

# Exercises: Functions

## Exercises

1. Find the values of the given functions at the stated arguments. For example, if the function is  $s: \mathbb{N} \rightarrow \mathbb{N}$ , where  $s(n) = n + 1$ , then  $s(0) = 1$  and  $s(5) = 6$ .
  - (a)  $f: \mathbb{N} \rightarrow \mathbb{N}$ , where  $f(n) = n + 3$ . Find the values of  $f(0)$ ,  $f(3)$  and  $f(7)$ .
  - (b)  $g: \mathbb{Z} \rightarrow \mathbb{N}$ , where  $g(n) = |n|$ . (This means the absolute value of  $n$ .) Find the values of  $g(0)$ ,  $g(-1)$ ,  $g(2)$ .
  - (c)  $h: \mathbb{R} \rightarrow \{1\}$ , where  $h(n) = 1$ . Find the values of  $h(1)$ ,  $h(\pi)$ ,  $h(-e^{-\pi})$ .
2. For each of the following functions, specify a suitable codomain  $X$ . There may be more than one suitable answer.
  - (a)  $f: \mathbb{N} \rightarrow X$ , where  $f(x) = x^2$
  - (b)  $g: \{1, 2\} \rightarrow X$ , where  $g(x) = 2x$
  - (c)  $h: \mathbb{N} \rightarrow X$ , where  $h(x) = -x$
  - (d)  $i: \mathbb{N} \times \mathbb{Z} \rightarrow X$ , where  $i(x, y) = x \times y$
3. In each of the following questions you will be given two functions and an input, then asked to compose the functions and describe what the output is.
  - (a) Let  $f: \mathbb{Z} \rightarrow \mathbb{Z}$  and  $g: \mathbb{Z} \rightarrow \mathbb{Z}$  be functions defined by  $f(m) = m^2$  and  $g(n) = n^3$ , for  $m, n \in \mathbb{Z}$ .
    - i. For a given integer  $m \in \mathbb{Z}$ , describe  $(g \circ f)(m)$ .
    - ii. If  $m = 3$ , what is the value of  $(g \circ f)(m)$ ?
    - iii. Is  $(g \circ f)(m) = (f \circ g)(m)$  always true?
  - (b) Let  $\alpha: \mathbb{R} \rightarrow \mathbb{R}$  and  $\beta: \mathbb{R} \rightarrow \mathbb{R}$  be functions defined by  $\alpha(m) = \frac{m}{2}$  and  $\beta(n) = \text{abs}(n)$ , for  $m \in \mathbb{R}$  and  $n \in \mathbb{Z}$ . (The symbols  $\alpha$  and  $\beta$  are pronounced ‘alpha’ and ‘beta’. The notation for the function  $\beta$  is for the absolute value, sometimes written  $|n|$  instead of  $\text{abs}(n)$ .)
    - i. For a given integer  $m \in \mathbb{Z}$ , describe  $(\beta \circ \alpha)(m)$ . If  $m = 3$ , what is the value of  $(\beta \circ \alpha)(m)$ ?

- ii. If  $m = -\pi$ , what is the value of  $(\beta \circ \alpha)(m)$ ?
  - iii. If  $m = -\pi$ , what is the value of  $(\alpha \circ \beta)(m)$ ?
  - iv. Is  $(\beta \circ \alpha)(m) = (\alpha \circ \beta)(m)$  always true?
- (c) Let  $p: \{a, b, c\} \rightarrow \{1, 2, 3\}$  and  $q: \{1, 2, 3\} \rightarrow \{x, y, z\}$  be functions defined by  $p(a) = 1$ ,  $p(b) = 2$ ,  $p(c) = 3$  and  $q(1) = x$ ,  $q(2) = y$ ,  $q(3) = z$ .
- i. For each  $m \in \{a, b, c\}$ , what is the value of  $(q \circ p)(m)$ ?
  - ii. Is it possible to define the composite function  $p \circ q$ ?
  - iii. Can you define a function  $r: \{x, y, z\} \rightarrow \{a, b, c\}$  such that  $(r \circ q)(1) = b$ ,  $(r \circ q)(2) = b$ , and  $(r \circ q)(3) = a$ ?
4. (Challenge) In the following questions we will investigate a link between sets of functions  $f: X \rightarrow \{0, 1\}$  and subsets of a set  $X$ . Let  $X$  be a set. Then the **power set of  $X$**  is defined as the set whose elements are the subsets  $A \subseteq X$ . I.e.,

$$\mathbb{P}(X) = \{A \mid A \subseteq X\}.$$

For examples and further explanation, see the notes on power set.

- (a) For each of the following sets, count how many different subsets they have. I.e., for each set  $B$  below, count the number of elements in the set  $\mathbb{P}(B)$ .
  - i.  $X = \emptyset$ ,
  - ii.  $Y = \{0\}$ ,
  - iii.  $Z = \{0, 1\}$ ,
  - iv.  $W = \{0, 1, 2\}$ .
- (b) For each of the following sets, count how many different functions there are into the set  $\{0, 1\}$ . I.e., for each set  $B$  below, count the number of elements in the set  $\text{Fun}(B, \{0, 1\})$ .
  - i.  $X = \emptyset$ ,
  - ii.  $Y = \{0\}$ ,
  - iii.  $Z = \{0, 1\}$ ,
  - iv.  $W = \{0, 1, 2\}$ .
- (c) Do you notice anything about the numbers in the parts above? I.e., is there any link between the number of subset of a set  $B$  and the number of functions  $B \rightarrow \{0, 1\}$ ?
- (d) Given a subset  $A \subset B$ , can you use this to define a function  $f_A: B \rightarrow \{0, 1\}$ ? I.e., can you define a function  $\varphi: \mathbb{P}(B) \rightarrow \text{Fun}(B, \{0, 1\})$ ?
- (e) Given a function  $f: B \rightarrow \{0, 1\}$ , can you use this to define a subset  $A_f \subseteq B$ ? I.e., can you define a function  $\psi: \text{Fun}(B, \{0, 1\}) \rightarrow \mathbb{P}(B)$ ?

## Solutions

1. (a)  $f(0) = 3$ ,  $f(3) = 6$ ,  $f(7) = 10$
- (b)  $g(0) = 0$ ,  $g(-1) = 1$ ,  $g(2) = 2$

- (c)  $h(1) = 1, h(\pi) = 1, h(-e^{-\pi}) = 1$
2. (a) One possible codomain is  $X = \mathbb{N}$ .
- (b) One possible codomain is  $X = \{2, 4\}$ . Another is  $X = \mathbb{N}$ .
- (c) One possible codomain is  $X = \mathbb{Z}$ .
- (d) One possible codomain is  $\mathbb{Z}$ .
3. (a) i.  $(g \circ f)(m) = g(f(m)) = g(m^2) = (m^2)^3 = m^6$   
 ii.  $(g \circ f)(m) = (g \circ f)(3) = 3^6 = 729$   
 iii. The other composite is given by

$$\begin{aligned}
 (f \circ g)(m) &= f(m^3) \\
 &= (m^3)^2 \\
 &= m^6 \\
 &= (g \circ f)(m).
 \end{aligned}$$

So no matter the value of  $m \in \mathbb{Z}$ , we always have  $(g \circ f)(m) = (f \circ g)(m)$ . We can then say that  $g \circ f = f \circ g$ .

- (b) i.  $(\beta \circ \alpha)(m) = \beta(\alpha(m)) = \beta(\frac{m}{2}) = \text{abs}(\frac{m}{2})$ . There's nothing much more we can do to simplify that.
- ii. If  $m = -\pi$ , then  $\alpha(m) = \alpha(-\pi) = \frac{-\pi}{2}$ . Since  $-\pi \approx -3.14159$ , then  $\frac{-\pi}{2} \approx -1.57$ . (In general, don't truncate numbers like  $\pi$  until you have to - so just keep using the symbol  $\pi$  until you actually need a numerical value out of it.) So  $\beta(\alpha(-\pi)) \approx \text{abs}(-1.57) \approx 1.57$ .
- iii. If  $m = -\pi$ , then  $\beta(m) = \text{abs}(-\pi) = \pi$ . So  $\alpha(\beta(-\pi)) \approx (1.57)$ .
- iv. We can just think about this first without putting in any values. One of the functions,  $\alpha$ , halves the value we put into it. The other function,  $\beta$ , takes the absolute value of the number. It doesn't matter what order we do this in, so we always have  $\beta \circ \alpha = \alpha \circ \beta$ .
- (c) i.  $(q \circ p)(a) = x, (q \circ p)(b) = y, (q \circ p)(c) = z$   
 ii. Since the sets  $\{a, b, c\}$  and  $\{x, y, z\}$  are not equal, we cannot define the other composite  $p \circ q$  since the domains and codomains don't match up in the correct way.  
 iii. Define  $r: \{x, y, z\} \rightarrow \{a, b, c\}$  by  $r(x) = b, r(y) = b$ , and  $r(z) = a$ .
4. (Challenge) This was originally a guided classroom exercise, so is far harder than I'd expect you to tackle by yourself. However, you might find it interesting to try and work through it.
- (a) i.  $\mathbb{P}(X) = \{\emptyset\}$ , so  $|\mathbb{P}(X)| = 1$ ;  
 ii.  $\mathbb{P}(Y) = \{\emptyset, \{0\}\}$ , so  $|\mathbb{P}(Y)| = 2$ ;  
 iii.  $\mathbb{P}(Z) = \{\emptyset, \{0\}, \{1\}, \{0, 1\}\}$ , so  $|\mathbb{P}(Z)| = 4$ ;

- iv.  $\mathbb{P}(W) = \{\emptyset, \{0\}, \{1\}, \{2\}, \{0, 1\}, \{0, 2\}, \{1, 2\}, \{0, 1, 2\}\}$ , so  $|\mathbb{P}(W)| = 8$ .
- (b)
  - i.  $X = \emptyset$ : there is one function  $\emptyset \rightarrow \{0, 1\}$  - this can be tricky to see since the domain is empty;
  - ii.  $Y = \{0\}$ : there are two possible functions here - send 0 to 0, or send 0 to 1;
  - iii.  $Z = \{0, 1\}$ : there are four possible functions here - send both elements to 0, send both elements to 1, send both elements to themselves, or swap the values;
  - iv.  $W = \{0, 1, 2\}$ : this time there are eight possible functions, which you may want to try and write down.
- (c) Given a set  $B$ , the number of subsets of  $B$  is the same as the number of functions  $B \rightarrow \{0, 1\}$ . I.e., for any set  $B$  we seem to be seeing that  $|\mathbb{P}(B)| = |\text{Fun}(B, \{0, 1\})|$ .
- (d) Let  $A$  be a subset of  $B$ . We need to define a function  $\varphi(A): B \rightarrow \{0, 1\}$ . One way of doing this is to define  $\varphi(A)(b) = 0$  if  $b \notin A$  and  $\varphi(A)(b) = 1$  if  $b \in A$ . So what we are doing is sending an element  $b \in B$  to 1 (True) if the element is also in the subset  $A$ , otherwise we send it to 0 (False).
- (e) Can you define a function  $\psi: \text{Fun}(B, \{0, 1\}) \rightarrow \mathbb{P}(B)$ ? Let  $f: B \rightarrow \{0, 1\}$  be a function. We need to define a subset  $\psi(f) = A_f \subseteq B$ . We can define this by

$$A_f = \{b \in B \mid f(b) = 1\}.$$

This is similar to how we defined the function in the previous part.