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Analytic study of force free magnetic fields in a tube

1 Introduction

The idea of this project is that we want to investigate how force free magnetic fields relax under the MHD equations without diffusion. We believe that the Lorentz Force is the dominant force in the Corona, so it dictates the form the field will take over time, to minimise the Lorentz force. Force-free fields are these fields. However, this is purely out of curiosity as in actuality, the corona is in such activity that the fields never have time to morph to a force-free field.

First of all, we want to find a magnetic field with a nugget of twist and overlap instances of it in a way to create a braided field (ie. very intertwined) and see how it evolves. Preliminary observations suggest our field should stay relatively stable and not really change. But let us try to demonstrate this hypothesis. Let us start.

Given a tubular coordinate system with constant radius, we fill it with a magnetic field following field lines curving uniformly around the central axis of the tube. We would like to investigate the constraint on the field, if we have it be force free (ie: $\nabla \times \boldsymbol{B} = \alpha \boldsymbol{B}$).

Please note that this paper will be utilizing knowledge of differential form in Differential Calculus; so it it advised that the reader have elementary understanding of this concept.

2 Acknowledgments

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3 Starting Concepts

We remind the reader that the tubular coordinate system is defined by

$$\boldsymbol{T} : [0, L] \times [0, 1] \times [0, 2\pi] \to \mathbb{R}^3 \quad (s, \rho, \theta) = \boldsymbol{r}(s) + \rho R(s)(\boldsymbol{d}_1(s)\cos(\theta) + \boldsymbol{d}_2(s)\sin(\theta)), \tag{1}$$

with

$$\frac{d}{ds}\begin{bmatrix} \boldsymbol{d}_1(s) \\ \boldsymbol{d}_2(s) \\ \boldsymbol{d}_3(s) \end{bmatrix} = \begin{bmatrix} 0 & 0 & -u_2 \\ 0 & 0 & u_1 \\ u_2 & u_1 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{d}_1(s) \\ \boldsymbol{d}_2(s) \\ \boldsymbol{d}_3(s) \end{bmatrix}. \tag{2}$$

and that Given R(s), we can derive the metric tensor of the coordinate system $\boldsymbol{T}(s,\rho,\theta)$. Knowing that the metric tensor has entries $g_{ij} = (\frac{\partial \boldsymbol{T}}{\partial x_i})(\frac{\partial \boldsymbol{T}}{\partial x_j})$, we get

$$g_{ij} = \begin{bmatrix} (1+\rho R(u_1(s)\sin(\theta) - u_2(s)\cos(\theta)))^2 + \rho^2 R'^2 & \rho R R' & 0 \\ \rho R R' & R^2 & 0 \\ 0 & 0 & \rho^2 R^2 \end{bmatrix} \tag{3}$$

and $g = \sqrt{\det(g_{ij})} = \rho R^2(1+\rho R(u_1(s)\sin(\theta) - u_2(s)\cos(\theta)))$. Note that $u_1(s)$ and $u_2(s)$ control the curvature of the tube; and the curvature is define by $\kappa(s) = \sqrt{u_1(s)^2 + u_2(s)^2}$.

We can express our unit tangent field $\boldsymbol{N}$ as follows:

$$\begin{aligned} \boldsymbol{N} &= \boldsymbol{e}_s N_s + \boldsymbol{e}_\rho N_\rho + \boldsymbol{e}_\theta N_\theta \\ &= \frac{l}{\lambda}\boldsymbol{e}_s + r\frac{\rho R'}{\lambda R} s\boldsymbol{e}_\rho + \frac{t}{\lambda}\boldsymbol{e}_\theta \\ &= \frac{1}{\lambda}\tilde{\boldsymbol{N}} \\ \text{with} \qquad & \lambda = \sqrt{l^2 + r^2\rho^4 R'^2 + t^2 R^2}, \end{aligned} \tag{4}$$

where l, r, t are functions of s, ρ and θ that control the shape of the field inside the tube. Note that these parameters work simultaneously as our field is normalized and the normalisation contains l, r and t.

1. l : controls "inclination": gives more freedom with the angle of the field with the $\boldsymbol{e}_s$ direction. It allows the field to have a mushroom shape or to ripple/fold inside/under itself. Having $l(s,\rho)$ could help the equations simplify even with no ripple/folding (overall magnetic field should be the same). In this way, r and t have less to 'fulfill' and the conditions are more evenly shared amongst the 3 functions.

2. r : controls "bulging": dictates how much the field is pointing inward or outward.

3. t : controls "twisting": dictates to what degree the field is rotating around the central axis of the tube.

Note that our coordinate vectors are not orthogonal. This is due to the fact that our metric tensor of T is not diagonal. Additionally, our coordinate vectors are not unit. So this has to be taken into account in our calculation of the curl. In fact, we will use differential forms to calculate the curl of $\tilde{\boldsymbol{N}}$. (Note that in this report, we use interchangeably the coordinates: $x_1 = s$, $x_2 = \rho$, $x_3 = \theta$.)

Let us note one more thing. Given any distribution of $\phi_0(\rho,\theta)$ at the base of the tube and the unit tangent field $\boldsymbol{N}$, we can use the divergence condition anywhere to get a the ϕ everywhere: $\phi(s,\rho,\theta) = \phi_0(\rho,\theta)e^{\int_0^s \nabla\times\boldsymbol{N}dt}$ along field lines, in a way that the resulting field is divergence free; ie. make it a magnetic field.

4 Restrictions

Now the restrictions we will be making are a combination of:

1. u_1 and u_2: these control the curvature of the tube.

2. R: This is the Radius of the curve at any point.
3. l: this gives control on inclination of field.
4. r: this allows bulging of the field.
5. t: this allows twisting of the field.
6. s dependence.
7. ρ dependence.
8. θ dependence: may lead us to having to deal with Fourrier series to be able to deal with the alignement of the field at $2\pi \equiv 0$.
9. $[\nabla\phi]_N = 0$: this entails that the intensity of the magnetic field is constant along field lines. Gets rid of the latter terms of the 3 ForceFree-ness equations.
10. Lie condition: $\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} = \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_i}$
11. Separation of variables: Careful, most of the time this causes a restrictions on the solutions and can cause incongruencies such as $r_\rho(\rho) = e^{\int_0^s f(t,\rho)dt}$ which is clearly not ok.

Ideally, we would like to find a field with an α that dies away from the origin (allowing us to overlap several fields) and with a strong nugget of twist at the origin (allowing us to braid field lines). Being able to impose something like: $t(s,\rho) = e^{-s^2-\rho^2}$ would be nice. However, this may prove extremely difficult to solve exactly (even by employing numerical techniques), so it would be acceptable to have a small amount of error. If this proves too difficult as will, than we would settle for finding any field that works.

5 Expression of α

First of all, let us notice that we can write:

$$
\begin{aligned}
&\nabla \times \boldsymbol{B} = \alpha \boldsymbol{B} \\
\Leftrightarrow &\nabla \times \phi \boldsymbol{N} = \alpha\phi\boldsymbol{N} \\
\Leftrightarrow &\phi\nabla \times \boldsymbol{N} + \nabla\phi \times \boldsymbol{N} = \alpha\phi\boldsymbol{N} \\
\Leftrightarrow &\boldsymbol{N} \cdot [\phi(\nabla \times \boldsymbol{N}) + (\nabla\phi) \times \boldsymbol{N}] = \alpha\phi\boldsymbol{N} \cdot \boldsymbol{N} \\
\Leftrightarrow &\phi\boldsymbol{N} \cdot \nabla \times \boldsymbol{N} + 0 = \alpha\phi \\
\Leftrightarrow &\boldsymbol{N} \cdot \nabla \times \boldsymbol{N} = \alpha
\end{aligned} \tag{5}
$$

Which we can simplify to:

$$
\begin{aligned}
\alpha &= \boldsymbol{N} \cdot \nabla \times \boldsymbol{N} \\
&= \frac{\tilde{\boldsymbol{N}}}{\lambda} \cdot \nabla \times \frac{1}{\lambda}\tilde{\boldsymbol{N}} \\
&= \frac{\tilde{\boldsymbol{N}}}{\lambda}\big[\frac{1}{\lambda}(\nabla \times \tilde{\boldsymbol{N}}) + (\nabla\frac{1}{\lambda}) \times \tilde{\boldsymbol{N}}\big] \\
&= \frac{1}{\lambda^2}\tilde{\boldsymbol{N}} \cdot (\nabla \times \tilde{\boldsymbol{N}})
\end{aligned} \tag{6}
$$

6 Alternative conditions of ForceFree-ness

By using standard algebraical manipulation, we can find more clear expressions for the conditions that must hold. First of all, we can use the following method:

$$\begin{aligned}
\nabla \times (\phi \boldsymbol{N}) &= \alpha\phi\boldsymbol{N} \\
\nabla\phi \times \boldsymbol{N} + \phi\nabla \times \boldsymbol{N} &= \alpha\phi\boldsymbol{N} \\
(\boldsymbol{e}_i \times \boldsymbol{N}) \cdot (\nabla\phi \times \boldsymbol{N}) + \phi(\boldsymbol{e}_i \times \boldsymbol{N}) \cdot (\nabla \times \boldsymbol{N}) &= \alpha\phi(\boldsymbol{e}_i \times \boldsymbol{N}) \cdot \boldsymbol{N} \\
(\boldsymbol{e}_i \times \boldsymbol{N}) \cdot (\nabla\phi \times \boldsymbol{N}) + \phi(\boldsymbol{e}_i \times \boldsymbol{N}) \cdot (\nabla \times \boldsymbol{N}) &= 0 \\
\nabla\phi \cdot (\boldsymbol{N} \times (\boldsymbol{e}_i \times \boldsymbol{N})) + \phi(\boldsymbol{e}_i \times \boldsymbol{N}) \cdot (\nabla \times \boldsymbol{N}) &= 0 \\
\nabla\phi \cdot (\boldsymbol{e}_i - (\boldsymbol{e}_i \cdot \boldsymbol{N})\boldsymbol{N}) + \phi(\boldsymbol{e}_i \times \boldsymbol{N}) \cdot (\nabla \times \boldsymbol{N}) &= 0 \\
-\phi(\boldsymbol{e}_i \times \boldsymbol{N}) \cdot (\nabla \times \boldsymbol{N}) + [\nabla\phi]_N(\boldsymbol{e}_i \cdot \boldsymbol{N}) &= [\nabla\phi]_i
\end{aligned} \tag{7}$$

This gives for our 3 directions:

$$\begin{cases}
[\nabla\phi]_\rho = -\phi(\boldsymbol{e}_\rho \times \boldsymbol{N}) \cdot (\nabla \times \boldsymbol{N}) + [\nabla\phi]_N(\boldsymbol{e}_\rho \cdot \boldsymbol{N}) \\
[\nabla\phi]_s = -\phi(\boldsymbol{e}_s \times \boldsymbol{N}) \cdot (\nabla \times \boldsymbol{N}) + [\nabla\phi]_N(\boldsymbol{e}_s \cdot \boldsymbol{N}) \\
[\nabla\phi]_\theta = -\phi(\boldsymbol{e}_\theta \times \boldsymbol{N}) \cdot (\nabla \times \boldsymbol{N}) + [\nabla\phi]_N(\boldsymbol{e}_\theta \cdot \boldsymbol{N})
\end{cases} \tag{8}$$

We can combine this with

$$[\nabla\phi]_N = -\phi(\nabla \cdot \boldsymbol{N}) \tag{9}$$

to get:

$$\begin{cases}
[\nabla\phi]_\rho &= -\phi(\boldsymbol{e}_\rho \times \boldsymbol{N}) \cdot (\nabla \times \boldsymbol{N}) - \phi(\nabla \cdot \boldsymbol{N})(\boldsymbol{e}_\rho \cdot \boldsymbol{N}) = \phi f_1 \\
[\nabla\phi]_s &= -\phi(\boldsymbol{e}_s \times \boldsymbol{N}) \cdot (\nabla \times \boldsymbol{N}) - \phi(\nabla \cdot \boldsymbol{N})(\boldsymbol{e}_s \cdot \boldsymbol{N}) = \phi f_2 \\
[\nabla\phi]_\theta &= -\phi(\boldsymbol{e}_\theta \times \boldsymbol{N}) \cdot (\nabla \times \boldsymbol{N}) - \phi(\nabla \cdot \boldsymbol{N})(\boldsymbol{e}_\theta \cdot \boldsymbol{N}) = \phi f_3
\end{cases} \tag{10}$$

7 Attempts at finding solutions

In this section, we will be searching for solutions in a non-straight curvy-linear cylinder by making assumptions on the metric or the field itself.

7.1 Curvy-Linear coordinates: Constant radius and no twist

Let us start by taking R constant and having no twist ($\psi = 0$). In this case, the metric is diagonal, and the divergence of the tangent field is 0. So $\phi = \phi_0 e^{\int_0^s 0dt} = \phi_0$

First of all, express $\tilde{\boldsymbol{N}}$ and $\nabla \times \tilde{\boldsymbol{N}}$ in terms of differential forms:

$$\begin{aligned}
n_i &= g_{ij}N^j \\
n &= n_1 dx^1 + n_2 dx^2 + n_3 dx^3 \\
&= (1 + \rho R(u_1(s)\sin(\theta) - u_2(s)\cos(\theta)))^2 dx^1
\end{aligned} \tag{11}$$

$$\begin{aligned}
dn &= \frac{\partial n_1}{\partial x_2} dx^2 \wedge dx^1 + \frac{\partial n_2}{\partial x_1} dx^1 \wedge dx^2 + \frac{\partial n_1}{\partial x_3} dx^3 \wedge dx^1 + \frac{\partial n_3}{\partial x_1} dx^1 \wedge dx^3 + \frac{\partial n_3}{\partial x_2} dx^2 \wedge dx^3 + \frac{\partial n_2}{\partial x_3} dx^3 \wedge dx^2 \\
&= \frac{\partial n_1}{\partial x_2} dx^2 \wedge dx^1 + \frac{\partial n_1}{\partial x_3} dx^3 \wedge dx^1
\end{aligned} \tag{12}$$

Now we can start to manipulate the forms; we calculate the inner product of $\boldsymbol{N}$ and $\boldsymbol{\nabla} \times \boldsymbol{N}$:

$$\begin{aligned} n \wedge dn &= n_3 \frac{\partial n_1}{\partial x_2} dx^3 \wedge dx^2 \wedge dx^1 + n_2 \frac{\partial n_1}{\partial x_3} dx^2 \wedge dx^3 \wedge dx^1 \\ &= (n_2 \frac{\partial n_1}{\partial x_3} - n_3 \frac{\partial n_1}{\partial x_2}) dx^1 \wedge dx^2 \wedge dx^3 \\ \Leftrightarrow \tilde{\boldsymbol{N}} \cdot (\boldsymbol{\nabla} \times \tilde{\boldsymbol{N}}) &= 0 \\ \Leftrightarrow \alpha &= 0 \end{aligned} \tag{13}$$

So we are in-fact just looking at a potential field. This isn't really interesting because we know how to solve these using Laplace's equation. But we will just go on for fun.

Now let us study the ForceFree-ness. In this case, equation:

$$\begin{cases} [\nabla\phi]_\rho = -\phi(\boldsymbol{e}_\rho \times \boldsymbol{N}) \cdot (\nabla \times \boldsymbol{N}) \\ [\nabla\phi]_\theta = -\phi(\boldsymbol{e}_\theta \times \boldsymbol{N}) \cdot (\nabla \times \boldsymbol{N}) \\ [\nabla\phi]_s = -\phi(\boldsymbol{e}_s \times \boldsymbol{N}) \cdot (\nabla \times \boldsymbol{N}) = 0 \end{cases} \tag{14}$$

So let us study the individual parts: First of all, express $\boldsymbol{N}$ in terms of differential forms:

$$\begin{aligned} n_i &= g_{ij} N^j \\ n &= n_1 dx^1 + n_2 dx^2 + n_3 dx^3 \\ &= \frac{g_{11}}{\lambda} dx^1 \\ &= \frac{(1 + \rho R(u_1(s)\sin(\theta) - u_2(s)\cos(\theta)))^2}{\rho R^2 (1 + \rho R(u_1(s)\sin(\theta) - u_2(s)\cos(\theta)))} dx^1 \\ &= \frac{1 + \rho R(u_1(s)\sin(\theta) - u_2(s)\cos(\theta))}{\rho R^2} dx^1 \end{aligned} \tag{15}$$

Now we can start to manipulate the forms; we calculate $\boldsymbol{\nabla} \times \boldsymbol{N}$:

$$\begin{aligned} dn &= \frac{\partial n_1}{\partial x_2} dx^2 \wedge dx^1 + \frac{\partial n_2}{\partial x_1} dx^1 \wedge dx^2 + \frac{\partial n_1}{\partial x_3} dx^3 \wedge dx^1 + \frac{\partial n_3}{\partial x_1} dx^1 \wedge dx^3 + \frac{\partial n_3}{\partial x_2} dx^2 \wedge dx^3 + \frac{\partial n_2}{\partial x_3} dx^3 \wedge dx^2 \\ &= \frac{\partial n_1}{\partial x_2} dx^2 \wedge dx^1 + \frac{\partial n_1}{\partial x_3} dx^3 \wedge dx^1 \end{aligned} \tag{16}$$

$$\boldsymbol{\nabla} \times \boldsymbol{N} = \frac{1}{g} \left[\frac{\partial n_1}{\partial x_3} \boldsymbol{e}_\rho - \frac{\partial n_1}{\partial x_2} \boldsymbol{e}_\theta \right] \tag{17}$$

A ——— we calculate $e_s \times \boldsymbol{N}$:

$$\begin{aligned} e_s \wedge n &= [(1 + \rho R(u_1(s)\sin(\theta) - u_2(s)\cos(\theta)))^2 dx^1] \wedge [n_1 dx^1] \\ &= 0 \end{aligned} \tag{18}$$

B ——— we calculate $e_\rho \times \boldsymbol{N}$:

$$\begin{aligned} e_\rho \wedge n &= [R^2 dx^2] \wedge [n_1 dx^1] \\ &= -R^2 n_1 dx^1 \wedge dx^2 \\ \Leftrightarrow \boldsymbol{e}_\rho \times \boldsymbol{N} &= -\frac{R^2 n_1}{g} \boldsymbol{e}_\theta \end{aligned} \tag{19}$$

Now: $(\boldsymbol{e}_\rho \times \boldsymbol{N}) \cdot (\nabla \times \boldsymbol{N}) = \frac{\rho^2 R^4}{g^2} n_1 \frac{\partial n_1}{\partial x_2}$.

C ——— we calculate $e_\theta \times \boldsymbol{N}$:

$$\begin{aligned} e_\theta \wedge n &= [\rho^2 R^2 dx^3] \wedge [n_1 dx^1] \\ &= \rho^2 R^2 n_1 dx^3 \wedge dx^1 \\ \Leftrightarrow \boldsymbol{e}_\rho \times \boldsymbol{N} &= \frac{\rho^2 R^2 n_1}{g} \boldsymbol{e}_\rho \end{aligned} \tag{20}$$

Now: $(\boldsymbol{e}_\rho \times \boldsymbol{N}) \cdot (\nabla \times \boldsymbol{N}) = \frac{\rho^2 R^4}{g^2} n_1 \frac{\partial n_1}{\partial x_3}$.

So our equations redue to:

$$\begin{cases} [\nabla\phi]_\rho = -\phi \left[\frac{\rho^2 R^4}{g^2} n_1 \frac{\partial n_1}{\partial x_2}\right] \\ [\nabla\phi]_\theta = -\phi \left[\frac{\rho^2 R^4}{g^2} n_1 \frac{\partial n_1}{\partial x_3}\right] \\ [\nabla\phi]_s = 0 \end{cases} \tag{21}$$

which is

$$\begin{cases} \frac{\partial}{\partial\theta}\phi(\rho,\theta) = -\frac{\phi(\rho,\theta)(\cos(\theta)u_1(s)+\sin(\theta)u_2(s))}{\rho(\rho\sin(\theta)u_1(s)-\rho\cos(\theta)u_2(s)+1)} \\ \frac{\partial}{\partial\rho}\phi(\rho,\theta) = \frac{\phi(\rho,\theta)}{\rho^3(\rho\sin(\theta)u_1(s)-\rho\cos(\theta)u_2(s)+1)} \end{cases} \tag{22}$$

This implies that:

$$\begin{cases} u_1(s) = C_1 \\ u_2(s) = C_2 \end{cases} \tag{23}$$

Which in turn makes our equation:

$$\begin{cases} \frac{\partial}{\partial\theta}\phi(\rho,\theta) = \phi(\rho,\theta)\frac{-(\cos(\theta)C_1+\sin(\theta)C_2)}{\rho(\rho\sin(\theta)C_1-\rho\cos(\theta)C_2+1)} \\ \frac{\partial}{\partial\rho}\phi(\rho,\theta) = \phi(\rho,\theta)\frac{1}{\rho^3(\rho\sin(\theta)C_1-\rho\cos(\theta)C_2+1)} \end{cases} \tag{24}$$

This PDE for ϕ clearly cannot be solved using separation of variables, therefor I don't know how to solve it.

7.2 Curvy-Linear coordinates: Use of Lie condition to get force-free conditions independent of ϕ_0, no twist, no bulging, no mushrooming.

So our metric is no longer diagonal. So we must recalculate the curl and divergence expressions.

$$\lambda = \sqrt{(1+\rho R(u_1\sin(\theta) - u_2\cos(\theta)))^2 + (\rho R')^2} \tag{25}$$

$$g = \rho R^2[(1+\rho R(u_1\sin(\theta) - u_2\cos(\theta)))^2 + (\rho R')^2] - \rho^4 R^4 R'^2 \tag{26}$$

and:

$$N = \frac{1}{\lambda}(\boldsymbol{e}_s + \frac{\rho R'}{R}\boldsymbol{e}_\rho) \qquad \Rightarrow \qquad n \;= (\lambda + \frac{\rho^2 R'^2}{\lambda})dx^1 + 2\frac{\rho R R'}{\lambda}dx^2 \tag{27}$$

Also remember that combining:

$$\begin{cases} [\nabla\phi]_\rho &= -\phi(\boldsymbol{e}_\rho \times \boldsymbol{N}) \cdot (\nabla \times \boldsymbol{N}) + [\nabla\phi]_N(\boldsymbol{e}_\rho \cdot \boldsymbol{N}) \\ [\nabla\phi]_s &= -\phi(\boldsymbol{e}_s \times \boldsymbol{N}) \cdot (\nabla \times \boldsymbol{N}) + [\nabla\phi]_N(\boldsymbol{e}_s \cdot \boldsymbol{N}) \\ [\nabla\phi]_\theta &= -\phi(\boldsymbol{e}_\theta \times \boldsymbol{N}) \cdot (\nabla \times \boldsymbol{N}) + [\nabla\phi]_N(\boldsymbol{e}_\theta \cdot \boldsymbol{N}) \end{cases} \tag{28}$$

and:

$$[\nabla\phi]_N = -\phi(\nabla \cdot \boldsymbol{N}) \tag{29}$$

we get:

$$\begin{cases} [\nabla\phi]_\rho & = -\phi(\boldsymbol{e}_\rho \times \boldsymbol{N})\cdot(\nabla\times\boldsymbol{N}) - \phi(\nabla\cdot\boldsymbol{N})(\boldsymbol{e}_\rho\cdot\boldsymbol{N}) = \phi f_1 \\ [\nabla\phi]_s & = -\phi(\boldsymbol{e}_s \times \boldsymbol{N})\cdot(\nabla\times\boldsymbol{N}) - \phi(\nabla\cdot\boldsymbol{N})(\boldsymbol{e}_s\cdot\boldsymbol{N}) = \phi f_2 \\ [\nabla\phi]_\theta & = -\phi(\boldsymbol{e}_\theta \times \boldsymbol{N})\cdot(\nabla\times\boldsymbol{N}) - \phi(\nabla\cdot\boldsymbol{N})(\boldsymbol{e}_\theta\cdot\boldsymbol{N}) = \phi f_3 \end{cases} \tag{30}$$

Additionally, we can alternativly, investigate a very precise case where ϕ to fulfills the Lie Condition: $\frac{\partial}{\partial\rho}\frac{\partial\phi}{\partial s} = \frac{\partial}{\partial s}\frac{\partial\phi}{\partial\rho}$, $\frac{\partial}{\partial\rho}\frac{\partial\phi}{\partial\theta} = \frac{\partial}{\partial\theta}\frac{\partial\phi}{\partial\rho}$, $\frac{\partial}{\partial\theta}\frac{\partial\phi}{\partial s} = \frac{\partial}{\partial s}\frac{\partial\phi}{\partial\theta}$. In this case, we can eliminate ϕ from our ForceFree-ness conditions all together. In this way our $\boldsymbol{N}$ would be force free no matter what base field $\phi_0(\rho,\theta)$ we use. Then $\phi(s,\rho,\theta) = \phi_0(\rho,\theta)e^{\int_0^t \nabla\cdot\boldsymbol{N}dt}$ along field lines of $\boldsymbol{N}$. So we get in one case:

$$\begin{gathered} \frac{\partial}{\partial\rho}\frac{\partial\phi}{\partial s} = \frac{\partial}{\partial s}\frac{\partial\phi}{\partial\rho} \\ \frac{\partial f_1}{\partial\rho}\phi + f_1\frac{\partial\phi}{\partial\rho} = \frac{\partial f_2}{\partial s}\phi + f_2\frac{\partial\phi}{\partial s} \\ \frac{\partial f_1}{\partial\rho}\phi + f_1 f_2\phi = \frac{\partial f_2}{\partial s}\phi + f_2 f_1\phi \\ \frac{\partial f_1}{\partial\rho} = \frac{\partial f_2}{\partial s} \end{gathered} \tag{31}$$

Applied in different order, we get the alternative conditions:

$$\begin{cases} \frac{\partial f_1}{\partial\rho} = \frac{\partial f_2}{\partial s} \\ \frac{\partial f_3}{\partial s} = \frac{\partial f_1}{\partial\theta} \\ \frac{\partial f_2}{\partial\theta} = \frac{\partial f_3}{\partial\rho} \end{cases} \tag{32}$$

Now we must find explicit expressions for each of f_1, f_2, f_3. First of all,

$$\nabla\cdot\boldsymbol{N} = \frac{\lambda\frac{\partial g}{\partial s} - g\frac{\partial\lambda}{\partial s}}{g\lambda^2} + \rho R'\frac{\lambda\frac{\partial g}{\partial\rho} - g\frac{\partial\lambda}{\partial\rho}}{Rg\lambda^2} + \frac{R'}{R\lambda} \tag{33}$$

Next,

$$\begin{cases} \boldsymbol{e}_s\cdot\boldsymbol{N} = \lambda + \frac{\rho^2 R'^2}{\lambda} \\ \boldsymbol{e}_\rho\cdot\boldsymbol{N} = 2\frac{\rho R R'}{\lambda} \\ \boldsymbol{e}_\theta\cdot\boldsymbol{N} = 0 \end{cases} \tag{34}$$

Also:

$$\begin{aligned} (\boldsymbol{e}_i\times\boldsymbol{N})\cdot(\nabla\times\boldsymbol{N}) &= (\boldsymbol{e}_i\times\boldsymbol{N})\cdot(C_{N_j}\boldsymbol{e}_j) \\ &= C_{N_j}(\boldsymbol{e}_j\times\boldsymbol{e}_i)\cdot\boldsymbol{N} \\ &= C_{N_j} g(dx^j\wedge dx^i)\wedge\left[(g_{11}N_1 + g_{12}N_2)dx^1 + (g_{21}N_1 + g_{22}N_1)dx^2\right] \end{aligned} \tag{35}$$

which allows us to calculate:

$$\begin{cases} (\boldsymbol{e}_s\times\boldsymbol{N})\cdot(\nabla\times\boldsymbol{N}) = -C_{N_3}(\lambda + \frac{\rho^2 R'^2}{\lambda}) \\ (\boldsymbol{e}_\rho\times\boldsymbol{N})\cdot(\nabla\times\boldsymbol{N}) = C_{N_3}\frac{2\rho R R'}{\lambda} \\ (\boldsymbol{e}_\theta\times\boldsymbol{N})\cdot(\nabla\times\boldsymbol{N}) = C_{N_2}(\lambda + \frac{\rho^2 R'^2}{\lambda}) - C_{N_1}\frac{2\rho R R'}{\lambda} \end{cases} \tag{36}$$

Also:

$$\begin{aligned} dn &= -\frac{\partial n_2}{\partial\theta}dx^2\wedge dx^3 + \frac{\partial n_1}{\partial\theta}dx^3\wedge dx^1 + (\frac{\partial n_2}{\partial s} - \frac{\partial n_1}{\partial\rho})dx^1\wedge dx^2 \\ \nabla\times\boldsymbol{N} &= -\frac{1}{g}\frac{\partial n_2}{\partial\theta}\boldsymbol{e}_s + \frac{1}{g}\frac{\partial n_1}{\partial\theta}\boldsymbol{e}_\rho + \frac{1}{g}(\frac{\partial n_2}{\partial s} - \frac{\partial n_1}{\partial\rho})\boldsymbol{e}_\theta \end{aligned} \tag{37}$$

So:

$$\begin{cases} f1 = (\frac{\lambda\frac{\partial g}{\partial s} - g\frac{\partial \lambda}{\partial s}}{g\lambda^2} + \rho R' \frac{\lambda\frac{\partial g}{\partial \rho} - g\frac{\partial \lambda}{\partial \rho}}{Rg\lambda^2} + \frac{R'}{R\lambda})(\lambda + \frac{\rho^2 R'^2}{\lambda}) - \frac{1}{g}(\frac{\partial n_2}{\partial s} - \frac{\partial n_1}{\partial \rho})(\lambda + \frac{\rho^2 R'^2}{\lambda}) \\ f2 = (\frac{\lambda\frac{\partial g}{\partial s} - g\frac{\partial \lambda}{\partial s}}{g\lambda^2} + \rho R' \frac{\lambda\frac{\partial g}{\partial \rho} - g\frac{\partial \lambda}{\partial \rho}}{Rg\lambda^2} + \frac{R'}{R\lambda})(2\frac{\rho R R'}{\lambda}) + \frac{1}{g}(\frac{\partial n_2}{\partial s} - \frac{\partial n_1}{\partial \rho})\frac{2\rho R R'}{\lambda} \\ f3 = \frac{1}{g}\frac{\partial n_1}{\partial \theta}(\lambda + \frac{\rho^2 R'^2}{\lambda}) + \frac{1}{g}\frac{\partial n_2}{\partial \theta}\frac{2\rho R R'}{\lambda} \end{cases} \tag{38}$$

I can't solve these but it is a set of simultaneous ODEs. So if there is a solution, we should be able to solve it with enough computing power.

7.3 Curvy-Linear coordinates: Constant radius and θ and ρ independence

Ok so here we will denote $t(s) = \psi'(s)$. First of all, express $\tilde{\boldsymbol{N}}$ and $\nabla \times \tilde{\boldsymbol{N}}$ in terms of differential forms:

$$\begin{aligned} n_i &= g_{ij} N^j \\ n &= n_1 dx^1 + n_2 dx^2 + n_3 dx^3 \\ &= n_1 dx^1 + n_3 dx^3 \\ &= (1 + \rho R(u_1(s)\sin(\theta) - u_2(s)\cos(\theta)))^2 dx^1 + 0 + \psi' \rho^2 R^2 dx^3 \end{aligned} \tag{39}$$

$$\begin{aligned} dn &= \frac{\partial n_1}{\partial x_2} dx^2 \wedge dx^1 + \frac{\partial n_2}{\partial x_1} dx^1 \wedge dx^2 + \frac{\partial n_1}{\partial x_3} dx^3 \wedge dx^1 + \frac{\partial n_3}{\partial x_1} dx^1 \wedge dx^3 + \frac{\partial n_3}{\partial x_2} dx^2 \wedge dx^3 + \frac{\partial n_2}{\partial x_3} dx^3 \wedge dx^2 \\ &= \frac{\partial n_1}{\partial x_2} dx^2 \wedge dx^1 + (\frac{\partial n_1}{\partial x_3} - \frac{\partial n_3}{\partial x_1}) dx^3 \wedge dx^1 + \frac{\partial n_3}{\partial x_2} dx^2 \wedge dx^3 \end{aligned} \tag{40}$$

Now we can start to manipulate the forms; we calculate the inner product of $\boldsymbol{N}$ and $\boldsymbol{\nabla} \times \boldsymbol{N}$:

$$\begin{aligned} n \wedge dn &= n_3 \frac{\partial n_1}{\partial x_2} dx^3 \wedge dx^2 \wedge dx^1 + n_1 \frac{\partial n_3}{\partial x_2} dx^1 \wedge dx^2 \wedge dx^3 \\ &= (n_1 \frac{\partial n_3}{\partial x_2} - n_3 \frac{\partial n_1}{\partial x_2}) dx^1 \wedge dx^2 \wedge dx^3 \end{aligned} \tag{41}$$

Now we can reintroduce our explicit expression of N:

$$\begin{aligned} n_1 \frac{\partial n_3}{\partial x_2} &= (1 + \rho R(u_1(s)\sin(\theta) - u_2(s)\cos(\theta)))^2 \cdot (2\psi' \rho R^2) \\ n_3 \frac{\partial n_1}{\partial x_2} &= (\psi') \cdot 2\rho^2 R^2 (1 + \rho R(u_1(s)\sin(\theta) - u_2(s)\cos(\theta)))(R(u_1(s)\sin(\theta) - u_2(s)\cos(\theta))) \end{aligned} \tag{42}$$

But remember, the $\boldsymbol{\nabla} \times \boldsymbol{N}$ term still has the $dx^1 \wedge dx^2 \wedge dx^3$ term which represents a volume term. So when doing this inner product, we need to divide by $\sqrt{||g_{ij}||} = \rho R^2 (1 + \rho R(u_1(s)\sin(\theta) - u_2(s)\cos(\theta)))$. So we get:

$$\begin{aligned} \langle \tilde{\boldsymbol{N}}, \boldsymbol{\nabla} \times \tilde{\boldsymbol{N}} \rangle &= \frac{(n_1 \frac{\partial n_3}{\partial x_2} - n_3 \frac{\partial n_1}{\partial x_2})}{\rho R^2 (1 + \rho R(u_1(s)\sin(\theta) - u_2(s)\cos(\theta)))} \\ &= \frac{(1 + \rho R(u_1(s)\sin(\theta) - u_2(s)\cos(\theta)))^2 \cdot (2\psi' \rho R^2)}{\rho R^2 (1 + \rho R(u_1(s)\sin(\theta) - u_2(s)\cos(\theta)))} - \\ &\quad \frac{(\psi') \cdot 2\rho^2 R^2 (1 + \rho R(u_1(s)\sin(\theta) - u_2(s)\cos(\theta)))(R(u_1(s)\sin(\theta) - u_2(s)\cos(\theta)))}{\rho R^2 (1 + \rho R(u_1(s)\sin(\theta) - u_2(s)\cos(\theta)))} \\ &= 2\psi'(1 + \rho R(u_1(s)\sin(\theta) - u_2(s)\cos(\theta))) - 2\psi' \rho (R(u_1(s)\sin(\theta) - u_2(s)\cos(\theta))) \\ &= 2\psi' \end{aligned} \tag{43}$$

Now let us figure out $\frac{1}{\lambda^2}$:

$$
\begin{aligned}
\lambda^2 &= || \begin{bmatrix} (1+\rho R(u_1(s)\sin(\theta) - u_2(s)\cos(\theta))) \\ -\rho R\psi'(s)\sin(\theta) \\ \rho R\psi'(s)\cos(\theta) \end{bmatrix} \cdot \begin{bmatrix} \boldsymbol{d}_3 \\ \boldsymbol{d}_1 \\ \boldsymbol{d}_2 \end{bmatrix} ||^2 \\
&= (1+\rho R(u_1(s)\sin(\theta) - u_2(s)\cos(\theta)))^2 + (\rho R\psi'(s)\sin(\theta))^2 + (\rho R\psi'(s)\cos(\theta))^2 \\
&= (1+\rho R(u_1(s)\sin(\theta) - u_2(s)\cos(\theta)))^2 + \rho^2 R^2 \psi'(s)^2
\end{aligned} \tag{44}
$$

So finally:

$$
\alpha = \frac{2\psi'}{(1+\rho R(u_1(s)\sin(\theta) - u_2(s)\cos(\theta)))^2 + \rho^2 R^2 \psi'(s)^2} \tag{45}
$$

Notice that along the central curve, when $\rho = 0$, the curl is $2\psi'$. This seems very strange as you would expect it to be only ψ'. The reason is because the wrapping around the central axis is the same at opposite points of it. So it gets double contribution.

Also due to our choice of field lines, it is clear that:

$$
\begin{aligned}
0 &= \boldsymbol{e}_\rho \cdot \boldsymbol{\nabla} \times \tilde{\boldsymbol{N}} \\
0 &= \frac{\partial \boldsymbol{T}}{\partial \rho} \cdot \boldsymbol{\nabla} \times \tilde{\boldsymbol{N}} \\
&\Leftrightarrow \\
0 &= (g_{22} \cdot dx^2) \wedge (\frac{\partial n_1}{\partial x_2} dx^2 \wedge dx^1 + (\frac{\partial n_1}{\partial x_3} - \frac{\partial n_3}{\partial x_1}) dx^3 \wedge dx^1 + \frac{\partial n_3}{\partial x_2} dx^2 \wedge dx^3) \\
0 &= R^2 (\frac{\partial n_1}{\partial x_3} - \frac{\partial n_3}{\partial x_1}) dx^1 \wedge dx^2 \wedge dx^3 \\
&\Leftrightarrow \\
0 &= R^2 (\frac{\partial n_1}{\partial x_3} - \frac{\partial n_3}{\partial x_1}) \frac{1}{\sqrt{||g_{ij}||}} \\
0 &= 2\rho R(u_1\cos(\theta) + u_2 sin(\theta))(1+\rho R(u_1\sin(\theta) - u_2\cos(\theta))) - \rho^2 R^2 \psi(s)'' \\
0 &= 2(u_1\cos(\theta) + u_2 sin(\theta))(1+\rho R(u_1\sin(\theta) - u_2\cos(\theta))) - \rho R\psi'(s)' \\
0 &= 2u_1\cos(\theta) + 2u_2 sin(\theta) + 2\rho R(u_1\cos(\theta) + u_2 sin(\theta))(u_1\sin(\theta) - u_2\cos(\theta)) - \rho R\psi(s)'' \\
0 &= 2\kappa\cos(\gamma)\cos(\theta) + 2\kappa\sin(\gamma)\sin(\theta) + \\
&\quad 2\rho R(\kappa\cos(\gamma)\cos(\theta) + \kappa\sin(\gamma)\sin(\theta))(\kappa\cos(\gamma)\sin(\theta) - \kappa\sin(\gamma)\cos(\theta)) \\
&\quad - \rho R\psi(s)'' \\
0 &= 2\kappa\cos(\theta-\gamma) + 2\rho R\kappa^2(\cos(\theta-\gamma)\sin(\theta-\gamma)) - \rho R\psi(s)'' \\
\psi(s)'' &= \frac{2\kappa\cos(\theta-\gamma) + 2\rho R\kappa^2(\cos(\theta-\gamma)\sin(\theta-\gamma))}{\rho R}
\end{aligned} \tag{46}
$$

Note that we used $u_1 = \kappa\cos(\gamma)$ and $u_2 = \kappa\sin(\gamma)$

This gives:

$$
\psi(s)'' = \frac{2\kappa(s)\cos(\theta_0 + \psi(s) - \gamma(s))}{\rho R}(1 + \rho R\kappa(s)(\sin(\theta_0 + \psi(s) - \gamma(s))) \tag{47}
$$

Note that this is impossible because by postulate: $\psi(s)$ depends only on s. So this tells us that a force-free magnetic field, whose field lines follow the unit tangent vector space $\boldsymbol{T}$, cannot exist in a curvilinear constant-width tubular manifold.

7.4 Solving in straight cylinder: Constant Radius and ρ dependent twist only: using field lines and orthogonal lines.

Note that in this case the metric tensor is as follows:

$$g_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & R^2 & 0 \\ 0 & 0 & \rho^2 R^2 \end{bmatrix} \quad , \quad \sqrt{|g_{ij}|} = \rho R^2 \tag{48}$$

So in this system:

1. $l = 0$
2. $r = 0$
3. $t = f^{unct}(\rho)$.

Now let us express $\tilde{\boldsymbol{N}}$ and its curl in form:

$$\begin{aligned} n &= 1 \cdot dx^1 + 0dx^2 + n_3 dx^3 \\ &= dx^1 + R^2\rho^2 t dx^3 \\ dn &= \frac{\partial n_1}{\partial x_2} dx^2 \wedge dx^1 + \frac{\partial n_2}{\partial x_1} dx^1 \wedge dx^2 + \frac{\partial n_1}{\partial x_3} dx^3 \wedge dx^1 + \frac{\partial n_3}{\partial x_1} dx^1 \wedge dx^3 + \frac{\partial n_3}{\partial x_2} dx^2 \wedge dx^3 + \frac{\partial n_2}{\partial x_3} dx^3 \wedge dx^2 \\ &= \frac{\partial n_3}{\partial x_1} dx^1 \wedge dx^3 + \frac{\partial n_3}{\partial x_2} dx^2 \wedge dx^3 \\ n \wedge dn &= \frac{\partial n_3}{\partial x_2} dx^1 \wedge dx^2 \wedge dx^3 \end{aligned} \tag{49}$$

However, note that:

$$\frac{\partial n_3}{\partial x_2} = R^2 \frac{\partial(\rho^2 t)}{\partial \rho} = R^2 \rho (2t + \rho \frac{\partial t}{\partial \rho}) \tag{50}$$

So:

$$\begin{aligned} \alpha &= \frac{1}{\lambda^2} \tilde{\boldsymbol{N}} \cdot (\nabla \times \tilde{\boldsymbol{N}}) \\ &= \frac{1}{\rho} \cdot R^2 \rho (2t + \rho \frac{\partial t}{\partial \rho}) \cdot \frac{1}{1 + (\rho R t)^2} \\ &= R^2 \frac{2t + \rho \frac{\partial t}{\partial \rho}}{1 + (\rho R t)^2} \end{aligned} \tag{51}$$

However, let us look at this a little more abstractly, we get:

$$\begin{aligned} \nabla\phi \times \boldsymbol{N} + \phi \nabla \times \boldsymbol{N} &= \alpha\phi \boldsymbol{N} \\ \frac{\partial \boldsymbol{T}}{\partial \rho} \cdot \nabla\phi \times \boldsymbol{N} + \phi \frac{\partial \boldsymbol{T}}{\partial \rho} \cdot \nabla \times \boldsymbol{N} &= \alpha\phi \frac{\partial \boldsymbol{T}}{\partial \rho} \cdot \boldsymbol{N} \\ \nabla\phi \cdot \boldsymbol{N} \times \boldsymbol{e}_\rho + \phi \cdot \boldsymbol{e}_\rho \cdot \nabla \times \boldsymbol{N} &= \alpha\phi \boldsymbol{e}_\rho \cdot \boldsymbol{N} \\ -\nabla\phi \cdot \boldsymbol{e}_\rho \times \boldsymbol{N} + \phi \cdot \boldsymbol{e}_\rho \cdot \nabla \times \boldsymbol{N} &= 0 \\ \phi \cdot \boldsymbol{e}_\rho \cdot \nabla \times \boldsymbol{N} &= \nabla\phi \cdot \boldsymbol{P} \end{aligned} \tag{52}$$

Now let us study this using form:

$$\begin{aligned} \boldsymbol{N} &= \frac{1}{\sqrt{1 + (\rho R t)^2}} (\boldsymbol{e}_s + t \boldsymbol{e}_\theta) \\ \Leftrightarrow n &= \frac{1}{\sqrt{1 + (\rho R t)^2}} dx^1 + \frac{\rho^2 R^2 t}{\sqrt{1 + (\rho R t)^2}} dx^3 \\ \Leftrightarrow dn &= \frac{\partial n_1}{\partial x_2} dx_2 \wedge dx_1 + (\frac{\partial n_1}{\partial x_3} - \frac{\partial n_3}{\partial x_1}) dx_3 \wedge dx_1 + \frac{\partial n_3}{\partial x_2} dx_2 \wedge dx_3 \end{aligned} \tag{53}$$

and:

$$e_\rho = R^2 dx^2 \tag{54}$$

So:

$$\begin{aligned} e_\rho \wedge dn &= R^2 dx^2 \wedge (\frac{\partial n_1}{\partial x_2} dx^2 \wedge dx^1 + (\frac{\partial n_1}{\partial x_3} - \frac{\partial n_3}{\partial x_1}) dx^3 \wedge dx^1 + \frac{\partial n_3}{\partial x_2} dx^2 \wedge dx^3) \\ &= R^2 (\frac{\partial n_1}{\partial x_3} - \frac{\partial n_3}{\partial x_1}) dx^1 \wedge dx^2 \wedge dx^3 \end{aligned} \tag{55}$$

Therefor:

$$\begin{aligned} \boldsymbol{e}_\rho \cdot \nabla \times \boldsymbol{N} &= \frac{1}{\rho} (\frac{\partial n_1}{\partial x_3} - \frac{\partial n_3}{\partial x_1}) \\ &= 0 \end{aligned} \tag{56}$$

Note that $[\nabla \phi]_i = \frac{\partial \phi}{\partial y_i}$, where we choose arbitrary coordinates $\{y_i\}$.

However, just like $\boldsymbol{N} = \frac{d\boldsymbol{L}}{dl} = \frac{d\boldsymbol{f}(l)}{dl}$ along the field lines $\boldsymbol{f}(l)$ where l is the arc-length of $\boldsymbol{f}(l)$, $\boldsymbol{e}_\rho \times \boldsymbol{N} = R \frac{d\boldsymbol{P}}{dt}$ along lines $\boldsymbol{y}(t)$ where t is the arc length of $\boldsymbol{y}(t)$. Note first that thanks to the R normalisation, $\boldsymbol{P}$ is unit length; second that the sets of field lines $\boldsymbol{f}(l)$ and $y(t)$ are orthogonal. So in this case, $[\boldsymbol{e}_\rho \times \boldsymbol{N}]_i = R \frac{dy_i(t)}{dt}$ along the field lines $\boldsymbol{f}(t)$. Finally, we can argue the following:

$$\begin{aligned} \nabla \phi \cdot (\boldsymbol{e}_\rho \times \boldsymbol{N}) &= R \frac{d\phi}{dy_i} \frac{dy_i}{dt} \\ &= R \frac{d\phi}{dt} \end{aligned} \tag{57}$$

So on each of the lines $\boldsymbol{y}t$, we must have:

$$0 = R \frac{d\phi}{dt} \tag{58}$$

Therefor, ϕ is constant on each of these lines.

Also:

$$\begin{aligned} \nabla \cdot \boldsymbol{N} &= \frac{1}{R^2 \rho} \frac{\partial}{\partial x_i} (R^2 \rho N_i) \\ &= \frac{1}{\rho} (\frac{\partial}{\partial s} \frac{\rho}{\sqrt{1 + (\rho R t)^2}}) + \frac{1}{\rho} (\frac{\partial}{\partial \theta} \frac{\rho^3 R^2 t}{\sqrt{1 + (\rho R t)^2}}) \\ &= 0 \end{aligned} \tag{59}$$

Remember t only depends on ρ.

Ok so this means that what we did all along was mostly useless, because ϕ is constant along all the field lines. So to solve our problem we just have to derive conditions on ϕ at the base of our tube. Note however, equation (58) tells us that ϕ is the same along orthogonal lines to the field lines. We guess that these reduce to ϕ having a radial distribution on the base, but we will see.

Also:

$$\begin{aligned}
\boldsymbol{P} &= \frac{1}{R}\boldsymbol{e}_\rho \times \boldsymbol{N} \\
\Leftrightarrow P &= \frac{1}{R}(R^2 dx^2) \wedge (\frac{1}{\sqrt{1+(\rho R t)^2}}dx^1 + \frac{\rho^2 R^2 t}{\sqrt{1+(\rho R t)^2}}dx^3) \\
&= -R\frac{1}{\sqrt{1+(\rho R t)^2}}dx^1 \wedge dx^2 + R\frac{\rho^2 R^2 t}{\sqrt{1+(\rho R t)^2}}dx^2 \wedge dx^3 \\
\Leftrightarrow \boldsymbol{P} &= \frac{1}{R^2\rho}R\frac{\rho^2 R^2 t}{\sqrt{1+(\rho R t)^2}}\boldsymbol{e}_s - \frac{1}{R^2\rho}R\frac{1}{\sqrt{1+(\rho R t)^2}}\boldsymbol{e}_\theta \\
&= \frac{\rho R t}{\sqrt{1+(\rho R t)^2}}\boldsymbol{e}_s - \frac{1}{\rho R}\frac{1}{\sqrt{1+(\rho R t)^2}}\boldsymbol{e}_\theta
\end{aligned} \tag{60}$$

now let us investigate the streamlines $\boldsymbol{y}(s,\rho,\theta)$ in $\boldsymbol{P}$:

$$\begin{cases}
\frac{d}{dt}y_s = P_s = \frac{\rho R t}{\sqrt{1+(\rho R t)^2}} \\
\frac{d}{dt}y_\rho = P_\rho = 0 \\
\frac{d}{dt}y_\theta = P_\theta = -\frac{1}{\rho R\sqrt{1+(\rho R t)^2}}
\end{cases} \tag{61}$$

We don't really care about the first part, all it tells us is that the lines go up. The second part insures that ρ is held constant. The third part however is always negative so there is a uniform twisting of l_θ around the central axis in the opposite direction of the field lines.

Yet because:

1. all the streamlines of $\boldsymbol{P}$ and $\boldsymbol{N}$ are orthogonal
2. ϕ is the same on the streamlines of $\boldsymbol{P}$ and $\boldsymbol{N}$ with same starting point $[s=0,\rho_0,\theta_0]$
3. ρ is held constant for the streamlines of these fields

all field lines starting on a certain ring of constant ρ must cross. Yet ϕ is uniquely defined. So ϕ must be radially distributed on the base of the cylinder.

Now $\nabla\phi$ must be of the form:

$$\begin{cases}
[\nabla\phi]_s = 0 \\
[\nabla\phi]_\rho = \gamma \\
[\nabla\phi]_\theta = 0
\end{cases} \tag{62}$$

Now let us study the dot product with $\boldsymbol{e}_\rho \times \boldsymbol{N}$:

$$\begin{aligned}
\nabla\phi \times \boldsymbol{N} + \phi\nabla \times \boldsymbol{N} &= \alpha\phi\boldsymbol{N} \\
(\boldsymbol{e}_\rho \times \boldsymbol{N})\cdot(\nabla\phi \times \boldsymbol{N}) + \phi(\boldsymbol{e}_\rho \times \boldsymbol{N})\cdot(\nabla \times \boldsymbol{N}) &= \alpha\phi(\boldsymbol{e}_\rho \times \boldsymbol{N})\cdot\boldsymbol{N} \\
(\boldsymbol{e}_\rho \times \boldsymbol{N})\cdot(\nabla\phi \times \boldsymbol{N}) + \phi(\boldsymbol{e}_\rho \times \boldsymbol{N})\cdot(\nabla \times \boldsymbol{N}) &= 0 \\
\nabla\phi\cdot(\boldsymbol{N}\times(\boldsymbol{e}_\rho \times \boldsymbol{N})) + \phi(\boldsymbol{e}_\rho \times \boldsymbol{N})\cdot(\nabla \times \boldsymbol{N}) &= 0 \\
\nabla\phi\cdot\boldsymbol{e}_\rho + \phi(\boldsymbol{e}_\rho \times \boldsymbol{N})\cdot(\nabla \times \boldsymbol{N}) - [\nabla\phi]_N(\boldsymbol{e}_\rho\cdot\boldsymbol{N}) &= 0 \\
-\phi(\boldsymbol{e}_\rho \times \boldsymbol{N})\cdot(\nabla \times \boldsymbol{N}) + 0 &= [\nabla\phi]_\rho
\end{aligned} \tag{63}$$

Using form, let us try to find: $(\boldsymbol{e}_\rho \times \boldsymbol{N})\cdot(\nabla \times \boldsymbol{N})$:

First:

$$\begin{aligned}
dn &= \frac{\partial n_1}{\partial x_2}dx_2 \wedge dx_1 + (\frac{\partial n_1}{\partial x_3} - \frac{\partial n_3}{\partial x_1})dx_3 \wedge dx_1 + \frac{\partial n_3}{\partial x_2}dx_2 \wedge dx_3 \\
\Leftrightarrow \nabla \times \boldsymbol{N} &= \frac{1}{R^2\rho}\frac{\partial n_3}{\partial x_2}\boldsymbol{e}_s - \frac{1}{R^2\rho}\frac{\partial n_1}{\partial x_2}\boldsymbol{e}_\theta \\
&= \frac{\rho\frac{\partial t}{\partial\rho} + R^2\rho^2 t^3 + 2t}{(1+(R\rho t)^2)^{3/2}}\boldsymbol{e}_s + \frac{t(\rho\frac{\partial t}{\partial\rho} + t)}{(1+(R\rho t)^2)^{3/2}}\boldsymbol{e}_\theta
\end{aligned} \tag{64}$$

Second:

$$\begin{aligned} e_\rho \wedge n &= -R^2 \frac{1}{\sqrt{1+(\rho Rt)^2}} dx^1 \wedge dx^2 + R^2 \frac{\rho^2 R^2 t}{\sqrt{1+(\rho Rt)^2}} dx^2 \wedge dx^3 \\ \Leftrightarrow \boldsymbol{e}_\rho \times \boldsymbol{N} &= \frac{\rho R^2 t}{\sqrt{1+(\rho Rt)^2}} \boldsymbol{e}_s - \frac{1}{\rho\sqrt{1+(\rho Rt)^2}} \boldsymbol{e}_\theta \end{aligned} \tag{65}$$

So:

$$\begin{aligned} (\boldsymbol{e}_\rho \times \boldsymbol{N}) \cdot (\nabla \times \boldsymbol{N}) &= \frac{\rho R^2 t}{(1+(R\rho t)^2)^2} \left[\rho \frac{\partial t}{\partial \rho} + R^2 \rho^2 t^3 + 2t - \rho \frac{\partial t}{\partial \rho} - t \right] \\ &= \frac{\rho R^2 t^2}{(1+(R\rho t)^2)^2} \left[R^2 \rho^2 t^2 + 1 \right] \\ &= \frac{\rho R^2 t^2}{(1+(R\rho t)^2)} \end{aligned} \tag{66}$$

So in retrospect, we get (with t as a function of ρ) :

$$\begin{cases} \alpha = R^2 \frac{2t+\rho\frac{\partial t}{\partial \rho}}{1+(\rho Rt)^2} \\ [\nabla\phi]_s = 0 \qquad , \qquad [\nabla\phi]_\rho = -\phi \frac{\rho R^2 t^2}{(1+(R\rho t)^2)} \qquad , \qquad [\nabla\phi]_\theta = 0 \end{cases} \tag{67}$$

In this case, we can choose the distribution $t_0(\rho)$ at the base of the tube and the amplitude of the field at a given ρ, then solve for the rest of the field such that it is force free. We have created a program that does this and solves the equations numerically.
For example, given the base distribution: $t_0(\rho) = \sin(8\pi\rho)$ and $\phi(0) = 1$. In this case we get:

This is great but it does not quite help us in our original goal, as we cannot build nots with these solutions. Indeed, topologically, all fields of this type can be continuously deformed into a straight cylinder; and additionally, overlapping them is difficult due to the varying α. But this solution is a victory none the less.

7.5 Solving in straight Cylinder: Constant Radius with twist depending on ρ and s

Please note that this section is erroneous. Indeed, I assumed that $\nabla\phi \cdot \boldsymbol{N} = 0$ (see eqn.70). This is unreasonable, as the twist of the field depends on s. However, this section uses Fourier Analysis and is quite interesting none-the-less. Therefor, I have decided to include it.

For the First condition, we can simply reuse the general expression of alpha from before:

$$\alpha = \frac{\rho \frac{\partial t}{\partial \rho} + 2t}{1+(\rho Rt)^2} \tag{68}$$

Before going further, let us establish the following:

$$\begin{aligned} dn &= \frac{\partial n_1}{\partial x_2} dx^2 \wedge dx^1 + \frac{\partial n_1}{\partial x_3} dx^3 \wedge dx^1 + \frac{\partial n_3}{\partial x_1} dx^1 \wedge dx^3 + \frac{\partial n_3}{\partial x_2} dx^2 \wedge dx^3 \\ &= \frac{\partial n_1}{\partial x_2} dx^2 \wedge dx^1 + (\frac{\partial n_1}{\partial x_3} - \frac{\partial n_3}{\partial x_1}) dx^3 \wedge dx^1 + \frac{\partial n_3}{\partial x_2} dx^2 \wedge dx^3 \\ \Leftrightarrow \nabla \times \boldsymbol{N} &= \frac{1}{R^2\rho} \frac{\partial n_3}{\partial x_2} \boldsymbol{e}_s + \frac{1}{R^2\rho} (\frac{\partial n_1}{\partial x_3} - \frac{\partial n_3}{\partial x_1}) \boldsymbol{e}_\rho - \frac{1}{R^2\rho} \frac{\partial n_1}{\partial x_2} \boldsymbol{e}_\theta \\ &= \frac{\rho\frac{\partial t}{\partial \rho} + R^2\rho^2 t^3 + 2t}{(1+(R\rho t)^2)^{3/2}} \boldsymbol{e}_s + \frac{\rho\frac{\partial t}{\partial s} - \rho t \frac{\partial t}{\partial s}}{(1+(\rho Rt)^2)^{3/2}} \boldsymbol{e}_\rho + \frac{t(\rho\frac{\partial t}{\partial \rho} + t)}{(1+(R\rho t)^2)^{3/2}} \boldsymbol{e}_\theta \end{aligned} \tag{69}$$

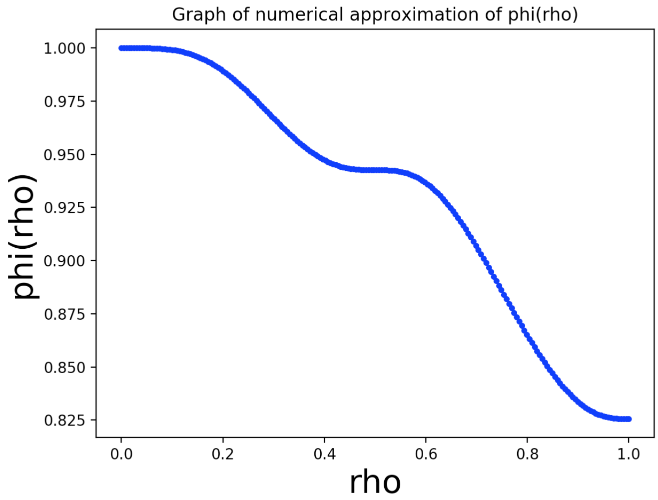

Figure 1: Graph of the amplitude of the field $\phi(\rho)$

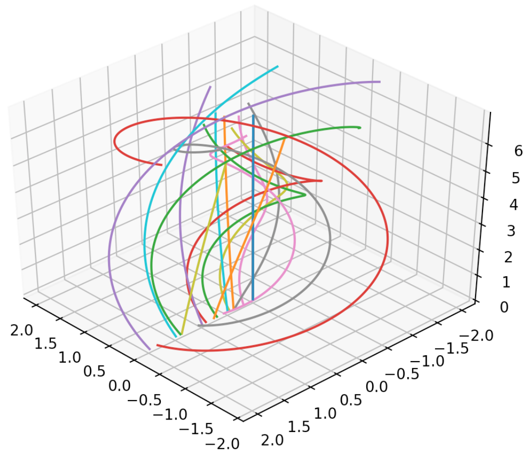

Figure 2: Visualisation of the field lines from side

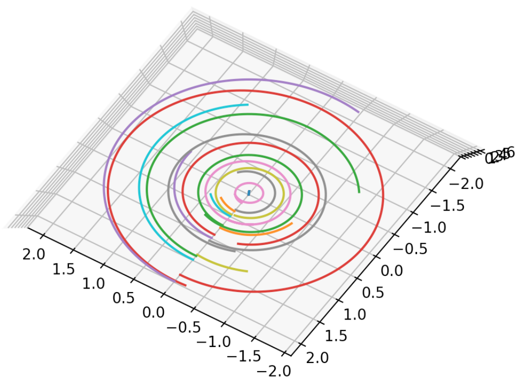

Figure 3: Visualisation of the field lines from top

Now, Let us study the dot product with $\boldsymbol{e}_\rho \times \boldsymbol{N}$, $\boldsymbol{e}_s \times \boldsymbol{N}$, and $\boldsymbol{e}_\theta \times \boldsymbol{N}$

A ——— with $\boldsymbol{e}_\rho \times \boldsymbol{N}$:

$$\begin{aligned}
\nabla\phi \times \boldsymbol{N} + \phi\nabla \times \boldsymbol{N} &= \alpha\phi\boldsymbol{N} \\
(\boldsymbol{e}_\rho \times \boldsymbol{N}) \cdot (\nabla\phi \times \boldsymbol{N}) + \phi(\boldsymbol{e}_\rho \times \boldsymbol{N}) \cdot (\nabla \times \boldsymbol{N}) &= \alpha\phi(\boldsymbol{e}_\rho \times \boldsymbol{N}) \cdot \boldsymbol{N} \\
(\boldsymbol{e}_\rho \times \boldsymbol{N}) \cdot (\nabla\phi \times \boldsymbol{N}) + \phi(\boldsymbol{e}_\rho \times \boldsymbol{N}) \cdot (\nabla \times \boldsymbol{N}) &= 0 \\
\nabla\phi \cdot (\boldsymbol{N} \times (\boldsymbol{e}_\rho \times \boldsymbol{N})) + \phi(\boldsymbol{e}_\rho \times \boldsymbol{N}) \cdot (\nabla \times \boldsymbol{N}) &= 0 \\
\nabla\phi\boldsymbol{e}_\rho + \phi(\boldsymbol{e}_\rho \times \boldsymbol{N}) \cdot (\nabla \times \boldsymbol{N}) &= 0 \\
\phi(\boldsymbol{e}_\rho \times \boldsymbol{N}) \cdot (\nabla \times \boldsymbol{N}) &= -[\nabla\phi]_\rho
\end{aligned} \tag{70}$$

Now:

$$\begin{aligned}
e_\rho \wedge n &= -R^2\frac{1}{\sqrt{1+(\rho Rt)^2}}dx^1 \wedge dx^2 + R^2t\sqrt{1+(\rho Rt)^2}dx^2 \wedge dx^3 \\
\Leftrightarrow \boldsymbol{e}_\rho \times \boldsymbol{N} &= \frac{\rho R^2 t}{\sqrt{1+(\rho Rt)^2}}\boldsymbol{e}_s - \frac{1}{\rho\sqrt{1+(\rho Rt)^2}}\boldsymbol{e}_\theta
\end{aligned} \tag{71}$$

So:

$$\begin{aligned}
(\boldsymbol{e}_\rho \times \boldsymbol{N}) \cdot (\nabla \times \boldsymbol{N}) &= \frac{\rho R^2 t}{(1+(R\rho t)^2)^2}\left[\rho\frac{\partial^2\theta^c(s,\rho,\theta)}{\partial s\partial\rho} + R^2\rho^2t^3 + 2t \right.\\
&\qquad \left. - \rho\frac{\partial^2\theta^c(s,\rho,\theta)}{\partial s\partial\rho} - t\right] \\
&= \frac{\rho R^2(\frac{\partial\theta^c(s,\rho_0,\theta_0)}{\partial s})^2}{(1+(R\rho t)^2)^2}\left[R^2\rho^2t^2 + 1\right] \\
&= \frac{\rho R^2(\frac{\partial\theta^c(s,\rho_0,\theta_0)}{\partial s})^2}{1+(R\rho t)^2}
\end{aligned} \tag{72}$$

B ——— Similarly with $\boldsymbol{e}_s \times \boldsymbol{N}$, we get:

$$\phi(\boldsymbol{e}_s \times \boldsymbol{N}) \cdot (\nabla \times \boldsymbol{N}) = -[\nabla\phi]_s \tag{73}$$

Now:

$$\begin{aligned}
e_s \wedge n &= \frac{\rho^2 R^2 \frac{d\theta^c(s,\rho,\theta)}{ds}}{\sqrt{1+(\rho Rt)^2}}dx^1 \wedge dx^3 \\
\Leftrightarrow \boldsymbol{e}_s \times \boldsymbol{N} &= -\frac{\rho\frac{d\theta^c(s,\rho,\theta)}{ds}}{\sqrt{1+(\rho Rt)^2}}\boldsymbol{e}_\rho
\end{aligned} \tag{74}$$

So:

$$\begin{aligned}
(\boldsymbol{e}_s \times \boldsymbol{N}) \cdot (\nabla \times \boldsymbol{N}) &= -R^2\frac{\rho\frac{\partial\theta^c(s,\rho,\theta)}{\partial s}}{\sqrt{1+(\rho Rt)^2}} \cdot \frac{\rho\frac{\partial^2\theta^c(s,\rho,\theta)}{\partial s^2} - \rho t\frac{\partial^2\theta^c(s,\rho,\theta)}{\partial s\partial\theta}}{(1+(\rho Rt)^2)^{3/2}} \\
&= R^2\rho^2\frac{t^2\frac{\partial^2\theta^c(s,\rho,\theta)}{\partial s\partial\theta} - \frac{\partial\theta^c(s,\rho,\theta)}{\partial s}\frac{\partial^2\theta^c(s,\rho,\theta)}{\partial s^2}}{(1+(\rho Rt)^2)^2}
\end{aligned} \tag{75}$$

C ——— Similarly with $\boldsymbol{e}_\theta \times \boldsymbol{N}$, we get:

$$\phi(\boldsymbol{e}_\theta \times \boldsymbol{N}) \cdot (\nabla \times \boldsymbol{N}) = -[\nabla\phi]_\theta \tag{76}$$

Now:

$$\begin{aligned}
e_\theta \wedge n &= (\rho^2R^2dx^3) \wedge (\frac{1}{\sqrt{1+(\rho Rt)^2}}dx^1 + \frac{\rho^2R^2\frac{d\theta^c(s,\rho,\theta)}{ds}}{\sqrt{1+(\rho Rt)^2}}dx^3) \\
&= \frac{\rho^2R^2}{\sqrt{1+(\rho Rt)^2}}dx^3 \wedge dx^1 \\
\Leftrightarrow \boldsymbol{e}_\theta \times \boldsymbol{N} &= \frac{\rho}{\sqrt{1+(\rho Rt)^2}}\boldsymbol{e}_\rho
\end{aligned} \tag{77}$$

So:

$$(\boldsymbol{e}_\theta \times \boldsymbol{N}) \cdot (\nabla \times \boldsymbol{N}) = \frac{\rho}{\sqrt{1+(\rho R t)^2}} \boldsymbol{e}_\rho \cdot \frac{\rho \frac{\partial^2 \theta^c(s,\rho,\theta)}{\partial s^2} - \rho t \frac{\partial^2 \theta^c(s,\rho,\theta)}{\partial s \partial \theta}}{(1+(\rho R t)^2)^{3/2}} \boldsymbol{e}_\rho$$
$$= R^2\rho^2 \frac{(\frac{\partial^2 \theta^c(s,\rho,\theta)}{\partial s^2} - t\frac{\partial^2 \theta^c(s,\rho,\theta)}{\partial s \partial \theta})}{(1+(\rho R t)^2)^2} \tag{78}$$

So finally, we get he set of condition:

$$\begin{cases} [\nabla \phi]_s = -\phi \cdot R^2\rho^2 \frac{t^2 \frac{\partial^2 \theta^c(s,\rho,\theta)}{\partial s \partial \theta} - \frac{\partial \theta^c(s,\rho,\theta)}{\partial s} \frac{\partial^2 \theta^c(s,\rho,\theta)}{\partial s^2}}{(1+(\rho R t)^2)^2} \\ [\nabla \phi]_\rho = -\phi \cdot \rho R^2 \frac{t^2}{1+(R\rho t)^2} \\ [\nabla \phi]_\theta = -\phi \cdot R^2\rho^2 \frac{(\frac{\partial^2 \theta^c(s,\rho,\theta)}{\partial s^2} - t\frac{\partial^2 \theta^c(s,\rho,\theta)}{\partial s \partial \theta})}{(1+(\rho R t)^2)^2} \end{cases} \tag{79}$$

Which is:

$$\begin{cases} [\nabla \phi]_s = -\phi \cdot R^2\rho^2 \frac{\psi^2 \frac{\partial \psi}{\partial \theta} - \psi \frac{\partial \psi}{\partial s}}{(1+(\rho R \psi)^2)^2} & (A) \\ [\nabla \phi]_\rho = -\phi \cdot \rho R^2 \frac{\psi^2}{1+(R\rho\psi)^2} & (B) \\ [\nabla \phi]_\theta = -\phi \cdot R^2\rho^2 \frac{(\frac{\partial \psi}{\partial s} - \psi \frac{\partial \psi}{\partial \theta})}{(1+(\rho R \psi)^2)^2} & (C) \end{cases} \tag{80}$$

Ok so the problem now, is that we haven't included the 2π periodicity constraint on ψ or ϕ. Let us start of with assuming that ϕ is 2π periodic with respect to the θ parameter. So let us assume then that our magnetic field takes the form:

$$\boldsymbol{B} = \phi(s,\rho,\theta) \cdot \boldsymbol{N}(s,\rho,\theta)$$
$$\phi(s,\rho,\theta) = \sum_{n=0}^{\infty} \Big[\phi_n^1(s,\rho) \cdot \cos(n\theta) + \phi_n^2(s,\rho) \cdot \sin(n\theta) \Big] \tag{81}$$

So maybe given restrictions on the function ϕ, we can derive conditions on ψ. So now, we get:

A ————————

$$0 = \sum_{n=0}^{\infty} \Big[\cos(n\theta)(\frac{\partial \phi_n^1(s,\rho)}{\partial s} + \phi_n^1(s,\rho) R^2\rho^2 \frac{\psi^2 \frac{\partial \psi}{\partial \theta} - \psi \frac{\partial \psi}{\partial s}}{(1+(\rho R \psi)^2)^2}) +$$
$$\sin(n\theta)(\frac{\partial \phi_n^2(s,\rho)}{\partial s} + \phi_n^2(s,\rho) R^2\rho^2 \frac{\psi^2 \frac{\partial \psi}{\partial \theta} - \psi \frac{\partial \psi}{\partial s}}{(1+(\rho R \psi)^2)^2}) \Big] \tag{82}$$
$$\Leftrightarrow \begin{cases} 0 = \frac{\partial \phi_n^1(s,\rho)}{\partial s} + \phi_n^1(s,\rho) R^2\rho^2 \frac{\psi^2 \frac{\partial \psi}{\partial \theta} - \psi \frac{\partial \psi}{\partial s}}{(1+(\rho R \psi)^2)^2} \\ 0 = \frac{\partial \phi_n^2(s,\rho)}{\partial s} + \phi_n^2(s,\rho) R^2\rho^2 \frac{\psi^2 \frac{\partial \psi}{\partial \theta} - \psi \frac{\partial \psi}{\partial s}}{(1+(\rho R \psi)^2)^2} \end{cases}$$

B ——————

$$0 = \sum_{n=0}^{\infty} \Big[\cos(n\theta)(\frac{\partial \phi_n^1(s,\rho)}{\partial \rho} + \phi_n^1(s,\rho)\rho R^2 \frac{\psi^2}{1+(R\rho\psi)^2}) + \sin(n\theta)(\frac{\partial \phi_n^2(s,\rho)}{\partial \rho} + \phi_n^2(s,\rho)\rho R^2 \frac{\psi^2}{1+(R\rho\psi)^2})\Big] \tag{83}$$

$$\Leftrightarrow \begin{cases} 0 = \frac{\partial \phi_n^1(s,\rho)}{\partial \rho} + \phi_n^1(s,\rho)\rho R^2 \frac{\psi^2}{1+(R\rho\psi)^2} \\ 0 = \frac{\partial \phi_n^2(s,\rho)}{\partial \rho} + \phi_n^2(s,\rho)\rho R^2 \frac{\psi^2}{1+(R\rho\psi)^2} \end{cases}$$

C ——————

$$0 = \sum_{n=0}^{\infty} \Big[\cos(n\theta)(n\phi_n^2(s,\rho) + \phi_n^1(s,\rho)R^2\rho^2 \frac{(\frac{\partial\psi}{\partial s} - \psi\frac{\partial\psi}{\partial\theta})}{(1+(\rho R\psi)^2)^2}) + \sin(n\theta)(-n\phi_n^1(s,\rho) + \phi_n^2(s,\rho)R^2\rho^2 \frac{(\frac{\partial\psi}{\partial s} - \psi\frac{\partial\psi}{\partial\theta})}{(1+(\rho R\psi)^2)^2})\Big] \tag{84}$$

$$\Leftrightarrow \begin{cases} 0 = n\phi_n^2(s,\rho) + \phi_n^1(s,\rho)R^2\rho^2 \frac{(\frac{\partial\psi}{\partial s} - \psi\frac{\partial\psi}{\partial\theta})}{(1+(\rho R\psi)^2)^2} \\ 0 = -n\phi_n^1(s,\rho) + \phi_n^2(s,\rho)R^2\rho^2 \frac{(\frac{\partial\psi}{\partial s} - \psi\frac{\partial\psi}{\partial\theta})}{(1+(\rho R\psi)^2)^2} \end{cases}$$

Combining the equations from C together, we can derive the conditions for $n \neq 0$:

$$\begin{cases} S = R^2\rho^2 \frac{(\frac{\partial\psi}{\partial s} - \psi\frac{\partial\psi}{\partial\theta})}{(1+(\rho R\psi)^2)^2} \\ \phi_n^2 = \phi_n^1(s,\rho)(-\frac{S}{n}) \\ \phi_n^1(s,\rho) = \phi_n^2(\frac{S}{n}) \end{cases} \Rightarrow \begin{cases} \phi_n^2 = -(\frac{S}{n})^2\phi_n^2 \\ \phi_n^1 = -(\frac{S}{n})^2\phi_n^1 \end{cases} \Rightarrow \begin{cases} \phi_n^2 = 0 \\ \phi_n^1 = 0 \end{cases} \tag{85}$$

And for $n = 0$, we get:

$$0 = R^2\rho^2 \frac{(\frac{\partial\psi}{\partial s} - \psi\frac{\partial\psi}{\partial\theta})}{(1+(\rho R\psi)^2)^2} \qquad \frac{\partial\psi}{\partial s} = \psi\frac{\partial\psi}{\partial\theta} \tag{86}$$

But looking back at our Curl condition... it has a $\boldsymbol{e}_\rho$ condition which we are saying is impossible because our field lines don't bulge. So we need in any case, we need $\frac{\partial\psi}{\partial s} = \psi\frac{\partial\psi}{\partial\theta}$.

So:

$$\phi(s,\rho,\theta) = \phi(s,\rho) \tag{87}$$

$$\begin{cases} \frac{\partial\psi}{\partial s} = \psi\frac{\partial\psi}{\partial\theta} \\ 0 = \frac{\partial\phi}{\partial s} + \phi\rho^2R^2\frac{\psi^2\frac{\partial\psi}{\partial\theta} - \psi\frac{\partial\psi}{\partial s}}{(1+(\rho R\psi)^2)^2} = \frac{\partial\phi}{\partial s} + 0 \\ 0 = \frac{\partial\phi}{\partial\rho} + \phi\rho R^2\frac{\psi^2}{1+(R\rho\psi)^2} \end{cases} \tag{88}$$

Then ϕ is independent of s and θ.

Let us quickly look at the divergence condition:

$$\begin{aligned}
0 =& \nabla \cdot \boldsymbol{B} \\
=& \frac{1}{R^2 \rho} \frac{\partial}{\partial x_i} (R^2 \rho N_i \phi) \\
=& \frac{1}{\rho} (\frac{\partial}{\partial s} \frac{\phi \rho}{\sqrt{1 + (\rho R t)^2}}) + \frac{1}{\rho} (\frac{\partial}{\partial \theta} \frac{\phi \rho^3 R^2 \frac{d\theta^c(s,\rho,\theta)}{ds}}{\sqrt{1 + (\rho R t)^2}}) \\
=& -\frac{\partial \psi}{\partial s} \psi \phi + \psi^2 \frac{\partial \phi}{\partial s} + \frac{\partial \phi}{\partial s} + \phi \frac{\partial \psi}{\partial \theta} + r^2 R^2 \psi^3 \frac{\partial \phi}{\partial \theta} + \psi \frac{\partial \phi}{\partial \theta} \\
\frac{\partial \psi}{\partial s} \psi \phi =& \phi \frac{\partial \psi}{\partial \theta} \\
\frac{\partial \psi}{\partial s} \psi =& \frac{\partial \psi}{\partial \theta}
\end{aligned} \tag{89}$$

Ok so let us try to solve this using separation of variables:

$$\begin{aligned}
\psi(s, \rho, \theta) &= S(s, \rho) \cdot T(\theta, \rho) \\
&= S \cdot T
\end{aligned} \tag{90}$$

$$\begin{aligned}
\frac{\partial \psi}{\partial s} \psi &= \frac{\partial \psi}{\partial \theta} \\
\Leftrightarrow \dot{S} S T^2 &= S T' \\
\dot{S} T^2 &= T' \\
\dot{S} =& \frac{T'}{T^2} = C_1(\rho) \\
\Leftrightarrow S(s, \rho) = C_1(\rho) s \qquad \& \qquad & T(\theta, \rho) = \frac{1}{C_2(\rho) + C_1(\rho) \theta}
\end{aligned} \tag{91}$$

$$\Leftrightarrow \psi = \frac{C_1(\rho) s}{C_2(\rho) + C_1(\rho) \theta} \tag{92}$$

This is not possible as there is a singularity around $\theta \equiv 0 \equiv 2\pi$.
So let us try to express ϕ as a fourrier series of θ and use that.

$$\psi = \sum_{n=0}^{\infty} C_n^1(s, \rho) \cos(n\theta) + C_n^2(s, \rho) \sin(n\theta) \tag{93}$$

$$\begin{aligned}
\frac{\partial \psi}{\partial s} \psi &= \frac{\partial \psi}{\partial \theta} \\
\Leftrightarrow \left(\sum_{n=0}^{\infty} -C_n^1 n \sin(n\theta) + C_n^2 n \cos(n\theta) \right) &= \left(\sum_{n=0}^{\infty} C_n^1 \cos(n\theta) + C_n^2 \sin(n\theta) \right) \left(\sum_{n=0}^{\infty} \frac{\partial C_n^1}{\partial s} \cos(n\theta) + \frac{\partial C_n^2}{\partial s} \sin(n\theta) \right)
\end{aligned} \tag{94}$$

Ok... We don't know how to solve this, so let us bac up and look at the wider picture. We know that $0 = \frac{\partial \phi}{\partial \rho} + \phi \rho R^2 \frac{\psi^2}{1 + (R\rho\psi)^2}$, but also that ϕ is independent of s and θ. This implies that: $\frac{\psi^2}{1 + (R\rho\psi)^2}$ is independent of s and θ. This in turn implies that ψ is independent of s and θ. So we boil down to our original solution that adds nothing to the problem.

7.6 Solving in straight cylinder: Constant Radius and θ independent

In this section, we will be search for solutions in a non-straight curvy-linear cylinder by making assumptions on the metric or the field itself.

The tensor metric is now as follows:

$$g_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & R^2 & 0 \\ 0 & 0 & \rho^2 R^2 \end{bmatrix} \quad , \quad \sqrt{|g_{ij}|} = \rho R^2 \tag{95}$$

First we establish that our system is independent of θ. So we get:

$$\begin{aligned} \boldsymbol{N} &= \frac{1}{\lambda}\tilde{\boldsymbol{N}} \\ &= \frac{1}{\lambda}(\frac{d\xi^c(s,\rho,\theta)}{ds}\boldsymbol{e}_s + \frac{\partial \rho^c(s,\rho)}{\partial s}\boldsymbol{e}_\rho + \frac{\partial \theta^c(s,\rho)}{\partial s}\boldsymbol{e}_\theta) \\ &= \frac{1}{\lambda}(l(s,\rho)\boldsymbol{e}_s + r(s,\rho)\boldsymbol{e}_\rho + t(s,\rho)\boldsymbol{e}_\theta) \\ &= \frac{l}{\sqrt{l^2+(Rr)^2+(\rho R t)^2}}\boldsymbol{e}_s + \frac{r}{\sqrt{l^2+(Rr)^2+(\rho R t)^2}}\boldsymbol{e}_\rho + \frac{t}{\sqrt{l^2+(Rr)^2+(\rho R t)^2}}\boldsymbol{e}_\theta \end{aligned} \tag{96}$$

To find the α term, let us express $\tilde{\boldsymbol{N}}$ and its curl in form:

$$\begin{aligned} n &= n_1 dx^1 + n_2 dx^2 + n_3 dx^3 \\ dn &= \frac{\partial n_1}{\partial x_2}dx^2\wedge dx^1 + \frac{\partial n_2}{\partial x_1}dx^1\wedge dx^2 + \frac{\partial n_1}{\partial x_3}dx^3\wedge dx^1 + \frac{\partial n_3}{\partial x_1}dx^1\wedge dx^3 + \frac{\partial n_3}{\partial x_2}dx^2\wedge dx^3 + \frac{\partial n_2}{\partial x_3}dx^3\wedge dx^2 \\ &= \frac{\partial n_3}{\partial x_2}dx^2\wedge dx^3 - \frac{\partial n_3}{\partial x_1}dx^3\wedge dx^1 + (\frac{\partial n_2}{\partial x_1} - \frac{\partial n_1}{\partial x_2})dx^1\wedge dx^2 \\ n\wedge dn &= \left[n_1\frac{\partial n_3}{\partial x_2} - n_2\frac{\partial n_3}{\partial x_1} + n_3\left(\frac{\partial n_2}{\partial x_1} - \frac{\partial n_1}{\partial x_2}\right)\right] dx^1\wedge dx^2\wedge dx^3 \end{aligned} \tag{97}$$

So:

$$\alpha = \frac{1}{R^2\rho(l^2+R^2r^2+\rho^2R^2t^2)}\left[l\frac{\partial t}{\partial \rho} - r\frac{\partial t}{\partial s} + n_3\frac{\partial r}{\partial s} - n_3\frac{\partial l}{\partial s}\right] \tag{98}$$

Before we go further, note that we are now dealing with $\boldsymbol{N}$, not $\tilde{\boldsymbol{N}}$:

$$\begin{aligned} n &= n_1 dx^1 + n_2 dx^2 + n_3 dx^3 \\ &= \frac{1}{\sqrt{l^2+(Rr)^2+(\rho R t)^2}}dx^1 + \frac{R^2 r}{\sqrt{l^2+(Rr)^2+(\rho R t)^2}}dx^2 + \frac{R^2\rho^2 t}{\sqrt{l^2+(Rr)^2+(\rho R t)^2}}dx^3 \\ dn &= \frac{\partial n_1}{\partial x_2}dx^2\wedge dx^1 + \frac{\partial n_2}{\partial x_1}dx^1\wedge dx^2 + \frac{\partial n_1}{\partial x_3}dx^3\wedge dx^1 + \frac{\partial n_3}{\partial x_1}dx^1\wedge dx^3 + \frac{\partial n_3}{\partial x_2}dx^2\wedge dx^3 + \frac{\partial n_2}{\partial x_3}dx^3\wedge dx^2 \\ &= \frac{\partial n_3}{\partial x_2}dx^2\wedge dx^3 - \frac{\partial n_3}{\partial x_1}dx^3\wedge dx^1 + (\frac{\partial n_2}{\partial x_1} - \frac{\partial n_1}{\partial x_2})dx^1\wedge dx^2 \\ \nabla\times\boldsymbol{N} &= \frac{1}{R^2\rho}\frac{\partial n_3}{\partial x_2}\boldsymbol{e}_s - \frac{1}{R^2\rho}\frac{\partial n_3}{\partial x_1}\boldsymbol{e}_\rho + \frac{1}{R^2\rho}(\frac{\partial n_2}{\partial x_1} - \frac{\partial n_1}{\partial x_2})\boldsymbol{e}_\theta \end{aligned} \tag{99}$$

Now, Let us study the dot product with $\boldsymbol{e}_\rho\times\boldsymbol{N}$, $\boldsymbol{e}_s\times\boldsymbol{N}$, and $\boldsymbol{e}_\theta\times\boldsymbol{N}$

A ——— with $\boldsymbol{e}_\rho\times\boldsymbol{N}$:

$$\begin{aligned}
e_\rho \wedge n &= (R^2 dx^2) \wedge (\frac{l}{\sqrt{l^2+(Rr)^2+(\rho Rt)^2}}dx^1 + \frac{R^2 r}{\sqrt{l^2+(Rr)^2+(\rho Rt)^2}}dx^2 + \frac{R^2\rho^2 t}{\sqrt{l^2+(Rr)^2+(\rho Rt)^2}}dx^3) \\
&= -\frac{lR^2}{\sqrt{l^2+(Rr)^2+(\rho Rt)^2}}dx^1 \wedge dx^2 + \frac{R^4\rho^2 t}{\sqrt{l^2+(Rr)^2+(\rho Rt)^2}}dx^2 \wedge dx^3) \\
\Leftrightarrow \boldsymbol{e}_\rho \times \boldsymbol{N} &= \frac{R^2\rho t}{\sqrt{l^2+(Rr)^2+(\rho Rt)^2}}\boldsymbol{e}_s - \frac{l}{\rho\sqrt{l^2+(Rr)^2+(\rho Rt)^2}}\boldsymbol{e}_\theta
\end{aligned} \tag{100}$$

Using Mathematica, we can now calculate: $(\boldsymbol{e}_\rho \times \boldsymbol{N}) \cdot (\nabla \times \boldsymbol{N})$.

B ——— with $\boldsymbol{e}_s \times \boldsymbol{N}$:

$$\begin{aligned}
e_s \wedge n &= dx^1 \wedge (\frac{l}{\sqrt{l^2+(Rr)^2+(\rho Rt)^2}}dx^1 + \frac{R^2 r}{\sqrt{l^2+(Rr)^2+(\rho Rt)^2}}dx^2 + \frac{R^2\rho^2 t}{\sqrt{l^2+(Rr)^2+(\rho Rt)^2}}dx^3) \\
&= \frac{R^2 r}{\sqrt{l^2+(Rr)^2+(\rho Rt)^2}}dx^1 \wedge dx^2 - \frac{R^2\rho^2 t}{\sqrt{l^2+(Rr)^2+(\rho Rt)^2}}dx^3 \wedge dx^1 \\
\Leftrightarrow \boldsymbol{e}_s \times \boldsymbol{N} &= -\frac{\rho t}{\sqrt{l^2+(Rr)^2+(\rho Rt)^2}}\boldsymbol{e}_\rho + \frac{r}{\rho\sqrt{l^2+(Rr)^2+(\rho Rt)^2}}\boldsymbol{e}_\theta
\end{aligned} \tag{101}$$

Using Mathematica, we can now calculate: $(\boldsymbol{e}_s \times \boldsymbol{N}) \cdot (\nabla \times \boldsymbol{N})$.

C ——— with $\boldsymbol{e}_\theta \times \boldsymbol{N}$:

$$\begin{aligned}
e_\theta \wedge n &= R^2\rho^2 dx^3 \wedge (\frac{l}{\sqrt{l^2+(Rr)^2+(\rho Rt)^2}}dx^1 + \frac{R^2 r}{\sqrt{l^2+(Rr)^2+(\rho Rt)^2}}dx^2 + \frac{R^2\rho^2 t}{\sqrt{l^2+(Rr)^2+(\rho Rt)^2}}dx^3) \\
&= \frac{lR^2\rho^2}{\sqrt{l^2+(Rr)^2+(\rho Rt)^2}}dx^3 \wedge dx^1 - \frac{R^4\rho^2 r}{\sqrt{l^2+(Rr)^2+(\rho Rt)^2}}dx^2 \wedge dx^3 \\
\Leftrightarrow \boldsymbol{e}_\theta \times \boldsymbol{N} &= -\frac{R^2\rho r}{\sqrt{l^2+(Rr)^2+(\rho Rt)^2}}\boldsymbol{e}_s + \frac{l\rho}{\sqrt{l^2+(Rr)^2+(\rho Rt)^2}}\boldsymbol{e}_\rho
\end{aligned} \tag{102}$$

Using Mathematica, we can now calculate: $(\boldsymbol{e}_\theta \times \boldsymbol{N}) \cdot (\nabla \times \boldsymbol{N})$.

Realizing that the next step implies finding an expression for $\boldsymbol{N} \cdot \nabla\phi$, we will abandon this route as it will most likely lead to a complex expression with no easy to find solution (if any).

8 Conclusion of the project

In conclusion, this project is inconclusive. Rather than offering a solution (as attempted), this paper rather provides techniques to study this problem: notably the proper use of form, the triplet of equation that summarizes this solution, useful restrictions on the system, use of the Lie condition, use of field lines and orthogonal lines and use of Fourier-form solutions.

The only solution we managed to find is a simple field with twist in a straight cylinder. Although this is a solution, it does not actually provide much insight as it can continuously deformed into a straight cylinder; and additionally, we cannot overlap these fields as the α varies. So it is impossible to compose knots with it. The other attempts at solving the problem result in either impossibilities or difficult to solve equations.

What this paper shows, is that this problem is not at all as straight forward as anticipated. These equations do not reduce nicely when you constrain them, and the resulting system of equations is often a system of PDEs which involves a variety of parameters and directional derivatives that are difficult to treat.

9 Personal thoughts on the project

This project was a success despite it being inconclusive.

First of all, I learned so much mathematical content in the realm of Differential Calculus and how to actually apply it: What is a manifold? What is a tensor metric? What is form? How to take a problem and represent it using maths? I also learned how to use Wolfram Alpha Mathematica which was a interesting experience as it is not a linear language. But this is an expected outcome of my project.

More importantly though, I gained assurance when communicating with other academics during our weekly presentations with the Magneto-Hydro-Dynamics research group. I practiced explaining my thoughts clearly and concisely and understanding the thoughts of others: what information is inferred, how to ask pertinent questions and so on.

A consequence of this project I had not anticipated at all however, was my improvement in organisation. Notably, I stopped writing things by hand, and typed everything in LaTeX (I have over 40 pages of notes) ; this way, everything I write down is saved and can be recovered, allowing to look back on past attempts, copy-past formulas is easier and print out and show other people in a clear layout. This way I became completely fluent in LaTeX which is a useful skill for the future. Furthermore, I learned to organise my time and my work pace in the context of actually work rather than university study and homework.

Lastly, I gained confidence to the point of being able to challenge my supervisor's arguments and converse with him critically. I realised that research is not going from point A to point B, but taking a bunch of side-roads, hitting dead-ends, foraging for tools to use and clear a new path. In fact my paper is just that: evidence that this topic is much more complex than one could think. These equations do not reduce to an easy-to-solve system of PDEs. I learned to work days just to hit a dead end and start again from scratch. But as time went on, I learned to spot early signs that would indicate that this method would not lead anywhere. As my supervisor put it so neatly: "It builds character"; and I agree completely.

References

[1] C. Prior, A.R. Yeates *"Twisted versus braided magnetic flux ropes in coronal geometry: I. construction and relaxation"* Astronomy & Astrophysics manuscript, January 20, 2016