Functions 1

Motivation

The concept of a function is one of the most fundamental ideas in mathematics. Examples include the trigonometrical functions, which convert an angle (a number) to a ratio of distances (also a number). However, the idea of a function is more general than this.

Functions are often first encountered as 'number machines', which take a number as an input, do something to it, and then return another number as an output. Our more general idea of a function will be that we specify a set of inputs (which need not necessarily be numbers) and describe what the function does to each of these inputs. In other words, for each input we will specify precisely one output from some other set of elements (which, again, need not necessarily be numbers). An input to a function is called an **argument**.

The rule associating values with arguments may be presented in a number of ways. For many functions, the only practicable method is to list all the arguments, together with the associated values. A practical example of this is an email directory, which associates each person with their email address.

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Examples Of Functions

Area of a circle

You may recall that if a circle has a radius of r, then its area, A, can be calculated as $A = \pi r^2$. We can then say that A is described as a function of r. The value of A depends on the input, or argument, r.

So if $r = 1$, then $A = \pi 1^2 = \pi$. Instead, if $r = 3$, then $A = \pi 3^2 = 9\pi$.

Cost on a menu

We can think of the items on a menu as forming a set, say M. If each of the items on the menu have an associated cost, then we can think of a function called `cost` which returns the price of any menu item $m \in M$.

Addition

A process we are quite familiar with is addition. This is really a function which takes two arguments (in this case numbers, say 3 and 5) and returns a single number as an output (in the case of our example, this output would be $3 + 5$, or more simply 8).

Notation Of Functions

The general notation for writing a function is as follows.

$$f\colon X \to Y$$

Here f is the name of the function. We read the above notation as "the function f from the set X to the set Y" or just "f from X to Y" and $f\colon X \to Y$ is called the **signature** of the function. We are in some sense saying that f is of *type* $X \to Y$. The set X is called the **domain** of f and is the set of inputs to which f can be applied. The set Y is called the **codomain** of f and can be thought of as the set of *potential outputs* that f can produce.

If we take an element $x \in X$ as an input and apply f to it, then we write this as $f(x) \in Y$, which we read as "f of x". Note that after we have applied f to the element $x \in X$, the output is an element $f(x) \in Y$.

Some functions can be specified by formulae. For example, $f(x) = x^3+4x+1$ for $f\colon \mathbb{Z} \to \mathbb{Z}$. The symbol x can be replaced with any valid expression. It might be a particular element of $\mathbb{Z}$, say 5, in which case the output is $f(5) = 5^3 + 4 \times 5 + 1$. Or it might be a more general expression, for example

$$f(a + 1) = (a + 1)^3 + 4(a + 1) + 1.$$

Note that brackets are need to show, for example, that the whole of the argument $a + 1$ is cubed and the whole of the argument is multiplied by 4.

Since the codomain is simply the set of *potential* outputs, it can be useful to have a name for the subset of the codomain which just consists of the outputs which are actually produced. If

$$f\colon X \to Y$$

is a function, then the subset

$$\mathrm{im}(f) = \{y \in Y \mid y = f(x) \text{ for some } x \in X\}$$

is called the **image** of f.

We will return to our examples and formalise them in this new notation.

Video Visit the URL below to view a video:
`https://www.youtube.com/embed/hvl6t7byaXQ`

Area of a circle

We had a formula which calculated the area of a circle given any radius r. We can think of this as a function

$$A\colon \{r \in \mathbb{R} \mid r \geq 0\} \to \{x \in \mathbb{R} \mid x \geq 0\}$$

which takes any non-negative real number, r, and returns the value $A(r) = \pi r^2$. Note here that the domain and codomain are in fact the same set, just using different symbols.

We could define the function with a different codomain, say as

$$\alpha\colon \{r \in \mathbb{R} \mid r \geq 0\} \to \mathbb{R}.$$

Note now that we won't necessarily be able to produce all of the outputs in the codomain. Since $\mathbb{R}$ contains negative numbers and area is always positive, there will be no input r which produces a negative output from the formula $\alpha(r) = \pi r^2$. This point is subtle. The two descriptions of the functions A and α do exactly the same thing but they are not the same function since their signatures are different. This subtlety will be important later when want to *compose* functions.

Cost on a menu

We had a set M, of items on a menu. We want to define a function

$$\texttt{cost}\colon M \to \mathbb{R}$$

which takes as input an item, $m \in M$, from the menu and returns how much it costs, which will be a number. We could do something similar to the example with the circle and restrict our outputs to just positive values, since it would be unlikely that a restaurant would want to charge a negative value for something!

The function `cost` applied to a menu item $m \in M$ would then return a numerical value, $\texttt{cost}(m)$, describing the cost.

Addition

In the case of addition we are actually taking two inputs. For this example we will think of adding two integers together. We can think of addition as being a function

$$\texttt{add}\colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$$

which takes an ordered pair of integers, $(m, n) \in \mathbb{Z} \times \mathbb{Z}$, and returns their sum, $\texttt{add}(m, n)$. This might seem an odd way to think about addition, since we usually see it written as $m + n$, but it is more similar to the way we define functions in a programming language.