# Chapter 1 - Measure Theory

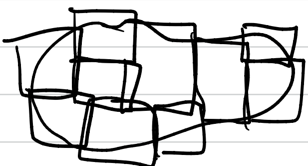
**Def** (i) For every rectangle $(a_1, b_1) \times \cdots \times (a_d, b_d) \subseteq R \subseteq [a_1, b_1] \times \cdots \times [a_d, b_d]$ where $-\infty < a_i \leq b_i < +\infty$, $\forall i$, we call **Volume** of $R$, denoted

$$\mathrm{Vol}(R) = \prod_{i=1}^{d} (b_i - a_i)$$

If $b_i - a_i = b_j - a_j$ $\forall i, j$ then $R$ is a **Cube**

(ii) $\forall A \subseteq \mathbb{R}^d$, we call **exterior measure** of $A$, denoted

$$m_*(A) = \inf \left\{ \sum_{k=1}^{\infty} vol(Q_k) \,\middle|\, Q_k \text{ closed cubes s.t. } A \in \bigcup_{k=1}^{\infty} Q_k \right\}$$



rm 1. $m_*(A) \neq \emptyset$ since $A \subseteq \bigcup_{k=1}^{\infty} [-k, k]^d = \mathbb{R}^d$

rm 2. $m_*(A) \in [0, \infty]$

Throughout the course, $\infty$ will be considered as a number such that:

- $\forall x \in \mathbb{R}$, $x + \infty = \infty$, $x - \infty = -\infty$, $-\infty < x < \infty$
- $\forall x \in \mathbb{R}$, $x \cdot \infty = \infty$, $-x \cdot \infty = -\infty$
- $\infty - \infty$ is not defined
- $0 \times \infty$ is sometimes defined as $0$

**Remark**

in the definition above, closed cubes can be equivalently replaced by open cubes or rectangles (proof in HW 1)

**Prop.**

If $A \subseteq \mathbb{R}^d$ is countable, then $m_*(A) = 0$

proof: since $A$ is countable, $\exists\ (x_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^d$ s.t. $A = \{x_n : n \in \mathbb{N}\}$

Since $\forall\ m \in \mathbb{N}$, $\{x_m\}$ is a cube of volume $0$, and $A = \bigcup_{m=1}^{\infty} \{x_m\}$, we obtain $m_*(A) \le \sum_{m=1}^{\infty} vol(\{x_m\}) = 0$ $\square$

**Prop.**

If $A \subseteq B \subseteq \mathbb{R}^d$, then $m_*(A) \le m_*(B)$

proof:

$$\overset{\tilde{A}}{\overbrace{\left\{ \sum_{k=1}^{\infty} Vol(Q_k) \,\middle|\, Q_k \text{ closed cubes s.t. } A \subseteq \bigcup_{k=1}^{\infty} Q_k \right\}}} \supseteq \overset{\tilde{B}}{\overbrace{\left\{ \sum_{k=1}^{\infty} Vol(Q_k) : Q_k \text{ closed cubes s.t. } B \subseteq \bigcup_{k=1}^{\infty} Q_k \right\}}}$$

Hence, $\inf \tilde{A} \le \inf \tilde{B}$, ie. $m_*(A) \le m_*(B)$ $\square$

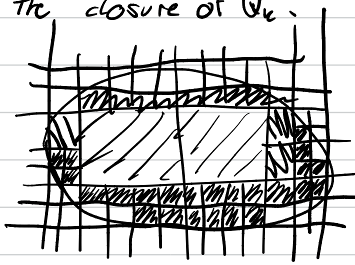
p.7 in textbook

**Prop.**

(i) Every open set $\mathcal{O} \subset \mathbb{R}^d$ can be written as $\mathcal{O} = \bigcup_{k=1}^{\infty} \overline{Q}_k$ where $Q_k$ are mutually disjoint open cubes and $\overline{Q}_k$ is the closure of $Q_k$.

(ii) Furthermore,

$$m_*(\mathcal{O}) = \sum_{k=1}^{\infty} vol\, Q_k$$



proof:

(i) For each $k \in \mathbb{N}$, let $\mathcal{C}_k$ be the set of all cubes with vertices in the grid $2^{-k}\mathbb{Z}^d$

Let $\mathcal{C}_1(\mathcal{O}) = \{Q \in \mathcal{C}_1 : \overline{Q} \subseteq \mathcal{O}\}$ and $\mathcal{O}_1 = \bigcup_{Q \in \mathcal{C}_1} \overline{Q}$

By induction, $\forall\ k \ge 2$, $\mathcal{C}_k(\mathcal{O}) = \{Q \in \mathcal{C}_k : Q \subseteq \mathcal{O} \text{ and } \overline{Q} \not\subseteq \mathcal{O}_{k-1}\}$ and $\mathcal{O}_{k-1} = \bigcup_{Q \in \mathcal{C}_1(\mathcal{O}) \cup \dots \cup \mathcal{C}_{k-1}(\mathcal{O})} \overline{Q}$

With this construction, we obtain that

$$\bigcup_{k=1}^{\infty} \mathcal{O}_k = \bigcup_{k=1}^{\infty}\left(\bigcup_{Q\in \mathcal{C}_k} Q\right) \subseteq \mathcal{O}$$

And that the interior of the cubes in $\bigcup_{k=1}^{\infty} \mathcal{C}_k$ are mutually disjoint.

Conversely, let $x \in \mathcal{O}$, $\forall k \in \mathbb{N}$, let $Q_{k,x} \in \mathcal{C}_k$ s.t. $x \in \overline{Q}_{k,x}$

Since $\mathcal{O}$ is open and $2^{-k} \to 0$ as $k \to \infty$, we have $\overline{Q}_{k,x} \subset \mathcal{O}$ for large $k$.

From our construction, we then obtain $\overline{Q}_{k,x} \subseteq \mathcal{O}_k$, hence

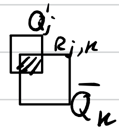
$$x \in \overline{Q}_{k,x} \subseteq \bigcup_{k=1}^{\infty} \mathcal{O}_k$$

(ii)

($\leq$) By definition of $m_*(\mathcal{O})$, since $\mathcal{O} = \bigcup_{k=1}^{\infty} \overline{Q}_k$, we have $m_*(\mathcal{O}) \leq \sum_{k=1}^{\infty} \mathrm{Vol}(Q_k)$

($\geq$) let $(Q_j')_{j\in\mathbb{N}}$ be closed cubes s.t. $\mathcal{O} \subseteq \bigcup_{j=1}^{\infty} Q_j'$

$\forall j, k \in \mathbb{N}$, let $R_{j,k} = Q_j' \cap \overline{Q}_k$, then $R_{j,k}$ is a rectangle




Since $\mathcal{O} \subseteq \bigcup_{j=1}^{\infty} Q_j'$, we obtain that $\forall k \in \mathbb{N}$, $\overline{Q}_k \subseteq \bigcup_{j=1}^{\infty} R_{j,k}$

It follows that $\mathrm{Vol}(\overline{Q}_k) \leq \sum_{j=1}^{\infty} \mathrm{Vol}(R_{j,k})$

will come back on this point

On the other hand, since the cubes $Q_k$ are mutually disjoint (i.e. $\overline{Q}_k$ have mutually disjoint interiors), we obtain that for $j$ fixed, the rectangles $(R_{j,k})_k$ have mutually disjoint interiors.

Furthermore, the rectangles $(R_{j,k})_k$ are included in $Q_j'$.
It follows that
$$\sum_{k=1}^{\infty} Vol(R_{j,k}) \leq \sum_{k=1}^{\infty} Vol(Q_j')$$

Thus, we obtain that
$$\sum_{k=1}^{\infty} Vol(\bar{Q}_k) \leq \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} Vol(R_{j,k})$$
$$= \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} Vol(R_{j,k})$$
$$\leq \sum_{j=1}^{\infty} Vol(Q_j')$$
Taking the inf on the cubes $(Q_j')_{j\in\mathbb{N}}$, we obtain
$$\sum_{k=1}^{\infty} Vol(\bar{Q}_k) \leq m_*(O)$$
Hence $\sum_{k=1}^{\infty} Vol(\bar{Q}_k) = m_*(O)$

In the above proof, we used the following facts:
Let $R$ and $\{R_k\}_{k\in\mathbb{N}}$ be rectangles, then
(iii) If $R \subseteq \bigcup_{k=1}^{\infty} R_k$ then $Vol(R) \leq \sum_{k=1}^{\infty} vol(R_k)$

(iv) If $\bigcup_{k=1}^{\infty} R_k \subseteq R$ and the rectangles $\{R_k\}_k$ are mutually disjoint,
Then $\sum_{k=1}^{\infty} vol(R_k) \leq Vol(R$

proof (iii)

If $Vol(R) = 0$, nothing to prove.
Otherwise, let $R_\varepsilon$ be a compact rectangle such that $R_\varepsilon \subseteq R$ and $Vol(R_\varepsilon) > Vol(R) - \varepsilon$ (by slightly decreasing the sides of $R$)

Now, let $R_{k,\varepsilon}$ be an open rectangle s.t. $R_k \subseteq R_{k,\varepsilon}$ and $Vol(R_{k,\varepsilon}) < Vol(R_k) + \varepsilon \cdot 2^{-k}$ (by increasing slightly the sides of $R_k$) so that

$$\sum_{k=1}^{\infty} Vol(R_{k,\varepsilon}) < \sum_{k=1}^{\infty} vol(R_k) + \varepsilon \cdot \underbrace{\sum_{k=1}^{\infty} 2^{-k}}_{=1}$$

so that the sum of $\varepsilon$ doesn't go to $\infty$

We have

$$R_\varepsilon \subseteq R \subseteq \bigcup_{k=1}^{\infty} R_k \subseteq \bigcup_{k=1}^{\infty} R_{k,\varepsilon}$$

with $R_\varepsilon$ compact and $R_{k,\varepsilon}$ open

Therefore, $\exists (k_i)_{1 \leq i \leq n}$, $n \in \mathbb{N}$ s.t. $R_\varepsilon \subseteq \bigcup_{i=1}^{n} R_{k_i,\varepsilon}$.

By extending the sides of the rectangles $R_\varepsilon$ and $R_{k_i,\varepsilon}$, we obtain

$$\sum_{i=1}^{n} Vol(R_{k_i,\varepsilon}) - Vol(R_\varepsilon) = \sum_{j=1}^{n'} vol(R'_j) \geqslant 0$$

for some rectangles $(R'_j)_{1 \leq j \leq n}$

It follows that

$$Vol(R-\varepsilon) < vol(R_\varepsilon) \leq \sum_{i=1}^{n'} Vol(R_{k_i,\varepsilon}) \leq \sum_{k=1}^{\infty} Vol(R_{k,\varepsilon}) < \sum_{k=1}^{\infty} Vol(R_k) + \varepsilon$$

And, as $\varepsilon \to 0$, we obtain $Vol(R) \leq \sum_{k=1}^{\infty} Vol(R_k)$ $\square$

## Proof (iv)

By extending the sides of the rectangles, we obtain

$$Vol(R) - \sum_{k=1}^{n} Vol(R_k) = \sum_{j=1}^{n'} Vol(R'_j) \geqslant 0$$

for some rectangles $(R'_j)_{1 \leq j \leq m}$

Hence, $\quad Vol(R) \geq \sum_{k=1}^{n} Vol(R_k)$

Parsing to the limit as $n \to \infty$, we obtain

$$Vol(R) \geq \sum_{k=1}^{\infty} Vol(R_k) \qquad \square$$