An Exploration of Fourier Series

Alex Derhacobian

June 16, 2022

1 Introduction

This paper is an exploration of Fourier series. Fourier series are used extensively in many fields of mathematics and physics. For example, Fourier series and Fourier analysis more broadly lay the groundwork for modern signal processing, approximation theory, and control theory. One of the key assumptions made in Fourier analysis is that the Fourier series of f is a good representation of f on the interval $[-\pi, \pi]$. In other words, Fourier series is assumed to be a good local approximation, but global approximation is a much higher bar to meet. This paper will investigate the conditions under which the Fourier series converges pointwise and uniformally to a continuous function.

In particular, we will focus on proving two results: First, will show the n-th partial sums of the Fourier series satisfy *pointwise convergence* to f. Then, we will show the Cesáro sum of the partial sums of the Fourier series satisfy *uniform convergence* to f. These are important results in Fourier analysis. For example, we can use the pointwise convergence of these partial sums to prove that a projection error goes to zero as the number of dimensions goes to infinity. Finally, we will explore a property of Fourier series known as the *Gibbs phenomenon.* Although pointwise convergence remains one of the most challenging problems in Fourier analysis, we will pursue a rigorous exploration of this concept.

2 Definitions and Preliminaries

2.1 Inner Product Spaces

First, it is important to establish some preliminary definitions and concepts. We assume the reader is familiar with linear algebra and real analysis. Otherwise, consult [1] and [2] as a preliminary reference. One structure that is crucial to our exploration is the orthonormal set. An orthonormal set is a set of vectors that are mutually orthogonal and have unit length. If this orthonormal set spans some vector space V, then we can trivially express any vector $x \in V$ as a linear combination of this orthonormal set $\{x_1, \ldots, x_n\}$ where $x = c_1x_1 + \ldots + c_nx_n$, $x \in \mathbb{R}^n$. But, with some simple manipulation, we can express every constant c_i as the inner product (x, x_i). So, we could otherwise express x as $(x, x_1)x_1 + \ldots + (x, x_n)x_n$. This form of expressing any element of an inner product space as the linear combination of an orthonormal set will become central to understanding the Fourier series.

2.2 The Trigonometric Orthonormal Set

Now let's define a particular countable orthonormal set: the trigonometric orthonormal set. Following the same procedure above, we will express bounded functions f from a to $a + 2\pi$ as the linear combinations of functions in the trigonometric set with coefficients described by these inner products. We will denote this set of bounded functions on $[a, a + 2\pi]$ as $\{f | f \in \mathcal{R}[a, a + 2\pi\}$. Formally, we define the trigonometric orthonormal set as the set $\{\phi_n(x)\}_{n=0}^{\infty}$ for any $x \in \mathbb{R}$ where

$$\phi_0(x) = \frac{1}{\sqrt{2n}} \qquad \phi_{2n-1}(x) = \frac{\cos nx}{\sqrt{\pi}}, \qquad \phi_{2n}(x) = \frac{\sin nx}{\sqrt{\pi}}. \tag{1}$$

We will prove that this set of function $\{\phi_n(x)\}$ produces an orthonormal set. First, we will show that for any $i, j \in [0, \infty)$, $\langle \phi(x)_i \phi(x)_j \rangle = 1$ when $i = j$. and $\langle \phi(x)_i \phi(x)_j \rangle = 0$ when $i \neq j$. We will prove the first condition.

$$\begin{aligned}
\|\phi(x)_0\|^2 &= \int_0^{2\pi} \left(\frac{1}{\sqrt{2\pi}}\right)^2 dx = \frac{2\pi}{2\pi} - 0 = 1, \\
\|\phi(x)_{2n-1}\|^2 &= \int_0^{2\pi} \frac{cos^2(nx)}{\pi} dx = \frac{nx + \sin(nx)\cos(nx)}{2n\pi}\Big|_0^2 \pi = 1, \\
\|\phi(x)_{2n}\|^2 &= \int_0^{2\pi} \frac{sin^2(nx)}{\pi} dx = \frac{1}{\pi}\left(\frac{1}{2}\left(nx - \frac{1}{n}\sin(nx)\right)\right)\Big|_0^{2\pi} = 1.
\end{aligned} \tag{2}$$

Now, we will show that any two distinct vectors in this set are orthogonal. We will consider all possible inner products. We have

$$\begin{aligned}
\langle \phi_0, \phi_{2n-1} \rangle &= \int_0^{2\pi} \frac{\cos(nx)}{\sqrt{2\pi}} dx = \frac{\sin(nx)}{n\sqrt{2\pi}}\Big|_0^{2\pi} = 0, \\
(\phi_0, \phi_{2n}) &= \int_0^{2\pi} \frac{\sin(nx)}{\sqrt{2\pi}} dx = -\frac{\cos(nx)}{n\sqrt{2\pi}}\Big|_0^{2\pi} = 0.
\end{aligned} \tag{3}$$

Now, in the case where $n = m$, we have

$$\langle \phi_{2n-1}, \phi_{2m} \rangle = \int_0^{2\pi} \frac{\cos(nx)\sin(nx)}{\pi} dx = -\frac{1}{2\pi}\left\{\frac{\cos(2nx)}{2n}\right\}_0^{2\pi} = 0. \tag{4}$$

When $n \neq m$, we have

$$\langle \phi_{2n-1}, \phi_{2m} \rangle = \int_0^{2\pi} \frac{\cos(nx)\sin(mx)}{\pi} dx = \frac{-1}{2\pi}\left\{\frac{\cos[(m+n)x]}{m+n} - \frac{\cos[(m-n)x]}{m-n}\right\}_0^{2\pi} = 0. \tag{5}$$

Therefore, we have shown that the trigonometric set $\{\phi_n(x)\}_{n=0}^{\infty}$ is an orthonormal set.

2.3 The Fourier Series

Using these inner products, we can express any vector in an inner product space over the trigonometric set as a linear combination of the trigonometric set. Now, we will explore the behavior of this method when we push the n to infinity. To explore this domain, we will use the Fourier series. Let us define this more rigorously.

Definition 2.1 (The Fourier Series)**.** We let $X = \{x_1, x_2, \dots\}$ be a countable orthonormal set in an inner product space V and let $x \in V$. The Fourier series of x in V relative to the set X is

$$\sum_{n=1}^{\infty} \langle x, x_n \rangle x_n. \tag{6}$$

Now, we will consider the Fourier series for a particular type of orthonormal set. We will extend this general definition to the Fourier series of a function f in $R[a, a + 2\pi]$ relative to the trigonometric set. This Fourier series is

$$\sum_{n=0}^{\infty} \langle f, \phi_n \rangle \phi_n(x). \tag{7}$$

We can expand this definition of the Fourier series as

$$\sum_{n=0}^{\infty} (f, \phi_n)\phi_n(x) = \frac{1}{\sqrt{2\pi}} \int_a^{a+2\pi} \frac{f(t)}{\sqrt{2\pi}} dt + \sum_{n=1}^{\infty} \left(\frac{\cos nx}{\sqrt{\pi}} \int_a^{a+2\pi} f(t)\frac{\cos nt}{\sqrt{\pi}} dt + \frac{\sin nx}{\sqrt{\pi}} \int_a^{a+2\pi} f(t)\frac{\sin nt}{\sqrt{\pi}} dt\right). \tag{8}$$

We can simplify this to be

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx), \tag{9}$$

with

$$a_n = \frac{1}{\pi} \int_a^{a+2\pi} f(t)\cos(nt) dt, \qquad b_n = \frac{1}{\pi} \int_a^{a+2\pi} f(t)\sin(nt) dt \qquad \text{for } n = 1, 2, \dots,$$

where a_n and b_n are known as the n-th Fourier coefficients. Note that this is an example of a trigonometric polynomial. Now that we have established the preliminary definitions of the Fourier series, we will shift our discussion towards the proving the convergence of the Fourier Series.

3 The Dirichlet Kernel

An important preliminary to the proof of the pointwise convergence of the partial sums of the Fourier series is the Dirichlet Kernel, denoted as $D_n(x)$. The Dirichlet kernel helps us prove pointwise convergence since the convolution of $D_n(x)$ with any function f of period 2π is the n-th degree Fourier series approximation to f.

Definition 3.1 (The Dirichlet kernel)**.** The Dirichlet kernel $D_n(t)$ is defined as

$$D_0(t) = \frac{1}{2}, \qquad D_n(t) = \frac{1}{2} + \sum_{k=1}^{n} \cos kt \qquad \text{for } n = 1, 2, \dots \tag{10}$$

There are some essential properties of the Dirichlet kernel we will prove that will be useful in showing pointwise convergence.

Lemma 3.1. $\frac{1}{\pi}\int_{-\pi}^{\pi} D_n(x)dx = 1$.

Proof.

$$\begin{aligned}
\frac{1}{\pi}\int_{-\pi}^{\pi} D_n(x)dx &= \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{1}{2} + \sum_{k=1}^{n} \cos kx \, \mathrm{d}x \qquad (11)\\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} \left(\frac{1}{2} + \cos(x) + \cos(2x) + \ldots + \cos(nx)\right) \mathrm{d}x\\
&= \frac{1}{\pi}\left[\int_{-\pi}^{\pi} \frac{1}{2}\,\mathrm{d}x + \int_{-\pi}^{\pi} \cos(x)\,\mathrm{d}x + \int_{-\pi}^{\pi} \cos(2x)\,\mathrm{d}x + \ldots + \int_{-\pi}^{\pi} \cos(nx)\,\mathrm{d}x\right]\\
&= \frac{1}{\pi}\left[\pi + \sum_{k=0}^{n} \big(2\sin(k\pi)\big)\right]\\
&= \frac{1}{\pi}\left[\pi + \sum_{k=0}^{n} (0)\right] = 1.
\end{aligned}$$

□

Another important property of the Dirichlet Kernel is that it takes the form of a trigonometric identity. This result will deem useful in our proof of pointwise convergence.

Lemma 3.2.

$$D_n(t) = \frac{\sin[(n+\frac{1}{2})t]}{2\sin(t/2)} \tag{12}$$

Proof. First, we note the trigonometric identity $\sin(u+v) - \sin(u-v) = 2\sin(v)\cos(u)$. We take $u = kt$ and $v = t/2$. From this identity, it follows that

$$\sin\left[\left(k+\frac{1}{2}\right)t\right] - \sin\left[\left(k-1+\frac{1}{2}\right)t\right] = 2\sin\left(\frac{t}{2}\right)\cos kt. \tag{13}$$

Now, we compute the telescoping sum. We have

$$\begin{aligned}
\sin\left[\left(n+\frac{1}{2}\right)t\right] - \sin\left(\frac{t}{2}\right) &= \sum_{k=1}^{n}\left\{\sin\left[\left(k+\frac{1}{2}\right)t\right] - \sin\left[\left(k-1+\frac{1}{2}\right)t\right]\right\} \qquad (14)\\
&= \sum_{k=1}^{n} 2\sin\left(\frac{t}{2}\right)\cos kt
\end{aligned}$$

We can simplify this to

$$\begin{aligned}
\sin\left[\left(n+\frac{1}{2}\right)t\right]-\sin\left(\frac{t}{2}\right) &= \sum_{k=1}^{n} 2\sin\left(\frac{t}{2}\right)\cos kt, \\
\sin\left[\left(n+\frac{1}{2}\right)t\right] &= \sin\left(\frac{t}{2}\right)+2\sin\left(\frac{t}{2}\right)\sum_{k=1}^{n}\cos kt, \\
\frac{\sin\left[\left(n+\frac{1}{2}\right)t\right]}{2\sin\left(\frac{t}{2}\right)} &= \frac{1}{2}+\sum_{k=1}^{n}\cos kt.
\end{aligned} \tag{15}$$

$\square$

Since the right hand side matches equation (10), our proof is complete.

Now, we will use the Dirichlet Kernel to define the n-th partial sum of the Fourier series. We define $S_n(x)$ as the n-th partial sum of the Fourier series of f at x.

4 The Partial Sum of the Fourier Series

Understanding $S_n(x)$ is essential, since eventually we will take n to infinity to prove the pointwise convergence of the Fourier series. Additionally, the relevance of S_n motivates our discussion of the Dirichlet Kernel, since the convolution of the Dirichlet kernel with any function f is $S_n(x)$ of f. Moreover, we will prove pointwise convergence by showing the limit of $S_n(x)$ exists for all x. Towards this end, we will first derive an integral representation of $S_n(x)$.

Lemma 4.1. $S_n(X) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x-y)D_n(y)\,\mathrm{d}y$, where $S_n(x)$ is the n-th partial sum of the Fourier series of f at x.

Proof. From the definition of the Fourier series in Definition 2.1, it follows that

$$\begin{aligned}
S_n(x) &= \frac{1}{2\pi}\int_a^{a+2\pi} f(t)dt + \frac{1}{\pi}\sum_{k=1}^{n}\left[\cos kx\int_a^{a+2\pi} f(t)\cos kt\,\mathrm{d}t + \sin kx\int_a^{a+2\pi} f(t)\sin nt\,\mathrm{d}t\right] \\
&= \frac{1}{2\pi}\int_a^{a+2\pi} f(t)dt + \frac{1}{\pi}\sum_{k=1}^{n}\int_a^{a+2\pi} f(t)\left[\cos kx\cos kt\,\mathrm{d}t + \sin kx\sin kt\right]\mathrm{d}t \\
&= \frac{1}{2\pi}\int_a^{a+2\pi} f(t)dt + \frac{1}{\pi}\sum_{k=1}^{n}\int_a^{a+2\pi} f(t)\cos[k(x-t)]\,\mathrm{d}t.
\end{aligned} \tag{16}$$

Now, suppose that $f(a) = f(a+2\pi)$. Then, we are able to extend f to a periodic function on $\mathbb{R}$ of period 2π. Note that if $f(a) \neq f(a+2\pi)$, we can redefine f at $a+2\pi$ such that $f(a) = f(a+2\pi)$. It is important to realize that this operation would not affect the Fourier series of f. We perform the change of variable $u = x-t$. Then, our formula becomes

$$\begin{aligned}
S_n(x) &= \frac{1}{2\pi}\int_{x-(a+2\pi)}^{x-a} f(x-u)du + \frac{1}{\pi}\sum_{k=1}^{n}\int_{x-(a+2\pi)}^{x-a} f(x-u)\cos ku\,\mathrm{d}u \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)dt + \frac{1}{\pi}\sum_{k=1}^{n}\int_{-\pi}^{\pi} f(x-t)\cos kt\,\mathrm{d}t \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x-t)\left(\frac{1}{2}+\sum_{k=0}^{n}\cos kt\right)\mathrm{d}t.
\end{aligned} \tag{17}$$

But since we defined $D_n(x)$ as in Definition 3.1, we can express $S_n(x)$ as

$$S_n(t) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x-t)D_n(t)dt$$

$\square$

This result demonstrates an important relation between the Dirichlet kernel and the n-th partial sum of the Fourier series. It shows that the convolution of any function f of period 2π with the Dirichlet Kernel of degree n is the n-th partial sum of the Fourier series.

Now that we have a integral representation of $S_n(x)$, there is one more result that is crucial to the proof of pointwise convergence. In the general case, if $\{x_1, x_2, \dots\}$ was an orthonormal set in a vector space V, then $\lim_{n\to\infty}(x, x_n) = 0$. This follows from Bessel's Inequality (see Theorem 75.6 in [1] for a complete proof of Bessel's Inequality). But extending this to $V = \mathcal{R}[a, a + 2\pi]$ results in the Riemann-Lebesgue Lemma. In this context, this implies that the Fourier coefficients of f tend to 0 as n grows. Otherwise, the Fourier series would diverge. The Riemman-Lebesgue Lemma shows that this condition is satisfied.

Lemma 4.2 (The Riemann-Lebesgue Lemma). : If $f : [-\pi, \pi] \to \mathbb{R}$ is continuous and 2π-periodic, then

$$\lim_{n\to\infty} \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)\cos(nt)\,\mathrm{d}t = \lim_{n\to\infty} \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)\sin(nt)\,\mathrm{d}t = 0. \tag{18}$$

Note that this results hints at the fact that as n approaches infinity, then the contributions of each term of the Fourier series diminishes. For the complete proof of this lemma, please see Theorem 75.8 in [1] and Theorem 1.4 in [3]. Intuitively, this should suggest that some form of convergence is worth exploration.

5 Pointwise Convergence of $S_n(x)$

Now that we have established the necessary preliminaries, we will now leverage the properties of the Dirichlet kernel and its relation to $S_n(x)$ to prove that $S_n(x)$ converge pointwise to $f(x)$ as n tends to infinity for a function f in $C^1[-\pi, \pi]$, where $C^1[-\pi, \pi]$ is the set of differentiable functions on $[-\pi, \pi]$. This result is important because it means that we can show that the Fourier series converges to $f(x)$ globally. Demonstrating pointwise convergence is one method by which to show this, although pointwise convergence is weaker than uniform convergence. Nonetheless, this result is an important problem in classical Fourier analysis.

Theorem 5.1. If f is a periodic function which is continuously differentiable on $[-\pi, \pi]$, then the partial sum $S_n(x)$ converges to $f(x)$ pointwise as n grows to ∞, that is, $\lim_{n\to\infty} S_n(x) = f(x)$ for all $x \in [-\pi, \pi]$

Proof. In Theorem 2, we defined S_n as

$$S_n(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x - t) D_n(t)\,\mathrm{d}t. \tag{19}$$

Therefore, to prove that the partial $S_n(x)$ converges to $f(x)$ pointwise as n grows to ∞, we will show that

$$\lim_{n\to\infty} \frac{1}{\pi} \int_{-\pi}^{\pi} f(x - t) D_n(t)\,\mathrm{d}t = f(x). \tag{20}$$

By Lemma 3.1, we know that

$$f(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) D_n(t) dt. \tag{21}$$

Therefore, an equivalent proof to the proof of Theorem 5.1 is to show that

$$\lim_{n\to\infty} \frac{1}{\pi} \int_{-\pi}^{\pi} [f(x - y) - f(x)] D_n(t)\,\mathrm{d}t = 0. \tag{22}$$

Let $\epsilon > 0$. Since

$$\lim_{t\to\infty} \frac{2\sin(t/2)}{t} = 1, \tag{23}$$

then there must exists constants k and δ_1 where $0 < \delta_1 < \pi$ such that

$$\left|\frac{t}{2\sin(t/2)}\right| < k \qquad \text{for all } 0 < |t| < \delta_1 \tag{24}$$

Then, for any positive integer n, if $0 < |t| < \delta_1$, from theorem 3, we have

$$\left|tD_n(t)\right| = \left|\frac{t}{2\sin(t/2)} \sin\left[\left(n + \frac{1}{2}\right)t\right]\right| < k. \tag{25}$$

If $t = 0$, $tD_n(t) = 0$ trivially. Therefore,

$$\left|tD_n(t)\right| < k \qquad \text{for all } |t| \le \delta_1 \text{ and } n \in \mathbb{P}. \tag{26}$$

Since we define $f(x)$ in terms of D_n, then by the Lipschitz condition (see Definition 76.5 in [1]), there exists $\delta_2 > 0$ and $M > 0$ such that

$$|f(x) - f(y)| \le M|x - y| \qquad \text{for } |x - y| \le \delta_2. \tag{27}$$

Now, we let $\delta = \min\{\delta_1, \delta_2, \epsilon/(4Mk)\}$. Then, for any positive integer n,

$$\begin{aligned}\left|\int_{-\delta}^{\delta}[f(x-t) - f(x)]D_n(t)dt\right| &\le \int_{-\delta}^{\delta}|f(x-t) - f(x)||D_n(t)|dt \\ &\le \int_{-\delta}^{\delta} M|t||D_n(t)|\,\mathrm{d}t \\ &\le \int_{-\delta}^{\delta} Mk\,\mathrm{d}t = 2Mk\delta \le \frac{\epsilon}{2}.\end{aligned} \tag{28}$$

Now that we have established a bound on Eq. (28), we will complete our proof of convergence using I_n define as

$$I_n = \int_{-\pi}^{-\delta}[f(x-t) - f(x)]D_n(t)dt + \int_{\delta}^{\pi}[f(x-t) - f(x)]D_n(t)dt. \tag{29}$$

By the Riemann Localization Theorem, we know that for any function $g \in \mathcal{R}[-\pi, \pi]$, and some $\delta \in (0, \phi)$, then

$$\lim_{\mathrm{n}\to\infty}\left[\int_{-\pi}^{-\delta} g(t)D_n(t)dt + \int_{\delta}^{\pi} g(t)D_n(t)dt\right] = 0. \tag{30}$$

Although we omit the proof of the Riemann Localization theorem, please see Theorem 76.4 in [1] for the complete proof. In our case we assign $g(t) = f(x-t) - f(x)$. Then, by the definition of the limit, we know that for any positive ϵ, there must exist a positive integer N such that for all $n \ge N$, we have

$$|I_n| < \frac{\epsilon}{2}. \tag{31}$$

If $n \ge N$, then from Eq. (32) and Eq. (29), we have

$$\left|\int_{-\pi}^{\pi}[f(x-t) - f(x)]D_n(t)dt\right| \le \left|\int_{-\delta}^{\delta}[f(x-t) - f(x)]D_n(t)dt\right| + |I_n| < \epsilon. \qquad \square$$

Our proof is complete.We have shown that the partial sums of the Fourier Series converge pointwise to $f(x)$ as n tends to infinity. Now, we will shift our discussion towards demonstrating uniform convergence, which will utilize the Fejér Kernel.

6 The Fejér Kernel

An important phenomenon is that the Fourier series can diverge for some continuous functions. Therefore, it is worth exploring the conditions under which the Fourier series converges for all continuous functions. For this purpose, we require the Fejér kernel. The Fejér kernel is important because it is a preliminary concept to understanding the notion of uniform convergence. The Fejér kernel will help us in showing that the sequence of Cesáro sums of the sequence of partial sums of the Fourier series converge uniformly to f on $[-\pi, \pi]$. This is a noteworthy result, since it is a much stronger convergence than the pointwise convergence we have shown above. We begin with some preliminary definitions.

Definition 6.1 (The Fejér kernel). The Fejér kernel K_n is defined as

$$K_n(t) = \frac{1}{n+1}\sum_{k=0}^{n} D_k(t) \qquad \text{for all } t \in \mathbb{R} \text{ and } n = 0, 1, \ldots \tag{32}$$

Now, we will investigate some important properties of the Fejér kernel which will help us in proving the uniform convergence of Cesáro sums mentioned above. First, we will show that, similar to the Dirichlet kernel, the Fejér kernel can be expressed as a trigonometric identity.

Lemma 6.1.

$$K_n(t) = \frac{1}{2(n+1)} \left[\frac{\sin[(n+1)t/2]}{\sin(t/2)}\right]^2 \tag{33}$$

for all t with $\sin(t/2) \neq 0$

Proof. By Lemma 3.2:, we can express the Dirichlet kernel as a trigonometric identity. So we have

$$\begin{aligned} K_n(t) &= \frac{1}{n+1} \sum_{k=0}^{n} D_k(t) \\ &= \frac{1}{n+1} \sum_{k=0}^{n} \frac{\sin[(k+\frac{1}{2})t]}{2\sin(t/2)} \\ &= \frac{1}{2(n+1)\sin^2(t/2)} \sum_{k=0}^{n} \sin\left[\left(k+\frac{1}{2}\right)t\right] \sin\frac{t}{2}. \end{aligned} \tag{34}$$

Now, we use the identity that for any u, v, $\cos(u-v) - \cos(u+v) = 2\sin u \sin v$. So assigning $u = (k+\frac{1}{2})t$ and $v = t/2$, we get

$$\frac{\cos kt - \cos[(k+1)t]}{2} = \sin\left[\left(k+\frac{1}{2}\right)t\right] \sin\frac{t}{2}. \tag{35}$$

Computing the telescoping sum, we get

$$\begin{aligned} \frac{1-\cos[(n+1)t]}{2} &= \sum_{k=0}^{n} \frac{\cos kt - \cos[(k+1)t]}{2} \\ &= \sum_{k=0}^{n} \sin\left[\left(k+\frac{1}{2}\right)t\right] \sin\frac{t}{2}. \end{aligned} \tag{36}$$

By the half angle formula, we get

$$\frac{1-\cos[(n+1)t]}{2} = \sin^2\left[(n+1)\frac{t}{2}\right].$$

□

Lemma 6.1 follows from this result. A trivial result that follows from Lemma 6.1 is that for all t, $K_n(t)$ is greater than 0. This result will be useful in our proof of uniform convergence in the next section. Now, we establish another important property that the Fejér and Dirichlet Kernels have in common.

Lemma 6.2.

$$\frac{1}{\pi}\int_{-\pi}^{\pi} K_N(x)dx = 1. \tag{37}$$

Proof. By Lemma 3.1, we know that $\frac{1}{\pi}\int_{-\pi}^{\pi} D_n(x) = 1$. So we have

$$\frac{1}{\pi}\int_{-\pi}^{\pi} K_n(x)dx = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{1}{n+1}\sum_{k=0}^{n} D_k(x)dx = \frac{1}{\pi}\cdot\frac{1}{n+1}\sum_{k=0}^{n}\int_{-\pi}^{\pi} D_k(x)dx = \frac{1}{\pi}\cdot\frac{1}{n+1}\sum_{k=0}^{n}\pi = 1.$$

□

7 Cesáro Summation

Now that we have established the necessary preliminary properties and definitions of the Fejér Kernel, we return to our initial goal of proving the uniform convergence of the sequence of Cesáro sums of the sequence of partial sums of the Fourier series to f on $[-\pi, \pi]$. For sake of clarity, we formally define this concept

Definition 7.1 (Cesáro Sums). The n-th Cesáro sum of the partial sums of Fourier series is

$$\sigma_n(x) = \frac{1}{n+1}\sum_{k=0}^{n} S_k(x) \tag{38}$$

To prove uniform convergence, we will need leverage our preliminary knowledge of the Fejér Kernel. But in order to establish a relationship between the Fejér Kernel and $\sigma_n(x)$, we can perform some simple manipulations of the definition of $\sigma_n(x)$ to get an alternative form for $\sigma_n(x)$ in terms of $K_n(x)$.

Lemma 7.1.

$$\sigma_n(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x-y)K_n(y)\,\mathrm{d}y \tag{39}$$

Proof. By the definition of $\sigma_N(x)$, and subsequently, the definition of $S_n(x)$, we have

$$\begin{aligned}
\sigma_n(x) &= \frac{1}{n+1}\sum_{k=0}^{n} S_k(x) && (40)\\
&= \frac{1}{n+1}\sum_{k=0}^{n}\frac{1}{\pi}\int_{-\pi}^{\pi} f(x-t)D_k(t)dt\\
&= \frac{1}{n+1}\left[\frac{1}{\pi}\left(\int_{-\pi}^{\pi} f(x-t)D_1(t)dt + \int_{-\pi}^{\pi} f(x-t)D_2(t)dt + \ldots + \int_{-\pi}^{\pi} f(x-t)D_n(t)dt\right)\right]\\
&= \frac{1}{\pi}\left(\int_{-\pi}^{\pi} f(x-t)\cdot\frac{1}{n+1}\sum_{k=0}^{n} D_k(t)dt\right).
\end{aligned}$$

By the Definition 6.1 of the Fejér Kernel, we can simplify this to

$$\sigma_n(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x-t)K_n(t)dt.$$

□

8 Uniform Convergence of $\sigma_n(x)$

Now that we have established a closed-form of $\sigma_n(x)$ in terms of $K_n(x)$, we will use this result prove the convergence of $\sigma_n(x)$ to $f(x)$ uniformly on $[-\pi, \pi]$.

Theorem 8.1. If f is a continuous periodic function on $[-\pi, \pi]$, then $\sigma_n(x)$ converges to $f(x)$ uniformly on $[-\pi, \pi]$, that is, for each $\epsilon > 0$, there is $N \in \mathbb{P}$ such that $|\sigma_n(x) - f(x)| < \epsilon$ for all $x \in [-\pi, \pi]$ and $n \geq N$.

Proof. Let $\epsilon > 0$. We can choose M such that $|f(x)| \leq M$ for all $x \in \mathbb{R}$. We notice that since f is uniformly continuous, then there exists some δ with $\pi \geq \delta \geq 0$ for which

$$|f(x) - f(y)| \leq \frac{\epsilon}{4} \qquad |x-y| \leq \delta \tag{41}$$

As we had proven in Lemma 6.1, we know that $K_n(x) \geq 0$ for all $x \in [-\pi, \pi]$. Therefore, there exists a positive integer N such that if $n \geq N$ and $\delta \leq |t| \leq \pi$, then

$$K_n(t) \leq \frac{\epsilon}{16M} \tag{42}$$

By Lemma 7.1 and Lemma 6.1,

$$\begin{aligned}
\sigma_n(x) - f(x) &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x-t)K_n(t)\,\mathrm{d}t - \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)K_n(t)dt && (43)\\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} [f(x-t) - f(x)]K_n(t)\,\mathrm{d}t.
\end{aligned}$$

It follows that

$$|\sigma_n(x) - f(x)| \leq \frac{1}{\pi}\int_{-\pi}^{\pi} |f(x-t) - f(x)|K_n(t)\,\mathrm{d}t \tag{44}$$

If $|t| \leq \delta$, then $|(x-t) - x| \leq \delta$ trivially. Therefore, it follows that

$$\begin{aligned}
\frac{1}{\pi}\int_{-\pi}^{\pi} |f(x-t) - f(x)|K_n(t)\,\mathrm{d}t &\leq \frac{\epsilon}{4\pi}\int_{-\delta}^{\delta} K_n(t)\,\mathrm{d}t && (45)\\
&\leq \frac{\epsilon}{4\pi}\int_{-\pi}^{\pi} K_n(t)\,\mathrm{d}t\\
&= \frac{\epsilon}{4} \qquad n = 1, 2, \ldots
\end{aligned}$$

If $n \geq N$ and $\delta \leq |t| \leq \pi$, then

$$K_n(t) \leq \frac{\epsilon}{16M}. \tag{46}$$

Therefore, it follows that

$$\frac{1}{\pi} \cdot \int_{-\pi}^{-\delta} |f(x-t) - f(x)| K_n(t)\,\mathrm{d}t + \int_{\pi}^{\delta} |f(x-t) - f(x)| K_n(t)\,\mathrm{d}t \leq \frac{\epsilon}{16M\pi} \int_{-\pi}^{\pi} 2M\,\mathrm{d}t = \frac{\epsilon}{4} \tag{47}$$

Therefore, if $n \geq N$ and $x \in \mathbb{R}$, then

$$|\sigma_n(x) - f(x)| \leq \frac{\epsilon}{2} < \epsilon. \qquad \square$$

Our proof is complete, and we have shown the uniform convergence of the Cesáro sums of the partial sums to $f(x)$.

9 Differences between $D_n(x)$ and $K_n(x)$

An important consideration is why the proof of uniform convergence does not work for the Dirichlet Kernel. To answer this, we must first realize that while both the Dirichlet and the Fejér kernel have oscillating behavior, the Dirichlet Kernel is different in that the integral of the absolute value of D_n diverges as n goes to infinity. This is due to its asymptotic behavior at multiples of 2π. On the other hand, the Fejér does not have this asymptotic behavior. More formally, $\|D_n\|_{L_1} \geq \Omega(\log(n))$. To prove this, we see that

$$\int_0^{2\pi} |D_n(x)| dx \geq \int_0^{\pi} \frac{|\sin[(2n+1)x]}{x} dx \geq \sum_{k=0}^{2n} \int_{kn}^{(k+1)\pi} \frac{\sin(s)}{s} ds \geq \left| \sum_{k=0}^{2n} \int_0^{\pi} \frac{\sin(s)}{(k+1)\pi} ds \right| = \frac{2}{\pi} H_{2n+1} \geq \frac{2}{\pi} \log(2n+1),$$

where H_n are the first-order harmonic numbers (see [4] for complete proof). On the other hand, the K_n is bounded, making K_n the optimal kernel choice. This is at the heart of the reason why the Dirichlet kernel cannot be used to prove uniform convergence. σ_n is guaranteed to converge when S_n does not. Therefore, we can always ensure uniform convergence with the Fejér kernel, but this is not guaranteed with the Dirichlet kernel.

10 Exploration: The Gibbs Phenomenon

One extension of the Fourier series is the Gibbs phenomenon, discovered by Henry Wilbraham in 1848 and later rediscovered by J. Willard Gibbs in 1899. On a high level, the Gibbs phenomenon is an overshoot of the Fourier series at discontinuities. Let us understand the contextual motivation for the Gibbs phenomenon. We know that the uniform limit of a continuous function is continuous, so for a function with discontinuities, the Fourier series cannot converge uniformly. The Gibbs phenomenon expresses the behavior of the Fourier series at such a discontinuity. For example, at a jump discontinuity, the Fourier series converges to the average of the limits from the left and from the right at that. But on either side of the jump discontinuity, the Fourier series has high-frequency oscillations. But, the Fourier series overshoots and undershoots the larger and smaller sides of the jump discontinuity, respectively. But what does any of this have to do with the Gibbs phenomenon? Gibbs rediscovered that the amount of overshoot/undershoot is proportional to the size of the jump discontuity. The exact proportion, also known as the Wilbraham-Gibbs constant, is given by

$$\frac{1}{\pi} \int_0^{\pi} \frac{sin(t)}{t} dt - \frac{1}{2} = 0.089489\ldots \tag{48}$$

Now, let us explore a more mathematically rigorous definition of the Gibb's phenomenon. These results were inspired by Pinsky in [5]. Let $f \in C^1$ be a piecewise continuous differentiable function with period L. Consider x_0 where there exists a jump discontinuity. Formally,

$$|f(x_0^+) - f(x_0^-)| = \alpha > 0 \tag{49}$$

Let us denote $S_n(x)$ as the n-th partial Fourier series of f at x. Formally,

$$S_n(x) = \frac{1}{2} a_0 + \sum_{n=1}^{N} \left(a_n \cos\left(\frac{2\pi nx}{L}\right) + b_n \sin\left(\frac{2\pi nx}{L}\right) \right), \tag{50}$$

where

$$a_n \coloneqq \frac{2}{L}\int_0^L f(x)\cos\left(\frac{2\pi nx}{L}\right)dx \qquad b_n \coloneqq \frac{2}{L}\int_0^L f(x)\sin\left(\frac{2\pi nx}{L}\right)dx. \tag{51}$$

Then, it follows that

$$\lim_{n\to\infty} S_n(x_0+\frac{L}{2n}) = f(x_0^+) + \alpha * 0.0895 \qquad \lim_{n\to\infty} S_n(x_0-\frac{L}{2n}) = f(x_0^-) - \alpha * 0.0895. \tag{52}$$

We know that

$$\lim_{n\to\infty} S_n(x_0) = \frac{f(x_0^-) - f(x_0^+)}{2}. \tag{53}$$

So, if any sequence of reals x_n converges to x_0 as n goes to infinity, then

$$\limsup_{n\to\infty} S_n(x_n) \le f(x_0^+) + \alpha \cdot 0.0895 \qquad \liminf_{n\to\infty} S_n(x_n) \ge f(x_0^-) - \alpha \cdot 0.0895. \tag{54}$$

We can see that the overshoot and undershoot is in fact quantified as a proportion of the jump relative to the Wilbraham-Gibbs constant. Now, we will consider an application of the Gibbs phenomenon. Consider a square wave with period $L = 2\pi$ and some jump discontinuity at $x_0 = 0$ where the distance between $f(x_0^+)$ and $f(x_0^-)$ is $\frac{\pi}{2}$. Let us assign $S_n(x)$ as

$$S_n(x) = \sin(x) + \frac{1}{3}\sin(3x) + \ldots + \frac{1}{n-1}\sin((n-1)x). \tag{55}$$

Now, if we substitute $2\pi/2N$ for x, we get

$$S_n(\frac{2\pi}{2N}) = \sin(\frac{\pi}{n}) + \frac{1}{3}\sin(\frac{3\pi}{2N}) + \ldots + \frac{1}{n-1}\sin\left(\frac{(n-1)\pi}{n}\right). \tag{56}$$

Now, we introduction the function $\text{sinc}(x) = \sin(x)/x$. So, we can express the above as

$$S_n(\frac{2\pi}{2n}) = \frac{\pi}{2}\left[\frac{2}{n}\text{sinc}\left(\frac{1}{n}\right) + \frac{2}{n}\text{sinc}\left(\frac{3}{n}\right) + \ldots + \frac{2}{n}\text{sinc}\left(\frac{n-1}{n}\right)\right]. \tag{57}$$

We notice that the expression inside brackets approximates the Riemann integral of $\text{sinc}(x)$ from 0 to 1 using the midpoint of the rectangles. Since $\text{sinc}(x)$ is continuous, we can take the limit of the above to find the true value of the integral of $\text{sinc}(x)$ on $[0, 1]$. So, we have

$$\lim_{n\to\infty} S_n(\frac{2\pi}{2n}) = \frac{\pi}{2}\int_0^1 \text{sinc}(x)dx = \frac{1}{2}\int_{x=0}^{x=1}\frac{\sin(\pi x)}{\pi x}d(\pi x) = \frac{1}{2}\int_0^\pi \frac{\sin(t)}{t}dt = \frac{\pi}{4} + \frac{\pi}{2}\cdot(0.08948987\ldots). \tag{58}$$

As we can see, the jump discontinuity is still a 0.089 proportion of the size of the jump discontinuity. Although this is just one concrete example of the Gibbs phenomenon, Gibbs phenomenon appears extensively in signal processing and sound engineering.

11 Conclusion

We have explored the pointwise convergence of the partial sums of the Fourier series to a continuous function, as well as the uniform convergence of the Cesaro sums of S_n. Finally, we explored the Gibbs Phenomenon, a property of the behavior of Fourier series at jump discontinuities. These results deem useful in many domains, including physics, chemistry, and electrical engineering. The proofs in this paper were based on results in [1]. Please see the References for further reading.

References

[1] Richard Johnsonbaugh and William E Pfaffenberger. *Foundations of mathematical analysis.* Courier Corporation, 2012.

[2] Sheldon Axler. *Linear Algebra Done Right.* Springer, 2017.

[3] Kourosh Raeen. *A study of the Gibbs phenomenon in Fourier series and wavelets.* PhD thesis, University of New Mexico, 2008.

[4] Bruckner J. B. amp; Bruckner A. M. Thomson, B. S. *Real analysis*, volume 1. Princeton University Press, 2008.

[5] M. A. Pinksy. *Introduction to Fourier analysis and wavelets*, volume 1. American Mathematical Society, 2002.