Recall Riemann Integral

Def Let $[a,b]$ be closed, bdd interval in $\mathbb{R}$, and $f:[a,b]\to\mathbb{R}$ be a bdd function.
We say that f is Riemann integrable if

$$\underline{\int_a^b} f = \overline{\int_a^b} f$$

where
$$\underline{\int_a^b} f = \sup\left\{ \sum_{i=1}^{n} \inf_{[x_{i-1},x_i]} (x_i - x_{i-1}) \;\middle|\; a = x_0 < x_1 < \cdots < x_n = b \right\}$$

$$\overline{\int_a^b} f = \inf\left\{ \sum_{i=1}^{n} \sup_{[x_{i-1},x_i]} (x_i - x_{i-1}) \;\middle|\; a = x_0 < x_1 < \cdots < x_n = b \right\}$$

Prop $\underline{\int_a^b} f \leq \overline{\int_a^b} f$

pf: suffices to show that $\forall\, x_i, x_i'$ st.
$x_0 = a < x_1 < \cdots < x_n = b$, $x_0' = a < x_1' < \cdots < x_n' = b$

$$\sum_{i=1}^{n} \inf_{[x_{i-1},x_i]} f(x_i - x_{i-1}) \leq \sum_{i=1}^{m} \sup_{[x_{i-1}',x_i']} f(x_i' - x_{i-1}')$$

Consider $(x_i'')_{1\le i\le n''}$ st. $a = x_1'' < x_2'' < \cdots < x_n'' = b$ and $\{x_i''\} = \{x_i\}\cup\{x_i'\}$

and observe that
- $\inf_{A\cup B} f \leq \inf_A f$ and $\inf_{A\cup B} f \leq \inf_B f$
- $\sup_{A\cup B} f \geq \sup_A f$ and $\sup_{A\cup B} f \geq \sup_B f$

Which gives
$$\sum_{i=1}^{n} \inf_{[x_{i-1},x_i]}[f]\,(x_i - x_{i-1}) \leq \sum_{i=1}^{n} \inf_{[x_{i-1}'',x_i'']}[f]\,(x_i'' - x_{i-1}'') \leq \sum_{i=1}^{n} \sup_{[x_{i-1}'',x_i'']}[f]\,(x_{i-1}'' - x_i'')$$
$$\leq \sum_{i=1}^{n} \sup_{[x_{i-1}',x_i']}[f]\,(x_{i-1}' - x_i')$$

Prop Every cts function $f : [a,b) \to \mathbb{R}$ is Riemann integrable.

proof: For each $n \in \mathbb{N}$, consider $x_{i,n} = a + \frac{b-a}{n}$ and observe that

$$\sum_{i=1}^{n} \inf_{[x_{i-1,n},\, x_{i,n}]} [f] \underbrace{(x_{i,n} - x_{i-1,n})}_{\frac{b-a}{n}} = \frac{b-a}{n} \sum_{i=1}^{n} \inf_{[x_{i-1,n},\, x_{i,n}]} [f]$$

similarly,

$$\sum_{i=1}^{n} \sup_{[x_{i-1,n},\, x_{i,n}]} [f] (x_{i,n} - x_{i-1,n}) = \frac{b-a}{n} \sum_{i=1}^{n} \sup_{[x_{i-1,n},\, x_{i,n}]} [f]$$

And, since f is cts, $\exists\, y_{i,n}$ & $z_{i,n} \in [x_{i-1,n}, x_{i,n}]$ such that $f(y_{i,n}) = \min\limits_{[x_{i-1,n},\, x_{i,n}]} f$ and $f(z_{i,n}) = \max\limits_{[x_{i-1,n},\, x_{i,n}]} f$

Hence

$$\sum_{i=1}^{n} \sup_{[x_{i-1,n},\, x_{i,n}]} [f] (x_{i,n} - x_{i-1,n}) - \sum_{i=1} \inf_{[x_{i-1,n},\, x_{i,n}]} [f] (x_{i,n} - x_{i-1,n})$$

$$= \frac{b-a}{n} \sum_{i=1}^{n} \big(f(z_{i,n}) - f(y_{i,n}) \big)$$

Moreover, $|z_{i,n} - y_{i,n}| \le \frac{b-a}{n}$ since $y_{i,n}, z_{i,n} \in [x_{i-1,n}, x_i]$. Hence by uniform continuity of f in $[a,b]$, $\forall\, \varepsilon > 0$, if n is large enough, then $f(z_{i,n}) - f(y_{i,n}) < \varepsilon$.

On the other hand, $\overline{\int_a^b} f - \underline{\int_a^b} f \le \sum\limits_{i=1}^{n} \big(\sup\limits_{[x_{i-1}, x_{i,n}]} f\big)(x_{i,n} - x_{i-1,n})$
$\qquad - \sum\limits_{i=1}^{n} \big(\inf\limits_{[x_{i-1}, x_{i,n}]} f\big)(x_{i,n} - x_{i-1,n})$

Thus $\overline{\int_a^b} f \le \underline{\int_a^b} f$ hence $\overline{\int_a^b} = \underline{\int_a^b} \le \frac{b-a}{\cancel{n}} \times \cancel{n}\, \varepsilon \to 0$ as $\varepsilon \to 0$ ∎

Example of Non-cts function

let $f:[0,1] \to \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{o.w.} \end{cases}$

Since $\mathbb{Q}$ is countable, $\exists \{x_n\}_{n\in\mathbb{N}}$ in $[0,1]$ s.t. $\mathbb{Q} = \{x_n : n \in \mathbb{N}\}$

Let $f_n = [0,1] \to \begin{cases} 1 & \text{if } x \in \{x_1, \dots x_n\} \\ 0 & \text{o.w.} \end{cases}$

Then $0 \leq f_n \leq f_{n+1} \leq 1$ and $\lim_{n\to\infty} f_n(x) = f(x) \ \forall x \in [0,1]$

And f_n is R ⓘ with $\int_0^1 f_n = 0$

(exercise: Prove by choosing $(x_i)_i$ close to $(q_j)_{1\leq j\leq n}$)

BUT, f is not R ⓘ, indeed for every $[x_{i-1}, x_i]$, $\sup_{[x_{i-1},x_i]} f = 1$ by density of $\mathbb{Q}$ and $\inf_{[x_{i-1},x_i]} f = 0$ by density of $\mathbb{R}\setminus\mathbb{Q}$

Hence $\underline{\int_a^b} f = 0$ and $\overline{\int_a^b} f = 1 \Rightarrow$ Not R ⓘ

Problem !!!

Aims of Lebesgue's Integral (1904)

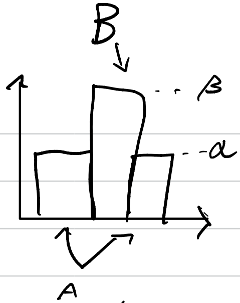
→ extend Riemann integral to more general function (like the one above)

→ Provide limit theorems for sequences of integrable functions that is: $\int \lim_{n\to\infty} f_n = \lim_{n\to\infty} \int f_n$

Idea: Calculate the integral by partitioning the Range instead of the domain of the functions (in Riemann)

Ex) Let $A, B \subseteq \mathbb{R}^d$, $d \geqslant 1$ s.t. $A \cap B = \emptyset$

consider $f : A \cup B \to \begin{cases} \alpha & \text{if } x \in A \\ \beta & \text{if } x \in B \end{cases}$ $\alpha, \beta \in \mathbb{R}$



We want the integral of f to be the sum of $\alpha\, m(A)$ and $\beta\, m(B)$ where $m(A)$ & $m(B)$ will be "measures" of A and B (to be defined rigorously)