MATH/MTHE 281
Homework # 1

Date Due: **Friday Jan. 16 at 11:59pm**, No late penalties until Wednesday Jan. 21st at 11:59pm

E1.1: Let $[(j_1, k_1)], [(j_2, k_2)]$ be two elements of $\mathbb{Z}$. Show that addition

$$[(j_1, k_1)] + [(j_2, k_2)] = [(j_1 + j_2, k_1 + k_2)]$$

is well defined, i.e., prove that for any $(j_1', k_1') \in [(j_1, k_1)]$ and $(j_2', k_2') \in [(j_2, k_2)]$, we have that $(j_1' + j_2', k_1' + k_2')$ is equivalent to $(j_1 + j_2, k_1 + k_2)$.

E1.3: Show that the relations $<$ and $\leq$ on $\mathbb{Z}$ have the following properties:

1. $[(0, j)] < [(0, 0)]$ for all $j \in \mathbb{Z}_{>0}$;
2. $[(0, j)] < [(k, 0)]$ for all $j, k \in \mathbb{Z}_{>0}$;
5. $[(j, 0)] < [(k, 0)]$, $j, k \in \mathbb{Z}_{\geq 0}$, if and only if $j < k$;

E1.4: Show that a subset $A \subset \mathbb{Q}$ is bounded if and only it is has a lower bound and an upper bound.

E1.5: For each of the following subsets of $\mathbb{R}$ find the least upper bound and greatest lower bound if they exist. You do not have to justify.

(a) $\{1, 3\}$

(b) $[0, 4]$

(c) $\left\{\frac{1}{n} \;\middle|\; \in \mathbb{Z}_{>0}\right\}$

(d) $\left\{\frac{n}{n+1} \;\middle|\; n \in \mathbb{Z}_{>0}\right\}$

(e) $\left\{n + \frac{(-1)^n}{n} \;\middle|\; n \in \mathbb{Z}_{>0}\right\}$

(f) $\bigcap_{n=1}^{\infty} \left(1 - \frac{1}{n}, 1 + \frac{1}{n}\right)$

E1.6: Find the least upper bound and the greatest lower bound of the sets below. For any given $\epsilon > 0$ find a number in the set that is greater than $\sup A - \epsilon$ and a number in the set that is smaller than $\inf A + \epsilon$.

(a) $A = \left\{\frac{4+x}{x} \;\middle|\; x \geq 1\right\}$

(b) $A = \left\{\frac{\sqrt{x-1}}{x} \;\middle|\; x \geq 2\right\}$

(c) $A = \{x \mid x^2 - x < 6\}$

E1.9: Suppose A is a non-empty bounded subset of $\mathbb{R}$ and let $-A$ denote the set $\{-x \mid x \in A\}$. Show that $\sup(-A) = -\inf A$.

E1.11: Prove that if a subset S of $\mathbb{R}$ has a maximal element s (that is, there exists an $s \in S$ such that $x \leq s$ for all x in S) then $s = \sup S$.

E1.13: Suppose that A and B are two non-empty bounded subsets of $\mathbb{R}$ and let

$$A + B = \{x + y \mid x \in A \text{ and } y \in B\}.$$

Show that

$$\sup(A + B) = \sup A + \sup B.$$