

# Exercises: Matrix Equations

## Exercises 2

1. Let

$$A = \begin{pmatrix} -3 & 5 \\ 4 & 3 \end{pmatrix}.$$

- (a) Find the determinant of  $A$ .
- (b) Find the inverse of  $A$ .
- (c) Use  $A^{-1}$  to find a solution for the simultaneous equations

$$-3x + 5y = 58,$$

$$4x + 3y = -29.$$

- (d) Use  $A^{-1}$  to find a solution for the simultaneous equations

$$-3x + 5y = -2,$$

$$4x + 3y = 1.$$

2. Let

$$B = \begin{pmatrix} 7 & -3 \\ 3 & 5 \end{pmatrix}.$$

- (a) Find the determinant of  $B$ .
- (b) Find the inverse of  $B$ .
- (c) Use  $B^{-1}$  to find a solution for the simultaneous equations

$$7a - 3b = 5,$$

$$3a + 5b = 7.$$

- (d) Use  $B^{-1}$  to find a solution for the simultaneous equations

$$7u - 3v = 44,$$

$$3u + 5v = -44.$$

3. Consider the following pair of simultaneous equations.

$$17x - 5y = 21$$

$$6x + 22y = 13$$

- (a) Write these equations as a single matrix equation.  
 (b) Use matrix inverses and matrix multiplication to solve these equations for  $x$  and  $y$ .

4. Consider the following pair of simultaneous equations.

$$91x + 14y = 121$$

$$65x + 10y = -72$$

- (a) Write these equations as a single matrix equation.  
 (b) Use determinants to show that there are no solutions for these equations for  $x$  and  $y$ .

5. Calculate the matrix product

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix},$$

and hence check that the formula for the inverse of a matrix, given below, is correct.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

6. If  $A$  and  $B$  are invertible matrices of the same size, simplify  $(B^{-1}A^{-1})(AB)$ .  
 Hence show that  $(AB)^{-1} = B^{-1}A^{-1}$ .

## Solutions

1. (a)

$$\det(A) = (-3 \times 3) - (5 \times 4) = -9 - 20 = -29.$$

- (b)

$$A^{-1} = \frac{1}{-29} \begin{pmatrix} 3 & -5 \\ -4 & -3 \end{pmatrix}$$

- (c)

$$A^{-1} \begin{pmatrix} 58 \\ -29 \end{pmatrix} = \frac{1}{-29} \begin{pmatrix} 3 & -5 \\ -4 & -3 \end{pmatrix} \begin{pmatrix} 58 \\ -29 \end{pmatrix} = \begin{pmatrix} -11 \\ 5 \end{pmatrix}$$

- (d)

$$A^{-1} \begin{pmatrix} -2 \\ 1 \end{pmatrix} = \frac{1}{-29} \begin{pmatrix} 3 & -5 \\ -4 & -3 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{11}{29} \\ \frac{-5}{29} \end{pmatrix}$$

2. (a)

$$\det(B) = (7 \times 5) - (-3 \times 3) = 35 - (-9) = 44.$$

(b)

$$B^{-1} = \frac{1}{44} \begin{pmatrix} 5 & 3 \\ -3 & 7 \end{pmatrix}$$

(c)

$$B^{-1} \begin{pmatrix} 5 \\ 7 \end{pmatrix} = \frac{1}{44} \begin{pmatrix} 5 & 3 \\ -3 & 7 \end{pmatrix} \begin{pmatrix} 5 \\ 7 \end{pmatrix} = \begin{pmatrix} \frac{46}{44} \\ \frac{34}{44} \end{pmatrix}$$

(d)

$$B^{-1} \begin{pmatrix} 44 \\ -44 \end{pmatrix} = \frac{1}{44} \begin{pmatrix} 5 & 3 \\ -3 & 7 \end{pmatrix} \begin{pmatrix} 44 \\ -44 \end{pmatrix} = \begin{pmatrix} 2 \\ -10 \end{pmatrix}$$

3. (a) We can formulate the simultaneous equations as a single matrix equation:

$$\begin{pmatrix} 17 & -5 \\ 6 & 22 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 21 \\ 13 \end{pmatrix}.$$

(b) The determinant of the coefficient matrix is

$$\det \left( \begin{pmatrix} 17 & -5 \\ 6 & 22 \end{pmatrix} \right) = (17 \times 22) - (-5 \times 6) = 404,$$

hence there are solutions to the pair of simultaneous equations. The inverse of the coefficient matrix is given by

$$\begin{pmatrix} 17 & -5 \\ 6 & 22 \end{pmatrix}^{-1} = \frac{1}{404} \begin{pmatrix} 22 & 5 \\ -6 & 17 \end{pmatrix}.$$

The solutions are hence given by

$$\frac{1}{404} \begin{pmatrix} 22 & 5 \\ -6 & 17 \end{pmatrix} \begin{pmatrix} 21 \\ 13 \end{pmatrix} = \begin{pmatrix} \frac{527}{404} \\ \frac{95}{404} \end{pmatrix}.$$

4. (a) We can formulate the simultaneous equations as a single matrix equation:

$$\begin{pmatrix} 91 & 14 \\ 65 & 10 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 121 \\ -72 \end{pmatrix}.$$

(b) The determinant of the coefficient matrix is

$$\det \left( \begin{pmatrix} 91 & 14 \\ 65 & 10 \end{pmatrix} \right) = (91 \times 10) - (14 \times 65) = 0,$$

hence there are no solutions to the pair of simultaneous equations.

5.

$$\begin{aligned} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} &= \begin{pmatrix} ad - bc & -ba + ba \\ cd - dc & -bc + ad \end{pmatrix} \\ &= \begin{pmatrix} ad - bc & 0 \\ 0 & ad - bc \end{pmatrix} \\ &= (ad - bc) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

$$= (ad - bc)I$$

The inverse of a matrix should satisfy  $AA^{-1}$ . The calculation shows that

$$\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

almost works but we still need to divide out by a factor of  $(ad - bc)$ .

6.

$$\begin{aligned} (B^{-1}A^{-1})(AB) &= B^{-1}(A^{-1}A)B \\ &= B^{-1}IB \\ &= B^{-1}B \\ &= I \end{aligned}$$

Similarly,  $AB(B^{-1}A^{-1}) = I$ . The inverse of a matrix  $C$  is a matrix  $C^{-1}$  such that  $CC^{-1} = I = C^{-1}C$ . Since  $B^{-1}A^{-1}$  does this job for  $AB$ , then we have found that

$$(AB)^{-1} = B^{-1}A^{-1}.$$