Proposition For every Rectangle R, $m_*(R) = Vol(R)$

proof:

($\leq$) Using the equivalent definition of $m_*(R)$ with rectangles $(Q_1, A1)$
We obtain that $m_*(R) \leq Vol(R)$

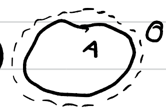
($\geq$) Conversely, let $(R_n)_{k\in\mathbb{N}}$ be rectangles s.t. $R \subseteq \bigcup_{k=1}^{\infty} R_k$
Then $vol(R) \leq \sum_{k=1}^{\infty} vol(R_n)$ (Proven last lecture (iii))

By taking the infimum on the rectangles, we obtain $Vol(R) \leq m_*(R)$ $\blacksquare$

Proposition
For every set $A \subseteq \mathbb{R}^d$, $m_*(A) = \inf\{m_*(O) \mid O \text{ open s.t. } A \subseteq O\}$

proof:

($\leq$) By monotonicity, if $A \subseteq O$, we have $m_*(A) \leq m_*(O)$
hence $m_*(A) \leq \inf\{m_*(O) \mid O \text{ open s.t. } A \subseteq O\}$



($\geq$) If $m_*(A) = \infty$ nothing else to prove.
Otherwise, want to show that $m_*(A) \geq \inf\{m_*(O) \mid O \text{ open}, A \subseteq O\}$

By using the equivalent definition of $m_*(A)$ with open cubes,
$\exists$ open cubes $(Q_k)_{k\in\mathbb{N}}$ s.t. $A \subseteq \bigcup_{k=1}^{\infty} Q_k$ and $\sum_{k=1}^{\infty} vol(Q_k) \leq m_*(A) + \varepsilon$

let $O_\varepsilon = \bigcup_{k=1}^{\infty} Q_n$, then O_ε is open, $A \subseteq O$, and $m_*(O_\varepsilon) \leq \sum_{k=1}^{\infty} Vol(Q_k)$
by definition of $m_*(O_\varepsilon)$.

It follows that $\inf\{m_*(O) \mid O \text{ open}, A \subseteq O\} \leq \sum_{k=1}^{\infty} Vol(Q_k) \leq m_*(A) + \varepsilon$
letting $\varepsilon \to 0$, we obtain that

$$\inf\{m_*(O) \mid O \text{ open}, A \subseteq O\} \leq m_*(A)$$

$\blacksquare$

Proposition (Countable subadditivity)

For every $(A_k)_{k\in\mathbb{N}} \subseteq \mathbb{R}^d$, $m_*\left(\bigcup_{k=1}^{\infty} A_k\right) \leq \sum_{k=1}^{\infty} m_*(A_k)$

proof:

If $m_*(A_k)$ for some k, nothing to prove.

Otherwise, by definition, $\forall \varepsilon > 0$, $\exists \left(Q_{k,j,\varepsilon}\right)_{j\in\mathbb{N}}$ closed cubes such that $A_k \subseteq \bigcup_{j=1}^{\infty} Q_{k,j,\varepsilon}$ and $\sum_{j=1}^{\infty} Vol(Q_{k,j,\varepsilon}) \leq m_*(A_k) + \varepsilon \cdot 2^{-k}$

Then, $\bigcup_{k=1}^{\infty} A_k \subseteq \bigcup_{k=1}^{\infty} \bigcup_{j=1}^{\infty} Q_{k,j,\varepsilon}$, which gives

$$m_*\left(\bigcup_{k=1}^{\infty} A_k\right) \leq \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} Vol(Q_{k,j,\varepsilon}) \leq \sum_{k=1}^{\infty} m_*(A_k) + \overbrace{\varepsilon \sum_{k=1}^{\infty} 2^{-k}}^{=\varepsilon}$$

letting $\varepsilon \to 0$, we get $m_*\left(\bigcup_{k=1}^{\infty} A_k\right) \leq \sum_{k=1}^{\infty} m_*(A_k)$

Proposition

Let $A_1, A_2 \subseteq \mathbb{R}^d$ be such that $d(A_1, A_2) > 0$ (ie. $\inf\{|x-y| \,/\, x\in A_1, y\in A_2\} > 0$)

Then $m_*(A_1 \cup A_2) = m_*(A_1) + m_*(A_2)$

proof:

($\leq$) By subadditivity, we have $m_*(A_1 \cup A_2) \leq m_*(A_1) + m_*(A_2)$

($\geq$) Let $(Q_k)_{k\in\mathbb{N}}$ be closed cubes such that $A_1 \cup A_2 \subseteq \bigcup_{k=1}^{\infty} Q_k$.
By subdividing the cubes, we may assume that $diam(Q_k) < \delta$, where

$$diam(Q_k) = \sup\{|x-y| \,;\, x, y \in Q_k\}$$

For each $i \in \{1,2\}$, let $K_i = \{k \in \mathbb{N} : Q_k \cap A_i \neq \emptyset\}$

Then, $A_i \subseteq \bigcup_{k\in K_i} Q_k$, hence $m_*(A_i) \leq \sum_{k\in K_i} vol(Q_k)$.

By summing, we obtain

$$m_*(A_2) + m_*(A_1) \leq \sum_{k\in K_1} vol(Q_k) + \sum_{k\in K_2} vol(Q_k)$$

Moreover, if δ is chosen s.t. $\delta < d(A_1, A_2)$, then $K_1 \cap K_2 = \emptyset$. Hence,

$$\sum_{k \in K_1} vol(Q_k) + \sum_{k \in K_2} vol(Q_k) = \sum_{k \in K_1 \cup K_2} vol(Q_k) \leq \sum_{k=1}^{\infty} vol(Q_k)$$

By taking the infimum on coverings $(Q_k)_{k \in \mathbb{N}}$, we obtain

$$m_*(A_1) + m_*(A_2) \leq m_*(A_1 \cup A_2) \quad \blacksquare$$