CHAPTER 3

Integration Theory

3.1. Lebesgue Integration

Definition 3.1. Let $\varphi : A \to \mathbb{R}$ be a simple function on a measurable set $A \subseteq \mathbb{R}^d$ and let $\varphi = \sum_{k=1}^{n} c_k \chi_{A_k}$ be its canonical form. Then the **Lebesgue integral** of φ over A the number

$$\int_A \varphi \coloneqq \sum_{k=1}^{n} c_k m(A_k).$$

Moreover, for every measurable set $B \subseteq A$, the integral of φ over B is the number

$$\int_B \varphi \coloneqq \int_A \varphi \chi_B.$$

Lemma 3.1 (Independence of representation)**.** Let $c_1, \dots, c_n \in \mathbb{R}$ and $A_1, \dots, A_n \subseteq A$ be measurable, mutually disjoint, and of finite measure. Then $\varphi \coloneqq \sum_{k=1}^{n} c_k \chi_{A_k}$ is simple and $\int_A \varphi = \sum_{k=1}^{n} c_k m(A_k)$.

Proof. Without loss of generality suppose we may assume that $c_k \neq 0$ for all $k \in \{1, \dots, n\}$. Let $c'_1 < \cdots < c'_{n'}$ be the numbers such that $\{c'_1, \dots, c'_{n'}\} = \{c_1, \dots, c_n\}$. For each $k \in \{1, \dots, n'\}$ let $A'_k \coloneqq \varphi^{-1}(\{c'_k\}) = \bigcup_{j \in J_k} A_j$, where $J_k \coloneqq \{j \in \{1, \dots, n\} : c_j = c'_k\}$. Then $\sum_{k=1}^{n'} c'_k \chi_{A'_k}$ is the canonical form of φ. Thus we complete the proof using

$$\int_A \varphi = \sum_{k=1}^{n'} c'_k m(A'_k) = \sum_{k=1}^{n'} c'_k \sum_{j \in J_k} m(A_j) = \sum_{k=1}^{n'} \sum_{j \in J_k} c'_k m(A_j) = \sum_{j=1}^{n} c_j m(A_j),$$

by finite additivity since the sets $(A_j)_j$ are mutually disjoint. ∎

Proposition 3.1. Let $\varphi, \psi : A \to \mathbb{R}$ be simple functions.

1. $\forall \alpha, \beta \in \mathbb{R} : \alpha\varphi + \beta\psi$ is simple and $\int_A (\alpha\varphi + \beta\psi) = \alpha \int_A \varphi + \beta \int_A \psi$;
2. $\forall B_1, B_2 \subseteq A$ which are measurable and disjoint, $\int_{B_1 \cup B_2} \varphi = \int_{B_1} \varphi + \int_{B_2} \varphi$;
3. If $\varphi \leq \psi$ in A then $\int_A \varphi \leq \int_A \psi$; and
4. $|\varphi|$ is a simple function and $\left|\int_A \varphi\right| \leq \int_A |\varphi|$.

Proof.

1. Let $\varphi = \sum_{k=1}^{n} c_k \chi_{A_k}$ and $\psi = \sum_{k=1}^{n'} c'_k \chi_{A'_k}$ be the canonical forms of φ and ψ. Let $c_0 = c'_0 = 0$ and $A_0 \coloneqq \varphi^{-1}(\{0\}) \cap \tilde{A}$, $A'_0 = \psi^{-1}(\{0\}) \cap \tilde{A}$, where $\tilde{A} \coloneqq \bigcup_{k=1}^{n} A_k \cup \bigcup_{k=1}^{n} A'_k$ so that $\varphi = \sum_{k=0}^{n} c_0 \chi_{A_k}$, $\psi = \sum_{k=0}^{n'} c'_k \chi_{A'_k}$, the sets $(A_k)_k$, $(A'_k)_k$ are measurable, mutually disjoint, and of finite measure, and $\bigcup_{k=0}^{n} A_k = \bigcup_{k=0}^{n} A'_k = \tilde{A}$.

Let $A_{i,j} \coloneqq A_i \cap A'_j$ for all i, j so that the set $(A_{i,j})_{i,j}$ are measurable, mutually disjoint, of finite measure, and $\varphi = \sum_{i,j} c_i \chi_{A_{i,j}}$, $\psi = \sum_{i,j} c'_j \chi_{A_{i,j}}$. It follows that

$$\begin{aligned}\int(\alpha\varphi+\beta\psi) &= \int\sum_{i,j}(\alpha c_i+\beta c'_j)\chi_{A_{i,j}} = \sum_{i,j}(\alpha c_i+\beta c'_j)m(A_{i,j}) \\ &= \alpha\sum_{i,j}c_i m(A_{i,j})+\beta\sum_{i,j}c'_j m(A_{i,k}) = \alpha\int_A\varphi+\beta\int_A\psi. \qquad \text{(cf. Lemma 3.1)}\end{aligned}$$

2. $\int_{B_1\cup B_2}\varphi = \int_A \varphi\chi_{B_1\cup B_2} = \int_A \varphi\chi_{B_1} + \int_A \varphi\chi_{B_2} = \int_{B_1}\varphi + \int_{B_2}\varphi$, since B_1 and B_2 are disjoint we have $\chi_{B_1\cup B_2} = \chi_{B_1} + \chi_{B_2}$. (Exercise is to prove simple function)

3. If $\varphi \le \psi$ then $\psi - \phi \ge 0$ in A. Let $\psi - \varphi = \sum_{k=1}^n c_k \chi_{A_k}$ be the canonical form of $\psi - \varphi$. Since $\psi - \varphi \ge 0$ in A, we have $c_k \ge 0$ for each k and so $\int_A(\psi-\varphi) = \sum_{k=1}^n c_k m(A_k) \ge 0$, which implies by linearity $\int_A \psi \ge \int_A \varphi$.

4. Such follows from (3) noting that $-|\varphi| \le \varphi \le |\varphi|$. ■

Definition 3.2. We denote by $\mathrm{supp}(f)$ the **support** of a measurable function $f : A \to \overline{\mathbb{R}}$; that is,

$$\mathrm{supp}(f) \coloneqq \{x \in A : f(x) \neq 0\}.$$

If $\mathrm{supp}(f) \subseteq E \subseteq A$, we say that f is *supported* in E; and if $m(\mathrm{supp}(f)) < \infty$, we say that f has *finite support.*

Proposition 3.2. Let $f : A \to \mathbb{R}$ be measurable, bounded ($\exists\ M > 0 : |f| < M$ in A), and with finite support. Then there exists a number $\ell \in \mathbb{R}$ such that

$$\lim_{k\to\infty}\int\varphi_k = \ell$$

for all sequences of simple functions $(\varphi_k)_{k\in\mathbb{N}}$ satisfying

1. There exists a measurable set $E \subseteq A$ such that $m(E) < \infty$ and φ_k is supported in E for all $k \in \mathbb{N}$;

2. There exists an $C > 0$ such that $|\varphi_k| < C$ in A for all $k \in \mathbb{N}$; and

3. $(\varphi_k)_k$ converges pointwise a.e. in A to f.

We call ℓ the **integral** of f over A and denote it by $\int_A f$.

Proof. Since $m(E) < \infty$ by Egorov's theorem and (3), we obtain that for all $\varepsilon > 0$ there exists a closed set $F_\varepsilon \subseteq E$ such that φ_k converges uniformly in F_ε and $m(E \setminus F_\varepsilon) < \varepsilon$. Hence, there exists an $k_\varepsilon \in \mathbb{N}$ such that $|\varphi_k - f| < \varepsilon$ in F_ε whenever $k \ge k_\varepsilon$. Let $k_1, k_2, \ge k_\varepsilon$. Then

$$\left|\int_A(\varphi_{k_1}-\varphi_{k_2})\right| \le \int_A|\varphi_{k_1}-\varphi_{k_2}| = \underbrace{\int_{F_\varepsilon}|\varphi_{k_1}-\varphi_{k_2}|}_{\text{unif. conv.}} + \underbrace{\int_{E\setminus F_\varepsilon}|\varphi_{k_1}-\varphi_{k_2}|}_{m(E\setminus F_\varepsilon)<\varepsilon, |\varphi_k|<C} + \underbrace{\int_{A\setminus E}|\varphi_{k_1}-\varphi_{k_2}|}_{m(\mathrm{supp}(\varphi_k))<\infty \text{ in } E}.$$

■

Remark 3.1. By the simple approximation lemma, such a sequence of functions $(\varphi_k)_k$ exists.

Remark 3.2. For $B \subseteq A$ we call the integral of f over B the number: $\int_B f = \int_A f\chi_B$.

Corollary 3.1. Let f be as in Proposition 3.2. If $f = 0$ a.e. in A then $\int_A f = 0$.