

Exercises: Functions

Exercises

1. Find the values of the given functions at the stated arguments. For example, if the function is $s: \mathbb{N} \rightarrow \mathbb{N}$, where $s(n) = n + 1$, then $s(0) = 1$ and $s(5) = 6$.
 - (a) $f: \mathbb{N} \rightarrow \mathbb{N}$, where $f(n) = n + 3$. Find the values of $f(0)$, $f(3)$ and $f(7)$.
 - (b) $g: \mathbb{Z} \rightarrow \mathbb{N}$, where $g(n) = |n|$. (This means the absolute value of n .) Find the values of $g(0)$, $g(-1)$, $g(2)$.
 - (c) $h: \mathbb{R} \rightarrow \{1\}$, where $h(n) = 1$. Find the values of $h(1)$, $h(\pi)$, $h(-e^{-\pi})$.
2. For each of the following functions, specify a suitable codomain X . There may be more than one suitable answer.
 - (a) $f: \mathbb{N} \rightarrow X$, where $f(x) = x^2$
 - (b) $g: \{1, 2\} \rightarrow X$, where $g(x) = 2x$
 - (c) $h: \mathbb{N} \rightarrow X$, where $h(x) = -x$
 - (d) $i: \mathbb{N} \times \mathbb{Z} \rightarrow X$, where $i(x, y) = x \times y$
3. In each of the following questions you will be given two functions and an input, then asked to compose the functions and describe what the output is.
 - (a) Let $f: \mathbb{Z} \rightarrow \mathbb{Z}$ and $g: \mathbb{Z} \rightarrow \mathbb{Z}$ be functions defined by $f(m) = m^2$ and $g(n) = n^3$, for $m, n \in \mathbb{Z}$.
 - i. For a given integer $m \in \mathbb{Z}$, describe $(g \circ f)(m)$.
 - ii. If $m = 3$, what is the value of $(g \circ f)(m)$?
 - iii. Is $(g \circ f)(m) = (f \circ g)(m)$ always true?
 - (b) Let $\alpha: \mathbb{R} \rightarrow \mathbb{R}$ and $\beta: \mathbb{R} \rightarrow \mathbb{R}$ be functions defined by $\alpha(m) = \frac{m}{2}$ and $\beta(n) = \text{abs}(n)$, for $m \in \mathbb{R}$ and $n \in \mathbb{Z}$. (The symbols α and β are pronounced ‘alpha’ and ‘beta’. The notation for the function β is for the absolute value, sometimes written $|n|$ instead of $\text{abs}(n)$.)
 - i. For a given integer $m \in \mathbb{Z}$, describe $(\beta \circ \alpha)(m)$. If $m = 3$, what is the value of $(\beta \circ \alpha)(m)$?

- ii. If $m = -\pi$, what is the value of $(\beta \circ \alpha)(m)$?
 - iii. If $m = -\pi$, what is the value of $(\alpha \circ \beta)(m)$?
 - iv. Is $(\beta \circ \alpha)(m) = (\alpha \circ \beta)(m)$ always true?
- (c) Let $p: \{a, b, c\} \rightarrow \{1, 2, 3\}$ and $q: \{1, 2, 3\} \rightarrow \{x, y, z\}$ be functions defined by $p(a) = 1$, $p(b) = 2$, $p(c) = 3$ and $q(1) = x$, $q(2) = y$, $q(3) = z$.
- i. For each $m \in \{a, b, c\}$, what is the value of $(q \circ p)(m)$?
 - ii. Is it possible to define the composite function $p \circ q$?
 - iii. Can you define a function $r: \{x, y, z\} \rightarrow \{a, b, c\}$ such that $(r \circ q)(1) = b$, $(r \circ q)(2) = b$, and $(r \circ q)(3) = a$?

Solutions

1. (a) $f(0) = 3$, $f(3) = 6$, $f(7) = 10$
 (b) $g(0) = 0$, $g(-1) = 1$, $g(2) = 2$
 (c) $h(1) = 1$, $h(\pi) = 1$, $h(-e^{-\pi}) = 1$
2. (a) One possible codomain is $X = \mathbb{N}$.
 (b) One possible codomain is $X = \{2, 4\}$. Another is $X = \mathbb{N}$.
 (c) One possible codomain is $X = \mathbb{Z}$.
 (d) One possible codomain is \mathbb{Z} .
3. (a) i. $(g \circ f)(m) = g(f(m)) = g(m^2) = (m^2)^3 = m^6$
 ii. $(g \circ f)(m) = (g \circ f)(3) = 3^6 = 729$
 iii. The other composite is given by

$$\begin{aligned}
 (f \circ g)(m) &= f(m^3) \\
 &= (m^3)^2 \\
 &= m^6 \\
 &= (g \circ f)(m).
 \end{aligned}$$

So no matter the value of $m \in \mathbb{Z}$, we always have $(g \circ f)(m) = (f \circ g)(m)$. We can then say that $g \circ f = f \circ g$.

- (b) i. $(\beta \circ \alpha)(m) = \beta(\alpha(m)) = \beta(\frac{m}{2}) = \text{abs}(\frac{m}{2})$. There's nothing much more we can do to simplify that.
- ii. If $m = -\pi$, then $\alpha(m) = \alpha(-\pi) = \frac{-\pi}{2}$. Since $-\pi \approx -3.14159$, then $\frac{-\pi}{2} \approx -1.57$. (In general, don't truncate numbers like π until you have to - so just keep using the symbol π until you actually need a numerical value out of it.) So $\beta(\alpha(-\pi)) \approx \text{abs}(-1.57) \approx 1.57$.
- iii. If $m = -\pi$, then $\beta(m) = \text{abs}(-\pi) = \pi$. So $\alpha(\beta(-\pi)) \approx (1.57)$.

- iv. We can just think about this first without putting in any values. One of the functions, α , halves the value we put into it. The other function, β , takes the absolute value of the number. It doesn't matter what order we do this in, so we always have $\beta \circ \alpha = \alpha \circ \beta$.
- (c)
 - i. $(q \circ p)(a) = x$, $(q \circ p)(b) = y$, $(q \circ p)(c) = z$
 - ii. Since the sets $\{a, b, c\}$ and $\{x, y, z\}$ are not equal, we cannot define the other composite $p \circ q$ since the domains and codomains don't match up in the correct way.
 - iii. Define $r: \{x, y, z\} \rightarrow \{a, b, c\}$ by $r(x) = b$, $r(y) = b$, and $r(z) = a$.