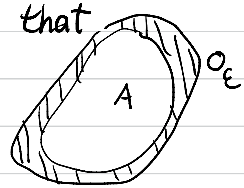
Def We say that a set $A \subseteq \mathbb{R}^d$ is *(Lebesgue)-Measurable* if $\forall \varepsilon > 0$, $\exists$ an open set $O_\varepsilon \subseteq \mathbb{R}^d$ such that $m_*(O_\varepsilon \setminus A) < \varepsilon$, and $A \subseteq O_\varepsilon$.



We then denote $m_*(A) = \underbrace{m(A)}_{\text{Lebesgue measure of } A}$

Prop If $m_*(A) = 0$, then A is measurable

proof:

Since $m_*(A) = \inf\{m_*(O) : O \text{ open}, A \subseteq O\}$, $\forall \varepsilon > 0$, $\exists O_\varepsilon$ open s.t. $A \subseteq O_\varepsilon$ and

$$m_*(O_\varepsilon) \leq m_*(A) + \varepsilon = \varepsilon$$

By monotonicity, since $O_\varepsilon \setminus A \subseteq O_\varepsilon$, it follows that $m_*(O_\varepsilon \setminus A) < \varepsilon$.

Prop A countable union of measurable sets is measurable.

proof:

Let $(A_k)_{k \in \mathbb{N}}$ be measurable sets. Given $\varepsilon > 0$, since A_k is measurable, $\exists O_{k,\varepsilon}$ open s.t. $A_k \subseteq O_{k,\varepsilon}$ and $m_*(O_{k,\varepsilon} \setminus A) < \varepsilon 2^{-k}$

Then, define $O_\varepsilon = \bigcup_{k=1}^\infty O_{k,\varepsilon}$. O_ε is open as union of open sets, $\bigcup_{k=1}^\infty A_k \subseteq O_\varepsilon$ Since $A_k \subseteq O_{k,\varepsilon}$ and

$m_*\left(O_\varepsilon \setminus \left(\bigcup_{k=1}^\infty A_k\right)\right) \leq m_*\left(\bigcup_{k=1}^\infty (O_{k,\varepsilon} \setminus A_k)\right)$ by monotonicity since

$$\left(\bigcup_{k=1}^\infty O_{k,\varepsilon}\right) \setminus \left(\bigcup_{k=1}^\infty A_k\right) \subseteq \bigcup_{k=1}^\infty O_{k,\varepsilon} \setminus A_k$$

And, $m_*\left(\bigcup_{k=1}^{\infty}(O_{k,\varepsilon}\setminus A_k)\right) \leq \sum_{k=1}^{\infty} m_*(O_{k,\varepsilon}\setminus A_k)$ by ctbl subadditivity

$$\leq \varepsilon \sum_{k=1}^{\infty} 2^{-k} = \varepsilon$$

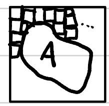
Prop. Closed sets and open sets are Measurable.

proof:

Measurability of open sets follows directly from the definition.

(i) suppose A is closed and bounded (i.e. compact since $A \subseteq \mathbb{R}^d$)

Then, $\exists\, Q$ open cube s.t. $A \subseteq Q$, with $O = Q \setminus A$ open.



Q

Thus, $\exists\, (Q_k)_{k\in\mathbb{N}}$ mutually disjoint open cubes s.t. $O = \bigcup_{k=1}^{\infty} \overline{Q}_k$.

let $O_n = Q \setminus \bigcup_{k=1}^{n} \overline{Q}_k$, then O_n is open since Q open and $\overline{Q}_k$ closed, i.e. we are removing closed sets from an open set, what remains is still open.

Moreover, $m_*(\underbrace{O_n \setminus A}_{= O \setminus \bigcup_{k=1}^{n} \overline{Q}_k \subseteq \bigcup_{k=n+1}^{\infty} \overline{Q}_k}) = m_*\left(\bigcup_{k=n+1}^{\infty} \overline{Q}_k\right)$ by monotonicity

And, $m_*\left(\bigcup_{k=n+1}^{\infty} \overline{Q}_k\right) \leq \sum_{k=n+1}^{\infty} m_*(\overline{Q}_k)$ by sub additivity

$$= \sum_{k=n+1}^{\infty} \mathrm{vol}(\overline{Q}_k) \longrightarrow 0 \text{ as } n \to \infty$$

since $\sum_{k=1}^{\infty} \mathrm{vol}(Q_k) = m_*(O) \leq m_*(Q) < \infty$

$\boxed{(i)}$

(ii) Suppose A unbounded

Then, write $A = \bigcup_{k=1}^{\infty} \underbrace{A \cap [-k,k]^d}_{A_k}$. Since A and $[-k,k]^d$ are closed, and $[-k,k]^d$ is bounded, we obtain that A_k are compact, hence measurable by (i).

Therefore A is measurable as it is the countable union of measurable sets.

Prop. For every measurable set $A \subseteq \mathbb{R}^d$, $\mathbb{R}^d \setminus A$ is measurable.

proof:

For each $k \in \mathbb{N}$, since A is measurable, $\exists\, \mathcal{O}_k$ open s.t. $A \subseteq \mathcal{O}_k$ and $m_*(\mathcal{O}_k \setminus A) < 1/k$.

Let $F_k = \mathbb{R}^d \setminus \mathcal{O}_k$, then F_k is closed. Define $F = \bigcup_{k=1}^{\infty} F_k$.
Since the sets $(F_k)_k$ are closed, so measurable, it follows that F is measurable as ctbl union of measurable sets.

Let $N = (\mathbb{R}^d \setminus A) \setminus F = (\mathbb{R}^d \setminus A) \setminus \left(\bigcup_{k=1}^{\infty} (\mathbb{R}^d \setminus \mathcal{O}_k) \right)$

$$= \bigcap_{k=1}^{\infty} \mathcal{O}_k \setminus A$$

Then $m_*(N) \leq m_*(\mathcal{O}_k \setminus A)$ by monotonicity
$\leq 1/k \to 0$

So $\mathbb{R}^d \setminus A = F \cup N$, N is measurable since $m_*(N) = 0$, F is measurable, hence their union is measurable