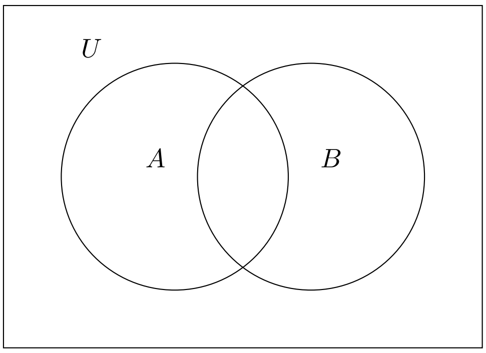
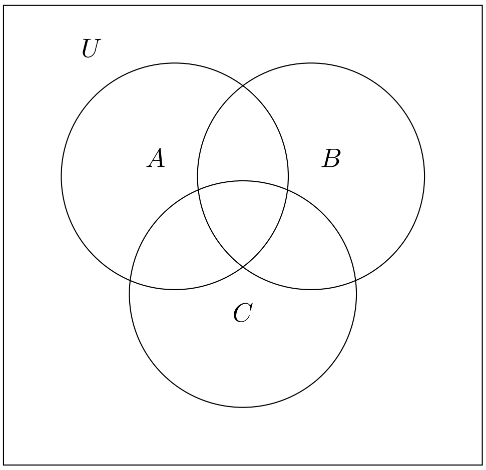
Operations on Sets

Venn Diagrams

Once we have defined some sets, we can use them to create more, by forming intersections, unions, and differences. We will define these in the next section, and illustrate the definitions with diagrams known as **Venn diagrams**. In these diagrams, sets are shown as roughly circular areas, with the area inside the curve representing elements in the set. Elements may be written in the appropriate parts of the diagram if necessary. The universal set is usuallly drawn as a rectangle, which must contain all of the other sets. Venn diagrams must be drawn so that there are sections for elements in and out of each set in all combinations. This is hard for more than 3 sets; the diagrams below are suitable for 2 and 3 sets.





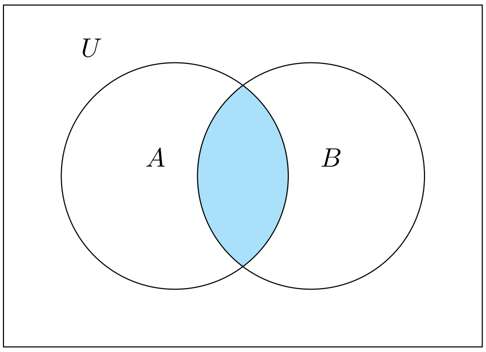
Operations on Sets

Intersection

If A and B are two sets then their **intersection** $A \cap B$, pronounced "A intersect B", is the set of all elements common to A and B. It is defined by

$$A \cap B = \{x \mid x \in A \ \wedge \ x \in B\}.$$

The intersection is shaded in the Venn diagram below.



Some examples are shown below.

$$\begin{aligned} \{1,3,5,7,9\} \cap \{3,6,9\} &= \{3,9\} \\ \{1,2,3,4\} \cap \{5,6,7,8\} &= \varnothing \\ \{a,b,c\} \cap \{b,c\} &= \{b,c\} \end{aligned}$$

Two sets A and B are said to be **disjoint** if they have no elements in common. This is the same as saying that their interesection is the empty set, or $A \cap B = \varnothing$. Several sets are said to be **mutually disjoint** if every pair of them are disjoint. For example, the sets $A = \{1,2\}$, $B = \{3,4\}$, and $C = \{5,6\}$ are mutually disjoint because

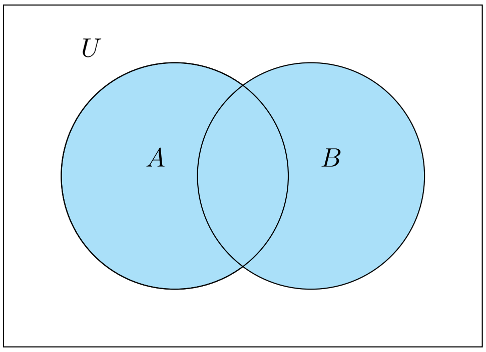
$$A \cap B = A \cap C = B \cap C = \varnothing.$$

Union

For two sets A and B, their **union** $A \cup B$, pronounced "A union B", is the set of all elements which are in A or in B (or in both). It is defined by

$$A \cup B = \{x \mid x \in A \ \vee \ x \in B\}.$$

The union is shaded in the Venn diagram below.



Some examples are shown below.

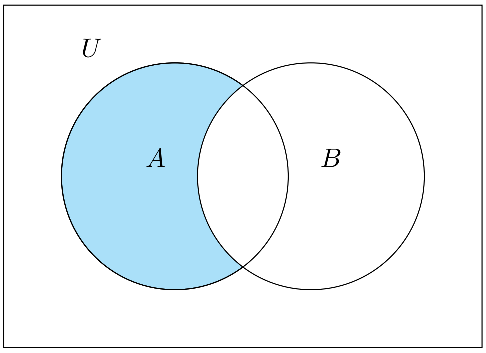
$$\{1,3,5,7,9\} \cup \{3,6,9\} = \{1,3,5,6,7,9\}$$
$$\{1,2,3,4\} \cup \{5,6,7,8\} = \{1,2,3,4,5,6,7,8\}$$
$$\{a,b,c\} \cup \{b,c\} = \{a,b,c\}$$

Set Difference

If A and B are two sets then the **relative complement** of A in B, written $A \setminus B$ and often pronounced "A set difference B" or "A without B", is the set of all elements which are in A but are not in B. It is defined by

$$A \setminus B = \{x \mid x \in A \ \wedge \ x \notin B\}.$$

The relative complement of A in B is shaded in the Venn diagram below.



Some examples are shown below.

$$\{1,3,5,7,9\} \setminus \{3,6,9\} = \{1,5,7\}$$
$$\{1,2,3,4\} \setminus \{5,6,7,8\} = \{1,2,3,4\}$$
$$\{a,b,c\} \setminus \{b,c\} = \{a\}$$

For any set A, the relative complement of A in A is the empty set. I.e., $A \setminus A = \varnothing$.

Some texts will write the relative complement of A in B as $A - B$ and call it "A set minus B". For every pair of sets A and B, the union $A \cup B$ can be written as the union of three mutually disjoint sets:

$$A \cup B = (A \setminus B) \cup (A \cap B) \cup (B \setminus A).$$

This is easy to see when looking at a Venn diagram.

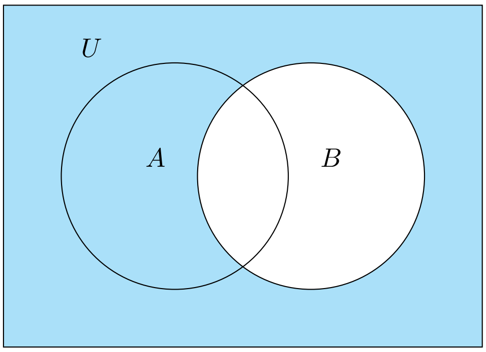
Complement

If U is the universal set we have in mind for some application, and $A \subseteq U$, then the **complement** of A, written A^c, is the set of everything not in A (but in the universal set). It can be defined as

$$A^c = U \setminus A.$$

Note that $A \setminus B = A \cap B^c$.

The complement of B is shaded in the Venn diagram below.



An example using $U = \mathbb{Z}$ and $E = \{x \in \mathbb{Z} \mid x \text{ is even}\}$ is that $E^c = \{x \in \mathbb{Z} \mid x \text{ is odd}\}$.

Some texts will use the notation A' for the complement of A.

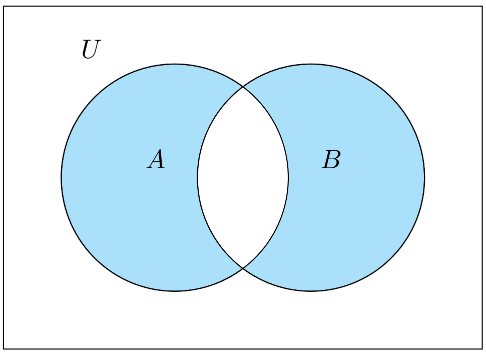
Video Visit the URL below to view a video:
`https://www.youtube.com/embed/rhIroGv9Dqw`

Symmetric Difference

Just as we could combine connectives in logic together, we can do the same with operations on sets. One example that comes up often enough to get its own name is the **symmetric difference**. Given two sets, A and B, the symmetric difference of A and B is defined by

$$A \,\Delta\, B = (A \cup B) \setminus (A \cap B).$$

Essentially, we want the elements which are either in A, are in B, but not those elements which are in *both*.



	a	b
1	$(1, a)$	$(1, b)$
3	$(3, a)$	$(3, b)$
5	$(5, a)$	$(5, b)$

Cartesian Product

A very important way of forming new sets from old is by the **Cartesian product**. If A and B are two sets, then a typical element of the Cartesian product, $A \times B$, of A and B is the *ordered pair* (a, b), where $a \in A$ and $b \in B$. Formally defined, we have

$$A \times B = \{z \mid z = (x, y) \text{ for some } x \in A \text{ and } y \in B\}.$$

Often we would shorten this definition to

$$A \times B = \{(x, y) \mid x \in A \ \wedge \ y \in B\}.$$

For example, if $A = \{1, 3, 5\}$ and $B = \{a, b\}$, then

$$A \times B = \{(1, a), (1, b), (3, a), (3, b), (5, a), (5, b)\}.$$

We can picture this using a table.

It is important to note that $(1, a) \neq (a, 1)$, since this is the set of *ordered* pairs.

A more visual example is to think of a grid drawn on a pair of x and y axes, like you might have when plotting points for a graph. All of these points are elements of the set $\mathbb{Z} \times \mathbb{Z}$. For example, $(1, 2), (-37, 2), (-100, -100) \in \mathbb{Z} \times \mathbb{Z}$.

We can form the Cartesian product of more than two sets at once, though there is some subtely involved. Suppose that we have three sets A_1, A_2, and A_3. Then we can consider the following three sets.

$$\begin{aligned}
&(A_1 \times A_2) \times A_3 \\
&A_1 \times A_2 \times A_3 \\
&A_1 \times (A_2 \times A_3)
\end{aligned}$$

The first set has elements which look like $((a_1, a_2), a_3)$, the second set has elements which look like (a_1, a_2, a_3), and the last set has elements which look like $(a_1, (a_2, a_3))$. They can be regarded as the ''same' set in a particular way. When working with such sets by hand we don't have to worry too much about which way we write the elements, as it's obvious that we can just move or remove the brackets involved. More care has to be taken if using such ordered **tuples** in a programming language. A program might accept an input of type $((a, b), c)$ but then not know how to handle an input of type $(a, (b, c))$.

Power Set

There is no reason why sets should not have other sets as members. This situtation arises in the following definition. If A is any set, then the **power set** $\mathbb{P}(A)$ is the set of subsets of A. So

$$\mathbb{P}(A) = \{X \mid X \subseteq A\}.$$

For example, if $A = \{1\}$, then

$$\mathbb{P}(A) = \{\varnothing, \{1\}\}.$$

If $B = \{a, b\}$, then

$$\mathbb{P}(B) = \{\varnothing, \{a\}, \{b\}, \{a, b\}\}.$$

If $C = \{x, y, z\}$, then

$$\mathbb{P}(C) = \{\varnothing, \{x\}, \{y\}, \{z\}, \{x, y\}, \{x, z\}, \{y, z\}, \{x, y, z\}\}.$$

Some texts use the notation $P(A)$ or 2^A to denote the power set of A.

Cardinality

An important property of a set is its **cardinality**. This is the number of elements in the set. It can be defined for all sets, but we shall only be concerned with finite sets, where the cardinality is a natural number. If A is a finite set we write $\mid A \mid$ for the number of elements in A.

Some examples are shown below (be careful with the last one).

$$\mid \{1, 2, 3\} \mid = 3$$
$$\mid \{a, b, c\} \mid = 3$$
$$\mid \{1, 5, 1\} \mid = 2$$

Some properties of cardinality are listed below.

$\mid \varnothing \mid = 0$
$\mid A \cap B \mid \leq \mid A \mid$
$\mid A \cap B \mid \leq \mid B \mid$
$\mid A \cup B \mid \geq \mid A \mid$
$\mid A \cup B \mid \geq \mid B \mid$
If $A \cup B = \varnothing$, then $\mid A \cup B \mid = \mid A \mid + \mid B \mid$
$\mid A \mid = \mid A \cap B \mid + \mid A \setminus B \mid$
$\mid A \cup B \mid = \mid A \mid + \mid B \mid - \mid A \cap B \mid$
$\mid A \times B \mid = \mid A \mid\mid B \mid$
$\mid \mathbb{P}(A) \mid = 2^{\mid A \mid}$

One of the most useful formulas in the above list is this one:

$$\mid A \cup B \mid = \mid A \mid + \mid B \mid - \mid A \cap B \mid.$$

This is known as the **Inclusion-Exclusion** principle and allows us to count the number of elements in the union of two sets A and B. We find this by counting the size of the sets A and B individually, but since they can overlap this would cause us to count the elements in their intersection, $A \cap B$, twice. Try applying this formula in the following exercises.

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Given two sets A and B, we can denote the **set of functions** from A to B by $\text{Fun}(A, B)$. Then we can think about the cardinality of this set:

$$\mid \text{Fun}(A, B) \mid = \mid B \mid^{\mid A \mid}.$$

Sets and Boolean

Video Visit the URL below to view a video:
https://www.youtube.com/embed/SfMMThPhaZk

Another example, not mentioned in the video, is that of the symmetric difference: $A \Delta B$. This corresponds to the `XOR` operation (exclusive OR).

Concept Checks

Test Yourself Visit the URL below to try a numbas exam:
https://numbas.mathcentre.ac.uk/question/74768/set-operations/embed/