Case of Measurable, Bounded, with Finite Support Functions

Proposition - Definition

Let $f: A \to \mathbb{R}$ be measurable, bdd, and with finite support. Then, $\exists \ell \in \mathbb{R}$ s.t. $\lim_{k\to\infty} \int_A \varphi_k = \ell$ for every sequence of simple functions $(\varphi_k)_{k\in\mathbb{N}}$ such that

(i) $\exists E \subset A$ of finite measure s.t. $\operatorname{supp} \varphi_k \subseteq E \ \forall k$
(ii) $\exists M > 0$ s.t. $|\varphi_k| \leq M$ in $A \ \forall k$
(iii) $\varphi_k(x) \to f(x)$ for a.e. $x \in A$

If $f = 0$ a.e. in A, then $\ell = 0$

proof:

By Egorov's Thm and (i) and (iii), $\forall \varepsilon > 0$, $\exists F_\varepsilon \subseteq E$ closed such that $m(E \setminus F_\varepsilon) < \varepsilon$ and $(\varphi_k)_k$ converges uniformly to f in F_ε. Since $(\varphi_k)_k$ converges unif. to f in F_ε, $\exists k_\varepsilon \in \mathbb{N}$ s.t. $\forall k \geq k_\varepsilon$, $|\varphi_k - f| < \varepsilon$ in F_ε. Then, $\forall k, k' \geq k_\varepsilon$,

$$\left|\int_A \varphi_k - \int_A \varphi_{k'}\right| = \int_A |\varphi_k - \varphi_{k'}| = \underbrace{\int_{F_\varepsilon} |\varphi_k - \varphi_{k'}|}_{\substack{\leq |\varphi_k - f| + |\varphi_{k'} - f| \\ < 2\varepsilon/4m(E)}} + \underbrace{\int_{E\setminus F_\varepsilon} |\varphi_k - \varphi_{k'}|}_{\substack{\leq |\varphi_k| + |\varphi_{k'}| \\ \leq 2M \\ \text{by (ii)}}} + \underbrace{\int_{A\setminus E} |\varphi_k - \varphi_{k'}|}_{\substack{= 0 \text{ since } \varphi_k = \varphi_{k'} = 0 \\ \text{in } A\setminus E \\ \text{by (i)}}}$$

shady $\Big\{$

$$\leq \varepsilon\, m(F_\varepsilon) + 2M\, m(E \setminus F_\varepsilon) \leq 2\varepsilon\, m(E) + 2M(\varepsilon/4M)$$

$$< \varepsilon/2 + \varepsilon/2 = \varepsilon$$

Therefore, $\left(\int_A \varphi_k\right)_k$ is cauchy thus converging in $\mathbb{R}$.

case $f = 0$ a.e. in A: let k_ε and F_ε as above

$$\forall k \geq k_\varepsilon, \quad \left|\int_A \varphi_k\right| \leq \int_A |\varphi_k| = \underbrace{\int_{F_\varepsilon} |\varphi_k|}_{\substack{\leq \frac{\varepsilon}{4(m(E)+1)} m(F_\varepsilon) \\ \leq \varepsilon/4}} + \underbrace{\int_{E\setminus F_\varepsilon} |\varphi_k|}_{\substack{\leq M\, m(E\setminus F_\varepsilon) \\ \leq \varepsilon/4}} + \underbrace{\int_{A\setminus E} |\varphi_k|}_{=0} \leq \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}$$

Uniqueness of the limit. Let $(\varphi_k)_k$ and $(\varphi'_k)_k$ be the sequences of simple functions satisfying (i)-(ii)-(iii). Then $(\varphi_k - \varphi'_k)_k$ is a sequence of simple functions satisfying (i)-(ii)-(iii) with $f=0$. From above, it follows that $\int_A (\varphi_k - \varphi'_k) \to 0$ i.e. $\lim_{k\to\infty} \int_A \varphi_k = \lim_{k\to\infty} \int_A \varphi'_k$. ■

Proposition

Let $f,g: A \to \mathbb{R}$ be measurable, bounded with finite support.

(i) $\forall \alpha, \beta \in \mathbb{R}$, $\alpha f + \beta g$ is measurable, bdd with finite support and

$$\int_A (\alpha f + \beta g) = \alpha \int_A f + \beta \int_A g$$

(ii) $\forall E_1, E_2 \subseteq A$ measurable and disjoint, $\int_{E_1 \cup E_2} f = \int_{E_1} f + \int_{E_2} f$

(iii) if $f \leq g$ in A, then $\int_A f \leq \int_A g$

(iv) $|f|$ is measurable, bounded with finite support and $\left| \int_A f \right| \leq \int_A |f|$

proof.

(i) & (ii) follow from the analog results for simple functions along with the definition of $\int f$, $\int g$ and $\int (\alpha f + \beta g)$.

(iii) $f \leq g$ in $A \Rightarrow g - f \geq 0$ in A. Let $(\varphi_k)_{k\in\mathbb{N}}$ be a sequence of simple functions satisfying (i)-(ii)-(iii) for $g-f$ of the previous definition (by simple approx. thm), which can be chosen such that $\varphi_k \geq 0$ on A (since $f-g \geq 0$ on A). It follows that $\int_A \varphi_k \geq 0$, which gives

$$\int_A \varphi_k \longrightarrow \int_A (g-f) \geq 0 \quad \text{i.e.} \quad \int_A g \geq \int_A f$$

(iv) follows from (iii) and $-|f| \leq f \leq |f|$

■

Theorem (Bounded convergence)

Let $(f_k)_{k\in\mathbb{N}}$ $f_k: A \to \mathbb{R}$ be measurable such that

(i) $\exists E \subseteq A$ measurable of finite measure such that $\operatorname{supp} f_k \subseteq E$ $\forall k \in \mathbb{N}$

(ii) $\exists M \geq 0$ s.t. $|f_k| \leq M$ a.e. in E $\forall k \in \mathbb{N}$

(iii) $\exists f: A \to \mathbb{R}$ s.t. $\lim\limits_{k\to\infty} f_k(x) = f(x)$ for a.e. $x \in E$

Then, f is measurable, bounded (by M), with finite support, and

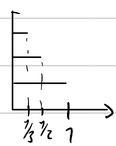
$$\int_A f = \lim_{k\to\infty} \int_A f_k$$

Ex) $f_k(x) = (\cos(x))^k$ in $A = [0,\pi]$ tends (pointwise) to zero if $x \in (0,\pi)$, or to 1 if $x = 0$, -1 if $x = \pi$.

Hence it converges to 0 almost everywhere, whence f_n satisfies (i)-(ii)-(iii) above, so $\lim\limits_{k\to\infty} \int_{[0,\pi]} f_k = \int_{[0,\pi]} 0 = 0$

Counter example

$f_k(x) = k\chi_{(0,1/k)}(x)$ in $[0,1]$



$\longrightarrow 0$ pointwise, But (ii) is not satisfied.

$$\int_{[0,1]} f_k = k\, m((0,1/k)) = k \times \frac{1}{k} = 1 \not\to \int_A 0 = 0$$

proof of thm:

Let F_ε & k_ε be as in the proof of the first prop-def above.
Wlog, we may assume (ii) & (iii) are satisfied Everywhere in A (by subtracting a set of measure zero).

$$\forall k \geq k_\varepsilon,\ \left|\int_A f_n - \int_A f\right| \leq \int_A |f_k - f| = \int_{F_\varepsilon} |f_k - f| + \int_{E\setminus F_\varepsilon} |f_k - f| + \int_{A\setminus E} |f_k - f|$$

$$\leq \frac{\varepsilon}{4(m(E)+1)} m(F_\varepsilon) + 2M\, m(E\setminus F_\varepsilon) + 0$$

$$\leq \frac{\varepsilon}{4} + \frac{2M\varepsilon}{4M} = \frac{\varepsilon}{2}$$

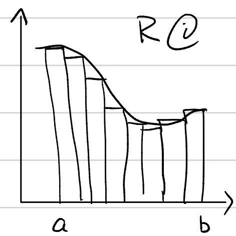
Proposition

If $A=[a,b]$, $a<b \in \mathbb{R}$, then every bounded function $f:[a,b]\to\mathbb{R}$ that is Riemann integrable is also measurable and its Lebesgue integral $\int_A^{(L)} f$ is equal to its Rieman integral $\int_A^{(R)} f$.

proof:

Since we have that f is Riemann integrable, $\exists (\varphi_k)_{k\in\mathbb{N}}, (\psi_k)_{k\in\mathbb{N}}$ step functions such that $\varphi_k \le f \le \psi_k$ and $\lim \int_A \varphi_k = \lim \int_A \psi_k = \int_A^{(R)} f$

Define $\tilde{\varphi}_k = \max\{\varphi_1, \dots, \varphi_k\}$ and $\tilde{\psi}_k = \min\{\psi_1, \dots, \psi_k\}$ so that $\tilde{\varphi}_k \le f \le \tilde{\psi}_k$, $\tilde{\varphi}_k$ is increasing, $\tilde{\psi}_k$ is decreasing.



$$0 \le \int_A^{(R)} (\tilde{\psi}_k - f) = \int_A \tilde{\psi} - \int_A^{(R)} f \le \int_A \psi_k - \int_A^{(R)} f \to 0$$

i.e. $\lim_{k\to\infty} \int_A \tilde{\psi}_k = \int_A^{(R)} f$ and similarly, $0 \le \int_A^{(R)} (f - \varphi_k) \to 0$,

i.e. $\lim_{k\to\infty} \int_A \tilde{\varphi}_k = \int_A^{(R)} f$. Since $(\tilde{\varphi}_n)$ and $(\tilde{\psi}_n)$ are bounded (by ψ_1) and monotone, there exists $\tilde{\varphi}(x) = \lim_{n\to\infty} \tilde{\varphi}_n(x)$ and $\tilde{\psi}(x) = \lim_{n\to\infty} \tilde{\psi}_n(x)$ $\forall x \in A$.

Moreover, by passing to the limit in $\tilde{\varphi}_n \le f \le \tilde{\psi}_n$, we obtain $\tilde{\varphi} \le f \le \tilde{\psi}$.

$\forall B \subseteq A$ measurable, $\int_B (\tilde{\psi} - \tilde{\varphi}) \le \int_B (\tilde{\psi}_n - \tilde{\varphi}_n) \le \int_A (\tilde{\psi}_n - \tilde{\varphi}_n) \to 0$

i.e. $\int_B (\tilde{\psi} - \tilde{\varphi}) = 0$ which gives that $\tilde{\psi} - \tilde{\varphi} = 0$ a.e. in B (A4 Q2)

i.e. $\tilde{\psi} = \tilde{\varphi} = f$ a.e in A, and in particular, f is measurable as a.e. pointwise limit of measurable function.

By the Bounded Convergence Theorem, we also obtain that

$$\int_A^{(L)} f = \lim_{n\to\infty} \int_A \tilde{\varphi}_n = \lim_{n\to\infty} \int_A \tilde{\psi}_n = \int_A^{(R)} f$$

$\blacksquare$

Case of Non-negative Measurable Functions

Definition

Let $f: A \to [0,\infty]$ be measurable. Then, we define the integral of f over A as

$$\int_A f = \int_A f(x)\,dx = \sup\left\{ \int_A h \;\middle|\; \begin{array}{l} h: A \to [0,\infty] \text{ measurable, bdd, with} \\ \text{finite supp. such that } h \leq f \text{ in } A \end{array} \right\}$$

For every $B \subseteq A$ measurable, we define $\int_B f = \int_A f \chi_B$ (with the convention that $0 \cdot \infty := 0$).
We say that f is integrable over B if $\int_B f < \infty$.

Proposition

Let $f, g: A \to [0,\infty]$ be measurable. Then,

(i) $\forall \alpha, \beta \geq 0$, $\alpha f + \beta g$ is non-negative, measurable, and $\int_A (\alpha f + \beta g) = \alpha \int_A f + \beta \int_A g$

(ii) $\forall B_1, B_2 \subseteq A$ disjoint, measurable, $\int_{B_1 \cup B_2} f = \int_{B_1} f + \int_{B_2} f$

(iii) if $f = g$ a.e. in A, then $\int_A f = \int_B g$

(iv) if $f \leq g$ a.e. in A, then $\int_A f \leq \int_A g$

proof:

(i) $\forall \alpha > 0$,

$$\int_A \alpha f = \sup\left\{ \underbrace{\int_A h}_{\alpha \int_A \tilde{h} \text{ where } \tilde{h} = h/\alpha} \;\middle|\; \begin{array}{l} h: A \to [0,\infty] \text{ measurable, bounded, with} \\ \text{finite supp. such that } h \leq \alpha f \text{, i.e. } \tilde{h} \leq f \end{array} \right\}$$

$$= \sup\left\{ \alpha \int_A \tilde{h} \;\middle|\; \begin{array}{l} \tilde{h}: A \to [0,\infty] \text{ measurable, bounded, with finite} \\ \text{support, such that } \tilde{h} \leq f \end{array} \right\}$$

$$= \alpha \int_A f$$

$$\int_A (f+g) = \sup \overbrace{\left\{ \int_A h \;\middle|\; \begin{array}{l} h: A \to [0,\infty] \text{ measurable, bdd, with finite support} \\ \text{such that } h \le f+g \end{array} \right\}}^{C :=}$$

$$\int_A f + \int_A g = \sup \overbrace{\left\{ \underbrace{\int_A h_1 + \int_A h_2}_{= \int_A (h_1 + h_2)} \;\middle|\; \begin{array}{l} h_1, h_2 : A \to [0,\infty] \text{ measurable, bdd, with} \\ \text{finite support such that } h_1 \le f,\ h_2 \le g \end{array} \right\}}^{D :=}$$

<u>$D \subseteq C$</u>. If $\int_A (h_1 + h_2) \in D$, then $\int_A (h_1 + h_2) \in C$, i.e. $h_1 + h_2 \le f+g$ when $h_1 \le f$ and $h_2 \le g$.

<u>$C \subseteq D$</u>. Let $\int_A h \in C$, $h_1 := \min(h, f)$, and $h_2 := h - h_1 = \max(0, h-f)$. Then, $h_1, h_2 \geq 0$ (since $h, f, 0 \geq 0$), are measurable (since h, f measurable) with finite support (since $0 \le h_1, h_2 \le h$ which has finite supp.), and $h_1 \le f$, $h_2 \le g$ since $g \geq 0$ and $g \geq h-f$. Thus, $\int_A h = \int_A (h_1 + h_2) \in D$.

This proves $C = D$, i.e. $\int_A (f+g) = \int_A f + \int_A g$. $\boxed{\text{i}}$

(ii) $\displaystyle\int_{B_1 \cup B_2} f = \int_A f \chi_{B_1 \cup B_2} = \int_A f(\chi_{B_1} + \chi_{B_2}) \overset{(i)}{=} \int_A f \chi_{B_1} + \int_A f \chi_{B_2} = \int_{B_1} f + \int_{B_2} f$ $\boxed{\text{ii}}$

(iii) Let $N := \{x \in A : f(x) \neq g(x)\}$, so $f = g$ a.e. in A, we have $m(N) = 0$

Then,
$$\begin{cases} \int_A f = \int_{A \setminus N} f + \int_N f \\ \int_A g = \underbrace{\int_{A \setminus N} g}_{= \int_{A \setminus N} f} + \int_N g \end{cases}$$

And, $\int_N g = \int_N f = 0$ since every bdd, measurable function $h \geq 0$ with finite support satisfies $\int_N h \le \int_N \sup h = \sup h \times m(N) = 0$

This proves $\int_A f = \int_A g = \int_{A \setminus N} f$ $\boxed{\text{iii}}$

(iv) wlog, by using (iii), we may assume that $f \le g$ in A. By observing that if $h \le f$, then $h \le g$, we obtain $\int_A f \le \int_A g$. $\boxed{\text{iv}}$

Example $\int_A \chi_B = m(B) \ \forall B \subseteq A$ measurable.

This follows directly from the construction if $m(B) < \infty$.
Otherwise, consider χ_{B_k} where $B_k = B \cap B(0,k)$ so that $0 \leq \chi_{B_k} \leq \chi_B$ has finite support and is bounded by 1. We obtain

$$\int_A \chi_B \geq \int_A \chi_{B_k} = m(B_k) \to m(B) \qquad \text{[By continuity of measure]}$$

Chebyshev's Inequality.

Let $f: A \to [0,\infty]$ be measurable, then

$$\forall c > 0, \quad m(f^{-1}([c,\infty])) \leq \frac{1}{c}\int_A f$$

proof:

Observe that $f \geq c \cdot \chi_{f^{-1}([c,\infty])}$ since $f \geq 0$ and $f^{-1}([c,\infty]) = \{x \in A : f(x) \geq c\}$

Then, $\int_A f \geq c \int_A \chi_{f^{-1}([c,\infty])} = c \cdot m(f^{-1}([c,\infty]))$

■

Corollary 1

Let $f: A \to [0,\infty]$ be measurable. Then $f = 0$ a.e. in $A \Leftrightarrow \int_A f = 0$

proof:

($\Rightarrow$) is shown in the previous proposition.

($\Leftarrow$) If $\int_A f = 0$, then Chebyshev's inequality gives that

$$m(f^{-1}([1/k,\infty])) = 0 \quad \forall k \in \mathbb{N}.$$

By continuity of measure, it follows that $m(f^{-1}((0,\infty])) = 0$
i.e. $f = 0$ a.e. in A

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