Chapter 5. Differentiation and integration

Problem 1. Let F be an integrable function on $[a,b] \subset \mathbb{R}$. Does this imply that $F(x) = \int_a^x f$ is differentiable, at least for a.e. $x \in [a,b]$ and $F' = f$.

Problem 2. Under which conditions on $F: [a,b] \to \mathbb{R}$ do we have that F' exists a.e. in $[a,b]$ and $\int_a^b F' = F(b) - F(a)$.

Theorem

A monotone function $f: [a,b] \to \mathbb{R}$ is differentiable a.e in (a,b).

Moreover, f' is integrable and $\int_a^b f' \begin{cases} \leq f(b) - f(a) & \text{if } f \text{ is increasing} \\ \geq f(b) - f(a) & \text{if } f \text{ is decreasing} \end{cases}$

counterexample to equality:

$f(x) = \begin{cases} 0 & \text{if } x \in [0, 1/2] \\ 1 & \text{if } x \in [1/2, 1] \end{cases}$, f' exists everywhere except at $1/2$ and $\int f' = 0 < f(1) - f(0) = 1$

Another example that is continuous, is the cantor-lebesgue function.
(it is a.e. constant)

Proof: wlog, we may assume that f is increasing, o.w. consider $-f$. For every $x \in (a,b)$, let $\underline{D} f(x) = \liminf_{t \to 0,\ t \neq 0} \frac{f(x+t) - f(x)}{t}$ and $\overline{D} f(x) = \limsup_{t \to 0,\ t \neq 0} \frac{f(x+t) - f(x)}{t}$

Since f is increasing, we have $0 \leq \underline{D}f \leq \overline{D}f \leq \infty$. f is not differentiable at $x \Leftrightarrow$ either $\underline{D}f(x) < \overline{D}f(x)$ or $\underline{D}f(x) = \overline{D}f(x) = \infty$.

Observe that $\underline{D}f(x) < \overline{D}f(x) \Leftrightarrow \exists\, \alpha, \beta \in \mathbb{Q}$ s.t. $\underline{D}f(x) < \alpha < \beta < \overline{D}f(x)$ by density of $\mathbb{Q}$ in $\mathbb{R}$. This gives that

$$A_{\alpha,\beta} := \{x \in (a,b) : \underline{D}f(x) < \overline{D}f(x)\} = \bigcup_{\alpha,\beta \in \mathbb{Q},\, \alpha < \beta} \{x \in (a,b) : \underline{D}f(x) < \alpha \text{ and } \overline{D}f(x) > \beta\}$$

We claim that $m_*(A_{\alpha,\beta}) = 0$. Assuming this, we obtain that $\underline{D}f = \overline{D}f$ a.e. in (a,b). Denote $Df = \underline{D}f = \overline{D}f$. Consider

$$D_{1/n} f(x) = \begin{cases} n\left(f(x+\frac{1}{n}) - f(x)\right) & \text{if } a < x < b - 1/n \\ 0 & \text{if } b - 1/n \leq x < b \end{cases} \quad \forall n \in \mathbb{N}$$

f is measurable, as every monotone function is measurable (The inverse image of intervals are intervals). Thus, $D_{1/n} f$ is measurable $\forall n \in \mathbb{N}$. Moreover, $D_{1/n} f \geq 0$ since f is increasing, and $D_{1/n} f \to Df$ a.e. in (a,b), as $Df = \overline{D}f = \underline{D}f$ a.e.

By applying Fatou's lemma, we obtain $\int_{(a,b)} Df \leq \liminf_{n \to \infty} \int_{(a,b)} D_{1/n} f$.

$$\int_{(a,b)} D_{1/n} f = n \int_a^{b-1/n} \left(f(x + 1/n) - f(x)\right) = n\left(\int_{a+1/n}^{b} f(x)\,dx - \int_a^{b-1/n} f(x)\,dx\right)$$

$$= n\left(\int_{b-1/n}^{b} \underbrace{f(x)}_{\leq f(b)}\,dx - \int_a^{a+1/n} \underbrace{f(x)}_{\geq f(a) \text{ since } f \text{ increasing}}\,dx\right) \leq n\left(f(b)\int_{b-1/n}^{b} 1 - f(a)\int_a^{a+1/n} 1\right)$$

Definition of $D_{1/n}$

Therefore, Df is integrable and $\int_{(a,b)} Df \le f(b)-f(a)$. It that Df is finite a.e. in (a,b) which gives that $f' = Df$ a.e. in (a,b) and $\int_{(a,b)} f' \le f(a)-f(b)$

We are left to prove $m_*(A_{\alpha,\beta}) = 0$. Let $\varepsilon > 0$, then there exists an open set $O_\varepsilon \subseteq (a,b)$ such that $m_*(O_\varepsilon) \le m_*(A_{\alpha,\beta}) + \varepsilon$ and $A_{\alpha,\beta} \subseteq O_\varepsilon$.

Claim:

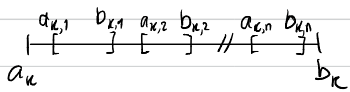
(i) There exists a finite collection of disjoint intervals $[a_1,b_1], [a_2,b_2], \ldots [a_n,b_n] \subseteq O_\varepsilon$ such that $m_*(A_{\alpha,\beta} \setminus \bigcup_{k=1}^{n} [a_k,b_k]) < \varepsilon$ and $f(b_k)-f(a_k) < \alpha(b_k-a_k)$

(ii) There exists, for each $k \in \{1,\ldots,n\}$, a finite collection of disjoint intervals $[a_{k,1},b_{k,1}], [a_{k,2},b_{k,2}], \ldots, [a_{k,n},b_{k,n}] \subseteq (a,b)$ such that $m_*(A_{\alpha,\beta} \cap (a_k,b_k) \setminus \bigcup_{j=1}^{n_k} [a_{k,j},b_{k,j}]) < \varepsilon/m$ and $f(b_{k,j})-f(a_{k,j}) > \beta(b_{k,j}-a_{k,j})$

Proof that $m(A_{\alpha,\beta})=0$, assuming (i) & (ii):

$$\beta \sum_{k=1}^{n} \sum_{j=1}^{n_k} (b_{k,j}-a_{k,j}) < \sum_{k=1}^{n} \sum_{j=1}^{n_k} (f(b_{k,j})-f(a_{k,j})) < \sum_{k=1}^{n} (f(b_k)-f(a_k))$$

since f is monotone



Moreover,
$$\sum_{k=1}^{n} (b_k-a_k) = m\left(\bigcup_{k=1}^{n} [a_k,b_k]\right) \le m(O_\varepsilon) < m_*(A_{\alpha,\beta}) + \varepsilon$$

since $\bigcup [a_k,b_k] \subseteq O_\varepsilon$

And,
$$m_*(A_{\alpha,\beta}) \le \underbrace{m_*\left(A_{\alpha,\beta} \setminus \bigcup_{k=1}^{n} [a_k,b_k]\right)}_{<\varepsilon} + \underbrace{m_*\left(A_{\alpha,\beta} \cap \bigcup_{k=1}^{n} [a_k,b_k]\right)}_{\le \sum_{k=1}^{n} m_*(A_{\alpha,\beta} \cap [a_k,b_k])}$$ by subadditivity

$$m_*(A_{\alpha,\beta}\cap[a_k,b_k]) \le \underbrace{m_*\Big(A_{\alpha,\beta}\cap[a_k,b_k]\setminus\bigcup_{j=1}^{n_k}[a_{k,j},b_{k,j}]\Big)}_{<\varepsilon/n} + \underbrace{m_*\Big(A_{\alpha,\beta}\cap\bigcup_{j=1}^{n_k}[a_{k,j},b_{k,j}]\Big)}_{\le \sum_{j=1}^{n_k} m([a_{k,j},b_{k,j}]) = \sum_{j=1}^{n_k}(b_{k,j}-a_{k,j})}$$

It follows that,

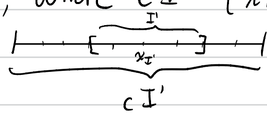
$$\beta\, m_*(A_{\alpha,\beta}) \le \beta\Big(2\varepsilon + \sum_{k=1}^{n}\sum_{j=1}^{n_k}(b_{k,j}-a_{k,j})\Big) \le 2\beta\varepsilon + \alpha m_*(A_{\alpha,\beta}) + \alpha\varepsilon$$

which gives, $(\beta-\alpha)\, m_*(A_{\alpha,\beta}) \le (\alpha+2\beta)\varepsilon$, i.e.

$$m_*(A_{\alpha,\beta}) \le \frac{\alpha+2\beta}{\beta-\alpha}\,\varepsilon \to 0 \quad \text{as } \varepsilon\to 0 \qquad \checkmark$$

Vitali Converging Lemma

Let $c>3$, F be a collection of bounded and closed intervals $[a,b]$ of positive length. Then, there exists an at most countable collection $F'\subseteq F$ of disjoint intervals such that for each $I\in F$, there exists $I'\in F$ such that $I\cap I'\neq\emptyset$ and $I\subseteq cI'$, where $cI' = \{x\in\mathbb{R} : x_{I'} + \frac{1}{c}(x-x_{I'})\in I'\}$, $x_{I'}$ middle point of I':



. In particular, $\bigcup_{I\in F} I \subseteq \bigcup_{I'\in F'} cI'$.

Counter example for $c<3$. Consider $F=\{[-1,0],[0,1]\}$

Proof of the claims assuming VCL

(i) Define $F=\{[a',b']\subseteq O_\varepsilon : f(b')-f(a')<\alpha(b'-a')\}$. Let $F'\subset F$ be the collection given by VCL, $F'=(I_k)_{k\in K}$, $K=\mathbb{N}$ or $\{1,\dots,n_K\}$. Let $x\in A_{\alpha,\beta}\setminus\bigcup_{k=1}^{n} I_k$ for some $n\in K$. Since $\bigcup_{k=1}^{n} I_k$ is closed, $\exists\,\delta_0>0$ such that $[x-\delta_0, x+\delta_0]\subset O_\varepsilon\setminus\bigcup_{k=1}^{n} I_k$.

By definition of $A_{\alpha,\beta}$, since $\underline{D}f(x)<\alpha$, $\exists$ some arbitrarily small $\delta\in(0,\delta_0)$ such that $[x-\delta,x]$ or $[x,x+\delta]\in F$. Let $I_x=[x-\delta,x]$ or $[x,x+\delta]$, since $I_x\subset O_\varepsilon\setminus\bigcup_{k=1}^{n} I_k$, VCL gives that $\exists\ n_x\geq n$ s.t. $I_x\cap I_{n_x}\neq\emptyset$ and $I_x\subseteq c\,I_{n_x}$. It follows that $A_{\alpha,\beta}\setminus\bigcup_{k=1}^{n} I_k\subseteq\bigcup_{k\in K\setminus\{1,\dots,n\}} cI_k$, hence by monotonicity and subadditivity, we obtain that

$$m_*\Big(A_{\alpha,\beta}\setminus\bigcup_{k=1}^{n} I_k\Big)\leq m\Big(\bigcup_{k\in K\setminus\{1,\dots,n\}} cI_k\Big)$$
$$\leq c\sum_{k\in K\setminus\{1,\dots,n\}} m(I_k)$$

In case K finite, we obtain $m_*\big(A_{\alpha,\beta}\setminus\bigcup_{k=1}^{n} I_k\big)=0$. In case K infinite, for n large, we have $m_*\big(A_{\alpha,\beta}\setminus\bigcup_{k=1}^{n} I_k\big)<\varepsilon$ since $\sum_{n\in K} m(I_k)<\infty$

Proof of VCL

If F is empty, nothing to prove, so assume not. Choose an arbitrary interval $I_1\in F$. By induction, we define $F_n=\{I\in F: I\cap\bigcup_{k=1}^{n} I_k\neq\emptyset\}$ and if $F_n\neq\emptyset$, let $I_{n+1}\in F_n$ be such that $\underset{(\text{length})}{\ell(I_{n+1})}>(1-\varepsilon)\sup_{I\in F_n}\ell(I)$, where $\varepsilon\in(0,1)$ is fixed. Let $F'=\{I_n\}$, by construction, F' is at most countable, and the intervals $(I_n)_n$ are disjoint. Let $I\in F$, observe that $F_{n+1}\subseteq F_n$ $\forall n$. Let $F_0=F$, then $n_I=\sup\{n: I\in F_n\}$ $(\in\{0,1,2,\dots\}\cup\{\infty\})$ exists.

Case $n_I = \infty$ In this case, $I \in F_n \ \forall n$, then $\ell(I) < \frac{1}{1-\varepsilon}\ell(I_n) \ \forall n$.

Since $(I_n)_n$ are disjoint and included in $[a,b]$, we have $\sum_{n=1}^{\infty} \ell(I_n) \leq \ell([a,b]) < \infty$

hence $\ell(I_n) \to 0$ as $n \to \infty$, which implies $\ell(I) = 0$, contradiction.

Case $n_I < \infty$ In this case, $I \in F_{n_I+1}$, so $I \cap I_{n+1} \neq \emptyset$, ie. $\exists y \in I \cap I_{n+1}$.

$\forall x \in I$, $|x - x_{I_{n+1}}| \leq \underbrace{|x-y|}_{\leq \ell(I) < \frac{1}{1-\varepsilon}\ell(I_{n+1})} + \underbrace{|y - x_{I_{n+1}}|}_{\leq 1/2\, \ell(I_{n+1})}$, where $x_{I_{n+1}}$ is the middle point of I_{n+1}.

Hence,

$$|x - x_{I_{n+1}}| < \left(\frac{1}{1-\varepsilon} + \frac{1}{2}\right)\ell(I_{n+1})$$

ie. $x \in 2\left(\frac{1}{1-\varepsilon} + \frac{1}{2}\right) I_{n+1} = \left(\frac{2}{1-\varepsilon} + 1\right) I_{n+1}$

This shows $I \subseteq \left(\frac{2}{1-\varepsilon} + 1\right) I_{n+1}$, letting $\varepsilon \to 0$, we obtain $I \subseteq c I_{n+1}$

by observing that $\frac{2}{1-\varepsilon} + 1 \to 3$ as $\varepsilon \to 0$

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Proposition

A monotone function $f : [a,b] \to \mathbb{R}$ has at most countable points of discontinuity.

Proof:

Let D be the set of points of discontinuity of f. Then $\forall x \in D$, $\underbrace{\lim_{y \to x^-} f(y)}_{\ell_{x_-}} < \underbrace{\lim_{y \to x^+} f(y)}_{\ell_{x_+}}$.

By density of $\mathbb{Q}$ in $\mathbb{R}$, $\exists q_x \in (\ell_{x_-}, \ell_{x_+})$. Since f is monotone,

we obtain that $\begin{array}{l} D \to \mathbb{Q} \\ x \mapsto q_x \end{array}$ is injective, so D is at most countable.

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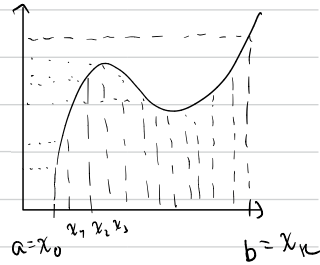
Functions of Bounded Variations

Definition

Let $f:[a,b] \longrightarrow \mathbb{R}$, we define $T_f(a,b) = \sup\left\{ \sum_{i=1}^{k} |f(x_i) - f(x_{i-1})| : a = x_0 < x_1 < \cdots < x_k = b \right\}$

$T_f(a,b)$ is called the total variation of f.

If $T_f(a,b) < \infty$, we say that f is of bounded variation.



Example

(i) Monotone functions are of bounded variation. $\forall x_0 = a < x_1 < x_2 < \cdots < x_k = b$,

$$\sum_{i=1}^{k} |f(x_i) - f(x_{i-1})| = \begin{cases} \sum_{i=1}^{k} (f(x_i) - f(x_{i-1})) = f(b) - f(a) & \text{if } f \text{ is increasing} \\ \sum_{i=1}^{k} (f(x_i) - f(x_{i-1})) = f(a) - f(b) & \text{if } f \text{ is decreasing} \end{cases}$$

(ii) Lipschitz continuous functions ie. s.t. $\exists\, c > 0\ \forall x, y \in [a,b]$ s.t. $|f(x) - f(y)| < c|x-y|$ are of bounded variation.

$\forall x_0 = a < x_1 < \ldots < x_k = b$, $\sum_{i=1}^{k} |f(x_i) - f(x_{i-1})| \le c \sum_{i=1}^{k} (x_i - x_{i-1}) = c(b-a)$.

In particular, recall that if $f:[a,b] \to \mathbb{R}$ is continuous in $[a,b]$ an differentiable in (a,b), then $|f(x) - f(y)| \le \sup_{(a,b)} \{ |f'| \cdot |x-y| \}$. Moreover, $\sup |f'| < \infty$, then f is Lipschitz continuous.

Counterexample: $f(x) = \begin{cases} x\cos\left(\frac{\pi}{2x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ is not of bounded variation in $[0,1]$

Proof left as an exercise. (A6 Q2)

Proposition $\forall\, a < c < b$, $T_f(a,b) = T_f(a,c) + T_f(c,b)$

Proof

$(\leq)$ $\forall\, a = x_0 < x_1 < \dots < x_k = c$, $c = x'_0 < x'_1 < \dots < x'_{k'} = b$,

$$\sum_{i=1}^{k} |f(x_i) - f(x_{i-1})| + \sum_{i=1}^{k'} |f(x'_i) - f(x'_{i-1})| \leq T_f(a,b),$$

Since $a = x_0 < x_1 < \dots < x_k = x'_0 < x'_1 < \dots < x'_{k'} = b$

By taking sup on $(x_i)_i$ and $(x'_i)_i$, we obtain $T_f(a,c) + T_f(c,b) \leq T_f(a,b)$

$(\geq)$ $\forall\, x_0 = a < x_1 < \dots < x_k = b$, letting $i_0 \in \{0, \dots, k-1\}$ be s.t. $x_{i_0} \leq c < x_{i_0+1}$,

$$\sum_{i=1}^{k} |f(x_i) - f(x_{i-1})| \leq \underbrace{\sum_{i=1}^{i_0} |f(x_i) - f(x_{i-1})| + |f(c) - f(x_{i_0})|}_{\leq T_f(a,c)} + \underbrace{|f(x_{i_0+1}) - f(c)| + \sum_{i=i_0+2}^{k} |f(x_i) - f(x_{i-1})|}_{\leq T_f(b,c)}$$

by Δ-ineq

Hence $T_f(a,b) \leq T_f(a,c) + T_f(c,b)$

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