Definition

We call Borel σ-algebra the intersection of all σ-algebras containing the open sets of $\mathbb{R}^d$. } i.e. the smallest

We call Borel set any element of the Borel σ-algebra

In particular, Borel sets are Lebesgue measurable.

Definition

We call Cantor-Lebesgue (or Cantor staircase) function

$$\varphi : [0,1] \to [0,1]$$

Defined by

- $\varphi(0)$, $\varphi(1) = 1$
- $\forall x \in [0,1] \setminus C$ $\quad$ (C = cantor set $= \bigcap_{k=1}^{\infty} C_k$)

$$\varphi(x) = \begin{cases} 1/2 & \text{if } x \in \left(\frac{1}{2}, \frac{2}{3}\right) = [0,1] \setminus C_1 \\ 1/4 & \text{if } x \in (1/9, 2/9) \\ 3/4 & \text{if } x \in (2/3 + 1/9, 2/3 + 2/9) \\ \vdots & \\ i/2^k & \text{if } x \in J_{i,k} \text{ where } J_{i,k} \text{ is the } \\ & j^{th} \text{ interval of } [0,1] \setminus C_k = \bigcup_{i=1}^{2^k - 1} J_{i,k} \end{cases}$$

(the 2nd and 3rd lines together: $[0,1] \setminus C_2$)

- $\forall x \in C \setminus \{0,1\}$ $\quad \varphi(x) = \sup\{\varphi(y) : y \in (0,x) \setminus C\}$

Proposition φ is increasing, continuous, and surjective from $[0,1]$ to $[0,1]$

proof: Monotonicity follows directly from the construction

<u>continuity:</u> Since the intervals $J_{i,k}$ are open and φ is constant on each $J_{i,k}$, we obtain that φ is continuous at each point of $[0,1] \setminus C = \bigcup_{k=1}^{\infty} \bigcup_{i=1}^{2^k-1} J_{i,k}$.

Let $x \in C$, for every $k \in \mathbb{N}$, $\exists\, i \in \{1, \dots, 2^k\}$ such that

$$\begin{cases} x \text{ is between } 0 \text{ and } J_{i,k} \text{ if } i=1 \\ x \text{ is between } J_{i-1,k} \text{ and } J_{i,k} \text{ if } 2 \leq i \leq 2^k-1 \\ x \text{ is between } J_{2^k-1,k} \text{ and } 1 \text{ if } i = 2^k \end{cases}$$

Hence, $\exists\, x_{k,-}, x_{k,+} \in [0,1]$ such that

$$\begin{cases} x_{k,-} \text{ and } x_{k,+} \in J_{1,k} \text{ if } i=1 \\ x_{k,-} \in J_{i-1,k} \text{ and } x_{k,+} \in J_{i,k} \text{ if } 2 \leq i \leq 2^k-1 \\ x_{k,-} \in J_{2^k-1,k} \text{ and } x_{k,+} = 1 \text{ if } i = 2^k \end{cases}$$

and,

$$\begin{cases} x_{k,-} < x < x_{k,+} \text{ if } x \notin \{0,1\} \\ x_{k,-} = 0 = x < x_{k,+} \text{ if } x = 0 \\ x_{k,-} < x = x_{k,+} = 1 \text{ if } x = 1 \end{cases}$$

In all cases, $\varphi(x_{k,+}) = \frac{i}{2^k} = \frac{i-1}{2^k} + \frac{1}{2^k} = \varphi(x_{k,-}) + \frac{1}{2^k}$

It follows that $\forall y \in [x_{k,-}, x_{k,+}]$, by monotonicity,
$-1/2^k = \varphi(x_{k,-}) - \varphi(x_{k,+}) \leq \varphi(x) - \varphi(y) \leq \varphi(x_{k,+}) - \varphi(x_{k,-}) = 1/2^k$
That is, $|\varphi(x) - \varphi(y)| \leq 1/2^k$

For every $\varepsilon > 0$, let $k \in \mathbb{N}$ large enough so that $1/2^k < \varepsilon$ and $\delta > 0$ small enough so that

$$I_{\delta_\varepsilon} = \begin{cases} [0, \delta_\varepsilon) \subset [x_{k_\varepsilon,-}, x_{k_\varepsilon,+}] & \text{if } i=1 \\ (x-\delta_\varepsilon, x+\delta_\varepsilon) \subset [x_{k_\varepsilon,-}, x_{k_\varepsilon,+}] & \text{if } 2 \le i \le 2^{k_\varepsilon}-1 \\ (1-\delta_\varepsilon, 1] \subset [x_{k_\varepsilon,-}, x_{k_\varepsilon,+}] & \text{if } i = 2^{k_\varepsilon} \end{cases}$$

Then, $\forall y \in I_{\delta_\varepsilon}$ $|\varphi(x) - \varphi(y)| < \varepsilon \Rightarrow \varphi$ continuous at x,

Surjectivity: Since φ is cts, $\varphi(0) = 0$, $\varphi(1) = 1$, the intermediate value theorem gives that $\forall c \in [0,1]$, $\exists x \in [0,1]$ such that $\varphi(x) = c$ i.e. φ surjective.

Proposition

There exists a lebesgue measurable set in $\mathbb{R}^d$ that is not a Borel set.

proof:

Define $\psi : \begin{array}{l} [0,1] \longrightarrow [0,1] \\ x \longmapsto \varphi(x) + x \end{array}$ so that ψ is strictly increasing (since φ is increasing and $x \mapsto x$ is strictly increasing), continuous (since $x \mapsto x$ and φ are cts), and bijective (injective since strictly increasing and surjective by the intermediate value theorem since $\psi(0) = 0$, $\psi(1) = 2$)

Moreover, ψ^{-1} is cts (A2 Q4 → use fact that ψ strictly increasing).

Since C is closed and ψ^{-1} continuous, we have that $\psi(C) = (\psi^{-1})^{-1}(C)$ is closed, hence measurable. Moreover, $m(\psi(C)) = m([0,2]) - m([0,2] \setminus \psi(C))$ where

$$[0,2] \setminus \psi(C) = \psi([0,1] \setminus C) = \psi\Big(\bigcup_{k=1}^{\infty} \bigcup_{i=1}^{2^{k-1}} (J_{i,k})\Big) = \psi\Big(\underbrace{J_{1,1} \cup J_{1,2} \cup J_{3,2} \cup \dots}_{(1/3,2/3) \cup (1/9,2/9) \cup (1/3+1/9,\, 2/3+1/9) \dots}\Big)$$

Since ψ is bijective, and intervals are disjoint, $= \psi(J_{1,1}) \cup \psi(J_{1,2}) \cup \psi(J_{3,2}) \cup \dots$

It follows that

$$
\begin{aligned}
m([0,2]\setminus C) &= m(\Psi(J_{1,1})) + m(\Psi(J_{1,2})) + m(\Psi(J_{3,2})) + \cdots \quad \text{by additivity} \\
&= m(J_{1,1}) + m(J_{1,2}) + m(J_{3,2}) + \cdots \quad \text{since } \psi \text{ has slope 1 on the intervals} \\
&\qquad \text{i.e. they are sent to intervals of the same length.} \\
&= 1/3 + 1/9 + 1/9 + \cdots \\
&= \frac{1}{3} + \left(\frac{1}{9} + \frac{1}{9}\right) + \left(\frac{1}{3^3} + \frac{1}{3^3} + \frac{1}{3^3} + \frac{1}{3^3}\right) + \cdots \\
&= \sum_{k=1}^{\infty} \frac{2^{k-1}}{3^k} = \frac{1}{3} \sum_{k=1}^{\infty} \left(\frac{2}{3}\right)^{k-1} = \frac{1}{3} \sum_{k=0}^{\infty} \left(\frac{2}{3}\right)^k = \frac{1}{3} \frac{1}{1-2/3} = 1
\end{aligned}
$$

Which gives $m(\Psi(C)) = m([0,2]) - m([0,2]\setminus \Psi(C)) = 2 - 1 = 1$

Since $m(\Psi(C)) > 0$, $\exists\, E \subset \Psi(C)$ that is not measurable. Then, define $D = \Psi^{-1}(E)$, since $D \subseteq C$ and $m(C) = 0$, we have $m(D) = 0$ by monotonicity, hence D is measurable.

Assume by contradiction that D is Borel. The inverse image by a continuous function of a Borel set is a Borel set. Thus, $E = (\Psi^{-1})^{-1}(D)$ is a Borel set, by continuity of Ψ^{-1}. But E is not lebesgue measurable: contradiction.

$\square$