

Cryptography: Public Keys and Euclidean Algorithm

Alex Corner

Sheffield Hallam University

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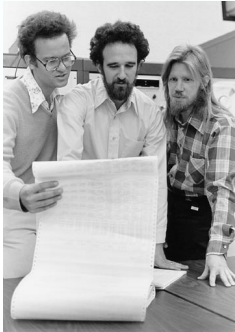
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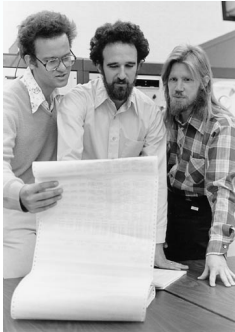
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- ▶ This is where **public key cryptography** comes into play.

Diffie–Hellman Key Exchange



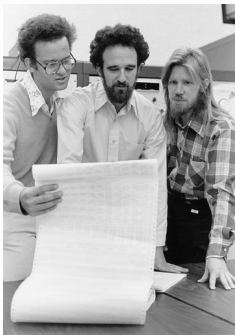
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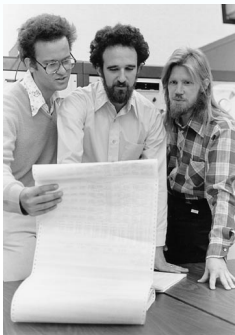
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Diffie–Hellman–Merkle–Ellis–Cocks–Williamson Key Exchange



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 - ▶ Alice then finds $(\alpha^y)^x \bmod p$ and Bob finds $(\alpha^x)^y \bmod p$: this means that Alice and Bob now both have a number $\alpha^{xy} \bmod p$ without ever sharing their secret values x and y .

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- ▶ Alice and Bob now share a secret value and can use this to generate their private key.

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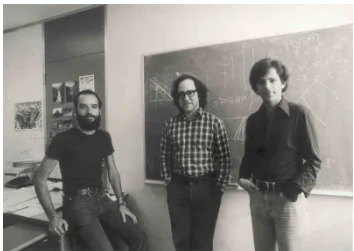
- ▶ Key exchange was a great innovation, but another idea about public key cryptography was quick to follow.
- ▶ What if we let everybody know the encryption key, but kept another key secret which was only used for decryption?
- ▶ Various cryptographers had this idea and knew it would rely on finding a mathematical function which was **one-way**: many inputs could produce the same output.

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- ▶ The scheme depends on large prime numbers, modular arithmetic, and the inherent difficulty in factorising large products.

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- ▶ This is easy to compute for prime numbers: $\varphi(p) = p - 1$. And for products of primes: $\varphi(pq) = (p - 1)(q - 1)$.

RSA: Implementation

- ▶ We'll look at how RSA actually works next week!

Euclidean Algorithm

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The least number before 0 is the highest common factor. Here this is 1, so e and a are coprime.

Euclidean Algorithm: Example

Here's another example of the Euclidean Algorithm to show that $\text{hcf}(72, 26) = 2$.

$$72 = 2 \times 26 + 20$$

$$26 = 1 \times 20 + 6$$

$$20 = 3 \times 6 + 2$$

$$6 = 3 \times 2 + 0$$

Diophantine Equations

Often in mathematical applications we want to find solutions to equations such as

$$72a + 5b = c.$$

When we implement RSA we will need to do something similar, except we will need each of a , b , and c to be integers. These types of equations where we are interested in integer solutions are called **Diophantine equations**.

Extended Euclidean Algorithm

Extended Euclidean Algorithm

$(1, 0)$	72		5	$(0, 1)$
		$\times 14$		
$(0, 14)$	70		4	$(2, -28)$
		$\times 2$		
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$(1, -14)$	2		1	$(-2, 29)$
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$(-4, 58)$	2			
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At each stage, the pair of coordinates (a, b) corresponding to a number c in the table tells us that $72a + 5b = c$. E.g., the pair $(2, -28)$ corresponds to the number 4 in the table, so we know that $72 \times 2 + 5 \times (-28) = 4$.

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$(0, 25)$	175		5	$(1, -25)$
		$\times 1$		
$(1, -25)$	5		2	$(-1, 26)$
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$(-2, 52)$	4			
$(3, -77)$	1			

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Yes! Take $x = 3$ and $y = -77$. Then

$$180 \times 3 + 7 \times (-77) = 1.$$

Extended Euclidean Algorithm: Examples

Let's do some more examples!

Tutorials

In the tutorial this week we will:

- ▶ Create a spreadsheet to handle Extended Euclidean Algorithm calculations.
- ▶ Practice performing the Euclidean Algorithm by hand and checking it with the spreadsheet.