Verification of Quantum Systems

Alex Della Schiava

Università degli Studi di Udine (DMIF) - V&V Techniques

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Error in Quantum Computing

- **Error** is a common matter of discussion in the field of quantum computing.
- Herein, the issue is tackled following the work done in:

  M. Lewis, S. Soudjani, P. Zuliani (2021), Formal Verification of Quantum Programs: Theory, Tools and Challenges

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- Three main causes to quantum error can be identified:
  1. **Randomness.** Due to the probabilistic nature of quantum mechanics.
  2. **Hardware.** It is hard to work on *closed quantum systems* avoiding external *interference*.
  3. **Defective software.** Typical, well known issue concerning faulty implementations.

Quantum Computing meets Formal Verification

- Focus on **defective software**: common issue for both *classical* and *quantum* computing.

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- **Quantum case.** Is it possible to somehow translate these results into the quantum context?

Quantum Computing meets Formal Verification

- Focus on **defective software**: common issue for both *classical* and *quantum* computing.
- **Classical case.** Plenty of successful solutions thanks to the work done in *formal verification*.
- **Quantum case.** Is it possible to somehow translate these results into the quantum context?
- The answer is **yes**. This seminar shall try to motivate this answer.

Outline

1. Background
2. Quantum **while**-programs
3. Quantum Deductive Verification
4. Model Checking on Quantum CTL
5. Extra: Case study on Deutsch's Algorithm

Classical Verification Techniques

The results here presented are heavily inspired by the following classical verification techniques:

- Model Checking for CTL
  - Automatic, exhaustive, provides counterexamples.
  - But... state explosion problem, no fairness conditions in CTL.
- Deductive Verification
  - Use of inference rules to prove validity of the desired properties with respect to a system/program.
  - Floyd-Hoare logic.
  - Weakest precondition.

Quantum Mechanics

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- Writing $|\psi\rangle$ denotes a *ket*, that is, a column vector in some vector space (Hilbert space in our case).

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Example

$$|\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \quad \langle\psi| = \begin{pmatrix} \alpha^* & \beta^* \end{pmatrix}$$

where $\alpha, \beta \in \mathbb{C}$.

Quantum Mechanics

Quantum mechanics can be formulated in terms of:

State vectors

$$|\psi\rangle$$

- Simple vectors.
- More intuitive.
- Example: $|\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$

Density operators (matrices)

$$\rho = |\psi\rangle \langle\psi|$$

- Positive operators.
- Trace: $\mathsf{tr}(\rho) = 1$.
- Can describe *mixed states*:

$$\rho = \sum_i p_i |\psi_i\rangle \langle\psi_i|$$

Quantum Mechanics

Postulate 1

A quantum system is fully described by a state vector in a given Hilbert space with norm 1.

Example: a qubit.

$$|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$$

- Hilbert space: $\mathcal{H}^2$.
- Basis: $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$; $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$
- Norm: $\sqrt{\langle\psi|\psi\rangle} = \sqrt{|\alpha|^2 + |\beta|^2} = 1$

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$$\rho = |\psi\rangle\,\langle\psi|$$

- Matrix representation:

$$\rho = \begin{pmatrix} |\alpha|^2 & \alpha \cdot \beta^* \\ \alpha^* \cdot \beta & |\beta|^2 \end{pmatrix}$$

- Trace: $|\alpha|^2 + |\beta|^2 = 1$.

Quantum Mechanics

Postulate 2

The evolution of closed quantum systems is described by means of *unitary operators*.

- $U^\dagger$ is the complex conjugate transposed (*adjoint*) of U;
- Unitary operator U: $UU^\dagger = I$.

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- Resulting state:

$$|\psi'\rangle = U|\psi\rangle$$

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$$\begin{aligned}\rho' &= U|\psi\rangle\langle\psi|U^\dagger \\ &= U\rho U^\dagger\end{aligned}$$

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Quantum evolution is **reversible**: just apply $U^\dagger$ to the resulting state.

Postulate 3

A quantum projective measurement M is described by a collection of projectors $\{P_m\}$. The subscripts m denote the possible outcomes of the measurement.

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- Probability of measuring m:

$$p(m) = \langle\psi| P_m |\psi\rangle$$

$$p(m) = \mathsf{tr}(P_m \rho)$$

- Post-measurement state $|\psi'\rangle$:

$$|\psi'\rangle = \frac{P_m |\psi\rangle}{\sqrt{p(m)}}$$

$$\rho' = \frac{P_m \rho P_m^\dagger}{p(m)}$$

Quantum Mechanics

Postulate 4

The state space of a *composite* quantum system is given by the tensor product $\otimes$ of the state spaces of its components.

Example

Consider a composite quantum system A with two qubits $|\psi\rangle, |\varphi\rangle \in \mathcal{H}^2$. The state space of A corresponds to:

$$\mathcal{H}^2 \otimes \mathcal{H}^2 = \mathcal{H}^4$$

- **Note:** one can still interact with a single component, with $U, I \in \mathcal{H}^2$:

$$U \otimes I$$

- U is applied to $|\psi\rangle$ while $|\varphi\rangle$ is unaltered.

Quantum Mechanics

Now that the machinery of quantum mechanics has been formalized…

…time to break it.

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Definition (Partial Density Operators)

Given a Hilbert space $\mathcal{H}$, a partial density operator $\rho \in \mathcal{D}^{-}(\mathcal{H})$ is a density operator such that $\mathsf{tr}(\rho) \leq 1$, where $\mathcal{D}^{-}(\mathcal{H})$ is the set of all partial density operators in $\mathcal{H}$.

Partial density operators describe **non-normalized states**.

- What are they good for? This will be clearer when talking about quantum programs semantics.

Quantum **while**-programs

- An imperative, deterministic quantum programming language.
- First introduced in:

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Association for Computing Machinery, Sections 3,4,5.

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    Association for Computing Machinery, Sections 3,4,5.
- (1)*Quantum data,* (2)*quantum control*
  1. Manipulate quantum variables.
  2. The state of a program is a quantum state: computational paths support *quantum branching*.

Quantum **while**-programs

- **Variables** represent quantum systems.
- Can be restricted to two types (w.l.o.g.):
  - **Boolean.**
  - **Integer.**

  Their domains are defined in terms of Hilbert spaces:

$$\mathcal{H}_{\text{Boolean}} = \mathcal{H}^2; \quad \mathcal{H}_{\text{Integer}} = \mathcal{H}^\infty$$

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$$\mathcal{H}_{\text{Boolean}} = \mathcal{H}^2; \quad \mathcal{H}_{\text{Integer}} = \mathcal{H}^\infty$$

- **Quantum registers.** Finite sequences of variables: $\bar{q} = q_1, \ldots, q_n$. With domain:

$$\mathcal{H}_{\bar{q}} = \bigotimes_{i=1}^{n} \mathcal{H}_{q_i}$$

Syntax

The syntax of quantum **while**-programs is the following:

$$
\begin{aligned}
S ::= {} & \texttt{skip} \,\|\, q := |0\rangle \;\|\, \bar{q} := U\bar{q} \,\|\, S_1; S_2 \;\| \\
::= {} & \texttt{measure}\, M[\bar{q}] : \bar{S} \,\|\, \texttt{while}\, M[\bar{q}] = 1 \,\texttt{do}\, S
\end{aligned}
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Some important remarks:

- $q := |0\rangle$ represents initialization. Assignment is not possible due to the No-cloning theorem.

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Some important remarks:

- $q := |0\rangle$ represents initialization. Assignment is not possible due to the No-cloning theorem.
- `measure` and `while` work by means of measurement. Coherently with Postulate 3 the state of $\bar{q}$ is altered.

Operational semantics

- A **state** of a program S is represented by a partial density operator ρ in the Hilbert space:

$$\mathcal{H}_S = \bigotimes_{q \in \mathcal{V}(S)} \mathcal{H}_q$$

where $\mathcal{V}(S)$ is the set of variables of S.

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- A **configuration** of a program is a pair $\langle S, \rho \rangle$, where:
  - S is the program still to be executed;
  - ρ is the current state.

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- A **configuration** of a program is a pair $\langle S, \rho \rangle$, where:
  - S is the program still to be executed;
  - ρ is the current state.
- The operational semantics is defined by means of the *transition relation*:

$$\langle S, \rho \rangle \to \langle S', \rho' \rangle$$

  $\langle \downarrow, \rho \rangle$ denotes a terminating configuration.

Transition rules: the Loop case.

- $\texttt{while}\, M[\bar{q}] = 1\, \texttt{do}\, S$

$$\frac{}{\langle \texttt{while}, \rho\rangle \to \langle \downarrow, M_0 \rho M_0^\dagger\rangle} \quad (\textit{Loop 0})$$

$$\frac{}{\langle \texttt{while}, \rho\rangle \to \langle S; \texttt{while}, M_1 \rho M_1^\dagger\rangle} \quad (\textit{Loop 1})$$

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Why $M_0 \rho M_0^\dagger$? Postulate 3 says that the post-measurement state should be:

$$\rho_0' = \frac{M_0 \rho M_0^\dagger}{p(0)}, \text{ with } p(0) = \mathsf{tr}(M_0 \rho M_0 \dagger),$$

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Idea. Use *partial density operators* instead.

- Introduce *non-determinism* to avoid probabilistic transition rules:

$$\frac{}{\langle \texttt{while}, \rho\rangle \xrightarrow{p_m} \langle S_m, \rho_m\rangle}$$

Denotational semantics

- Given a program S, its denotational semantics is defined by means of the function $[\![S]\!] : \mathcal{D}^-(\mathcal{H}_S) \to \mathcal{D}^-(\mathcal{H}_S)$:

$$[\![S]\!](\rho) = \sum \{|\rho' : \langle S, \rho\rangle \xrightarrow{*} \langle \downarrow, \rho'\rangle|\}$$

  where $\xrightarrow{*}$ stands for n steps for any n.
- The sum of all terminating states.
- **Observation:** what effect does it have on the trace?

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$$\text{Probability of diverging} = \mathsf{tr}(\rho) - \mathsf{tr}([\![S]\!](\rho))$$

Quantum Deductive Verification

- This section refers to the work done in:

M. Ying (2011), Floyd-Hoare Logic for Quantum Programs
Association for Computing Machinery, Sections 6-9.

E. D'Hondt, P. Panangaden (2006), Quantum Weakest Preconditions
Mathematical Structures in Computer Science, 16(3), 429-451.

Quantum Deductive Verification

Definition (Quantum Predicate)

A quantum predicate M is a *projective measurement*. As such, it has a *spectral decomposition* of the form:

$$M = \sum_m m P_m$$

where the possible outcomes m are given by the *eigenvalues*. In the case of quantum predicates, eigenvalues are bounded by 1.

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- **Idea:** define satisfiability as the *expectation value* $\mathsf{tr}(M\rho)$.
- That is, the probability of ρ satisfying M.
- Natural translation of the satisfies relation:

$$\rho \models_r M \iff \mathsf{tr}(M\rho) \geq r,$$

where $r \in [0, 1]$.

Quantum Floyd-Hoare Logic (QHL)

To do:

1. Define the notion of quantum correctness;
2. Define the set of axioms and inference rules for quantum **while**-programs.

Definition (Quantum Hoare Triple)

Given a quantum program S, a state ρ and two quantum predicates P, Q, the quantum Hoare triple $\{P\}S\{Q\}$ denotes that:

$$\mathsf{tr}(P\rho) \leq \mathsf{tr}(Q[\![S]\!](\rho))$$

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- This definition is perhaps clearer when translated into the *quantum satisfies* relation:

$$\rho \models_r P \Longrightarrow [\![S]\!](\rho) \models_r Q$$

$\forall r \in [0, 1]$.

Quantum Floyd-Hoare Logic (QHL)

Definition (Quantum Total Correctness)

The quantum Hoare triple $\{P\}S\{Q\}$ is valid in terms of total correctness (formally $\models tot\{P\}S\{Q\}$) if $\forall \rho \in \mathcal{D}^{-}(\mathcal{H}_S)$

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The quantum Hoare triple $\{P\}S\{Q\}$ is valid in terms of partial correctness (formally $\models par\{P\}S\{Q\}$) if $\forall \rho \in \mathcal{D}^-(\mathcal{H}_S)$

$$\mathsf{tr}(P\rho) \leq \mathsf{tr}(Q[\![S]\!](\rho)) + \big(\mathsf{tr}(\rho) - \mathsf{tr}([\![S]\!](\rho)\big)$$

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$$\mathsf{tr}(P\rho) \leq \mathsf{tr}(Q[\![S]\!](\rho)) + \big(\mathsf{tr}(\rho) - \mathsf{tr}([\![S]\!](\rho)\big)$$

Remember that $\big(\mathsf{tr}(\rho) - \mathsf{tr}([\![S]\!](\rho)\big)$ represents the probability that S will not terminate.

Quantum Partial Correctness Proof System

- It is now possible to define an *axiomatic base* for quantum **while**-programs.

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Quantum Partial Correctness Proof System

- It is now possible to define an *axiomatic base* for quantum **while**-programs.
- As in the classical case, two proof systems can be defined respectively for partial and total correctness.
- **The Unitary transformation case:**

$$(Unitary) \quad \frac{}{\{U^\dagger P U\}\ \bar{q} := U\bar{q}\ \{P\}}$$

- Notice how, for any $\bar{q}$:

$$\mathsf{tr}(U^\dagger P U \bar{q}) = \mathsf{tr}(P U \bar{q} U^\dagger) = \mathsf{tr}(P[\![\bar{q} := U\bar{q}]\!])$$

Quantum Weakest Precondition

- What does one mean by **weakest**?
- A way to compare the **strength** of quantum predicates is required.

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Definition (Löwner partial order)

Given two predicates P, Q, the writing $P \sqsubseteq Q$ is used to denote that for any $\rho \in \mathcal{D}^-(\mathcal{H})$:

$$\mathsf{tr}(P\rho) \leq \mathsf{tr}(Q\rho)$$

- Translating into the r-satisfies relation:

$$\rho \models_r P \Longrightarrow \rho \models_r Q$$

Quantum Weakest Precondition

Definition (Quantum Weakest Precondition)

The weakest precondition of a predicate Q with respect to a program S is a quantum predicate $\mathsf{qwp}.S(Q) \in \mathcal{P}(\mathcal{H}_S)$ such that:

- $\models_{tot} \{\mathsf{qwp}.S(Q)\}S\{Q\}$
- For any $P \in \mathcal{P}(\mathcal{H}_S)$, $\models_{tot} \{P\}S\{Q\} \Rightarrow P \sqsubseteq \mathsf{qwp}.S(Q)$

- The function can be defined over the statements of quantum **while**-programs.
- Easily verify a quantum Hoare triple $\{P\}S\{Q\}$:

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- The function can be defined over the statements of quantum **while**-programs.
- Easily verify a quantum Hoare triple $\{P\}S\{Q\}$:
  1. Compute $\mathsf{qwp}.S(Q)$;
  2. Check if $P \sqsubseteq \mathsf{qwp}.S(Q)$.

Model Checking on Quantum CTL

- On the lines of the work done in:

  P. Baltazar, R. Chadha, P. Mateus (2008), Quantum Computation Tree Logic - Model Checking and Complete Calculus
  International Journal of Quantum Information

- Quantum CTL (QCTL) is a temporal logic built on dEQPL (*decidable fragment of the Exogenous Quantum Propositional Logic*).
- Herein, a restricted version of dEQPL is presented:
  - Finite Hilbert spaces;
  - Closed formulae only.
- QCTL is obtained by simply enriching dEQPL with temporal modalities.

dEQPL

- **Idea:** Replace propositional letters with qubits.

dEQPL

- **Idea:** Replace propositional letters with qubits.
- Set of n qubits $\mathsf{qB} = \{q_1, \ldots, q_n\}$.
- A valuation over qB is a state $|\psi\rangle$ in the Hilbert space:

$$\bigotimes_{i=1}^{n} \mathcal{H}^2 = \mathcal{H}^{2^n} = \mathcal{H}_{\mathsf{qB}}$$

generated by the computational basis: $\{|v_i\rangle\}_{i=1}^{2^n}$.

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- Each vector of the basis represents a possible valuation over qB:

$$|v_1\rangle = |0^n\rangle \,; |v_2\rangle = |0^{(n-1)}1\rangle \,; \dots$$

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- Each vector of the basis represents a possible valuation over qB:

$$|v_1\rangle = |0^n\rangle\,; |v_2\rangle = |0^{(n-1)}1\rangle\,; \ldots$$

- **Important:** it is possible to describe superpositions of valuations:

$$|\psi\rangle = \frac{1}{\sqrt{2}}\,|0^n\rangle + \frac{1}{\sqrt{2}}\,|1^n\rangle$$

dEQPL - Syntax

Syntax. Three syntactic categories:

- **Classical formulae** (where $q \in \mathsf{qB}$):

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- **Quantum formulae:**

$$\gamma \coloneqq t \leq t \parallel \perp\!\!\!\perp \parallel \gamma \sqsupset \gamma$$

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Semantics. Given a set of n qubits qB dEQPL formulae are interpreted over a state $|\psi\rangle \in \mathcal{H}_{\mathsf{qB}}$.

- **Terms denotations:**

$$\begin{aligned}
[\![\mathsf{Re}(|\top\rangle_A)]\!]_{|\psi\rangle} &= \mathsf{Re}(\langle v_A|\psi\rangle) && \text{(Real part)}\\
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- $\mathsf{Re}(\langle v_A|\psi\rangle)$ and $\mathsf{Im}(\langle v_A|\psi\rangle)$ denote the real and imaginary of the logical amplitude of $\langle v_A|\psi\rangle$.
- Any $A \subseteq \mathsf{qB}$ is mapped onto a unique valutation $|v_A\rangle$.
  - *e.g.* $A = \{q_1, q_3\}$ is mapped onto $|v_A\rangle = |1010\ldots0\rangle$.
- From $|\psi\rangle = \frac{1}{\sqrt{2}}|0^n\rangle + \frac{1}{\sqrt{2}}|1^n\rangle$ one gets $\langle 0^n|\psi\rangle = \frac{1}{\sqrt{2}}$.

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2. Calculate the probability of $|\psi\rangle$ *collapsing* into any of the valuations in the extent:

$$\mu_{|\psi\rangle}(\mathcal{E}(\alpha)) = \sum_{v \in \mathcal{E}(\alpha)} \| \langle v|\psi\rangle \|^2$$

dEQPL - Semantics

Quantum formulae. Semantics are defined by means of the $\Vdash_d$ relation.

$$
\begin{aligned}
&|\psi\rangle \Vdash_d (t_1 \leq t_2) \iff [\![t_1]\!]_{|\psi\rangle} \leq [\![t_2]\!]_{|\psi\rangle} \\
&|\psi\rangle \nVdash_d \perp\!\!\!\perp \\
&|\psi\rangle \Vdash_d (\gamma_1 \sqsupset \gamma_2) \iff (|\psi\rangle \nVdash_d \gamma_1) \vee (|\psi\rangle \Vdash_d \gamma_2)
\end{aligned}
$$

where $\Vdash_d$ is used to denote dEQPL satisfaction.

dEQPL - Model Checking

Given a set of qubits $\mathsf{qB} = \{q_1, \ldots, q_n\}$, a dEQPL formula γ and a state $|\psi\rangle \in \mathcal{H}_{\mathsf{qB}}$, check whether:

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- Terms of the form $\int \alpha$ are responsible for the exponential factor.
- Computing $\mathcal{E}(\alpha)$ requires an iteration over all 2^n valuations:

$$\mathcal{E}(\alpha) = \{v \in 2^{\mathsf{qB}} : v \Vdash_c \alpha\}$$

QCTL

Syntax. Enrich dEQPL with *temporal modalities*:

$$\theta := \gamma \parallel \theta \sqsupset \theta \parallel \mathsf{EX}\theta \parallel \mathsf{AF}\theta \parallel \mathsf{E}[\theta \mathsf{U} \theta]$$

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Definition (Quantum Kripke Structure)

Given a finite set of qubits qB, a *quantum Kripke structure* is a pair $\mathcal{T} = (S, R)$ where:

- $S \subset \mathcal{H}_{\mathsf{qB}}$ is the set of *states*. Each state $|\psi\rangle$ is a unit vector in $\mathcal{H}_{\mathsf{qB}}$.
- $R \subseteq S \times S$ is a *transition relation* such that for all $|\psi\rangle \in S$, there exists $|\psi\rangle' \in S$ such that $(|\psi\rangle, |\psi\rangle') \in R$.

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- No labelling function needed!

Semantics.

- Given a quantum Kripke structure $\mathcal{T} = (S, R)$, a state $|\psi\rangle$ and a QCTL formulae θ the semantics of QCTL are defined by the relation $\Vdash_{\mathsf{Q}}$:

$$
\begin{aligned}
\mathcal{T}, |\psi_i\rangle \Vdash_{\mathsf{Q}} \gamma &\iff |\psi_i\rangle \Vdash_d \gamma \\
\mathcal{T}, |\psi_i\rangle \Vdash_{\mathsf{Q}} \theta_1 \sqsupset \theta_2 &\iff \mathcal{T}, |\psi_i\rangle \nVdash_{\mathsf{Q}} \theta_1 \vee \mathcal{T}, |\psi_i\rangle \Vdash_{\mathsf{Q}} \theta_2 \\
\mathcal{T}, |\psi_i\rangle \Vdash_{\mathsf{Q}} \mathsf{EX}\theta &\iff \exists\, |\psi'\rangle \in S,\ (|\psi_i\rangle, |\psi'\rangle) \in R \text{ and } \mathcal{T}, |\psi'\rangle \Vdash_{\mathsf{Q}} \theta \\
\mathcal{T}, |\psi_i\rangle \Vdash_{\mathsf{Q}} \mathsf{AF}\theta &\iff \forall\pi = |\psi_i\rangle\, |\psi_{i+1}\rangle\, |\psi_{i+2}\rangle \ldots, \exists\, j \geq i, \\
& \qquad (\mathcal{T}, |\psi_j\rangle \Vdash_{\mathsf{Q}} \theta) \\
\mathcal{T}, |\psi_i\rangle \Vdash_{\mathsf{Q}} \mathsf{E}[\theta_1 \,\mathsf{U}\, \theta_2] &\iff \exists\pi = |\psi_i\rangle\, |\psi_{i+1}\rangle\, |\psi_{i+2}\rangle \ldots, \exists\, j \geq i \\
& \qquad (\mathcal{T}, |\psi_j\rangle \Vdash_{\mathsf{Q}} \theta_2, \forall\, k,\ i \leq k < j,\ \mathcal{T}, |\psi_k\rangle \Vdash_{\mathsf{Q}} \theta_1)
\end{aligned}
$$

Model Checking.

- Given a QCTL formula θ and a quantum Kripke structure $\mathcal{T} = (S, R)$ compute:

$$Sat_{\mathcal{T}}(\theta) \coloneqq \{|\psi\rangle \in S : \mathcal{T}, |\psi\rangle \Vdash_{\mathsf{Q}} \theta\}$$

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- **Idea:** Drawing inspiration from *symbolic Model Checking*, QCTL formulae are characterized by sets of states.
- Temporal operators? Fixpoint characterization:
  - Consider the complete lattice $(\wp(S), \subseteq)$.
  - Bottom and top elements are respectively represented by $\perp\!\!\!\perp$ and $\top\!\!\!\top$.
  - Describe temporal modalities by suiting monotonic predicate transformers.

QCTL

- S is finite, the same results from fixpoint theory can be exploited.
- QCTL formulae are characterized as follows:

$$
\begin{aligned}
Sat_{\mathcal{T}}(\gamma) &= \{|\psi\rangle \in S : \psi \Vdash_{\mathsf{Q}} \gamma\} \\
Sat_{\mathcal{T}}(\theta_1 \sqsupset \theta_2) &= \big(S \setminus Sat_{\mathcal{T}}(\theta_1)\big) \cup Sat_{\mathcal{T}}(\theta_2) \\
Sat_{\mathcal{T}}(\mathsf{EX}\,\theta) &= \{|\psi\rangle \in S : \exists\, |\psi'\rangle \big((|\psi\rangle, |\psi'\rangle) \in R \wedge |\psi'\rangle \in Sat_{\mathcal{T}}(\theta)\big)\} \\
Sat_{\mathcal{T}}(\mathsf{AF}\,\theta) &= \mu Z.(\theta \vee \mathsf{AX} Z) \\
Sat_{\mathcal{T}}(\mathsf{E}[\theta_1 \mathsf{U} \theta_2]) &= \mu Z.(\theta_2 \vee (\theta_1 \wedge \mathsf{EX} Z))
\end{aligned}
$$

where $\mu Z.\tau(Z)$ represents the least fixpoint of τ.

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$$\mathcal{O}(|\theta|^2 \cdot (|S| + |R|) \cdot 2^n)$$

where $n = |\mathsf{qB}|$.

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  where $n = |\mathsf{qB}|$.
- **Last remark.** Time is polynomial w.r.t. the dimension of the model...
  - **... but** simulating a quantum model with classical machinery requires exponential space.
  - More space is required to encode all possible 2^{2^n} superpositions of states.

Conclusion

Other techniques:

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Quantum Hoare Logic is being constantly improved and extended:

- Theorem provers have been developed (*e.g.* QHLProver);
- Current problem: consider the Hoare triple $\{P\}S\{Q\}$
  - The dimensions of both P and Q grow exponentially w.r.t. the number of qubits in S.
  - Is there a way to *reason locally* on the satisfaction of quantum predicates?

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> *"Hilbert space is a big place."*
>
> – Carlton Caves

The End

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Unless... there is more time!

Case Study: Deutsch's Algorithm

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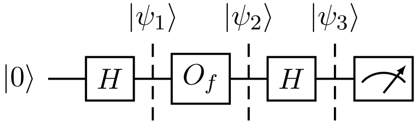
- The algorithm can be described by the following circuit:



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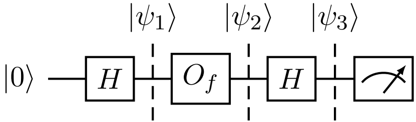

- H denotes a Hadamard gate, briefly it acts on $\mathcal{H}^2$ as follows:

$$H|0\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} = |+\rangle \qquad H|1\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}} = |-\rangle$$

$$H\frac{|0\rangle + |1\rangle}{\sqrt{2}} = |0\rangle \qquad H\frac{|0\rangle - |1\rangle}{\sqrt{2}} = |1\rangle$$

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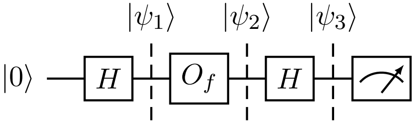

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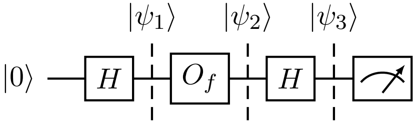

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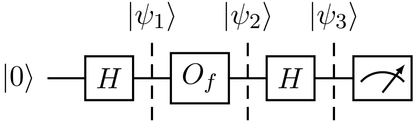

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$$|\psi_1\rangle = H\,|0\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

$$|\psi_2\rangle = O_f\,|\psi_1\rangle = \frac{(-1)^{f(0)}\,|0\rangle + (-1)^{f(1)}\,|1\rangle}{\sqrt{2}}$$

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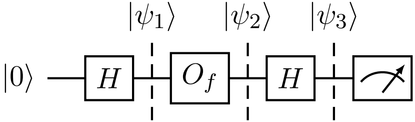

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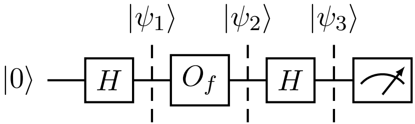

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- $|\psi_2\rangle$ can be represented as follows:

$$|\psi_2\rangle = \begin{cases} \pm\frac{|0\rangle+|1\rangle}{\sqrt{2}}, \text{ if } f(0) = f(1) \\ \pm\frac{|0\rangle-|1\rangle}{\sqrt{2}}, \text{ if } f(0) \neq f(1) \end{cases}$$

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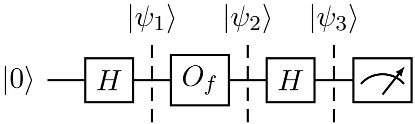

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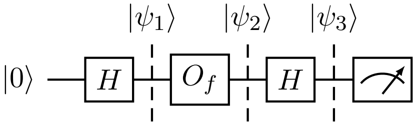

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- Now the measurement $M = \{|0\rangle\langle 0|, |1\rangle\langle 1|\}$ will give rise to:
  - $|0\rangle$ with probability 1 if $f(0) = f(1)$.
  - $|1\rangle$ with probability 1 if $f(0) \neq f(1)$.

Quantum Weakest Precondition

- First, describe Deutsch's algorithm into the quantum **while**-language:

$$\begin{aligned} Deutsch \equiv [&q := 0; \\ &q := Hq; \\ &q := O_f q; \\ &q := Hq; \\ &\texttt{measure}\, M[q] : \texttt{skip}; \texttt{skip}] \end{aligned}$$

- The Postcondition:

$$Post = (1 - f(0) \oplus f(1)) \left|0\right\rangle \left\langle 0\right| + f(0) \oplus f(1) \left|1\right\rangle \left\langle 1\right|$$

- *Post* states a property that is wished to be proven valid.

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- *Post* states a property that is wished to be proven valid.
- Now, running backwards through the statements of *Deutsch*:

$$\mathsf{qwp}.[\texttt{measure}\, M[q] : \texttt{skip}; \texttt{skip}](Post) = M_0^\dagger Post M_0 + M_1^\dagger Post M_1$$
$$= |0\rangle \langle 0| c |0\rangle \langle 0|0\rangle \langle 0| + |1\rangle \langle 1| b |1\rangle \langle 1|1\rangle |1\rangle$$
$$= c |0\rangle \langle 0| + b |1\rangle \langle 1| = Post$$

where $c = 1 - f(0) \oplus f(1)$ and $b = f(0) \oplus f(1)$.

Case Study: Deutsch's Algorithm

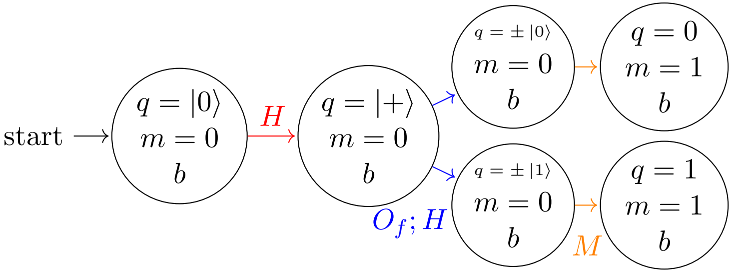
- Eventually one obtains I as the weakest precondition of the first statement.
- Applying the function on the composition of statements:

$$\mathsf{qwp}.[Deutsch](Post) = I$$

- The Hoare triple $\{I\}Deutsch\{Post\}$ is totally correct.
- Any other precondition P is such that $P \sqsubseteq I$.
- **Meaning:** Under any assumption, *Deutsch's algorithm is always correct.*

Case Study: Deutsch's Algorithm

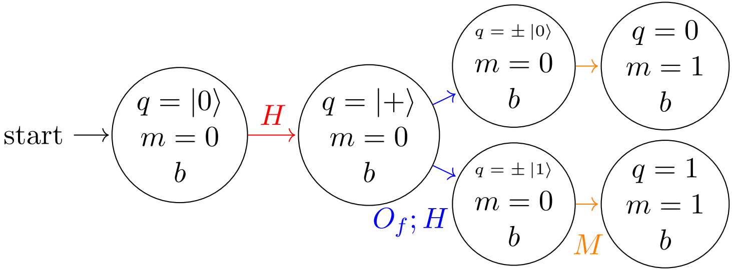
QCTL: The quantum Kripke structure:



where:

- q is the qubit
- m denotes whether q has been measured $(= 1)$ or not $(= 0)$
- $b = f(0) \oplus f(1)$.

Case Study: Deutsch's Algorithm



- Zuliani et al. describe the correctness of Deutsch's algorithm through the following QCTL formula:

$$\theta = \mathsf{A}\Big[(\boxminus(\Box m))\,\mathsf{U}\Big(\Box m \sqcap (\Box b \equiv (\textstyle\int q = 1))\Big)\Big]$$

- Shorthands:
  - $\Box x$ states $x = 1$;
  - $\boxminus$: quantum negation.
  - $\sqcap$: quantum conjunction.

The End

(for real this time)

Thank you for your attention.