A Higher-Dimensional Eckmann–Hilton Argument

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Plan

Aim: Describe the 3-fold generalisation of the Eckmann–Hilton argument for 3-degenerate 4-categories...

...with a mind to develop this in generality for $(n-1)$-degenerate n-categories.

Section n: degenerate n-categories $(1 \leq n \leq 4)$

The Concept of Degeneracy

category		monoid
0-cells$\}$ trivial		-
1-cells	$\longrightarrow$	elements
composition	$\longrightarrow$	multiplication
identity	$\longrightarrow$	identity

Idea: In a k-degenerate n-category, the bottom k dimensions are trivial.

n-category		k-d-n-category
0-cells		-
1-cells		-
$\vdots$	$\}$ trivial	$\vdots$
$(k-1)$-cells		-
k-cells	$\longrightarrow$	0-cells
$\vdots$	$\vdots$	$\vdots$
$(n-1)$-cells	$\longrightarrow$	$(n-k-1)$-cells
n-cells	$\longrightarrow$	$(n-k)$-cells

Idea: A k-degenerate n-category 'is' an $(n-k)$-category with extra structure.

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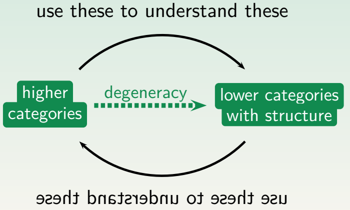
A Question of Totalities



Objects	Totality
monoids	category
categories	2-category
monoidal categories	2-category
2-categories	3-category

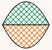
Idea: **d-Cat** is a full sub-2-category of **Cat**. We can 'truncate' **d-Cat** to a category: discard the natural transformations.

Problem: **d-2-Cat** is a full sub-3-category of **2-Cat**. We can't 'truncate' **d-2-Cat** to a 2-category: this is fixed using icons.

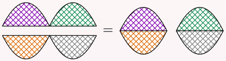
1-degenerate 2-categories

2-category		monoidal category
0-cells } trivial		-
1-cells	$\longrightarrow$	objects
2-cells	$\longrightarrow$	morphisms
hor. composites	$\longrightarrow$	$\otimes$

vert. composites	$\longrightarrow$	composition



identity	$\longrightarrow$	unit object
interchange	$\longrightarrow$	interchange



$$(a * b) \circ (c * d) = (a \circ c) * (b \circ d)$$

A Question of Totalities

use these to understand these

higher categories $\xrightarrow{\text{degeneracy}}$ lower categories with structure

use these to understand these

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categories	2-category
monoidal categories	2-category
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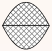

identity	$\longrightarrow$	unit object
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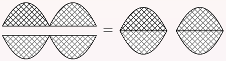


$$(a * b) \circ (c * d) = (a \circ c) * (b \circ d)$$

2-degenerate 2-categories

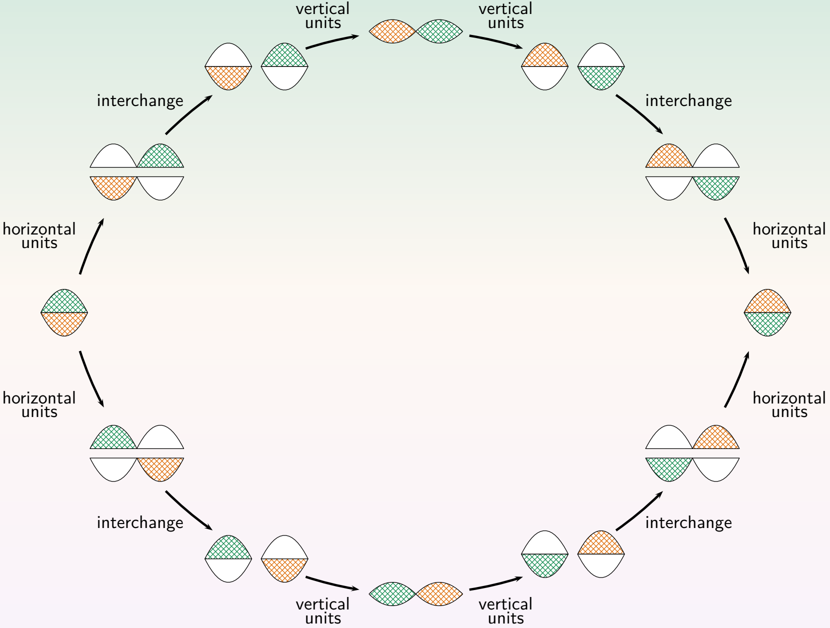
2-category		comm. monoid
0-cells } trivial		-
1-cells		-
2-cells	$\longrightarrow$	elements
hor. composites	$\longrightarrow$	multiplication $*$
vert. composites	$\longrightarrow$	multiplication $\circ$
identity	$\longrightarrow$	identity
interchange	$\longrightarrow$	interchange

$$(a * b) \circ (c * d) = (a \circ c) * (b \circ d)$$

Idea: $\circ$ and $*$ are the same and commutative.
This is the Eckmann–Hilton argument:

$$\begin{aligned} a \circ b &= (1 * a) \circ (b * 1) \\ &= (1 \circ b) * (a \circ 1) \\ &= b * a \\ &= (b \circ 1) * (1 \circ a) \\ &= (b * 1) \circ (1 * a) \\ &= b \circ a \end{aligned}$$

Eckmann–Hilton Argument



The Periodic Table of n-Categories

set	category	2-category	3-category	4-category
monoid	monoidal category	?	?	
commutative monoid	braided monoidal category	?		
commutative monoid	symmetric monoidal category			
commutative monoid				

This is what we were interested in: 2-degenerate 3-categories.

We want to repeat the story for 3-degenerate 4-categories.

1-degenerate 3-categories

monoidal 2-categories

3-degenerate 3-categories

commutative monoids

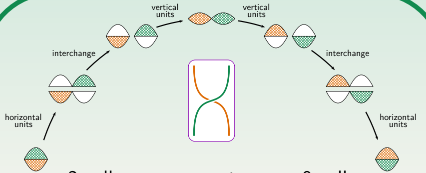
1-degenerate 3-categories

monoidal 2-categories

3-degenerate 3-categories

commutative monoids

2-degenerate 3-categories

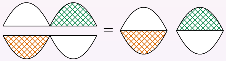


2-cells	$\longrightarrow$	0-cells
3-cells	$\longrightarrow$	1-cells
composites	$\longrightarrow$	$\otimes$s
int E–H ge	$\longrightarrow$	braiding

braided monoidal categories

Coherence in 3 flavours

	vertical composition	horizontal units	interchange
GPS	strict	strict	weak
JK	strict	weak	strict
CC	weak	strict	strict



3-degenerate 4-categories

symmetric monoidal categories

semi-strict 4-categories

composition of 1-cells
composition of 2-cells
interchange
} **strict**

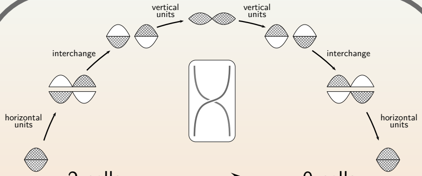
composition of 3-cells

0-composition	weak
1-composition	strict
2-composition	weak

Mix of strict and weak enrichment:

- Vertically weak tricategories: **Bicat$_s$-Cat**.
- Semi-strict 4-categories: **Bicat$_s$-Cat-wCat**.
  - Use iterated icons, 2-monads, iterated distributive laws on **Cat-Gph-Gph**.
- Triply-degenerate: we can produce a symmetric monoidal category.
- Future: Obtain full coherence result and comparison of totalities like before.

2-degenerate 3-categories



2-cells	⟶	0-cells
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braided monoidal categories

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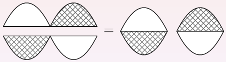
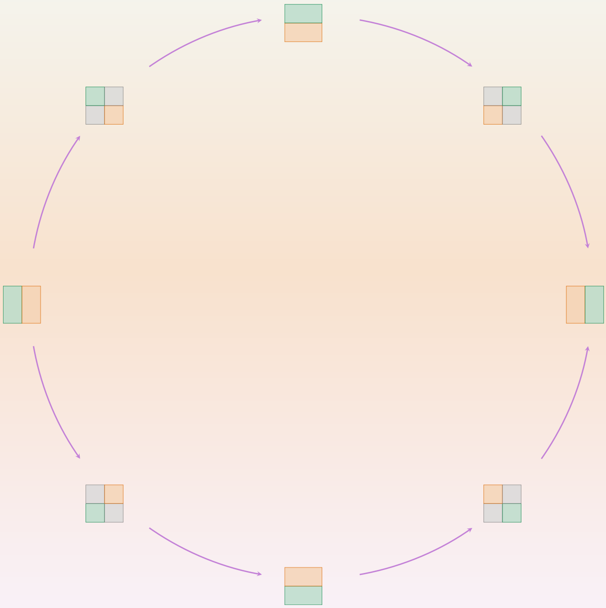


The Eckmann–Hilton Sphere



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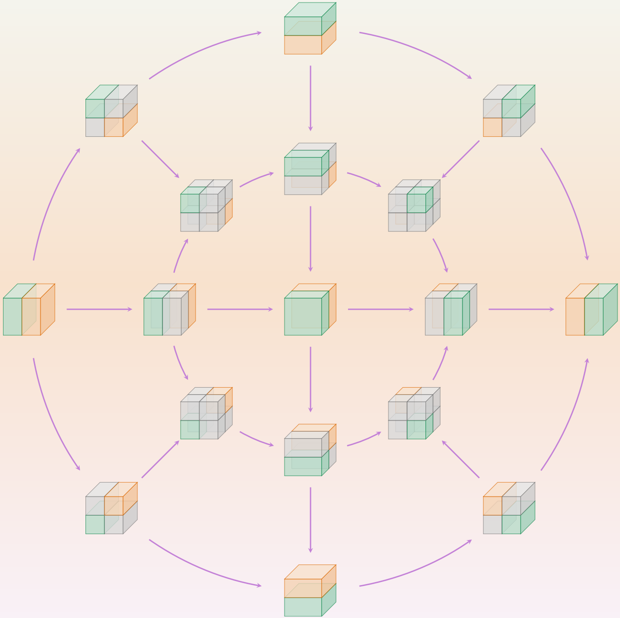

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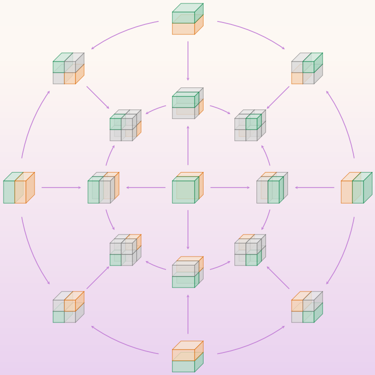
monoid	monoidal category	monoidal 2-category	monoidal 3-category	monoidal 4-category
commutative monoid	braided monoidal category	braided monoidal 2-category	braided monoidal 3-category	braided monoidal 4-category
commutative monoid	symmetric monoidal category	sylleptic monoidal 2-category	sylleptic monoidal 3-category	sylleptic monoidal 4-category
commutative monoid	symmetric monoidal category	symmetric monoidal 2-category	...	...

Eckmann–Hilton and Monoidal ∞-Categories?

Summary

We have a heirarchy of results in progress demonstrating that all of the following produce symmetric monoidal categories:

- 3-tuply monoidal categories: two weak, one strict, strict interchanges
- 3-degenerate 4-categories: produced using iconic constructions
- $(n-1)$-degenerate n-categories: produced using iconic constructions



Thank you!