

Statistics and Probability

Alex Corner

Sheffield Hallam University

Statistics

Some terminology

Some terminology

- ▶ **Population:** *All* relevant data.

Some terminology

- ▶ **Population:** *All* relevant data.
- ▶ **Sample:** *Some* of the relevant data, hopefully enough to be *representative* of the population.

Some terminology

- ▶ **Population:** *All* relevant data.
- ▶ **Sample:** *Some* of the relevant data, hopefully enough to be *representative* of the population.
 - ▶ This should be a **random sample** with no **bias** towards a group or individual in the population.

Some terminology

- ▶ **Population:** *All* relevant data.
- ▶ **Sample:** *Some* of the relevant data, hopefully enough to be *representative* of the population.
 - ▶ This should be a **random sample** with no **bias** towards a group or individual in the population.
 - ▶ This can be difficult to achieve and careful thought needs to be given as to how data is collected.

Some terminology

- ▶ **Population:** *All* relevant data.
- ▶ **Sample:** *Some* of the relevant data, hopefully enough to be *representative* of the population.
 - ▶ This should be a **random sample** with no **bias** towards a group or individual in the population.
 - ▶ This can be difficult to achieve and careful thought needs to be given as to how data is collected.
- ▶ Sample/population **data** can be:
 - ▶ **Discrete:** E.g., number of attacks on a port in a given hour of the day.

Some terminology

- ▶ **Population:** *All* relevant data.
- ▶ **Sample:** *Some* of the relevant data, hopefully enough to be *representative* of the population.
 - ▶ This should be a **random sample** with no **bias** towards a group or individual in the population.
 - ▶ This can be difficult to achieve and careful thought needs to be given as to how data is collected.
- ▶ Sample/population **data** can be:
 - ▶ **Discrete:** E.g., number of attacks on a port in a given hour of the day. Usually integer-valued, e.g., 0, 1, 2, etc.

Some terminology

- ▶ **Population:** *All* relevant data.
- ▶ **Sample:** *Some* of the relevant data, hopefully enough to be *representative* of the population.
 - ▶ This should be a **random sample** with no **bias** towards a group or individual in the population.
 - ▶ This can be difficult to achieve and careful thought needs to be given as to how data is collected.
- ▶ Sample/population **data** can be:
 - ▶ **Discrete:** E.g., number of attacks on a port in a given hour of the day. Usually integer-valued, e.g., 0, 1, 2, etc.
 - ▶ **Continuous:** E.g., amount of time elapsed since previous attack.

Some terminology

- ▶ **Population:** *All* relevant data.
- ▶ **Sample:** *Some* of the relevant data, hopefully enough to be *representative* of the population.
 - ▶ This should be a **random sample** with no **bias** towards a group or individual in the population.
 - ▶ This can be difficult to achieve and careful thought needs to be given as to how data is collected.
- ▶ Sample/population **data** can be:
 - ▶ **Discrete:** E.g., number of attacks on a port in a given hour of the day. Usually integer-valued, e.g., 0, 1, 2, etc.
 - ▶ **Continuous:** E.g., amount of time elapsed since previous attack. Often classified into groups, e.g., 0 – 5 minutes, 5 – 10 minutes, 10 – 15 minutes, etc.

Grouping Data

- ▶ Suppose that we have data from a random sample of 100 people, giving their age at their last birthday.

Grouping Data

- ▶ Suppose that we have data from a random sample of 100 people, giving their age at their last birthday.

66	70	10	54	62	13	11	15	69	26	49	11	3
67	10	54	42	32	56	39	60	79	33	12	47	24
19	47	63	32	7	70	55	46	11	20	15	39	37
28	72	46	64	61	51	56	53	61	11	80	53	28
76	6	5	39	58	29	52	54	47	60	62	51	72
41	57	32	12	33	17	40	20	10	27	47	71	68
44	7	23	17	81	23	12	33	16	46	71	48	58
79	80	43	31	72	68	36	41	11				

- ▶ It's a little hard to get a feel for the data, so we could count the frequency in each bin, order the data numerically, or calculate various **statistics**.

Grouping Data

- ▶ Suppose that we have data from a random sample of 100 people, giving their age at their last birthday.

66	70	10	54	62	13	11	15	69	26	49	11	3
67	10	54	42	32	56	39	60	79	33	12	47	24
19	47	63	32	7	70	55	46	11	20	15	39	37
28	72	46	64	61	51	56	53	61	11	80	53	28
76	6	5	39	58	29	52	54	47	60	62	51	72
41	57	32	12	33	17	40	20	10	27	47	71	68
44	7	23	17	81	23	12	33	16	46	71	48	58
79	80	43	31	72	68	36	41	11				

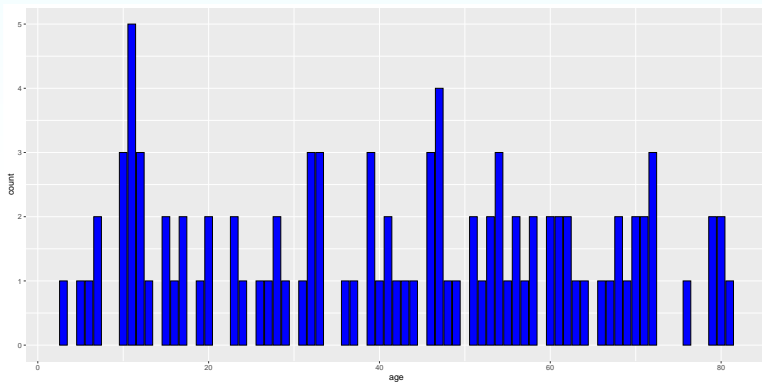
- ▶ It's a little hard to get a feel for the data, so we could count the frequency in each bin, order the data numerically, or calculate various **statistics**.
- ▶ In doing so, we lose some of the information but might get a clearer overview.

Grouping Data

- Here we have placed the data into bins of **class width 5**.

Group Number	Age Range	Tally	Number in Group (Frequency)
1	$0 \leq x < 5$		1
2	$5 \leq x < 10$		4
3	$10 \leq x < 15$		12
4	$15 \leq x < 20$		6
5	$20 \leq x < 25$		5
6	$25 \leq x < 30$		5
7	$30 \leq x < 35$		7
8	$35 \leq x < 40$		5
9	$40 \leq x < 45$		6
10	$45 \leq x < 50$		9
11	$50 \leq x < 55$		8
12	$55 \leq x < 60$		6
13	$60 \leq x < 65$		8
14	$65 \leq x < 70$		5
15	$70 \leq x < 75$		7
16	$75 \leq x < 80$		3
17	$80 \leq x < 85$		3
18	$85 \leq x < 90$		0
19	$90 \leq x < 95$		0
20	$95 \leq x < 100$		0
21	$100 \leq x < 105$		0

Charts: Bar Chart



Charts: Histogram



Charts: Histogram



Measures of Central Tendency

Measures of Central Tendency

- ▶ **Arithmetic Mean:** Add up all of the values and divide by how many there are.

Measures of Central Tendency

- ▶ **Arithmetic Mean:** Add up all of the values and divide by how many there are.
- ▶ **Median:** Order the data and look at the middle value(s).

Measures of Central Tendency

- ▶ **Arithmetic Mean:** Add up all of the values and divide by how many there are.
- ▶ **Median:** Order the data and look at the middle value(s).
- ▶ **Mode:** Tally the data and select the one with the highest frequency.

(Arithmetic) Mean

(Arithmetic) Mean

- In a sample of five values (6, 9, 2, 4, 3) the mean value is:

$$\frac{6 + 9 + 2 + 4 + 3}{5} = \frac{24}{5} = 4.8.$$

(Arithmetic) Mean

- ▶ In a sample of five values (6, 9, 2, 4, 3) the mean value is:

$$\frac{6 + 9 + 2 + 4 + 3}{5} = \frac{24}{5} = 4.8.$$

- ▶ The one hundred values from the previous histogram example add up to 4165 and so the mean value is $\frac{4165}{100} = 41.65$.

(Arithmetic) Mean

- ▶ In a sample of five values (6, 9, 2, 4, 3) the mean value is:

$$\frac{6 + 9 + 2 + 4 + 3}{5} = \frac{24}{5} = 4.8.$$

- ▶ The one hundred values from the previous histogram example add up to 4165 and so the mean value is $\frac{4165}{100} = 41.65$.
- ▶ For n data values x_1, x_2, \dots, x_n , the formula for the mean is:

$$\bar{x} = \frac{x_1 + x_2 + \dots + x_n}{n}.$$

(Arithmetic) Mean

- ▶ In a sample of five values (6, 9, 2, 4, 3) the mean value is:

$$\frac{6 + 9 + 2 + 4 + 3}{5} = \frac{24}{5} = 4.8.$$

- ▶ The one hundred values from the previous histogram example add up to 4165 and so the mean value is $\frac{4165}{100} = 41.65$.
- ▶ For n data values x_1, x_2, \dots, x_n , the formula for the mean is:

$$\bar{x} = \frac{x_1 + x_2 + \dots + x_n}{n}.$$

Median

Median

- ▶ Here we arrange the data values in ascending (or descending) order. Then the **median** is:
 1. the middle item if there is an odd number of data items;
 2. the average of the middle two items if there is an even number of data items.

Median

- ▶ Here we arrange the data values in ascending (or descending) order. Then the **median** is:
 1. the middle item if there is an odd number of data items;
 2. the average of the middle two items if there is an even number of data items.
- ▶ E.g., if the ordered data is 2, 3, 4, 6, 9, then the middle item is 4 and so the median is 4.

Median

- ▶ Here we arrange the data values in ascending (or descending) order. Then the **median** is:
 1. the middle item if there is an odd number of data items;
 2. the average of the middle two items if there is an even number of data items.
- ▶ E.g., if the ordered data is 2, 3, 4, 6, 9, then the middle item is 4 and so the median is 4.
- ▶ From our histogram example, the sample of one hundred data values, the ordered data is:

3	5	6	7	7	10	10	10	11	11	11	11	11
12	12	12	13	15	15	16	17	17	19	20	20	23
23	24	26	27	28	28	29	31	32	32	32	33	33
33	36	37	39	39	39	40	41	41	42	43	44	46
46	46	47	47	47	47	48	49	51	51	52	53	53
54	54	54	55	56	56	57	58	58	60	60	61	61
62	62	63	64	66	67	68	68	69	70	70	71	71
72	72	72	76	79	79	80	80	81				

Median

- ▶ Here we arrange the data values in ascending (or descending) order. Then the **median** is:
 1. the middle item if there is an odd number of data items;
 2. the average of the middle two items if there is an even number of data items.
- ▶ E.g., if the ordered data is 2, 3, 4, 6, 9, then the middle item is 4 and so the median is 4.
- ▶ From our histogram example, the sample of one hundred data values, the ordered data is:

3	5	6	7	7	10	10	10	11	11	11	11	11
12	12	12	13	15	15	16	17	17	19	20	20	23
23	24	26	27	28	28	29	31	32	32	32	33	33
33	36	37	39	39	39	40	41	41	42	43	44	46
46	46	47	47	47	47	48	49	51	51	52	53	53
54	54	54	55	56	56	57	58	58	60	60	61	61
62	62	63	64	66	67	68	68	69	70	70	71	71
72	72	72	76	79	79	80	80	81				

- ▶ The middle two values are 43 and 44, hence the median is 43.5.

Mode

Mode

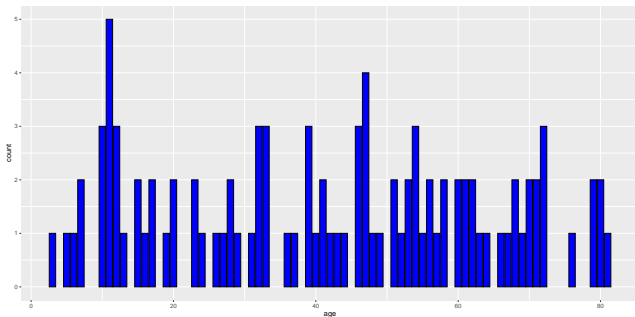
- ▶ The **mode** is the data value which occurs most in the sample.

Mode

- ▶ The **mode** is the data value which occurs most in the sample.
- ▶ For grouped data we can talk of a **modal group** or **modal class**, sometimes specifying the mid-point in order to give a single value.

Mode

- ▶ The **mode** is the data value which occurs most in the sample.
- ▶ For grouped data we can talk of a **modal group** or **modal class**, sometimes specifying the mid-point in order to give a single value.
- ▶ For the one hundred data values in the histogram example the mode is clearly indicated on the bar chart as the tallest bar: 11. We can also tally values to find it.



Probability

Definition of Probability

Definition of Probability

- ▶ A is a statement which is either true or false.

Definition of Probability

- ▶ A is a statement which is either true or false.
- ▶ N is the number of trials, or observations.

Definition of Probability

- ▶ A is a statement which is either true or false.
- ▶ N is the number of trials, or observations.
- ▶ S is the number of successes (so $N - S$ is the number of failures).

Definition of Probability

- ▶ A is a statement which is either true or false.
- ▶ N is the number of trials, or observations.
- ▶ S is the number of successes (so $N - S$ is the number of failures).
- ▶ For a single trial: Probability that A is true, or the **probability of success**, is:

$$P(A) = \frac{S}{N}.$$

Definition of Probability

- ▶ A is a statement which is either true or false.
- ▶ N is the number of trials, or observations.
- ▶ S is the number of successes (so $N - S$ is the number of failures).
- ▶ For a single trial: Probability that A is true, or the **probability of success**, is:

$$P(A) = \frac{S}{N}.$$

- ▶ Since $0 \leq S \leq N$, then $0 \leq P(A) \leq 1$.

Definition of Probability

- ▶ A is a statement which is either true or false.
- ▶ N is the number of trials, or observations.
- ▶ S is the number of successes (so $N - S$ is the number of failures).
- ▶ For a single trial: Probability that A is true, or the **probability of success**, is:

$$P(A) = \frac{S}{N}.$$

- ▶ Since $0 \leq S \leq N$, then $0 \leq P(A) \leq 1$.
- ▶ The *probability of failure* is:

$$P(\bar{A}) = \frac{N - S}{N} = 1 - \frac{S}{N} = 1 - P(A).$$

Probability: Example 1

Tossing a coin:

Probability: Example 1

Tossing a coin:

- ▶ Toss a coin 4 times: we *expect* 2 heads.

Probability: Example 1

Tossing a coin:

- ▶ Toss a coin 4 times: we *expect* 2 heads.
- ▶ Toss a coin 100 times: we *expect* 50 heads.

Probability: Example 1

Tossing a coin:

- ▶ Toss a coin 4 times: we *expect* 2 heads.
- ▶ Toss a coin 100 times: we *expect* 50 heads.
- ▶ Toss a coin once: what do we *expect*?

Probability: Example 1

Tossing a coin:

- ▶ Toss a coin 4 times: we *expect* 2 heads.
- ▶ Toss a coin 100 times: we *expect* 50 heads.
- ▶ Toss a coin once: what do we *expect*?

$$P(\text{one head in one trial}) = \frac{1}{2}.$$

Probability: Example 1

Tossing a coin:

- ▶ Toss a coin 4 times: we *expect* 2 heads.
- ▶ Toss a coin 100 times: we *expect* 50 heads.
- ▶ Toss a coin once: what do we *expect*?

$$P(\text{one head in one trial}) = \frac{1}{2}.$$

- ▶ $A = \text{'a head shows when a coin is tossed'}$, so $P(A) = \frac{1}{2}$.

Probability: Example

Group Number	Age Range	Tally	Number in Group (Frequency)
1	$0 \leq x < 5$		1
2	$5 \leq x < 10$		4
3	$10 \leq x < 15$		12
4	$15 \leq x < 20$		6
5	$20 \leq x < 25$		5
6	$25 \leq x < 30$		5
7	$30 \leq x < 35$		7
8	$35 \leq x < 40$		5
9	$40 \leq x < 45$		6
10	$45 \leq x < 50$		9
11	$50 \leq x < 55$		8
12	$55 \leq x < 60$		6
13	$60 \leq x < 65$		8
14	$65 \leq x < 70$		5
15	$70 \leq x < 75$		7
16	$75 \leq x < 80$		3
17	$80 \leq x < 85$		3
18	$85 \leq x < 90$		0
19	$90 \leq x < 95$		0
20	$95 \leq x < 100$		0
21	$100 \leq x < 105$		0

Age at last birthday:

Probability: Example

Group Number	Age Range	Tally	Number in Group (Frequency)
1	$0 \leq x < 5$		1
2	$5 \leq x < 10$		4
3	$10 \leq x < 15$		12
4	$15 \leq x < 20$		6
5	$20 \leq x < 25$		5
6	$25 \leq x < 30$		5
7	$30 \leq x < 35$		7
8	$35 \leq x < 40$		5
9	$40 \leq x < 45$		6
10	$45 \leq x < 50$		9
11	$50 \leq x < 55$		8
12	$55 \leq x < 60$		6
13	$60 \leq x < 65$		8
14	$65 \leq x < 70$		5
15	$70 \leq x < 75$		7
16	$75 \leq x < 80$		3
17	$80 \leq x < 85$		3
18	$85 \leq x < 90$		0
19	$90 \leq x < 95$		0
20	$95 \leq x < 100$		0
21	$100 \leq x < 105$		0

Age at last birthday:

- Probability that one of the people selected at random is in the 15-19 age group.

Probability: Example

Group Number	Age Range	Tally	Number in Group (Frequency)
1	$0 \leq x < 5$		1
2	$5 \leq x < 10$		4
3	$10 \leq x < 15$		12
4	$15 \leq x < 20$		6
5	$20 \leq x < 25$		5
6	$25 \leq x < 30$		5
7	$30 \leq x < 35$		7
8	$35 \leq x < 40$		5
9	$40 \leq x < 45$		6
10	$45 \leq x < 50$		9
11	$50 \leq x < 55$		8
12	$55 \leq x < 60$		6
13	$60 \leq x < 65$		8
14	$65 \leq x < 70$		5
15	$70 \leq x < 75$		7
16	$75 \leq x < 80$		3
17	$80 \leq x < 85$		3
18	$85 \leq x < 90$		0
19	$90 \leq x < 95$		0
20	$95 \leq x < 100$		0
21	$100 \leq x < 105$		0

Age at last birthday:

- ▶ Probability that one of the people selected at random is in the 15-19 age group.
- ▶ The total number of outcomes is 100 and the number of 'successes' is 6.

Probability: Example

Group Number	Age Range	Tally	Number in Group (Frequency)
1	$0 \leq x < 5$		1
2	$5 \leq x < 10$		4
3	$10 \leq x < 15$		12
4	$15 \leq x < 20$		6
5	$20 \leq x < 25$		5
6	$25 \leq x < 30$		5
7	$30 \leq x < 35$		7
8	$35 \leq x < 40$		5
9	$40 \leq x < 45$		6
10	$45 \leq x < 50$		9
11	$50 \leq x < 55$		8
12	$55 \leq x < 60$		6
13	$60 \leq x < 65$		8
14	$65 \leq x < 70$		5
15	$70 \leq x < 75$		7
16	$75 \leq x < 80$		3
17	$80 \leq x < 85$		3
18	$85 \leq x < 90$		0
19	$90 \leq x < 95$		0
20	$95 \leq x < 100$		0
21	$100 \leq x < 105$		0

Age at last birthday:

- ▶ Probability that one of the people selected at random is in the 15-19 age group.
- ▶ The total number of outcomes is 100 and the number of 'successes' is 6.
- ▶ The probability of success is: $P(\text{Success}) = \frac{6}{100}$.

Probability: Example

Group Number	Age Range	Tally	Number in Group (Frequency)
1	$0 \leq x < 5$		1
2	$5 \leq x < 10$		4
3	$10 \leq x < 15$		12
4	$15 \leq x < 20$		6
5	$20 \leq x < 25$		5
6	$25 \leq x < 30$		5
7	$30 \leq x < 35$		7
8	$35 \leq x < 40$		5
9	$40 \leq x < 45$		6
10	$45 \leq x < 50$		9
11	$50 \leq x < 55$		8
12	$55 \leq x < 60$		6
13	$60 \leq x < 65$		8
14	$65 \leq x < 70$		5
15	$70 \leq x < 75$		7
16	$75 \leq x < 80$		3
17	$80 \leq x < 85$		3
18	$85 \leq x < 90$		0
19	$90 \leq x < 95$		0
20	$95 \leq x < 100$		0
21	$100 \leq x < 105$		0

Age at last birthday:

- ▶ Probability that one of the people selected at random is in the 15-19 age group.
- ▶ The total number of outcomes is 100 and the number of 'successes' is 6.
- ▶ The probability of success is: $P(\text{Success}) = \frac{6}{100}$.
- ▶ The probability of failure is: $P(\text{Failure}) = \frac{94}{100}$.

Probability: Example 3

Rolling dice:

Probability: Example 3

Rolling dice:

- ▶ Two dice are rolled and the sum of the spots is calculated.

Probability: Example 3

Rolling dice:

- ▶ Two dice are rolled and the sum of the spots is calculated.
- ▶ $A =$ 'The sum is 7 when two dice are rolled.'

Probability: Example 3

Rolling dice:

- ▶ Two dice are rolled and the sum of the spots is calculated.
- ▶ $A =$ 'The sum is 7 when two dice are rolled.'

(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

Probability: Example 3

Rolling dice:

- ▶ Two dice are rolled and the sum of the spots is calculated.
- ▶ $A =$ 'The sum is 7 when two dice are rolled.'

(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

- ▶ There are 36 possible outcomes when rolling two dice. But only six of these outcomes make A true.

Probability: Example 3

Rolling dice:

- ▶ Two dice are rolled and the sum of the spots is calculated.
- ▶ $A =$ 'The sum is 7 when two dice are rolled.'

(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

- ▶ There are 36 possible outcomes when rolling two dice. But only six of these outcomes make A true.
- ▶ Number of trials $N = 36$, number of successes $S = 6$.

$$P(A) = \frac{6}{36} = \frac{1}{6}.$$



And, Or: Intersection, Union

- ▶ Last semester you used $A \cdot B$ to mean A AND B . We will sometimes use $A \wedge B$ instead.

And, Or: Intersection, Union

- ▶ Last semester you used $A \cdot B$ to mean A AND B . We will sometimes use $A \wedge B$ instead.
- ▶ Its analogue in set theory is **intersection**: $A \cap B$.

And, Or: Intersection, Union

- ▶ Last semester you used $A \cdot B$ to mean A AND B . We will sometimes use $A \wedge B$ instead.
- ▶ Its analogue in set theory is **intersection**: $A \cap B$.
- ▶ Last semester you used $A + B$ to mean A OR B . We will sometimes use $A \vee B$ instead.

And, Or: Intersection, Union

- ▶ Last semester you used $A \cdot B$ to mean A AND B . We will sometimes use $A \wedge B$ instead.
- ▶ Its analogue in set theory is **intersection**: $A \cap B$.
- ▶ Last semester you used $A + B$ to mean A OR B . We will sometimes use $A \vee B$ instead.
- ▶ Its analogue in set theory is **union**: $A \cup B$.

Law of multiplication

Law of multiplication

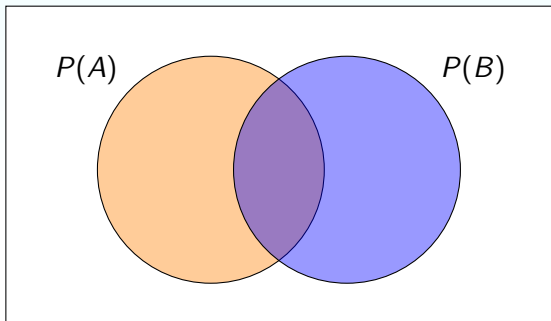
- ▶ Consider two events A and B .

Law of multiplication

- ▶ Consider two events A and B .
- ▶ $P(A)$ and $P(B)$ are denoted by the sets in the Venn diagram.

Law of multiplication

- ▶ Consider two events A and B .
- ▶ $P(A)$ and $P(B)$ are denoted by the sets in the Venn diagram. U



- ▶ The intersection of the sets is $P(A \cap B)$: this is the probability A and B are both true.

Law of multiplication

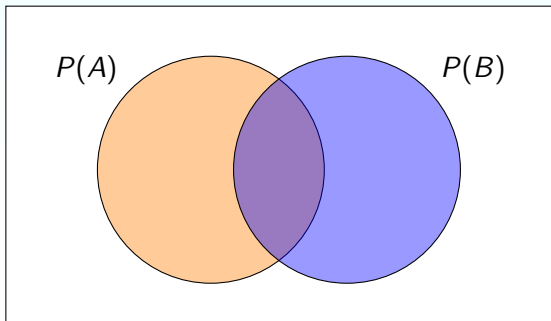
- ▶ Consider two events A and B .
- ▶ $P(A)$ and $P(B)$ are denoted by the sets in the Venn diagram. U



- ▶ The intersection of the sets is $P(A \wedge B)$: this is the probability A and B are both true.
- ▶ If A and B are mutually exclusive (i.e., can't both be true at the same time), then $P(A \wedge B) = 0$.

Law of multiplication

- ▶ Consider two events A and B .
- ▶ $P(A)$ and $P(B)$ are denoted by the sets in the Venn diagram. U



- ▶ The intersection of the sets is $P(A \wedge B)$: this is the probability A and B are both true.
- ▶ If A and B are mutually exclusive (i.e., can't both be true at the same time), then $P(A \wedge B) = 0$.
- ▶ If A and B are **independent** (i.e., the outcome of one does not affect the other), then $P(A \wedge B) = P(A) \times P(B)$.

Probability: Example 4

Probability: Example 4

- ▶ A pack of 52 playing cards has 26 red cards and 26 black cards.

Probability: Example 4

- ▶ A pack of 52 playing cards has 26 red cards and 26 black cards.
- ▶ One card is drawn from the full pack at random, then replaced in the pack.

Probability: Example 4

- ▶ A pack of 52 playing cards has 26 red cards and 26 black cards.
- ▶ One card is drawn from the full pack at random, then replaced in the pack.
- ▶ Then a second card is drawn from the pack.

Probability: Example 4

- ▶ A pack of 52 playing cards has 26 red cards and 26 black cards.
- ▶ One card is drawn from the full pack at random, then replaced in the pack.
- ▶ Then a second card is drawn from the pack.
- ▶ Calculate the probability that both cards are red.

Probability: Example 4

- ▶ A pack of 52 playing cards has 26 red cards and 26 black cards.
- ▶ One card is drawn from the full pack at random, then replaced in the pack.
- ▶ Then a second card is drawn from the pack.
- ▶ Calculate the probability that both cards are red.
- ▶ Let $A =$ 'The first card is red' and $B =$ 'The second card is red'.

Probability: Example 4

- ▶ A pack of 52 playing cards has 26 red cards and 26 black cards.
- ▶ One card is drawn from the full pack at random, then replaced in the pack.
- ▶ Then a second card is drawn from the pack.
- ▶ Calculate the probability that both cards are red.
- ▶ Let $A =$ 'The first card is red' and $B =$ 'The second card is red'.
- ▶ $P(A) = \frac{26}{52} = \frac{1}{2}$ and $P(B) = \frac{26}{52} = \frac{1}{2}$. The two events are independent, so:

$$P(A \wedge B) = P(A) \times P(B) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}.$$

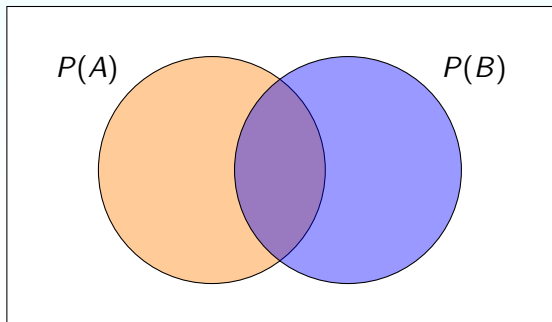
Law of Addition

Law of Addition

- ▶ Consider two events A and B .

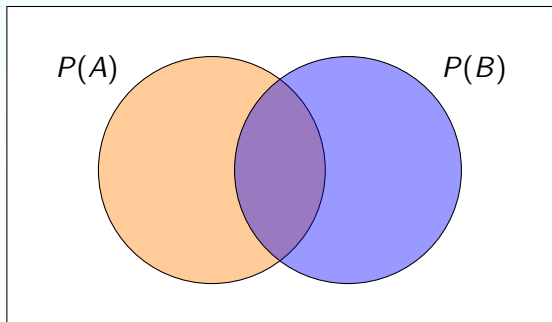
Law of Addition

- ▶ Consider two events A and B .
- ▶ $P(A)$ and $P(B)$ are denoted by the sets in the Venn diagram. U



Law of Addition

- ▶ Consider two events A and B .
- ▶ $P(A)$ and $P(B)$ are denoted by the sets in the Venn diagram. U



- ▶ The union of the sets is $P(A \cup B)$: this is the probability that A or B are true.

Law of Addition

- ▶ Consider two events A and B .
- ▶ $P(A)$ and $P(B)$ are denoted by the sets in the Venn diagram. U



- ▶ The union of the sets is $P(A \cup B)$: this is the probability that A or B are true.
- ▶ If A and B are mutually exclusive, then $P(A \cup B) = P(A) + P(B)$.

Law of Addition

- ▶ Consider two events A and B .
- ▶ $P(A)$ and $P(B)$ are denoted by the sets in the Venn diagram. U

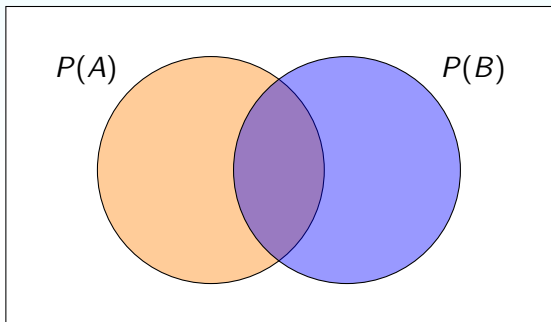


- ▶ The union of the sets is $P(A \vee B)$: this is the probability that A or B are true.
- ▶ If A and B are mutually exclusive, then $P(A \vee B) = P(A) + P(B)$.
- ▶ In general,

$$P(A \vee B) = P(A) + P(B)$$

Law of Addition

- ▶ Consider two events A and B .
- ▶ $P(A)$ and $P(B)$ are denoted by the sets in the Venn diagram. U



- ▶ The union of the sets is $P(A \vee B)$: this is the probability that A or B are true.
- ▶ If A and B are mutually exclusive, then $P(A \vee B) = P(A) + P(B)$.
- ▶ In general,

$$P(A \vee B) = P(A) + P(B) - P(A \wedge B).$$

Probability: Example 4 continued

Probability: Example 4 continued

- ▶ Same setup as our previous example, but now we want the probability that at least one card is red. ($P(A) = P(B) = \frac{1}{2}$)

Probability: Example 4 continued

- ▶ Same setup as our previous example, but now we want the probability that at least one card is red. ($P(A) = P(B) = \frac{1}{2}$)
- ▶ We want to find $P(A \vee B) = P(A) + P(B) - P(A \wedge B)$. Since A and B are independent, then $P(A \wedge B) = P(A) \times P(B)$.

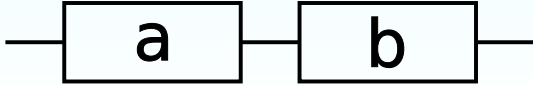
Probability: Example 4 continued

- ▶ Same setup as our previous example, but now we want the probability that at least one card is red. ($P(A) = P(B) = \frac{1}{2}$)
- ▶ We want to find $P(A \vee B) = P(A) + P(B) - P(A \wedge B)$. Since A and B are independent, then $P(A \wedge B) = P(A) \times P(B)$. So:

$$P(A) + P(B) - P(A) \times P(B) = \frac{1}{2} + \frac{1}{2} - \frac{1}{2} \times \frac{1}{2} = \frac{3}{4}.$$

Example: Network 1

- ▶ A short network link consists of two sections (a) and (b), in series as shown below.



Example: Network 1

- ▶ A short network link consists of two sections (a) and (b), in series as shown below.



- ▶ The link functions only if both (a) and (b) both function.

Example: Network 1

- ▶ A short network link consists of two sections (a) and (b), in series as shown below.



- ▶ The link functions only if both (a) and (b) both function.
- ▶ The two sections are independent. If one fails, the other is not affected.

Example: Network 1

- ▶ A short network link consists of two sections (a) and (b), in series as shown below.



- ▶ The link functions only if both (a) and (b) both function.
- ▶ The two sections are independent. If one fails, the other is not affected.
- ▶ The probability that (a) functions is $P(A) = 0.8$, while for (b) the probability is $P(B) = 0.9$.

Example: Network 1

- ▶ A short network link consists of two sections (a) and (b), in series as shown below.

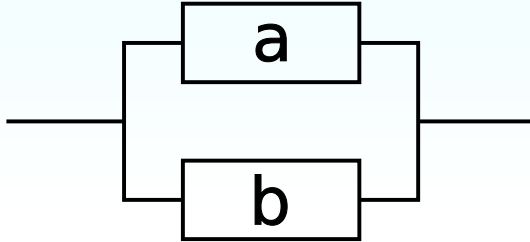


- ▶ The link functions only if both (a) and (b) both function.
- ▶ The two sections are independent. If one fails, the other is not affected.
- ▶ The probability that (a) functions is $P(A) = 0.8$, while for (b) the probability is $P(B) = 0.9$.
- ▶ The probability that the network functions (both sections function) is:

$$P(A \wedge B) = P(A) \times P(B) = 0.8 \times 0.9 = 0.72.$$

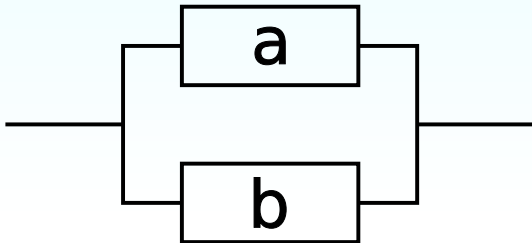
Example: Network 2

- ▶ A short network link consists of two sections (a) and (b), in parallel as shown below.



Example: Network 2

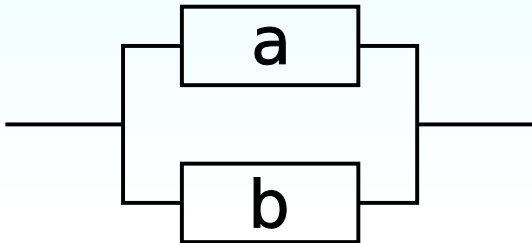
- ▶ A short network link consists of two sections (a) and (b), in parallel as shown below.



- ▶ The link functions if either of (a) or (b) function (or both).

Example: Network 2

- ▶ A short network link consists of two sections (a) and (b), in parallel as shown below.



- ▶ The link functions if either of (a) or (b) function (or both).
- ▶ The two sections are independent. If one fails, the other is not affected.

Example: Network 2

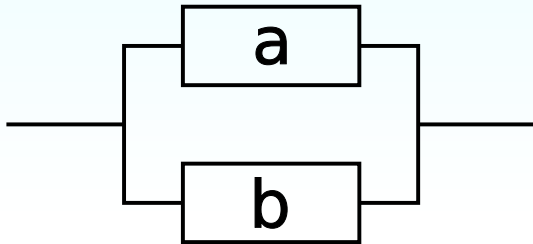
- ▶ A short network link consists of two sections (a) and (b), in parallel as shown below.



- ▶ The link functions if either of (a) or (b) function (or both).
- ▶ The two sections are independent. If one fails, the other is not affected.
- ▶ The probability that (a) functions is $P(A) = 0.8$, while for (b) the probability is $P(B) = 0.9$.

Example: Network 2

- ▶ A short network link consists of two sections (a) and (b), in parallel as shown below.



- ▶ The link functions if either of (a) or (b) function (or both).
- ▶ The two sections are independent. If one fails, the other is not affected.
- ▶ The probability that (a) functions is $P(A) = 0.8$, while for (b) the probability is $P(B) = 0.9$.
- ▶ The probability that the network functions is:

$$P(A \vee B) = P(A) + P(B) - P(A \wedge B) = 0.8 + 0.9 - 0.72 = 0.98.$$

Conditional Probability

- ▶ When two events are *not independent*, then the process is not as simple.

Conditional Probability

- ▶ When two events are *not independent*, then the process is not as simple.
- ▶ Consider a single trial consisting of two events A and B .

Conditional Probability

- ▶ When two events are *not independent*, then the process is not as simple.
- ▶ Consider a single trial consisting of two events A and B .
- ▶ The **conditional probability** $P(B|A)$ means 'the probability that B will be true *given that* A is already true'.

Probability: Example 5

- ▶ A pack of 52 playing cards has 26 red cards and 26 black cards.

Probability: Example 5

- ▶ A pack of 52 playing cards has 26 red cards and 26 black cards.
- ▶ One card is drawn from the full pack at random, then *kept*.

Probability: Example 5

- ▶ A pack of 52 playing cards has 26 red cards and 26 black cards.
- ▶ One card is drawn from the full pack at random, then *kept*.
- ▶ Then a second card is drawn from the pack.

Probability: Example 5

- ▶ A pack of 52 playing cards has 26 red cards and 26 black cards.
- ▶ One card is drawn from the full pack at random, then *kept*.
- ▶ Then a second card is drawn from the pack.
- ▶ Calculate the probability that both cards are red.

Probability: Example 5

- ▶ A pack of 52 playing cards has 26 red cards and 26 black cards.
- ▶ One card is drawn from the full pack at random, then *kept*.
- ▶ Then a second card is drawn from the pack.
- ▶ Calculate the probability that both cards are red.
- ▶ Let $A =$ 'The first card is red' and $B =$ 'The second card is red'.

Probability: Example 5

- ▶ A pack of 52 playing cards has 26 red cards and 26 black cards.
- ▶ One card is drawn from the full pack at random, then *kept*.
- ▶ Then a second card is drawn from the pack.
- ▶ Calculate the probability that both cards are red.
- ▶ Let $A =$ 'The first card is red' and $B =$ 'The second card is red'.
- ▶ $P(A) = \frac{26}{52} = \frac{1}{2}$ as before but $P(B)$ is a conditional probability because it depends on the result of event A .

Probability: Example 5

- ▶ A pack of 52 playing cards has 26 red cards and 26 black cards.
- ▶ One card is drawn from the full pack at random, then *kept*.
- ▶ Then a second card is drawn from the pack.
- ▶ Calculate the probability that both cards are red.
- ▶ Let $A =$ 'The first card is red' and $B =$ 'The second card is red'.
- ▶ $P(A) = \frac{26}{52} = \frac{1}{2}$ as before but $P(B)$ is a conditional probability because it depends on the result of event A .
- ▶ If the first card was red, then there are now only 25 cards left in the remaining 51 cards, so:

$$P(B|A) = \frac{25}{51}.$$

Probability: Example 5

- ▶ A pack of 52 playing cards has 26 red cards and 26 black cards.
- ▶ One card is drawn from the full pack at random, then *kept*.
- ▶ Then a second card is drawn from the pack.
- ▶ Calculate the probability that both cards are red.
- ▶ Let $A =$ 'The first card is red' and $B =$ 'The second card is red'.
- ▶ $P(A) = \frac{26}{52} = \frac{1}{2}$ as before but $P(B)$ is a conditional probability because it depends on the result of event A .
- ▶ If the first card was red, then there are now only 25 cards left in the remaining 51 cards, so:

$$P(B|A) = \frac{25}{51}.$$

- ▶ This means the probability that the second card is red given the first card was red is:

$$P(A \wedge B) = P(B|A) \times P(A) = \frac{25}{51} \times \frac{1}{2} = \frac{25}{102}.$$

Conditional Probability

- ▶ The **conditional probability** can be expressed as

$$P(B|A) = \frac{P(A \wedge B)}{P(A)}.$$

Conditional Probability

- ▶ The **conditional probability** can be expressed as

$$P(B|A) = \frac{P(A \wedge B)}{P(A)}.$$

- ▶ This can be rearranged as

$$P(A \wedge B) = P(B|A) \times P(A).$$

Conditional Probability

- ▶ The **conditional probability** can be expressed as

$$P(B|A) = \frac{P(A \wedge B)}{P(A)}.$$

- ▶ This can be rearranged as

$$P(A \wedge B) = P(B|A) \times P(A).$$

- ▶ If A and B are independent, then

$$P(B|A) = P(B).$$

Probability: Example 6

- Calculate the probability that the first card is black and the second card is red.

Probability: Example 6

- ▶ Calculate the probability that the first card is black and the second card is red.
- ▶ Both cards are red:

$$P(A \wedge B) = P(B|A) \times P(A) = \frac{25}{51} \times \frac{1}{2} = \frac{25}{102}.$$

Probability: Example 6

- ▶ Calculate the probability that the first card is black and the second card is red.
- ▶ Both cards are red:

$$P(A \wedge B) = P(B|A) \times P(A) = \frac{25}{51} \times \frac{1}{2} = \frac{25}{102}.$$

- ▶ First card is black, second card is red:

$$P(\bar{A} \wedge B) = P(B|\bar{A}) \times P(\bar{A}) = \frac{26}{51} \times \frac{1}{2} = \frac{26}{102}.$$

Probability Trees

- ▶ For small examples, we can often visualise things in the form of a **probability tree**.

Probability Trees

- For small examples, we can often visualise things in the form of a **probability tree**.

