34. Area Between Curves.

Consider the region bounded by $f(x) = \frac{x}{3}$ (a straight line through the origin) and $g(x) = \sqrt{x}$ (a parabola opening to the right).

(a) **Sketching and Intersection Points:** First, find where the curves intersect by setting $f(x) = g(x)$:

$$\frac{x}{3} = \sqrt{x} \implies \frac{x^2}{9} = x \implies x^2 - 9x = 0 \implies x(x-9) = 0$$

The intersection points are at $x = 0$ (where $y = 0$) and $x = 9$ (where $y = 3$). On the interval $(0, 9)$, test a point like $x = 1$: $g(1) = 1$ and $f(1) = 1/3$. Since $1 > 1/3$, the curve $g(x) = \sqrt{x}$ is on top.

(b) **Vertical Slices (Δx):** When using vertical slices, we integrate with respect to x. The height of a slice is the upper function minus the lower function: Height $= g(x) - f(x)$. The width is Δx (or dx). The bounds are from $x = 0$ to $x = 9$.

$$\text{Area} = \int_0^9 \left(\sqrt{x} - \frac{x}{3}\right) dx$$

(Evaluation not required for this part).

(c) **Horizontal Slices (Δy):** When using horizontal slices, we integrate with respect to y. We must express the functions as x in terms of y.

- $y = \frac{x}{3} \implies x = 3y$ (This is the line on the right).
- $y = \sqrt{x} \implies x = y^2$ (This is the parabola on the left).

The width of a slice is the right function minus the left function: Width $= 3y - y^2$. The thickness is Δy (or dy). The bounds are from $y = 0$ to $y = 3$.

$$\text{Area} = \int_0^3 (3y - y^2) dy$$

(Evaluation not required for this part).

---

35. Integration by Parts.

Evaluate $\int xe^{-4x} dx$.

Step 1: Choose parts. Use the formula $\int u\,dv = uv - \int v\,du$. Choose u to be the part that gets simpler when differentiated, and dv to be the part that is easy to integrate.

- Let $u = x \implies du = dx$
- Let $dv = e^{-4x} dx \implies v = \int e^{-4x} dx = -\frac{1}{4}e^{-4x}$

Step 2: Apply formula.

$$\int xe^{-4x} dx = (x)\left(-\frac{1}{4}e^{-4x}\right) - \int \left(-\frac{1}{4}e^{-4x}\right) dx$$

$$= -\frac{1}{4}xe^{-4x} + \frac{1}{4}\int e^{-4x} dx$$

Step 3: Evaluate the remaining integral.

$$= -\frac{1}{4}xe^{-4x} + \frac{1}{4}\left(-\frac{1}{4}e^{-4x}\right) + C$$

$$\boxed{-\frac{1}{4}xe^{-4x} - \frac{1}{16}e^{-4x} + C}$$

Step 4: Verification (Differentiation). Differentiate the result to check:

$$\frac{d}{dx}\left[-\frac{1}{4}xe^{-4x} - \frac{1}{16}e^{-4x}\right]$$

Using Product Rule on the first term:

$$\left[-\frac{1}{4}(1)e^{-4x} + \left(-\frac{1}{4}x\right)(-4e^{-4x})\right] - \frac{1}{16}(-4e^{-4x})$$

$$= -\frac{1}{4}e^{-4x} + xe^{-4x} + \frac{1}{4}e^{-4x}$$

$$= xe^{-4x} \quad \text{(Correct)}$$

---

36. Splitting Fractions Standard Integrals.

Evaluate $\int \frac{2+x}{1+x^2}dx$.

Step 1: Split the numerator. Separate the integral into two simpler parts:

$$I = \int \frac{2}{1+x^2}dx + \int \frac{x}{1+x^2}dx$$

Step 2: Evaluate the first part. Recognize that $\int \frac{1}{1+x^2}dx = \arctan(x)$.

$$\int \frac{2}{1+x^2}dx = 2\arctan(x)$$

Step 3: Evaluate the second part. Use substitution for $\int \frac{x}{1+x^2}dx$. Let $w = 1+x^2$, then $dw = 2xdx \implies xdx = \frac{1}{2}dw$.

$$\int \frac{1}{w} \cdot \frac{1}{2}dw = \frac{1}{2}\ln|w| = \frac{1}{2}\ln(1+x^2)$$

(Note: $1+x^2$ is always positive, so absolute value bars are optional).

Step 4: Combine.

$$\boxed{2\arctan(x) + \frac{1}{2}\ln(1+x^2) + C}$$

---

37. Mixed Integration Techniques.

(a) **Evaluate $\int xe^{-3x^2}dx$.**

Method: Substitution. Let $w = -3x^2$. Then $\frac{dw}{dx} = -6x \implies xdx = -\frac{1}{6}dw$.

Substitute into the integral:

$$\int e^w \left(-\frac{1}{6}\right) dw = -\frac{1}{6}\int e^w dw$$

$$= -\frac{1}{6}e^w + C$$

Substitute back $w = -3x^2$:

$$\boxed{-\frac{1}{6}e^{-3x^2} + C}$$

(b) **Evaluate $\int \arccos(x)dx$.**

Method: Integration by Parts. Let $u = \arccos(x)$ and $dv = dx$.

- $du = -\frac{1}{\sqrt{1-x^2}}dx$
- $v = x$

Apply formula:

$$I = x\arccos(x) - \int x\left(-\frac{1}{\sqrt{1-x^2}}\right)dx$$

$$I = x\arccos(x) + \int \frac{x}{\sqrt{1-x^2}}dx$$

Now, use substitution for the remaining integral. Let $w = 1 - x^2$, then $dw = -2xdx \implies xdx = -\frac{1}{2}dw$.

$$\int w^{-1/2}\left(-\frac{1}{2}\right)dw = -\frac{1}{2}\int w^{-1/2}dw$$

$$= -\frac{1}{2}(2w^{1/2}) = -\sqrt{w} = -\sqrt{1-x^2}$$

Combine with the first part:

$$\boxed{x\arccos(x) - \sqrt{1-x^2} + C}$$