Chapter 1

Complex Numbers

1.1 Recap

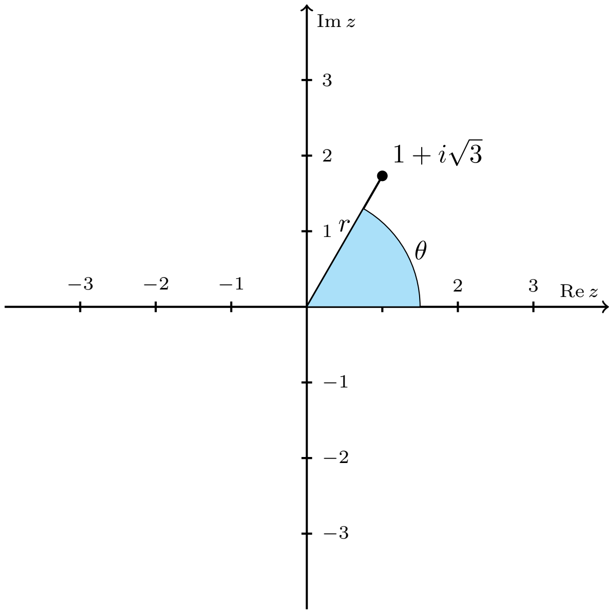
- i is the **imaginary number** such that $i^2 = -1$.
- A **complex number** is of the form
$$z = a + ib$$
where a, $b \in \mathbb{R}$.
- The **real part** of z is $\mathrm{Re}(z) = a$ and the **imaginary** part of z is $\mathrm{Im}(z) = b$.

1.2 Argand diagrams, modulus, argument

The way we have seen complex numbers so far is in their **rectangular form**. For example, the complex number $z = 1 + i\sqrt{3}$ which has real part $\mathrm{Re}(z) = 1$ and imaginary part $\mathrm{Im}(z) = \sqrt{3}$. We can think of this as going 1 unit in the 'real' direction and $\sqrt{3}$ units in the 'imaginary' direction, like the sides of a rectangle. But another way to represent a complex number is to use a distance and an angle, all relative to the centre point of a pair of axes.

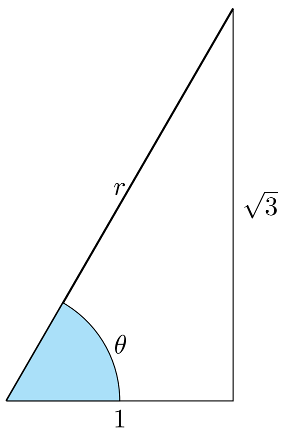
On an **Argand diagram** we have a pair of axes. The horizontal axis is called the **real axis** and the vertical axis is called the **imaginary axis**. We can essentially 'plot' a complex number on these axes. For example, the complex number $z = 1 + i\sqrt{3}$ can be thought of as being at the point $(1, \sqrt{3})$ as on the diagram below. Another way of specifying where this point is is to say how far it is from the central point $(0, 0)$, or $0 + i0$, it is along with how far anti-clockwise it is around from the positive real axis.

When we specify a complex number z by its distance and angle, we call the distance from the centre the **modulus**, written $r = |z|$, and the angle is called the **principal argument**, written $\theta = \arg(z)$.



Example 1

For our example number $z = 1 + i\sqrt{3}$, we can work out the modulus and argument. We think of the modulus, r, as the hypotenuse of a right-angled triangle, as below. Then the argument is the angle formed between the modulus and the real axis.



Using Pythagoras' Theorem, we can find the modulus to be

$$r = |z| = \sqrt{1^2 + \sqrt{3}^2} = 2.$$

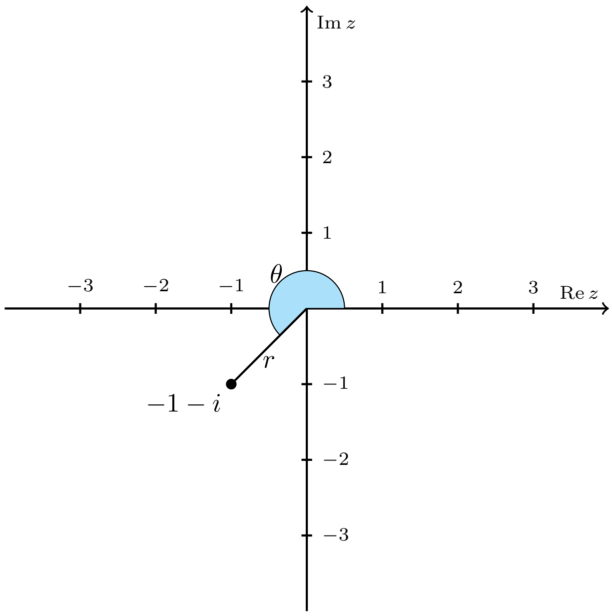
Then using some trigonometry we can find the modulus to be

$$\theta = \arg(z) = \arctan\left(\frac{\sqrt{3}}{1}\right) = \frac{\pi}{3}.$$

You can play around with a complex number on an Argand diagram in the GeoGebra applet below. It will show you the modulus and the argument of the complex number if you drag it around the diagram.

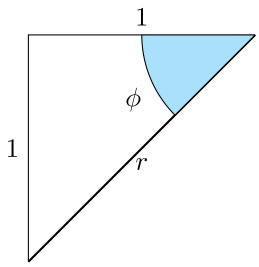
Example 2

In this example we will represent $z = -1 - i$ on an Argand diagram. Then we will find the modulus, $r = |z|$, and the argument, $\theta = \arg(z)$.



In this case we find the modulus in much the same way as before, by picturing a triangle. Using Pythagoras' Theorem again, we find that

$$r = |z| = \sqrt{1^2 + 1^2} = \sqrt{2}.$$



Notice that in the triangle we have used a different greek letter for the angle, ϕ rather than θ, since we can't use inverse tan directly as we did before to find the angle. In general, we measure the argument of a complex number between 0 and 2π. That is, for any complex number z, we take its principal argument to be $\arg(z) \in [0, 2\pi)$.

In this case we can still find the angle ϕ in the triangle. But since we measure the whole argument as going round from the positive real axis, we need to make sure we count the first π radians it has already turned through. Using some trigonometry again, we can find that

$$\phi = \arctan\left(\frac{1}{1}\right) = \frac{\pi}{4}.$$

Adding the angles together, we can then find that

$$\theta = \arg(z) = \pi + \phi = \pi + \frac{\pi}{4} = \frac{5\pi}{4}.$$

Note that some textbooks measure the argument of a complex to be between $-\pi$ and π, which is to say that $\theta \in [-\pi, \pi]$. We will take the convention that the argument is always between 0 and 2π.

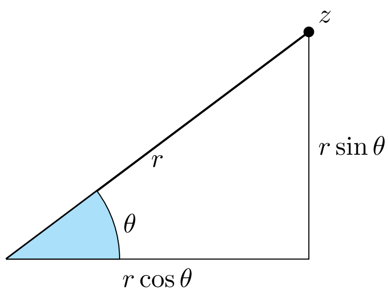
Try some exercises below so that you are happy calculating the modulus and argument of a complex number.

Test Yourself Visit the URL below to try a numbas exam:
`https://numbas.mathcentre.ac.uk/question/123470/polar-form-of-complex-number/embed/`

1.3 Polar form

We introduced Argand diagrams in order to show that a complex number can be represented in another way, by how far from the centre it is and what angle it has moved through. Not only does this provide a nice graphical representation of a complex number, but it also gives us a very useful alternative notation that can help make calculations with complex numbers more simple.

The idea of the **polar form** of a complex number is represented by the triangle below.



If we know the modulus, r, and the argument, θ, of a complex number $z = a + ib$, then we can use some basic trigonometry to work out that the rectangular coordinates are given by

$$a = r\cos\theta$$

and

$$b = r\sin\theta.$$

Using this idea we can then instead write

$$z = r(\cos\theta + i\sin\theta).$$

This is the polar form of z.

For example, for $z = 1 + i\sqrt{3}$ we found that $r = 2$ and $\theta = \frac{\pi}{3}$. Hence we can write

$$z = 1 + i\sqrt{3} = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right).$$

An important thing to note here is not to multiply in the 2 and evaluate the cos and sin parts, otherwise we'll just end up back with the rectangular coordinates that we started with.

As another example, for $z = -1 - i$, we found that $r = \sqrt{2}$ and $\theta = \frac{5\pi}{4}$, so

$$z = -1 - i = \sqrt{2}\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right).$$

1.3.1 Multiplying in polar form

One of the advantages of using the polar form of complex numbers is that it makes multiplication much simpler. If we have two complex numbers

$$z_1 = r_1 (\cos\theta_1 + i\sin\theta_1)$$

and

$$z_2 = r_2 (\cos\theta_2 + i\sin\theta_2),$$

then their product is simply given by

$$z_1 z_2 = r_1 r_2 (\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)).$$

That is, the modulus of $z_1 z_2$ is the the product of the modulus of z_1 and the modulus of z_2, so

$$|z_1 z_2| = r_1 r_2,$$

and the argument of $z_1 z_2$ is the sum of the arguments of z_1 and z_2, so

$$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2).$$

As long as we have our complex numbers written in polar form, multiplication can be done very quickly.

Example 3

Previously we mentioned that the argument of a complex number must lie in the interval $[0, 2\pi)$. But what happens if we multiply two complex numbers whose arguments sum to a number bigger than 2π? Let's find out.

Take $z_1 = 1 + i$ and $z_2 = \sqrt{3} - i$. In polar form, these can be written as

$$z_1 = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

and

$$z_2 = 2\left(\cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6}\right).$$

Using our multiplication formula, this would give the argument of $z_1 z_2$ to be

$$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2) = \frac{\pi}{4} + \frac{11\pi}{6} = \frac{25\pi}{12},$$

which is greater than 2π. In this case, since $\frac{25\pi}{12} > 2\pi$, we reduce the value by 2π to be in the correct range. So we can instead write

$$\arg(z_1 z_2) = \frac{25\pi}{12} - 2\pi = \frac{\pi}{12}.$$

Similarly, if we ever end up with a negative angle, we can add on multiples of 2π until we end up back in the correct range of $[0, 2\pi)$.

Altogether, we can then say that

$$z_1 z_2 = 2\sqrt{2}\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right).$$

1.3.2 Why does this work?

You might wonder why multiplication suddenly works this way if we're in polar form. It relies on a few trigonometric identities:

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta,$$

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta.$$

Let's start with $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$. Then we get the following calculation using the formulae above and a few places where $i^2 = -1$:

$$\begin{aligned}
z_1 z_2 &= r_1(\cos\theta_1 + i\sin\theta_1)\, r_2(\cos\theta_2 + i\sin\theta_2) \\
&= r_1 r_2(\cos\theta_1\cos\theta_2 + i\cos\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) \\
&= r_1 r_2\left((\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)\right) \\
&= r_1 r_2\left(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\right).
\end{aligned}$$

1.4 Euler's formula, exponential form

So far we have seen two ways to notate a complex number: rectangular form and polar form. There is a third way which is very closely related to the polar form. **Euler's formula** says that

$$re^{i\theta} = r(\cos\theta + i\sin\theta).$$

The way of writing the complex number on the left-hand side is known as the **exponential form**.

1.4.1 Why does this work?

It can seem slightly suprising to see this identity and you may wonder where it comes from. Here we will again show how this comes about, which makes use of the following Maclaurin series for e, cos, and sin:

$$\begin{aligned}
e^x &= 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \dots, \\
\cos x &= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \dots, \\
\sin x &= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7.
\end{aligned}$$

There are some similarities in the three series above, each involving powers of x and fractions where the denominator is a factorial. Our identity involves $e^{i\theta}$ on the left-hand side, so let's expand that:

$$\begin{aligned}
e^{i\theta} &= 1 + (i\theta) + \frac{1}{2!}(i\theta)^2 + \frac{1}{3!}(i\theta)^3 + \frac{1}{4!}(i\theta)^4 + \frac{1}{5!}(i\theta)^5 + \dots \\
&= 1 + i\theta - \frac{1}{2!}\theta^2 - \frac{1}{3!}i\theta^3 + \frac{1}{4!}\theta^4 + \frac{1}{5!}i\theta^5 + \dots \\
&= \left(1 - \frac{1}{2!}\theta^2 + \frac{1}{4!}\theta^4 + \dots\right) + i\left(\theta - \frac{1}{3!}\theta^3 + \frac{1}{5!}\theta^5 - \dots\right) \\
&= \cos\theta + i\sin\theta.
\end{aligned}$$

This new exponential form for a complex number makes the multiplication rule for the polar form even more obvious, since now it is simply addition of indices:

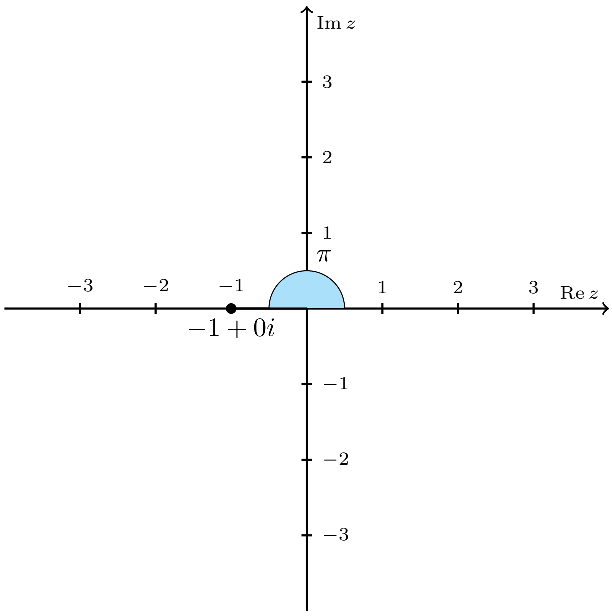
$$r_1 e^{i\theta_1} r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1+\theta_2)}.$$

1.4.2 Special case

There is one special case of Euler's formula that gets a lot of attention. It is often called **Euler's identity** and is the special case of taking $\theta = \pi$:

$$e^{i\pi} = -1.$$

To see why this works, consider where this number would be on an Argand diagram, given that its modulus is $r = 1$ and argument is $\theta = \pi$.



1.5 Application: Phasors

Complex numbers find applications in a number of engineering disciplines, especially when sinusoidal waveforms appear. We'll take a brief look at such an application in electrical engineering.

In electrical engineering, we often see voltages in a.c. circuits expressed as sinusoids such as

$$v(t) = V \cos(\omega t + \phi),$$

where V is the maximum voltage, ω is the angular frequency, and ϕ is the phase. Another way to represent these sinusoids is using complex numbers. We know that the polar form of a complex number with modulus r and argument θ is given by

$$z = r(\cos\theta + i\sin\theta),$$

which is similar to $v(t)$ above. If we took $r = V$ and $\theta = \omega t + \phi$, then this would give

$$z = V(\cos(\omega t + \phi) + i\sin(\omega t + \phi)) = Ve^{\omega t+\phi} = Ve^{\omega t}e^{\phi}.$$

Notice then that the real part of z is the same as the voltage $v(t)$.

The **phasor** form of the voltage $v(t)$ is determined by the maximum voltage, V, and the phase, ϕ. For example, if

$$v_1(t) = 2\cos\left(\omega t + \frac{\pi}{6}\right)$$

then the maximum voltage is $V_1 = 2$ and the phase is $\phi_1 = \frac{\pi}{6}$. The phasor form of $v_1(t)$ is then written

$$V_1 = 2\angle\frac{\pi}{6}.$$

Since we already know the angular frequency we don't represent it in the phasor form. From this form we can translate it to rectangular form:

$$2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) = \sqrt{3} + i.$$

Using phasors in this way to represent voltages or currents is useful as it allows multiple voltages/currents to be added by translating phasors to rectangular form, adding them, and then translating them back into phasors.

Example 4

Suppose we have two voltages

$$v_1(t) = 2\cos\left(\omega t + \frac{\pi}{6}\right)$$

and

$$v_2(t) = 3\sin\left(\omega t - \frac{\pi}{4}\right).$$

Now suppose that we want to find their sum and represent it as a single sinusoid:

$$v_1(t) + v_2(t) = A\cos\left(\omega t + \phi\right).$$

We could use a lot of trigonometric identities to slowly figure this out, but here we'll use phasors to simplify the process.

First, notice that the second voltage is in terms of sin, rather than cos. We can write this in terms of cos instead since $\sin(x) = \cos\left(x - \frac{\pi}{2}\right)$. So

$$v_2(t) = 3\cos\left(\omega_t - \frac{\pi}{4} - \frac{\pi}{2}\right) = 3\cos\left(\omega t + \frac{\pi}{4}\right).$$

We have already seen above that we can write $v_1(t)$ as a phasor and corresponding rectangular form:

$$V_1 = 2\angle\frac{\pi}{6} = \sqrt{3} + i.$$

Similarly, the phasor for the voltage $v_2(t)$ is given along with its rectangular form as

$$V_2 = 3\angle\frac{\pi}{4} = 3\frac{\sqrt{2}}{2} + i\left(3\frac{\sqrt{2}}{2}\right).$$

Now let's add them together:

$$\begin{aligned}
V_1 + V_2 &= \left(2\angle\frac{\pi}{6}\right) + \left(3\angle\frac{\pi}{4}\right)\\
&= \sqrt{3} + i + 3\frac{\sqrt{2}}{2} + i\left(3\frac{\sqrt{2}}{2}\right)\\
&= 3.85 + i3.12\\
&= 4.96\angle 0.68.
\end{aligned}$$

Since we now have a single phasor, we can then write the sum of voltages as

$$v_1(t) + v_2(t) = 4.96\cos\left(\omega t + 0.68\right).$$