Exercises: Matrix Equations

Exercises 2

1. Let
$$A = \begin{pmatrix} -3 & 5 \\ 4 & 3 \end{pmatrix}.$$
   (a) Find the determinant of A.
   (b) Find the inverse of A.
   (c) Use A^{-1} to find a solution for the simultaneous equations
$$-3x + 5y = 58,$$
$$4x + 3y = -29.$$
   (d) Use A^{-1} to find a solution for the simultaneous equations
$$-3x + 5y = -2,$$
$$4x + 3y = 1.$$

2. Let
$$B = \begin{pmatrix} 7 & -3 \\ 3 & 5 \end{pmatrix}.$$
   (a) Find the determinant of B.
   (b) Find the inverse of B.
   (c) Use B^{-1} to find a solution for the simultaneous equations
$$7a - 3b = 5,$$
$$3a + 5b = 7.$$
   (d) Use B^{-1} to find a solution for the simultaneous equations
$$7u - 3v = 44,$$
$$3u + 5v = -44.$$

3. Consider the following pair of simultaneous equations.

$$17x - 5y = 21$$

$$6x + 22y = 13$$

   (a) Write these equations as a single matrix equation.

   (b) Use matrix inverses and matrix multiplication to solve these equations for x and y.

4. Consider the following pair of simultaneous equations.

$$91x + 14y = 121$$

$$65x + 10y = -72$$

   (a) Write these equations as a single matrix equation.

   (b) Use determinants to show that there are no solutions for these equations for x and y.

5. Calculate the matrix product

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix},$$

and hence check that the formula for the inverse of a matrix, given below, is correct.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

6. If A and B are invertible matrices of the same size, simplify $\left(B^{-1}A^{-1}\right)(AB)$. Hence show that $(AB)^{-1} = B^{-1}A^{-1}$.

Solutions

1. (a)

$$\det(A) = (-3 \times 3) - (5 \times 4) = -9 - 20 = -29.$$

   (b)

$$A^{-1} = \frac{1}{-29} \begin{pmatrix} 3 & -5 \\ -4 & -3 \end{pmatrix}$$

   (c)

$$A^{-1} \begin{pmatrix} 58 \\ -29 \end{pmatrix} = \frac{1}{-29} \begin{pmatrix} 3 & -5 \\ -4 & -3 \end{pmatrix} \begin{pmatrix} 58 \\ -29 \end{pmatrix} = \begin{pmatrix} -11 \\ 5 \end{pmatrix}$$

   (d)

$$A^{-1} \begin{pmatrix} -2 \\ 1 \end{pmatrix} = \frac{1}{-29} \begin{pmatrix} 3 & -5 \\ -4 & -3 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{11}{29} \\ \frac{-5}{29} \end{pmatrix}$$

2. (a)

$$\det(B) = (7 \times 5) - (-3 \times 3) = 35 - (-9) = 44.$$

(b)

$$B^{-1} = \frac{1}{44}\begin{pmatrix} 5 & 3 \\ -3 & 7 \end{pmatrix}$$

(c)

$$B^{-1}\begin{pmatrix} 5 \\ 7 \end{pmatrix} = \frac{1}{44}\begin{pmatrix} 5 & 3 \\ -3 & 7 \end{pmatrix}\begin{pmatrix} 5 \\ 7 \end{pmatrix} = \begin{pmatrix} \frac{46}{44} \\ \frac{34}{44} \end{pmatrix}$$

(d)

$$B^{-1}\begin{pmatrix} 44 \\ -44 \end{pmatrix} = \frac{1}{44}\begin{pmatrix} 5 & 3 \\ -3 & 7 \end{pmatrix}\begin{pmatrix} 44 \\ -44 \end{pmatrix} = \begin{pmatrix} 2 \\ -10 \end{pmatrix}$$

3. (a) We can formulate the simultaneous equations as a single matrix equation:

$$\begin{pmatrix} 17 & -5 \\ 6 & 22 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 21 \\ 13 \end{pmatrix}.$$

(b) The determinant of the coefficient matrix is

$$\det\left(\begin{pmatrix} 17 & -5 \\ 6 & 22 \end{pmatrix}\right) = (17 \times 22) - (-5 \times 6) = 404,$$

hence there are solutions to the pair of simultaneous equations. The inverse of the coefficient matrix is given by

$$\begin{pmatrix} 17 & -5 \\ 6 & 22 \end{pmatrix}^{-1} = \frac{1}{404}\begin{pmatrix} 22 & 5 \\ -6 & 17 \end{pmatrix}.$$

The solutions are hence given by

$$\frac{1}{404}\begin{pmatrix} 22 & 5 \\ -6 & 17 \end{pmatrix}\begin{pmatrix} 21 \\ 13 \end{pmatrix} = \begin{pmatrix} \frac{527}{404} \\ \frac{95}{404} \end{pmatrix}.$$

4. (a) We can formulate the simultaneous equations as a single matrix equation:

$$\begin{pmatrix} 91 & 14 \\ 65 & 10 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 121 \\ -72 \end{pmatrix}.$$

(b) The determinant of the coefficient matrix is

$$\det\left(\begin{pmatrix} 91 & 14 \\ 65 & 10 \end{pmatrix}\right) = (91 \times 10) - (14 \times 65) = 0,$$

hence there are no solutions to the pair of simultaneous equations.

5.

$$\begin{aligned}
\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} &= \begin{pmatrix} ad - bc & -ba + ba \\ cd - dc & -bc + ad \end{pmatrix} \\
&= \begin{pmatrix} ad - bc & 0 \\ 0 & ad - bc \end{pmatrix} \\
&= (ad - bc)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\end{aligned}$$

$$= (ad - bc)I$$

The inverse of a matrix should satisfy AA^{-1}. The calculation shows that

$$\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

almost works but we still need to divide out by a factor of $(ad - bc)$.

6.

$$\begin{aligned}(B^{-1}A^{-1})(AB) &= B^{-1}(A^{-1}A)B \\ &= B^{-1}IB \\ &= B^{-1}B \\ &= I\end{aligned}$$

Similarly, $AB(B^{-1}A^{-1}) = I$. The inverse of a matrix C is a matrix C^{-1} such that $CC^{-1} = I = C^{-1}C$. Since $B^{-1}A^{-1}$ does this job for AB, then we have found that

$$(AB)^{-1} = B^{-1}A^{-1}.$$