Chapter 2 - Measurable Functions

Denote $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$

Proposition and Definition

Let A be a measurable subset of $\mathbb{R}^d$ and $f: A \to \overline{\mathbb{R}}$.

TFAE:

(i) $\forall c \in \mathbb{R}$, $f^{-1}((c, +\infty])$ is measurable

(ii) $\forall c \in \mathbb{R}$, $f^{-1}([c, +\infty])$ is measurable

(iii) $\forall c \in \mathbb{R}$, $f^{-1}([-\infty, c))$ is measurable

(iv) $\forall c \in \mathbb{R}$, $f^{-1}([-\infty, c])$ is measurable

If any of these statements are satisfied, then f is measurable

proof:

(i) $\Rightarrow$ (ii) follows from
$$f^{-1}([c, +\infty]) = f^{-1}\left(\bigcap_{k=1}^{\infty} (c - 1/k, +\infty]\right) = \bigcap_{k=1}^{\infty} f^{-1}((c - 1/k, +\infty])$$
and the fact that countable countable intersections of measurable sets are measurable.

(ii) $\Rightarrow$ (iii) follows from $f^{-1}([-\infty, c)) = f^{-1}(\overline{\mathbb{R}} \setminus [c, \infty]) = A \setminus f^{-1}([c, \infty])$ and the fact that complements between measurable sets are measurable.

(iii) $\Rightarrow$ (iv) similar as (i) $\Rightarrow$ (ii)

(iv) $\Rightarrow$ (i) follows from $f^{-1}((c, \infty]) = A \setminus f^{-1}([-\infty, c])$

$\square$

Proposition

Continuous functions $f: A \to \overline{\mathbb{R}}$ are measurable

proof:

$\forall c \in \mathbb{R}$, $f^{-1}((c,\infty]) = f^{-1}((c,\infty))$ since ∞ is not attained (f continuous)

Since f continuous and (c,∞) open, we obtain that $f^{-1}((c,\infty))$ is open so measurable.

Proposition

Let $A \subseteq \mathbb{R}^d$ be measurable set, $f: A \to \overline{\mathbb{R}}$ measurable. Then, for every Borel set $B \subseteq \mathbb{R}$, $f^{-1}(B)$ is measurable

<u>Remark</u>: There exists a measurable function f and measurable set B such that $f^{-1}(B)$ is not measurable. Indeed, take $f = \varphi^{-1}$, $B = D$ where φ & D are as in the last proof of chapter 1 (last class). Then $f^{-1}(D) = (\varphi^{-1})^{-1}(D) = \varphi(D) = E$ is not measurable. But f is measurable as continuous.

proof of prop:

Define $\Omega = \{B \subseteq \mathbb{R} : f^{-1}(B) \text{ is measurable}\}$. We show that Ω is a σ-algebra containing the open sets. By definition of Borel sets, this suffices to prove that Ω contains the Borel sets.

(i) $$f^{-1}(\mathbb{R}) = f^{-1}\left(\bigcup_{k=1}^{\infty} [-k,k]\right) = \bigcup_{k=1}^{\infty} f^{-1}([-k,k])$$

$$= \bigcup_{k=1}^{\infty} f^{-1}([-\infty,k] \cap [-k,\infty])$$

$$= \bigcup_{k=1}^{\infty} f^{-1}([-\infty,k]) \cap f^{-1}([-k,\infty])$$

Since countable unions and intersections preserve measurability and f is measurable, it follows that $f^{-1}(\mathbb{R})$ is measurable.

(ii) $\forall B_1, B_2 \in \Omega$, $f^{-1}(B_1 \setminus B_2) = f^{-1}(B_1) \setminus f^{-1}(B_2)$ is measurable since $f^{-1}(B_1)$ and $f^{-1}(B_2)$ are measurable.

(iii) $\forall (B_k)_{k\in\mathbb{N}} \subseteq \Omega$, $f^{-1}\left(\bigcup_{k=1}^{\infty} B_k\right) = \bigcup_{k=1}^{\infty} f^{-1}(B_k)$ is measurable since $f^{-1}(B_k)$ are measurable.

(iv) Let $O \subseteq \mathbb{R}$ open. Then, $\exists (I_k)_{k\in\mathbb{N}}$ open, mutually disjoint intervals in $\mathbb{R}$ such that $O = \bigcup_{k=1}^{\infty} I_k$. It follows that

By thm 1.3 textbook

$$f^{-1}(O) = f^{-1}\left(\bigcup_{k=1}^{\infty} I_k\right) = \bigcup_{k=1}^{\infty} f^{-1}(I_k)$$

Let $a_k, b_k \in \overline{\mathbb{R}}$ be s.t. $I_k = (a_k, b_k)$, we are left to show that $f^{-1}((a_k, b_k))$ is measurable.

(1) Case $a_k > -\infty$, $b_k < \infty$:

$$f^{-1}((a_k, b_k)) = f^{-1}\left([a_k, \infty] \cap [-\infty, b_k)\right) = f^{-1}((a_k, \infty]) \cap f^{-1}([-\infty, b_k)) \text{ measurable}$$

(2) Case $a_k = -\infty$:

$$f^{-1}((a_k, b_k)) = f^{-1}([-\infty, b_k)) = f^{-1}\left(\bigcup_{k=1}^{\infty} (-k, b_k)\right) = \bigcup_{k=1}^{\infty} f^{-1}((-k, b_k)) \text{ and use (1)}$$

(3) Case $b_k = \infty$:

$$f^{-1}((a_k, b_k)) = f^{-1}((a_k, +\infty)) = f^{-1}\left(\bigcup_{k=1}^{\infty} (a_k, k)\right) = \bigcup_{k=1}^{\infty} f^{-1}((a_k, k)) \text{ and use (1)}$$

(4) case $a_k = -\infty$, $b = +\infty$:

$$f^{-1}((a_k, b_k)) = f^{-1}((-\infty, \infty)) = f^{-1}(\mathbb{R}) \text{ and use (i)}$$

In all cases, by using preservation of measurability by countable unions and intersections, we obtain that $f^{-1}((a_k, b_k))$ is measurable.

This proves that Ω is a σ-algebra containing all open sets, hence contains also Borel sets.