Operads, Equivariance, and Pseudo-Commutativity

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Slides: alex-corner.github.io/slides/corner-pssl110.pdf

Operads

Data $n \in \mathbb{N} = \{0, 1, 2, \ldots\}$

- sets $P(n)$ - thought of as 'n-ary operations'
- functions - 'composition' or 'multiplication'

$$\mu^P : P(n) \times P(k_1) \times \ldots \times P(k_n) \to P(k_1 + \ldots + k_n)$$

- identity element $e \in P(1)$

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Axioms

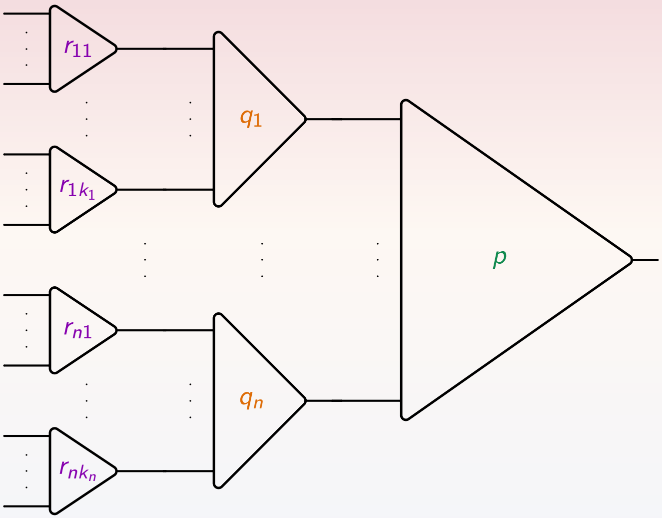
- multiplication:

$$\mu(\mu(p; q_1, \ldots, q_n); r_{11}, \ldots, r_{1k_1}, \ldots, r_{n1}, \ldots, r_{nk_n})$$
$$=$$
$$\mu(p; \mu(q_1; r_{11}, \ldots, r_{1k_1}), \ldots, \mu(q_n; r_{n1}, \ldots, r_{nk_n}))$$

- identity:

$$\mu(p; e, \ldots, e) = p = \mu(e; p)$$

Composition Axiom



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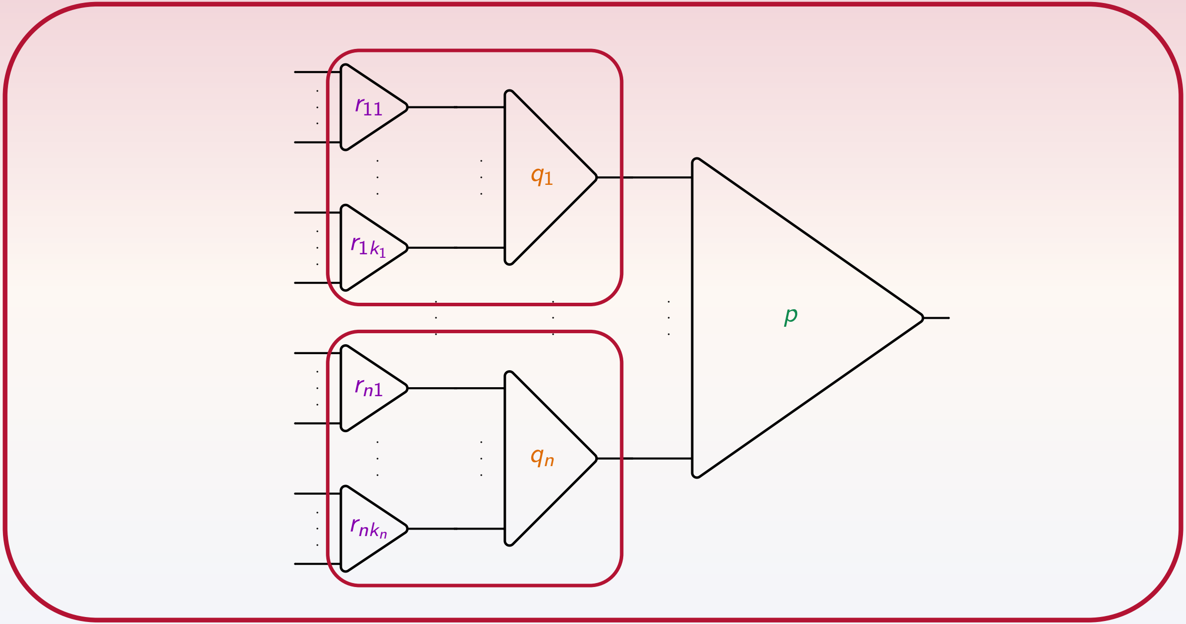

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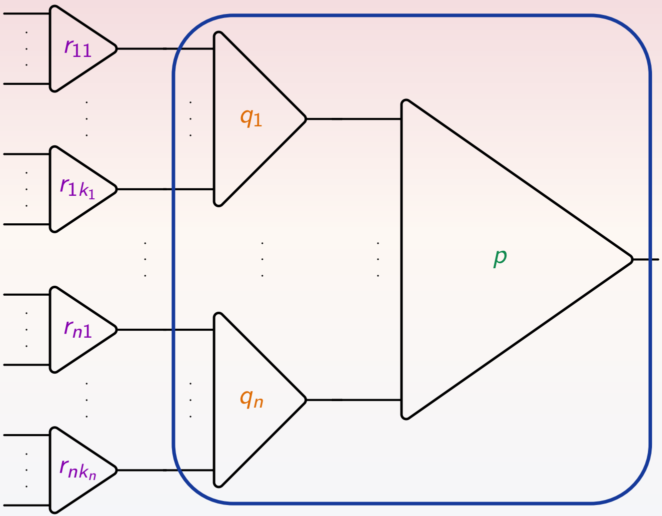

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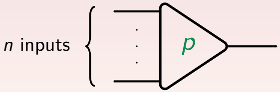

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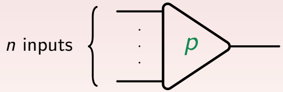

Symmetric Operads

- for each $n \in \mathbb{N}$, an action:

$$P(n) \times \Sigma_n \to P(n)$$

- satisfying:

$$\mu(p \cdot g; q_1, \ldots, q_n) = \mu(p; q_{g^{-1}(1)}, \ldots, q_{g^{-1}(n)}) \cdot \delta(g)$$

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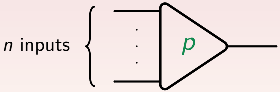

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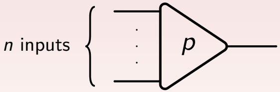

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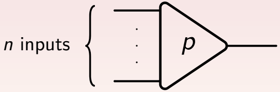

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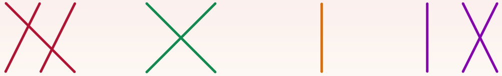

Σ_6

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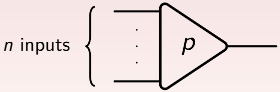

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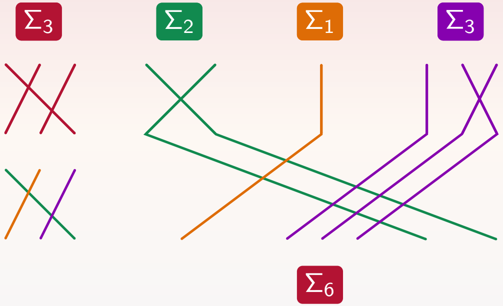

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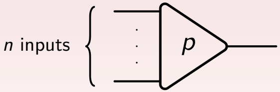

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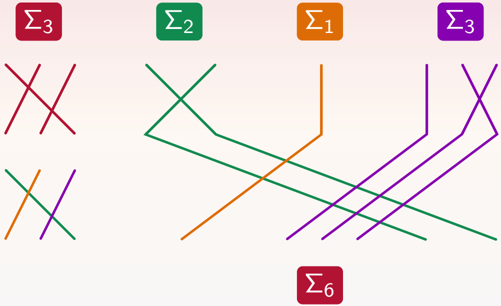

Block Sum $\beta(p_1, \ldots, p_n) = \mu(e_n; p_1, \ldots, p_n)$

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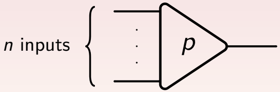

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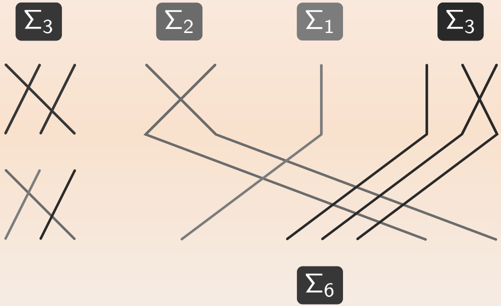

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Action Operads

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- **groups** $\Lambda(n)$
- plain operad structure (μ, e)
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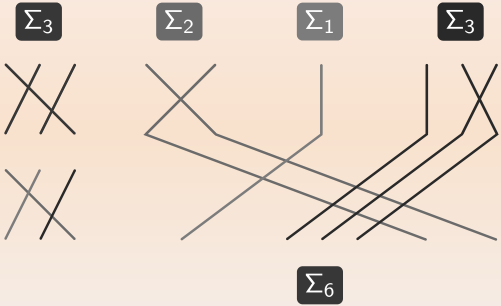

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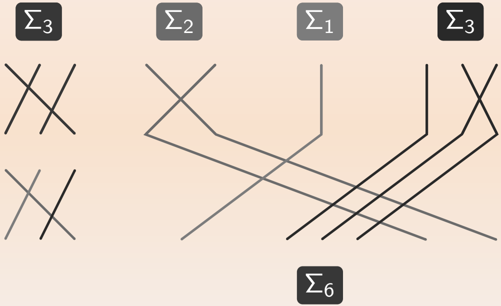

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- $\Lambda(n) = \Sigma_n$
- operad structure: β and δ, $e_1 = (1) \in \Sigma_1$
- π_n is the identity
- equivariance follows from symmetric equivariance

Action Operads - Examples

Examples

- Σ: Σ_n, symmetric groups
- T: 1, trivial groups
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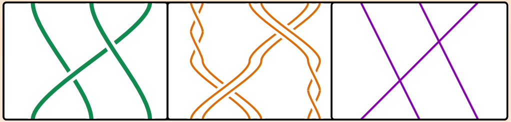

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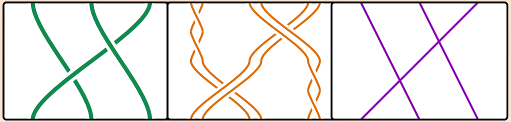

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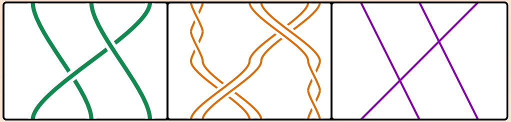

Non-Examples

- A: A_n, alternating groups
- H: H_n, hyperoctahedral groups
- C: C_n, cyclic groups
- R: C_2, π_n order-reversing
- Most skew-simplicial groups, despite similarities. — Fiedorowicz–Loday

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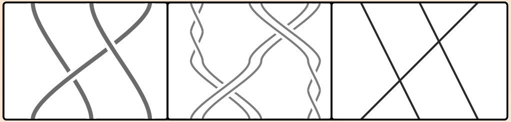

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- an operad P
- actions $P(n) \times \Lambda(n) \to P(n)$
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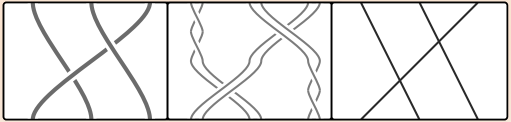

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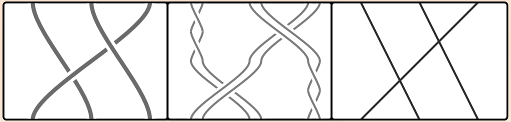

Non-Examples

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- Most skew-simplicial groups, despite similarities.

Λ-Operads

Data

- an action operad Λ
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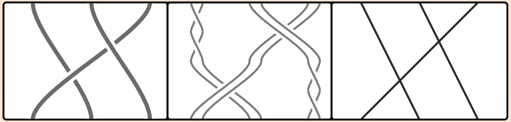

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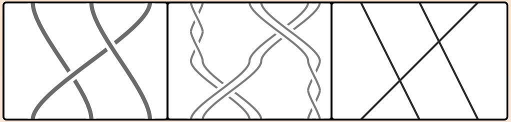

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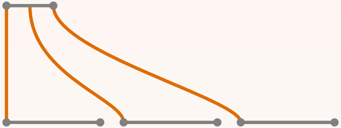

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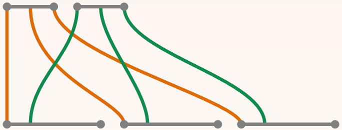

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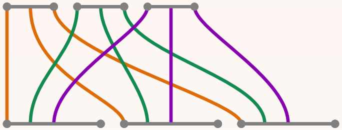

Pseudo-Commutative 2-Monads

- $T : \mathcal{C} \to \mathcal{C}$ a 2-monad
- equipped with strength and costrength (2d, strict)
- equipped with invertible modification:

Hyland–Power

$$\begin{tikzcd}
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- T is **pseudo-commutative** if

lots of axioms

- T ps-commutative $\Rightarrow$ T-$\mathbf{Alg}_{ps}$ is pseudo-closed
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- can characterise Λ-operads with ps-commutativity
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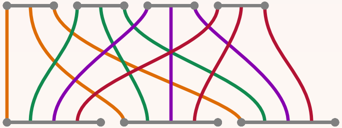

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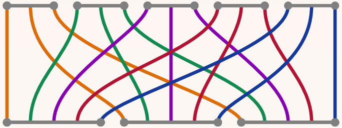

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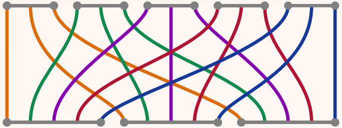

- these permutations correspond to matrix transpose

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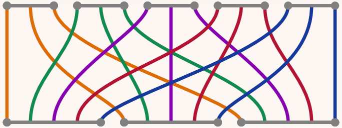

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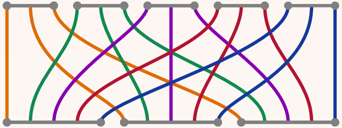

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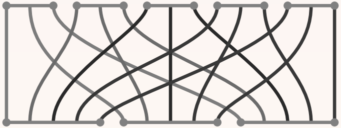

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Examples/Non-Examples

Examples

Guillou–May–Merling–Osorno

- Σ: symmetric strict monoidal categories
- B: braided strict monoidal categories

C–G

Non-Examples

- J: cactus/coboundary strict monoidal categories

C–G

- arXiv:1508.04050 – Operads, tensor products, and the categorical Borel construction
- arXiv:1312.5910 – Operads with general groups of equivariance, and some 2-categorical aspects of operads in **Cat**