Cryptography: Public Keys and Euclidean Algorithm

Alex Corner

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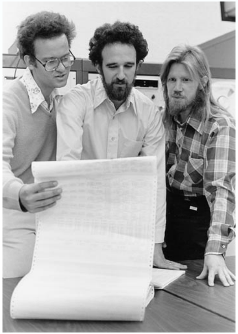
- This is where **public key cryptography** comes into play.

Diffie–Hellman Key Exchange



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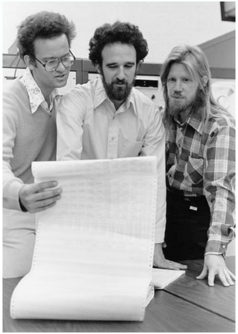

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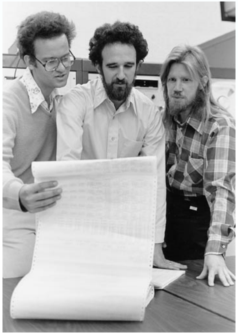

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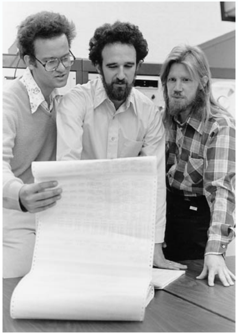

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  - Alice then finds $(\alpha^y)^x \mod p$ and Bob finds $(\alpha^x)^y \mod p$: this means that Alice and Bob now both have a number $\alpha^{xy} \mod p$ without ever sharing their secret values x and y.

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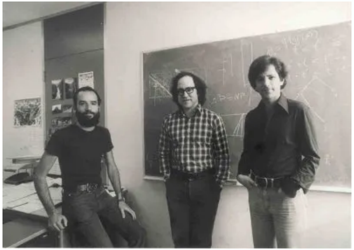
- Alice and Bob now share a secret value and can use this to generate their private key.

RSA: Development



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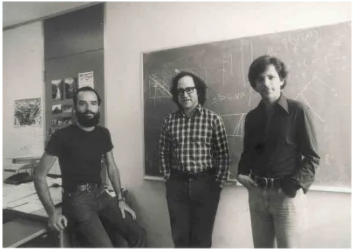

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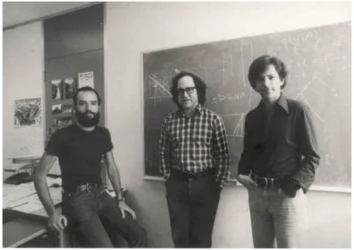

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- Various cryptographers had this idea and knew it would rely on finding a mathematical function which was **one-way**: many inputs could produce the same output.

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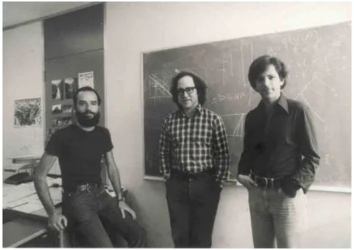

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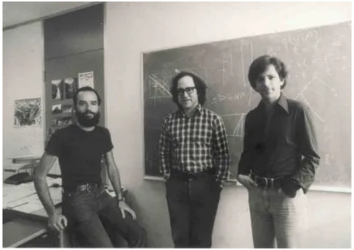

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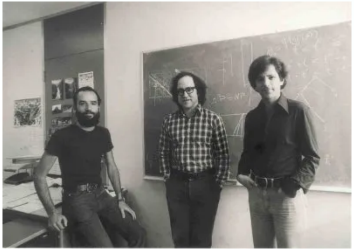

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- The scheme depends on large prime numbers, modular arithmetic, and the inherent difficulty in factorising large products.

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- This is easy to compute for prime numbers: $\varphi(p) = p - 1$. And for products of primes: $\varphi(pq) = (p - 1)(q - 1)$.

RSA: Implementation

- We'll look at how RSA actually works next week!

Euclidean Algorithm

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The least number before 0 is the highest common factor. Here this is 1, so e and a are coprime.

Euclidean Algorithm: Example

Here's another example of the Euclidean Algorithm to show that $\mathrm{hcf}(72, 26) = 2$.

$$\begin{aligned} 72 &= 2 \times 26 + 20 \\ 26 &= 1 \times 20 + 6 \\ 20 &= 3 \times 6 + 2 \\ 6 &= 3 \times 2 + 0 \end{aligned}$$

Diophantine Equations

Often in mathematical applications we want to find solutions to equations such as

$$72a + 5b = c.$$

When we implement RSA we will need to do something similar, except we will need each of a, b, and c to be integers. These types of equations where we are interested in integer solutions are called **Diophantine equations**.

Extended Euclidean Algorithm

Extended Euclidean Algorithm

$$
\begin{array}{cccccc}
(1,0) & 72 & & 5 & (0,1) \\
 & & \times 14 & & \\
(0,14) & 70 & & 4 & (2,-28) \\
\cline{4-5}
 & & \times 2 & & \\
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(1,-14) & 2 & & 1 & (-2,29) \\
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At each stage, the pair of coordinates (a, b) corresponding to a number c in the table tells us that $72a + 5b = c$. E.g., the pair $(2, -28)$ corresponds to the number 4 in the table, so we know that $72 \times 2 + 5 \times (-28) = 4$.

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There is only ever a solution to $xa + yb = c$ if c is a multiple of $\mathrm{hcf}(x, y)$.

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 & & \times 25 & & \\
(0,25) & 175 & & 5 & (1,-25) \\
\cline{4-5}
 & & \times 1 & & \\
\hline
(1,-25) & 5 & & 2 & (-1,26) \\
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Yes! Take $x = 3$ and $y = -77$. Then

$$180 \times 3 + 7 \times (-77) = 1.$$

Extended Euclidean Algorithm: Examples

Let's do some more examples!

Tutorials

In the tutorial this week we will:

- Create a spreadsheet to handle Extended Euclidean Algorithm calculations.
- Practice performing the Euclidean Algorithm by hand and checking it with the spreadsheet.