Statistics and Probability

Alex Corner

Sheffield Hallam University

Statistics

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  - **Continuous**: E.g., amount of time elapsed since previous attack. Often classified into groups, e.g., $0-5$ minutes, $5-10$ minutes, $10-15$ minutes, etc.

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67	10	54	42	32	56	39	60	79	33	12	47	24
19	47	63	32	7	70	55	46	11	20	15	39	37
28	72	46	64	61	51	56	53	61	11	80	53	28
76	6	5	39	58	29	52	54	47	60	62	51	72
41	57	32	12	33	17	40	20	10	27	47	71	68
44	7	23	17	81	23	12	33	16	46	71	48	58
79	80	43	31	72	68	36	41	11				

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28	72	46	64	61	51	56	53	61	11	80	53	28
76	6	5	39	58	29	52	54	47	60	62	51	72
41	57	32	12	33	17	40	20	10	27	47	71	68
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- In doing so, we lose some of the information but might get a clearer overview.

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- Here we have placed the data into bins of **class width** 5.

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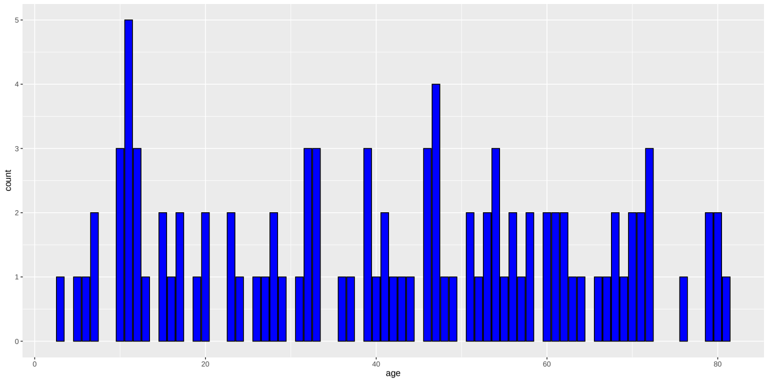
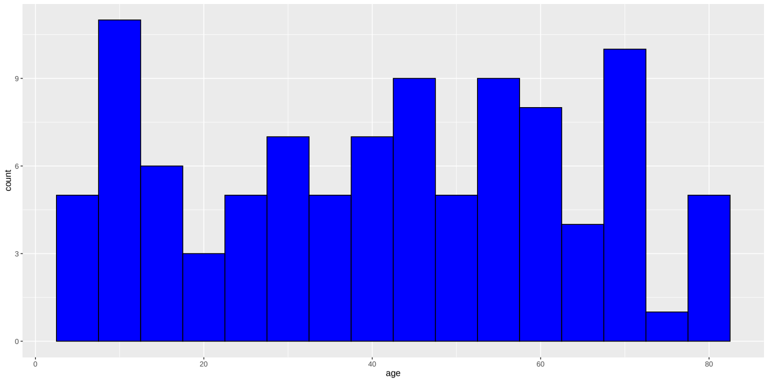
Charts: Bar Chart



Charts: Histogram



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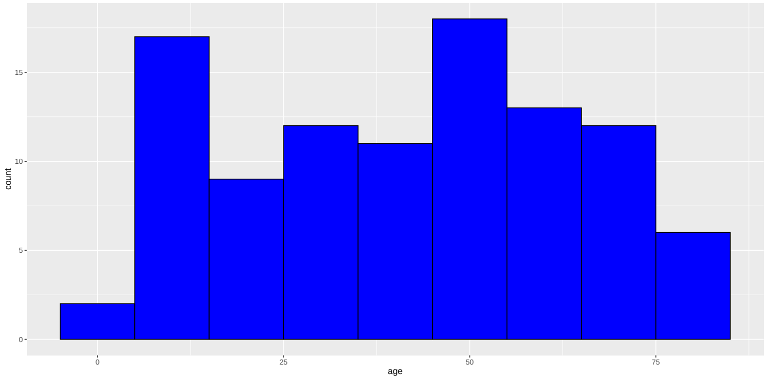

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- **Median**: Order the data and look at the middle value(s).
- **Mode**: Tally the data and select the one with the highest frequency.

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- In a sample of five values (6, 9, 2, 4, 3) the mean value is:

$$\frac{6+9+2+4+3}{5} = \frac{24}{5} = 4.8.$$

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- From our histogram example, the sample of one hundred data values, the ordered data is:

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23	24	26	27	28	28	29	31	32	32	32	33	33
33	36	37	39	39	39	40	41	41	42	**43**	**44**	46
46	46	47	47	47	47	48	49	51	51	52	53	53
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33	36	37	39	39	39	40	41	41	42	**43**	**44**	46
46	46	47	47	47	47	48	49	51	51	52	53	53
54	54	54	55	56	56	57	58	58	60	60	61	61
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- The middle two values are 43 and 44, hence the median is 43.5.

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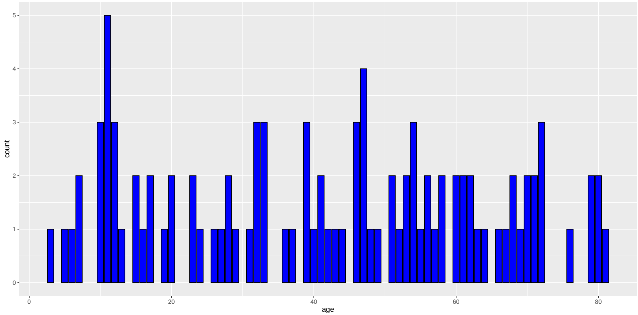
- For grouped data we can talk of a **modal group** or **modal class**, sometimes specifying the mid-point in order to give a single value.
- For the one hundred data values in the histogram example the mode is clearly indicated on the bar chart as the tallest bar: 11. We can also tally values to find it.



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- The *probability of failure* is:

$$P(\overline{A}) = \frac{N - S}{N} = 1 - \frac{S}{N} = 1 - P(A).$$

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- $A =$ 'a head shows when a coin is tossed', so $P(A) = \frac{1}{2}$.

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(5, 1)	**(5, 2)**	(5, 3)	(5, 4)	(5, 5)	(5, 6)
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- There are 36 possible outcomes when rolling two dice. But only six of these outcomes make A true.
- Number of trials $N = 36$, number of successes $S = 6$.

$$P(A) = \frac{6}{36} = \frac{1}{6}.$$

KLAUS TEUBER
CATAN
GAME RULES
& ALMANAC

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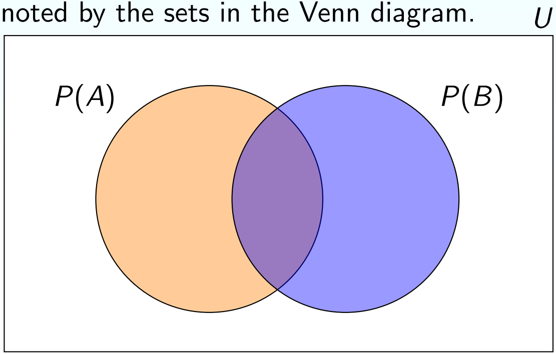

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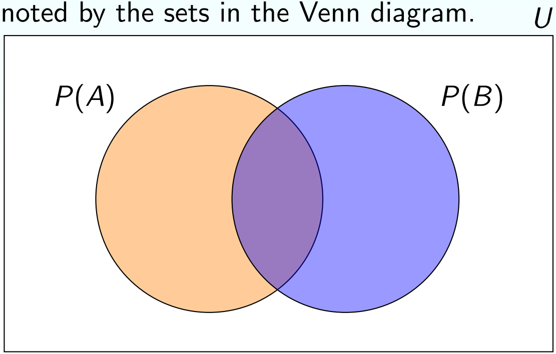

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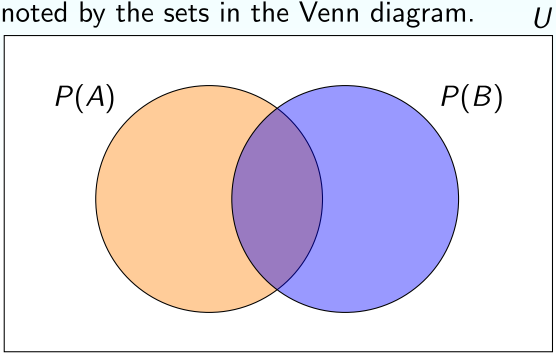

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- $P(A) = \frac{26}{52} = \frac{1}{2}$ and $P(B) = \frac{26}{52} = \frac{1}{2}$. The two events are independent, so:

$$P(A \wedge B) = P(A) \times P(B) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}.$$

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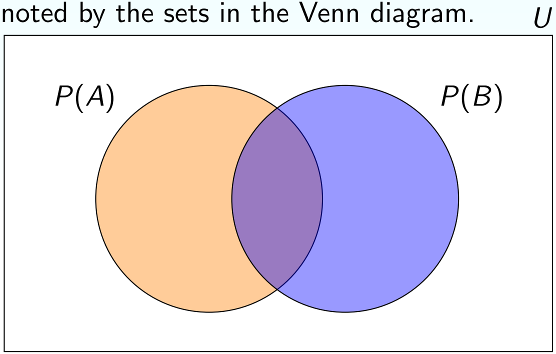

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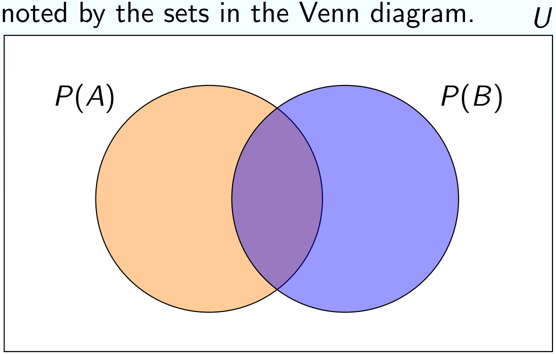

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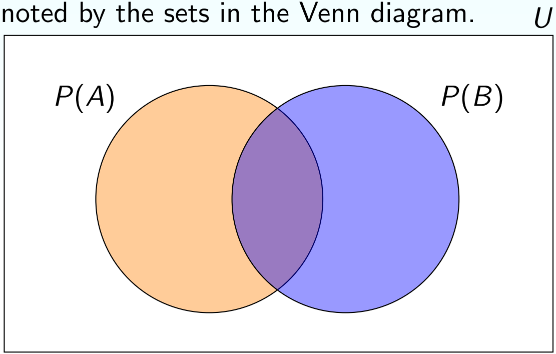

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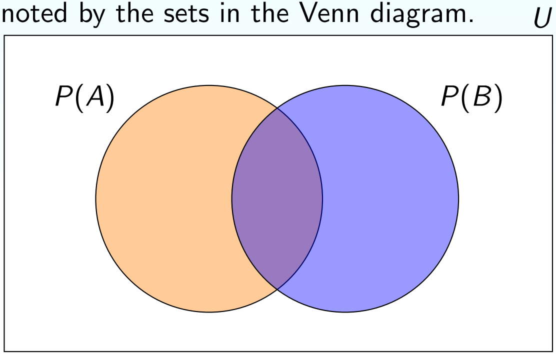

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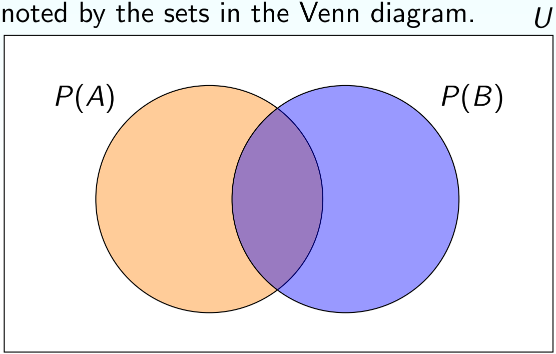

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$$P(A) + P(B) - P(A) \times P(B) = \frac{1}{2} + \frac{1}{2} - \frac{1}{2} \times \frac{1}{2} = \frac{3}{4}.$$

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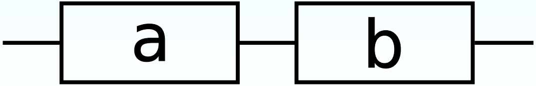
- A short network link consists of two sections (a) and (b), in series as shown below.



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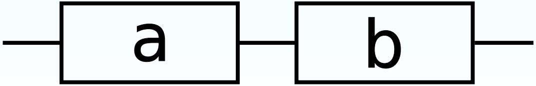

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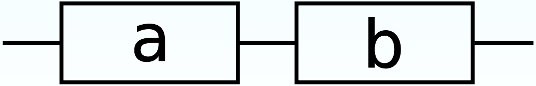

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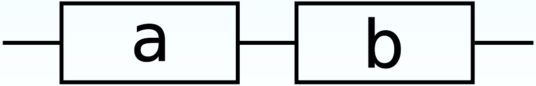

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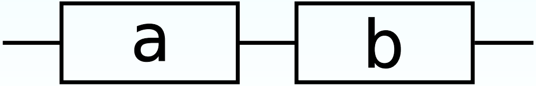

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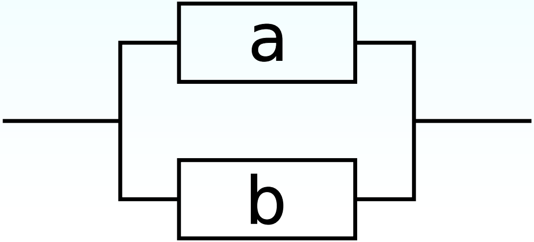

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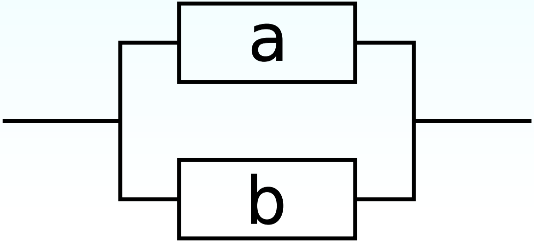

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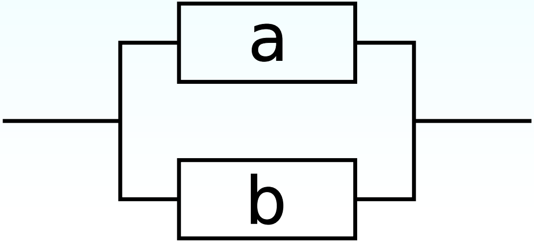

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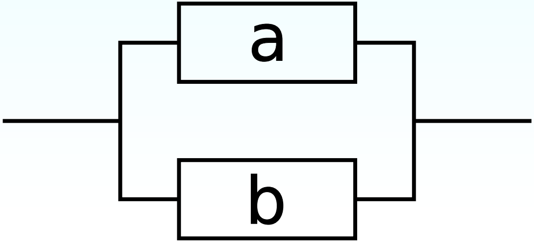

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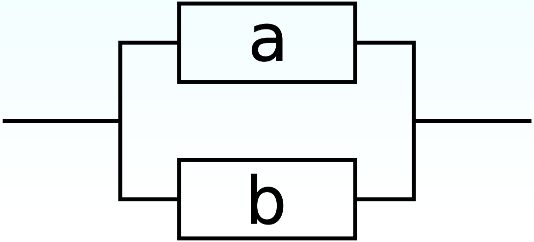

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