Questions

(1) Does there exist non-measurable sets? yes: Axiom of choice

(2) Does there exist sets of measure 0 that are uncountable?

(3) Can all measurable sets be obtained as combinations of countable union & complements of open sets.

Question 1

Recall

Axiom of choice

let Ω be a collection of nonempty sets. Then there exists a function

$$f : \Omega \to \bigcup_{S\in\Omega} S$$

such that $\forall s \in \Omega$, $f(s) \in S$

Proposition

$\forall$ set $A \subseteq \mathbb{R}^d$ such that $m_*(A) > 0$, $\exists\, B \subseteq A$ which is not measurable.

proof:

Write $A = \bigcup_{k_1,\dots,k_d \in \mathbb{Z}} A_{k_1,\dots,k_d}$ where $A_{k_1,\dots,k_d} = A \cap [k_1, k_1+1] \times \dots \times [k_d, k_d+1]$

By countable additivity, $m_*(A) \le \sum_{k_1,\dots,k_d \in \mathbb{Z}} m_*(A_{k_1,\dots,k_d})$

Since $m_*(A) > 0$, it follows that $\exists\, k_1,\dots,k_d$ s.t. $m_*(A_{k_1,\dots,k_d}) > 0$.
Thus, wlog, we may assume $A \subseteq [k_1, k_1+1] \times \dots \times [k_d, k_d+1]$

By translation invariance of m_* (Q_3, A1), we may assume $k_1,\dots,k_d = 0$

For every $x \in A$, define $S_x = A \cap (x + \mathbb{Q}^d)$ where

$$x + \mathbb{Q}^d = \{(x_1 + q_1, \dots, x_d + q_d) : q_1, \dots, q_d \in \mathbb{Q}\}$$

Apply the axiom of choice with $\Omega = \{S_x : x \in A\}$:

$\exists f : \Omega \to \bigcup_{x \in A} S_x$ s.t. $f(S_x) \in S_x$, i.e. $f(S_x) \in A$ and $f(S_x) = x + q(x)$, $q(x) \in \mathbb{Q}^d$

We will show that $f(\Omega)$ is not measurable

Observe that $\forall x \in A$, $q(x) = f(S_x) - x \in [-1,1]^d$ since $x \& f(S_x) \in A \subseteq [0,1]^d$.

We first show that $A \subseteq \bigcup_{q \in \mathbb{Q}^d \cap [-1,1]^d} B_q$ where $B_q = f(\Omega) + q$

Indeed, let $x \in A$, then $x = f(S_x) - q(x) \in B_{-q(x)}$, with $-q(x) \in [-1,1]^d$ since $q(x) \in [-1,1]^d$.

Moreover, the sets $(B_q)_q$ are mutually disjoint, indeed, let $f(S_x) + q \in B_q$, $f(S_{x'}) + q' \in B_{q'}$ be such that

$$f(S_x) + q = f(S_{x'}) + q'$$

Then, $x + q(x) + q = x' + q(x') + q'$, which gives

$$x - x' = q(x') - q(x) + q' - q \in \mathbb{Q}^d \Rightarrow S_x = S_{x'}$$

And,

$$f(S_x) = f(S_{x'})$$

Which implies that $q = q'$. By subadditivity, since $A \subseteq \bigcup_{q \in \mathbb{Q}^d \cap [-1,1]} B_q$, we obtain

$$m_*(A) \leq \sum_{q \in \mathbb{Q}^d \cap [-1,1]} m_*(B_q)$$

Moreover, $m_*(B_q) = m_*(f(\Omega))$ by translation inverriance (A1Q3)

Since $m_*(A) > 0$, we obtain that $\sum_{q\in\mathbb{Q}^d\cap[-1,1]^d} m_*(f(\Omega)) > 0$,

Which implies that $m_*(f(\Omega)) > 0$ and

$$\sum_{q\in\mathbb{Q}^d\cap[-1,1]} m_*(f(\Omega)) = \infty$$

If we assume by contradiction that $f(\Omega)$ is measurable, then the sets $(B_q)_q$ are measurable and by ctbl additivity, we obtain

$$m\Big(\bigcup_{q\in\mathbb{Q}^d\cap[-1,1]^d} B_q\Big) = \sum_{q\in\mathbb{Q}^d\cap[-1,1]^d} m(B_q) = \sum_{q\in\mathbb{Q}^d\cap[-1,1]} f(\Omega) = \infty$$

but,

$$\bigcup_{q\in\mathbb{Q}^d\cap[-1,1]} B_q \subseteq f(\Omega) + [-1,1]^d \subseteq A + [-1,1]^d \subseteq [0,1]^d + [-1,1]^d$$

Which is bounded, hence $m\Big(\bigcup_{q\in\mathbb{Q}^d\cap[-1,1]^d} B_q\Big) < \infty$, contradiction.

Question 2: existence of an uncountable set of measure 0.

Recall the cantor set $C = \bigcap_{k=1}^{\infty} C_k$ where $C_1 = [0, 1/3] \cup [2/3, 1]$

And, $\forall k \geq 2$, $C_k = \bigcup_{j=1}^{2^k} I_{j,k}$ where $\forall j \in \{1, \dots, 2^{k-1}\}$,
$I_{2j-1,k}$ and $I_{2j,k}$ are the first and last third of $I_{j,k-1}$ (closed)

Proposition

C is closed, uncountable, and $m_*=0$

proof

C is closed since it is the intersection of closed sets

By monotonicity, since $C \subseteq C_k$ $\forall k \in \mathbb{N}$ we obtain

$$m_*(C) \leq m_*(C_k)$$

Since $C_k = \bigcup_{j=1}^{2^k} I_{j,k}$ and the intervals $(I_{j,k})_j$ are mutually disjoint. By ctbl additivity,

$$m(C_k) = \sum_{j=1}^{2^k} m(I_{j,k}) = \sum_{j=1}^{2^k} 3^{-k}$$

(The length of $I_{j,k}$ is divided by 3 at each iteration of k)

$$= 2^k \cdot 3^{-k}$$

$$= (2/3)^k \to 0 \text{ as } k \to \infty$$

So $m(C) = 0$, and cantor set uncountable

Question 3

Can all measurable sets be obtained as combinations of countable union & complements of open sets.

Definition

A collection Ω of subsets of $\mathbb{R}^d$ is called a σ-algebra if

(i) $\mathbb{R}^d \in \Omega$

(ii) $\forall A, B \in \Omega$, $A \setminus B \in \Omega$

(iii) $\forall (A_k)_{k \in \mathbb{N}} \in \Omega$, $\bigcup_{k=1}^{\infty} A_k \in \Omega$

Ex) The collection of $\mathcal{A}$ measurable sets in $\mathbb{R}^d$ is a σ-algebra

Proposition

Any intersection of σ-algebras is a σ-algebra

proof

let $(\Omega_j)_{j \in J}$ be a family of σ-algebras and $\Omega = \bigcap_{j \in J} \Omega_j$

Then, (i) $\mathbb{R}^d \in \Omega_j \ \forall_j$ hence $\mathbb{R}^d \in \Omega$

(ii) $\forall A, B \in \Omega$, $A \setminus B \in \Omega_j$ since $A, B \in \Omega_j \subseteq \Omega \ \forall_j$ hence $A \setminus B \in \Omega$

(iii) $\forall (A_k)_{k \in \mathbb{N}} \in \Omega$, $\bigcup_{k=1}^{\infty} A_k \in \Omega_j$ since $A_k \in \Omega_j \subseteq \Omega \ \forall_j$ hence $\bigcup_{k=1}^{\infty} A_k \in \Omega$

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