

Statistics and Probability

Alex Corner

Sheffield Hallam University

Statistics

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19	47	63	32	7	70	55	46	11	20	15	39	37
28	72	46	64	61	51	56	53	61	11	80	53	28
76	6	5	39	58	29	52	54	47	60	62	51	72
41	57	32	12	33	17	40	20	10	27	47	71	68
44	7	23	17	81	23	12	33	16	46	71	48	58
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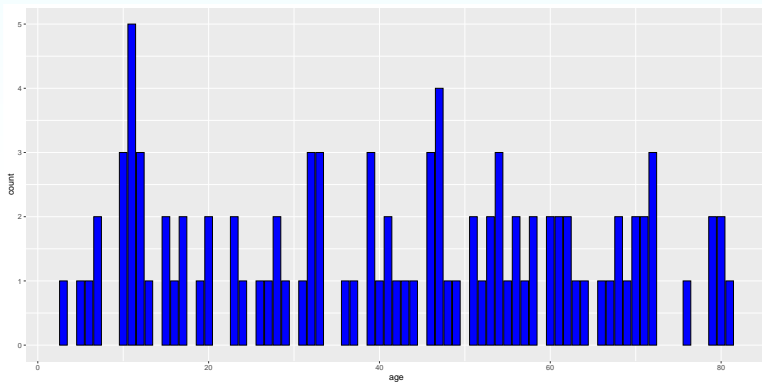
- ▶ It's a little hard to get a feel for the data, so we could count the frequency in each bin, order the data numerically, or calculate various **statistics**.
- ▶ In doing so, we lose some of the information but might get a clearer overview.

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- Here we have placed the data into bins of **class width 5**.

Group Number	Age Range	Tally	Number in Group (Frequency)
1	$0 \leq x < 5$		1
2	$5 \leq x < 10$		4
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Charts: Bar Chart



Charts: Histogram



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- ▶ **Mode:** Tally the data and select the one with the highest frequency.

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- ▶ The middle two values are 43 and 44, hence the median is 43.5.

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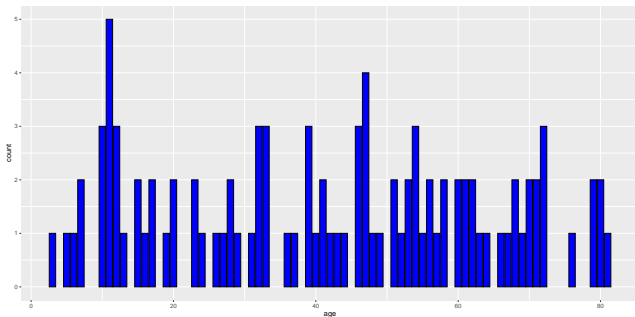
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- ▶ For grouped data we can talk of a **modal group** or **modal class**, sometimes specifying the mid-point in order to give a single value.
- ▶ For the one hundred data values in the histogram example the mode is clearly indicated on the bar chart as the tallest bar: 11. We can also tally values to find it.



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- ▶ The *probability of failure* is:

$$P(\bar{A}) = \frac{N - S}{N} = 1 - \frac{S}{N} = 1 - P(A).$$

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- ▶ $A = \text{'a head shows when a coin is tossed'}$, so $P(A) = \frac{1}{2}$.

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- ▶ The probability of failure is: $P(\text{Failure}) = \frac{94}{100}$.

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- ▶ There are 36 possible outcomes when rolling two dice. But only six of these outcomes make A true.
- ▶ Number of trials $N = 36$, number of successes $S = 6$.

$$P(A) = \frac{6}{36} = \frac{1}{6}.$$



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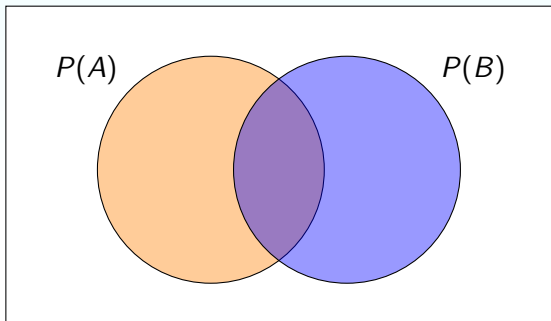
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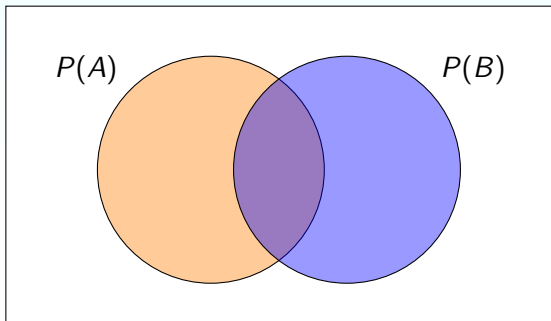
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- ▶ $P(A) = \frac{26}{52} = \frac{1}{2}$ and $P(B) = \frac{26}{52} = \frac{1}{2}$. The two events are independent, so:

$$P(A \wedge B) = P(A) \times P(B) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}.$$

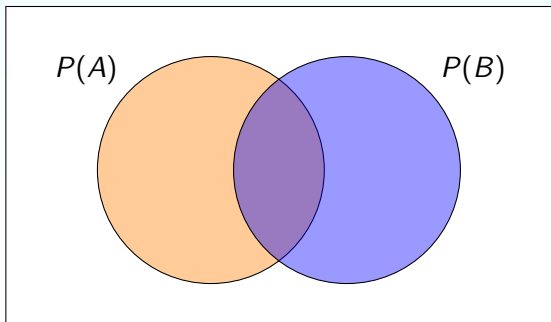
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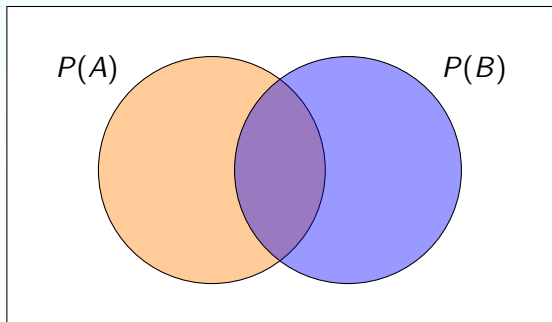
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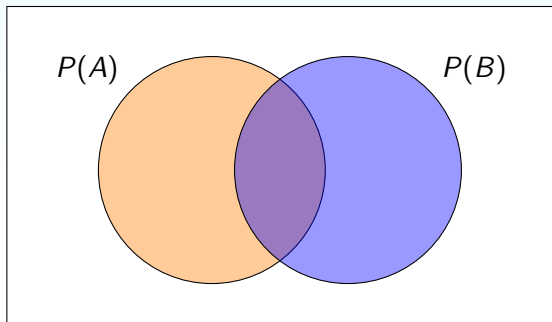


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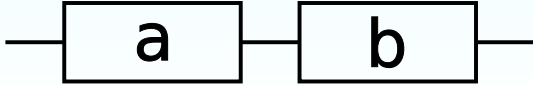
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$$P(A) + P(B) - P(A) \times P(B) = \frac{1}{2} + \frac{1}{2} - \frac{1}{2} \times \frac{1}{2} = \frac{3}{4}.$$

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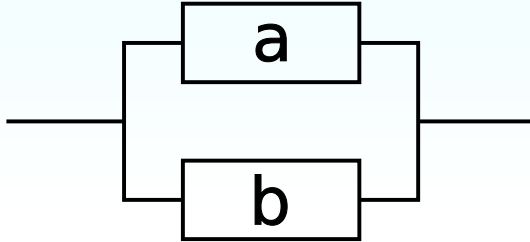


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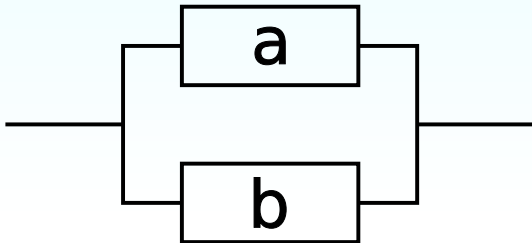
Example: Network 2

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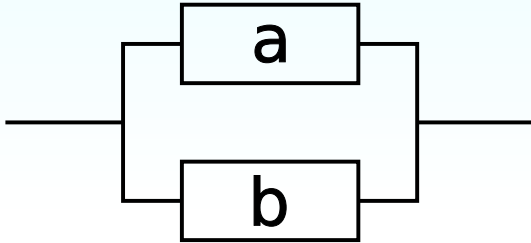
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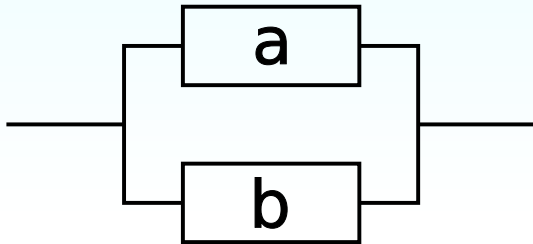
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- ▶ This means the probability that the second card is red given the first card was red is:

$$P(A \wedge B) = P(B|A) \times P(A) = \frac{25}{51} \times \frac{1}{2} = \frac{25}{102}.$$

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