Exercises: Matrix Equations

Exercises 2

1. Let
$$A = \begin{pmatrix} -3 & 5 \\ 4 & 3 \end{pmatrix}.$$
   (a) Find the determinant of A.
   (b) Find the inverse of A.
   (c) Use A^{-1} to find a solution for the simultaneous equations
$$-3x + 5y = 58,$$
$$4x + 3y = -29.$$
   (d) Use A^{-1} to find a solution for the simultaneous equations
$$-3x + 5y = -2,$$
$$4x + 3y = 1.$$

2. Let
$$B = \begin{pmatrix} 7 & -3 \\ 3 & 5 \end{pmatrix}.$$
   (a) Find the determinant of B.
   (b) Find the inverse of B.
   (c) Use B^{-1} to find a solution for the simultaneous equations
$$7a - 3b = 5,$$
$$3a + 5b = 7.$$
   (d) Use B^{-1} to find a solution for the simultaneous equations
$$7u - 3v = 44,$$
$$3u + 5v = -44.$$

3. Consider the following pair of simultaneous equations.

$$17x - 5y = 21$$

$$6x + 22y = 13$$

   (a) Write these equations as a single matrix equation.

   (b) Use matrix inverses and matrix multiplication to solve these equations for x and y.

4. Consider the following pair of simultaneous equations.

$$91x + 14y = 121$$

$$65x + 10y = -72$$

   (a) Write these equations as a single matrix equation.

   (b) Use determinants to show that there are no solutions for these equations for x and y.

5. Calculate the matrix product

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix},$$

and hence check that the formula for the inverse of a matrix, given below, is correct.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

6. If A and B are invertible matrices of the same size, simplify $\left(B^{-1}A^{-1}\right)(AB)$. Hence show that $(AB)^{-1} = B^{-1}A^{-1}$.

7. (Challenge) Let

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}.$$

The inverse of A is given by

$$A^{-1} = \frac{1}{12} \begin{bmatrix} 1 & -3 & 4 \\ 7 & 3 & -8 \\ -5 & 3 & 4 \end{bmatrix}.$$

   (a) Show that $A^{-1}A = I$.

   (b) Using A^{-1}, show that the solution to the simultaneous equations given below is $x = -3$, $y = 9$, and $z = 3$.

$$3x + 2y + z = 12$$

$$x + 2y + 3z = 24$$

$$3x + y + 2z = 6$$

Solutions

1. (a)
$$\det(A) = (-3 \times 3) - (5 \times 4) = -9 - 20 = -29.$$
   (b)
$$A^{-1} = \frac{1}{-29} \begin{pmatrix} 3 & -5 \\ -4 & -3 \end{pmatrix}$$
   (c)
$$A^{-1} \begin{pmatrix} 58 \\ -29 \end{pmatrix} = \frac{1}{-29} \begin{pmatrix} 3 & -5 \\ -4 & -3 \end{pmatrix} \begin{pmatrix} 58 \\ -29 \end{pmatrix} = \begin{pmatrix} -11 \\ 5 \end{pmatrix}$$
   (d)
$$A^{-1} \begin{pmatrix} -2 \\ 1 \end{pmatrix} = \frac{1}{-29} \begin{pmatrix} 3 & -5 \\ -4 & -3 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{11}{29} \\ \frac{-5}{29} \end{pmatrix}$$

2. (a)
$$\det(B) = (7 \times 5) - (-3 \times 3) = 35 - (-9) = 44.$$
   (b)
$$B^{-1} = \frac{1}{44} \begin{pmatrix} 5 & 3 \\ -3 & 7 \end{pmatrix}$$
   (c)
$$B^{-1} \begin{pmatrix} 5 \\ 7 \end{pmatrix} = \frac{1}{44} \begin{pmatrix} 5 & 3 \\ -3 & 7 \end{pmatrix} \begin{pmatrix} 5 \\ 7 \end{pmatrix} = \begin{pmatrix} \frac{46}{44} \\ \frac{34}{44} \end{pmatrix}$$
   (d)
$$B^{-1} \begin{pmatrix} 44 \\ -44 \end{pmatrix} = \frac{1}{44} \begin{pmatrix} 5 & 3 \\ -3 & 7 \end{pmatrix} \begin{pmatrix} 44 \\ -44 \end{pmatrix} = \begin{pmatrix} 2 \\ -10 \end{pmatrix}$$

3. (a) We can formulate the simultaneous equations as a single matrix equation:
$$\begin{pmatrix} 17 & -5 \\ 6 & 22 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 21 \\ 13 \end{pmatrix}.$$
   (b) The determinant of the coefficient matrix is
$$\det \left(\begin{pmatrix} 17 & -5 \\ 6 & 22 \end{pmatrix} \right) = (17 \times 22) - (-5 \times 6) = 404,$$
   hence there are solutions to the pair of simultaneous equations. The inverse of the coefficient matrix is given by
$$\begin{pmatrix} 17 & -5 \\ 6 & 22 \end{pmatrix}^{-1} = \frac{1}{404} \begin{pmatrix} 22 & 5 \\ -6 & 17 \end{pmatrix}.$$
   The solutions are hence given by
$$\frac{1}{404} \begin{pmatrix} 22 & 5 \\ -6 & 17 \end{pmatrix} \begin{pmatrix} 21 \\ 13 \end{pmatrix} = \begin{pmatrix} \frac{527}{404} \\ \frac{95}{404} \end{pmatrix}.$$

4. (a) We can formulate the simultaneous equations as a single matrix equation:

$$\begin{pmatrix} 91 & 14 \\ 65 & 10 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 121 \\ -72 \end{pmatrix}.$$

(b) The determinant of the coefficient matrix is

$$\det\left(\begin{pmatrix} 91 & 14 \\ 65 & 10 \end{pmatrix}\right) = (91 \times 10) - (14 \times 65) = 0,$$

hence there are no solutions to the pair of simultaneous equations.

5.

$$\begin{aligned}
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} &= \begin{pmatrix} ad - bc & -ba + ba \\ cd - dc & -bc + ad \end{pmatrix} \\
&= \begin{pmatrix} ad - bc & 0 \\ 0 & ad - bc \end{pmatrix} \\
&= (ad - bc) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
&= (ad - bc)I
\end{aligned}$$

The inverse of a matrix should satisfy AA^{-1}. The calculation shows that

$$\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

almost works but we still need to divide out by a factor of $(ad - bc)$.

6.

$$\begin{aligned}
(B^{-1}A^{-1})(AB) &= B^{-1}(A^{-1}A)B \\
&= B^{-1}IB \\
&= B^{-1}B \\
&= I
\end{aligned}$$

Similarly, $AB(B^{-1}A^{-1}) = I$. The inverse of a matrix C is a matrix C^{-1} such that $CC^{-1} = I = C^{-1}C$. Since $B^{-1}A^{-1}$ does this job for AB, then we have found that

$$(AB)^{-1} = B^{-1}A^{-1}.$$

7. (a) Let

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$$

and

$$A^{-1} = \frac{1}{12} \begin{bmatrix} 1 & -3 & 4 \\ 7 & 3 & -8 \\ -5 & 3 & 4 \end{bmatrix}.$$

We will calculate $A \cdot A^{-1}$ and check if we obtain the identity matrix $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Step 1: Write Out the Multiplication The product $A \cdot A^{-1}$ is calculated as:

$$A \cdot A^{-1} = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix} \cdot \frac{1}{12} \begin{bmatrix} 1 & -3 & 4 \\ 7 & 3 & -8 \\ -5 & 3 & 4 \end{bmatrix}.$$

This can be rewritten as:

$$A \cdot A^{-1} = \frac{1}{12} \begin{bmatrix} 3 & 2 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -3 & 4 \\ 7 & 3 & -8 \\ -5 & 3 & 4 \end{bmatrix}.$$

Step 2: Perform Matrix Multiplication To find each element in the resulting matrix, we multiply the corresponding row from A by the corresponding column from A^{-1} and sum the products.

First Row

$$(1,1) : 3 \cdot 1 + 2 \cdot 7 + 1 \cdot (-5) = 3 + 14 - 5 = 12,$$

$$(1,2) : 3 \cdot (-3) + 2 \cdot 3 + 1 \cdot 3 = -9 + 6 + 3 = 0,$$

$$(1,3) : 3 \cdot 4 + 2 \cdot (-8) + 1 \cdot 4 = 12 - 16 + 4 = 0.$$

Second Row

$$(2,1) : 1 \cdot 1 + 2 \cdot 7 + 3 \cdot (-5) = 1 + 14 - 15 = 0,$$

$$(2,2) : 1 \cdot (-3) + 2 \cdot 3 + 3 \cdot 3 = -3 + 6 + 9 = 12,$$

$$(2,3) : 1 \cdot 4 + 2 \cdot (-8) + 3 \cdot 4 = 4 - 16 + 12 = 0.$$

Third Row

$$(3,1) : 3 \cdot 1 + 1 \cdot 7 + 2 \cdot (-5) = 3 + 7 - 10 = 0,$$

$$(3,2) : 3 \cdot (-3) + 1 \cdot 3 + 2 \cdot 3 = -9 + 3 + 6 = 0,$$

$$(3,3) : 3 \cdot 4 + 1 \cdot (-8) + 2 \cdot 4 = 12 - 8 + 8 = 12.$$

Step 3: Write Out the Result Putting it all together, we get:

$$A \cdot A^{-1} = \frac{1}{12} \begin{bmatrix} 12 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 12 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Conclusion Thus, we have verified that

$$A \cdot A^{-1} = I,$$

which confirms that A^{-1} is indeed the inverse of A.

(b) To solve the following system using matrix A and show that $x = -3$, $y = 9$, and $z = 3$, we can represent the system as a matrix equation and use the inverse of A to find the solution.

Given the system:

$$3x + 2y + z = 12,$$

$$x + 2y + 3z = 24,$$

$$3x + y + 2z = 6,$$

we can write this in matrix form as:

$$A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ 24 \\ 6 \end{bmatrix},$$

where

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}.$$

To solve for $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$, we can use the inverse matrix A^{-1}:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1} \begin{bmatrix} 12 \\ 24 \\ 6 \end{bmatrix}.$$

We know that

$$A^{-1} = \frac{1}{12} \begin{bmatrix} 1 & -3 & 4 \\ 7 & 3 & -8 \\ -5 & 3 & 4 \end{bmatrix}.$$

Step 1: Multiply A^{-1} by the constant matrix Let

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{12} \begin{bmatrix} 1 & -3 & 4 \\ 7 & 3 & -8 \\ -5 & 3 & 4 \end{bmatrix} \begin{bmatrix} 12 \\ 24 \\ 6 \end{bmatrix}.$$

Now we perform the matrix multiplication:

$$x = \frac{1}{12}(1 \cdot 12 + (-3) \cdot 24 + 4 \cdot 6) = \frac{1}{12}(12 - 72 + 24) = \frac{1}{12} \cdot (-36) = -3,$$

$$y = \frac{1}{12}(7 \cdot 12 + 3 \cdot 24 + (-8) \cdot 6) = \frac{1}{12}(84 + 72 - 48) = \frac{1}{12} \cdot 108 = 9,$$

$$z = \frac{1}{12}((-5) \cdot 12 + 3 \cdot 24 + 4 \cdot 6) = \frac{1}{12}(-60 + 72 + 24) = \frac{1}{12} \cdot 36 = 3.$$

Conclusion Thus, the solution to the system is

$$x = -3, \quad y = 9, \quad z = 3.$$

This confirms that the values $x = -3$, $y = 9$, and $z = 3$ satisfy the system of equations.