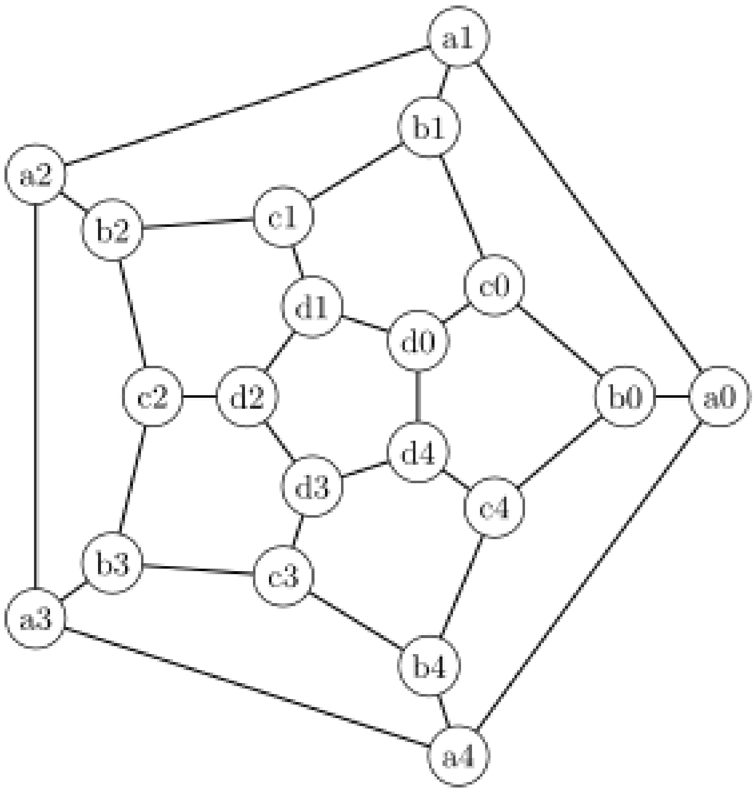
# Hamiltonian Graphs

## The Icosian Game

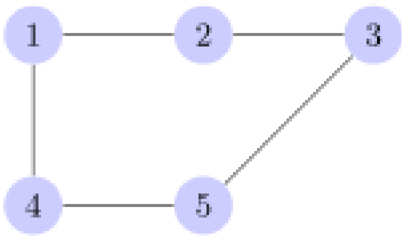
The icosian game was invented in 1857 by the mathematician and astronomer William Rowan Hamilton. The point of the game is to find a cycle in the following graph which passes through each vertex *exactly once*.

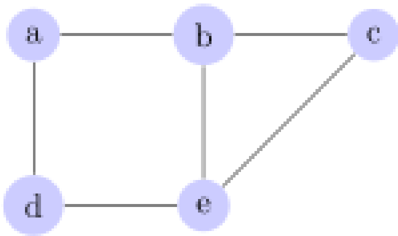



A **Hamiltonian cycle** in a graph is a cycle that includes every vertex. A graph is **Hamiltonian** if it has such a cycle. Remember that a cycle has no repeated vertices. So a Hamiltonian graph has a walk that includes each vertex *exactly once*, and finishes at its starting point. The cycle does not need to use every edge; it will use exactly two edges at each vertex.

A graph is **semi-Hamiltonian** if it is not Hamiltonian, but there is a path which visits every vertex exactly once (starting and finishing at different vertices). A graph is **non-Hamiltonian** if it is neither Hamiltonian nor semi-Hamiltonian.

Both of our examples from the Eulerian section are Hamiltonian. Just go around the outside of the graph: $1 \to 2 \to 3 \to 5 \to 4 \to 1$ and $a \to d \to e \to c \to b \to a$.



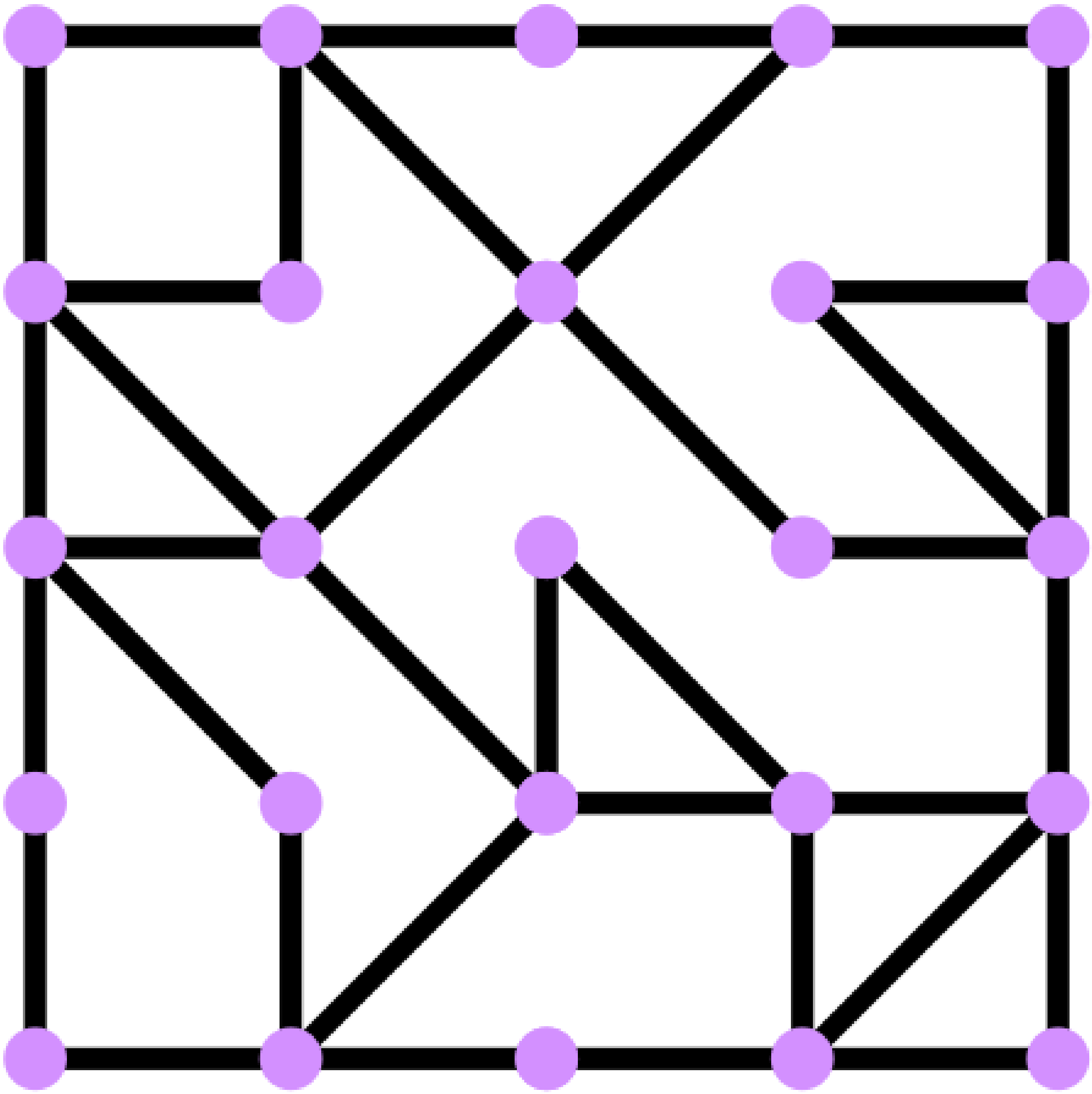

Deciding whether a graph is Hamiltonian is in general much harder than deciding whether it is Eulerian.

For smallish Hamiltonian graphs, you can prove they are Hamiltonian by explicitly finding a suitable cycle (like the examples above). But if you can't find one, is it because the graph isn't Hamiltonian, or have you just not looked hard enough? You can often find a proof that there is no Hamiltonian cycle, but how to do this will depend on the particular graph.

In general, we could check whether or not a graph is Hamiltonian by listing the vertices in all possible orders. For each order, see whether all vertices are adjacent in the list (and the last and first) are connected by an edge in the graph. If they are, then stop and report that the graph is Hamiltonian. If no cycle is found for any ordering of the vertices, the graph is not Hamiltonian. But how many possible orders are there for the list of vertices, and how long will it take to check them all?

No one has found an efficient algorithm to decide whether a graph is Hamiltonian. It is generally believed that it is impossible to find an efficient algorithm, but no one has yet proved that there is no such algorithm. (For anyone who wants a more precise statement of this, the problem of deciding whether a graph is Hamiltonian is NP-complete. Look up what this means on the Internet or in any book on computational complexity!)

Here is an example to play with.



- Is it Eulerian?
- It isn't Hamiltonian; how do you know?
- How many vertices (and edges incident with them) do you need to remove to make it Hamiltonian?

**Video** Visit the URL below to view a video:
`https://www.youtube.com/embed/3pdBoawo2Jo`