Chapter 4 - Fubini's and Tonelli's Theorems

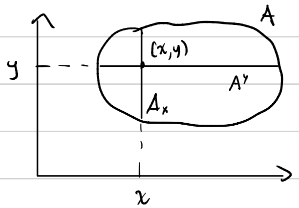
Let $d_1, d_2 \in \mathbb{N}$ be s.t. $d = d_1 + d_2$, we denote $(x,y) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} = \mathbb{R}^d$

For every set $A \subseteq \mathbb{R}^d$, we denote

$$A_x = \{ y \in \mathbb{R}^{d_2} : (x,y) \in A \}$$
$$A^y = \{ x \in \mathbb{R}^{d_1} : (x,y) \in A \}$$

For every function $f : A \to \overline{\mathbb{R}}$, we denote

$$f_x : \begin{array}{l} A_x \to \overline{\mathbb{R}} \\ y \longmapsto f(x,y) \end{array} \qquad f^y : \begin{array}{l} A_y \to \overline{\mathbb{R}} \\ x \longmapsto f(x,y) \end{array}$$



Remarks

(i) The sets A_x and A^y are not necessarily measurable when A is measurable. For example,

Let $D \subseteq [0,1]$ non measurable subset, then $D \times \{0\}^{d-1} \subseteq [0,1] \times \{0\}^{d-1}$,
but $(D \times \{0\}^{d-1})_0 = D$ is not measurable in $\mathbb{R}$.

(ii) It is not always true that $\int_{\mathbb{R}^{d_1}} \int_{A_x} f(x,y)\, dy\, dx = \int_{\mathbb{R}^{d_2}} \int_{A^y} f(x,y)\, dx\, dy$
even when these integrals are well defined.
for example, let $D = [0,1]^2$

$$\int_0^1 \int_0^1 \frac{x^2 - y^2}{(x^2+y^2)^2}\, dy\, dx = \int_0^1 \left[\frac{y}{x^2+y^2} \right]_{y=0}^{y=1} dx = \int_0^1 \frac{1}{x^2+1}\, dx = \arctan x \Big|_{x=0}^{x=1}$$
$$= \pi/4$$

But,

$$\int_0^1 \int_0^1 \frac{x^2-y^2}{(x^2+y^2)^2}\, dx\, dy = \int_0^1 \left[\frac{-x}{x^2+y^2} \right]_{x=0}^{x=1} dy = \int_0^1 \frac{-1}{x^2+y^2}\, dy = -\arctan y \Big|_{y=0}^{y=1} = \frac{-\pi}{4}$$

Fubini's Theorem

Let $f: \mathbb{R}^d \to \overline{\mathbb{R}}$ be integrable. Then,

(i) For a.e. $y \in \mathbb{R}^{d_2}$, f^y is integrable in $\mathbb{R}^{d_1}$

(ii) $y \longmapsto \int_{\mathbb{R}^{d_1}} f^y$ is integrable over $\mathbb{R}^{d_2}$

(iii) $\int_{\mathbb{R}^{d_2}} \Big(\underbrace{\int_{\mathbb{R}^{d_1}} f^y}_{= \int_{\mathbb{R}^{d_1}} f(x,y)\, dx} \Big) dy = \int_{\mathbb{R}^d} f$

Remark

The roles of x and y can be inverted, giving $\int_{\mathbb{R}^{d_1}} \left(\int_{\mathbb{R}^{d_2}} f(x,y)\, dy \right) dx = \int_{\mathbb{R}^d} f$

Proof of Fubini's:

Let $F := \{ f: \mathbb{R}^d \to \overline{\mathbb{R}}$ integrable satisfying (i), (ii), (iii) $\}$

Step 1: $\forall f_1, \dots, f_n \in F$, $\forall \alpha_1, \dots, \alpha_n \in \mathbb{R}$, we have $\alpha_1 f_1, \dots, \alpha_n f_n \in F$

follows from linearity of the integral

Step 2: $\forall f: \mathbb{R}^d \to \overline{\mathbb{R}}$ integrable, $\forall (f_k)_{k \in \mathbb{N}}$ s.t. $(f_k)_k$ is monotone, (i.e. $f_k \leq f_{k+1}$ or $f_k \geq f_{k+1}$ $\forall k$) and converging pointwise to f in $\mathbb{R}^d$, then $f \in F$.

proof: Without loss of generality, we may assume $(f_k)_k$ is increasing (otherwise, consider $-f_k$), and non negative (o.w. consider $\underbrace{f_k - f_1}_{\text{well defined since both are integrable}}$)

Since $(f_k)_k$ is increasing, nonnegative, and pointwise converging to f, the MCT gives $\int_{\mathbb{R}^d} f = \lim_{k \to \infty} \int_{\mathbb{R}^d} f_k$. By (i), the functions $(f_k^y)_k$ are measurable for a.e. $y \in \mathbb{R}^{d_2}$. Since $(f_k^y)_k$ is increasing, non negative, and pointwise converging to f^y, the MCT gives that f^y is measurable and, $\int_{\mathbb{R}^{d_1}} f^y = \lim \int_{\mathbb{R}^{d_1}} f_k^y$ for a.e. $y \in \mathbb{R}^{d_2}$. By (ii), the functions $\left(y \mapsto \int_{\mathbb{R}^{d_1}} f_k^y\right)_k$ are measurable. Since $\left(y \mapsto \int_{\mathbb{R}^{d_1}} f_k^y\right)_k$ is increasing, non negative, and converging pointwise a.e. in $\mathbb{R}^{d_2}$, MCT gives that $y \mapsto \int_{\mathbb{R}^{d_1}} f^y$ is measurable and that

$\int_{\mathbb{R}^{d_2}} \left(\int_{\mathbb{R}^{d_1}} f^y \right) dy = \lim_{k \to \infty} \int_{\mathbb{R}^{d_2}} \left(\int_{\mathbb{R}^{d_1}} f_k^y \right) dy$. By (iii),

$\int_{\mathbb{R}^d} f_k = \int_{\mathbb{R}^{d_2}} \left(\int_{\mathbb{R}^{d_1}} f_k \right) dy$. By passing to the limit as $k \to \infty$, we obtain

$$\int_{\mathbb{R}^d} f = \int_{\mathbb{R}^{d_2}} \left(\int_{\mathbb{R}^{d_1}} f^y \right) dy$$ i.e. (iii) is true for f.

Since f is integrable in $\mathbb{R}^d$, it follows that $y \mapsto \int_{\mathbb{R}^{d_1}} f^y$ is integrable in $\mathbb{R}^{d_2}$, i.e. (ii) is true for f.
It follows that for a.e. $y \in \mathbb{R}^{d_2}$, $\int_{\mathbb{R}^{d_1}} f^y < \infty$, i.e. f^y is integrable over $\mathbb{R}^{d_1}$, i.e. (i) is true for f.

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Step 3: $\forall f: \mathbb{R}^d \to \overline{\mathbb{R}}$, if $f = 0$ a.e. in $\mathbb{R}^d$, and for a.e. $y \in \mathbb{R}^{d_2}$, we have $f^y = 0$ a.e. in $\mathbb{R}^{d_1}$, then $f \in F$.

proof: For a.e $y \in \mathbb{R}^{d_2}$, if $f^y = 0$ a.e. in $\mathbb{R}^{d_1}$, then f^y is integrable (ie. (i) is true) and, $\int_{\mathbb{R}^{d_1}} f^y = 0$. It follows that $y \mapsto \int_{\mathbb{R}^{d_1}} f^y$ is integrable (ie. (ii) is true) and $\int_{\mathbb{R}^{d_2}} \left(\int_{\mathbb{R}^{d_1}} f^y \right) dy = 0$. Moreover, since $f = 0$ a.e. in $\mathbb{R}^d$, it follows that

$$\int_{\mathbb{R}^{d_1}} f = 0 = \int_{\mathbb{R}^{d_2}} \left(\int_{\mathbb{R}^{d_1}} f^y \right) dy$$

Step 4: $\forall A \subseteq \mathbb{R}^d$ measurable of finite measure, $\chi_A \in F$

- <u>Case $A = Q$ open cube.</u>

Write $Q = Q_1 \times Q_2$ where Q_1 and Q_2 are open cubes in $\mathbb{R}^{d_1}$ and $\mathbb{R}^{d_2}$ respectively. In this case, $\chi_A(x,y) = \chi_{Q_1}(x)\chi_{Q_2}(y)$,

$$\chi_A^y(x) = \begin{cases} \chi_{Q_1}(x) & \text{if } y \in Q_2 \\ 0 & \text{if } y \notin Q_2 \end{cases}$$

is integrable over $\mathbb{R}^{d_1}$,

$$\int_{\mathbb{R}^{d_1}} \chi_A^y = \left\{ \begin{matrix} m(Q_1) & \text{if } y \in Q_2 \\ 0 & \text{if } y \notin Q_2 \end{matrix} \right\} = m(Q_1)\chi_{Q_2}(y)$$

is in turn integrable, and $\int_{\mathbb{R}^{d_2}} \left(\int_{\mathbb{R}^{d_1}} \chi_A^y \right) dy = m(Q_1)m(Q_2)$.

Hence,

$$\int_{\mathbb{R}^d} \chi_A = m(A) = \mathrm{vol}(Q_1 \times Q_2) = \mathrm{Vol}(Q_1)\mathrm{Vol}(Q_2) = m(Q_1)m(Q_1) = \int_{\mathbb{R}^{d_2}} \left(\int_{\mathbb{R}^{d_1}} \chi_A^y \right) dy$$

- <u>Case A open set</u>

In this case, A can be written as $A = \bigcup_{k=1}^{\infty} \overline{Q_k}$ where $(Q_k)_k$ are open, mutually disjoint cubes. Then,

$$\chi_A = \sum_{k=1}^{\infty} \chi_{Q_k} + \chi_{\tilde{A}},$$

where $\tilde{A} \subseteq \bigcup_{k=1}^{\infty} \partial Q_k$ since the sets $\tilde{A}$ and Q_k are disjoint and $\bigcup_{k=1}^{\infty} Q_k \cup \tilde{A} = A$. Since $\chi_{Q_k} \in F$ $\forall k \in \mathbb{N}$, by step 1, we obtain that $\sum_{k=1}^{n} \chi_{Q_k} \in F$ $\forall n \in \mathbb{N}$. By step 2, since $\sum_{k=1}^{n} \chi_{Q_k} \nearrow \sum_{k=1}^{\infty} \chi_{Q_k}$, it follows that $\sum_{k=1}^{\infty} \chi_{Q_k} \in F$.

Moreover, $m(\tilde{A}) \leq \sum_{k=1}^{\infty} m(\partial Q_k)$ by subadditivity, hence $m(\tilde{A}) = 0$, that is, $\chi_{\tilde{A}} = 0$ a.e. in $\mathbb{R}^d$, and for a.e. $y \in \mathbb{R}^{d_2}$, $m(\tilde{A}^y) = 0$.

except on the boundary of cubes.

Thus, $\chi^y_{\tilde{A}} = 0$ a.e. in $\mathbb{R}^{d_1}$ and step 3 gives that $\chi_{\tilde{A}} \in F$. Finally, Step 1 gives $\chi_A \in F$.

- Case A is a G_δ-set.

  In this case, $A = \bigcap_{k=1}^{\infty} O_k$, O_k open. Define $f_n = \chi_{\bigcap_{k=1}^{n} O_k}$.

  Then, $f_n \searrow \chi_A$, hence step 2 gives $\chi_A \in F$.

- Case A is of measure 0.

  Write $A = G \setminus N$ where G is a G_δ-set and $m(N) = 0$. Then, additivity gives $m(G) = m(A) + m(N) = 0 + 0 = 0$. Since $m(A) = 0$, we have $\chi_A = 0$ a.e. in $\mathbb{R}^d$. Moreover, $\chi_G \in F$, we have that for a.e. $y \in \mathbb{R}^{d_2}$, χ^y_G is integrable, $y \mapsto \underbrace{\int_{\mathbb{R}^{d_2}} \chi^y_Q}_{= m(G^y)}$ is integrable and $\int_{\mathbb{R}^{d_2}} m(G^y) = \int_{\mathbb{R}^d} \chi_G = m(G) = 0$, hence we obtain that $m(G^y) = 0$ for a.e. $y \in \mathbb{R}^{d_2}$. Since $A^y \subseteq G^y$, by monotonicity, it follows that $m(A^y) = 0$, ie. $[\chi^y_A = 0 \text{ a.e. in } \mathbb{R}^{d_1}]$ for a.e. $y \in \mathbb{R}^{d_2}$

- Case A measurable

Write $A = G \setminus N$ where G is a G_δ-set and $N \subseteq G$ is of measure 0. Then,

$\chi_A = \chi_G - \chi_N$, and since $\chi_G, \chi_N \in F$, step 1 gives that $\chi_A \in F$.

Step 5: Every simple function is in F.

Proof: Apply step 1 and 4.

Step 6: Every non-negative, integrable function is in F.

Proof: Apply the simple approximation theorem: $\exists\ (\varphi_k)_k$ simple functions $\in F$ by step 5.

s.t. $\varphi_k \nearrow f$ (converges pointwise to f, increasing sequence). Then, step 2 gives that $f \in F$.

Step 7: Every integrable function is in F.

Proof: write $f = f^+ - f^-$ and use step 1 $\blacksquare$

Tonelli's Theorem

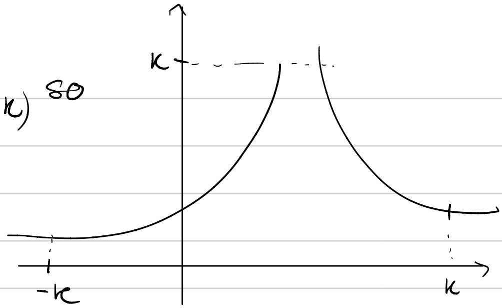
Let $f : \mathbb{R}^n \to [0, \infty]$ be measurable. Then,

(i) for a.e. $y \in \mathbb{R}^{d_2}$, f^y is measurable.

(ii) $y \longmapsto \int_{\mathbb{R}^{d_1}} f^y$ is measurable

(iii) $\int_{\mathbb{R}^{d_2}} \left(\int_{\mathbb{R}^{d_1}} f^y \right) dy = \int_{\mathbb{R}^d} f$

Proof: Consider $f_k = \min(f, k)\,\chi_{B(0,k)}$ so that f_k is integrable, increasing in k and pointwise converging to f.



By Fubini's Theorem, we have

(i') For a.e. $y \in \mathbb{R}^{d_2}$, f_k^y is integrable

(ii') $y \mapsto \int_{\mathbb{R}^{d_1}} f_k^y$ is integrable

(iii') $\int_{\mathbb{R}^{d_2}} \left(\int_{\mathbb{R}^{d_1}} f_k^y \right) dy = \int_{\mathbb{R}^d} f_k$

By applying the MCT, and passing to the limit in (i'), (ii'), (iii'), successively in $\mathbb{R}^{d_1}$, $\mathbb{R}^{d_2}$, and $\mathbb{R}^d$, like in step 2 of Fubini's, we obtain (i), (ii), (iii).

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