Predicate Logic

Predicates and Notation

One of the problems with propositional logic is that most statements are not universally true or false. A simple statement like "John is in bed" can be ambiguous if we know two people called John and only one of them is in bed. This problem becomes more obvious with algebraic statements like "$x > 3$". If we don't know what x is, we don't know whether "$x > 3$" is true or false.

These types of statements are called **predicates**. The truth value of a predicate depends on one or more unspecified elements which are called **free variables**.

The predicates above have just one free variable but this needn't be the case. For example we might have $x > y$ or $x + y = z$, with two and three free variables respectively. Whereas a proposition might be written as p or q, a predicate will generally be written something like $P(x)$, $Q(x, y, z)$, or $R(x, z)$, the capital letter representing the **context** and the lower case letters being a list of the free variables in the sentence. The values of the free variables in the predicate then determine the truth value of the statement.

For example, we might have $P(x) =$ "$x > 3$". The free variable is x. If we have $x = 1$, then $P(1) =$ "$1 > 3$" is a proposition (which is false). If we have $x = 4$, then $P(4) =$ "$4 > 3$" is also a proposition (this time true). This method of replacing each occurrence of a free variable in a predicate is called **substitution**. If we do this for each free variable in a predicate then we obtain a proposition.

Some more examples of predicates are given below.

$E(x)$	"x is over 18"
$V(x)$	"x is over 30"
$C(x)$	"x is a computer"
$P(x)$	"x is a person"
$G(x)$	"$x > 3$"
$H(x, y)$	"$x > y$"
$G(x, y)$	"y is a grandfather of x"
$S(x, y, z)$	"$x + y = z$"

Quantifiers

We just saw that substitution allows us to turn predicates into propositions. Another method is called **quantification**. This allows us to take a predicate

and precede it with a **quantifier**. We will consider two such quantifiers. The first quantifier is the phrase "for all x", while the second is the phrase "there exists an x such that".

Consider a predicate $P(x)$. The first quantifier allows us to obtain the proposition "For all x, $P(x)$ is true". This is true provided that $P(x)$ is true for all possible substitutions of x. The second quantifier gives the proposition "There exists an x such that $P(x)$ is true". This is true provided that $P(x)$ is true for at least one possible value of x.

These two quantifiers get their own special symbols.

$$\forall \text{ "for all"}$$
$$\exists \text{ "there exists"}$$

They also have names: $\forall$ is the **universal quantifier** and $\exists$ is the **existential quantifier**. You will often see expressions in this course with a dot, $\bullet$, written between the quantified variable and the relevant predicate. For example, $\forall\ x\ \bullet\ P(x)$. However, this is not standard in all texts.

Let's look at an example. Take the predicate $E(x) =$ "x is over 18". We can quantify this predicate using the quantifiers we just saw, which gives the following. So we can get $\forall\ x\ \bullet\ E(x) =$ "For all x, $E(x)$ is true" = "Everyone is over 18" or $\exists\ x\ \bullet\ E(x) =$ "There exists x such that $E(x)$ is true" = "Someone is over 18".

Quantifying a predicate changes each occurrence of the free variable x into a **bound variable**.

Examples of Quantified Predicates

Manipulating quantifiers and predicates is fairly straightforward, even when we negate them as in the next section. However, it can get complicated very quickly when we try to actually interpret what a given predicate means in plain English.

Some examples of fairly simple-sounding statements are given below along with their notational expression. Note that the last two are the same. We will see how to switch between the two in the next section.

"All P are Q"	$\forall\ x\ \bullet\ P(x) \Rightarrow Q(x)$
"All P with property Q are R"	$\forall\ x\ \bullet\ P(x)\ \wedge\ Q(x) \Rightarrow R(x)$
"Some P is/are Q"	$\exists\ x\ \bullet\ P(x)\ \wedge\ Q(x)$
"No P is Q"	$\neg\ (\exists\ x\ \bullet\ P(x)\ \wedge\ Q(x))$
"No P is Q"	$\forall\ x\ \bullet\ P(x) \Rightarrow \neg\ Q(x)$

Let's have a look at a more concrete example. We'll use the following predicates.

$$A(x) = \text{"}x \text{ is an animal"}$$
$$G(x) = \text{"}x \text{ is grey"}$$
$$D(x) = \text{"}x \text{ is a dog"}$$
$$L(x) = \text{"I like } x\text{"}$$

We'll translate some sentences involving these predicates into predicate logic.

"All dogs are animals"	$\forall\ x\ \bullet\ D(x) \Rightarrow A(x)$
"I like some dogs"	$\exists\ x\ \bullet\ D(x)\ \wedge\ L(x)$
"I don't like dogs"	$\forall\ x\ \bullet\ D(x) \Rightarrow \neg\ L(x)$
"I don't like dogs''	$\neg\ (\forall\ x\ \bullet\ D(x)\ \wedge\ L(x)$
"There are dogs I don't like"	$\exists\ x\ \bullet\ D(x)\ \wedge\ \neg\ L(x)$
"I like grey dogs"	$\forall\ x\ \bullet\ D(x)\ \wedge\ G(x) \Rightarrow L(x)$

If the predicate involves several free variables then they can be bound by preceding them by several quantifiers. However, we must be careful about the order in which these quantifiers occur, particularly if some are existential and some are universal. This can best be explained by an example. Consider the following propositions.

$$\forall\ x\ \bullet\ \exists\ y\ \bullet\ y - x = 3$$
$$\exists\ y\ \bullet\ \forall\ x\ \bullet\ y - x = 3$$

The predicate in both of the statements ($y - x = 3$) is the same. The quantifiers bind the same variables each time as well: y is quantified by $\exists$ and x is quantified by $\forall$.

The first statement reads "For all x, there exists y such that $y - x = 3$", which with a bit of thinking can be seen to be true by choosing $y = x + 3$.

The second statement reads "There exists y such that for all x, $y - x = 3$". This can be shown to be false by choosing a particular value of y, say $y = a$. Since this should work for *all* values of x then it must work for $x = a$ as well. But then we have $x - y = a - a = 0 \neq 3$. So the second statement is false.
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Negation of Quantified Expressions

We have seen a couple of cases where an expression is given by two different quantified predicates. In each case, these have used negation. If you think about the following two statements, you might realise (informally) that they mean the same thing.

$$\neg\ (\exists\ x\ \bullet\ P(x)) = \text{"There does not exist } x \text{ with } P(x) \text{ true"}$$
$$\forall\ x\ \bullet\ \neg\ P(x) = \text{"For all } x, P(x) \text{ is false"}$$

A general rule to formalise this is that:

$$\neg\ (\exists\ x\ \bullet\ P(x)) = \forall\ x\ \bullet\ \neg\ P(x).$$

Similarly, the following two statements mean the same.

$$\neg\ (\forall\ x\ \bullet\ P(x)) = \text{"It is not true that } P(x) \text{ is true for all } x\text{"}$$
$$\exists\ x\ \bullet\ \neg\ P(x) = \text{"There exists } x \text{ such that } P(x) \text{ is false"}$$

A general rule to formalise this is that:

$$\neg\ (\forall\ x\ \bullet\ P(x)) = \exists\ x\ \bullet\ \neg\ P(x).$$

Let's go back to the example where we said that

$$\neg\ (\exists\ x\ \bullet\ P(x)\ \wedge\ Q(x)) = \forall\ x\ \bullet\ P(x) \Rightarrow \neg\ Q(x)$$

and show that it holds.

$$\begin{aligned}
\neg\ (\exists\ x\ \bullet\ P(x)\ \wedge\ Q(x)) &= \forall\ x\ \bullet\ \neg\ (P(x)\ \wedge\ Q(x)) && \text{(negation rules)}\\
&= \forall\ x\ \bullet\ \neg\ P(x)\ \vee\ \neg\ Q(x) && \text{(de Morgan's)}\\
&= \forall\ x\ \bullet\ P(x) \Rightarrow \neg\ Q(x) && \text{(implication)}
\end{aligned}$$

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