Matrix Equations

Motivation

You may be familiar with methods of solving simultaneous equations, such as the following pair of equations.

$$x + 3y = 7$$

$$-x + 4y = 14$$

The straightforward way to solve this is to rearrange one of the equations and substitute it into the other. For example, if we rearrange the first equation to be $x = 7 - 3y$ and substitute this into the second, then we get

$$\begin{aligned} 14 &= -x + 4y \\ &= -(7 - 3y) + 4y \\ &= -7 + 7y \\ &= 7(-1 + y), \end{aligned}$$

from which we can deduce that $-1 + y = 2$ and so $y = 3$. Substituting this back in to the first equation gives

$$\begin{aligned} 7 &= x + 3y \\ &= x + 3 \times 3 \\ &= x + 9 \end{aligned}$$

from which we can see that $x = -2$.

What we will do now is write this pair of equations as a single **matrix equation**.

$$\begin{pmatrix} 1 & 3 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 \\ 14 \end{pmatrix}$$

Notice that the square matrix represents the coefficients of x and y in the original equations (the numbers in front of x and y) while the column matrix on the right hand side of the equation represents the constant values from the original equations. This is a matrix equation $AX = B$ where

$$A = \begin{pmatrix} 1 & 3 \\ -1 & 4 \end{pmatrix}, X = \begin{pmatrix} x \\ y \end{pmatrix}, B = \begin{pmatrix} 7 \\ 14 \end{pmatrix}.$$

If we could find a matrix C which when postmultiplied by A returned the identity (so that $CA = I$), then we could immediately read off the solution to these equations. We will consider such matrices in the next section.

Matrix Inverses

The matrix equivalent of division is multiplication by the **matrix inverse**. If A is a square matrix and C is a matrix of the same size, then C is called the inverse of A if

$$CA = I = AC.$$

The usual notation for the inverse matrix of A is A^{-1}, which makes the above equation into

$$A^{-1}A = I = AA^{-1}.$$

This is similar to how we can write $\frac{1}{3} = 3^{-1}$ and have

$$\frac{1}{3}3 = 1 = 3\frac{1}{3}.$$

We will only calculate the inverses of 2×2 matrices but many programming languages have built-in commands to do this for larger matrices. (Some languages are built around matrix manipulation, such as MATLAB.)

If we have a 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

then its inverse is described by

$$A^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

For an example, take

$$A = \begin{pmatrix} 1 & 3 \\ -1 & 4 \end{pmatrix}.$$

The determinant of A is given by $\det(A) = (1 \times 4) - (3 \times -1) = 7$. So

$$A^{-1} = \frac{1}{7} \begin{pmatrix} 4 & -3 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{4}{7} & -\frac{3}{7} \\ \frac{1}{7} & \frac{1}{7} \end{pmatrix}.$$

We can check this by seeing that $A^{-1}A = I = A^{-1}A$.

$$A^{-1}A = \begin{pmatrix} \frac{4}{7} & -\frac{3}{7} \\ \frac{1}{7} & \frac{1}{7} \end{pmatrix} \begin{pmatrix} 4 & -3 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

$$AA^{-1} = \begin{pmatrix} 4 & -3 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{4}{7} & -\frac{3}{7} \\ \frac{1}{7} & \frac{1}{7} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

Not every square matrix has an inverse. If $\det(A) = 0$ then A has no inverse; it is then called a **singular matrix**. This will be important in the next section.

Solving a Matrix Equation

Let's go back to our original pair of equations in matrix equation form:

$$\begin{pmatrix} 1 & 3 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 \\ 14 \end{pmatrix}$$

If we can find an inverse for the square matrix on the left, then we can find the values for x and y. But in the previous section we already found that

$$\begin{pmatrix} 1 & 3 \\ -1 & 4 \end{pmatrix}^{-1} = \frac{1}{7} \begin{pmatrix} 4 & -3 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{4}{7} & -\frac{3}{7} \\ \frac{1}{7} & \frac{1}{7} \end{pmatrix}.$$

So we know that the left hand side of the matrix equation can be multiplied by this inverse to get

$$\begin{pmatrix} \frac{4}{7} & -\frac{3}{7} \\ \frac{1}{7} & \frac{1}{7} \end{pmatrix} \begin{pmatrix} 1 & 3 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = I \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

If we also multiply the right hand side of the matrix equation by the inverse, then we will get

$$\begin{pmatrix} \frac{4}{7} & -\frac{3}{7} \\ \frac{1}{7} & \frac{1}{7} \end{pmatrix} \begin{pmatrix} 7 \\ 14 \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}.$$

Putting everything together, what we have found is that

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$$

from which we can read off $x = -2$ and $y = 3$, just as we had before.

Now this might seem like a lot of effort to go to for these two simple equations but the real power here is when we have many more equations in many more unknowns. We can go through this same process, albeit on a larger scale, to find solutions to systems of equations.

Another benefit is that the calculation of the inverse of a matrix A always includes the factor

$$\frac{1}{\det(A)}.$$

This means that if $\det(A) = 0$, then an inverse doesn't exist (we can't divide by 0). For a matrix A that represents the coefficients from some system of equations, if the inverse doesn't exist then no solution exists to the original system of equations either! So this provides a quick check that we can do before we start trying to find a solution – there's no point wasting time trying to find solutions if we can check beforehand and find that there aren't any.

Video Visit the URL below to view a video:
`https://www.youtube.com/embed/TZfSztQkj1A`

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