Theorem

A function $f:[a,b]\to\mathbb{R}$ is of bounded variation $\iff$ it can be written as the difference between two increasing functions. In particular, f is differentiable a.e. in (a,b) and f is integrable over (a,b).

Proof: $(\Rightarrow)$ Write $f(x) = (f(x)+T_f(a,x)) - T_f(a,x)$.

$T_f(a,\cdot)$ is increasing: $\forall x<y$, $T_f(a,y) - T_f(a,x) = T_f(x,y) \geq 0$

ie. $T_f(a,y) \geq T_f(a,x)$. $f+T_f(a,\cdot)$ is increasing: $\forall x<y$,

$$(f(y)+T_f(a,y)) - (f(x)+T_f(a,x)) = f(y)-f(x) + T_f(x,y) \geq -|f(y)-f(x)| + T_f(x,y) \geq 0$$

$(\Leftarrow)$ If $f_1, f_2 : [a,b]\to\mathbb{R}$ are increasing, then $\forall\ a = x_0 < \dots < x_k = b$,

$$\sum_{i=1}^{k} |(f_1-f_2)(x_i) - (f_1-f_2)(x_{i-1})| \leq \sum_{i=1}^{k} |f_1(x_i)-f_1(x_{i-1})| + \sum_{i=1}^{k} |f_2(x_i)-f_2(x_{i-1})|$$

$$= \sum_{i=1}^{k} (f_1(x_i)-f_1(x_{i-1})) + \sum_{i=1}^{k} (f_2(x_i)-f_2(x_{i-1})) \qquad [\text{since } f_1, f_2 \nearrow]$$

$$= f_1(b)-f_1(a)+f_2(b)-f_2(a)$$

Hence $T_{f_1-f_2}(a,b) \leq f_1(b)-f_1(a)+f_2(b)-f_2(a) < \infty$, i.e. f_1-f_2 is of bdd variation. $\blacksquare$

End of Final Exam Material

Absolutely Continuous Functions

Definition: We say that a function $f:[a,b]\to\mathbb{R}$ is absolutely continuous on $[a,b]$ if $\forall\ \varepsilon>0$, $\exists\ \delta>0$ s.t. $\forall$ finite collection of disjoint, open intervals, $(a_1,b_1),\dots,(a_k,b_k) \subseteq [a,b]$, if $\sum_{i=1}^{k} b_i - a_i < \delta$, then $\sum_{i=1}^{k}|f(b_i)-f(a_i)| < \varepsilon$.

Remark: Absolute continuity implies uniform continuity (case $k=1$)

Example

Lipschitz continuous functions are absolutely continuous.

Proof:

$\forall (a_1, b_1), \dots, (a_k, b_k)$ disjoint in $[a,b]$, if $\sum_{i=1}^{k} (b_i - a_i) < \delta = \varepsilon / c$, then

$$\sum_{i=1}^{k} |f(b_i) - f(a_i)| \leq c \sum_{i=1}^{k} (b_i - a_i) < \varepsilon .$$

$\blacksquare$

Counter example.

The cantor-lebesgue function is not absolutely cts.

Proof:

Let φ be the cantor-lebesgue function and $C = \bigcap_{k=1}^{\infty} C_k$ be the Cantor set,

$C_k = \bigcup_{j=1}^{2^k} [a_{k,j}, b_{k,j}]$, where the intervals $[a_{k,j}, b_{k,j}]$ are disjoint. Then,

$$\sum_{j=1}^{2^k} |\varphi(b_{k,j}) - \varphi(a_{k,j})| = \sum_{j=1}^{2^k} 2^{-k} = 1, \text{ but } \sum_{j=1}^{2^k} (b_{k,j} - a_{k,j}) = \sum_{j=1}^{2^k} 3^{-k} = \frac{2^k}{3^k} \to 0$$

$\blacksquare$

Proposition

If $f : [a,b] \to \mathbb{R}$ is absolutely continuous on $[a,b]$, then it can be written as the difference between two increasing, absolutely continuous function. In particular, f is of bounded variation.

Proof:

<u>Step 1</u> $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $\forall (a_1, b_1), \dots, (a_k, b_k)$ open, disjoint in $[a,b]$, if

$\sum_{i=1}^{k} (b_i - a_i) < \delta$, then $\sum_{i=1}^{k} T_f(a_i, b_i) < \varepsilon$.

Proof of Step 1:

Since f is abs. cts on $[a,b]$, $\forall\ \varepsilon>0, \exists \delta>0$ s.t. $\forall\ (a_1,b_1),\dots,(a_k,b_k)$ open disjoint in $[a,b]$, if $\sum_{i=1}^{k}(b_i-a_i)<\delta$, then $\sum_{i=1}^{k}|f(b_i)-f(a_i)|<\varepsilon$. Observe that $\forall\ x_{i,0}=a_i<\dots<x_{i,k_i}=b_i$, $(x_{i,0},x_{i,1}),\dots,(x_{i,k_i-1},x_{i,k_i})$ are open, disjoint in (a_i,b_i), and $\sum_{i=1}^{k}\sup_{a=x_{i,0}<\dots<x_{i,k_i}=b}\sum_{i=1}^{k}\sum_{j=1}^{k_i}|f(x_{i,j})-f(x_{i,j-1})|=\sum_{i=1}^{k}T_f(a_i,b_i)$ and $\sum_{i=1}^{k}\sum_{j=1}^{k_i}(x_{i,j}-x_{i,j-1})=\sum_{i=1}^{k}(b_i-a_i)$. We obtain the claim in step 1 $\square$

Step 2: Show that $T_f(a,b)<\infty$ i.e. f is of bounded variation.

Proof:

Consider $x_{i,k}=a+\frac{i}{k}(b-a)$. By Step 1, $\exists \delta>0$ s.t. $\forall (x,y)\subset[a,b]$, if $x-y<\delta$, Then $T_f(x,y)<1$. By choosing k large enough so that $\frac{1}{k}<\delta$, we obtain $T_f(x_{i-1,k},x_{i,k})<1$. It follows that $\sum_{i=1}^{k}T_f(x_{i-1,k},x_{i,k})=\sum_{i=1}^{k}1=k<\infty$ $\square$

Conclusion

By remarking that $T_f(a_i,b_i)=T_f(a,b_i)-T_f(a,a_i)$, which are finite by Step 2, Step 1 gives that $T_f(a,\cdot)$ is absolutely continuous on $[a,b]$. Now, write $f(x)=(f(x)+T_f(a,x))-T_f(a,x)$. It remains to show that $f+T_f(a,\cdot)$ is absolutely continuous. More generally, $\forall f_1,f_2$ abs. cts on $[a,b]$, f_1+f_2 is abs. cts on

$[a,b]$: Indeed, if f_1, f_2 are abs. cts on $[a,b]$, $\forall \varepsilon > 0, \exists \delta_1, \delta_2 > 0$ s.t.

$\forall (a_1, b_1), \dots, (a_k, b_k)$ disjoint, open in $[a,b]$, if

$$\sum_{i=1}^{k} (b_i - a_i) < \begin{matrix} \delta_1 \\ \delta_2 \end{matrix} \quad \text{then} \quad \begin{matrix} \sum_{i=1}^{k} |f_1(b_i) - f_1(a_i)| < \varepsilon/2 \\ \sum_{i=1}^{k} |f_2(b_i) - f_2(a_i)| < \varepsilon/2 \end{matrix}$$

Then,

$$\sum_{i=1}^{k} \left|(f_1+f_2)(b_i) + (f_1+f_2)(a_i)\right| = \sum_{i=1}^{k} |f_1(b_i) + f_1(a_i)| + \sum_{i=1}^{k} |f_2(b_i) - f_2(a_i)| < \varepsilon,$$

provided $\sum_{i=1}^{k} (b_i - a_i) < \min(\delta_1, \delta_2)$

$\blacksquare$