

# Statistics and Probability

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# Statistics

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19	47	63	32	7	70	55	46	11	20	15	39	37
28	72	46	64	61	51	56	53	61	11	80	53	28
76	6	5	39	58	29	52	54	47	60	62	51	72
41	57	32	12	33	17	40	20	10	27	47	71	68
44	7	23	17	81	23	12	33	16	46	71	48	58
79	80	43	31	72	68	36	41	11				

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41	57	32	12	33	17	40	20	10	27	47	71	68
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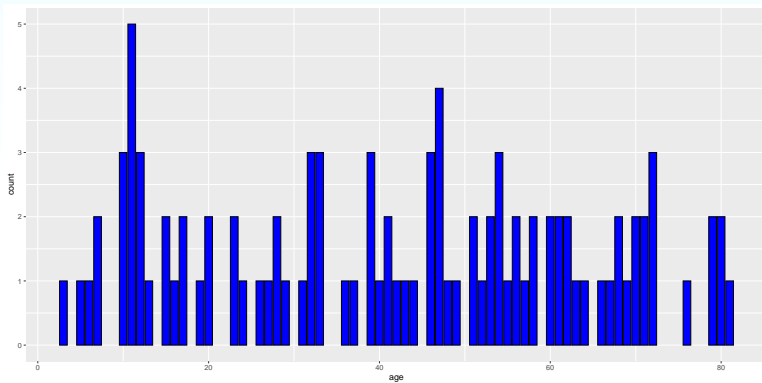
- ▶ It's a little hard to get a feel for the data, so we could count the frequency in each bin, order the data numerically, or calculate various **statistics**.
- ▶ In doing so, we lose some of the information but might get a clearer overview.

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- Here we have placed the data into bins of **class width 5**.

Group Number	Age Range	Tally	Number in Group (Frequency)
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## Charts: Bar Chart





# Charts: Histogram



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- ▶ **Median:** Order the data and look at the middle value(s).
- ▶ **Mode:** Tally the data and select the one with the highest frequency.

## (Arithmetic) Mean

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12	12	12	13	15	15	16	17	17	19	20	20	23
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- ▶ The middle two values are 43 and 44, hence the median is 43.5.



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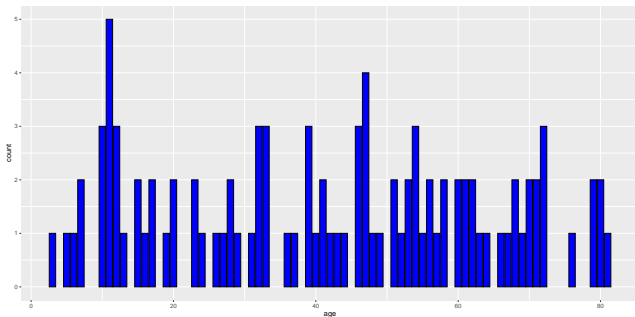
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- ▶ For grouped data we can talk of a **modal group** or **modal class**, sometimes specifying the mid-point in order to give a single value.
- ▶ For the one hundred data values in the histogram example the mode is clearly indicated on the bar chart as the tallest bar: 11. We can also tally values to find it.



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- ▶ The *probability of failure* is:

$$P(\bar{A}) = \frac{N - S}{N} = 1 - \frac{S}{N} = 1 - P(A).$$

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- ▶  $A = \text{'a head shows when a coin is tossed'}$ , so  $P(A) = \frac{1}{2}$ .

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(4, 1)	(4, 2)	<b>(4, 3)</b>	(4, 4)	(4, 5)	(4, 6)
(5, 1)	<b>(5, 2)</b>	(5, 3)	(5, 4)	(5, 5)	(5, 6)
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(4, 1)	(4, 2)	<b>(4, 3)</b>	(4, 4)	(4, 5)	(4, 6)
(5, 1)	<b>(5, 2)</b>	(5, 3)	(5, 4)	(5, 5)	(5, 6)
<b>(6, 1)</b>	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

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- ▶ There are 36 possible outcomes when rolling two dice. But only six of these outcomes make  $A$  true.
- ▶ Number of trials  $N = 36$ , number of successes  $S = 6$ .

$$P(A) = \frac{6}{36} = \frac{1}{6}.$$



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- ▶ Its analogue in set theory is **intersection**:  $A \cap B$ .



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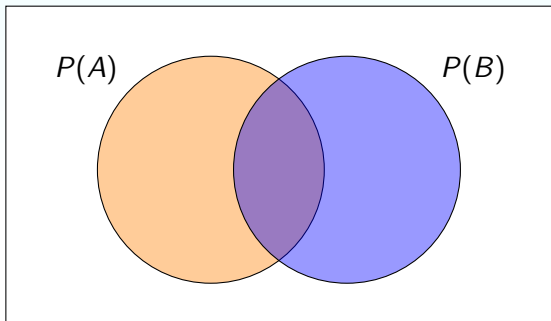
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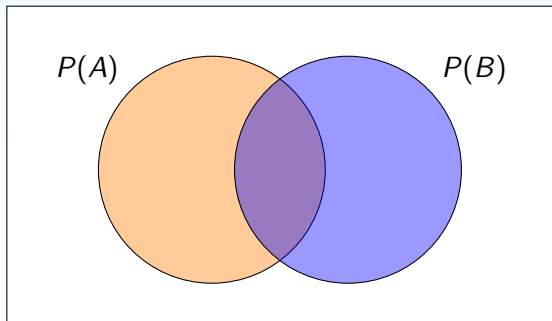
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- ▶  $P(A) = \frac{26}{52} = \frac{1}{2}$  and  $P(B) = \frac{26}{52} = \frac{1}{2}$ . The two events are independent, so:

$$P(A \wedge B) = P(A) \times P(B) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}.$$

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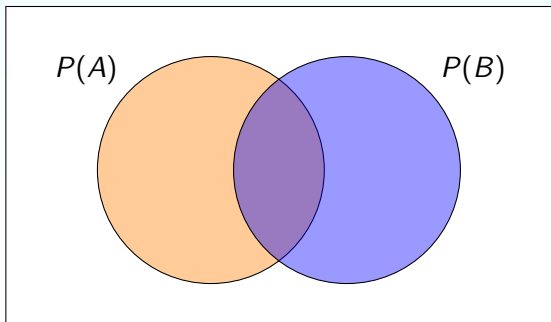


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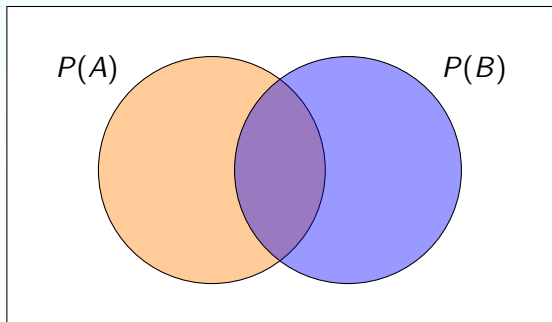
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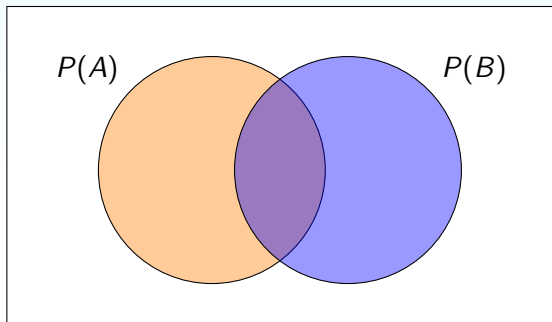


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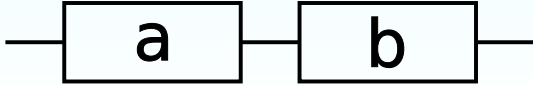
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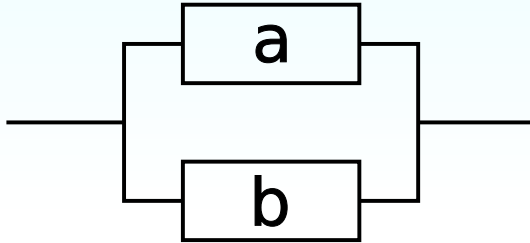


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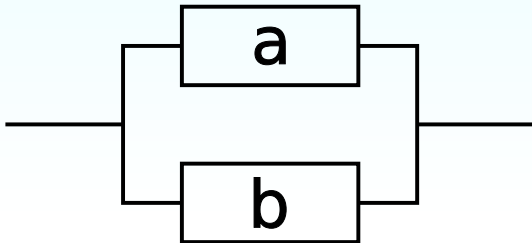
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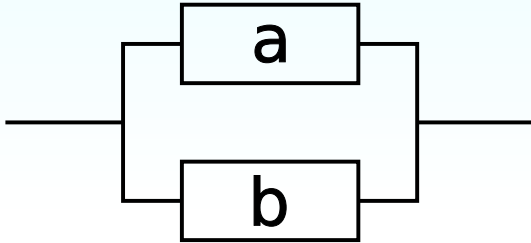
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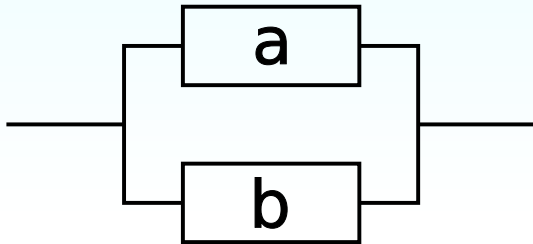
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