Basic Ideas

Motivation

A **set** is simply a collection of things. The things in the set are called the **elements** or **members** of the set. There is no formal requirement for the elements of a set to be related to each other, be in a particular order or in any kind of structured way. It's like you've got a bag with some things in, it just matters whether the thing belongs in the bag or not.

In practice, we generally consider sets where the elements share some common property.

We'll start off with some smaller examples of sets, which we can describe by just listing all of the elements. By convention, the elements in the list are separated by commas and enclosed in braces (curly brackets). So the following are sets.

$$\{1, 2, 3, 5\}$$
$$\{\text{Michael}, \text{Dan}, \text{Nathan}\}$$
$$\{\text{Yellow}, 6, \mathcal{C}, \text{Europe}\}$$

The order in which the elements are listed is irrelevant. So, in particular, the sets $\{1, 2, 3, 5\}$ and $\{5, 3, 1, 2\}$ are the regarded to be the same set.

Defining Sets

With larger sets we often leave it to the reader to 'fill in the gaps'. For example, the set

$$\{1, 2, 3, \ldots, 99, 100\}$$

suggests that we are going up one at a time from 1 to 100, so the set contains all of the integers from 1 to 100. Some programming languages use similar syntax: in Haskell a list of numbers from 1 to 100 is defined using `[1..100]`.

The sets we have seen so far are **finite** sets, having a finite number of elements. In a similar fashion to the previously described set, the set

$$\{1, 2, 3, \ldots\}$$

suggests that we are going up one at a time, starting from 1, but there is not an end to the list. This is an **infinite** set, having an infinite amount of elements.

Using the previous notation must be done with care, however. Consider the set

$$\{2, 4, \ldots, 64\}.$$

Does this set contain all of the even numbers between 2 and 64, or is it a set containing powers of 2?

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`https://www.youtube.com/embed/WKcmyRMvPss`

There are some sets which we will be used regularly and which have their own special names and notation.

$\varnothing = \{\,\}$	The empty set
$\mathbb{B} = \{0, 1\}$	Truth values
$\mathbb{N} = \{0, 1, 2, 3, \ldots\}$	Natural numbers
$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$	Integers

The first of these, the **empty set**, is a set which contains no elements - this may seem strange but it will turn up all the time. You may also come across some definitions of the natural numbers which don't include 0. For this module we will consider the natural numbers to include 0, while the set $\mathbb{N}_+ = \{1, 2, 3, \ldots\}$ will be called the **positive integers**. It can be useful to include 0 in our idea of the natural numbers since many lists and arrays start with the 0th entry. This happens in lists in Python, for example.

There is another infinite set which we often use, called the **real numbers** or $\mathbb{R}$. This contains all of the numbers that we usually consider to make up the 'number line', so all of the integers, all of the fractions, and then everything 'in between' such as π, e, $\sqrt{2}$, etc. The real numbers cannot be listed in the same way that the natural numbers or integers can be.

Frequently, the elements of a set cannot simply be listed. For example, the set

$$\{m \in \mathbb{Z} \mid m > 3\} = \{4, 5, 6, 7, 8, \ldots\}$$

has an infite number of elements and there's no we can actually write them all down. So instead we have described the condition that the elements of the set must satisfy instead. We read the symbol '$|$' as 'such that'. So $\{x \in \mathbb{Z} \mid x^2 = 4\}$ is read out loud as 'the elements x in $\mathbb{Z}$ such that $x^2 = 4$'. We can also use this for finite sets, so

$$\{x \mid x \text{ is a student registered on this module}\}$$

is the set of people who are registered on this module. We *could* list these but sometimes we care about the common property of the elements in a set, rather than what they specifically are.

With our previous example of the set $\{2, 4, \ldots, 64\}$, we can make it clear what we mean by using a predicate.

$$\{x \mid x \text{ is an even integer and } x > 1 \text{ and } x \leq 64\}$$
$$\{x \mid x \text{ is a power of 2 and } x > 1 \text{ and } x \leq 64\}$$

It should be now be clear, with a bit of thought, that these two sets are not the same.

Note that sometimes when writing things by hand we use a slightly different symbol to indicate the start of the predicate:

$$\{x \;:\; x \text{ is an even integer and } x > 1 \text{ and } x \leq 64\}$$
$$\{x \;:\; x \text{ is a power of 2 and } x > 1 \text{ and } x \leq 64\}$$

Set Membership

We consider elements of a set to only be members once. So $\{a, a\}$ and $\{a\}$ are the same set. A bigger example is that the following sets are the same.

$$\{a, a, b, a, c, d, b, c, a, b\} = \{a, b, c, d\}$$

There is a special notation for **set membership**. The symbol '$\in$ is used to mean 'is a member of'. Here are some examples of its use.

$$\begin{aligned} 5 &\in \mathbb{Z} \\ 0 &\in \mathbb{N} \\ \sqrt{\pi} &\in \mathbb{R} \\ a &\in \{a, b, c, d\} \\ 1 &\in \{x \mid x \in \mathbb{Z} \ \text{ AND } \ x^2 = 1\} \end{aligned}$$

The symbol '$\in$' tends to be said out loud as 'in', 'is an element of', or 'is in the set'. So $x \in X$ might be said aloud as 'x in X', 'x is an element of X', or 'x is in the set X'. Often we do the slightly confusing thing of using a capital letter, such as X, to give a name to a set, before then describing a generic element using the lower case version of that letter, such as x.

We also use '$\notin$' to mean that an element is not a member of a set. This is said aloud as 'not in' or similar variations on the above.

$$\begin{aligned} 3.5 &\notin \mathbb{Z} \\ 0 &\notin \varnothing \\ 2 &\notin \{x \mid x \in \mathbb{Z} \ \text{ AND } \ x^2 = 1\} \end{aligned}$$

Universal Set

Usually when we are talking about sets we have an agreed upon **universal set** or **universe of discourse**, sometimes denoted U or $\mathcal{U}$. This idea can be illustrated by thinking about what the elements of the following set are.

$$\{x \mid 1 \ \leq \ x \ \leq \ 5\}$$

Without knowing what the universal set is, the elements are ambiguous. The set could refer to all of the integers between 1 and 5, or maybe all of the fractions between them, or even all of the real numbers between 1 and 5.

If we specify that the universe we are working in is the integers, $\mathbb{Z}$, then we know that the set above is described by

$$\{x \mid 1 \ \leq \ x \ \leq \ 5\} = \{x \in \mathbb{Z} \mid 1 \ \leq \ x \ \leq \ 5\} = \{1, 2, 3, 4, 5\},$$

avoiding any ambiguity. Often it is obvious what universe we are working in but it can be helpful to clarify if there might be any ambiguous situations.

Subsets

A concept related to, but distinct from, set membership is that of **set inclusion**. Consider the sets $A = \{1, 6, 4\}$ and $B = \{1, 2, 3, 4, 6, 8\}$. Then every member of A is also a member of B. We say that A is a **subset** of B and write $A \subseteq B$. We must not write $A \in B$ or $1 \subseteq B$. If A is not a subset of B, we can write $A \subsetneq B$. The formal definition of the statement $A \subseteq B$ is that 'for any x, if $x \in A$, then $x \in B$' or, using a predicate,

$$\forall\, x \;\bullet\; (x \in A \Rightarrow x \in B).$$

(We'll see what this notation means when we look at Propositional Logic.)

A wordier definition, which might be easier to understand is this: We say that a set B is a subset of a set A if every element $b \in B$ is also an element of A. I.e., if $b \in B$, then $b \in A$.

The following give some examples of how the subset symbol ($\subseteq$) is used. We will later see how we can illustrate subsets, along with other concepts involving sets, using Venn diagrams.

$$\begin{aligned} \{2,1,5\} &\subseteq \{1,3,5,2\} \\ \{1,2,3\} &\subseteq \{1,2,3\} \\ \{0,1\} &\subseteq \mathbb{N} \subseteq \mathbb{Z} \\ \{1,4\} &\subsetneq \{1,2,3,5,7\} \end{aligned}$$

Two sets are said to be **equal** if they contain the same elements. This can be defined formally by the statement

$$A = B \Leftrightarrow A \subseteq B \text{ AND } B \subseteq A.$$

That is, A is a subset of B and B is a subset of A. Checking two small sets are equal might be easy enough if we can just list their elements. If they are large, or described by some predicate, then we have to check two things. We need to check that for any element $x \in A$, x is also an element of B; and also that for any element $y \in B$, y is also an element of A. For example, the two sets

$$\{x \;\in\; \mathbb{Z} \mid x^2 \geq 9\}, \{x \;\in\; \mathbb{Z} \mid x \leq -3 \text{ or } x \geq 3\}$$

actually describe the same set. Some other examples are below.

$$\begin{aligned} \{1,2,3\} = \{3,2,1\} &= \{1,3,2,3,1\} \\ \{1,2,3\} &\neq \{1,6\} \end{aligned}$$

If A is a subset of B, but A is not *equal* to B, then we say that A is a **proper subset** of B, and write $A \subset B$.

Some textbooks will use the symbol '$\subset$' where we have used '$\subseteq$' and this can cause confusion. (The notation we are using has similarities with the notation $<$ for 'less than', and $\leq$ for 'less than or equal to'.)

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