Monoidal categories and operads

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Plan

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1. Introduction: What is a category?

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3. Operads: many inputs, one output

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2. Monoidal categories: operations in parallel
3. Operads: many inputs, one output
4. Action Operads: moving the inputs around

Sets, graphs, categories

Sets are simply collections of elements, e.g., $\{a, b, c\}$.

b

a

c

Sets, graphs, categories

Directed graphs have a set of vertices, with some arcs between them.

$$
\begin{tikzcd}
 & b \arrow[dd] \\
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\end{tikzcd}
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Sets, graphs, categories

Categories have objects, with morphisms between these objects which can be *composed*.

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$$\begin{tikzcd} & b \arrow[dd, "g"] \\ a \arrow[ur, "f"] \arrow[dr, dashed, "g \circ f"'] & \\ & c \end{tikzcd}$$

And every object has an identity morphism: $\mathrm{id}_a \colon a \longrightarrow a$.

Categories: axioms

Composition is associative: $(h \circ g) \circ f = h \circ (g \circ f)$.

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- Associativity: $(ab)c = a(bc)$ for all $a, b, c \in M$.
- Units: $ae = a = ea$ for all $a \in M$.

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- There are special associativity isomorphisms: $\alpha_{a,b,c}\colon (a \otimes b) \otimes c \to a \otimes (b \otimes c)$.
- There's a few axioms about how associativity works and how it interacts with the unit object.

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$$(a \otimes b) \otimes (c \otimes d)$$

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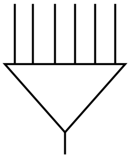

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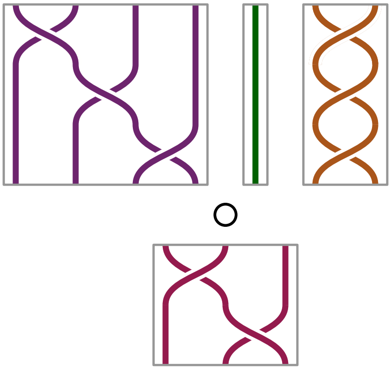
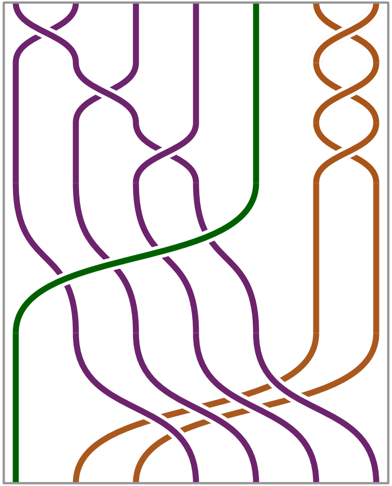
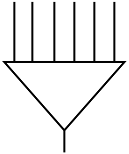
- The group multiplication in Br_n is to stack braids
- The *operad* multiplication **B** is similar, but we 'plug braids in to other braids'

$$\mu\colon \mathbf{B}(3) \times \mathbf{B}(4) \times \mathbf{B}(1) \times \mathbf{B}(2) \to \mathbf{B}(7)$$

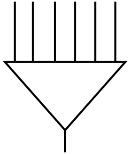


Operad of Braid Groups





Plain



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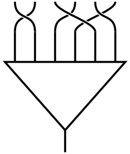
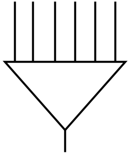


Braided



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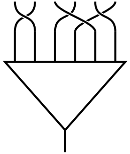
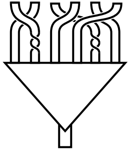
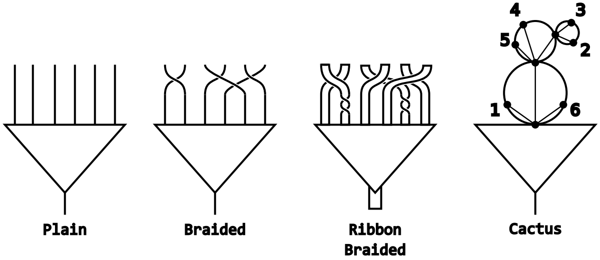
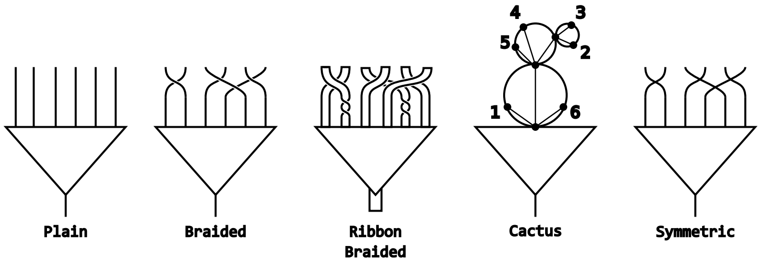


Braided



Ribbon Braided


4
3
5
2
1
6
Plain
Braided
Ribbon
Braided
Cactus


4
3
5
2
1
6
Plain
Braided
Ribbon
Braided
Cactus
Symmetric

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