Cryptography: Public Keys and Euclidean Algorithm

Alex Corner

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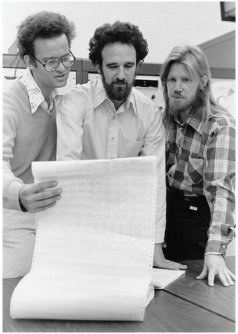
- This is where **public key cryptography** comes into play.

Diffie–Hellman Key Exchange



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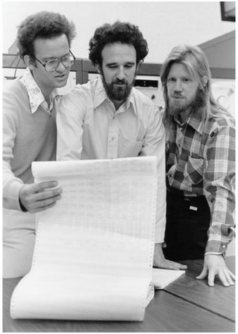

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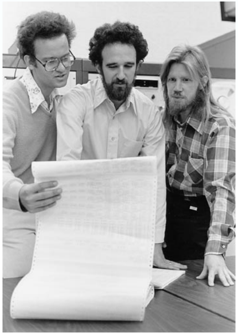

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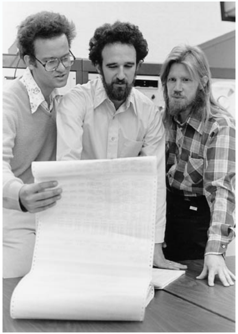

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  - Alice then finds $(\alpha^y)^x \mod p$ and Bob finds $(\alpha^x)^y \mod p$: this means that Alice and Bob now both have a number $\alpha^{xy} \mod p$ without ever sharing their secret values x and y.

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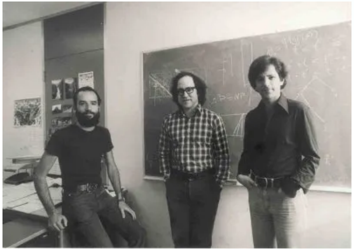
- Alice and Bob now share a secret value and can use this to generate their private key.

RSA: Development



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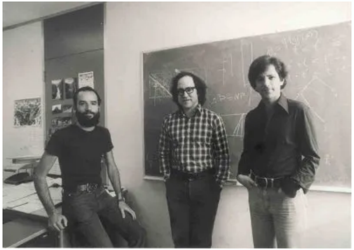

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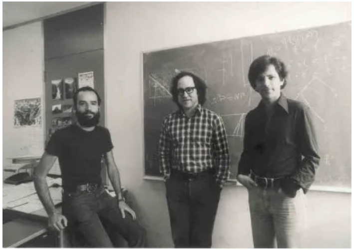

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- What if we let everybody know the encryption key, but kept another key secret which was only used for decryption?
- Various cryptographers had this idea and knew it would rely on finding a mathematical function which was **one-way**: many inputs could produce the same output.

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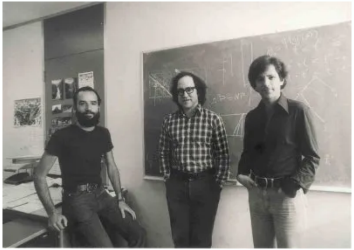

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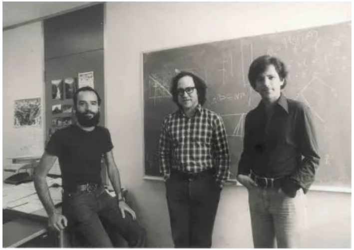

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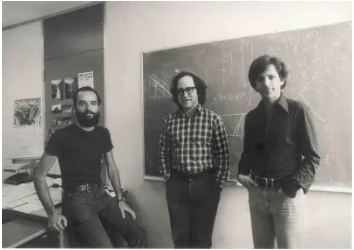

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- The scheme depends on large prime numbers, modular arithmetic, and the inherent difficulty in factorising large products.

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- This is easy to compute for prime numbers: $\varphi(p) = p - 1$. And for products of primes: $\varphi(pq) = (p - 1)(q - 1)$.

RSA: Implementation

- We'll look at how RSA actually works next week!

Euclidean Algorithm

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The least number before 0 is the highest common factor. Here this is 1, so e and a are coprime.

Euclidean Algorithm: Example

Here's another example of the Euclidean Algorithm to show that $\mathrm{hcf}(72, 26) = 2$.

$$\begin{aligned} 72 &= 2 \times 26 + 20 \\ 26 &= 1 \times 20 + 6 \\ 20 &= 3 \times 6 + 2 \\ 6 &= 3 \times 2 + 0 \end{aligned}$$

Diophantine Equations

Often in mathematical applications we want to find solutions to equations such as

$$72a + 5b = c.$$

When we implement RSA we will need to do something similar, except we will need each of a, b, and c to be integers. These types of equations where we are interested in integer solutions are called **Diophantine equations**.

Extended Euclidean Algorithm

Extended Euclidean Algorithm

$$
\begin{array}{cc c cc}
(1,0) & 72 & & 5 & (0,1) \\
 & & \times 14 & & \\
(0,14) & 70 & & 4 & (2,-28) \\ \cline{4-5}
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(1,-14) & 2 & & 1 & (-2,29) \\
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At each stage, the pair of coordinates (a, b) corresponding to a number c in the table tells us that $72a + 5b = c$. E.g., the pair $(2, -28)$ corresponds to the number 4 in the table, so we know that $72 \times 2 + 5 \times (-28) = 4$.

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There is only ever a solution to $xa + yb = c$ if c is a multiple of $\text{hcf}(x, y)$.

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 & & \times 25 & & \\
(0,25) & 175 & & 5 & (1,-25) \\
\cline{4-5}
 & & \times 1 & & \\
\hline
(1,-25) & 5 & & 2 & (-1,26) \\
 & & \times 2 & & \\
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Yes! Take $x = 3$ and $y = -77$. Then

$$180 \times 3 + 7 \times (-77) = 1.$$

Extended Euclidean Algorithm: Examples

Let's do some more examples!

Extended Euclidean Algorithm: Examples

A board game company is designing a game. Each copy of the game includes 1204 tokens. There are two types of box which are used to store the tokens, the type A boxes can hold 42 tokens and the type B boxes can hold 119 tokens.

1. The company wants to order boxes for each game in order to store the tokens. That is, they want to order a number of type A boxes and a number of type B boxes which will exactly hold the 1204 tokens of the game. Formulate this problem in the form $sa + tb = n$, where $a \in \mathbb{Z}$ is the number of type A boxes, $b \in \mathbb{Z}$ is the number of type B boxes, n is the total number of tokens in the board game, and $s, t \in \mathbb{Z}$.

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2. What further restriction should we put on a and b in order for this problem to make sense?
3. Use the Extended Euclidean Algorithm in order to find the highest common factor of 42 and 119.
4. Are there any solutions to the problem in part (a)?

Tutorials

In the tutorial this week we will:

- Create a spreadsheet to handle Extended Euclidean Algorithm calculations.
- Practice performing the Euclidean Algorithm by hand and checking it with the spreadsheet.