46. General Solutions to Second-Order DEs.
Consider $ax''(t) + bx'(t) + cx(t) = 0$. We solve by assuming $x(t) = e^{\lambda t}$.

(a) **Real Distinct Roots:** Suppose the characteristic equation yields roots $\lambda = 2$ and $\lambda = -\frac{1}{3}$. Since the roots are real and distinct, the general solution is a linear combination of exponentials:

$$\boxed{x(t) = c_1 e^{2t} + c_2 e^{-\frac{1}{3}t}}$$

(b) **Complex Roots:** Suppose the roots are $\lambda = -7 \pm 2i$.

The solution is of the form $e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$, where $\lambda = \alpha \pm \beta i$. Here, $\alpha = -7$ and $\beta = 2$.

$$\boxed{x(t) = e^{-7t}\left(c_1 \cos(2t) + c_2 \sin(2t)\right)}$$

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47. Damped Spring-Mass System.

(a) **Derive Equation:** Newton's Second Law: $F_{net} = ma$. Forces acting on the mass:

- Spring Force: $-kx$
- Damping Force: $-cv = -cx'$

$$ma = -kx - cx' \implies mx'' + cx' + kx = 0$$
$$\boxed{mx''(t) + cx'(t) + kx(t) = 0}$$

(b) **Find Real Solutions:** Given $m = 1, c = 4, k = 29$. Equation: $x'' + 4x' + 29x = 0$. Characteristic Equation: $\lambda^2 + 4\lambda + 29 = 0$.

$$\lambda = \frac{-4 \pm \sqrt{16 - 4(29)}}{2} = \frac{-4 \pm \sqrt{-100}}{2} = \frac{-4 \pm 10i}{2} = -2 \pm 5i$$

Since roots are complex, the two fundamental real solutions are:

$$\boxed{x_1(t) = e^{-2t}\cos(5t), \quad x_2(t) = e^{-2t}\sin(5t)}$$

(c) **Solve Initial Value Problem (IVP):** Initial conditions: $x(0) = 0$ and $x'(0) = 4$. General Solution: $x(t) = e^{-2t}(c_1 \cos(5t) + c_2 \sin(5t))$.

1. Apply $x(0) = 0$:

$$0 = e^0(c_1(1) + c_2(0)) \implies c_1 = 0$$

So, $x(t) = c_2 e^{-2t}\sin(5t)$.

2. Differentiate $x(t)$: Using Product Rule:

$$x'(t) = c_2\left[-2e^{-2t}\sin(5t) + 5e^{-2t}\cos(5t)\right]$$

3. Apply $x'(0) = 4$:

$$4 = c_2[-2(0) + 5(1)] \implies 4 = 5c_2 \implies c_2 = \frac{4}{5}$$

Final Solution:

$$\boxed{x(t) = \frac{4}{5}e^{-2t}\sin(5t)}$$

48. Comparing Harmonic Oscillators.

Spring/Mass System: $mx'' + cx' + kx = 0$

Note: in all 4 equations, there is no x' term, thus all systems are undamped ($c = 0$). So, we have the form:

$$mx'' + kx = 0$$

$$\implies x'' + \frac{k}{m}x = 0$$

Now, assuming $x(t) = e^{\lambda t}$:

$$\lambda^2 e^{\lambda t} + \frac{k}{m}e^{\lambda t} = 0$$

Since $e^{\lambda t} \neq 0$:

$$\lambda^2 + \frac{k}{m} = 0$$

Which gives solutions:

$$\lambda = \pm\sqrt{-\frac{k}{m}} = \pm i\sqrt{\frac{k}{m}}$$

Thus,

$$x(t) = e^{i\sqrt{\frac{k}{m}}t}, \quad x(t) = e^{-i\sqrt{\frac{k}{m}}t}$$

So, by Euler's identity, the solutions are of the form:

$$x(t) = k_1 \cos\left(\sqrt{\frac{k}{m}}t\right) + k_2 \sin\left(\sqrt{\frac{k}{m}}t\right)$$

(a) **Shortest Period?** The period is given by:

$$T = \frac{2\pi}{\sqrt{k/m}} = \frac{2\pi\sqrt{m}}{\sqrt{k}}$$

Thus:

$$\uparrow m \implies \uparrow T \quad \text{(Increasing mass increases period)}$$

$$\uparrow k \implies \downarrow T \quad \text{(Increasing stiffness decreases period)}$$

So, for the smallest period, we want **small** m and **large** k.

Checking the equations:

- (i) $4x'' + x = 0 \implies m = 4, k = 1$
- (ii) $x'' + 4x = 0 \implies m = 1, k = 4$
- (iii) $16x'' + x = 0 \implies m = 16, k = 1$
- (iv) $x'' + 16x = 0 \implies m = 1, k = 16$

Equation (iv) has the smallest m and largest k.

$$\implies \text{Equation (iv), where } T = \frac{2\pi}{\sqrt{16/1}} = \frac{2\pi}{4} = \boxed{\frac{1}{2}\pi}$$

(b) **Largest Amplitude?** Since velocity is initially zero, the amplitude is simply the initial displacement $x(0)$. System (i) starts at $x = 36$.

$$\boxed{36 \text{ m} \quad \text{(System i)}}$$

(c) **Highest Velocity?** First, can simplify the form of each solution using the initial displacements and velocities.

For example, using (ii):

$$\sqrt{\frac{k}{m}} = \sqrt{\frac{4}{1}} = 2 \quad \text{and} \quad x(0) = 25$$

So,

$$x(t) = k_1 \cos(2t) + k_2 \sin(2t)$$

$$\implies 25 = k_1 \cos(0) + k_2 \sin(0) \implies k_1 = 25$$

And taking the derivative:

$$x'(t) = -2k_1 \sin(2t) + 2k_2 \cos(2t)$$

$$\implies 0 = -2k_1 \sin(0) + 2k_2 \cos(0) \implies 0 = 2k_2 \implies k_2 = 0$$

Therefore, $x(t) = 25 \cos(2t)$.

This process holds for all equations, so we have:

i) $x(t) = 36 \cos(\frac{1}{2}t) \implies x'(t) = -18 \sin(\frac{1}{2}t)$

ii) $x(t) = 25 \cos(2t) \implies x'(t) = -50 \sin(2t)$

iii) $x(t) = 9 \cos(\frac{1}{4}t) \implies x'(t) = -\frac{9}{4} \sin(\frac{1}{4}t)$

iv) $x(t) = \cos(4t) \implies x'(t) = -4 \sin(4t)$

And we know $\sin(\omega t) \in [-1, 1]$ for all ω, thus max velocity is 50 m/s, achieved by eqn (ii).

$$\boxed{50 \text{ m/s} \quad \text{(System ii)}}$$