Adjacency and Reachability

There are a number of different matrices associated with a graph or a digraph. The ones we will look at rely on some definitions we have seen previously, but we will repeat here. We will use the terminology of digraphs but this can easily be translated into similar definitions for graphs.

If there is an arc from node i to node j then we say that node j is **adjacent** to node i. A node y is **reachable** from a node x if there is a path from x to y. Any node is reachable from itself by a path of length zero.

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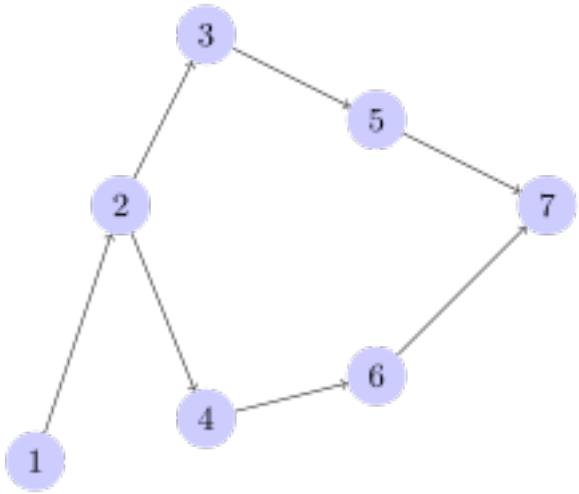
`https://www.youtube.com/embed/g5rFg9q8QqU`

Adjacency Matrices

The **adjacency matrix** of a graph is a square matrix, usually called A, with one row and column for each node. The entry in the ij-th position (row i, column j) is 1 if there is an arc from node i to node j and is otherwise 0. (More generally, if we allow several arcs from i to j, the ij-th entry is the number of such arcs.)



We described which vertices in this graph were adjacent to which others in the previous chapter, which produced a list. However, an adjacency matrix can

capture this information in a much more succint and readily usable way. The adjacency matrix of the above graph is as follows.

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The adjacency matrix specifies the graph or digraph fully, bearing in mind that the positions of the nodes are not part of its structure. It can often be useful for people to see the structure of a graph from a suitable diagram (as long as it's not too big), but the adjacency matrix is much better for manipulation in a computer program.

For an undirected graph, an edge between vertices i and j gives a 1 in both the ij- and ji-entries. So the adjacency matrix for a graph is **symmetric**, i.e., is its own transpose.

If we multiply A by itself, using matrix multiplication, we get a matrix which tells us how many walks there are of length 2 from any node to any other node. This matrix is, of course, written A^2. For example, using the matrix A for the digraph as given above, we get the following matrix.

$$A^2 = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

With such a small diagram we can verify that A^2 does indeed show how many walks there of length 2. For example, the first row shows that there is a single walk of length 2 from node 1 to node 3 and similarly from node 1 to node 4.

We can repeat this process, obtaining A^3, A^4, A^5 and so on, with each A^n describing the number of walks of length n. The identity matrix I gives the number of walks of length 0. Then the matrix

$$I + A + A^2 + A^3 + \ldots + A^r$$

gives the number of walks of length at most r.

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`https://www.youtube.com/embed/RcV0II7LK0A`

Reachability Matrices

What we have described above allows us to generate another matrix, called the **reachability matrix**. In this matrix the entry in the ij-th position is 1 if there

is a walk from node i to node j and is 0 otherwise. In other words, the ij-th entry is 1 if node j is reachable from node i. Note that this matrix doesn't count the *number* of walks between nodes, just the *existence* of walks.

To save ourselves from unnecessarily heavy arithmetic we may use simpler arithmetical systems. One such system is called **Boolean arithmetic**. In this we count 0, 1, 1, 1, 1, ... In other words, any number bigger than 0 is called 1. This is because if there are several walks between a pair of nodes then we often do not care exactly how many there are.

In Boolean arithmetic, multiplication is the same as in normal arithmetic: $0 \times 0 = 0$, $0 \times 1 = 0$, $1 \times 0 = 0$, and $1 \times 1 = 1$. Addition is the same as normal except that $1 + 1 = 1$. There is no subtraction or division.

Using Boolean arithmetic we get a formula for the reachability matrix, usually called R. If there are n nodes in the digraph and A is the corresponding adjacency matrix, then the reachability matrix is given by

$$R = I + A + A^2 + A^3 + \ldots + A^{n-1}.$$

This is because the longest path has length $n - 1$.

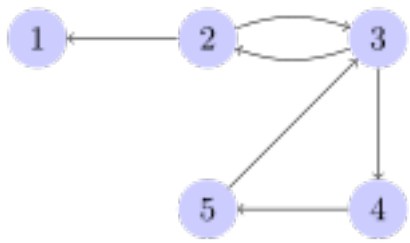
Another formula which gives the same answer, in Boolean arithmetic but not in ordinary arithmetic, is

$$R = (I + A)^{n-1}.$$

This can be quicker to work out, as we can repeatedly square the matrix, thereby going up in powers of 2, instead of one step at a time. In fact, if we work out $(I + A)^k$ (using Boolean arithmetic) for any integer $k > n - 1$ then that will give the same as $(I + A)^{n-1}$.

Video Visit the URL below to view a video:
`https://www.youtube.com/embed/LZV5I2dMvTc`

Examples



The digraph above has adjacency matrix

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

There are 5 nodes, so $n = 5$ and the reachability matrix will be given by $R = (I + A)^{n-1} = (I + A)^4$, in Boolean arithmetic.

$$I + A \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

$$(I + A)^2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$

$$(I+A)^4 = (I+A)^2(I+A)^2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

This last matrix is the reachability matrix. You should also be able to write down the reachability matrix straight from the diagram and see that it is the same as the one we calculated.