Prop Every countable intersection of measurable set is measurable.

proof:

Let $(A_k)_{k\in\mathbb{N}}$ be measurable sets. Write

$$\bigcap_{k=1}^{\infty} A_k = \mathbb{R}^d \setminus \left(\bigcup_{k=1}^{\infty} \mathbb{R}^d \setminus A_k\right)$$

The result follows from the previous propositions on coutable unions and complements of measurable sets.

Prop (Countable additivity) Let $(A_k)_{k\in\mathbb{N}}$ be mutually disjoint measurable sets. Then,

$$m\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} m(A_k)$$

In particular, $\forall A, B$ measurable s.t. $A \subseteq B$,

$$m(B) = m(A) + m(B \setminus A)$$

proof:

(i) case A_k bounded. let $\varepsilon > 0$

Since $\mathbb{R}^d \setminus A_k$ is measurable, $\exists\, O_{k,\varepsilon}$ open s.t. $\mathbb{R}^d \setminus A_k \subseteq O_{k,\varepsilon}$ and $m_*\big(\underbrace{O_{k,\varepsilon} \setminus (\mathbb{R}^d \setminus A_k)}_{= O_{k,\varepsilon} \cap A_k}\big) \leq \varepsilon 2^{-k}$

Let $K_{k,\varepsilon} = \mathbb{R}^d \setminus O_{k,\varepsilon}$, then $K_{k,\varepsilon} \subseteq A_k$ and

$$m_*(\underbrace{A_k \setminus K_{k,\varepsilon}}_{= A_k \cap O_{k,\varepsilon}}) \leq \varepsilon 2^{-k}$$

Moreover, $K_{k,\varepsilon}$ is closed as it is the complement of $O_{k,\varepsilon}$, and bounded as $K_{k,\varepsilon} \subseteq A_k$, hence compact!

$$\sum_{k=1}^{\infty} m(A_k) \le \sum_{k=1}^{\infty} m(\underbrace{K_{k,\varepsilon}) + m(A_k \setminus K_{k,\varepsilon}}_{})$$

$K_{k,\varepsilon} \cup (A_k \setminus K_{k,\varepsilon})$ $=$ A_k [by ctbl subadditivity] for each k

$$= \sum_{k=1}^{\infty} m(K_{k,\varepsilon}) + \varepsilon \sum_{k=1}^{\infty} 2^{-k}$$

$$\sum_{k=1}^{\infty} m(A_k) \le \sum_{k=1}^{\infty} m(K_{k,\varepsilon}) + \varepsilon$$

Now, we want to show that $\sum_{k=1}^{\infty} m(K_{k,\varepsilon}) \le m\left(\bigcup_{k=1}^{\infty} K_{k,\varepsilon}\right)$

First, a lemma

Lemma

let $K_1, K_2 \subseteq \mathbb{R}^d$ be disjoint, compact. Then $d(K_1, K_2) > 0$

proof: Assume not, t.w. contradiction

Then, $d(K_1, K_2) = \inf\{|x-y| : x \in K_1, y \in K_2\} = 0$

ie. $\exists\, (x_n) \in K_1,\ (y_n) \in K_2\ \forall n \in \mathbb{N}$ s.t.

$$|x_n - y_n| \to 0 \text{ as } n \to \infty$$

By compactness of K_1, K_2, $\exists\, (x_{n_k})_{n\in\mathbb{N}}, (y_{n_k})_{n \in \mathbb{N}}$ subsequences of x_n & y_n s.t. x_{n_k} converges to $x \in K_1$ and y_{n_k} converges to $y \in K_2$

Then, by continuity of $(x,y) \mapsto |x-y|$, it follows that $\lim_{k\to\infty} |x_{n_k} - y_{n_k}| = |x-y| = 0$

ie. $y = x$, contradiction since $K_1 \cap K_2 = \emptyset$ $\blacksquare$

Observe that since $K_{k,\varepsilon} \subseteq A_k$ and the sets $(A_k)_{k\in\mathbb{N}}$ are mutually disjoint, we have that the sets $(K_{k,\varepsilon})_k$ are also mutually disjoint.
Hence, by lemma above, every pair of sets are at positive distance.

For each $n \in \mathbb{N}$, it follows that $m\left(\bigcup_{k=1}^{n} K_{k,\varepsilon}\right) = m\left(K_{1,\varepsilon} \cup \bigcup_{k=2}^{n} K_{k,\varepsilon}\right)$

$$= m(K_{1,\varepsilon}) + m\left(\bigcup_{k=2}^{n} K_{k,\varepsilon}\right)$$

induction:
$$= \sum_{k=1}^{n} m(K_{k,\varepsilon})$$

By monotonicity, we have $m\left(\bigcup_{k=1}^{n} K_{k,\varepsilon}\right) \leq m\left(\bigcup_{k=1}^{\infty} K_{k,\varepsilon}\right)$

It follows that $\sum_{k=1}^{\infty} m(K_{k,\varepsilon}) = \lim_{n\to\infty} \sum_{k=1}^{n} m(K_{k,\varepsilon}) \leq m\left(\bigcup_{k=1}^{\infty} K_{k,\varepsilon}\right)$

Finally, we obtain

$$\sum_{k=1}^{\infty} m(A_k) \leq m\left(\bigcup_{k=1}^{\infty} K_{k,\varepsilon}\right) + \varepsilon$$

$$\leq m\left(\bigcup_{k=1}^{\infty} A_k\right) + \varepsilon$$ [by monotonicity since $\bigcup K_{k,\varepsilon} \subseteq \bigcup A_k$]

As $\varepsilon \to 0$, we obtain $\sum_{k=1}^{\infty} m(A_k) \leq m\left(\bigcup_{k=1}^{\infty} A_k\right)$

Conversely, by subadditivity, we have $m\left(\bigcup_{k=1}^{\infty} A_k\right) \leq \sum_{k=1}^{\infty} m(A_k)$

(ii) Case A_k unbounded:

Write $A_k = \bigcup_{j=1}^{\infty} A_{k,j}$ where $A_{k,1} = A_k \cap B(0,1)$
and $A_{k,j} = A_k \cap B(0,j) \setminus B(0,j-1)$ $\forall j \geq 2$

Then the set $(A_{k,j})_{k,j}$ are measurable since $(B(0,j))_j$ are open and complements and finite intersection preserve measurability.
Moreover, the sets $(A_{k,j})_{k,j}$ are bounded and mutually disjoint since the sets $B(0,1)$ and $(B(0,j) \setminus B(0,j-1))_{j\geq 2}$ are bdd & mutually dis.

It follows that $m\Big(\underbrace{\bigcup_{k=1}^{\infty}\bigcup_{j=1}^{\infty} A_{k,j}}_{=\bigcup_{k=1}^{\infty} A_k}\Big) = \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} m(A_{k,j})$

And, $m(\underbrace{A_k}_{=\bigcup_{j=1}^{\infty} A_{kj}}) = \sum_{k=1}^{\infty} m(A_{kj})$

Hence, $m\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} m(A_k)$

$\blacksquare$

Proposition. (Continuity of measure)
Let $(A_k)_{k\in\mathbb{N}}$ be measurable subsets of $\mathbb{R}^d$, then

(i) if $A_k \subseteq A_{k+1}$ $\forall k\in\mathbb{N}$, then $m\left(\bigcup_{k=1}^{\infty} A_k\right) = \lim_{k\to\infty} m(A_k)$

(ii) if $m(A_1) < \infty$, then if $A_{k+1} \subseteq A_k$ $\forall k\in\mathbb{N}$, then
$m\left(\bigcap_{k=1}^{\infty} A_k\right) = \lim_{k\to\infty} m(A_k)$

Counter example if $m(A_1) = \infty$
$A_k = [k,\infty)$, $\bigcap_{k=1}^{\infty} A_k = \emptyset$, $m(A_k) = \infty$, $m\left(\bigcap_{k=1}^{\infty} A_k\right) = 0$

proof:
(i) Write $\bigcup_{k=1}^{\infty} A_k = \bigcup_{k=1}^{\infty} \tilde{A}_k$ where $\tilde{A}_1 = A_1$ and $\tilde{A}_k = A_k \setminus A_{k-1}$ $\forall k \geq 2$

Then, the sets $(\tilde{A}_k)_k$ are disjoint.

- If $m(A_{k_0}) = \infty$ for some $k_0\in\mathbb{N}$ then since $A_{k_0} \subseteq A_k$ $\forall k \geq k_0$, by monotonicity, we obtain $m(A_k) = \infty$ $\forall k \geq k_0$, i.e.
$\lim_{k\to\infty} m(A_k) = 0$ and $m\left(\bigcup_{k=1}^{\infty} A_k\right) = \infty$
so nothing to prove.

- Otherwise, $m(A_k) < \infty \ \forall k \in \mathbb{N}$, then we write

solution from prof

$$\underbrace{m\left(\bigcup_{k=1}^{\infty} A_k\right)}_{=\bigcup_{k=1}^{\infty} \tilde{A}_k} = \sum_{k=1}^{\infty} \underbrace{m(\tilde{A}_k)}_{=m(A_k)-m(A_{k-1})} \quad \text{by countable add.}$$

well defined since the sets have finite measure by assumption

$$= \lim_{n\to\infty} \sum_{k=2}^{n} \left(m(A_k) - m(A_{k-1})\right) + m(A_1)$$

$$= \lim_{n\to\infty} m(A_n)$$

soln from book

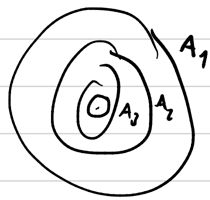
$$m\left(\bigcup_{k=1}^{\infty} A_k\right) = m\left(\bigcup_{k=1}^{\infty} \tilde{A}_k\right)$$

$$= \sum_{k=1}^{\infty} m(\tilde{A}_k) \quad \text{by ctbl additivity}$$

$$= \lim_{N\to\infty} \sum_{k=1}^{N} m(\tilde{A}_k)$$

$$= \lim_{N\to\infty} m\left(\bigcup_{k=1}^{N} \tilde{A}_k\right) = \lim_{N\to\infty} m(A_k)$$

(ii) Define $\tilde{A}_k = A_1 \setminus A_k$. Then $\tilde{A}_k$ is measurable and $A_1 \setminus \bigcap_{k=1}^{\infty} A_k = \bigcup_{k=1}^{\infty} A_1 \setminus A_k = \bigcup_{k=1}^{\infty} \tilde{A}_k$ and $\tilde{A}_k \subseteq \tilde{A}_{k-1}$ (since $A_k \supseteq A_{k+1}$) $\forall k \in \mathbb{N}$.



By (i), we obtain $m\left(A_1 \setminus \bigcap_{k=1}^{\infty} A_k\right) = \lim_{k\to\infty} m(\tilde{A}_k)$

$$\overset{(1)}{=} \lim_{k\to\infty} \left(m(A_1) - m(A_k)\right)$$

(1) provided $m(A_k) < \infty$, which is true by monotonicity, $m(A_k) \le m(A_1) < \infty$ since $A_k \subseteq A_1$

$$= m(A_1) - m\left(\bigcap_{k=1}^{\infty} A_k\right)$$

provided $m\left(\bigcap_{k=1}^{\infty} A_k\right) < \infty$ which is true for the same reason as (1)

Hence, $\cancel{m(A_1)} - m\left(\bigcap_{k=1}^{\infty} A_k\right) = \cancel{m(A_1)} - \lim_{k\to\infty} m(A_k)$

$$\Rightarrow m\left(\bigcap_{k=1}^{\infty} A_k\right) = \lim_{k\to\infty} m(A_k)$$

$\square$

Def

(i) We call G_δ set a countable intersection of open set

(ii) We call F_σ set a countable union of closed sets

Remark The complement of G_δ set is an F_σ set and vice-versa

Proposition TFAE:

(i) A is measurable

(ii) There exists a G_δ set G and a set $N \subseteq G$ s.t. $m_*(N) = 0$ and $A = G \setminus N$

(iii) $\forall \varepsilon > 0$, $\exists$ closed set $F_\varepsilon \subseteq A$ s.t. $m_*(A \setminus F_\varepsilon) < \varepsilon$

(iv) $\exists$ F_σ set F and a set $N \subseteq A \setminus F$ s.t. $m_*(N) = 0$ and $A = F \cup N$

proof:

(i)$\Rightarrow$(ii) Since A is measurable, $\forall k \in \mathbb{N}$, $\exists$ O_k open s.t. $O_k \supseteq A$ and $m(O_k \setminus A) < 1/k$.
Let $G = \bigcap_{k=1}^{\infty} O_k$, then G is a G_δ set. $G \supseteq A$ since $O_k \supseteq A$ $\forall k \in \mathbb{N}$. Let $N = G \setminus A$ so that

$$m(G \setminus A) = m(\underbrace{N}_{= \cap O_k \setminus A}) \leq m(O_k \setminus A) \quad \forall k \text{ by monotonicity since } G \setminus A \subseteq O_k \setminus A$$
$$< \tfrac{1}{k} \to 0 \text{ as } k \to \infty \Rightarrow m(N) = 0$$

(ii)$\Rightarrow$(i) G is measurable as countable intersection of open sets by def. of G_δ sets. N is measurable since $m_*(N) = 0$.

Hence $G \setminus N = G \cap \underbrace{N^c}_{\text{measurable as complement of measurable set}}$ is measurable

(i) $\Leftrightarrow$ (iii)

Observe that (i) $\Longleftrightarrow$ $\mathbb{R}^d \setminus A$ is measurable

$\Longleftrightarrow$ $\forall\, \varepsilon > 0$, $\exists\, O_\varepsilon$ open s.t. $\mathbb{R}^d \setminus A \subseteq O_\varepsilon$ and $m_*(O_\varepsilon \setminus (\mathbb{R}^d \setminus A)) < \varepsilon \Rightarrow m_*(O_\varepsilon \cap A) < \varepsilon$

$\Longleftrightarrow$ $\forall\, \varepsilon > 0$, $\exists\, F_\varepsilon = \mathbb{R}^d \setminus O_\varepsilon$ closed s.t. $\mathbb{R}^d \setminus A \subseteq \mathbb{R}^d \setminus F_\varepsilon$ i.e. $F_\varepsilon \subseteq A$, and $m_*(A \setminus F_\varepsilon) < \varepsilon$

$\Longleftrightarrow$ (iii)

Similarly, (ii) $\Longleftrightarrow$ (ii) for $\mathbb{R}^d \setminus A$ $\Longleftrightarrow$ $\exists\, F_\sigma$ set F and a set $N \subseteq \mathbb{R}^d \setminus F = G$ such that $\mathbb{R}^d \setminus A = G \setminus N = (\mathbb{R}^d \setminus F) \setminus N$ i.e. $A = F \cup N$

$\Longleftrightarrow$ (iv) $\square$