Operads, Equivariance, and Pseudo-Commutativity

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Slides: alex-corner.github.io/slides/corner-pssl110.pdf

Operads

Data $n \in \mathbb{N} = \{0, 1, 2, \ldots\}$

- sets $P(n)$ - thought of as 'n-ary operations'
- functions - 'composition' or 'multiplication'

$$\mu^P : P(n) \times P(k_1) \times \ldots \times P(k_n) \to P(k_1 + \ldots + k_n)$$

- identity element $e \in P(1)$

Axioms

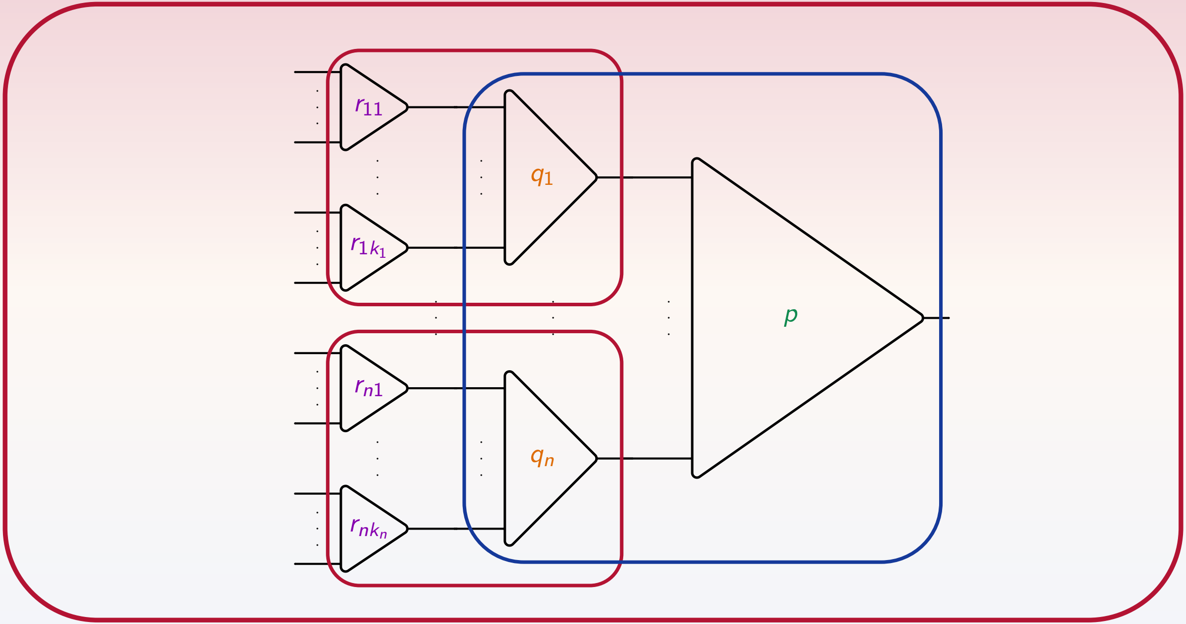
- multiplication:

$$\mu(\mu(p; q_1, \ldots, q_n); r_{11}, \ldots, r_{1k_1}, \ldots, r_{n1}, \ldots, r_{nk_n})$$
$$=$$
$$\mu(p; \mu(q_1; r_{11}, \ldots, r_{1k_1}), \ldots, \mu(q_n; r_{n1}, \ldots, r_{nk_n}))$$

- identity:

$$\mu(p; e, \ldots, e) = p = \mu(e; p)$$

Composition Axiom



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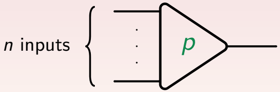
Operads - Symmetric Actions

Idea

- elements of $P(n)$ are n-ary operations:



Symmetric Operads

- for each $n \in \mathbb{N}$, an action:

$$P(n) \times \Sigma_n \to P(n)$$

- satisfying:

$$\mu(p \cdot g; q_1, \ldots, q_n) = \mu(p; q_{g^{-1}(1)}, \ldots, q_{g^{-1}(n)}) \cdot \delta(g)$$

$$\mu(p; q_1 \cdot g_1, \ldots, q_n \cdot g_n) = \mu(p; q_1, \ldots, q_n) \cdot \beta(g_1, \ldots, g_n)$$

Plain Operads

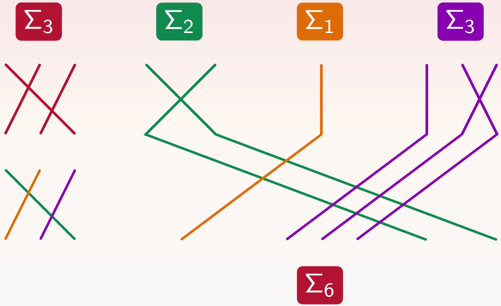
- for each $n \in \mathbb{N}$, a trivial action:

$$P(n) \times 1 \to P(n)$$

Operads - Symmetry Groups

Symmetry Groups

- sets: $P(n) = \Sigma_n$
- composition:



Block Sum $\beta(p_1, \ldots, p_n) = \mu(e_n; p_1, \ldots, p_n)$

Extension $\delta(p) = \mu(p; e_{k_1}, \ldots, e_{k_n})$

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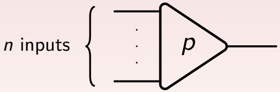

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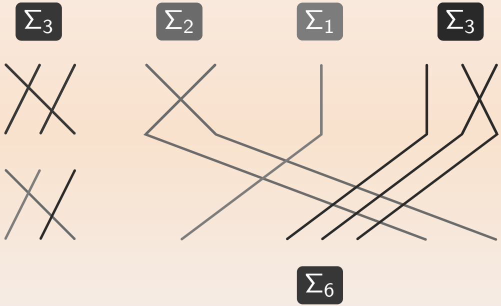

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Action Operads

Data

$n \in \mathbb{N} = \{0, 1, 2, \ldots\}$

- **groups** $\Lambda(n)$
- plain operad structure (μ, e)
- group homomorphisms $\pi_n \colon \Lambda(n) \to \Sigma_n$

Equivariance Axiom

$$\mu\left(g'; h'_1, \ldots h'_n\right) \cdot \mu\left(g; h_1, \ldots, h_n\right) = \mu\left(g' \cdot g; h'_{g(1)} \cdot h_1, \ldots, h'_{g(n)} \cdot h_n\right)$$

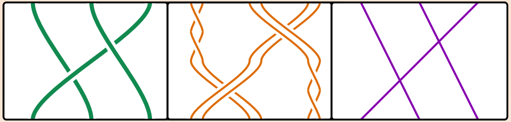
Symmetric Groups

- $\Lambda(n) = \Sigma_n$
- operad structure: β and δ, $e_1 = (1) \in \Sigma_1$
- π_n is the identity
- equivariance follows from symmetric equivariance

Action Operads - Examples

Examples

- Σ: Σ_n, symmetric groups
- T: 1, trivial groups
- B: B_n, braid groups — Fiedorowicz/Salvatore
- RB: RB_n, ribbon braid groups — Wahl/Zhang
- J: J_n, cactus groups — Henriques–Kamnitzer



Non-Examples

- A: A_n, alternating groups
- H: H_n, hyperoctahedral groups
- C: C_n, cyclic groups
- R: C_2, π_n order-reversing
- Most skew-simplicial groups, despite similarities. — Fiedorowicz–Loday

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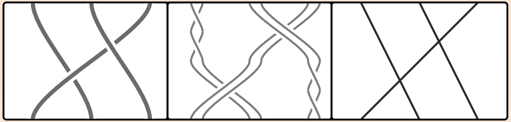

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Λ-Operads

Data

- an action operad Λ
- an operad P
- actions $P(n) \times \Lambda(n) \to P(n)$
- satisfying:

$$\mu^P(p \cdot g; q_1, \ldots, q_n) = \mu^P(p; q_{g^{-1}(1)}, \ldots, q_{g^{-1}(n)}) \cdot \delta^\Lambda(g)$$

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Monads

- each Λ-operad P induces a monad $\underline{P}$ on **Set**:

$$\underline{P}(X) = \coprod_{n \in \mathbb{N}} (P(n) \times_{\Lambda_n} X^n)$$

- each $P(n) \times_{\Lambda_n} X^n$ is a coequalizer:

$$(p \cdot g; x_1, \ldots, x_n) \sim \left(p; x_{g^{-1}(1)}, \ldots, x_{g^{-1}(n)}\right)$$

- we investigate Λ-operads in **Cat**: 2-monads

Commutative Monads

- $T\colon \mathcal{C} \to \mathcal{C}$ a monad
- a **strength** is a natural transformation + axioms

equivalent when $\mathcal{C}$ sym. mon. cat

$$t_{X,Y}\colon X \otimes TY \to T(X \otimes Y)$$

- a **costrength** is a natural transformation + axioms

$$t^*_{X,Y}\colon TX \otimes Y \to T(X \otimes Y)$$

- T is **commutative** if this commutes:

$$\begin{tikzcd}
 & TX \otimes TY \arrow[dl, "t_{TX,Y}"'] \arrow[dr, "t^*_{X,TY}"] & \\
T(TX \otimes Y) \arrow[d, "Tt^*_{X,Y}"'] & & T(X \otimes TY) \arrow[d, "Tt_{X,Y}"] \\
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- T commutative $\Rightarrow$ T-**Alg** is sym. mon. closed
- allows us to 'tensor' algebras for T

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Pseudo-Commutative 2-Monads

- $T : \mathcal{C} \to \mathcal{C}$ a 2-monad
- equipped with strength and costrength (2d, strict)
- equipped with invertible modification:

Hyland–Power

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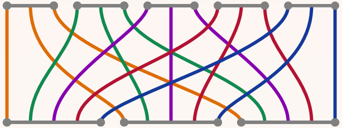
- T is **pseudo-commutative** if

lots of axioms

- T ps-commutative $\Rightarrow$ T-$\mathbf{Alg}_{ps}$ is pseudo-closed
- allows us to ‘tensor’ strict algebras for T

Pseudo-Commutative Λ-Operads

- can characterise Λ-operads with ps-commutativity
- requires elements $t_{m,n} \in \Lambda(mn)$ with $\pi(t_{m,n}) = \tau_{m,n}$
- $\tau_{5,3}$ is pictured below



- these permutations correspond to matrix transpose

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- a Λ-operad with these elements is ps-commutative if

lots of axioms

- then the corresponding 2-monad is ps-commutative

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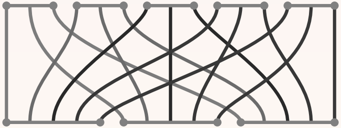

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Examples/Non-Examples

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Guillou–May–Merling–Osorno

- Σ: symmetric strict monoidal categories
- B: braided strict monoidal categories

C–G

Non-Examples

- J: cactus/coboundary strict monoidal categories

C–G

- arXiv:1508.04050 – Operads, tensor products, and the categorical Borel construction
- arXiv:1312.5910 – Operads with general groups of equivariance, and some 2-categorical aspects of operads in **Cat**