# Exercises: Functions

## Exercises

1. Find the values of the given functions at the stated arguments. For example, if the function is $s\colon \mathbb{N} \to \mathbb{N}$, where $s(n) = n + 1$, then $s(0) = 1$ and $s(5) = 6$.

   (a) $f\colon \mathbb{N} \to \mathbb{N}$, where $f(n) = n + 3$. Find the values of $f(0)$, $f(3)$ and $f(7)$.

   (b) $g\colon \mathbb{Z} \to \mathbb{N}$, where $g(n) = \mid n \mid$. (This means the absolute value of $n$.) Find the values of $g(0)$, $g(-1)$, $g(2)$.

   (c) $h\colon \mathbb{R} \to \{1\}$, where $h(n) = 1$. Find the values of $h(1), h(\pi), h(-e^{-\pi})$.

2. For each of the following functions, specify a suitable codomain $X$. There may be more than one suitable answer.

   (a) $f\colon \mathbb{N} \to X$, where $f(x) = x^2$

   (b) $g\colon \{1, 2\} \to X$, where $g(x) = 2x$

   (c) $h\colon \mathbb{N} \to X$, where $h(x) = -x$

   (d) $i\colon \mathbb{N} \times \mathbb{Z} \to X$, where $i(x, y) = x \times y$

3. In each of the following questions you will be given two functions and an input, then asked to compose the functions and describe what the output is.

   (a) Let $f\colon \mathbb{Z} \to \mathbb{Z}$ and $g\colon \mathbb{Z} \to \mathbb{Z}$ be functions defined by $f(m) = m^2$ and $g(n) = n^3$, for $m, n \in \mathbb{Z}$.

      i. For a given integer $m \in \mathbb{Z}$, describe $(g \circ f)(m)$.
      ii. If $m = 3$, what is the value of $(g \circ f)(m)$?
      iii. Is $(g \circ f)(m) = (f \circ g)(m)$ always true?

   (b) Let $\alpha\colon \mathbb{R} \to \mathbb{R}$ and $\beta\colon \mathbb{R} \to \mathbb{R}$ be functions defined by $\alpha(m) = \frac{m}{2}$ and $\beta(n) = \text{abs}(n)$, for $m \in \mathbb{R}$ and $n \in \mathbb{Z}$. (The symbols $\alpha$ and $\beta$ are pronounced ‘alpha’ and ‘beta’. The notation for the function $\beta$ is for the absolute value, sometimes written $|n|$ instead of $\text{abs}(n)$.)

      i. For a given integer $m \in \mathbb{Z}$, describe $(\beta \circ \alpha)(m)$. If $m = 3$, what is the value of $(\beta \circ \alpha)(m)$?

ii. If $m = -\pi$, what is the value of $(\beta \circ \alpha)(m)$?
iii. If $m = -\pi$, what is the value of $(\alpha \circ \beta)(m)$?
iv. Is $(\beta \circ \alpha)(m) = (\alpha \circ \beta)(m)$ always true?

(c) Let $p\colon \{a, b, c\} \to \{1, 2, 3\}$ and $q\colon \{1, 2, 3\} \to \{x, y, z\}$ be functions defined by $p(a) = 1$, $p(b) = 2$, $p(c) = 3$ and $q(1) = x$, $q(2) = y$, $q(3) = z$.

i. For each $m \in \{a, b, c\}$, what is the value of $(q \circ p)(m)$?
ii. Is it possible to define the composite function $p \circ q$?
iii. Can you define a function $r\colon \{x, y, z\} \to \{a, b, c\}$ such that $(r \circ q)(1) = b$, $(r \circ q)(2) = b$, and $(r \circ q)(3) = a$?

# Solutions

1. (a) $f(0) = 3$, $f(3) = 6$, $f(7) = 10$

   (b) $g(0) = 0$, $g(-1) = 1$, $g(2) = 2$

   (c) $h(1) = 1, h(\pi) = 1, h(-e^{-\pi}) = 1$

2. (a) One possible codomain is $X = \mathbb{N}$.

   (b) One possible codomain is $X = \{2, 4\}$. Another is $X = \mathbb{N}$.

   (c) One possible codomain is $X = \mathbb{Z}$.

   (d) One possible codomain is $\mathbb{Z}$.

3. (a) i. $(g \circ f)(m) = g(f(m)) = g(m^2) = (m^2)^3 = m^6$
   ii. $(g \circ f)(m) = (g \circ f)(3) = 3^6 = 729$
   iii. The other composite is given by

$$\begin{aligned}(f \circ g)(m) &= f(m^3)\\ &= (m^3)^2\\ &= m^6\\ &= (g \circ f)(m).\end{aligned}$$

   So no matter the value of $m \in \mathbb{Z}$, we always have $(g \circ f)(m) = (f \circ g)(m)$. We can then say that $g \circ f = f \circ g$.

   (b) i. $(\beta \circ \alpha)(m) = \beta(\alpha(m)) = \beta(\frac{m}{2}) = \text{abs}(\frac{m}{2})$. There's nothing much more we can do to simplify that.
   ii. If $m = -\pi$, then $\alpha(m) = \alpha(-\pi) = \frac{-\pi}{2}$. Since $-\pi \approx -3.14159$, then $\frac{-\pi}{2} \approx -1.57$. (In general, don't truncate numbers like $\pi$ until you have to - so just keep using the symbol $\pi$ until you actually need a numerical value out of it.) So $\beta(\alpha(-\pi)) \approx \text{abs}(-1.57) \approx 1.57$.
   iii. If $m = -\pi$, then $\beta(m) = \text{abs}(-\pi) = \pi$. So $\alpha(\beta(-\pi)) \approx (1.57)$.

iv. We can just think about this first without putting in any values. One of the functions, $\alpha$, halves the value we put into it. The other function, $\beta$, takes the absolute value of the number. It doesn't matter what order we do this in, so we always have $\beta \circ \alpha = \alpha \circ \beta$.

(c) i. $(q \circ p)(a) = x$, $(q \circ p)(b) = y$, $(q \circ p)(c) = z$

ii. Since the sets $\{a, b, c\}$ and $\{x, y, z\}$ are not equal, we cannot define the other composite $p \circ q$ since the domains and codomains don't match up in the correct way.

iii. Define $r\colon \{x, y, z\} \to \{a, b, c\}$ by $r(x) = b$, $r(y) = b$, and $r(z) = a$.