Exercises: Functions

Exercises

1. Find the values of the given functions at the stated arguments. For example, if the function is $s\colon \mathbb{N} \to \mathbb{N}$, where $s(n) = n + 1$, then $s(0) = 1$ and $s(5) = 6$.

   (a) $f\colon \mathbb{N} \to \mathbb{N}$, where $f(n) = n + 3$. Find the values of $f(0)$, $f(3)$ and $f(7)$.

   (b) $g\colon \mathbb{Z} \to \mathbb{N}$, where $g(n) = \mid n \mid$. (This means the absolute value of n.) Find the values of $g(0)$, $g(-1)$, $g(2)$.

   (c) $h\colon \mathbb{R} \to \{1\}$, where $h(n) = 1$. Find the values of $h(1), h(\pi), h(-e^{-\pi})$.

2. For each of the following functions, specify a suitable codomain X. There may be more than one suitable answer.

   (a) $f\colon \mathbb{N} \to X$, where $f(x) = x^2$

   (b) $g\colon \{1, 2\} \to X$, where $g(x) = 2x$

   (c) $h\colon \mathbb{N} \to X$, where $h(x) = -x$

   (d) $i\colon \mathbb{N} \times \mathbb{Z} \to X$, where $i(x, y) = x \times y$

3. In each of the following questions you will be given two functions and an input, then asked to compose the functions and describe what the output is.

   (a) Let $f\colon \mathbb{Z} \to \mathbb{Z}$ and $g\colon \mathbb{Z} \to \mathbb{Z}$ be functions defined by $f(m) = m^2$ and $g(n) = n^3$, for $m, n \in \mathbb{Z}$.

      i. For a given integer $m \in \mathbb{Z}$, describe $(g \circ f)(m)$.
      ii. If $m = 3$, what is the value of $(g \circ f)(m)$?
      iii. Is $(g \circ f)(m) = (f \circ g)(m)$ always true?

   (b) Let $\alpha\colon \mathbb{R} \to \mathbb{R}$ and $\beta\colon \mathbb{R} \to \mathbb{R}$ be functions defined by $\alpha(m) = \frac{m}{2}$ and $\beta(n) = \mathrm{abs}(n)$, for $m \in \mathbb{R}$ and $n \in \mathbb{Z}$. (The symbols α and β are pronounced 'alpha' and 'beta'. The notation for the function β is for the absolute value, sometimes written $|n|$ instead of $\mathrm{abs}(n)$.)

      i. For a given integer $m \in \mathbb{Z}$, describe $(\beta \circ \alpha)(m)$. If $m = 3$, what is the value of $(\beta \circ \alpha)(m)$?

ii. If $m = -\pi$, what is the value of $(\beta \circ \alpha)(m)$?
iii. If $m = -\pi$, what is the value of $(\alpha \circ \beta)(m)$?
iv. Is $(\beta \circ \alpha)(m) = (\alpha \circ \beta)(m)$ always true?

(c) Let $p\colon \{a, b, c\} \to \{1, 2, 3\}$ and $q\colon \{1, 2, 3\} \to \{x, y, z\}$ be functions defined by $p(a) = 1$, $p(b) = 2$, $p(c) = 3$ and $q(1) = x$, $q(2) = y$, $q(3) = z$.

i. For each $m \in \{a, b, c\}$, what is the value of $(q \circ p)(m)$?
ii. Is it possible to define the composite function $p \circ q$?
iii. Can you define a function $r\colon \{x, y, z\} \to \{a, b, c\}$ such that $(r \circ q)(1) = b$, $(r \circ q)(2) = b$, and $(r \circ q)(3) = a$?

4. (Challenge) In the following questions we will investigate a link between sets of functions $f\colon X \to \{0, 1\}$ and subsets of a set X. Let X be a set. Then the **power set of** X is defined as the set whose elements are the subsets $A \subseteq X$. I.e,

$$\mathbb{P}(X) = \{A \mid A \subseteq X\}.$$

For examples and further explanation, see the notes on power set.

(a) For each of the following sets, count how many different subsets they have. I.e., for each set B below, count the number of elements in the set $\mathbb{P}(B)$.

i. $X = \emptyset$,
ii. $Y = \{0\}$,
iii. $Z = \{0, 1\}$,
iv. $W = \{0, 1, 2\}$.

(b) For each of the following sets, count how many different functions there are into the set $\{0, 1\}$. I.e., for each set B below, count the number of elements in the set $\mathrm{Fun}(B, \{0, 1\}$.

i. $X = \emptyset$,
ii. $Y = \{0\}$,
iii. $Z = \{0, 1\}$,
iv. $W = \{0, 1, 2\}$.

(c) Do you notice anything about the numbers in the parts above? I.e., is there any link between the number of subset of a set B and the number of functions $B \to \{0, 1\}$?

(d) Given a subset $A \subset B$, can you use this to define a function $f_A\colon B \to \{0, 1\}$? I.e., can you define a function $\varphi\colon \mathbb{P}(B) \to \mathrm{Fun}(B, \{0, 1\})$?

(e) Given a function $f\colon B \to \{0, 1\}$, can you use this to define a subset $A_f \subseteq B$? I.e., can you define a function $\psi\colon \mathrm{Fun}(B, \{0, 1\} \to \mathbb{P}(B)$?

Solutions

1. (a) $f(0) = 3$, $f(3) = 6$, $f(7) = 10$

(b) $g(0) = 0$, $g(-1) = 1$, $g(2) = 2$

(c) $h(1) = 1, h(\pi) = 1, h(-e^{-\pi}) = 1$

2. (a) One possible codomain is $X = \mathbb{N}$.

   (b) One possible codomain is $X = \{2, 4\}$. Another is $X = \mathbb{N}$.

   (c) One possible codomain is $X = \mathbb{Z}$.

   (d) One possible codomain is $\mathbb{Z}$.

3. (a) i. $(g \circ f)(m) = g(f(m)) = g(m^2) = (m^2)^3 = m^6$

   ii. $(g \circ f)(m) = (g \circ f)(3) = 3^6 = 729$

   iii. The other composite is given by

$$\begin{aligned}(f \circ g)(m) &= f(m^3) \\ &= (m^3)^2 \\ &= m^6 \\ &= (g \circ f)(m).\end{aligned}$$

   So no matter the value of $m \in \mathbb{Z}$, we always have $(g \circ f)(m) = (f \circ g)(m)$. We can then say that $g \circ f = f \circ g$.

   (b) i. $(\beta \circ \alpha)(m) = \beta(\alpha(m)) = \beta(\frac{m}{2}) = \mathrm{abs}(\frac{m}{2})$. There's nothing much more we can do to simplify that.

   ii. If $m = -\pi$, then $\alpha(m) = \alpha(-\pi) = \frac{-\pi}{2}$. Since $-\pi \approx -3.14159$, then $\frac{-\pi}{2} \approx -1.57$. (In general, don't truncate numbers like π until you have to - so just keep using the symbol π until you actually need a numerical value out of it.) So $\beta(\alpha(-\pi)) \approx \mathrm{abs}(-1.57) \approx 1.57$.

   iii. If $m = -\pi$, then $\beta(m) = \mathrm{abs}(-\pi) = \pi$. So $\alpha(\beta(-\pi)) \approx (1.57)$.

   iv. We can just think about this first without putting in any values. One of the functions, α, halves the value we put into it. The other function, β, takes the absolute value of the number. It doesn't matter what order we do this in, so we always have $\beta \circ \alpha = \alpha \circ \beta$.

   (c) i. $(q \circ p)(a) = x$, $(q \circ p)(b) = y$, $(q \circ p)(c) = z$

   ii. Since the sets $\{a, b, c\}$ and $\{x, y, z\}$ are not equal, we cannot define the other composite $p \circ q$ since the domains and codomains don't match up in the correct way.

   iii. Define $r \colon \{x, y, z\} \to \{a, b, c\}$ by $r(x) = b$, $r(y) = b$, and $r(z) = a$.

4. (Challenge) In the following questions we will investigate a link between sets of functions $f \colon X \to \{0, 1\}$ and subsets of a set X. Let X be a set. Then the **power set of** X is defined as the set whose elements are the subsets $A \subseteq X$. I.e,

$$\mathbb{P}(X) = \{A \mid A \subseteq X\}.$$

   For examples and further explanation, see the notes on power set.

(a) For each of the following sets, count how many different subsets they have. I.e., for each set B below, count the number of elements in the set $\mathbb{P}(B)$.

   i. $\mathbb{P}(X) = \{\emptyset\}$, so $|\mathbb{P}(X)| = 1$;
   ii. $\mathbb{P}(Y) = \{\emptyset, \{0\}\}$, so $|\mathbb{P}(Y)| = 2$;
   iii. $\mathbb{P}(Z) = \{\emptyset, \{0\}, \{1\}, \{0, 1\}\}$, so $|\mathbb{P}(Z)| = 4$;
   iv. $\mathbb{P}(W) = \{\emptyset, \{0\}, \{1\}, \{2\}, \{0, 1\}, \{0, 2\}, \{1, 2\}, \{0, 1, 2\}\}$, so $|\mathbb{P}(W)| = 8$.

(b) For each of the following sets, count how many different functions there are into the set $\{0, 1\}$. I.e., for each set B below, count the number of elements in the set $\mathrm{Fun}(B, \{0, 1\}$.

   i. $X = \emptyset$: there is one function $\emptyset \to \{0, 1\}$ - this can be tricky to see since the domain is empty;
   ii. $Y = \{0\}$: there are two possible functions here - send 0 to 0, or send 0 to 1;
   iii. $Z = \{0, 1\}$: there are four possible functions here - send both elements to 0, send both elements to 1, send both elements to themselves, or swap the values;
   iv. $W = \{0, 1, 2\}$: this time there are eight possible functions, which you may want to try and write down.

(c) Do you notice anything about the numbers in the parts above? Given a set B, the number of subsets of B is the same as the number of functions $B \to \{0, 1\}$. I.e., for any set B we seem to be seeing that $|\mathbb{P}(B)| = |\mathrm{Fun}(B, \{0, 1\})|$.

(d) Can you define a function $\varphi \colon \mathbb{P}(B) \to \mathrm{Fun}(B, \{0, 1\})$? Let A be a subset of B. We need to define a function $\varphi(A) \colon B \to \{0, 1\}$. One way of doing this is to define $\varphi(A)(b) = 0$ if $b \notin A$ and $\varphi(A)(b) = 1$ if $b \in A$. So what we are doing is sending an element $b \in B$ to 1 (True) if the element is also in the subset A, otherwise we send it to 0 (False).

(e) Can you define a function $\psi \colon \mathrm{Fun}(B, \{0, 1\} \to \mathbb{P}(B)$? Let $f \colon B \to \{0, 1\}$ be a function. We need to define a subset $\psi(f) = A_f \subseteq B$. We can define this by

$$A_f = \{b \in B \mid f(b) = 1\}.$$

This is similar to how we defined the function in the previous part.