Exercises: Complexity

You may find graphing tools such as GeoGebra or Desmos useful for these exercises.

Exercises

1. Some of the following statements are true and some are false. Which are which?

   (a) $\log_{10} x + 3x^2$ is $O(x^3)$
   (b) $67x^4 + 2^x$ is $O(x^4)$
   (c) $\frac{3}{2}2^x$ is $O(2^x)$
   (d) $\log_{10} x + 3x^2$ is $\Theta(x^3)$

2. For each of the following functions of n, find the non-negative constant k such that the function is $\Theta\left(n^k\right)$, or explain why there is no such k. (In other words, find, where possible, k such that the function is $O\left(n^k\right)$ and it is not $O\left(n^j\right)$ for any $j < k$.) (Note that k doesn't have to be an integer in all cases.)

   (a) $5n^3 + 20n^2 + 3n + 2$
   (b) $6n^2 + \log_e n + 3n + 2$
   (c) $7n + \log_3 n + 2$
   (d) $6n^2 + \log_2 n + 2^n + 3n + 2$
   (e) 32
   (f) $4\log_2 n + 6\sqrt{n} + 2$
   (g) $5\log_2 n + 2$

3. Use a graphing tool like GeoGebra or Desmos to plot graphs of

   - $y = \log_e x$ (or $\ln x$),
   - $y = x$,
   - $y = x^2$,
   - $y = 100x^2$,
   - $y = e^x$.

   This will give you an idea of the relative growth rates of logarithmic, polynomial, and exponential functions.

Solutions

1. Some of the following statements are true and some are false. Which are which?

   (a) True. Though it is also $O(x^2)$.

   (b) False. The function 2^x grows much faster than this. It is $O(3^x)$ though.

   (c) True.

   (d) False. We definitely know the function is $O(n^3)$, but n^3 is not $O(\log_{10} x + 3x^2)$.

2. For each of the following functions of n, find the non-negative constant k such that the function is $\Theta\left(n^k\right)$, or explain why there is no such k. (In other words, find, where possible, k such that the function is $O\left(n^k\right)$ and it is not $O\left(n^j\right)$ for any $j < k$.) (Note that k doesn't have to be an integer in all cases.)

   (a) $5n^3 + 20n^2 + 3n + 2$ is $\Theta(n^3)$: $5n^3 < 5n^3 + 20n^2 + 3n + 2 < 6n^3$ as $n \to \infty$.

   (b) $6n^2 + \log_e n + 3n + 2$ is $\Theta(n^2)$.

   (c) $7n + \log_3 n + 2$ is $\Theta(n)$.

   (d) $6n^2 + \log_2 n + 2^n + 3n + 2$ is not $\Theta(n^k)$ for any k, since 2^n dominates.

   (e) 32 is $\Theta(1) = \Theta(n^0)$.

   (f) $4 \log_2 n + 6\sqrt{n} + 2$ is $\Theta(n^{\frac{1}{2}})$.

   (g) $5 \log_2 n + 2$ is $\Theta(n)$.

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   - $y = x$,
   - $y = x^2$,
   - $y = 100x^2$,
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   This will give you an idea of the relative growth rates of logarithmic, polynomial, and exponential functions.