19. Implicit Differentiation. Find the slope of $y^2 + xy - y^3 = 2$ at the point $(3, 2)$.

Step 1: Differentiate with respect to x.

$$\frac{d}{dx}(y^2) + \frac{d}{dx}(xy) - \frac{d}{dx}(y^3) = \frac{d}{dx}(2)$$

$$2y\frac{dy}{dx} + \left(1 \cdot y + x\frac{dy}{dx}\right) - 3y^2\frac{dy}{dx} = 0$$

Step 2: Isolate $\frac{dy}{dx}$.

$$\frac{dy}{dx}(2y + x - 3y^2) = -y$$

$$\frac{dy}{dx} = \frac{-y}{2y + x - 3y^2}$$

Step 3: Evaluate at $(3, 2)$.

$$\frac{dy}{dx} = \frac{-2}{2(2) + 3 - 3(2)^2} = \frac{-2}{4 + 3 - 12} = \frac{-2}{-5}$$

$$\boxed{\frac{2}{5}}$$

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20. Related Rates (Angle of Elevation/Depression).

- Height $y = 8$ m (constant).
- Horizontal distance $x = 30$ m at $t = 0$.
- Speed $\frac{dx}{dt} = -0.5$ m/s (distance is shrinking).
- Find $\frac{d\theta}{dt}$.

Step 1: Relate variables. From the right triangle formed by the observer and the object:

$$\tan(\theta) = \frac{8}{x}$$

Step 2: Differentiate w.r.t. time t.

$$\sec^2(\theta)\frac{d\theta}{dt} = -8x^{-2}\frac{dx}{dt} = \frac{-8}{x^2}\frac{dx}{dt}$$

Step 3: Solve for $\frac{d\theta}{dt}$. At $x = 30$:

$$\frac{d\theta}{dt} = \frac{1}{\sec^2(\theta)}\left(\frac{-8}{30^2}\right)(-0.5) = \cos^2(\theta)\left(\frac{4}{900}\right) = \frac{\cos^2(\theta)}{225}$$

Find $\cos(\theta)$ from the triangle geometry: hypotenuse $h = \sqrt{30^2 + 8^2} = \sqrt{964}$.

$$\cos(\theta) = \frac{\text{adj}}{\text{hyp}} = \frac{30}{\sqrt{964}}$$

$$\frac{d\theta}{dt} = \frac{1}{225}\left(\frac{30}{\sqrt{964}}\right)^2 = \frac{1}{225} \cdot \frac{900}{964} \approx 0.004$$

$$\boxed{0.004 \text{ rad/s}}$$

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21. Related Rates (Cone).

- Volume $V = \frac{1}{3}\pi r^2 h$. Volume is constant ($\frac{dV}{dt} = 0$).
- Given: $h = 2$, $r = 3$, $\frac{dh}{dt} = -0.1$.
- Find rate of change of circumference $\frac{dC}{dt}$ where $C = 2\pi r$.

Step 1: Differentiate Volume formula.

$$0 = \frac{\pi}{3}\left[2r\frac{dr}{dt}h + r^2\frac{dh}{dt}\right]$$

Step 2: Substitute known values.

$$0 = \frac{\pi}{3}\left[2(3)\frac{dr}{dt}(2) + (3)^2(-0.1)\right]$$

$$0 = 12\frac{dr}{dt} - 0.9 \implies 12\frac{dr}{dt} = 0.9 \implies \frac{dr}{dt} = \frac{0.9}{12} = 0.075$$

Step 3: Find $\frac{dC}{dt}$.

$$\frac{dC}{dt} = 2\pi\frac{dr}{dt} = 2\pi(0.075)$$

$$\boxed{0.15\pi \approx 0.471 \text{ m/s}}$$

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22. Optimization (Box Volume).

- Cardboard size: 15 cm $\times$ 8 cm.
- Cut squares of side x from each corner.

(a) **Volume Formula:** Dimensions of the box will be $l = 15 - 2x$, $w = 8 - 2x$, $h = x$.

$$V(x) = x(15 - 2x)(8 - 2x)$$

$$V(x) = 4x^3 - 46x^2 + 120x$$

Domain: $0 \le x \le 4$ (determined by shortest side $8 - 2x \ge 0$).

(b) **Maximize Volume:**

$$\frac{dV}{dx} = 12x^2 - 92x + 120$$

Set to 0:

$$3x^2 - 23x + 30 = 0$$

$$(3x-5)(x-6)=0$$

Solutions: $x=\frac{5}{3}$ or $x=6$. Since $x\leq 4$, we discard $x=6$.

Check Concavity: $V''(x)=24x-92$. $V''(\frac{5}{3})=24(\frac{5}{3})-92=40-92=-52<0$ (Max).

Dimensions: $x=\frac{5}{3}$, Length $=15-\frac{10}{3}=\frac{35}{3}$, Width $=8-\frac{10}{3}=\frac{14}{3}$.

Max Volume:

$$V\left(\frac{5}{3}\right)=\frac{5}{3}\left(\frac{35}{3}\right)\left(\frac{14}{3}\right)=\boxed{\frac{2450}{27}\approx 90.74\text{ cm}^3}$$

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23. Optimization (Rectangle in Parabola).

- Parabola: $y=20-4x^2$.
- Rectangle base on x-axis, top corners on curve.

Step 1: Area Formula. Width $=2x$, Height $=y=20-4x^2$.

$$A(x)=2x(20-4x^2)=40x-8x^3$$

Step 2: Differentiate and Solve.

$$\frac{dA}{dx}=40-24x^2=0$$

$$24x^2=40\implies x^2=\frac{40}{24}=\frac{5}{3}\implies x=\sqrt{\frac{5}{3}}$$

Step 3: Calculate Max Area.

$$A\left(\sqrt{\frac{5}{3}}\right)=40\sqrt{\frac{5}{3}}-8\left(\frac{5}{3}\right)\sqrt{\frac{5}{3}}$$

$$=\left(40-\frac{40}{3}\right)\sqrt{\frac{5}{3}}=\frac{80}{3}\sqrt{\frac{5}{3}}=\boxed{\frac{80\sqrt{15}}{9}\approx 34.43\text{ m}^2}$$