Matrix Operations

Adding and Subtracting

We'll return to some more basic definitions, this time how we can add and subtract matrices. If we have two matrices of the same size, then we can add them together *elementwise*; which means we just add the respective entries. For example, if we have the matrices

$$A = \begin{pmatrix} 1 & 0 & 4 \\ 3 & -1 & -6 \end{pmatrix}, B = \begin{pmatrix} 2 & 3 & -1 \\ 1 & 1 & 5 \end{pmatrix},$$

then

$$A + B = \begin{pmatrix} 1+2 & 0+3 & 4+(-1) \\ 3+1 & (-1)+1 & (-6)+5 \end{pmatrix} = \begin{pmatrix} 3 & 3 & 3 \\ 4 & 0 & -1 \end{pmatrix}$$

and

$$A - B = \begin{pmatrix} 1-2 & 0-3 & 4-(-1) \\ 3-1 & (-1)-1 & (-6)-5 \end{pmatrix} = \begin{pmatrix} -1 & -3 & 5 \\ 2 & -2 & -11 \end{pmatrix}.$$

Video Visit the URL below to view a video:
`https://www.youtube.com/embed/f-YTVlUzaJc`

Scaling a Matrix

If A is a matrix and $k \in \mathbb{R}$, then we can calculate the scaled matrix kA, simply by multiplying each element in the matrix by k. For example, if

$$A = \begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix},$$

then

$$\frac{1}{2}A = \begin{pmatrix} \frac{1}{2}\times 2 & \frac{1}{2}\times -1 \\ \frac{1}{2}\times 3 & \frac{1}{2}\times 4 \end{pmatrix} = \begin{pmatrix} 1 & -\frac{1}{2} \\ \frac{3}{2} & 2 \end{pmatrix}$$

and

$$2A = \begin{pmatrix} 2\times 2 & 2\times -1 \\ 2\times 3 & 2\times 4 \end{pmatrix} = \begin{pmatrix} 4 & -2 \\ 6 & 8 \end{pmatrix}.$$

Video Visit the URL below to view a video:
`https://www.youtube.com/embed/OK8wRz3ZBTM`

	Product A	**Product** B	**Product** C
January	100	300	500
February	200	500	400

Product A	45
Product B	50
Product C	80

Multiplying

Multiplication of matrices is not done elementwise as addition and subtraction are. We will look at a couple of small examples and more will be explained in the video.

Suppose, as an example, that we have the following tables where the first represents the number of products sold and the second represents the selling price (in pounds). We can calculate the sales revenue for each month as follows:

$$\begin{pmatrix} 100 & 300 & 500 \\ 200 & 500 & 400 \end{pmatrix} \begin{pmatrix} 45 \\ 50 \\ 80 \end{pmatrix} = \begin{pmatrix} (100 \times 45) + (300 \times 50) + (500 \times 80) \\ (200 \times 45) + (500 \times 50) + (400 \times 80) \end{pmatrix} = \begin{pmatrix} 59500 \\ 66000 \end{pmatrix}.$$

Another, smaller example is shown below. If we have matrices

$$A = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}, B = \begin{pmatrix} 5 & -1 \\ 0 & -2 \end{pmatrix}$$

then their matrix product, denoted AB, is given by

$$AB = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} 5 & -1 \\ 0 & -2 \end{pmatrix} = \begin{pmatrix} (1 \times 5) + (3 \times 0) & (1 \times -1) + (3 \times -2) \\ (2 \times 5) + (4 \times 0) & (2 \times -1) + (4 \times -2) \end{pmatrix} = \begin{pmatrix} 5 & -7 \\ 10 & -10 \end{pmatrix}.$$

Notice that in the resulting matrix, the entry in the ith row and jth column is found by a combination of the entries in the ith row of the first matrix, A, and the entries in the jth column of the second matrix, B.

In general, an $m \times k$ matrix can be multiplied by a $k \times n$ matrix, with the result being an $m \times n$ matrix. What this means is that the number of columns of the first matrix must be the same as the number of rows of the second matrix.

For another example, consider the matrices

$$A = \begin{pmatrix} 3 & 4 & 6 \\ 1 & 2 & 5 \end{pmatrix}, B = \begin{pmatrix} 2 & -1 \\ 0 & -3 \\ 1 & 4 \end{pmatrix}.$$

We can multiply these because A is a 2×3 matrix and B is a 3×2 matrix. In particular we can actually multiply these in either order to find AB or BA. However, note that AB will be a 2×2 matrix and BA will be a 3×3 matrix.

$$AB = \begin{pmatrix} 3 & 4 & 6 \\ 1 & 2 & 5 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 0 & -3 \\ 1 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} (3\times 2)+(4\times 0)+(6\times 1) & (3\times -1)+(4\times -3)+(6\times 4) \\ (1\times 2)+(2\times 0)+(5\times 1) & (1\times -1)+(2\times -3)+(5\times 4) \end{pmatrix}$$

$$= \begin{pmatrix} 12 & 9 \\ 7 & 13 \end{pmatrix}$$

$$BA = \begin{pmatrix} 2 & -1 \\ 0 & -3 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} 3 & 4 & 6 \\ 1 & 2 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} (2\times 3)+(-1\times 1) & (2\times 4)+(-1\times 2) & (2\times 6)+(-1\times 5) \\ (0\times 3)+(-3\times 1) & (0\times 4)+(-3\times 2) & (0\times 6)+(-3\times 5) \\ (1\times 3)+(4\times 1) & (1\times 4)+(4\times 2) & (1\times 6)+(4\times 5) \end{pmatrix}$$

$$= \begin{pmatrix} 5 & 6 & 7 \\ -3 & -6 & -15 \\ 7 & 12 & 26 \end{pmatrix}$$

Video Visit the URL below to view a video:
`https://www.youtube.com/embed/HQLpMxFYhIc`

For a brief introduction to an application of matrix multiplication, see the following video.

Video Visit the URL below to view a video:
`https://www.youtube.com/embed/4vsokrTsZCw`

Properties of Matrix Operations

As we have seen in the previous section, the order of multiplication for matrices is important, since matrix multiplication does not obey the *commutative law.* In general, $AB \neq BA$. This section will include some further terminology, as well as highlighting some of the properties that matrix addition and multiplication have. In the following, we will refer to matrices A, B, and C and assume they are compatible for adding or multiplying in the relevant contexts.

We say that the matrix A is **pre-multiplied** by B in the matrix product BA. We say that the matrix A is **post-multiplied** by B in the matrix product AB.

The following laws hold:

- Commutative law for addition: $A + B = B + A$;
- Associative law for addition: $(A + B) + C = A + (B + C)$;
- Associative law for multiplication: $(AB)C = A(BC)$;
- Distributive laws: $A(B + C) = AB + AC$ and $(B + C)A = BA + CA$.

A **zero** matrix, often denoted 0, is a matrix with all the elements being zero. For example,

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

These can be any size. Unlike numbers, if the product $AB = 0$ we cannot deduce that either A or B is a zero matrix. Try finding the following matrix product to see this.

$$\begin{pmatrix} 1 & 2 \\ -2 & -4 \end{pmatrix} \begin{pmatrix} -2 & 6 \\ 1 & -3 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

More Matrix Definitions

An **identity matrix** of size n is an $n \times n$ square matrix I, or I_n, with the property that $AI = A$ and $IB = B$ for any $a \times n$ matrix A and for any $n \times b$ matrix B. In particular, if C is any $n \times n$ matrix, then $CI = C = IC$.

An identity matrix has zeros everywhere except along the leading diagonal (top left to bottom right) where instead all of the elements are 1. For example,

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

are the 2×2 and 3×3 identity matrices, respectively.

A **diagonal matrix** is a square matrix with all the elements zero except along the leading diagonal, where they can take any value (including zero). For example,

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

is a diagonal matrix, as are identity matrices and zero matrices.

The **transpose** of a matrix A, denoted A^T or A', is obtained by interchanging the rows and columns of A. For example, if

$$A = \begin{pmatrix} -1 & 2 & 3 \\ 0 & 4 & 6 \end{pmatrix}$$

then the transpose is given by

$$A^T = \begin{pmatrix} -1 & 0 \\ 2 & 4 \\ 3 & 6 \end{pmatrix}.$$

Note that

$$(A + B)^T = A^T + B^T, (A - B)^T = A^T - B^T, (AB)^T = B^T A^T.$$

For a square matrix A, we can define a number called its **determinant**, written $\det(A)$ or $\mid A \mid$. For a 2×2 matrix, the definition is

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

For example,

$$\begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix} = (2 \times 4) - (1 \times 3) = 5.$$

This will be important in the next section.

Concept Checks

Test Yourself Visit the URL below to try a numbas exam:
`https://numbas.mathcentre.ac.uk/exam/16464/matrix-operations/embed/`