Graph encoding in Quantum Computing

Master Thesis in Artificial Intelligence & Cybersecurity

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March 16, 2023

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How to develop them?

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Probabilistic side of Quantum Mechanics: Measurement.

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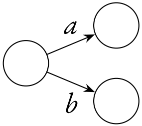

- $a, b \in [0, 1]$;
- $a + b = 1$.

Walker moves randomly.

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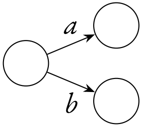

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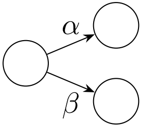

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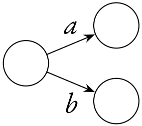

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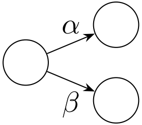

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Problem: Unitarity *restricts* the class of graphs amenable to quantum walks.

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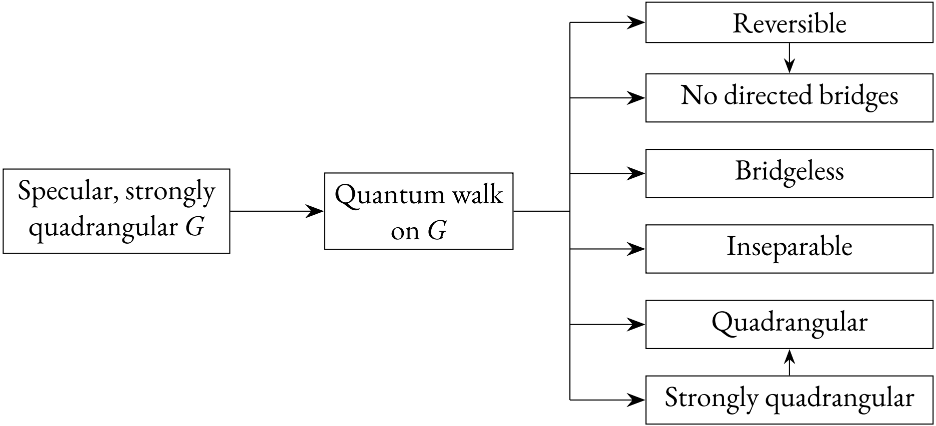
1. *Coined Quantum Walks* on *reversible graphs* are made explicit.
2. The discussion is extended to the phenomenon of **encoding bias**. That is, how does an encoding affect the behaviour of the resulting quantum walk?

Outline

Quantum walk amenable graphs

1. Over which graphs can quantum walks be performed? [2, 3, 4, 5]



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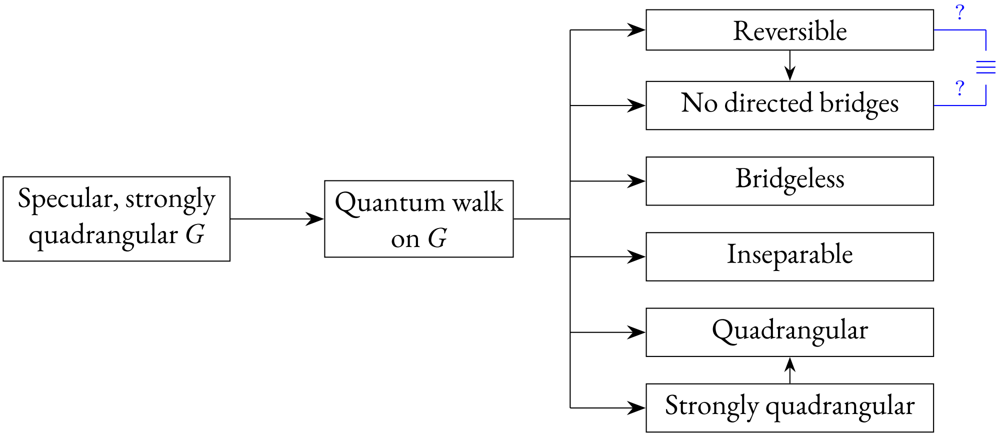

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- **Coined quantum walks on reversible graphs with self-loops.**
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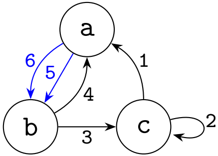

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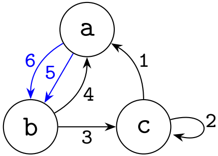


Figure: 2-regular multigraph $\mathcal{G}$.

2. The line graph of the regular multigraph is the underlying graph of the coined quantum walk.

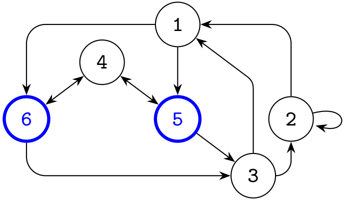
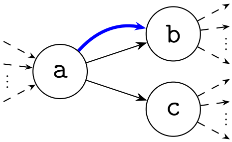


Figure: Line graph $\overrightarrow{\mathcal{G}}$.

Encoding bias

Encoding Bias. Phenomena of local bias affecting the behaviour of quantum walks caused by graph encoding.



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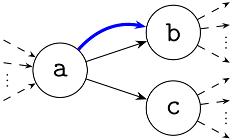

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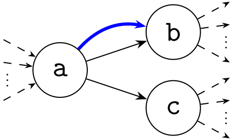

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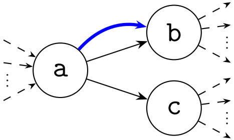

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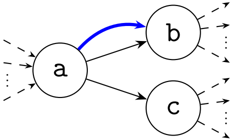

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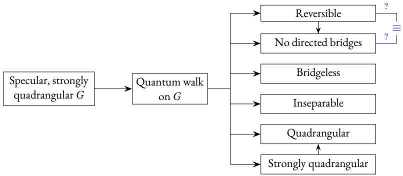
$\Uparrow$

Encoding Eulerian graphs into Eulerian graphs requires no operation.

Conclusions and future work

Two potential further developments:

- The relationship between graphs and quantum walks is intricate. Can it be simplified?



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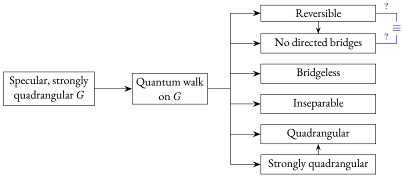

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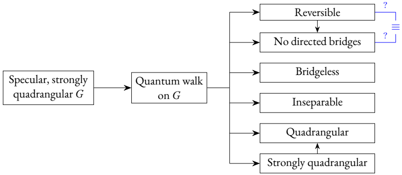

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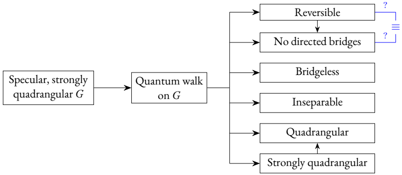

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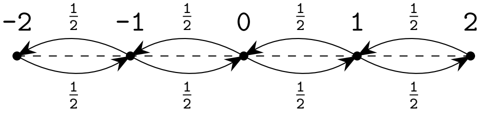
- The study on limits of Quantum Computing.

The line is not amenable to quantum walks

An *unbiased* random walk on the 5-line may be defined as:



$$\mathbf{P} = \begin{pmatrix} 0 & 1/2 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 1/2 & 0 \end{pmatrix}$$

Figure: Random walk on the line and the transition probability matrix.

A quantum walk on the 5-line is described by a unitary matrix with the same zero entries as $\mathbf{P}$.

Problem: There exists no such unitary matrix.

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Strong connectivity

For any two vertices u, v there exists a path from u to v.

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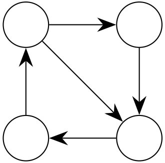

Figure: Strongly connected graph.

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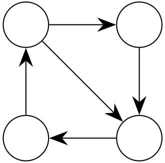

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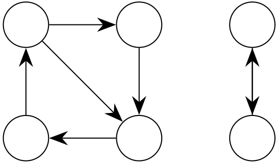

Figure: Reversible non-strongly connected graph

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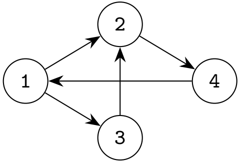

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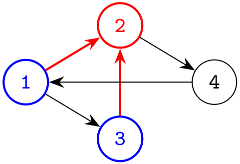

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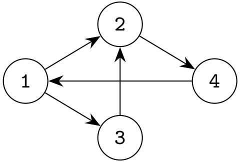

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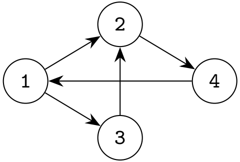

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Let S be any set of k vertices, where all $u \in S$ share at least one in- (out-) neighbour with some other $v \in S$. $\Rightarrow$ The union of the pairwise shared in- (out-) neighbours by the vertices in S must be at least k.

Decoding quadrangularity cont'd

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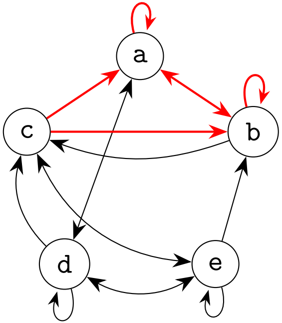
- Quadrangular graph not strongly quadrangular.
- The 3 vertices $\{\mathtt{a},\mathtt{b},\mathtt{c}\}$ share 2 out-neighbours:

$$\{\mathtt{a},\mathtt{b}\}.$$

- $2 < 3$: strong quadrangularity is violated.



Decomposable graphs

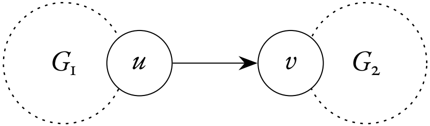


Figure: Graph with directed bridge (u, v).

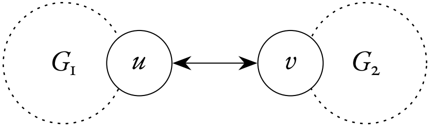


Figure: Graph with bridge $\{(u, v), (v, u)\}$.

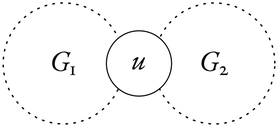


Figure: Separable graph with cut-vertex u.

Bibliography I

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