Definition

Let $P(x)$ be a statement depending on $x \in A \subseteq \mathbb{R}^d$, A measurable. Then, we say that $P(x)$ is true for almost every $x \in A$ if

$$m_*(\{x \in A : P(x) \text{ is false}\}) = 0$$

For short, we say "for a.e.".

Proposition

Let $(P_k(x))_{k \in \mathbb{N}}$ be a countable collection of statements depending on $x \in A$, A measurable subset of $\mathbb{R}^d$. Then

$$[\forall k \in \mathbb{N}, \text{ for a.e. } x \in A, \ P_k(x) \text{ is true}]$$
$$\Longleftrightarrow [\text{For a.e. } x \in A, \ \forall k \in \mathbb{N}, \ P_k(x) \text{ is true}]$$

proof:

$$\overbrace{\{x \in A : \underbrace{[\forall k \in \mathbb{N}, \ P_k(x) \text{ is True}]}_{\Leftrightarrow \exists k \in \mathbb{N} \text{ s.t. } P_k(x) \text{ is false}} \text{ is False}\}}^{B} = \bigcup_{k=1}^{\infty} \overbrace{\{x \in A : P_k(x) \text{ is false}\}}^{B_k}$$

By subadditivity, it follows that if $m_*(B_k) = 0$, then $m_*(B) = 0$.
By monotonicity, we also obtain that if $m_*(B) = 0$ then $m_*(B_k) = 0 \ \forall k$.

This proves our statement.

$\square$

Proposition

Let $f, g : A \to \overline{\mathbb{R}}$, with A measurable subset of $\mathbb{R}^d$. If f is measurable and $f = g$ a.e. in A, then g is measurable.

proof:

Define $N = \{x \in A : f(x) \neq g(x)\}$, then $m_*(N) = 0$ ie. measurable.

$$\forall c \in \mathbb{R},\ g^{-1}([-\infty, c)) = \underbrace{\left(g^{-1}([-\infty, c)) \cap N\right)}_{\substack{\subseteq N \text{ hence measurable} \\ \text{of measure } 0.}} \cup \underbrace{\left(g^{-1}([-\infty, c)) \setminus N\right)}_{\substack{= f^{-1}([-\infty, c)) \setminus N \\ \text{since } f = g \text{ in } A \setminus N}}$$

Since N & f are measurable, we obtain that $f^{-1}([-\infty, c)) \setminus N$ is measurable, hence $g^{-1}([-\infty, c)) \setminus N$ measurable since $f = g$ in $A \setminus N$

Thus, $g^{-1}([-\infty, c))$ is measurable by union of measurable sets. $\square$

Proposition

If $f : A \to \overline{\mathbb{R}}$ is measurable and $B \subseteq A$ is measurable, Then $f|_B$ is measurable.

proof: $\forall c \in \mathbb{R}$, $f^{-1}|_B([-\infty, c)) = f^{-1}([-\infty, c)) \cap B$

which is measurable since f & B are measurable $\square$

Proposition

let $(A_k)_k$ be mutually disjoint measurable sets in $\mathbb{R}^d$ and $\forall k \in \mathbb{N}$, $f_k : A_k \to \overline{\mathbb{R}}$ be measurable. Then,

$$f = \begin{array}{rcl} A = \bigcup_{k=1}^{\infty} A_k & \longrightarrow & \overline{\mathbb{R}} \\ x & \longmapsto & f_k(x) \quad \text{if } x \in A_k \end{array}$$

Is measurable.

proof: $\forall c \in \mathbb{R}$, $f^{-1}([-\infty, c)) = \bigcup_{k=1}^{\infty} f_k^{-1}([-\infty, c))$ is measurable since f_k is measurable $\forall k$, and measurability is preserved by countable unions.

Example

Piecewise continuous functions are measurable. In particular, for every measurable set $A \subseteq \mathbb{R}^d$, the function

$$\chi_A : \begin{array}{l} \mathbb{R}^d \longrightarrow \mathbb{R} \\ x \longmapsto \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{o.w.} \end{cases} \end{array}$$

is measurable. χ_A is called the **characteristic function** of A.

Proposition

Let A measurable subset of $\mathbb{R}^d$, B borel subset of $\mathbb{R}$, $f : B \to \mathbb{R}$ continuous and $g : A \to B$ measurable. Then, $f \circ g$ is measurable.

$$A \xrightarrow{g} B \xrightarrow{f} \mathbb{R}$$

Counter-example: g cts, f measurable $\not\Rightarrow$ $f \circ g$ measurable.

Let ψ, D, E be as in the construction at the end of chapter 1.

Consider $g = \psi^{-1}$, $f = \chi_D$. Then,

$$(f \circ g)^{-1}([1, \infty]) = g^{-1}(\underbrace{f^{-1}([1, \infty])}_{\text{points in } D \text{ by def of } \chi}) = g^{-1}(D) = \psi(D) = E$$

And E is not measurable. //

proof of Prop:

$\forall c \in \mathbb{R}$, $(f \circ g)^{-1}([-\infty, c)) = g^{-1}(f^{-1}([-\infty, c)))$

$$f^{-1}([-\infty, c)) = \underbrace{f^{-1}((-\infty, c))}_{\substack{\text{open in } B \text{ since } f \text{ cts, so } f^{-1} \text{ maps open to open.} \\ \text{Hence, it is a Borel set.}}} \text{ since } |f| < \infty$$

Since g is measurable, it follows that $g^{-1}(f^{-1}([-\infty, c)))$ is measurable. And because this is true $\forall c \in \mathbb{R}$, this proves that $f \circ g$ is measurable.

$\square$

Proposition

Let A measurable subset of $\mathbb{R}^d$ and $f, g : A \to \overline{\mathbb{R}}$ measurable. Then

(i) $\max(f, g)$ and $\min(f, g)$ are measurable.

(ii) Assume that $[g(x) > -\infty$ when $f(x) = \infty]$ and $[g(x) < \infty$ when $f(x) = -\infty]$. Then $f+g$ is measurable.

(iii) fg is measurable with the CONVENTION that $0 \times \infty = 0$.

proof:

(i) $\forall c \in \mathbb{R}$,
- $\max(f,g)^{-1}([-\infty, c)) = f^{-1}([-\infty, c)) \cap g^{-1}([-\infty, c))$
- $\min(f,g)^{-1}([-\infty, c)) = f^{-1}([-\infty, c)) \cup g^{-1}([-\infty, c))$

In both cases, since f and g are measurable, we obtain that $f^{-1}([-\infty, c))$ and $f^{-1}([-\infty, c))$ are measurable. Hence, $\max(f,g)$ & $\min(f,g)$ are measurable.

(ii) $\forall c \in \mathbb{R}$, $\forall x \in A$,

$$x \in (f+g)^{-1}([-\infty, c)) \iff f(x) + g(x) < c$$
$$\iff f(x) < c - g(x)$$

only if $f(x) \neq \infty$, always true if $f(x) = -\infty$ and $g(x) < \infty$

$$\iff \exists q \in \mathbb{Q} \text{ s.t. } f(x) < q < c - g(x) \quad \text{by density of } \mathbb{Q} \text{ in } \mathbb{R}$$
$$\iff \exists q \in \mathbb{Q} \text{ s.t. } f(x) < q \text{ and } g(x) < c - q$$
$$\iff \exists q \in \mathbb{Q} \text{ s.t. } x \in f^{-1}([-\infty, q)) \cap g^{-1}([-\infty, c-q))$$

Hence $(f+g)^{-1}([-\infty, c)) = \bigcup_{q \in \mathbb{Q}} \left(f^{-1}([-\infty, q)) \cap g^{-1}([-\infty, c-q))\right)$ measurable since f and g are measurable and measurability preserved over union and intersection

(iii)

- Case $|f| < \infty$ and $|g| < \infty$

Observe that $fg = \frac{1}{2}\left[(f+g)^2 - f^2 - g^2\right]$.
Since f, g measurable, we obtain that $f+g$ is measurable.
Moreover, the functions

$$x \longmapsto \frac{1}{2}x^2 \quad \& \quad x \longmapsto \frac{-1}{2}x^2$$

are continuous, hence we obtain that

$$\frac{1}{2}(f+g)^2, \quad -\frac{1}{2}f^2, \quad \frac{1}{2}g^2$$

are measurable, which in turn gives that fg is measurable.

- General case

$\forall c \in \mathbb{R}, \ \forall x \in A,$

$$\begin{aligned}(fg)(x) < c \iff & |f(x)| < \infty,\ |g(x)| < \infty \text{ and } (fg)(x) < c \\ & \text{OR} \\ & [f(x) = -\infty,\ g(x) > 0] \\ & \text{OR} \\ & [g(x) = -\infty,\ f(x) > 0] \\ & \text{OR} \\ & [f(x) = \infty,\ g(x) < 0] \\ & \text{OR} \\ & [g(x) = \infty,\ f(x) < 0] \\ & \text{OR} \\ & \{[|f(x)| = \infty,\ g(x) = 0] \text{ OR } [|g(x)| = \infty,\ f(x) = 0],\ \text{if } c > 0\}\end{aligned}$$

$$(fg)^{-1}([-\infty, c)) = (\tilde{f}\tilde{g})^{-1}((-\infty, c)) \cup \left[f^{-1}(\{-\infty\}) \cap g^{-1}((0,\infty])\right]$$
$$\cup \left[g^{-1}(\{-\infty\}) \cap f^{-1}((0,\infty])\right]$$
$$\cup \left[f^{-1}(\{\infty\}) \cap g^{-1}([-\infty,0))\right]$$
$$\cup \left[g^{-1}(\{\infty\}) \cap f^{-1}([-\infty,0))\right]$$
$$\begin{cases} \cup \left[f^{-1}(\{-\infty,\infty\}) \cap g^{-1}(\{0\})\right] \\ \cup \left[g^{-1}(\{-\infty,\infty\}) \cap f^{-1}(\{0\})\right] \end{cases} \text{if } c>0$$

where $\tilde{f} = f|_{f^{-1}(\mathbb{R}) \cap g^{-1}(\mathbb{R})}$

And, since f and g are measurable, we obtain that $f^{-1}(\mathbb{R})$ and $g^{-1}(\mathbb{R})$ are measurable (since $\mathbb{R}$ is open). Hence, $f^{-1}(\mathbb{R}) \cap g^{-1}(\mathbb{R})$ is measurable and $\tilde{f}, \tilde{g}$ are measurable. This gives that

- $(\tilde{f}\tilde{g})^{-1}((-\infty, c))$ is measurable
- $f^{-1}(\{-\infty\}) = f^{-1}\left(\bigcap_{k=1}^{\infty} [-\infty, k)\right) = \bigcap_{k=1}^{\infty} \left(f^{-1}([-\infty,k))\right)$ is measurable,
- $f^{-1}(\{\infty\}) = \bigcap_{k=1}^{\infty} \left(f^{-1}((k,\infty])\right)$ is measurable
- $f^{-1}(\{0\})$ is measurable since $\{0\}$ is closed
- $f^{-1}((0,\infty])$ is measurable since f is measurable
- Same sets above are also measurable for g.

Hence, since measurability is preserved by finite unions & intersections, it follows that $(fg)^{-1}([-\infty,\infty])$ is measurable

$\square$

Proposition

Let $(f_k)_{k\in\mathbb{N}}$, $f_k : A \to \overline{\mathbb{R}}$ measurable functions converging pointwise a.e. in A, that is

$$\lim_{n\to\infty} f_n(x) = f(x) \quad \text{for a.e. } x \in A.$$

Then, f is measurable.

proof: Let $N = \{x \in A \mid (f_k(x))_k \text{ does not converge to } f(x)\}$

Then, $m_*(N) = 0$. $\forall c \in \mathbb{R}$, $\forall x \in A \setminus N$, $f(x) < c \iff \lim_{k\to\infty} f_k(x) < c$

prove this (exercise) $\iff \exists n \in \mathbb{N}, K \in \mathbb{N}$ st. $\forall k > K$, $f_k(x) < c - \frac{1}{n}$

Hence, $f^{-1}([-\infty, c)) \setminus N = \bigcup_{n=1}^{\infty} \bigcup_{K=1}^{\infty} \bigcap_{k=K}^{\infty} f_k^{-1}([-\infty, c - 1/n)) \setminus N$

(everytime, $\exists$ = union, $\forall$ = intersection)

Since the sets $(f_k^{-1}([-\infty, c - 1/k)))$ are measurable by measurability of $(f_k)_k$, also since measurability preserved by ctbl union, intersection, complements, also since N is measurable of measure 0, we obtain that $f^{-1}([-\infty, c))$ is measurable, i.e. f is measurable.

$\square$

Proposition

Let $(f_k)_k$, $f_k : A \to \overline{\mathbb{R}}$ be measurable, then

$$\inf_{k\in\mathbb{N}} f_k, \quad \sup_{k\in\mathbb{N}} f_k, \quad \limsup_{k\to\infty} f_k, \quad \liminf_{k\to\infty} f_k \quad \text{are measurable}$$

proof: Follows from observing that
- $\inf_{k\in\mathbb{N}} f_k = \lim_{k\to\infty} \min(f_1, \dots, f_k)$
- $\sup_{k\in\mathbb{N}} f_k = \lim_{k\to\infty} \max(f_1, \dots, f_k)$
- $\liminf_{k\to\infty} f_k = \lim_{k\to\infty} \inf_{n\geq k} f_n$
- $\limsup_{k\to\infty} f_k = \lim_{k\to\infty} \sup_{n\geq k} f_n$ $\square$

Definition

We call **simple function** on A, a measurable function $\varphi: A \to \mathbb{R}$ such that φ takes a finite number of values (ie. $\varphi(A)$ is a finite set) and has **finite support** ie. $m(\{x\in A : \varphi(x)\neq 0\}) < \infty$.

In particular, we can write φ as

$$\varphi = \sum_{i=1}^{n} c_i \chi_{A_i}, \quad \text{where} \quad n \geq 0, \quad \overbrace{c_1, \dots, c_n}^{\text{distinct}} \in \mathbb{R}\setminus\{0\}$$

characteristic function

And, $A_1, \dots, A_n$ measurable, disjoint subset of A of finite measure s.t.

$$\{c_1, \dots, c_n\} = \varphi(A)\setminus\{0\} \quad \text{and} \quad A_i = \varphi^{-1}(\{c_i\})$$

$\sum_{i=1}^{n} c_i \chi_{A_i}$ is called the **canonical form** of φ.

Remark

When the sets A_i are rectangles, we call φ a **step function**.