lemma - Simple approximation.

Let $A \subseteq \mathbb{R}^d$ be a measurable set s.t. $m(A) < \infty$ and $f: A \to \mathbb{R}$ be measurable s.t. $\exists M > 0$ s.t. $\forall x \in A$, $|f(x)| < M$.

Then for each $\varepsilon > 0$, $\exists$ simple function $\varphi_\varepsilon, \psi_\varepsilon : A \to \mathbb{R}$ s.t.

$$\varphi_\varepsilon \leq f \leq \psi_\varepsilon \leq \varphi + \varepsilon$$

proof:

Let $n_\varepsilon \in \mathbb{N}$ be large enough so that $\frac{2M}{n_\varepsilon} < \varepsilon$

For each $k \in \{0, \dots, n_\varepsilon\}$, let $y_{k,\varepsilon} = M\left(\frac{2k}{n_\varepsilon} - 1\right)$

and, $\quad A_{k,\varepsilon} = f^{-1}([y_{k,\varepsilon}, y_{k+1,\varepsilon}))$

Then, the sets $(A_{k,\varepsilon})_k$ are measurable (inverse image of a borel set by a measurable function), disjoint (since the sets $([y_{k,\varepsilon}, y_{k+1,\varepsilon}))_k$ are also disjoint), $m(A_{k,\varepsilon}) < \infty$ (since $m(A) < \infty$)

And,
$$\bigcup_{k=0}^{n_\varepsilon - 1} A_{k,\varepsilon} = f^{-1}\Big(\underbrace{\bigcup_{k=0}^{n_\varepsilon - 1} [y_{k,\varepsilon}, y_{k+1,\varepsilon})}_{=[-M,M)}\Big) = A \quad \text{since } |f| < M \text{ in } A$$

Now, define $\varphi_\varepsilon = \sum_{k=0}^{n_\varepsilon - 1} y_{k,\varepsilon} \chi_{A_{k,\varepsilon}}$ and $\psi_\varepsilon = \sum_{k=1}^{n_\varepsilon - 1} y_{k+1,\varepsilon} \chi_{A_{k,\varepsilon}}$

Since $y_{k,\varepsilon} \leq f \leq y_{k+1,\varepsilon} + \varepsilon$ in $A_{k,\varepsilon}$ $\forall k \in \{0, \dots, n_\varepsilon - 1\}$, it follows that

$$\varphi_\varepsilon \leq f_\varepsilon < \psi < \varphi_\varepsilon + \varepsilon$$

Theorem - Simple Approximation

Let $A \subseteq \mathbb{R}^d$ be a measurable set, $f: A \to \mathbb{R}$ measurable function. Then, $\exists$ sequence of simple functions $(\varphi_k)_k$ on A such that

(i) $|\varphi_k(x)| \le |\varphi_{k+1}(x)| \le |f(x)| \quad \forall x \in A, k \in \mathbb{N}$

(ii) $\lim_{k\to\infty} \varphi_k(x) = f(x) \quad \forall x \in A$

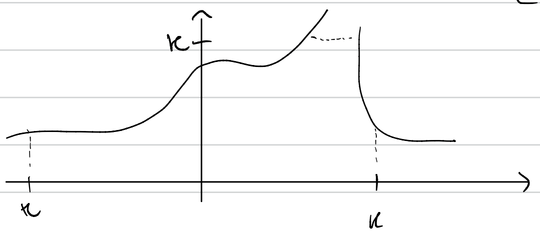
Moreover, if $f \geq 0$ in A, then we can choose $(\varphi_k)_k$ s.t. $\varphi_k \geq 0$ in A

proof:

- Case $f \geq 0$ in A

For each $k \in \mathbb{N}$, let $f_k = \min(f, k)\, \chi_{B(0,k)}$

Ball around 0.



Since $f_k \le k$ and $f_k = 0$ in $\mathbb{R}^n \setminus B(0,k)$, we can apply the simple approximation lemma to f_k which gives that $\exists$ simple function

$$\tilde{\varphi}_k : B(0,k) \to \mathbb{R}$$

Such that $\tilde{\varphi}_k \le f_k \le \tilde{\varphi}_k + 1/k$ in $B(0,k)$. By extending φ_k by 0 in $\mathbb{R}^d \setminus B(0,k)$, we may consider $\tilde{\varphi}_k$ as a function defined in $\mathbb{R}^d$ such that

$$\tilde{\varphi}_k \le f_k \le \tilde{\varphi}_k + 1/k \text{ in } \mathbb{R}^d \quad (\text{since } f = 0 \text{ in } \mathbb{R}^d \setminus B(0,k))$$

Then, define $\varphi_k = \max(\tilde{\varphi}_1, \dots, \tilde{\varphi}_k, 0)$ so that $\varphi_{k+1} \geq \varphi_k \geq 0 \ \forall k \in \mathbb{N}$

Moreover, at each $x \in A$,

- If $f(x) = \infty$, $\forall k > |x|$, $f_k(x) = k$ and

$$\varphi_k(x) \geq \tilde{\varphi}_k(x) \geq f_k(x) - \frac{1}{k} = k - 1/k$$

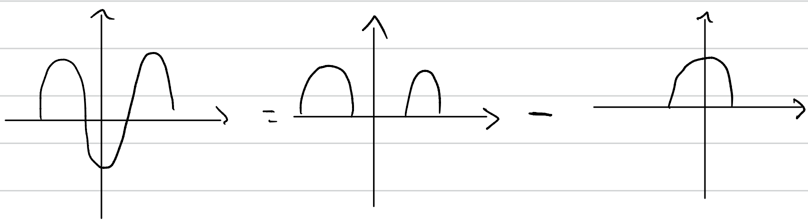
hence, $\lim_{k \to \infty} \varphi_k(x) = \infty = f(x)$

- If $f(x) < \infty$, $\forall k > \max(|x|, f(x))$, $f_k(x) = f(x)$ and

$$f(x) - \varphi_k(x) \leq \underbrace{f(x)}_{= f_k(x)} - \tilde{\varphi}(x) \leq 1/k \to 0 \quad \text{as } k \to \infty$$

Hence $\lim_{k \to \infty} \varphi_k(x) = f(x)$

- Case where f can change signs.



In this case, write $f = f_+ - f_-$ where $f_+ = \max(f, 0)$ and $f_- = \max(-f, 0)$. Apply the previous case to f_+ and f_-.
For (i), observe that $|f| = f_+ + f_-$

Egorov's Theorem

Let $A \subseteq \mathbb{R}^d$ be a measurable set s.t. $m(A) < \infty$, $(f_k)_{k\in\mathbb{N}}$, $f_k : A \to \mathbb{R}$ measurable converging pointwise to $f: A \to \mathbb{R}$.
Not $\overline{\mathbb{R}}$ Not $\overline{\mathbb{R}}$

Then $\forall \varepsilon > 0$, $\exists F_\varepsilon \subseteq A$ closed such that

(i) $m(A \setminus F_\varepsilon) < \varepsilon$

(ii) $(f_k)_{k\in\mathbb{N}}$ converges uniformly to $f \subset F_\varepsilon$, i.e. $\lim_{k\to\infty} \sup_{F_\varepsilon} |f_k - f| = 0$

proof: As a preliminary step, we show that $\forall n \in \mathbb{N}$, $\exists k_n \in \mathbb{N}$ and $\exists A_n \subset A$ measurable such that

(i) $m(A \setminus A_n) < \varepsilon 2^{-n-1}$ (ii) $\forall k > k_n$, $|f - f_k| < \frac{1}{n}$ in A_n

Since $(f_k)_k$ converges pointwise to $f \in A$, we have

same statement

Fix n, $\forall x \in A$, $\exists k_n \in \mathbb{N}$ s.t. $\forall k \geq k_n$, $|f(x) - f_k(x)| < 1/n$

ie.

$$A = \bigcup_{\overline{K}=1}^{\infty} \underbrace{\bigcap_{k=\overline{K}}^{\infty} (|f - f_k|)^{-1}((-\infty, 1/n))}_{A_{n,\overline{K}}}$$

Since $|\cdot|$ is continuous, f & f_k are measurable, and $(-\infty, 1/n)$ is Borel, it follows that $(|f - f_k|)^{-1}((-\infty, 1/n))$ is measurable $\forall k \in \mathbb{N}$.

Since countable intersection of measurable sets is measurable, it follows that $A_{n,\overline{K}}$ are measurable.

Observe that $A_{n,\overline{K}+1} \supseteq A_{n,\overline{K}}$ $\forall \overline{K} \in \mathbb{N}$, hence by continuity,

$$m(A) = \lim_{\overline{K}\to\infty} m(A_{n,\overline{K}})$$

Since $m(A)<\infty$, it follows that

$$\lim_{k\to\infty} m(A\setminus A_{n,k}) = m(A) = \lim_{k\to\infty} m(A_{n,k}) = 0$$

In particular, $\exists k_n \in \mathbb{N}$ s.t. $m(A\setminus A_{n,k}) < \varepsilon 2^{-n-1}$.
Moreover, by definition of $A_n := A_{n,k_n}$, we have $|f-f_k| < 1/n \quad \forall k \geq k_n$

Now, define $\tilde{A} = \bigcap_{n=1}^{\infty} A_n$. Since $\tilde{A}$ is measurable as countable intersection of measurable sets, $\exists F_\varepsilon \subseteq \tilde{A}$ closed such that $m(\tilde{A}\setminus F) < \varepsilon/2$.
We obtain that

$$m(A\setminus F_\varepsilon) = m\left((A\setminus\tilde{A}) \cup (\tilde{A}\setminus F_\varepsilon)\right) = m\left(\bigcup_{n=1}^{\infty}(A\setminus A_n) \cup (\tilde{A}\setminus F_\varepsilon)\right)$$

by ctbl subadd. $\longrightarrow \leq \sum_{n=1}^{\infty} m(A\setminus A_n) + m(\tilde{A}\setminus F_\varepsilon)$

$$< \varepsilon\left(\sum_{n=1}^{\infty} 2^{-n-1} + 1/2\right) = \varepsilon$$

And, $\forall n \in \mathbb{N}, \forall k \geq k_n$, $\sup_{F_\varepsilon} |f-f_k| \leq 1/n$ since $F_\varepsilon \subseteq \tilde{A} \subseteq A_n$, hence

$$\lim_{k\to\infty} \sup_{F_\varepsilon} |f-f_k| = 0$$

$\blacksquare$

Lusin's Theorem

Let $A \subseteq \mathbb{R}^d$ be measurable and $f: A \to \mathbb{R}$ (not $\overline{\mathbb{R}}$) be measurable.
Then, $\forall \varepsilon > 0$, $\exists F_\varepsilon \subseteq A$ closed such that

(i) $m(A\setminus F_\varepsilon) < \varepsilon$

(ii) $f|_{F_\varepsilon}$ is continuous in F_ε

Remark: "$f|_F$ is continuous" is not the same as "f is continuous on F"
e.g.

$f = \chi_{\mathbb{Q}}$ is nowhere continuous, but $f|_{\mathbb{Q}} = 1$ is continuous and so is $f|_{\mathbb{R}\setminus\mathbb{Q}} = 0$

lemma let $F_1, \dots, F_n$ be closed mutually disjoint subsets of $\mathbb{R}^d$ and $f_1, \dots, f_n$ continuous functions on $F_1, \dots, F_n$, respectively. Then,

$$f : \begin{array}{l} F = \bigcup_{k=1}^{\infty} F_k \longrightarrow \mathbb{R} \\ x \longmapsto f_k(x) \text{ if } x \in F_k \end{array}$$

proof of lemma:

let $(x_j)_{j\in\mathbb{N}}$ be a sequence in F such that $\lim_{j\to\infty} x_j = x$
Since $\{1, \dots, n\}$ is finite, there exists at least one $k \in \{1, \dots, n\}$ such that $x_j \in F_k$ for infinitely many $j \in \mathbb{N}$. On the other hand, for all such k, since F_k is closed, we must have $x \in F_k$. Because the sets $(F_k)_{1\le k\le n}$ are disjoint, there can only be one such k. We then obtain that $f(x_j) = f_k(x_j)$ for large j, hence $\lim_{j\to\infty} f(x_j) = f_k(x) = f(x)$ by continuity of f_k of F_k. $\square$

proof of theorem:

- Case where f is simple. Let $f = \sum_{j=1}^{n} c_j \chi_{A_j}$ be the canonical form of f. Let $c_0 = 0$ and $A_0 = f^{-1}\{0\}$. Then

$$f = \sum_{j=0}^{n} c_j \chi_{A_j} \text{ and } A = \bigcup_{j=0}^{n} A_j \, ,$$

The $(A_j)_j$ are measurable and disjoint. For each $j \in \{0, \dots, n\}$, since A_j is measurable, $\exists\, F_{j,\varepsilon} \subseteq A_j$ closed such that $m(A_j \setminus F_{j,\varepsilon}) < \varepsilon/(n+1)$. Then, letting $F_\varepsilon = \bigcup_{j=1}^{n} F_{j,\varepsilon}$, we obtain that F_ε is closed,

$$m(A\setminus F_\varepsilon) = m\left(\bigcap_{j=0}^{n} (A_j \setminus F_{j,\varepsilon})\right) \le \sum_{j=0}^{n} m(A_j \setminus F_{j,\varepsilon}) < \varepsilon$$

Moreover, by the lemma, since $f|_{F_{j,\varepsilon}} = c_j$ is constant and the sets $(F_{j,\varepsilon})_j$ are closed and mutually disjoint (as subsets of $(A_{j,\varepsilon})_j$), we obtain that

$$f|_F \text{ is continuous.}$$

- Case where f is measurable, $m(A) < \infty$

By the simple approximation theorem, $\exists (\varphi_k)_{k \in \mathbb{N}}$ simple functions converging pointwise to f in A. By Egorov's, $\exists F_\varepsilon \subseteq A$ closed such that $(\varphi_k)_k$ converges uniformly to f in F_ε and $m(A \setminus F_\varepsilon) < \varepsilon/2$. By the previous case, for each $k \in \mathbb{N}$, $\exists F_{\varepsilon,k} \subseteq A$ closed such that $m(A \setminus F_{\varepsilon,k}) < \varepsilon 2^{-k-1}$ and $\varphi_k|_{F_{\varepsilon,k}}$ is continuous. Let $F_\varepsilon = \bigcap_{k=0}^{\infty} F_{\varepsilon,k}$, F_ε is closed as intersection of closed sets. Moreover,

$$m\left(A \setminus \bigcap_{k=0}^{\infty} F_{\varepsilon,k}\right) = m\left(\bigcup_{k=0}^{\infty} A \setminus F_{\varepsilon,k}\right) \le \sum_{k=0}^{\infty} m(A \setminus F_{\varepsilon,k}) \quad \text{by countable subadd.}$$

$$< \varepsilon \sum_{k=0}^{\infty} 2^{-k-1} = \varepsilon$$

Since $(\varphi_k)_k$ converges uniformly to f in $F_\varepsilon \subseteq F_{\varepsilon,0}$ and $\varphi_k|_{F_\varepsilon}$ is continuous on F_ε (since $F_\varepsilon \subseteq F_{\varepsilon,k}$ $\forall k \in \mathbb{N}$), it follows that $f|_{F_\varepsilon}$ is continuous on F_ε.

- Case f is measurable, $m(A) = \infty$

consider $A = \bigcup_{k=0}^{\infty} A_k$ where $A_1 = A \cap \overline{B(0,1)}$ and $A_k = A \cap \overline{B(0,k)} \setminus A_{k-1}$

$\dot{x} = J\dot{q}$

$\dot{q} = J^{+}\dot{x}$

$J_{6\times7}$ $\dot{q}_{7\times1}$

But we don't want orientation

Take top 3 rows of J

$J_{3\times7}$ $\dot{q}_{3\times1}$

$\dot{q} = J^{+}\dot{x}$