Queen's University
Faculty of Engineering and Applied Science

MTHE 281 Notes

Introduction to Real Analysis and Differential Equations

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2.1 Introduction and Logistics

Welcome to MTHE 281! This course focuses on putting a formal basis to the calculus computations you have learned in the past. We will learn definitions and prove results rigorously.

Grading Scheme:

- Weekly homework (due Fridays, grace period until Wednesday).
- Two midterms (Week 5 and Week 10).
- Final Exam.

2.2 Construction of Natural Numbers ($\mathbb{N}$)

We begin by defining the natural numbers using set theory (Von Neumann construction). We define 0 as the empty set.

Definition 2.2.1: Natural Numbers (Inductive Definition)

We define the natural numbers inductively:

- $0 := \emptyset$
- Given n, we define its successor $n + 1 := n \cup \{n\}$.

Thus:

$$\begin{aligned} &0 = \emptyset \\ &1 = \{0\} = \{\emptyset\} \\ &2 = \{0, 1\} = \{\emptyset, \{\emptyset\}\} \\ &\vdots \end{aligned}$$

2.2.1 Arithmetic on $\mathbb{N}$

We define addition and multiplication recursively.

Definition 2.2.2: Addition

We define a map $a_k : \mathbb{N} \to \mathbb{N}$ (adding k) recursively:
- $a_k(0) = k$
- $a_k(n+1) = a_k(n) + 1$ (where $+1$ is the successor operation defined above).

We denote $a_k(n)$ as $n + k$.

Definition 2.2.3: Multiplication

We define a map $m_k : \mathbb{N} \to \mathbb{N}$ (multiplying by k) recursively:
- $m_k(0) = 0$
- $m_k(n+1) = m_k(n) + k$

We denote $m_k(n)$ as $n \cdot k$.

2.2.2 Ordering on $\mathbb{N}$

Definition 2.2.4: Order

For $j, k \in \mathbb{N}$, we say $j \leq k$ if and only if $j \subseteq k$ (as sets). We say $j < k$ if $j \subseteq k$ and $j \neq k$, or equivalently $j \in k$.

This is where we ended Week 1 Lecture 1

2.3 Construction of Integers ($\mathbb{Z}$)

The natural numbers lack additive inverses (negative numbers). We construct $\mathbb{Z}$ to solve the equation $n + x = m$.

Definition 2.3.1: Equivalence Relation on Pairs

Consider pairs $(j, k) \in \mathbb{N} \times \mathbb{N}$. We define an equivalence relation $\sim$:

$$(j_1, k_1) \sim (j_2, k_2) \iff j_1 + k_2 = j_2 + k_1$$

Definition 2.3.2: Integers

The set of integers $\mathbb{Z}$ is the set of equivalence classes of pairs in $\mathbb{N} \times \mathbb{N}$ under $\sim$.

$$\mathbb{Z} := (\mathbb{N} \times \mathbb{N}) / \sim$$

A generic integer is denoted $[(j, k)]$. Intuitively, $[(j, k)]$ represents the number "$j - k$".
- Positive integers $n \in \mathbb{N}$ correspond to $[(n, 0)]$.
- Negative integers $-n$ correspond to $[(0, n)]$.
- Zero corresponds to $[(0, 0)]$.

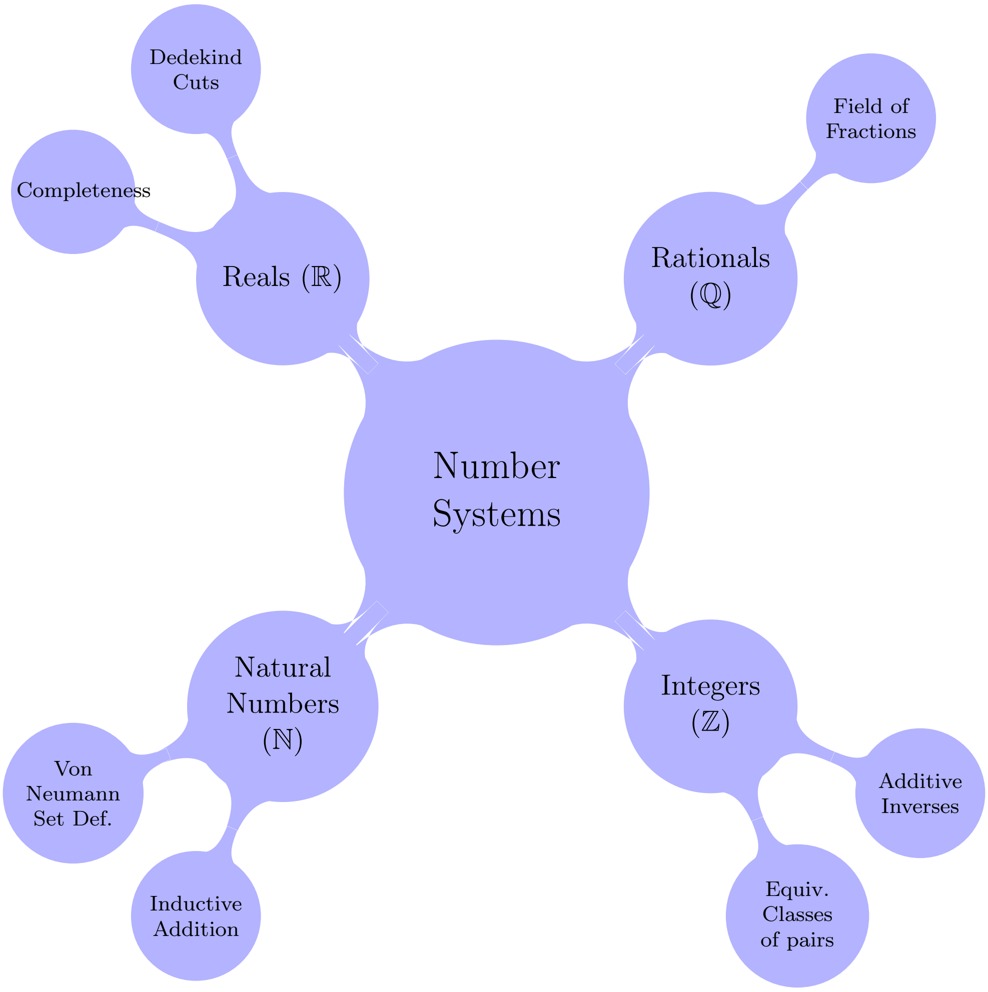

Figure 2.1. Hierarchy of Number Systems construction.

2.4 Construction of Rational Numbers ($\mathbb{Q}$)

The integers allow for subtraction, but not always division. We construct the rational numbers $\mathbb{Q}$ to solve equations of the form $bx = a$ where $b \neq 0$.

Definition 2.4.1: Equivalence Relation for Rationals

Consider the set of pairs $S = \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$. We define an equivalence relation $\sim$ on S:

$$(a, b) \sim (c, d) \iff ad = bc$$

This relation reflexivity, symmetry, and transitivity can be verified using integer arithmetic properties.

Definition 2.4.2: Rational Numbers

The set of rational numbers $\mathbb{Q}$ is the set of equivalence classes of S under $\sim$.

$$\mathbb{Q} := S/\sim$$

We denote the equivalence class $[(a, b)]$ as the fraction $\frac{a}{b}$.

Definition 2.4.3: Arithmetic on $\mathbb{Q}$

We define addition and multiplication as follows:

$$\frac{a}{b} + \frac{c}{d} := \frac{ad + bc}{bd}$$
$$\frac{a}{b} \cdot \frac{c}{d} := \frac{ac}{bd}$$

Using the properties of $\mathbb{Z}$, one can show these operations are well-defined (independent of the representative chosen) and satisfy the field axioms.

This is where we ended Week 1 Lecture 2