

Exercises: Matrices

Exercises 1

1. This question uses the following matrices. Also, I is used to denote an identity matrix of any suitable size.

$$A = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} \quad B = \begin{pmatrix} 0 & -2 \\ 3 & 1 \end{pmatrix} \quad C = \begin{pmatrix} 1 & -3 \\ -2 & 6 \end{pmatrix}$$

$$D = \begin{pmatrix} 1 & -1 & 2 \\ 8 & 3 & 2 \\ 1 & 0 & 1 \end{pmatrix} \quad E = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad F = \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix} \quad G = \begin{pmatrix} 1 & 0 \\ 2 & -6 \\ 1 & 2 \end{pmatrix}$$

Calculate the following, when they exist.

- | | | |
|---------------|---------------|--------------|
| (a) $A + B$ | (i) F^T | (q) GC |
| (b) $B - A$ | (j) AB | (r) A^2 |
| (c) $D + G$ | (k) IA | (s) E^2 |
| (d) $3C$ | (l) AI | (t) A^{-1} |
| (e) $2A + 3C$ | (m) CE | (u) B^{-1} |
| (f) $A + I$ | (n) DF | (v) E^{-1} |
| (g) $E - I$ | (o) $F^T D^T$ | (w) C^{-1} |
| (h) C^T | (p) CG | |

2. The matrix L shown below describes a football league part-way through the season. The first column represents the number of games won, the second the number of games drawn, and the final column the number of games lost.

- (a) Post-multiply L by the column $\begin{pmatrix} 3 & 1 & 0 \end{pmatrix}^T$. What does the result represent? (If you are unfamiliar with scoring in association football, see here.)
- (b) Pre-multiply L by the row $\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$. What does the result represent? Does it show that something is wrong?
- (c) Post-multiply the result of the previous part by the column $\begin{pmatrix} 0.5 & 0.5 & 0.5 \end{pmatrix}^T$. What does the result represent?

$$L = \begin{matrix} \text{Team A} \\ \text{Team B} \\ \text{Team C} \\ \text{Team D} \\ \text{Team E} \\ \text{Team F} \\ \text{Team G} \\ \text{Team H} \\ \text{Team I} \\ \text{Team J} \end{matrix} \begin{pmatrix} 11 & 5 & 4 \\ 11 & 2 & 7 \\ 11 & 1 & 6 \\ 10 & 3 & 6 \\ 9 & 4 & 7 \\ 9 & 3 & 9 \\ 9 & 3 & 8 \\ 6 & 2 & 12 \\ 5 & 1 & 12 \\ 4 & 1 & 15 \end{pmatrix}$$

Solutions

1. (a) $A + B = \begin{pmatrix} 1 & 0 \\ 5 & 0 \end{pmatrix}$
- (b) $B - A = \begin{pmatrix} -1 & -4 \\ 1 & 2 \end{pmatrix}$
- (c) $D + G$: Doesn't exist because D and G are different sizes.
- (d) $3C = \begin{pmatrix} 3 & -9 \\ -6 & 18 \end{pmatrix}$
- (e) $2A + 3C = \begin{pmatrix} 5 & -5 \\ -2 & 16 \end{pmatrix}$
- (f) $A + I = \begin{pmatrix} 2 & 2 \\ 2 & 0 \end{pmatrix}$
- (g) $E - I$: Doesn't exist since E is a 2×1 matrix and identity matrices are necessarily *square*.
- (h) $C^T = \begin{pmatrix} 1 & -2 \\ -3 & 6 \end{pmatrix}$
- (i) $F^T = \begin{pmatrix} -2 & 1 & 2 \end{pmatrix}$
- (j) $AB = \begin{pmatrix} 6 & 0 \\ -3 & -5 \end{pmatrix}$
- (k) $IA = A$
- (l) $AI = A$
- (m) $CE = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$
- (n) $DF = \begin{pmatrix} 1 \\ -9 \\ 0 \end{pmatrix}$
- (o) $F^T D^T = \begin{pmatrix} 1 & -9 & 0 \end{pmatrix}$
- (p) CG : Doesn't exist as the dimensions don't agree.
- (q) $GC = \begin{pmatrix} 1 & -3 \\ 14 & -42 \\ -3 & 9 \end{pmatrix}$

(r) $A^2 = \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$

(s) E^2 : Doesn't exist as the dimensions don't agree.

(t) $A^{-1} = \frac{1}{5} \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$

(u) $B^{-1} \frac{1}{6} \begin{pmatrix} 1 & 2 \\ -3 & 0 \end{pmatrix}$

(v) E^{-1} : Doesn't exist because E isn't a square matrix.

(w) C^{-1} : Doesn't exist because $\det(C) = 1 \times 6 - (-3 \times -2) = 0$.

2. The matrix L describes a football league part-way through the season. The first column represents the number of games won, the second the number of games drawn, and the final column the number of games lost.

(a)

$$\begin{array}{l} \text{Team A} \\ \text{Team B} \\ \text{Team C} \\ \text{Team D} \\ \text{Team E} \\ \text{Team F} \\ \text{Team G} \\ \text{Team H} \\ \text{Team I} \\ \text{Team J} \end{array} \begin{pmatrix} 11 & 5 & 4 \\ 11 & 2 & 7 \\ 11 & 1 & 6 \\ 10 & 3 & 6 \\ 9 & 4 & 7 \\ 9 & 3 & 9 \\ 9 & 3 & 8 \\ 6 & 2 & 12 \\ 5 & 1 & 12 \\ 4 & 1 & 15 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 38 \\ 35 \\ 34 \\ 33 \\ 31 \\ 30 \\ 30 \\ 20 \\ 16 \\ 13 \end{pmatrix}$$

(b)

$$(1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1) \begin{pmatrix} 11 & 5 & 4 \\ 11 & 2 & 7 \\ 11 & 1 & 6 \\ 10 & 3 & 6 \\ 9 & 4 & 7 \\ 9 & 3 & 9 \\ 9 & 3 & 8 \\ 6 & 2 & 12 \\ 5 & 1 & 12 \\ 4 & 1 & 15 \end{pmatrix} = (85 \quad 25 \quad 86)$$

These are the total numbers of Wins, Draws, and Losses. Somehow there have been fewer Wins than there have been Losses - which doesn't make sense!

(c)

$$(85 \quad 25 \quad 86) \begin{pmatrix} 0.5 \\ 0.5 \\ 0.5 \end{pmatrix} = 98$$

The total number of matches played so far. Each Win and Loss comes from one game, so counting them individually counts matches twice. Similarly, counting *every* draw counts it once for both teams in that match.