Exercises 1: Modular Arithmetic

Exercises

Calculators aren't 'needed' for these questions - you can always use them to check, but you should focus on getting to grips with modular arithmetic as a way of thinking before reaching for the technology. Try to avoid doing arithmetic with numbers bigger than 100 - see if you can reduce them first.

1. Use modular arithmetic to do the following calculations - you can always check your answer on a calendar after. There is a table in the notes which you may find useful.
   (a) What day of the week will 31st December fall on this year?
   (b) What day of the week will 14th February fall on next year?
2. (a) Calculate 100 modulo 24.
   (b) Calculate 1000 modulo 24.
   (c) What time of day will it be 3000 hours from now?
3. Find the value of $7^{137} \pmod{11}$.
4. Find the value of $7^{137} \pmod{8}$. (Hint: Don't just jump into this. If you think about this in the right way, it ends up being very easy.)
5. Find the last 2 digits of 3^{124}.

Solutions

1. (a) A Wednesday. Work out the number of days left until 31st December 2025. Then reduce this number modulo 7.
   (b) A Saturday. Do the same as above but for 14th February 2026.
2. (a) $100 = 4 \pmod{24}$
   (b) Calculate 1000 modulo 24 by multiplying the previous calculation by 10 and reducing it again modulo 24.
   (c) This can be calculated from the previous step by multiplying the result by 3 and reducing it again.

3. First break up the power into powers of 2: $7^{137} = 7^{128} \times 7^8 \times 7^1$. Then build it up in stages:

$$\begin{aligned}
7^1 &= 7 \pmod{11}, \\
7^2 &= 49 = 5 \pmod{11}, \\
7^4 &= (7^2)^2 = 5^2 = 25 = 3 \pmod{11}, \\
7^8 &= 3^2 = 9 \pmod{11}, \\
7^{16} &= 9^2 = 81 = 4 \pmod{11}, \\
7^{32} &= 4^2 = 16 = 5 \pmod{11}, \\
7^{64} &= 5^2 = 3 \pmod{11}, \\
7^{128} &= 3^2 = 9 \pmod{11}.
\end{aligned}$$

Then

$$7^{137} = 7^{128} \times 7^8 \times 7^1 = 9 \times 9 \times 7 = 81 \times 7 = 4 \times 7 = 28 = 6 \pmod{11}.$$

4. If you look at this the right way, it's very easy. Remember that we can use negative integers as well as positive integers. Since we are working modulo 8, then we have that

$$7 = -1 \pmod{8}.$$

From this we can then conclude that

$$7^{137} = (-1)^{137} = -1 = 7 \pmod{8}.$$

5. To find this, work out $3^{124} \pmod{100}$ in a similar way to before. The last two digits are 8 and 1.