PYU44C01 Assignment 2

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Problem 1

The script *ahackett_Assignment_2.py* was utilized to determine the rank of the matrices in question, and hence to determine whether or not an inverse existed. For matrix A1, the rank was determined to be 2. In order to be full rank, the rank of this matrix would have to be equal to the number of columns/rows, whichever is lower, in this case, three columns. Hence, this matrix is rank deficient and so neither a left nor a right inverse exist, since neither the number of columns, nor the number of rows, equal the matrix rank. (These are the conditions for the existence of a left or right inverse respectively) A pseudo-inverse, A^+ exists in general for every matrix, and is defined in this case through the singular value decomposition. If $A1 = U\Sigma V^T$ then $A^+ = V\Sigma^\dagger U^T$. Computing the SVD and hence the pseudo-inverse (Moore-Penrose inverse) gave:

$$\mathrm{A1}^+ = \begin{bmatrix} 0.05903166 & 0.05027933 & 0.03258845 & 0.03054004 \\ 0.02253259 & 0.02234637 & 0.10707635 & 0.04320298 \\ 0.03649907 & 0.02793296 & -0.0744879 & -0.01266294 \end{bmatrix}$$

For matrix A2, the rank of the matrix was found to be 3, hence the matrix was full rank. Since the rank of the matrix equaled the number of columns, there existed a left inverse of the matrix. Computing the left inverse directly via $A^{-1}_{left} = (A^TA)^{-1}A^T$ gave:

$$\mathrm{A2}^{-1}_{left} = \begin{bmatrix} 0.75757576 & 0.3030303 & 1 & -3.03030303 \\ -0.78181818 & -0.27272727 & -1 & 3.52727273 \\ -0.63030303 & -0.21212121 & -1 & 2.92121212 \end{bmatrix}$$

Computing the pseudo-inverse, via both the SVD method and by calling linalg.pinv() directly both produced:

$$\mathrm{A2}^{+} = \begin{bmatrix} 0.75757576 & 0.3030303 & 1 & -3.03030303 \\ -0.78181818 & -0.27272727 & -1 & 3.52727273 \\ -0.63030303 & -0.21212121 & -1 & 2.92121212 \end{bmatrix}$$

This demonstrated that the pseudo-inverse of a full rank rectangular matrix is equal to the left inverse of that matrix (should it exist), and equal to the right inverse of the matrix if that inverse exists. If both the right and the left inverses of a matrix exist, then the matrix is necessarily square and full rank, and as such the right inverse, left inverse, pseudo-inverse and inverse ('normal inverse') of the matrix all exist and are all equal (i.e. the matrix is square invertible/non-singular). This also demonstrated that the pseudo-inverse as calculated by linalg.pinv(), which is presumably performed via SVD, is equal to the left/right inverse where they exist, and exists generally as defined (Moore-Penrose) when these inverses do not exist.

Problem 2

The system of linear equations $A3 \cdot x = B3$ was investigated. The rank of the matrix was found to be 3, while it would be required to be 4 in order for the matrix to be full rank, as such, this system was found to have no exact solution. This was confirmed by computing the pseudo-inverse of the matrix A3 and seeing if it was exact, i.e. if $A3 \cdot A3^+ \cdot B3 = B3$ to within floating point accuracy. As the pseudo-inverse was found to be not exact, the system found to not be exactly solvable. The script first attempted to perform a QR decomposition, in order to solve via back substitution, but this failed as the matrix is singular. The script then employed the minimum norm method by using the singular value decomposition in order to produce a best possible answer. The solution vector x was found to be:

$$\mathrm{x} = \begin{bmatrix} 0.15026411 & -0.06327495 & -0.87787045 & 0.27272727 \end{bmatrix}^T$$

which was also the least squares solution as determined by linalg.lstsq().

Problem 3

The rank of A4 was found to be 4, hence the matrix was full rank. Basis vectors for the column space, the row space, the null space and the left null space of the matrix were found to be:

$$\mathrm{C}_b = \begin{bmatrix} 1 & 6 & -3 & 0 \\ 0 & 4 & 2 & -3 \\ 3 & 18 & 1 & -5 \\ 2 & 0 & 0 & 3 \\ 2 & 8 & 2 & 0 \end{bmatrix}, R_b = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\mathrm{N}_b = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}^T, LN_b = \begin{bmatrix} 1 & 15/8 & -3/4 & 5/8 & 0 \end{bmatrix}^T$$

The basis for the row space was found by placing the matrix into a reduced row echelon form, and constructing the basis from the row vectors of the matrix that were not all zero. The column space basis was found by placing the matrix into reduced row echelon form, and constructing the column space basis from the columns of the original matrix corresponding to the columns containing pivot elements in the reduced row echelon form (In this case, that amounted to all the columns of the original matrix, since the matrix had full column rank). The null space basis would be constructed from the columns of the reduced row echelon form of the matrix that do not contain a pivot element. Since all of these columns did contain a pivot element, the null space basis is the trivial zero vector, and hence the nullity of the matrix is 0. This was confirmed using the Rank-Nullity Theorem, since the rank (4) plus the nullity (0) gave the number of columns (4), as required. Finally, the left null space basis could not be constructed directly from the reduced row echelon form, but since the left null space basis vector acts on the rows of A4 as opposed to its columns, the $dim(row) \times dim(row)$ identity matrix was used to augment A4 before placing it into reduced row echelon form, where it acted to record all the elementary row operations which would alter the column space. This row operation matrix was then left multiplied with A4, and the rows of the row operation matrix that gave a row of zeros in the result when (left) operating on A4 were used to construct the left null space basis.