Exercises: Matrices

Exercises 1

1. This question uses the following matrices. Also, I is used to denote an identity matrix of any suitable size.

$$A = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} \quad B = \begin{pmatrix} 0 & -2 \\ 3 & 1 \end{pmatrix} \quad C = \begin{pmatrix} 1 & -3 \\ -2 & 6 \end{pmatrix}$$

$$D = \begin{pmatrix} 1 & -1 & 2 \\ 8 & 3 & 2 \\ 1 & 0 & 1 \end{pmatrix} \quad E = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad F = \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix} \quad G = \begin{pmatrix} 1 & 0 \\ 2 & -6 \\ 1 & 2 \end{pmatrix}$$

Calculate the following, when they exist.

(a) $A + B$
(b) $B - A$
(c) $D + G$
(d) $3C$
(e) $2A + 3C$
(f) $A + I$
(g) $E - I$
(h) C^T
(i) F^T
(j) AB
(k) IA
(l) AI
(m) CE
(n) DF
(o) $F^T D^T$
(p) CG
(q) GC
(r) A^2
(s) E^2
(t) A^{-1}
(u) B^{-1}
(v) E^{-1}
(w) C^{-1}

2. The matrix L shown below describes a football league part-way through the season. The first column represents the number of games won, the second the number of games drawn, and the final column the number of games lost.

   (a) Post-multiply L by the column $\begin{pmatrix} 3 & 1 & 0 \end{pmatrix}^T$. What does the result represent? (If you are unfamiliar with scoring in association football, see here.)

   (b) Pre-multiply L by the row $\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$. What does the result represent? Does it show that something is wrong?

   (c) Post-multiply the result of the previous part by the column $\begin{pmatrix} 0.5 & 0.5 & 0.5 \end{pmatrix}^T$. What does the result represent?

$$L = \begin{matrix} \text{Team A} \\ \text{Team B} \\ \text{Team C} \\ \text{Team D} \\ \text{Team E} \\ \text{Team F} \\ \text{Team G} \\ \text{Team H} \\ \text{Team I} \\ \text{Team J} \end{matrix} \begin{pmatrix} 11 & 5 & 4 \\ 11 & 2 & 7 \\ 11 & 1 & 6 \\ 10 & 3 & 6 \\ 9 & 4 & 7 \\ 9 & 3 & 9 \\ 9 & 3 & 8 \\ 6 & 2 & 12 \\ 5 & 1 & 12 \\ 4 & 1 & 15 \end{pmatrix}$$

Solutions

1. (a) $A + B = \begin{pmatrix} 1 & 0 \\ 5 & 0 \end{pmatrix}$

   (b) $B - A = \begin{pmatrix} -1 & -4 \\ 1 & 2 \end{pmatrix}$

   (c) $D + G$: Doesn't exists because D and G are different sizes.

   (d) $3C = \begin{pmatrix} 3 & -9 \\ -6 & 18 \end{pmatrix}$

   (e) $2A + 3C = \begin{pmatrix} 5 & -5 \\ -2 & 16 \end{pmatrix}$

   (f) $A + I = \begin{pmatrix} 2 & 2 \\ 2 & 0 \end{pmatrix}$

   (g) $E - I$: Doesn't exist since E is a 2×1 matrix and identity matrices are necessarily *square*.

   (h) $C^T = \begin{pmatrix} 1 & -2 \\ -3 & 6 \end{pmatrix}$

   (i) $F^T = \begin{pmatrix} -2 & 1 & 2 \end{pmatrix}$

   (j) $AB = \begin{pmatrix} 6 & 0 \\ -3 & -5 \end{pmatrix}$

   (k) $IA = A$

   (l) $AI = A$

   (m) $CE = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

   (n) $DF = \begin{pmatrix} 1 \\ -9 \\ 0 \end{pmatrix}$

   (o) $F^T D^T = \begin{pmatrix} 1 & -9 & 0 \end{pmatrix}$

   (p) CG: Doesn't exist as the dimensions don't agree.

   (q) $GC = \begin{pmatrix} 1 & -3 \\ 14 & -42 \\ -3 & 9 \end{pmatrix}$

(r) $A^2 = \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$

(s) E^2: Doesn't exist as the dimensions don't agree.

(t) $A^{-1} = \frac{1}{5}\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$

(u) $B^{-1}\frac{1}{6}\begin{pmatrix} 1 & 2 \\ -3 & 0 \end{pmatrix}$

(v) E^{-1}: Doesn't exist because E isn't a square matrix.

(w) C^{-1}: Doesn't exist because $\det(C) = 1 \times 6 - (-3 \times -2) = 0$.

2. The matrix L describes a football league part-way through the season. The first column represents the number of games won, the second the number of games drawn, and the final column the number of games lost.

(a)

$$\begin{matrix} \text{Team A} \\ \text{Team B} \\ \text{Team C} \\ \text{Team D} \\ \text{Team E} \\ \text{Team F} \\ \text{Team G} \\ \text{Team H} \\ \text{Team I} \\ \text{Team J} \end{matrix} \begin{pmatrix} 11 & 5 & 4 \\ 11 & 2 & 7 \\ 11 & 1 & 6 \\ 10 & 3 & 6 \\ 9 & 4 & 7 \\ 9 & 3 & 9 \\ 9 & 3 & 8 \\ 6 & 2 & 12 \\ 5 & 1 & 12 \\ 4 & 1 & 15 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 38 \\ 35 \\ 34 \\ 33 \\ 31 \\ 30 \\ 30 \\ 20 \\ 16 \\ 13 \end{pmatrix}$$

(b)

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 11 & 5 & 4 \\ 11 & 2 & 7 \\ 11 & 1 & 6 \\ 10 & 3 & 6 \\ 9 & 4 & 7 \\ 9 & 3 & 9 \\ 9 & 3 & 8 \\ 6 & 2 & 12 \\ 5 & 1 & 12 \\ 4 & 1 & 15 \end{pmatrix} = \begin{pmatrix} 85 & 25 & 86 \end{pmatrix}$$

These are the total numbers of Wins, Draws, and Losses. Somehow there have been fewer Wins than there have been Losses - which doesn't make sense!

(c)

$$\begin{pmatrix} 85 & 25 & 86 \end{pmatrix} \begin{pmatrix} 0.5 \\ 0.5 \\ 0.5 \end{pmatrix} = 98$$

The total number of matches played so far. Each Win and Loss comes from one game, so counting them individually counts matches twice. Similarly, counting *every* draw counts it once for both teams in that match.