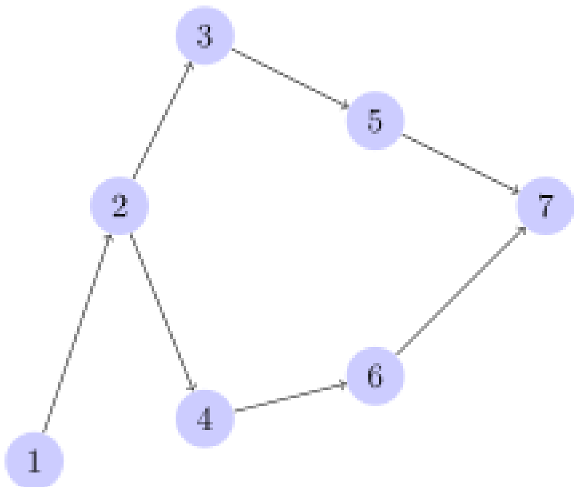
Chapter 1

Directed Graphs

Digraphs

A **digraph** (or **di**rected **graph**) is like a graph except that the edges have a definite direciton between a pair of vertices. They correspond to *ordered* pairs of vertices. The vertices are sometimes called **nodes** and the directed edges are called **arcs**. A diagram showing a digraph looks like a graph with arrows on the edges.



For most applications there will be at most one arc from one given node to another given node, but it is perfectly acceptable for there also to be an arc in the other direction between the same pair of nodes, and sometimes there may be more than one arc in the same direction. For some applications, we also allow an arc from a node to itself, and this is called a **loop** as before.

If there is an arc from node i to node j then we say that node j is **adjacent** to node i. We can describe this in the digraph above.

- 2 is adjacent to 1
- 3 is adjacent to 2

- 4 is adjacent to 2
- 5 is adjacent to 3
- 6 is adjacent to 4
- 7 is adjacent to 5 and 6

Walks, trails, paths, and cycles are defined as for a graph, except that they must follow the direction of the edges. A **semipath** is like a path, except that we ignore the direction of the arrows. A digraph with no cycles is called **acyclic**.

The **indegree** of a node is the number of arcs entering a node. The **outdegree** of a node is the number of arcs leaving a node.

Digraphs, like graphs, find many applications. Some are listed below:

- Road networks;
- Social media networks (such as Twitter, since following somebody need not mean that they follow you back);
- Flow charts;
- Scheduling theory;
- Deterministic finite automata (DFAs).

Connectivity in Graphs

There are different levels of connectivity between two nodes i and j in a digraph.

Strong

Node j is **strongly connected** to node i if there is a walk from node i to node j and also a walk from node j to node i. There is no need for the walks to follow similar routes. Equivalently we can say that there is a closed walk which visits both nodes. Perhaps surprisingly, there need not be a simple cycle which visits both nodes.

Unilateral

Node j is **unilaterally connected** to node i if at least one of the nodes is reachable from the other. In other words, there is either a walk from node i to node j or a walk from node j to node i, or both.

Weak

Node j is **weakly connected** to node i if there is a semipath from node i to node j. Of course in this case there will also be a semipath from node j to node i.

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Connectivity of Graphs

The above definitions of the levels of connectivity between nodes in a graph can also be applied to a whole digraph.

Strong

A digraph is **strongly connected** if every pair of nodes is strongly connected. In other words, it is strongly connected if every node is reachable from every other node.

Unilateral

A digraph is **unilaterally connected** if every pair of nodes is unilaterally connected.

Weakly Connected

A digraph is **weakly connected** if every pair of nodes is weakly connected. In other words, it is weakly connected if the graph which can be obtained by ignoring the directions of the arcs is connected.

So every digraph is either strongly connected, unilaterally connected, weakly connected, or not connected. If a digraph is strongly connected, it is also both unilaterally connected and weakly connected. Similarly a digraph which is unilaterally connected is also weakly connected.

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Sources and Sinks

A node is a **sink** if there are no arcs going out of it. (So if you enter a sink, you cannot escape.) A node is a **source** if there are no arcs going into it. (So a source can't be reached from anywhere else.)

A **strong component** is a collection of nodes which, together with all the arcs between them, form a strongly connected digraph, which is not part of a larger strongly connected digraph.

We can extend the definition of a **source** to mean a strong component with no arcs going into it. Similarly, a **sink** is a strong component with no arcs going out of it.

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