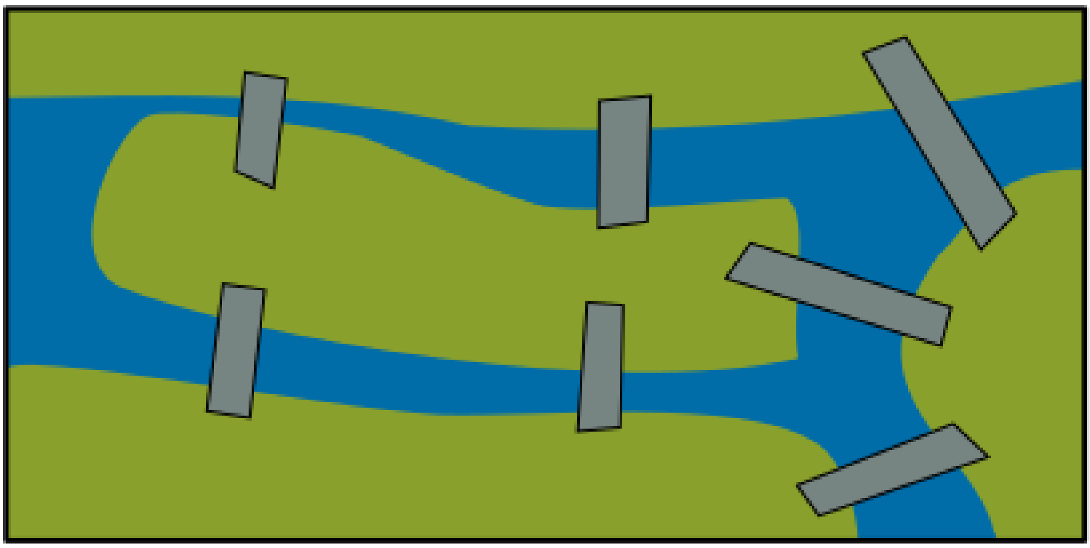
Chapter 1

*Eulerian Graphs

The Seven Bridges of Königsberg

This section starts with an old puzzle. Königsberg was a city in Prussia, through which ran the River Pregel. (It's still there, but it's now called Kaliningrad and is in Russia. And the bridges are a bit different now.) There were seven bridges in the center of the city which connected to two islands in the river. A crude diagram showing the layout of the bridges is drawn in the diagram below.



The problem was then as follows:

- Is it possible to walk round this area crossing each bridge exactly once?

This was often phrased in a manner of people going on their Sunday walk.

Leonhard Euler came up with a solution to this problem, which effectively kicked off the whole area of mathematics that we now call Graph Theory. For a lively narrative of the history of this problem, you can watch the Numberphile video below. The next section will describe Euler's results formally. (Note that

Euler is pronounced 'oy-ler' and not 'yoo-ler'. Knowing this will stop smart-ass mathematicians from trying to embarass you.)

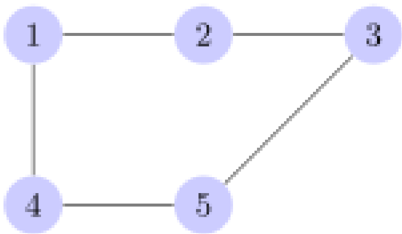
Video Visit the URL below to view a video:
`https://www.youtube.com/embed/W18FDEA1jRQ`

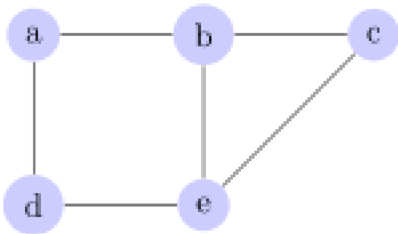
Eulerian Graphs

An **Eulerian trail** is a closed trail which includes each edge and each vertex of a graph. A graph is **Eulerian** if it has an Eulerian trail. A trail has no repeated edges. So an Eulerian graph has a walk that includes each edge *exactly once*, and finishes at its starting point. It may include vertices more than once.

A graph is **semi-Eulerian** if there is a trail which traverses every edge and vertex, but with different starting and ending vertices. A graph is **non-Eulerian** if it is neither Eulerian nor semi-Eulerian.

The graph on the left below is Eulerian: $1 \to 2 \to 3 \to 5 \to 4 \to 1$. The graph on the right below is semi-Eulerian: $e \to c \to b \to e \to d \to a \to b$.

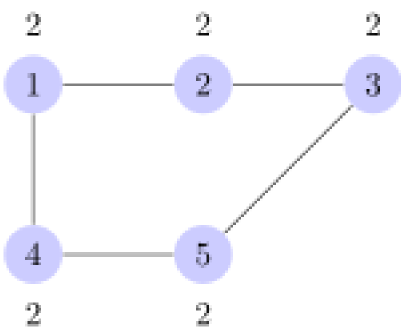




One way to show that a graph is Eulerian or semi-Eulerian is to show that it contains an appropriate trail. However, this can be difficult if the graph has many edges or is otherwise complicated. It can also be quite difficult to show that a graph is non-Eulerian. However, we have some results which will help us in determining these properties.

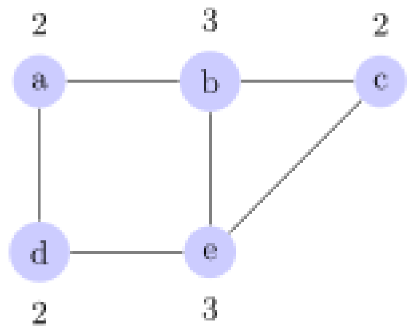
Theorem 1. A graph is Eulerian if and only if it is connected and every vertex has even degree.

We can see that this applies to our first example above. The vertices of the graph now display their degrees. Since all of them have degree 2, which is even, then this graph is Eulerian.



Theorem 2. A graph is semi-Eulerian if and only if it is connected and exactly two vertices have odd degree. A trail including all edges must start at one odd vertex and finish at the other.

We can use this result to look at our second example. Again, the degrees are listed next to the vertices. Since there are precisely two vertices with odd degree (b and e), then this graph must be semi-Eulerian.

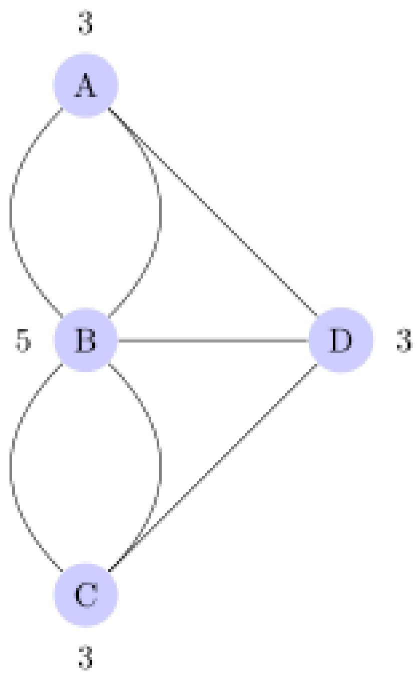


Proofs of both of these theorems can be found in most introductory graph theory textbooks.

Video Visit the URL below to view a video:
`https://www.youtube.com/embed/ZihDCF5siiY`

Bridges Revisited

We can use the theorems above to decide if there is a solution to the original puzzle about the bridges of Königsberg (this is also covered in the video above). We will denote each of the landmasses by a vertex. The top bank will be A, the small island will be B, the bottom bank will be C, and the right landmass will be D. We can then join vertices when there is a bridge connecting them. Doing this we obtain the following graph.



The degrees of the vertices are listed on the graph. We can see that all of them are odd. This means that neither of the theorems apply and so this graph must be non-Eulerian. In terms of the bridges problem, this means there is no route that passes over each bridge exactly once and returns to the starting point. And there is also no route which passes over each bridge exactly once but ends up somewhere different to where it started.