Challenges in Quantum Field Theory: Infinities, Renormalisation, and The Path Forward

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Abstract

Despite its success, quantum field theory (QFT) is faced with significant challenges that need to be overcome to understand the foundations of modern physics, and work towards a fundamental description of the universe. In this essay, the problem of mathematical rigour and renormalisation in QFT is framed, with a discussion of possible resolution through axiomatic reformulation. Following this, an outline is given of the wider issues of interpretation of QFT. Finally, I will conclude that all formulations of QFT have their strengths, and a combination of different approaches and interpretations is required to develop a deeper theory of the universe.

1 Introduction

In classical physics, the idea of a field introduced locality to systems that were originally perceived as involving 'action at a distance'. Similarly, following the formulation of quantum mechanics (QM) in the early 20th century, particle-like nature was translated over to fields, unifying core ideas from QM and relativity, resulting in quantum field theory (QFT). This has formed the basis for the standard model, the most empirically successful theory in physics [1]. Despite this enormous success, we still lack an understanding of the mathematical principles it is founded upon.

2 The Lagrangian formulation

To understand the problems in Lagrangian QFT, it is helpful to consider a brief overview of its formulation.

For a classical system of particles, there is a finite number of generalized coordinates $q_a(x,t)$, indexed by a label a, with x, t corresponding to points in space and time. In field theory, the system is described by $\phi_a(x,t)$, meaning we have a system with an infinite number of degrees of freedom. Their dynamics are governed by the Lagrangian, which can write in the form,

$$L = \int d^3x \mathcal{L}(\phi_a, \partial_\mu \phi_a), \tag{1}$$

where $\mathcal{L}$ is the Lagrangian density. For QFT, we want Lagrangians that are local and Lorentz invariant, so we can reconcile relativity. To describe a *quantum* field, we follow in the footsteps of QM formulation and use canonical quantization, which takes classical variables to operators. In the case of a field, we will obtain field operators which act as linear transformations on a Hilbert space representing the possible states in QFT. First, though, we must restate the classical Lagrangian as a Hamiltonian, with Hamiltonian density

$$\mathcal{H} = \pi^a(x)\dot{\phi}(x) - \mathcal{L}(x), \tag{2}$$

where

$$\pi^a(x) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}_a(x)}, \tag{3}$$

is the momentum conjugate to $\phi_a(x)$. The Hamiltonian is then simply

$$H = \int d^3x\mathcal{H}. \tag{4}$$

Analogous to the formulation of QM, we can define a quantum field as an operator-valued function of space obeying the commutation relations [2],

$$[\phi_a(\vec{x}), \phi_b(\vec{y})] = [\pi_a(\vec{x}), \pi_b(\vec{y})] = 0, \tag{5}$$

so

$$[\phi_a(\vec{x}), \pi_b(\vec{y})] = i\delta^{(3)}(\vec{x} - \vec{y})\delta^a_b. \tag{6}$$

For the purposes of deriving a simple QFT, we can consider the Fourier transform of the Klein-Gordon equation

$$\left(\frac{\partial^2}{\partial t^2} + (\vec{p}^2 + m^2)\right)\phi(\vec{p}, t) = 0. \tag{7}$$

The most general solution to the Klein-Gordon equation is a superposition of harmonic oscillators, each vibrating at a different frequency. Thus, with QFT, the universe can be described by an infinite lattice of infinite superpositions of harmonic oscillators.

Using this idea, the commutation relations (5,6), and stating in terms of creation, $a^\dagger$ and annihilation, a operators, we get the Hamiltonian corresponding to this picture [4]

$$H = \int \frac{d^3p}{(2\pi)^3}\omega_{\vec{p}}[a^\dagger_{\vec{p}} + \frac{1}{2}(2\pi)^3\delta^{(3)}_0], \tag{8}$$

which we can apply to the vacuum (or ground) state, $|0\rangle$ (the state where $|0\rangle$ is annihilated by all a_p),

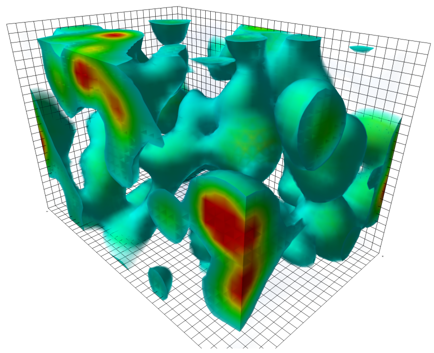

Figure 1: Visualisation of the energy fluctuations in a vacuum [3].

$$H|0\rangle = E_0|0\rangle = \left[\int d^3p\frac{1}{2}\delta^{(3)}_0\right]|0\rangle = \infty|0\rangle. \tag{9}$$

Considering this result, we can see that there are two sources of infinity in the field. The first arises due to the fact that the space we are considering is infinitely large, so we get an infinite energy as we are integrating over the whole volume. The other infinity arises because we have assumed that this picture is valid up to arbitrarily large energies. These are the infrared (IR) and ultraviolet (UV) divergences, respectively.

Importantly, when acting on the vacuum state with the creation operator n times, a state containing n particles is produced,

$$|\vec{p_1}, ..., \vec{p_n}\rangle = a^\dagger_{\vec{p_1}}...a^\dagger_{\vec{p_n}}|0\rangle. \tag{10}$$

The set of all such states can be used to construct the Fock space (i.e. an infinite sum of the Hilbert spaces), with number operator,

$$N = \int \frac{d^3p}{(2\pi)^3}a^\dagger_{\vec{p_n}}a_{\vec{p_n}}, \tag{11}$$

which counts the number of particles in a given state in the Fock space.

2.1 Renormalisation

Infinities have been problematic for QFT since its inception, and a 'war against infinities' was waged [5]. One of the main methods for resolving this issue was pioneered by Schwinger in the 1950s, called renormalisation [6]. For the case of the field described above (8), we can simply refine the Hamiltonian by subtracting the infinity,

$$H = \int \frac{d^3p}{(2\pi)^3} \omega_{\vec{p}} a^\dagger_{\vec{p}} a_{\vec{p}}. \tag{12}$$

This is a form of renormalisation called normal ordering, which Feynman described as "simply a way to sweep the difficulties of the divergences... under the rug" [7]. Despite having tamed the infinities, this result is conceptually striking. It implies that a total vacuum has non-zero energy, and the fields are 'restless' (figure 1), following from Heisenberg's uncertainty principle. This leads to the cosmological constant problem. Although this subject will not be discussed further here, it is important to note as a significant problem facing QFT and renormalisation [8].

Moving on to more complex situations, we can add an interaction term consisting of small perturbations to $\mathcal{L}$,

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2}m^2\phi^2 - \sum_{n\geq 3} \frac{\lambda_n}{n!}\phi^n, \tag{13}$$

where the coefficients λ_n are coupling constants upon which we can place restrictions to ensure all additional terms make small perturbations. By performing dimensional analysis, we see that $[\lambda_n] < 0$ for $n \geq 5$, thus the dimensionless parameter is $\lambda_n E^{n-4}$, where E is the energy scale of the interaction, meaning higher order contributions are small at low energies and high at large energies.

It can be seen, then, that interaction terms of the form ϕ^n for $n > 4$ are irrelevant at all but the largest energy scales, meaning that of all the infinite number of possible interaction terms only a few are needed below some energy cut-off, Λ. The energies above this cut-off are UV and those below are IR. From this we can define that a renormalizable theory is one that has no dependence on details above Λ. It can be inferred, then, that if a theory is non-renormalizable, it must be because there exists new phenomena above Λ and a new theory is needed. This is the basic idea of Wilson's renormalisation group [9].

The result of all this is that we can define effective field theories (EFTs), where new energy ranges bring different physics, so the infinities are "a reminder of a practical limitation" of QFT [10].

3 Questions from Lagrangian QFT

In the wake of this formulation, we are left with some questions that need to be addressed in order to form a satisfying QFT.

Firstly, if we need to define a new EFT for each new energy range, is this really a valid theory? Or is there some deeper fundamental theory that is capable of describing physics at all scales?

Can we even call the Lagrangian formulation a cohesive theory? According to the Stone–von Neumann theorem, (which applies nicely in QM) any two irreducible representations of the canonical commutation relations (CCRs) are unitarily equivalent [11]. In QFT, however, we are dealing with infinite degrees of freedom, and so different representations of CCRs are inequivalent, and apply individually to different physical situations. Further, Haag's theorem shows that CCRs cannot be unitarily equivalent unless the field is free at all times, yet interaction QFT is empirically successful [12]. Does this mean, then, that these are fundamentally different effects, or are we still missing something?

Finally, the question of renormalisation – is the idea of just sweeping infinities under the rug mathematically sound?

4 Axiomatic reformulations

As a result of these problems, attempts have been made to provide a mathematically sound framework for QFT. The two most famous attempts being Wightman QFT and Algebraic QFT (AQFT), which both aim to provide a set of rigorous axioms to create a cohesive model of QFT.

These two formulations both stem from Heisenberg's matrix mechanics, in which the algebraic relations between observables play a central role. Although the implementation of each formulation differs, they share the following core axioms: covariance (local observable algebras must transform covariantly under Poincaré symmetries), causality (spacelike separated observables are required to commute or anticommute), and the spectrum condition (energy must be positive in all Lorentz frames, so the vacuum state is a stable ground state) [13] [14]. In this formal approach, the fact that an infinity results from applying the Hamiltonian to the vacuum state implies that the state is not in the domain of the Hamiltonian. Thus, the system is renormalized by finding another Hilbert space representation of the CCRs that is unitarily inequivalent to the original Fock space [15].

For the Wightman approach, the entities upon which the axioms are imposed are smeared field operators acting on a fixed Hilbert space. Due to UV divergences, field operators can only be assigned to extended regions of spacetime. In practice, these regions can be made arbitrarily small.

Algebraic QFT, however, applies the axioms to directly observable, bounded, guage-invariant operators through the use of C* algebra (i.e a normalized and self adjoint algebra). A key insight employed by AQFT is the idea that the way algebras of local observables are linked with spacetime regions (nets of algebras) is exactly the thing that provides the observables with physical significance. So in this (unintuitive) setting, physical states are represented as functionals which map elements of local algebras to real numbers, adhering to the aforementioned fundamental axioms [16].

We are left with a dichotomy. As it stands, while the axiomatic constructions of QFT provide mathematical rigour and resolve some issues of the Lagrangian formalism, they are seldom useful to describe real physical phenomena. Meanwhile, Lagrangian QFT has been extremely successful in predicting experimental outcomes, yet is mathematically dubious. Is there a way to make these theories converge to a more foundational theory that is physically relevant *and* mathematically sound?

5 Rival Paradigms?

Due to the shared mathematical formalism of the axiomatic approaches, it is possible to construct a net of associated observable algebras within the Wightman formulation, thus it can be expressed as a sub-theory of Algebraic QFT [13].

Reconciling the Lagrangian formulation is more complicated. Wallace [17] argues that the different Lagrangian and axiomatic approaches to infinities in the UV limit make them fundamentally different theories. In the Lagrangian case of EFTs, UV infinities are cut-off. Meanwhile, in the Wightman case, fields are valid up to arbitrarily high energies, highlighting a potential contradiction. It has been argued [18], however, that this fact does *not* imply a contradiction, as Lagrangian QFT can merely be thought of as a low-energy limit sub-theory of some deeper algebraic QFT. In fact, there has been recent work aiming to extend the conventional renormalisation techniques to the algebraic framework of axiomatic QFT [19]. Furthermore, from a practical perspective, these completely different renormalisation approaches return the exact same empirical results [20] [21].

Thus, we have the axiomatic formulations which appear as 'idealized' models that are mathematically rigorous and philosophically pertinent, and the Lagrangian formulation which struggles in these respects, but is more practically useful. Is this characterisation fair? Fraser argues that even if the current axioms cannot produce physically realistic models, then they can be modified so that they do, and the idealization can be dispensed with [22]. Further, the cut-off variant of renormalisation is an idealization in itself, except in this case, if the idealization is dispensed with, the model is no longer valid.

Each of these paradigms offers a unique (if partial) perspective and set of tools. As Haag stated, we need "a synthesis of the knowledge gained in the different approaches" [23]. Going forward, the idea of recovering Lagrangian methods such as EFTs within an algebraic framework seems like a promising way to make sense of renormalisation, though the physical interpretation of this algebraic approach remains an issue.

6 Ontology and the problem of interpretation

The construction of the Fock space is the only known way to represent countable entities with infinite degrees of freedom, and would seem to imply a particle-like ontology. To investigate this, we must define the two key properties of particles: localizability and countability/ discreteness. It has been proven that the former property cannot hold in any relativistic theorem [24], whilst the latter property cannot hold due to the Unruh effect. This describes a situation where an accelerating observer would observe a thermal bath, whereas an inertial observer would observe none [25]. In QFT, the two observers would have unitarily inequivalent Fock spaces, thus would observe different particle numbers, undermining the notion of countability.

Fields, on the other hand, are non-localized and continuous, so with the failures of the particle ontology, would appear to provide a natural solution. There are various proposals for a field interpretation of QFT, here the focus will be the mainstream wave-functional interpretation [26]. In this view, the states can be interpreted as superpositions of field configurations, similar to how Fock space is a superposition of particle configurations. However, we are once more faced with the obstacle of Haag's theorem. For the Lagrangian case, this issue is essentially ignored with cut-offs, or through Wallace's 'naïve' approach [18], and is as yet unresolved in axiomatic QFT (although there have been suggestions based on Wightman smeared fields) [20] [27].

These difficulties in determining an ontology are in stark contrast with QFT's success. Ruetsche suggests that we instead opt for an approach free of an ontology of objects and properties, and take a 'coalescence approach', where each view has its value in certain situations, but none represent a fundamental truth [28]. This approach fits into the wider idea of ontic structural realism (OSR), which emphasises the importance of the relational structure of physical systems over individual components [29]. The extent of this emphasis varies from entirely relational views, to ones such as Lye's 'Extended OSR', which allows for structurally derived intrinsic properties as part of a larger relational framework [30]. Howard argues that even this framework struggles with the issue of inequivalent representations [31], whilst French refutes this claim, citing similar arguments as those used to defend the field ontology [32].

7 Conclusion

We have seen that Lagrangian QFT is severely lacking in mathematical justification for its success. Axiomatic formulations are mathematically rigorous and offer ways to address problems including renormalisation and lack of cohesion that plague the Lagrangian formulation, although they are still rarely relevant for practical purposes.

The issue of interpretation remains; although the particle ontology is seemingly not fundamental, it still is useful in many problems. The field and OSR ontologies represent avenues for uncovering a deeper truth, but require modification.

Overall, it would be premature to cast any mode of thinking aside – each of the formalisms and interpretations offer a glimpse at some underlying truth.

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