Exercises: Probability

1. A normal pack of 52 playing cards has 4 suits: clubs, diamonds, hearts, and spades, each with 13 cards. One card is drawn from the full pack at random, then replaced in the pack. Then a second card is drawn from the pack.

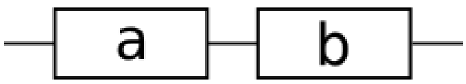
   (a) Calculate the probability that both cards are clubs.

   (b) Calculate the probability that at least one of the cards selected is a club.

   (c) Calculate the probability that neither of the cards selected is a club.

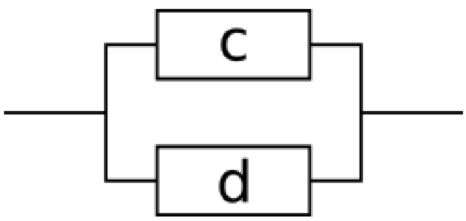
   (d) Use a different method to calculate the probability that at least one of the cards selected is a club.

2. A short network link consists of two sections, a and b, in series as shown below.



   The link functions only if both a and b function. The two sections are independent in that the functionality of one does not affect the functionality of the other. The probability that a continues to be functional during a particular time period is 0.9. The probability that b continues to be functional during a particular time period is 0.95. Determine the probability that the link continues to function during a particular time period.

3. A short network link consists of two sections, c and d, in parallel as shown below.



The link functions if either c or d (or both) function. The two sections are independent in that the functionality of one does not affect the functionality of the other. Consider the following statements.

- C: 'c continues to function during a particular time period'.
- D: 'd continues to function during a particular time period'.

The probability that c continues to function during a particular time period is 0.95. The corresponding probability for d is 0.97.

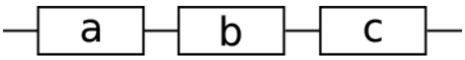
(a) Determine the probability that the link continues to function during a particular time period.

(b) Using the negation of each of the statements C and D define a statement which specifies the failure of the link.

(c) Determine the probability of link failure during a particular time period.

(d) Use your previous result to confirm the result of the first part.

4. A short network link consists of three sections, a, b, and c, in series as shown below.



The link functions only if all three of a, b, and c function. The three sections are independent in that the functionality of one does not affect the functionality of any other. The probability that a continues to be functional during a particular time period is 0.95. The corresponding probabilities for b and c are 0.96 and 0.91, respectively. Determine the probability that the link continues to function during a particular time period.

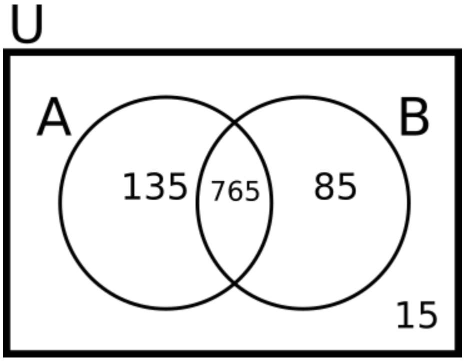
5. The following Venn diagram represents the occurrence of two events A and B. The out rectangle is the universal set and the two circular regions represent A and B as shown. The values in the diagram indicate the number of outcomes (total 1000) in each case.

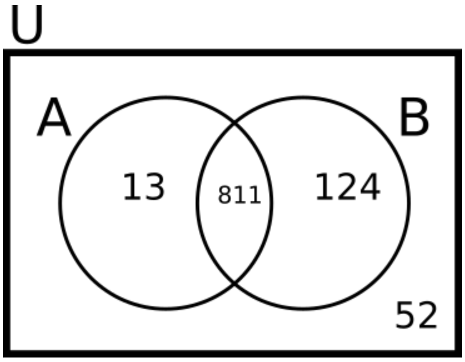
(a) Determine $P(A)$ and $P(B)$.

(b) Determine $P(B|A)$ and $P(A|B)$.

(c) Using your results of the previous parts, state whether A and B are independent or not.

6. The following Venn diagram represents the occurrence of two events A and B. The out rectangle is the universal set and the two circular regions represent A and B as shown. The values in the diagram indicate the number of outcomes (total 1000) in each case.





(a) Determine $P(A)$ and $P(B)$.

(b) Determine $P(B|A)$ and $P(A|B)$.

(c) Using your results of the previous parts, state whether A and B are independent or not.

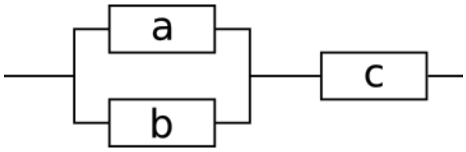
(d) Determine $P(\neg\, B|A)$ and compare it with $P(B|A)$.

(e) Determine $P(B|\neg\, A)$.

(f) Determine $P(\neg\, B|\neg\, A)$ and compare it with $P(B|\neg\, A)$.

(g) Determine $P(\neg\, A|\neg\, B)$.

7. A short network link consists of three sections a, b, and c as shown below.



The link functions if c functions and either a or b (or both) function. All sections are independent in the sense that the functionality of one does not affect the functionality of any other. The probability that a continues to function during a particular time period is 0.94. The corresponding probability for b is 0.92 and for c is 0.96.

(a) Write down statements, A, B, and C, defining the functionalities of a, b, and c respectively.

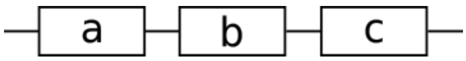
(b) Show that a statement which specifies the functionality of the link is $L = (A \vee B) \wedge C$.

(c) Calculate $P(A \vee B)$ and then determine $P((A \vee B) \wedge C)$.

(d) Use Boolean algebra to show that $L = (A \wedge C) \vee (B \wedge C)$ also specifies the functionality of the link.

(e) Calculate $P(A \wedge C)$ and $P(B \wedge C)$ and then determine $P((A \wedge C) \vee (B \wedge C))$.

8. A short network link consists of three sections a, b, and c as shown below.



The link functions only if all three of a, b, and c function. The functionality of section b depends upon that of a but is independent of c. Sections a and c are independent in that the functionality of one does not affect the functionality of any other. The probability that a continues to be functional during a particular time period is 0.95. The corresponding probabilities for b and c are 0.96 and 0.91. However, the probability that b **fails** during a particular time period, given that a has **already failed**, is 0.06.

(a) Determine the probability that the link fails during a particular time period.

(b) Use your previous result to determine the probability that the link continues to function during a particular time period.

Answers

1. A normal pack of 52 playing cards has 4 suits: clubs, diamonds, hearts, and spades, each with 13 cards. One card is drawn from the full pack at random, then replaced in the pack. Then a second card is drawn from the pack.

(a) Both are clubs: $\frac{13}{52} \times \frac{13}{52} = 0.00625$.

(b) At least one is a club: $(\frac{13}{52} \times \frac{39}{52}) + (\frac{39}{52} \times \frac{13}{52}) + (\frac{13}{52} \times \frac{13}{52}) = 0.4375$.

(c) Neither is a club: $1 - 0.4375 = 0.5625$.

(d) N/A

2. $P(A \wedge B) = 0.9 \times 0.95 = 0.855$

3. (a) $P(C \vee D) = P(C) + P(D) - P(C \wedge D) = 0.95 + 0.97 - 0.95 \times 0.97 = 0.9985$

   (b) The link failing corresponds to the following logical expression being true:
$$\neg\,(C \vee D) = \neg\, C \wedge \neg\, D.$$
(Or $\overline{C + D} = \overline{C} \cdot \overline{D}$.)

   (c) $P(\neg\,(C \vee D) = 1 - P(C \vee D) = 1 - 0.9985 = 0.0015$

   (d) N/A

4. $P(A \wedge B \wedge C) = 0.95 \times 0.96 \times 0.91 = 0.83$

5. (a) $P(A) = \frac{135+765}{1000} = 0.9$, $P(B) = \frac{765+85}{1000} = 0.85$

   (b) For this we will need $P(A \wedge B) = \frac{765}{1000} = 0.765$. Then $P(B|A) = \frac{P(A \wedge B)}{P(A)} = \frac{0.765}{0.9} = 0.85$ and $P(A|B) = \frac{0.765}{0.85} = 0.9$.

   (c) A and B are independent events if $P(A \wedge B) = P(A) \times P(B)$. Now $P(A \wedge B) = 0.765 = 0.9 \times 0.85 = P(A) \times P(B)$, so the events are independent.

6. The following Venn diagram represents the occurrence of two events A and B. The out rectangle is the universal set and the two circular regions represent A and B as shown. The values in the diagram indicate the number of outcomes (total 1000) in each case.

   (a) $P(A) = \frac{13+811}{1000} = 0.824$, $P(B) = \frac{811+124}{1000} = 0.935$

   (b) For this we will need $P(A \wedge B) = \frac{811}{1000} = 0.811$. Then $P(B|A) = \frac{P(A \wedge B)}{P(A)} = \frac{0.811}{0.824} = 0.984$ and $P(A|B) = \frac{P(A \wedge B)}{P(B)} = \frac{0.811}{0.935} = 0.867$.

   (c) A and B are independent if $P(A \wedge B) = P(A) \times P(B)$. Now $P(A \wedge B) = 0.811 = 0.77$, so the events are not independent.

   (d) $P(\neg\, B|A) = \frac{P(A \wedge \neg\, B)}{P(A)} = \frac{13}{13+811} = 0.0158$. Notice that this is $1 - P(B|A)$.

   (e) $P(B|\neg\, A) = \frac{P(\neg\, A \wedge B)}{P(\neg\, A)} = \frac{124}{124+52} = 0.705$

   (f) $P(\neg\, B|\neg\, A) = \frac{P(\neg\, A \wedge \neg\, B)}{P(\neg\, A)} = \frac{52}{52+124} = 0.295$, $P(B|\neg\, A) = 0.705$. These are complements again, in that $P(\neg\, B|\neg\, A) = 1 - P(B|\neg\, A)$.

   (g) $P(\neg\, A|\neg\, B) = \frac{52}{13+52} = 0.8$

7. (a)
   - A: 'a continues to function during a particular time period'.
   - B: 'b continues to function during a particular time period'.
   - C: 'c continues to function during a particular time period'.

   (b) There needs to be at least one route through the network link, either passing through a then c, or by passing through b then c. In order for this to be the case we need either, or both, of a and b to be functional at the same time that c is also function. Hence the logical expression of $(A \vee B) \wedge C$.

(c) $P(A \vee B) = P(A) + P(B) - P(A \wedge B) = 0.94 + 0.92 - 0.94 \times 0.92 = 0.9952$. (We can use $P(A \wedge B) = P(A) \times P(B)$ since the events are independent.) Then $P((A \vee B) \wedge C) = P(A \vee B) \times P(C) = 0.9952 \times 0.96 = 0.955$.

(d) The fact that $(A \wedge C) \vee (B \wedge C) = (A \vee B) \wedge C$ is a direct application of the distributivity law from Boolean algebra.

(e) N/A

8. Given information: $P(A) = 0.95$, $P(B) = 0.96$, $P(C) = 0.91$, $P(\neg B | \neg A) = 0.06$.

   (a) To appear.