# A dynamic model of a truck-trailer system with non-holonomically constrained motion

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## 1 Introduction

This brief document highlights the derivation of the equations of motion that describe the nonlinear non-holonomically constrained dynamics of a truck-trailer system. The purpose for creating this model is to test various path-tracking control algorithms in a realistic nonlinear setting. The model therefore only considers planar 2D motion and ignores 3D vibrations of the vehicle as these phenomena are not relevant for path-tracking. Additional dynamics may be considered in future iterations of the model when assessing other controllers such as those for preserving stability in hazardous scenarios.

## 2 Truck-trailer dynamics

### 2.1 System description

The entities of interest related to the truck-trailer system are shown in Fig. 1. The $x$ and $y$ axes represent the global frame of reference, while the $x_{\mathrm{A}}$ and $y_{\mathrm{A}}$ axes form a frame of reference that is fixed to the truck.

The centers-of-gravity (CG) of the truck and trailer are located at Points A and B. Other points of interest include the locations directly between each pair of wheels. Point 1 falls between the front wheels of the truck, and Point 2 between its rear wheels. Point 3 is located directly between the rear wheels of the trailer.

The angles of the truck's front wheels relative to the truck's body are denoted by the angle $\phi$. The angle of the trailer relative to the truck's body is represented by the angle $\gamma$. Other dimensions such as distances between points of interest are also evident in the figure.

### 2.2 Equations of motion

There are a total of eight unknown variables that must be solved for in order to simulate the motion of the truck trailer system. These include the following:

- $\mathbf{r}_{\mathrm{A}}$ – The position of the truck's CG measured in the global frame.
- $\theta$ – The angle of the truck's body measured counter-clockwise from the global $x$ axis.
- $\mathbf{v}_{\mathrm{A}}$ – The velocity of the truck's CG measured in its local frame.
- $\boldsymbol{\omega}_{\mathrm{A}}$ – The angular velocity of the truck.
- $\boldsymbol{\omega}_{\mathrm{B}}$ – The angular velocity of the trailer relative to the truck's body.
- $\lambda_1$ – The lateral force acting on the truck's front wheels.
- $\lambda_2$ – The lateral force acting on the truck's rear wheels.
- $\lambda_3$ – The lateral force acting on the trailer's rear wheels.

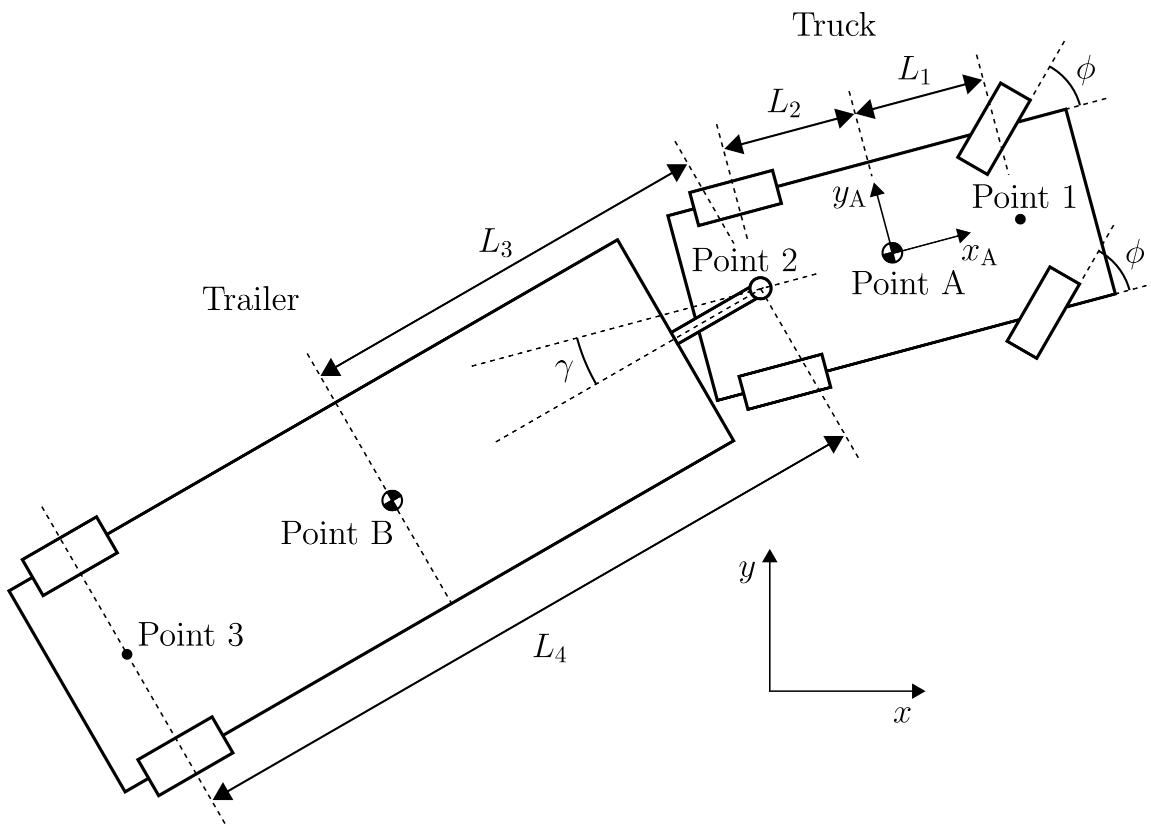


Figure 1: Schematic demonstrating the components, dimensions, reference frames, and points of interest relevant to the truck-trailer system.

To obtain these values, the following equations of motion must be solved:

$$\dot{\mathbf{r}}_{\mathrm{A}} = \mathbf{R}_{\theta}\mathbf{v}_{\mathrm{A}}, \tag{1}$$
$$\dot{\theta} = \omega_{\mathrm{A}}, \tag{2}$$
$$\mathbf{F}_{\mathrm{T}} + \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \mathbf{F}_{\mathrm{D}} = m_{\mathrm{A}}\mathbf{a}_{\mathrm{A}} + m_{\mathrm{B}}\mathbf{a}_{\mathrm{B}}, \tag{3}$$
$$\mathbf{r}_{2\to 1} \times (\mathbf{F}_{\mathrm{T}} + \mathbf{F}_1) - \mathbf{r}_{2\to \mathrm{A}} \times (m_{\mathrm{A}}\mathbf{a}_{\mathrm{A}}) = \mathbf{I}_{\mathrm{A}}\dot{\boldsymbol{\omega}}_{\mathrm{A}}, \tag{4}$$
$$\mathbf{r}_{2\to 3} \times \mathbf{F}_3 - \mathbf{r}_{2\to \mathrm{B}} \times (m_{\mathrm{B}}\mathbf{a}_{\mathrm{B}}) = \mathbf{I}_{\mathrm{B}}\left(\dot{\boldsymbol{\omega}}_{\mathrm{A}} + \dot{\boldsymbol{\omega}}_{\mathrm{B}}\right), \tag{5}$$
$$\mathbf{v}_1 \cdot \left(\mathbf{R}_{\frac{\pi}{2}}\mathbf{u}_{\phi}\right) = 0, \tag{6}$$
$$\mathbf{v}_2 \cdot \hat{j} = 0, \tag{7}$$
$$\mathbf{v}_3 \cdot \left(\mathbf{R}_{\frac{\pi}{2}}\mathbf{u}_{\gamma}\right) = 0, \tag{8}$$

Equations (1) and (2) relate the truck's position and angle measured in the global frame to its translational and angular velocities in the truck frame. Equation (3) applies a force balance to the truck-trailer system. Equations (4) and (5) apply moment balances to the truck and trailer around the contact joint. Gyroscopic terms are neglected since the truck and trailer only rotate along a 2D plane. Equations (6) to (8) represent the non-holonomic constraints that prevent truck and trailer tires from sliding along their local lateral directions.

The above expressions yield a system of differential algebraic equations (DAEs). It is possible to use a standard DAE solver to simulate the truck-trailer system; however, it is more accurate and less computationally intensive to eliminate the constraint equations through symbolic substitution. This approach replaces the system of DAEs with a system of ordinary differential equations (ODEs) that may be solved using any standard ODE solver in Python or MATLAB.

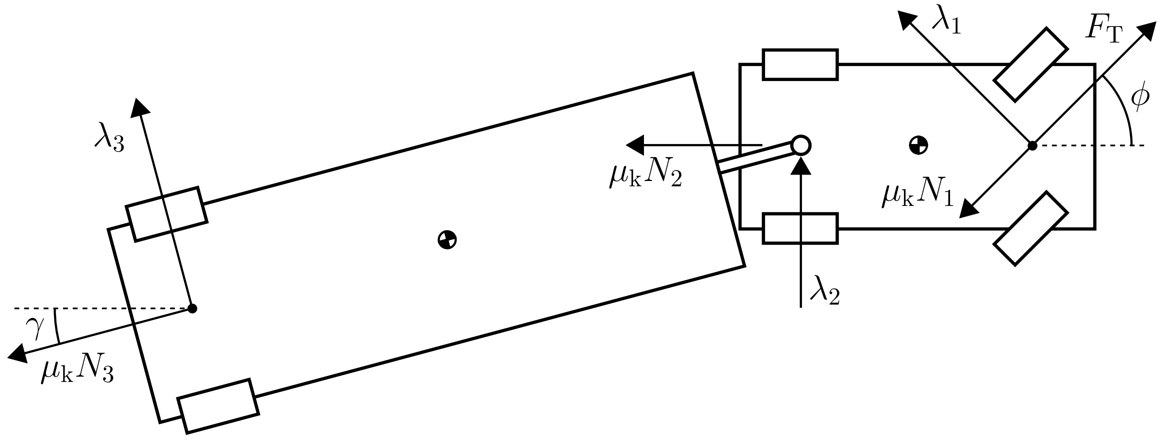


Figure 2: Schematic demonstrating the external forces acting on the truck-trailer system.

### 2.3 External forces

The external forces that act on the truck trailer system are shown in Fig. 2. The thrust force $F_{\mathrm{T}}$ acts along the longitudinal direction of the truck's front tires (*i.e.*, at the angle $\phi$), resulting in the following vector form:

$$\mathbf{F}_{\mathrm{T}} = F_{\mathrm{T}}\mathbf{u}_{\phi}. \tag{9}$$

The forces at the center-points of each tire pair comprise the friction forces acting along the longitudinal directions of the tires, and the lateral forces that prevent tire slip as follows:

$$\mathbf{F}_1 = -\mathrm{sgn}(\mathbf{v}_1 \cdot \mathbf{u}_{\phi})\mu_{\mathrm{k}} N_1 \mathbf{u}_{\phi} + \lambda_1 \mathbf{R}_{\frac{\pi}{2}}\mathbf{u}_{\phi}, \tag{10}$$

$$\mathbf{F}_2 = -\mathrm{sgn}(\mathbf{v}_2 \cdot \hat{\imath})\mu_{\mathrm{k}} N_2 \hat{\imath} + \lambda_2 \hat{\jmath}, \tag{11}$$

$$\mathbf{F}_3 = -\mathrm{sgn}(\mathbf{v}_3 \cdot \mathbf{u}_{\gamma})\mu_{\mathrm{k}} N_3 \mathbf{u}_{\gamma} + \lambda_3 \mathbf{R}_{\frac{\pi}{2}}\mathbf{u}_{\gamma}, \tag{12}$$

where $\mu_{\mathrm{k}}$ is the coefficient of rolling friction between the tires and the road, and the normal forces are defined as follows:

$$N_1 = \frac{L_2}{L_1 + L_2} m_{\mathrm{A}} g, \tag{13}$$

$$N_2 = \frac{L_1}{L_1 + L_2} m_{\mathrm{A}} g + \frac{L_4 - L_3}{L_4} m_{\mathrm{B}} g, \tag{14}$$

$$N_3 = \frac{L_3}{L_4} m_{\mathrm{B}} g, \tag{15}$$

where $m_{\mathrm{A}}$ and $m_{\mathrm{B}}$ denote the masses of the truck and trailer, respectively. The $\mathrm{sgn}(\cdot)$ function is used to regulate the direction of friction forces. To improve numerical stability, it is recommended to replace the $\mathrm{sgn}(\cdot)$ function with $\tanh(\cdot)$.

Finally, although not shown in Fig. 2, the aerodynamic drag force acting on the truck-trailer system is approximated as follows:

$$\mathbf{F}_{\mathrm{D}} = -\frac{1}{2} C_{\mathrm{D}} \rho A \left\| \mathbf{v}_{\mathrm{A}} \right\| \mathbf{v}_{\mathrm{A}}, \tag{16}$$

where $C_{\mathrm{D}}$ is the total drag coefficient of the truck-trailer system, $A$ is the reference drag area, and $\rho$ is the density of air. This approximation is only accurate when the truck and trailer are aligned, otherwise separate drag equations will be necessary for the truck and trailer. The approximation is valid since, if the truck is traveling fast enough for aerodynamic drag to be relevant, the trailer will be aligned with the truck. The two components are only misaligned during sharp turns; however the system will be moving too slowly for drag to be relevant when the trailer is not aligned with the truck.

## 2.4 Kinematics

The position and velocity vectors of points of interest within the truck-trailer system are described as follows:

$$\begin{aligned}
\mathbf{r}_{\mathrm{A}\to 1} &= \begin{bmatrix} L_1 & 0 & 0 \end{bmatrix}^{\mathrm{T}}, && (17)\\
\mathbf{r}_{\mathrm{A}\to 2} &= \begin{bmatrix} -L_2 & 0 & 0 \end{bmatrix}^{\mathrm{T}}, && (18)\\
\mathbf{r}_{2\to \mathrm{B}} &= \mathbf{R}_{\gamma}\begin{bmatrix} -L_3 & 0 & 0 \end{bmatrix}^{\mathrm{T}}, && (19)\\
\mathbf{r}_{2\to 3} &= \mathbf{R}_{\gamma}\begin{bmatrix} -L_4 & 0 & 0 \end{bmatrix}^{\mathrm{T}}. && (20)\\
\mathbf{v}_1 &= \mathbf{v}_{\mathrm{A}} + \boldsymbol{\omega}_{\mathrm{A}} \times \mathbf{r}_{\mathrm{A}\to 1}, && (21)\\
\mathbf{v}_2 &= \mathbf{v}_{\mathrm{A}} + \boldsymbol{\omega}_{\mathrm{A}} \times \mathbf{r}_{\mathrm{A}\to 2}, && (22)\\
\mathbf{v}_3 &= \mathbf{v}_2 + (\boldsymbol{\omega}_{\mathrm{A}} + \boldsymbol{\omega}_{\mathrm{B}}) \times \mathbf{r}_{2\to 3}. && (23)
\end{aligned}$$

Note that these vectors are defined in the reference frame that is fixed to the truck's CG.

The acceleration of the truck's CG measured in the local frame is defined as follows:

$$\mathbf{a}_{\mathrm{A}} = \dot{\mathbf{v}}_{\mathrm{A}} + \boldsymbol{\omega}_{\mathrm{A}} \times \mathbf{v}_{\mathrm{A}}. \tag{24}$$

The acceleration of the trailer's CG defined in the truck frame is a function of the acceleration of the truck and the rotation of the trailer relative to the truck as follows:

$$\begin{aligned}
\mathbf{a}_{\mathrm{B}} = \dot{\mathbf{v}}_{\mathrm{A}} + \boldsymbol{\omega}_{\mathrm{A}} \times \mathbf{v}_{\mathrm{A}} + \dot{\boldsymbol{\omega}}_{\mathrm{A}} \times \mathbf{r}_{\mathrm{A}\to 2} + \boldsymbol{\omega}_{\mathrm{A}} \times (\boldsymbol{\omega}_{\mathrm{A}} \times \mathbf{r}_{\mathrm{A}\to 2}) +& \\
(\dot{\boldsymbol{\omega}}_{\mathrm{A}} + \dot{\boldsymbol{\omega}}_{\mathrm{B}}) \times \mathbf{r}_{2\to \mathrm{B}} + (\boldsymbol{\omega}_{\mathrm{A}} + \boldsymbol{\omega}_{\mathrm{B}}) \times [(\boldsymbol{\omega}_{\mathrm{A}} + \boldsymbol{\omega}_{\mathrm{B}}) \times \mathbf{r}_{2\to \mathrm{B}}]&. \quad (25)
\end{aligned}$$

Finally, the remaining unit vectors and rotation matrices used in earlier equations are defined as follows:

$$\begin{aligned}
\mathbf{u}_{\phi} &= \begin{bmatrix} \cos\phi & \sin\phi & 0 \end{bmatrix}^{\mathrm{T}}, && (26)\\
\mathbf{u}_{\gamma} &= \begin{bmatrix} \cos\gamma & \sin\gamma & 0 \end{bmatrix}^{\mathrm{T}}, && (27)\\
\mathbf{R}_{\frac{\pi}{2}} &= \begin{bmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} & 0 \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}, && (28)\\
\mathbf{R}_{\gamma} &= \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}, && (29)\\
\mathbf{R}_{\theta} &= \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}, && (30)
\end{aligned}$$

## 2.5 Wheel dynamics

The parameters of interest when modelling the lateral rotation of the truck's front wheels are shown in Fig. 3. The torque delivered at the steering wheel is $T_{\psi}$, and this torque is increased by the steering system's gear ratio $N$. Due to the caster angle of the wheel lateral rotation axle, there exists an imaginary point a distance $\Delta L$ from the wheel center. The lateral force imparted onto the wheel from the ground generates a moment due to $\Delta L$ that restores the wheel to an orientation of $\phi = 0\,\mathrm{deg}$.

The resulting equations of motion describing the lateral rotation of the truck's front wheels are as follows:

$$\begin{aligned}
\dot{\phi} &= \omega_{\phi}, && (31)\\
\dot{\omega}_{\phi} &= \frac{1}{J_{\phi}}\left(NT_{\psi} - \lambda_1 \Delta L \cos\phi - b\omega_{\phi}\right), && (32)
\end{aligned}$$

where $J_{\phi}$ is the moment of inertia of the steering wheel system, $b$ is a linear damping coefficient, and $\omega_{\phi}$ is the lateral angular velocity of the truck's front wheels.

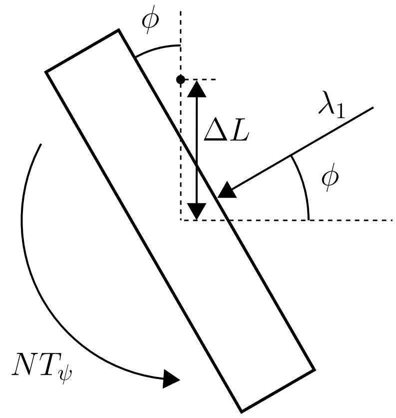


Figure 3: Schematic demonstrating the loads and dimensions relevant to modelling lateral rotations of the truck's front wheels.

# 3 Preliminary validation

To perform a preliminary validation of the truck-trailer simulator, the trajectory of the system is first assessed. A simulation is run with a truck thrust force of $F_\mathrm{T} = 1000\,\mathrm{N}$ and a steering wheel torque of $T_\psi = 1\,\mathrm{N}\cdot\mathrm{m}$. The position of the truck-trailer system at different instances in time is shown in Fig. 4. The first property to assess is that the positive steering wheel torque causes the truck to veer in the counter-clockwise direction. Another important observation is that the truck's body is always parallel to its path, which validates the non-holonomic constraints on its motion.

Additionally, the ability of the simulator to model steering wheel returnability is verified. Figure 5 plots various dynamic properties over a 300 sec simulation. One of these properties is the steering wheel angle. It is observed that, as the truck's velocity increases, which raises the centripetal forces acting laterally to the front wheels, the offset resulting from the caster angle causes the steering wheel angle to approach zero.

The longitudinal velocity of the truck also behaves as expected. Long-term, the longitudinal velocity approaches a steady value as the truck's thrust force balances with aerodynamic drag and friction forces. The truck's lateral velocity peaks at the start of the simulation when the steering wheel angle is large, and then approaches zero as centripetal forces cause the steering wheel angle to decrease. Finally, the angle of the trailer relative to the truck increases in magnitude at the start of the simulation since the truck is rotating but the non-holonomic constraints on the trailer's rear wheel are preventing it from rotating along with the truck.

Future steps for enhancing the model include the following:

- Obtaining data from a real truck-trailer system to tune the various simulation parameters.
- Adding 3D vibration physics to the equations of motion.
- Creating an unconstrained version of the model which relies on empirical tire forces to accelerate the vehicles. This model would enable simulating the truck-trailer system during periods of tire slip.
- Simulating aerodynamic drag separately on the truck and trailer to capture the impact of strong winds on trailer vibrational stability.

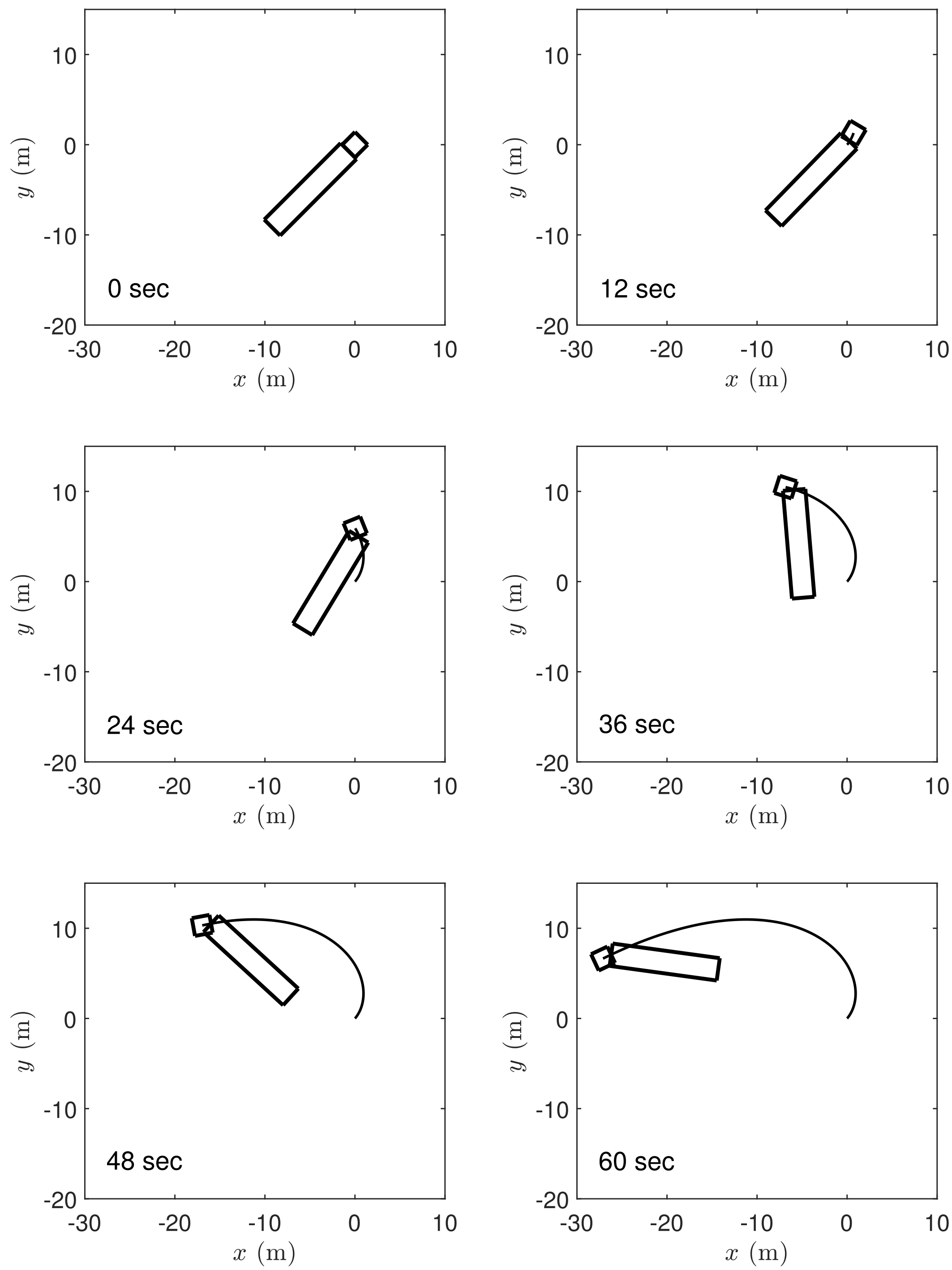


Figure 4: Location of the truck-trailer system at different instances in time. The solid black line traces the past trajectory of the truck's CG.

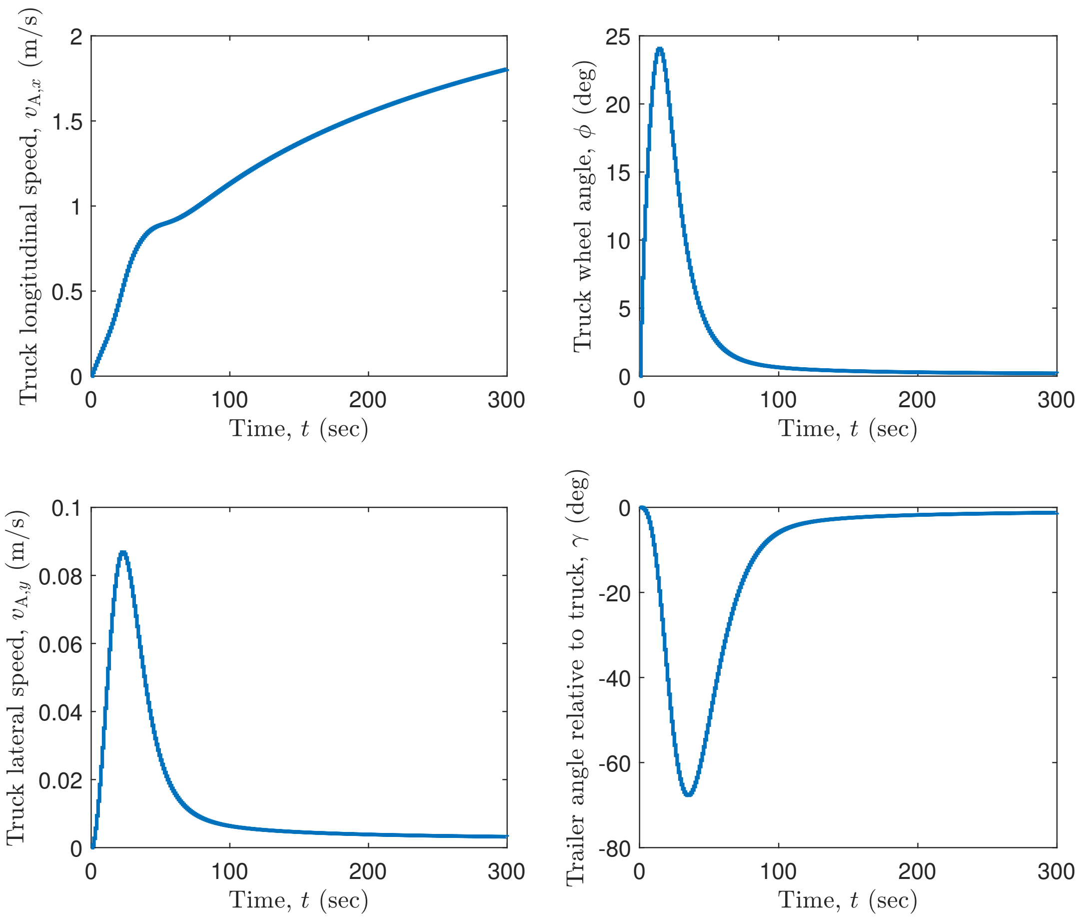


Figure 5: Evolution of various truck-trailer properties over time.