MTHE 212 Notes

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Chapter 1

Week 1

1.1 Lecture 1

1.1.1 Introduction to Vector Spaces

In mathematics, we often study families of objects and the rules governing their interactions. In this course, we focus on **linear structures**, specifically **Vector Spaces**.

A vector space essentially consists of a collection of elements (vectors) and two operations:

1. **Addition**: Combines two elements to produce a third.
2. **Scalar Multiplication**: Scales an element by a number (scalar).

Example.

Consider $\mathbb{R}^n$ (the set of column vectors with n real entries). Let $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \in \mathbb{R}^n$.

The standard operations are defined component-wise:

- Addition: $x + y = \begin{pmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{pmatrix}$.
- Scalar Multiplication: For $\alpha \in \mathbb{R}$, $\alpha x = \begin{pmatrix} \alpha x_1 \\ \vdots \\ \alpha x_n \end{pmatrix}$.

The operations in $\mathbb{R}^n$ satisfy 8 key properties:

1. Commutativity of addition: $x + y = y + x$.
2. Associativity of addition: $(x + y) + z = x + (y + z)$.
3. Additive identity: There exists $0 \in \mathbb{R}^n$ such that $x + 0 = x$.

4. Additive inverse: For every x, there exists $-x$ such that $x + (-x) = 0$.
5. Associativity of scalar multiplication: $\alpha(\beta x) = (\alpha\beta)x$.
6. Multiplicative identity: $1 \cdot x = x$.
7. Distributivity over vector sums: $\alpha(x + y) = \alpha x + \alpha y$.
8. Distributivity over scalar sums: $(\alpha + \beta)x = \alpha x + \beta x$.

1.1.2 Definition of a Vector Space

We abstract these properties to define a vector space generally.

Definition 1.1.1: Vector Space

A **vector space** is a set V equipped with two operations:

- Addition: $+ : V \times V \to V$
- Scalar Multiplication: $\cdot : F \times V \to V$ (where F is a field of scalars, usually $\mathbb{R}$ or $\mathbb{C}$)

satisfying the following axioms for all $u, v, w \in V$ and $\alpha, \beta \in F$:

1. $u + v = v + u$ (Commutativity)
2. $(u + v) + w = u + (v + w)$ (Associativity)
3. $\exists 0 \in V$ such that $u + 0 = u$ (Additive Identity)
4. $\forall u \in V, \exists - u \in V$ such that $u + (-u) = 0$ (Additive Inverse)
5. $1 \cdot u = u$ (Multiplicative Identity)
6. $\alpha(\beta u) = (\alpha\beta)u$ (Associativity of Scalar Mult.)
7. $\alpha(u + v) = \alpha u + \alpha v$ (Distributivity)
8. $(\alpha + \beta)u = \alpha u + \beta u$ (Distributivity)

The scalars come from a field F.

- If $F = \mathbb{R}$, we call V a **real vector space**.
- If $F = \mathbb{C}$, we call V a **complex vector space**.

1.1.3 Examples and Non-Examples

Example.

> Let $V = \{(x, 0) \in \mathbb{R}^2 \mid x \in \mathbb{R}\}$. This is the x-axis in $\mathbb{R}^2$. With standard component-wise operations, V is closed under addition and scalar multiplication, and inherits all axioms from $\mathbb{R}^2$. Thus, V is a vector space.

Example.

Let $V = \{(x, 1) \in \mathbb{R}^2 \mid x \in \mathbb{R}\}$. With standard operations:

- Addition: $(x, 1) + (y, 1) = (x + y, 2) \notin V$.
- Scalar Multiplication: $\alpha(x, 1) = (\alpha x, \alpha) \notin V$ (unless $\alpha = 1$).

Since V is not closed under these operations, it is **not** a vector space with standard operations. However, we could define *different* operations to make it a vector space (e.g., define addition such that the second component remains 1).

Example.

Consider the set of functions from a set S to a field F, denoted $F^S = \{f : S \to F\}$. Operations are defined pointwise:

- $(f + g)(x) = f(x) + g(x)$ for all $x \in S$.
- $(\alpha f)(x) = \alpha f(x)$ for all $x \in S$.

Under these operations, F^S is a vector space.

This is where we ended Week 1, Lecture 1

1.2 Lecture 2

1.2.1 Properties of Vector Spaces

We proved some basic propositions that follow directly from the axioms.

Proposition 1.2.1

For any $v \in V$ and scalar $a \in F$:

1. $a \cdot 0 = 0$ (where 0 is the zero vector).
2. $(-1) \cdot v = -v$ (the additive inverse of v).

Proof. 1. $a \cdot 0 = a \cdot (0 + 0) = a \cdot 0 + a \cdot 0$. Adding the additive inverse of $a \cdot 0$ to both sides gives $0 = a \cdot 0$.
2. We verify that $v + (-1)v = 0$. $v + (-1)v = 1 \cdot v + (-1)v = (1 + (-1))v = 0 \cdot v = 0$ (using the previous property $0 \cdot v = 0$ which is similarly proved). □

1.2.2 Subspaces

Often we find vector spaces inside other vector spaces.

Definition 1.2.2: Subspace

Let V be a vector space. A subset $U \subseteq V$ is a **subspace** if U is itself a vector space under the same operations as V, sharing the same zero vector.

To check if a subset is a subspace, we don't need to verify all 8 axioms. We can use the Subspace Test.

Proposition 1.2.3

Subspace Test

A subset $U \subseteq V$ is a subspace if and only if:

1. $0 \in U$ (Contains the zero vector).
2. U is closed under addition ($u, v \in U \implies u + v \in U$).
3. U is closed under scalar multiplication ($u \in U, \lambda \in F \implies \lambda u \in U$).

Example.

Let $U = \{(x_1, x_2, x_3, x_4) \in F^4 \mid 2x_1 - x_2 + x_4 = 0\}$.

- $0 \in U$ because $2(0) - 0 + 0 = 0$.
- Closed under addition: If $x, y \in U$, then $2(x_1 + y_1) - (x_2 + y_2) + (x_4 + y_4) = (2x_1 - x_2 + x_4) + (2y_1 - y_2 + y_4) = 0 + 0 = 0$. So $x + y \in U$.
- Closed under scalar multiplication (exercises).

Thus U is a subspace of F^4.

Example.

$C(\mathbb{R})$, the set of continuous real-valued functions, is a subspace of $\mathbb{R}^{\mathbb{R}}$ (all real functions). $D(\mathbb{R})$, the set of differentiable functions, is a subspace of $C(\mathbb{R})$.

Proposition 1.2.4

The intersection of any collection of subspaces is a subspace.

1.2.3 Sums of Subspaces

Definition 1.2.5: Sum of Subspaces

Let $V_1, \ldots, V_k$ be subspaces of V. The sum is defined as:

$$V_1 + \cdots + V_k = \{v_1 + \cdots + v_k \mid v_i \in V_i\}$$

Proposition 1.2.6

The sum $V_1 + \cdots + V_k$ is a subspace of V. Moreover, it is the **smallest** subspace containing all $V_1, \ldots, V_k$.

1.2.4 Direct Sums

Sometimes the representation of an element in a sum is not unique. If it is unique, we call it a direct sum.

Definition 1.2.7: Direct Sum

The sum $V_1 + \cdots + V_k$ is a **direct sum**, denoted $V_1 \oplus \cdots \oplus V_k$, if every element $v \in V_1 + \cdots + V_k$ can be written in a **unique** way as $v = v_1 + \cdots + v_k$ with $v_i \in V_i$.

Example.

Let $V_1 = \{(x, y, 0) \in \mathbb{F}^3\}$, $V_2 = \{(0, y, y) \in \mathbb{F}^3\}$, and $V_3 = \{(0, 0, z) \in \mathbb{F}^3\}$ be subspaces of $\mathbb{F}^3$.

Any $(x, y, z) \in \mathbb{F}^3$ can be decomposed as:

$$(x, y, z) = (x, 0, 0) + (0, y, y) + (0, 0, z - y)$$

so $V_1 + V_2 + V_3 = \mathbb{F}^3$.

However, the sum is **not direct**. Consider $(0, 1, 1)$:

$$(0, 1, 1) = (0, 0, 0) + (0, 1, 1) + (0, 0, 0)$$

$$(0, 1, 1) = (0, 1, 0) + (0, 0, 0) + (0, 0, 1)$$

Since the representation is not unique, the sum is not direct.

Proposition 1.2.8

The sum $V_1 + \cdots + V_k$ is direct if and only if the zero vector 0 has a unique representation:

$$0 = 0 + \cdots + 0$$

Proposition 1.2.9

For two subspaces U, W, the sum $U + W$ is direct if and only if $U \cap W = \{0\}$.

1.2.5 Span

Definition 1.2.10: Linear Combination

A linear combination of vectors $v_1, \ldots, v_k$ is a vector of the form $c_1 v_1 + \cdots + c_k v_k$ where $c_i \in F$.

Definition 1.2.11: Span

The span of $v_1, \ldots, v_k$, denoted $\text{span}(v_1, \ldots, v_k)$, is the set of all linear combinations of these vectors.

This is where we ended Week 1, Lecture 2