#### Lax monoidal model structures

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**BMC 2015** 

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- ► A lax monoidal category is like a monoidal category but there is a tensor product for each natural number and constraint cells need not be isomorphisms.
- Model categories give a setting in which to study the homotopy theory of given objects. Monoidal model categories give their homotopy categories a monoidal structure.
- Questions: Monoidal structures and model structures don't always play nice. (E.g. Gray-Cat.) Can we develop a sensible theory of lax monoidal model categories? What examples can we study in this framework?

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- 3. A model structure on lax monoids

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**Associators** 

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are replaced by morphisms

$$((X_{11} \otimes \ldots \otimes X_{1k_1}) \otimes \ldots \otimes (X_{n1} \otimes \ldots \otimes X_{nk_n})) \to (X_{11} \otimes \ldots \otimes X_{nk_n})$$

$$E_n(E_{k_1}(X_1),\ldots,E_{k_n}(\underline{X}_n)) \to E_{\Sigma k_i}(\underline{X}_1,\ldots,\underline{X}_n)$$

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- ► This sort of process is used in Trimble-like definitions of *n*-category.

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In a closed lax monoidal category, each

$$E_n: \mathbb{C}^n \to \mathbb{C}$$

is equipped with *n*-variable right adjoints, meaning each 1-variable restriction

$$E_n(X_1,\ldots,X_i,-,X_{i+1},\ldots,X_n)\colon C\to C$$

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The idea is to pass to a homotopy category where the weak equivalences are isomorphisms - 'localise at the weak equivalences'.

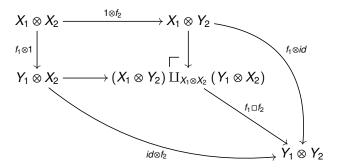
# Monoidal model categories

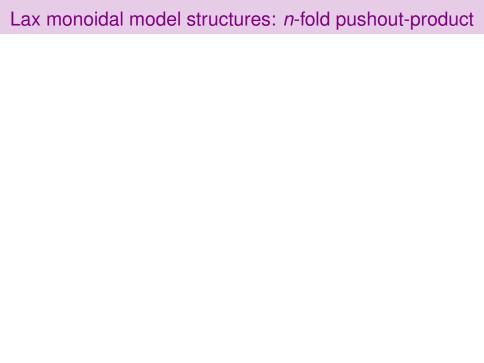
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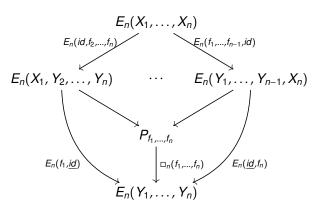
- A monoidal model category is both a model category and a monoidal category, where the structures interact nicely: the homotopy category is monoidal.
- ► The pushout-product axiom: If each  $f_i: X_i \to Y_i$  is in  $\mathbb{C}$  then the induced map  $f_1 \square f_2$  is in  $\mathbb{C}$ . If either  $f_i$  is also in  $\mathcal{W}$ , then so is  $f_1 \square f_2$ .





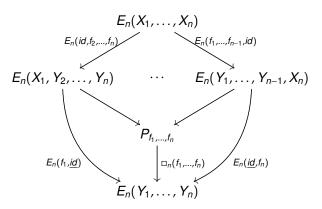
# Lax monoidal model structures: *n*-fold pushout-product

If each  $f_i$  is in  $\mathbb{C}$ , then the induced map  $\square_n(f_1,\ldots,f_n)$  is in  $\mathbb{C}$ . If any of the  $f_i$  is also in W, so is  $\square_n(f_1,\ldots,f_n)$ .



## Lax monoidal model structures: *n*-fold pushout-product

If each  $f_i$  is in  $\mathbb{C}$ , then the induced map  $\Box_n(f_1,\ldots,f_n)$  is in  $\mathbb{C}$ . If any of the  $f_i$  is also in  $\mathcal{W}$ , so is  $\Box_n(f_1,\ldots,f_n)$ .



We also say that  $E_n$  is a Quillen adjunction of n variables. The assignment  $\Box_n$  is functorial and makes Ar(C) a lax monoidal category - 'associators' are induced by  $\gamma$ .

# Lax monoidal model categories

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### **Proposition**

Let  $\mathcal{C}$  be a lax monoidal model category. Then the homotopy category  $Ho\mathcal{C}$  is a lax monoidal category.

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Identifying a *lax monoid axiom* will involve constructing a clear and explicit construction of pushouts of the form

$$\begin{array}{ccc}
FK & \xrightarrow{g} X \\
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where f is a generating trivial cofibration.

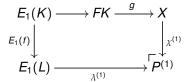
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where f is a generating trivial cofibration. The free lax monoid on K is given by

$$FK = \coprod_{n \in \mathbb{N}} E_n(K, \dots, K).$$

We can give a construction of P as a filtered colimit of maps formed by pushouts. Set  $P^{(0)} = X$ . Then define  $P^{(1)}$  as the following pushout.



Subsequent  $P^{(n)}$  are defined inductively as pushouts as follows.

$$E_{n}(\underline{K}) \longrightarrow FK \xrightarrow{g} X \xrightarrow{\chi^{(1)}} P^{(1)} \xrightarrow{\chi^{(2)}} \cdots \xrightarrow{\chi^{(n-1)}} P^{(n-1)}$$

$$\downarrow_{\chi^{(n)}} \qquad \downarrow_{\chi^{(n)}} \qquad \downarrow_{p(n)}$$

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We then take the colimit of the sequence

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This satisfies the universal property of the pushout P. A rather arduous induction argument shows that  $\operatorname{colim} P^{(i)}$  is a lax monoid. Similarly, the map  $s^{(0)}: X \to \operatorname{colim} P^{(i)}$  is a lax monoid map.

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The monoid axiom for monoidal model categories requires that every map in

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A crude speculation as to the lax monoid axiom is that we should have, for each n, that every map in

$$(E_n(\underline{\mathcal{W}}\cap \underline{\mathbb{C}}))$$
 –  $\operatorname{cof}_{\operatorname{reg}}$ 

is a weak equivalence.

