Exercises 3: Precedence and Implication

Exercises

1. Insert parentheses in the following Boolean expressions as specified by the precedence rules. Then rewrite them in the alternative notation using "+" and "." for "$\vee$" and "$\wedge$". Then simplify them as much as you can using the laws of Boolean algebra.

   (a) $p \vee q \wedge r \vee s \wedge \neg p \vee q \wedge \neg r \vee s$

   (b) $\neg \neg p \vee q \wedge \neg p \vee \neg q \wedge \neg r \vee \neg \neg s \wedge p$

   (c) $p \wedge q \vee r \Longrightarrow q \vee \neg p \wedge r$

2. Write the following statements in the form $p \Longrightarrow q$ or $p \Leftrightarrow q$ for suitable p and q:

   (a) If the cap fits, he will wear it.

   (b) You need to be dedicated to work here.

   (c) You should not cross the line while the red lights flash.

   (d) You may cross the line if and only if the light is green.

   (e) Finding the knife was enough to convict him.

   (f) You may board the plane only if you arrive before 10 a.m.

   (g) For the program to compile, it is necessary and sufficient that it is syntactically correct.

3. Construct truth tables for the following formulae and then show that two of them are equal:

   (a) $\neg q \Longrightarrow \neg p$

   (b) $p \Longrightarrow (q \Longrightarrow r)$

   (c) $(p \Longrightarrow q) \Longrightarrow r$

   (d) $p \wedge q \Longrightarrow r$

4. (Why is the implication truth table what it is?) It is easy to understand that $p \Longrightarrow q$ must be true when p and q are both true, and false when p is true but q is false. Some people find it harder to accept that $p \Longrightarrow q$ is

always true when p is false. If we are not sure of the correct value in these cases, there are four possibilities for the definition of $\Longrightarrow$. We'll call these connectives α, β, γ and δ, listed in the following truth table.

p	q	$p\ \alpha\ q$	$p\ \beta\ q$	$p\ \gamma\ q$	$p\ \delta\ q$
F	F	F	F	T	T
F	T	F	T	F	T
T	F	F	F	F	F
T	T	T	T	T	T

One way to resolve the difficulty is to use the fact that p must imply p OR q whatever q might be, so we want $p \Longrightarrow (p \lor q)$ to be a tautology.

Work out the truth table for this statement with the various possible interpretations of $\Longrightarrow$, or, in other works, complete the following truth table:

p	q	$p\ \alpha\ (p \lor q)$	$p\ \beta\ (p \lor q)$	$p\ \gamma\ (p \lor q)$	$p\ \delta\ (p \lor q)$
F	F				
F	T				
T	F				
T	T				

Solutions

1. (a)

$$p \lor q \land r \lor s \land \neg p \lor q \land \neg r \lor s$$

$$= p \lor (q \land r) \lor (s \land (\neg p)) \lor (q \land (\neg r)) \lor s$$

$$= p \lor (q \land r) \lor (q \land (\neg r)) \lor s \lor (s \land (\neg p)) \quad \text{(commutativity)}$$

$$= p \lor (q \land r) \lor (q \land (\neg r)) \lor s \lor (s \land (\neg p)) \quad \text{(commutativity)}$$

$$= p \lor (q \land r) \lor (q \land (\neg r)) \lor s \quad \text{(absorption)}$$

$$= p \lor (q \land (r \lor (\neg r))) \lor s \quad \text{(distributivity)}$$

$$= p \lor (q \land T) \lor s \quad \text{(cancellation)}$$

$$= p \lor q \lor s \quad \text{(identity)}$$

(b)

$$\neg \neg p \lor q \land \neg p \lor \neg q \land \neg r \lor \neg \neg s \land p$$

$$= (\neg (\neg p)) \lor (q \land (\neg p)) \lor ((\neg q) \land (\neg r)) \lor (\neg (\neg s) \land p)$$

$$= p \lor (q \land (\neg p)) \lor ((\neg q) \land (\neg r)) \lor (s \land p) \quad \text{(involution)}$$

$$= p \lor ((\neg p) \land q) \lor ((\neg q) \land (\neg r)) \lor (p \land s) \quad \text{(commutativity)}$$

$$= p \lor q \lor ((\neg\, q) \land (\neg\, r)) \lor (p \land s) \qquad \text{(negative absorption)}$$

$$= p \lor q \lor (\neg\, r) \lor (p \land s) \qquad \text{(negative absorption)}$$

$$= p \lor q \lor (p \land s) \lor (\neg\, r) \qquad \text{(commutativity)}$$

$$= p \lor (p \land s) \lor q \lor (\neg\, r) \qquad \text{(commutativity)}$$

$$= p \lor q \lor (\neg\, r) \qquad \text{(absorption)}$$

(c)

$$p \land q \lor r \Rightarrow q \lor \neg\, p \land r$$

$$= ((p \land q) \lor r) \Rightarrow (q \lor ((\neg\, p) \land r))$$

$$= \neg\, ((p \land q) \lor r) \lor (q \lor ((\neg\, p) \land r))) \qquad (\Rightarrow)$$

$$= (\neg\, (p \land q) \land \neg\, r)) \lor (q \lor ((\neg\, p) \land r))) \qquad \text{(de Morgan)}$$

$$= ((\neg\, p \lor \neg\, q) \land \neg\, r)) \lor (q \lor ((\neg\, p) \land r)) \qquad \text{(de Morgan)}$$

$$= ((\neg\, p \land \neg\, r) \lor (\neg\, q \land \neg\, r)) \lor (q \lor ((\neg\, p \land r))) \qquad \text{(distributivity)}$$

$$= (\neg\, p \land \neg\, r) \lor (q \lor (\neg\, q \land \neg\, r)) \lor ((\neg\, p \land r))) \qquad \text{(commutativity)}$$

$$= (\neg\, p \land \neg\, r) \lor (q \lor \neg\, r) \lor ((\neg\, p \land r))) \qquad \text{(negative absorption)}$$

$$= (\neg\, p \land \neg\, r) \lor q \lor \neg\, r \lor (r \land \neg\, p)) \qquad \text{(commutativity)}$$

2. (a) $p =$ "the cap fits", $q =$ "he will wear it": $p \Rightarrow q$
   (b) $p =$ "you are dedicated", $q =$ "you work here": $q \Rightarrow p$
   (c) $p =$ "do not cross the line", $q =$ "the red lights are flashing": $q \Rightarrow p$
   (d) $p =$ "you may cross the line", $q =$ "the light is green": $p \Leftrightarrow q$
   (e) $p =$ "the knife was found", $q =$ "he was convicted": $p \Rightarrow q$ (It is *sufficient* to have found the knife in order to convict him but it was not *necessary*.)
   (f) $p =$ "you may board the plane", $q =$ "you arrive before 10a.m.": $q \Rightarrow p$
   (g) $p =$ "the program compiles", $q =$ "the program is syntactically correct": $p \Leftrightarrow q$

3. $p \Rightarrow (q \Rightarrow r) = p \land q \Rightarrow r$

4.

p	q	$p \;\; \alpha \;\; (p \lor q)$	$p \;\; \beta \;\; (p \lor q)$	$p \;\; \gamma \;\; (p \lor q)$	$p \;\; \delta \;\; (p \lor q)$
F	F	F	F	T	T
F	T	F	F	F	T
T	F	T	T	T	T
T	T	T	T	T	T

The column for α is the same as for p, the column for β is the same as $p \lor q$, the column for γ is the same as $p \Rightarrow q$, and the column for δ is a tautology (all T).