Lax monoidal model structures

Alex Corner

University of Sheffield

BMC 2015

- A lax monoidal category is like a monoidal category but there is a tensor product for each natural number and constraint cells need not be isomorphisms.

- A lax monoidal category is like a monoidal category but there is a tensor product for each natural number and constraint cells need not be isomorphisms.
- Model categories give a setting in which to study the homotopy theory of given objects. Monoidal model categories give their homotopy categories a monoidal structure.

- A lax monoidal category is like a monoidal category but there is a tensor product for each natural number and constraint cells need not be isomorphisms.
- Model categories give a setting in which to study the homotopy theory of given objects. Monoidal model categories give their homotopy categories a monoidal structure.
- Questions: Monoidal structures and model structures don't always play nice. (E.g. **Gray**-**Cat**.) Can we develop a sensible theory of lax monoidal model categories? What examples can we study in this framework?

Plan

Plan

1. Lax monoidal categories

Plan

1. Lax monoidal categories
2. Lax monoidal model categories

Plan

1. Lax monoidal categories
2. Lax monoidal model categories
3. A model structure on lax monoids

Lax monoidal categories

Lax monoidal categories

A *lax monoidal category* C consists of

Lax monoidal categories

A *lax monoidal category* $\mathcal{C}$ consists of

- functors $E_n\colon \mathcal{C}^n \to \mathcal{C}$, for $n \in \mathbb{N}$

Lax monoidal categories

A *lax monoidal category* $\mathcal{C}$ consists of

- functors $E_n \colon \mathcal{C}^n \to \mathcal{C}$, for $n \in \mathbb{N}$
- constraints

$$\gamma \colon E_n(E_{k_1}, \ldots, E_{k_n}) \Rightarrow E_{\Sigma k_i} \qquad I \colon id_{\mathcal{C}} \Rightarrow E_1$$

Lax monoidal categories

A *lax monoidal category* $\mathcal{C}$ consists of

- functors $E_n \colon \mathcal{C}^n \to \mathcal{C}$, for $n \in \mathbb{N}$
- constraints

$$\gamma \colon E_n(E_{k_1}, \ldots, E_{k_n}) \Rightarrow E_{\Sigma k_i} \qquad I \colon id_{\mathcal{C}} \Rightarrow E_1$$

Associators

$$(x \otimes y) \otimes z \xrightarrow{\cong} x \otimes (y \otimes z) \qquad E_2(E_2(x, y), z) \xrightarrow{\cong} E_2(x, E_2(y, z))$$

Lax monoidal categories

A *lax monoidal category* $\mathcal{C}$ consists of

- functors $E_n \colon \mathcal{C}^n \to \mathcal{C}$, for $n \in \mathbb{N}$
- constraints

$$\gamma \colon E_n(E_{k_1}, \ldots, E_{k_n}) \Rightarrow E_{\Sigma k_i} \qquad I \colon id_{\mathcal{C}} \Rightarrow E_1$$

Associators

$$(x \otimes y) \otimes z \xrightarrow{\cong} x \otimes (y \otimes z) \qquad E_2(E_2(x, y), z) \xrightarrow{\cong} E_2(x, E_2(y, z))$$

are replaced by morphisms

$$((x_{11} \otimes \ldots \otimes x_{1k_1}) \otimes \ldots \otimes (x_{n1} \otimes \ldots \otimes x_{nk_n})) \to (x_{11} \otimes \ldots \otimes x_{nk_n})$$

$$E_n(E_{k_1}(\underline{X}_1), \ldots, E_{k_n}(\underline{X}_n)) \to E_{\Sigma k_i}(\underline{X}_1, \ldots, \underline{X}_n)$$

Lax monoidal categories: a construction

Lax monoidal categories: a construction

Given an operad P in a symmetric monoidal category $\mathcal{V}$ we can define

$$E_n^P(X_1, \ldots, X_n) \colon = P(n) \otimes X_1 \otimes \ldots X_n$$

Lax monoidal categories: a construction

Given an operad P in a symmetric monoidal category $\mathcal{V}$ we can define

$$E_n^P(X_1, \ldots, X_n) := P(n) \otimes X_1 \otimes \ldots X_n$$

Constraint cells are given by a combination of the operad structure of P and the monoidal constraints of $\mathcal{V}$.

Lax monoidal categories: a construction

Given an operad P in a symmetric monoidal category $\mathcal{V}$ we can define

$$E_n^P(X_1, \dots, X_n) := P(n) \otimes X_1 \otimes \dots X_n$$

Constraint cells are given by a combination of the operad structure of P and the monoidal constraints of $\mathcal{V}$.

$$\otimes(P_n, P_{k_1}, \underline{X}_1, \dots, P_{k_n}, \underline{X}_n) \cong \otimes(P_n, \underline{P}_{k_i}, \underline{\underline{X}}_i) \to \otimes(P_{\Sigma k_i}, \underline{\underline{X}}_i)$$
$$X \cong I \otimes X \to P_1 \otimes X$$

Lax monoidal categories: a construction

Given an operad P in a symmetric monoidal category $\mathcal{V}$ we can define

$$E_n^P(X_1, \ldots, X_n) :\, = P(n) \otimes X_1 \otimes \ldots X_n$$

Constraint cells are given by a combination of the operad structure of P and the monoidal constraints of $\mathcal{V}$.

$$\otimes(P_n, P_{k_1}, \underline{X}_1, \ldots, P_{k_n}, \underline{X}_n) \cong \otimes(P_n, \underline{P}_{k_i}, \underline{\underline{X}}_i) \to \otimes(P_{\Sigma k_i}, \underline{\underline{X}}_i)$$

$$X \cong I \otimes X \to P_1 \otimes X$$

- A lax monoid with respect to E_n^P is precisely an algebra for the operad P.

Lax monoidal categories: a construction

Given an operad P in a symmetric monoidal category $\mathcal{V}$ we can define

$$E_n^P(X_1, \ldots, X_n) \colon = P(n) \otimes X_1 \otimes \ldots X_n$$

Constraint cells are given by a combination of the operad structure of P and the monoidal constraints of $\mathcal{V}$.

$$\otimes(P_n, P_{k_1}, \underline{X}_1, \ldots, P_{k_n}, \underline{X}_n) \cong \otimes(P_n, \underline{P}_{k_i}, \underline{\underline{X}}_i) \to \otimes(P_{\Sigma k_i}, \underline{\underline{X}}_i)$$
$$X \cong I \otimes X \to P_1 \otimes X$$

- A lax monoid with respect to E_n^P is precisely an algebra for the operad P.
- This sort of process is used in Trimble-like definitions of n-category.

Closed lax monoidal categories

Closed lax monoidal categories

Monoidal model categories require the underlying monoidal structure to be closed, so that each functor

$$- \otimes B \colon \mathcal{C} \to \mathcal{C}$$

has a right adjoint.

Closed lax monoidal categories

Monoidal model categories require the underlying monoidal structure to be closed, so that each functor

$$- \otimes B \colon \mathcal{C} \to \mathcal{C}$$

has a right adjoint.

In a closed lax monoidal category, each

$$E_n \colon \mathcal{C}^n \to \mathcal{C}$$

is equipped with n-variable right adjoints, meaning each 1-variable restriction

$$E_n(X_1, \ldots, X_i, -, X_{i+1}, \ldots, X_n) \colon \mathcal{C} \to \mathcal{C}$$

has a right adjoint.

Model categories

Model categories

A model category is a complete and cocomplete category C with three distinguished classes of maps:

Model categories

A model category is a complete and cocomplete category $\mathcal{C}$ with three distinguished classes of maps:

- $\mathcal{W}$ = weak equivalences,

Model categories

A model category is a complete and cocomplete category $\mathcal{C}$ with three distinguished classes of maps:

- $\mathcal{W}$ = weak equivalences,
- $\mathbb{C}$ = cofibrations,

Model categories

A model category is a complete and cocomplete category $\mathcal{C}$ with three distinguished classes of maps:

- $\mathcal{W}$ = weak equivalences,
- $\mathbb{C}$ = cofibrations,
- $\mathbb{F}$ = fibrations.

Model categories

A model category is a complete and cocomplete category $\mathcal{C}$ with three distinguished classes of maps:

- $\mathcal{W}$ = weak equivalences,
- $\mathbb{C}$ = cofibrations,
- $\mathbb{F}$ = fibrations.

These satisfy various axioms regarding composition, factorisation, and lifting problems.

Model categories

A model category is a complete and cocomplete category $\mathcal{C}$ with three distinguished classes of maps:

- $\mathcal{W}$ = weak equivalences,
- $\mathbb{C}$ = cofibrations,
- $\mathbb{F}$ = fibrations.

These satisfy various axioms regarding composition, factorisation, and lifting problems.

The idea is to pass to a homotopy category where the weak equivalences are isomorphisms - ‘localise at the weak equivalences’.

Monoidal model categories

Monoidal model categories

- A monoidal model category is both a model category and a monoidal category, where the structures interact nicely: the homotopy category is monoidal.

Monoidal model categories

- A monoidal model category is both a model category and a monoidal category, where the structures interact nicely: the homotopy category is monoidal.
- The pushout-product axiom: If each $f_i \colon X_i \to Y_i$ is in $\mathbb{C}$ then the induced map $f_1 \square f_2$ is in $\mathbb{C}$. If either f_i is also in $\mathcal{W}$, then so is $f_1 \square f_2$.

$$\begin{tikzcd}
X_1 \otimes X_2 \arrow[rr, "1\otimes f_2"] \arrow[d, "f_1\otimes 1"'] & & X_1 \otimes Y_2 \arrow[d] \arrow[ddr, bend left, "f_1\otimes id"] & \\
Y_1 \otimes X_2 \arrow[r] \arrow[drrr, bend right, "id\otimes f_2"'] & (X_1 \otimes Y_2) \amalg_{X_1\otimes X_2} (Y_1 \otimes X_2) \arrow[rr, phantom] \arrow[drr, "f_1 \square f_2"] & & \\
& & & Y_1 \otimes Y_2
\end{tikzcd}$$

Lax monoidal model structures: *n*-fold pushout-product

If each f_i is in $\mathbb{C}$, then the induced map $\square_n(f_1, \ldots, f_n)$ is in $\mathbb{C}$. If any of the f_i is also in $\mathcal{W}$, so is $\square_n(f_1, \ldots, f_n)$.

$$\begin{tikzcd}
 & E_n(X_1, \ldots, X_n) \arrow[dl, "{E_n(id, f_2, \ldots, f_n)}"'] \arrow[dr, "{E_n(f_1, \ldots, f_{n-1}, id)}"] & \\
E_n(X_1, Y_2, \ldots, Y_n) \arrow[dr] \arrow[ddr, bend right, "{E_n(f_1, \underline{id})}"'] & \cdots & E_n(Y_1, \ldots, Y_{n-1}, X_n) \arrow[dl] \arrow[ddl, bend left, "{E_n(\underline{id}, f_n)}"] \\
 & P_{f_1, \ldots, f_n} \arrow[d, "{\square_n(f_1, \ldots, f_n)}"] & \\
 & E_n(Y_1, \ldots, Y_n) &
\end{tikzcd}$$

Lax monoidal model structures: n-fold pushout-product

If each f_i is in $\mathbb{C}$, then the induced map $\square_n(f_1, \ldots, f_n)$ is in $\mathbb{C}$. If any of the f_i is also in $\mathcal{W}$, so is $\square_n(f_1, \ldots, f_n)$.

$$
\begin{tikzcd}
 & & E_n(X_1, \ldots, X_n) \arrow[dll, "E_n(id, f_2, \ldots, f_n)"'] \arrow[drr, "E_n(f_1, \ldots, f_{n-1}, id)"] & & \\
E_n(X_1, Y_2, \ldots, Y_n) \arrow[drr] \arrow[ddrr, bend right, "E_n(f_1, \underline{id})"'] & & \cdots & & E_n(Y_1, \ldots, Y_{n-1}, X_n) \arrow[dll] \arrow[ddll, bend left, "E_n(\underline{id}, f_n)"] \\
 & & P_{f_1, \ldots, f_n} \arrow[d, "\square_n(f_1, \ldots, f_n)"] & & \\
 & & E_n(Y_1, \ldots, Y_n) & &
\end{tikzcd}
$$

We also say that E_n is a Quillen adjunction of n variables. The assignment $\square_n$ is functorial and makes $\mathrm{Ar}(\mathcal{C})$ a lax monoidal category - ‘associators’ are induced by γ.

Lax monoidal model categories

Lax monoidal model categories

Definition

Let $\mathcal{C}$ be a model category and let E be a closed lax monoidal structure on $\mathcal{C}$. Call $\mathcal{C}$ a lax monoidal model category if each E_n is a Quillen adjunction of n variables.

Lax monoidal model categories

Definition

Let $\mathcal{C}$ be a model category and let E be a closed lax monoidal structure on $\mathcal{C}$. Call $\mathcal{C}$ a lax monoidal model category if each E_n is a Quillen adjunction of n variables.

Proposition

Let $\mathcal{C}$ be a lax monoidal model category. Then the homotopy category $\mathrm{Ho}\mathcal{C}$ is a lax monoidal category.

Goals

Goals

- Identify an appropriate *lax monoid axiom* in order to show that the category of lax monoids inherits a model structure.

Goals

- Identify an appropriate *lax monoid axiom* in order to show that the category of lax monoids inherits a model structure.
- Establish appropriate axioms regarding enrichment in a lax monoidal model category.

Goals

- Identify an appropriate *lax monoid axiom* in order to show that the category of lax monoids inherits a model structure.
- Establish appropriate axioms regarding enrichment in a lax monoidal model category.
- Fit Trimble-like definitions of n-category into the framework of lax monoidal model categories, using the previous notions.

Goals

- Identify an appropriate *lax monoid axiom* in order to show that the category of lax monoids inherits a model structure.
- Establish appropriate axioms regarding enrichment in a lax monoidal model category.
- Fit Trimble-like definitions of n-category into the framework of lax monoidal model categories, using the previous notions.
- Investigate strictly unital higher categories, a la Batanin-Cisinski-Weber.

Goals

- Identify an appropriate *lax monoid axiom* in order to show that the category of lax monoids inherits a model structure.
- Establish appropriate axioms regarding enrichment in a lax monoidal model category.
- Fit Trimble-like definitions of *n*-category into the framework of lax monoidal model categories, using the previous notions.
- Investigate strictly unital higher categories, a la Batanin-Cisinski-Weber.

A model structure for lax monoids

A model structure for lax monoids

Identifying a *lax monoid axiom* will involve constructing a clear and explicit construction of pushouts of the form

$$\begin{tikzcd} FK \arrow[r, "g"] \arrow[d, "Ff"'] & X \arrow[d] \\ FL \arrow[r] & P \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

where f is a generating trivial cofibration.

A model structure for lax monoids

Identifying a *lax monoid axiom* will involve constructing a clear and explicit construction of pushouts of the form

$$\begin{tikzcd} FK \arrow[r, "g"] \arrow[d, "Ff"'] & X \arrow[d] \\ FL \arrow[r] & P \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

where f is a generating trivial cofibration. The free lax monoid on K is given by

$$FK = \coprod_{n \in \mathbb{N}} E_n(K, \ldots, K).$$

A model structure for lax monoids

A model structure for lax monoids

We can give a construction of P as a filtered colimit of maps formed by pushouts. Set $P^{(0)} = X$. Then define $P^{(1)}$ as the following pushout.

$$\begin{tikzcd} E_1(K) \arrow[r] \arrow[d, "E_1(f)"'] & FK \arrow[r, "g"] & X \arrow[d, "\chi^{(1)}"] \\ E_1(L) \arrow[rr, "\lambda^{(1)}"'] & & P^{(1)} \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

A model structure for lax monoids

Subsequent $P^{(n)}$ are defined inductively as pushouts as follows.

$$\begin{tikzcd} E_n(\underline{K}) \arrow[r] \arrow[d, "E_n(\underline{f})"'] & FK \arrow[r, "g"] & X \arrow[r, "\chi^{(1)}"] & P^{(1)} \arrow[r, "\chi^{(2)}"] & \cdots \arrow[r, "\chi^{(n-1)}"] & P^{(n-1)} \arrow[d, "\chi^{(n)}"] \\ E_n(\underline{L}) \arrow[rrrrr, "\lambda^{(n)}"'] & & & & & P^{(n)} \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

A model structure for lax monoids

Subsequent $P^{(n)}$ are defined inductively as pushouts as follows.

$$\begin{tikzcd}
E_n(\underline{K}) \arrow[r] \arrow[d, "E_n(\underline{f})"'] & FK \arrow[r, "g"] & X \arrow[r, "\chi^{(1)}"] & P^{(1)} \arrow[r, "\chi^{(2)}"] & \cdots \arrow[r, "\chi^{(n-1)}"] & P^{(n-1)} \arrow[d, "\chi^{(n)}"] \\
E_n(\underline{L}) \arrow[rrrrr, "\lambda^{(n)}"'] & & & & & P^{(n)} \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

We then take the colimit of the sequence

$$X = P^{(0)} \xrightarrow{\chi^{(1)}} P^{(1)} \xrightarrow{\chi^{(2)}} \cdots \xrightarrow{\chi^{(n)}} P^{(n)} \xrightarrow{\chi^{(n+1)}} \cdots .$$

A model structure for lax monoids

Subsequent $P^{(n)}$ are defined inductively as pushouts as follows.

$$\begin{tikzcd} E_n(\underline{K}) \arrow[r] \arrow[d, "E_n(\underline{f})"'] & FK \arrow[r, "g"] & X \arrow[r, "\chi^{(1)}"] & P^{(1)} \arrow[r, "\chi^{(2)}"] & \cdots \arrow[r, "\chi^{(n-1)}"] & P^{(n-1)} \arrow[d, "\chi^{(n)}"] \\ E_n(\underline{L}) \arrow[rrrrr, "\lambda^{(n)}"'] & & & & & P^{(n)} \end{tikzcd}$$

We then take the colimit of the sequence

$$X = P^{(0)} \xrightarrow{\chi^{(1)}} P^{(1)} \xrightarrow{\chi^{(2)}} \cdots \xrightarrow{\chi^{(n)}} P^{(n)} \xrightarrow{\chi^{(n+1)}} \cdots .$$

This satisfies the universal property of the pushout P. A rather arduous induction argument shows that $\mathrm{colim} P^{(i)}$ is a lax monoid. Similarly, the map $s^{(0)} \colon X \to \mathrm{colim} P^{(i)}$ is a lax monoid map.

Lax monoid axiom

Lax monoid axiom

The monoid axiom for monoidal model categories requires that every map in

$$((\mathcal{W} \cap \mathbb{C}) \otimes \mathcal{C}) - \mathsf{cof}_{\mathsf{reg}}$$

is a weak equivalence. That is, every map obtained as a transfinite composite of pushouts of tensors with trivial cofibrations, is a weak equivalence.

Lax monoid axiom

The monoid axiom for monoidal model categories requires that every map in

$$((\mathcal{W} \cap \mathbb{C}) \otimes \mathcal{C}) - \mathrm{cof}_{\mathrm{reg}}$$

is a weak equivalence. That is, every map obtained as a transfinite composite of pushouts of tensors with trivial cofibrations, is a weak equivalence.

A crude speculation as to the lax monoid axiom is that we should have, for each n, that every map in

$$(E_n(\underline{\mathcal{W} \cap \mathbb{C}})) - \mathrm{cof}_{\mathrm{reg}}$$

is a weak equivalence.

Thanks.