Corollary 2 (of Chebyshev's)

Let $f \geq 0$ be measurable on $A \subseteq \mathbb{R}^d$. If $\int_A f < \infty$ (i.e. f is integrable over A), then $f < \infty$ a.e. in A.

Proof:

By Chebyshev's inequality, we have $m(f^{-1}([n,\infty])) \leq \underbrace{\frac{1}{n}}_{\to 0} \underbrace{\int_A f}_{<\infty \text{ by assumption}}$

By continuity of measure, we obtain that $m(f^{-1}(\{\infty\})) = \lim_{n\to\infty} m(f^{-1}([m,\infty])) = 0$ i.e. $f < \infty$ a.e. in A.

$\blacksquare$

Fatou's lemma

Let $(f_n)_n$ be measurable, non negative on $A \subseteq \mathbb{R}^d$, then

$$\int_A \liminf_{n\to\infty} f_n \leq \liminf_{n\to\infty} \int_A f_n$$

Counter example: $f_n = n \chi_{(0,1/n)}$

i.e., in general the inequality is strict.

$\liminf_{n\to\infty} f_n(x) = 0 \quad \forall x \in (0,1)$ but $\int_A 0 = 0 < \lim_{n\to\infty} \int_{(0,1)} f_n = 1$

Proof (Fatou): Let $f = \liminf_{n\to\infty} f_n$, let $h: A \to [0,\infty]$ be measurable, bounded, of finite support and such that $h \leq f$.

Define $h_n = \min(h, \inf_{k \geq n} f_k)$ so that h_n is measurable (since h, f_k measurable), and non negative (since h and f_k are non negative).

Moreover, $\lim_{n\to\infty} h_n(x) = \min(h(x), f(x)) = h(x) \quad \forall x \in A$. Since $0 \leq h_n \leq h$ and h is bounded with finite support, we can apply the bounded convergence theorem, which gives

$$\int_A h = \lim_{n\to\infty} \underbrace{\int_A h_n}_{\leq \int_A f_n} \leq \liminf_{n\to\infty} \int_A f_n$$

By taking sup on h, we obtain $\int f \leq \liminf_{n\to\infty} \int_A f_n$

any h bdd, non negative, finite supp. s.t. $h \leq f$

$\blacksquare$

Monotone Convergence Theorem

Let $(f_n)_{n\in\mathbb{N}}$ measurable, nonnegative functions on $A \subseteq \mathbb{R}^d$. If f_n is increasing (i.e. $f_{n+1} \geq f_n$ in A $\forall n\in\mathbb{N}$), then

$$\int_A \lim_{n\to\infty} f_n = \lim_{n\to\infty} \int_A f_n$$

(limits exist since the sequence is increasing)

Proof:

Fatou's lemma gives that $\int_A \lim_{n\to\infty} f_n \leq \lim_{n\to\infty} \int_A f_n$. Moreover,

$$f_n \leq f_{n+1} \leq \lim_{n\to\infty} f_n$$

hence, $\int_A f_k \leq \int_A \lim_{n\to\infty} f_n$ $\forall k\in\mathbb{N}$, so $\lim_{n\to\infty} \int_A f_n \leq \int_A \lim_{n\to\infty} f_n$ ■

Corollary

Let $(u_k)_{k\in\mathbb{N}}$ be measurable, non negative functions on $A\subseteq\mathbb{R}^d$, then

$$\int_A \sum_{k=1}^{\infty} u_k = \sum_{k=1}^{\infty} \int_A u_k$$

Proof:

follows from the monotone convergence thm, by considering

$$f_n = \sum_{k=1}^{n} u_k \quad \text{(which is increasing)}$$

■

Case of Sign-changing functions

Definition

We say that a measurable function $f: A \to \mathbb{R}$ is Integrable over A if $f_+ = \max(f,0)$ and $f_- = \max(-f,0)$ are integrable. We then call integral of f over A the number $\int_A f = \int_A f(x)\,dx = \int_A f_+ - \int_A f_-$.

For every measurable subset $E \subseteq A$, we say that f is integrable over E if $f\chi_E$ is integrable over A and we denote $\int_E f = \int_A f\chi_E$.

Proposition

f is integrable $\Leftrightarrow$ $|f|$ is integrable

Proof: f is integrable $\Leftrightarrow$ f_+ and f_- are integrable $\Leftrightarrow$ $|f| = f_- + f_+$ is integrable $\blacksquare$

Remark

If $f, g : A \to \overline{\mathbb{R}}$, $f+g$ is not defined on $N := \{x \in A : f(x) = -g(x) = \pm\infty\}$.
However, if f and g are integrable over A, then $|f| < \infty$ and $|g| < \infty$ a.e. in A.
In this case, we still say that $f+g$ is integrable and we denote

$$\int_A (f+g) = \int_{A \setminus N} (f+g)$$

Proposition

If f, g are integrable over A, then

(i) $\forall \alpha, \beta \in \mathbb{R}$, $\alpha f + \beta g$ is integrable and $\int_A (\alpha f + \beta g) = \alpha \int_A f + \beta \int_A g$

(ii) $\forall B_1, B_2$ measurable, disjoint, $\int_{B_1 \cup B_2} f = \int_{B_1} f + \int_{B_2} f$

(iii) if $f = g$ a.e. in A, then $\int_A f = \int_A g$

(iv) if $f \le g$ a.e. in A, then $\int_A f \le \int_A g$

(v) $\left| \int_A f \right| \le \int_A |f|$

Proof: (i)-(iv) follows from properties of nonnegative measurable functions by writing $f = f_+ - f_-$ and $\int_A f = \int_A f_+ - \int_A f_-$
(v) follows from $-|f| \le f \le |f|$ and (iv)

Dominated Convergence Theorem

Let $(f_n)_{n\in\mathbb{N}}$ be measurable functions such that

(i) $\exists f: A \to \overline{\mathbb{R}}$ s.t. $\lim_{n\to\infty} f_n(x) = f(x)$ for a.e. $x \in A$.

(ii) $\exists g: A \to \overline{\mathbb{R}}$ nonnegative integrable over A s.t. $|f_n(x)| \leq g(x)$ for a.e. $x \in A$.

Then, f_n and f are integrable and $\int_A f = \lim_{n\to\infty} \int_A f_n$

Proof:

Since (i) & (ii) are satisfied a.e. in A, $\exists N$ measurable of measure 0 s.t. (i) & (ii) are true at all points of $\tilde{A} := A \setminus N$.

Since $|f_n| \leq g$ in $\tilde{A}$, we obtain that f_n is integrable over A. Since $f = \lim_{n\to\infty} f_n$, we also have $|f| \leq g$, hence f is integrable over A.

Consider $g_n = g \pm f_n$. Then by (i) & (ii), we obtain that g_n is nonnegative and $\lim_{n\to\infty} g_n = g \pm f$ in $\tilde{A}$. By Fatou's lemma, it follows that

$$\int_{\tilde{A}} \underbrace{\liminf_{n\to\infty} g_n}_{= g \pm f} \leq \liminf_{n\to\infty} \underbrace{\int_{\tilde{A}} g_n}_{= \int_{\tilde{A}} g \pm \int_{\tilde{A}} f_n}$$

$$\cancel{\int_{\tilde{A}} g} \pm \int_{\tilde{A}} f \leq \cancel{\int_{\tilde{A}} g} + \liminf_{n\to\infty} \left(\pm \int_{\tilde{A}} f_n \right)$$

which gives

$$\int_{\tilde{A}} f \leq \liminf_{n\to\infty} \int_{\tilde{A}} f_n$$

$$-\int_{\tilde{A}} f \leq \limsup_{n\to\infty} \int_{\tilde{A}} f_n \implies \limsup_{n\to\infty} \int_{\tilde{A}} f_n \leq \int_{\tilde{A}} f$$

Hence $\int_{\tilde{A}} f = \lim \int_{\tilde{A}} f_n$ i.e. $\int_A f = \lim \int_A f_n$ since $A \setminus \tilde{A}$ has measure 0