of cts functions is cts. More Precisely, if $A, B \subseteq \mathbb{R}$, $f: A \to \mathbb{R}$, $g: B \to \mathbb{R}$, with $f(A) \subseteq B$, then if f cts at $a \in A$ and g cts at $f(a)$, then $g \circ f$ cts at a.

Uniform continuity.

- A function $f: D \to \mathbb{R}$ is uniformly cts if $\forall \varepsilon > 0, \exists \delta > 0$ st. $\forall x, y \in D$ within distance $\leq \delta$, $f(x)$ and $f(y)$ are within distance $\leq \varepsilon$
  $\Leftrightarrow$ for any $A \subseteq D$ of diam $\leq \delta$, $f(A)$ has diam $\leq \varepsilon$
  $\Leftrightarrow \forall \varepsilon > 0, \exists \delta(\varepsilon) > 0$ st. $\forall x, y \in D$, $|x - y| < \delta(\varepsilon) \Rightarrow |f(x) - f(y)| < \varepsilon$
  so δ only depends on ε, not the choice of x, y.
- if $f: D \to \mathbb{R}$ is unif. cts and $(x_n), (y_n) \subseteq D$ are st. $d(x_n, y_n) \to 0$, then $d(f(x_n), f(y_n)) \to 0$
- unif. cts functions map cauchy seq to cauchy seq.
- Cts functions on compact domains are uniformly cts
- A function is not uniformly cts if $\forall \delta > 0, \exists \varepsilon > 0$ st. $|x - y| < \delta$ but $|f(x) - f(y)| \geq \varepsilon$.

$\Leftrightarrow \exists \varepsilon > 0$, $(x_n), (u_n)$ in D st. $\lim (x_n - u_n) = 0$, but $|f(x_n) - f(u_n)| \geq \varepsilon$ $\forall n \in \mathbb{N}$

- Lipschitz function: let $A \subseteq \mathbb{R}$, $f: A \to \mathbb{R}$, if $\exists$ a constant $K > 0$ st. $|f(x) - f(u)| \leq K |x - u|$ $\forall x, u \in A$
- If $f: A \to \mathbb{R}$ is Lipschitz, then f is uniformly cts on A
- Contiuous extensions: let $f: D \to \mathbb{R}$, $D \subseteq \tilde{D}$. $\tilde{f}: \tilde{D} \to \mathbb{R}$ is an extension of f if $\tilde{f}|_D = f$, ie. $\forall x \in D$, $f(x) = \tilde{f}(x)$
- let $f: D \to \mathbb{R}$ then f has at most one extension to $\bar{D}$
  $\Leftrightarrow f$ is uniformly cts on D
- Compositions of unif. cts. functions are unif. cts.
- compositions of Lipschitz functions are Lipschitz. In particular, if g, f are lipschitz with constants L_g, L_f, then $g \circ f$ has constant $L_f \cdot L_g$

Differentiation let I an interval, $f: I \to \mathbb{R}$

f is differentiable at $c \in I$ if $\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$ exists ← Derivative at c

- The straight line $\ell(x) := f'(c) \cdot (x - c) + f(c)$ is tangent to f at c.
- $g, h : D \to \mathbb{R}$, c a cluster pt, $h \neq 0$ $\forall x \in D \setminus \{c\}$, then we write $g(x) = o_c(h(x))$ if $\lim_{x \to c} \frac{g(x)}{h(x)} = 0$
- $f: I \to \mathbb{R}$ is differentiable at $c \in I \Leftrightarrow f$ is cts at c and there is a linear function $\ell(x) := a(x - c) + b$ st. $d(f(x), \ell(x)) = o_c(d(x, c))$.
  $\ell(x)$ is unique: $a = f'(c)$, $b = f(c)$

$\ell(x) := f'(c) \cdot (x - c) + f(c)$
$= \left(\lim_{x \to c} \frac{f(x) - f(c)}{x - c}\right)(x - c) + f(c)$

- Differentiability $\Rightarrow$ continuity
- let $f, g : D \to \mathbb{R}$ dif. at $c \in D$, then:
  (a) $\lambda \cdot f$ dif. at c $\forall \lambda \in \mathbb{R}$ and $(\lambda f)'(c) = \lambda f'(c)$
  (b) $f \pm g$ dif. at c and $(f \pm g)'(c) = f'(c) \pm g'(c)$
  (c) $f \cdot g$ dif. at c and $(f \cdot g)'(c) = f'(c) \cdot g(c) + f(c) \cdot g'(c)$
  (d) $\frac{f}{g}$ dif at c, $g \neq 0$ and $(f/g)'(c) = \frac{f'(c) \cdot g(c) + f(c) \cdot g'(c)}{g(c)^2}$
- Chain Rule let $f: I \to J$, $g: J \to \mathbb{R}$, I, J intervals. If f is diff. at $c \in I$ and g is diff. at $f(c) := d \in J$, then $g \circ f$ is diff. at c and $(g \circ f)' = g'(f(c)) \cdot f'(c)$
- Carathéodory's Thm $f: I \to \mathbb{R}$ is diff. at $c \in I$
  $\Leftrightarrow \exists! \, \varphi : I \to \mathbb{R}$ cts at c st. $f(x) - f(c) = \varphi(x)(x - c)$
  In which case, $f'(c) = \varphi(c)$
- Derivative of inverses: let $f: I \to J$ st. it has an inverse $g: J \to I$. If f is dif. at c and $f'(c) \neq 0$, then g is diff at $d := f(c)$, where $g'(d) = \frac{1}{f'(g(d))}$
- let $f: I \to J$ be cts. f is invertible (ie. bijective)
  $\Leftrightarrow f$ is strictly monotone and $J = f(I)$
- $f: I \to J$ is strictly monotone if $\forall x, y \in I$, if $x \lessgtr y$, then $f(x) \lessgtr f(y)$, where I, J are intervals

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- Interior point: x is an interior point of A, if $\exists \varepsilon > 0$ st. $B_\varepsilon(x) \subseteq A$. A° is the set of all interior points of A.
- The interior of an interval I is $I^\circ := I \setminus$ endpoints
- let $f: I \to \mathbb{R}$, $c \in I$, let I an interval & f cts at c. Then, (1) $\lim_{x \to c^-} f(x) = f(c) = \lim_{x \to c^+} f(x)$
  if c is not a left endpt; if c is not a right end pt.
  (2) $\sup_{x < c} f(x) = f(c) = \inf_{x > c} f(x)$ if f increasing
  $\sup_{x > c} f(x) = f(c) = \inf_{x < c} f(x)$ if f decreasing
- If $f: I \to f(I) = J$ is strictly monotone & cts, then so is f^{-1} ie. if f is strictly increasing, f^{-1} also is
- Derivative of inverses: let $f: I \to J := f(I)$ be strictly monotone and cts, so it has a cts inverse $f^{-1}: J \to I$. let $d \in J$. Suppose $f'(f^{-1}(d))$ exists, then f^{-1} is diff. at d
  $\Leftrightarrow f'(f^{-1}(d)) \neq 0$, where $(f^{-1})'(d) = \frac{1}{f'(f^{-1}(d))}$

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local extrema

$f: I \to \mathbb{R}$ has a local extremum at $c \in I^\circ$ if $\exists \delta > 0$ st. $f(c) \gtreqless f(x)$ $\forall x \in B_\delta(c) \subseteq I$

- If c is a local extremum of f and f is diff. at c, then $f'(c) = 0$
- Rolle's Thm let $f: [a, b] \to \mathbb{R}$ be cts on $[a, b]$ and diff. on (a, b). If $f(a) = f(b) \Longrightarrow \exists c \in (a, b)$ st. $f'(c) = 0$
- MVT let $f: [a, b] \to \mathbb{R}$ be cts on $[a, b]$ and diff. on (a, b). Then, $\exists c \in (a, b)$ s.t. $\frac{f(b) - f(a)}{b - a} = f'(c)$
- let $f: [a, b] \to \mathbb{R}$ be cts on $[a, b]$ & diff. on (a, b). If $f' \equiv 0$ on (a, b), then f is constant
- let $f, g: [a, b] \to \mathbb{R}$ be cts on $[a, b]$ & diff on (a, b). If $f'|_{(a,b)} = g'|_{(a,b)}$ then f and g differ by a constant, ie. $\exists C \in \mathbb{R}$ st. $f = C + g$.
- f is increasing $\Leftrightarrow f' \geq 0$, f is decreasing $\Leftrightarrow f' \leq 0$
- Derivative test for extrema let $f: [a, b] \to \mathbb{R}$ be cts on $[a, b]$, fix $c \in (a, b)$ and suppose f is diff on $(a, c) \cup (c, b)$
  (a) If $\exists B_\delta(c)$ st. $f' \geq 0$ on $(c - \delta, c)$ and $f' \leq 0$ on $(c, c + \delta)$ Then c is a local max
  (b) If $\exists B_\delta(c)$ st. $f' \leq 0$ on $(c - \delta, c)$ and $f' \geq 0$ on $(c, c + \delta)$ Then c is a local min

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- $\triangle$-ineq. (1) $|a + b| \leq |a| + b$
  (2) $||a| - |b|| \leq |a - b|$ (3) $|a - b| \leq |a| + |b|$
- $\forall a_1, a_2, a_3, \dots, a_n \in \mathbb{R}$, $|a_1 + a_2 + \dots + a_n| \leq |a_1| + |a_2| + \dots + |a_n|$
- $d(x, y) \leq d(x, z) + d(z, y)$
- Nested closed intervals: let $(I_0, I_1, I_2, \dots)$ be a seq of closed intervals, i.e. $I_n = [a_n, b_n]$, $I_{n+1} \subseteq I_n$. Then, $\bigcap_{n=0}^{\infty} I_n \neq \emptyset$, ie. $\exists x \in \mathbb{R}$ that is $\in$ of all I_n
- $x_n \to x \Leftrightarrow \forall \varepsilon > 0, \exists N \in \mathbb{N}$ st. $\forall n \geq N$, $x_n \in B_\varepsilon(x)$ ie. $|x - x_n| < \varepsilon$

TFAE: (i) $x_n \not\to x \in \mathbb{R}$

(ii) $\exists\, \varepsilon>0$ st. $\forall k\in\mathbb{N}$, $\exists\, n_k\in\mathbb{N}$ st. $n_k\geq k$ and $|x_{n_k}-x|\geq\varepsilon$

(iii) $\exists\, \varepsilon>0$ st. $|x_{n_k}-x|\geq\varepsilon$ $\forall k\in\mathbb{N}$

$x_n\not\to x \iff \forall\varepsilon>0$, $\exists^\infty n$ st. $|x_n-x|\geq\varepsilon$

If a_n, b_n converge and $\forall n\in\mathbb{N}$, $a_n\geq b_n$, then $\lim a_n\geq\lim b_n$

Every subseq. of a converging seq. also conv. to the same limit
~: if a sequence has a diverging subseq., then it diverges

All convergent sequences are bdd
~ Unbdd $\Rightarrow$ divergent

$x_n+y_n\to x+y$ $\quad \frac{x_n}{y_n}\to\frac{x}{y}$ $y\neq0$
$x_n\cdot y_n\to x\cdot y$

if $x_n\geq a\ \forall n\in\mathbb{N}$, then $x\geq a$

if x_n, y_n conv, and $x_n\geq y_n$ $\forall n\in\mathbb{N}$, then $x\geq y$

Squeeze
Let $\lim x_n=\lim z_n=l$
Then if $x_n\le y_n\le z_n$,
$\lim y_n=l$

$x_n\to x\Rightarrow|x_n|\to|x|$
$x_n\to x\Rightarrow\sqrt{x_n}\to\sqrt{x}$

$x_n\to+\infty$ iff $\forall E>0, \forall^\infty n$, $x_n\in(E,+\infty)$

Ratio $l=\lim\frac{x_{n+1}}{x_n}$
if: • $l<1$, $\lim x_n=0$
$\forall n$, $x_n\le C\cdot\lambda^n$, $C>0, \lambda\in(0,1)$
• $l>1$, $\lim x_n=+\infty$, $\forall n$,
$x_n\geq C\cdot\Lambda^n$, $C>0$, $\Lambda>1$
• $l=0$, inconclusive

$x_n=\frac{x_n}{x_{n-1}}\cdot\frac{x_{n-1}}{x_{n-2}}\cdot\ldots\cdot\frac{x_1}{x_0}\cdot x_0$

a seq is **monotone** if:
(1) increasing $x_0\le x_1\le x_2\le\ldots$
(2) decreasing $x_0\geq x_1\geq x_2\geq\ldots$

unbdd above $\Rightarrow \sup S=+\infty$
unbdd below $\Rightarrow \inf S=-\infty$

$r\in\mathbb{R}, r\neq1, n\in\mathbb{N}$
$1+r+r^2+\cdots+r^n=\frac{1-r^{n+1}}{1-r}$

$(1+\frac{1}{n})^n\to e$

$(x+y)^2\le4xy$

$2^{r-1}\le r!$

MCT: if x_n is bdd & Monotone, it converges:
• increasing: $\lim x_n=\sup x_n$
• decreasing: $\lim x_n=\inf x_n$
if unbdd, $\lim x_n=\sup x_n=+\infty$
or $\lim x_n=\inf x_n=-\infty$

$\sum\frac{1}{n}$ diverges
$\sum\frac{1}{n^2}<2$

$\binom{n}{k}=\frac{n!}{k!(n-k)!}$

Binomial
$(a+b)^n=\sum_{k=0}^{n}\binom{n}{k}a^kb^{n-k}$
$(1+b)^n=\sum_{k=0}^{n}\binom{n}{k}b^k=\binom{n}{0}+\binom{n}{1}b+\binom{n}{2}b^2+\cdots+\binom{n}{n}b^n$
$=1+bn+\frac{(n-1)nb^2}{2}+\cdots+b^n$

bernoulli:
$(1+b)^n\geq1+\binom{n}{k}b^k$
$(1+x)^r\geq1+rx$, $r\geq0$, $x\in\mathbb{R}$
$(1+x)^r\le1+rx$, $0\le r\le1$, $x\geq-1$

$\lim x_n=x\iff\forall n>N, N\in\mathbb{N}$, $|x_n-x|<\varepsilon$

Seq (x_n), subseq. $(x_n)_k$ st. $n_0<n_1<n_2<\ldots$

If $x_n\to x$, then all subseq. of x_n also conv. to x
~ if a seq has 2 subseq conv. to different lim., x_n diverges

B-W. Every bdd seq. has a convergent subseq.

Bounded seq diverges $\iff\geq2$ of its subseq. limits are different.

Unbdd $\iff$ diverges

x_n bdd converges iff set of subseq. limits has only one element.

$\limsup x_n$ $:=\lim_n(\sup\{x_n,x_{n+1},\ldots\})$
$:=\lim_n(\sup x_n:n\geq N)$
$:=\lim_n\sup_{n\geq N}x_n$

$\liminf x_n$ $:=\lim_n(\inf\{x_n,x_{n+1},x_{n+2},\ldots\})$
$:=\lim_n(\inf x_n:n\geq N)$
$:=\lim_n\inf_{n\geq N}x_n$

If x_n bdd, $S:=$ set of subseq limits
$\limsup x_n=\sup S=\max S$
$\liminf x_n=\inf S=\min S$

A bounded sequence converges iff $\liminf_n x_n=\limsup_n x_n$
diverges to ∞ if $\liminf x_n=\limsup x_n=\infty$

$\liminf_n x_n\neq\limsup x_n\Rightarrow$ diverges

Cauchy if $\forall\varepsilon>0$, $\exists N$ st $\forall n,m\geq N$,
$d(x_n,x_m)=|x_n-x_m|\le\varepsilon$
Cauchy $\iff$ convergent

Contractive
if $\exists\, C\in(0,1)$ st. $\forall^\infty n$
$|x_{n+2}-x_{n+1}|<C|x_{n+1}-x_n|$
$|x_n-x_{n-1}|<C|x_{n-1}-x_{n-2}|$

Contractive sequences are cauchy, hence converge

Surjective if $\forall y\in B, \exists x\in A(y=F(x))$
Fix any $y\in B$. ($G(x)=y$, try to express x in terms of y)
"consider $x=\ldots$" Show that $x\in A$. Then show that $f(x)=y$.
Disprove: $G:A\to B$, Argue B cannot be output of A

Injective
$\forall x,u\in A, F(x)=F(u)\Rightarrow u=x$
Proof: Fix any $x,u\in B$, with $G(u)=G(x)$
Using Algebra, we show $x=u$

$\frac{-b\pm\sqrt{b^2-4ac}}{2a}$, ax^2+bx+c

• A sequence converges if $\forall\varepsilon>0, \forall^\infty n, x_n\in B_\varepsilon(x)$
• $x=\sup A\iff x$ is an upper bdd & $\forall\varepsilon>0$, $\exists\, a\in A$ st. $x-\varepsilon<a$
• $x=\inf A\iff x$ is a lower bdd & $\forall\varepsilon>0$, $\exists\, a\in A$ st. $x+\varepsilon>a$
• AP. $\forall r\in\mathbb{R}, \exists\, n\in\mathbb{N}$ st. $n>r$
• $\mathbb{Q}$ and $\mathbb{R}\setminus\mathbb{Q}$ are dense in $\mathbb{R}$
• A contrapositive proof states that if the negation of the conclusion is true, then the negation of the premise also is.
• Every nonempty subset of $\mathbb{R}$ that's bdd above has a sup, analogous for inf (order - completeness of $\mathbb{R}$)
• $D\subseteq\mathbb{R}$ is dense in $\mathbb{R}\iff\forall a,b\in\mathbb{R}$, with $a<b$, $\exists\, d\in D$ st. $a<d<b$
• $\mathbb{Q}$ is dense in $\mathbb{R}$. Let $0<x<y$
i.e. $y-x>0$
Since $x,y\in\mathbb{R}$, $y-x\in\mathbb{R}\Rightarrow\exists\, n\in\mathbb{N}$ st.
$0<\frac{1}{n}<y-x\Rightarrow x+\frac{1}{n}<y$
$\Rightarrow nx+1<ny$
$nx<0$, $\exists\, m\in\mathbb{N}$ st. $m-1\le nx<m$
$\Rightarrow m\le nx+1<m+1\Rightarrow m\le nx+1<ny$
where $m>nx$, so $nx<m<ny\Rightarrow x<\frac{m}{n}<y$ □
• If $A\subseteq\mathbb{R}$ contains one of its upper bounds u, then $u=\sup A$