Algebra

Algebraic Expressions

Algebra can be thought of as generalised arithmetic. The numerical processes of addition, subtraction, multiplication, division etc. are applied to symbols representing numbers rather than numbers themselves. This leads to powerful abstract reasoning which permeates many areas of human endeavour, particularly science, engineering, computing, economics. Numbers are represented by symbols, usually letters, and grouped together to form algebraic expressions.

Consider the following set of instructions: ''Think of a number, double it, add 5, double the result, subtract 2'. We can represent this as an algebraic expression using the following table.

Instruction	**Algebraic Expression**	**Simplified Form**
Think of a number	n	n
Double it	$2n$	$2n$
Add 5	$2n+5$	$2n+5$
Double the result	$2(2n+5)$	$4n+10$
Subtract 2	$4n+10-2$	$4n+8$

The algebraic expression $4n+8$ is the equivalent of our set of instructions. The expression contains one **variable**, n. If we give a value to n the algebraic expression takes an arithmetic value. For example, if $n=3$, then $4n+8=4\times 3+8=20$. (Note that this is just the value that we would get if we applied the original instructions to the number 3.)

Note that in our construction of the algebraic expression we have not explicitly written a multiplication sign, thus doubling the original number is expressed as $2n$ rather than $2\times n$. In addition we have simplified the expression $2(2n+5)$ by multiplying each term in the bracket by the factor 2 outside the bracket. I.e.,

$$2(2n+5)=2\times 2n+2\times 5=4n+10.$$

Simplifying Algebraic Expressions

Algebraic expressions are simplified by collecting together like terms. For example, the expression $a+a^2-5a+4a$ may be simplified by collecting together the terms involving a. It becomes simply a^2.

Often algebraic expressions involve more than one variable, exactly the same principles apply in their simplification. For example, in simplifying the expression

$$x^2 - 4xy + 4y^2 + 6x^2 + 5xy + y^2 = 7x^2 + 5y^2 + xy.$$

Expanding

Often we have to deal with bracketed terms in algebra. These are simplified by multiplying them out.

For example, $3(a-2b)$ may be simplified by multiplying each term inside the bracket by the factor outside. This process is called **expanding** the bracket. Thus

$$3(a-2b) = 3 \times a - 3 \times (-2b) = 3a - 6b.$$

Often we need to cope with the product of two brackets. We use the same principles, but need to be careful we don't miss anything. As an example, consider the expression $(a+b)(a+b)$. To expand this we proceed as follows.

- Multiply first term in first bracket by first term in in second bracket to get: a^2,
- Multiply first term in first bracket by second term in in second bracket to get: ab,
- Multiply second term in first bracket by first term in in second bracket to get: ba,
- Multiply second term in first bracket by second term in in second bracket to get: b^2,
- Add together all terms to get: $a^2 + ab + ba + b^2$,
- Simplify to get: $a^2 + 2ab + b^2$.

Factorising

This is the inverse of expanding, and is harder. Mathematicians develop a fluency in this area, given time. We will look at it briefly.

Look at $3a - 12b$. Both of the terms in this expression have a factor of 3, so we can simplify the expression to $3(a-4b)$. This is the **factorised** version of $3a - 12b$. Easy enough, but now look at $3a^2 - 12ab$. Now the two terms have factors of 3 and a. We can write it as $3a(a-4b)$.

Now what about $10x^2y^2 - 15xy^3$. Each term has a factor of 5 ($10 = 2 \times 5$, $15 = 3 \times 5$), and in addition each term has factors of x and y^2. So the expression can be factorised as $5xy^2(2x - 3y)$.

When expressions consist of three terms factorisation is not so easy, and sometimes not possible at all. We will focus only on quadratic expressions, i.e., expressions like $x^2 + 5x - 6$, or $x^2 - 4x + 5$, or $2x^2 + 7x - 4$, where the highest power involved is 2. First of all recall that when linear brackets are expanded then a quadratic expression results.

For example,

$$(x+1)(x+2) = x^2 + 3x + 2,$$

$$(a-2)(2a+5) = 2a^2 + a - 10.$$

So we want to find out whether it is possible to write quadratic expressions as the product of two linear factors. This process is called **factorisation**.

Consider $x^2 + 4x + 3$. Can we write this as $(x+a)(x+b)$ where a and b are integers? Suppose we can, i.e.,

$$x^2 + 4x + 3 = (x+a)(x+b) = x^2 + ax + bx + ab = x^2 + (a+b)x + ab.$$

So we want a and b to satisfy $a+b = 4$ and $ab = 3$. The number 3 has factors 1 and 3. Adding these numbers together gives 4, so we let $a = 1$ and $b = 3$. Thus

$$x^2 + 4x + 3 = (x+1)(x+3).$$

Now let's try this one: factorist $x^2 - 7x + 12$. Proceeding as before:

$$x^2 - 7x + 12 = (x+a)(x+b) = x^2 + (a+b)x + ab.$$

So $a + b = -7$ and $ab = 12$. The factors of 12 are $(12,1)$ or $(6,2)$ or $(4,3)$ or $(-12,-1)$ or $(-6,-2)$ or $(-4,-3)$. Only the last pair add together to get -7, so we have $a = -4$ and $b = -3$. So

$$x^2 - 7x + 12 = (x-4)(x-3).$$