

# Exercises: Matrices

## Exercises 1

1. This question uses the following matrices. Also,  $I$  is used to denote an identity matrix of any suitable size.

$$A = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} \quad B = \begin{pmatrix} 0 & -2 \\ 3 & 1 \end{pmatrix} \quad C = \begin{pmatrix} 1 & -3 \\ -2 & 6 \end{pmatrix}$$

$$D = \begin{pmatrix} 1 & -1 & 2 \\ 8 & 3 & 2 \\ 1 & 0 & 1 \end{pmatrix} \quad E = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad F = \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix} \quad G = \begin{pmatrix} 1 & 0 \\ 2 & -6 \\ 1 & 2 \end{pmatrix}$$

Calculate the following, when they exist.

- |               |               |              |
|---------------|---------------|--------------|
| (a) $A + B$   | (i) $F^T$     | (q) $GC$     |
| (b) $B - A$   | (j) $AB$      | (r) $A^2$    |
| (c) $D + G$   | (k) $IA$      | (s) $E^2$    |
| (d) $3C$      | (l) $AI$      | (t) $A^{-1}$ |
| (e) $2A + 3C$ | (m) $CE$      | (u) $B^{-1}$ |
| (f) $A + I$   | (n) $DF$      | (v) $E^{-1}$ |
| (g) $E - I$   | (o) $F^T D^T$ | (w) $C^{-1}$ |
| (h) $C^T$     | (p) $CG$      |              |

2. The matrix  $L$  shown below describes a football league part-way through the season. The first column represents the number of games won, the second the number of games drawn, and the final column the number of games lost.

- (a) Post-multiply  $L$  by the column  $\begin{pmatrix} 3 & 1 & 0 \end{pmatrix}^T$ . What does the result represent? (If you are unfamiliar with scoring in association football, see here.)
- (b) Pre-multiply  $L$  by the row  $\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$ . What does the result represent? Does it show that something is wrong?
- (c) Post-multiply the result of the previous part by the column  $\begin{pmatrix} 0.5 & 0.5 & 0.5 \end{pmatrix}^T$ . What does the result represent?

$$L = \begin{matrix} \text{Team A} \\ \text{Team B} \\ \text{Team C} \\ \text{Team D} \\ \text{Team E} \\ \text{Team F} \\ \text{Team G} \\ \text{Team H} \\ \text{Team I} \\ \text{Team J} \end{matrix} \begin{pmatrix} 11 & 5 & 4 \\ 11 & 2 & 7 \\ 11 & 1 & 6 \\ 10 & 3 & 6 \\ 9 & 4 & 7 \\ 9 & 3 & 9 \\ 9 & 3 & 8 \\ 6 & 2 & 12 \\ 5 & 1 & 12 \\ 4 & 1 & 15 \end{pmatrix}$$

## Solutions

1. (a)  $A + B = \begin{pmatrix} 1 & 0 \\ 5 & 0 \end{pmatrix}$
- (b)  $B - A = \begin{pmatrix} -1 & -4 \\ 1 & 2 \end{pmatrix}$
- (c)  $D + G$ : Doesn't exist because  $D$  and  $G$  are different sizes.
- (d)  $3C = \begin{pmatrix} 3 & -9 \\ -6 & 18 \end{pmatrix}$
- (e)  $2A + 3C = \begin{pmatrix} 5 & -5 \\ -2 & 16 \end{pmatrix}$
- (f)  $A + I = \begin{pmatrix} 2 & 2 \\ 2 & 0 \end{pmatrix}$
- (g)  $E - I$ : Doesn't exist since  $E$  is a  $2 \times 1$  matrix and identity matrices are necessarily *square*.
- (h)  $C^T = \begin{pmatrix} 1 & -2 \\ -3 & 6 \end{pmatrix}$
- (i)  $F^T = \begin{pmatrix} -2 & 1 & 2 \end{pmatrix}$
- (j)  $AB = \begin{pmatrix} 6 & 0 \\ -3 & -5 \end{pmatrix}$
- (k)  $IA = A$
- (l)  $AI = A$
- (m)  $CE = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$
- (n)  $DF = \begin{pmatrix} 1 \\ -9 \\ 0 \end{pmatrix}$
- (o)  $F^T D^T = \begin{pmatrix} 1 & -9 & 0 \end{pmatrix}$
- (p)  $CG$ : Doesn't exist as the dimensions don't agree.
- (q)  $GC = \begin{pmatrix} 1 & -3 \\ 14 & -42 \\ -3 & 9 \end{pmatrix}$

(r)  $A^2 = \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$

(s)  $E^2$ : Doesn't exist as the dimensions don't agree.

(t)  $A^{-1} = \frac{1}{5} \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$

(u)  $B^{-1} \frac{1}{6} \begin{pmatrix} 1 & 2 \\ -3 & 0 \end{pmatrix}$

(v)  $E^{-1}$ : Doesn't exist because  $E$  isn't a square matrix.

(w)  $C^{-1}$ : Doesn't exist because  $\det(C) = 1 \times 6 - (-3 \times -2) = 0$ .

2. The matrix  $L$  describes a football league part-way through the season. The first column represents the number of games won, the second the number of games drawn, and the final column the number of games lost.

(a)

$$\begin{array}{l} \text{Team A} \\ \text{Team B} \\ \text{Team C} \\ \text{Team D} \\ \text{Team E} \\ \text{Team F} \\ \text{Team G} \\ \text{Team H} \\ \text{Team I} \\ \text{Team J} \end{array} \begin{pmatrix} 11 & 5 & 4 \\ 11 & 2 & 7 \\ 11 & 1 & 6 \\ 10 & 3 & 6 \\ 9 & 4 & 7 \\ 9 & 3 & 9 \\ 9 & 3 & 8 \\ 6 & 2 & 12 \\ 5 & 1 & 12 \\ 4 & 1 & 15 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 38 \\ 35 \\ 34 \\ 33 \\ 31 \\ 30 \\ 30 \\ 20 \\ 16 \\ 13 \end{pmatrix}$$

(b)

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 11 & 5 & 4 \\ 11 & 2 & 7 \\ 11 & 1 & 6 \\ 10 & 3 & 6 \\ 9 & 4 & 7 \\ 9 & 3 & 9 \\ 9 & 3 & 8 \\ 6 & 2 & 12 \\ 5 & 1 & 12 \\ 4 & 1 & 15 \end{pmatrix} = \begin{pmatrix} 85 & 25 & 86 \end{pmatrix}$$

These are the total numbers of Wins, Draws, and Losses. Somehow there have been fewer Wins than there have been Losses - which doesn't make sense!

(c)

$$\begin{pmatrix} 85 & 25 & 86 \end{pmatrix} \begin{pmatrix} 0.5 \\ 0.5 \\ 0.5 \end{pmatrix} = 98$$

The total number of matches played so far. Each Win and Loss comes from one game, so counting them individually counts matches twice. Similarly, counting *every* draw counts it once for both teams in that match.