Quantum Fisher Information Flow and Non-Markovianity in Open Systems

Xiao-Ming Lu,[1] Xiaoguang Wang,[1,*] and C. P. Sun[2,†]

[1]*Zhejiang Institute of Modern Physics, Department of Physics, Zhejiang University, Hangzhou 310027, China*
[2]*Institute of Theoretical Physics, Chinese Academy of Sciences, Beijing, 100080, China*

We propose to use the quantum Fisher information in characterizing the information flow of open quantum systems. This information-theoretic approach provides a quantitative measure to statistically distinguish Markovian and non-Markovian processes. A basic relation between the QFI flow and non-Markovianity is unveiled for quantum dynamics of open systems. For a class of time-local master equations, the exactly-analytic solution shows that the non-Markovianity is characterized by additive information sub-flows in different non-Markovian channels.

PACS numbers: 03.65.Yz, 03.65.Ta, 42.50.Lc

Introduction — Any system in the realistic world is open since it inevitably interacts with its environment. The quantum dynamic processes of open systems are simply classified into Markovian and non-Markovian ones according to the ways to lose energy or information [1]. In most situations, Markovian process uniquely determines its final steady state as an thermal equilibrium [2], which is independent of its initial one. In this sense a Markovian process is essentially an information erasure process, thus tends to continuously reduce the distinguishability between any two initial states.

However, Markovian description for open quantum system is only an approximation to most of actual processes, which are of non-Markovian. With many recent investigations about non-Markovian dynamics by making use of various analytical approaches and numerical simulations, a computable measure of "Markovianity" for quantum channels was introduced in Ref. [3]. Most recently, it was also recognized that difference between them can be measured through the continuous increment of the state distinguishability [4]. Then this increment is intuitively interpreted as the revival of information flow between the bath and the system though there no quantitative information measure is utilized. Another approach based on entanglement is proposed in Ref. [5]. In this letter, the quantum Fisher information (QFI) flow are used for directly characterizing the non-Markovianity of the quantum dynamics of open systems.

Actually, in the system-plus-bath approach for open quantum systems, the effective dynamics of the reduced density matrix ρ is induced by tracing over environment [1]. The simplest reduced dynamics is the quantum Markovian process described by dynamical semigroups [6]. Here, the reduced density matrix ρ at time $t + dt$ is determined completely by the one at time t. Contrarily, the general reduced dynamics may be of non-Markovian when the surrounding environment can retain a memory of the information about states at earlier times, and transfer it back to the system to affect its evolution. In this sense the Markovian process only happen when the environmental correlation time is relatively short so that memory effects can be neglected. These memory-based consideration for the Markovian approximation also means that the information-theoretical characterizing of the non-Markovianity is a quite natural fashion.

In this letter, we adopt the quantum Fisher information (QFI) to characterize the statistical distinguishability of reduced density matrix [7, 8]. An intuitive picture of the non-Markovian behavior then immediately follows from the dynamic return of the QFI, which is depicted by the QFI flow. For a class of the non-Markovian master equations in time-local forms, we exactly calculate the information flows for these cases. The analytic results show that the total QFI flow can be decomposed into the split contributions from different dissipative modes. On the other hand, because the QFI is equivalent to the Bures metric on the Riemannian manifold of density operators, the present approach is geometrical [8]. We also point out this approach is feasible to work for understanding the problems in the quantum metrology [9].

Quantum Fisher information in non-Markovian dynamics — We consider a class of quantum processes described by the following time-local master equation [4]

$$\frac{\partial}{\partial t}\rho(t) = \mathcal{K}(t)\rho(t), \tag{1}$$

where $\mathcal{K}(t)$ is a super-operator acting on the reduced density matrix $\rho(t)$ as [10, 11]

$$\mathcal{K}(t)\rho = -i\,[H,\rho] + \sum_i \gamma_i \left[A_i\rho A_i^\dagger - \frac{1}{2}\left\{A_i^\dagger A_i, \rho\right\}\right], \tag{2}$$

where $H(t)$ is the Hermitian Hamiltonian for the open quantum system without the couplings to the bath. $\{\cdot,\cdot\}$ denotes the anti-commutator. If all γ_i and A_i are time independent, and all γ_i are positive, equation (2) is the conventional master equation in the Lindblad form [10], which describes the conventional Markovian process. However, by making use of a variety of methods, like the time-convolutionless projection operator technique [12] and Feynman-Vervon influence functional theory [13], the parameters $\gamma_i = \gamma_i(t)$ and $A_i = A_i(t)$ in the time-local master equation may explicitly depend on time, and become negative sometimes. Obviously, the non-Markovian character resides in these time-dependence coefficients.

Taking some real number θ in the reduced density matrix $\rho(\theta;t)$ as the inference parameter, we write down the QFI

$$\mathcal{F}(\theta;t) = \mathrm{Tr}\left[L^2(\theta;t)\rho(\theta;t)\right], \quad (3)$$

where $L(\theta;t)$ is the so-called symmetric logarithmic derivative (SLD), which are Hermitian operators determined by [14]

$$\frac{\partial}{\partial\theta}\rho(\theta;t) = \frac{1}{2}\left[L(\theta;t)\rho(\theta;t) + \rho(\theta;t)L(\theta;t)\right]. \quad (4)$$

An important essential feature of the QFI is that we can obtain the achievable lower bound of the mean-square error of unbiased estimators for the parameter θ through the quantum Cramér-Rao theorem (QCR) $\mathrm{Var}(\theta;t) \geq 1/[M\mathcal{F}(\theta;t)]$, where M presents the times of the independent measurements [14]. The relations between the QFI and the statistical distinguishability of $\rho(\theta;t)$ and its neighbor has been pointed out in some previous works [7, 8, 14].

Flow of quantum Fisher information and its decomposition — Next we further use the QFI to characterize the non-Markovianity of the open quantum system by considering the QFI flow, which is the change rate $\mathcal{I}{=}\partial\mathcal{F}/\partial t$ of the QFI. As a central result in this letter, a proposition about the decomposition of the QFI flow is given as follows:

Proposition: For an open quantum system described by the time local master equation (1), the QFI flow $\mathcal{I} = \sum_i \mathcal{I}_i$ is explicitly written as a sum of subflows $\mathcal{I}_i = \gamma_i\mathcal{J}_i$:

$$\mathcal{J}_i \equiv -\mathrm{Tr}\left\{\rho\left[L, A_i\right]^\dagger\left[L, A_i\right]\right\} \leq 0. \quad (5)$$

Proof: From the differential of Eq. (4) with respect to time t, we have

$$\partial_t\partial_\theta\rho(\theta) = \frac{1}{2}\left[\dot{L}\rho + L\dot{\rho} + \dot{\rho}L + \rho\dot{L}\right].$$

It gives

$$\mathrm{Tr}\left[\rho\dot{L}L + \rho L\dot{L}\right] = \mathrm{Tr}[2L\partial_t\partial_\theta\rho(\theta)] - \mathrm{Tr}\left[2\dot{\rho}L^2\right].$$

From the differential of Eq. (3) with respect to time t, we obtain the QFI flow as

$$\mathcal{I} = \mathrm{Tr}\left[\mathcal{L}\frac{\partial}{\partial t}\rho\right]. \quad (6)$$

where the operator $\mathcal{L} = L(2\partial/\partial\theta - L)$ is defined. By using the concrete expression of the master equation (2), we split the QFI flow into those individuals corresponding to the different dissipative channels as $\mathcal{I} = \mathrm{Tr}\left[\mathcal{L}\mathcal{K}(t)\rho(t)\right]$ or

$$\mathcal{I} = \sum_i \gamma_i \mathrm{Tr}\left[\mathcal{L}A_i\rho A_i^\dagger - \frac{1}{2}\mathcal{L}\left\{A_i^\dagger A_i, \rho\right\}\right].$$

After some algebra, we get the decomposition $\mathcal{I} = \sum_i \gamma_i\mathcal{J}_i$, where $\mathcal{J}_i$ is just given in Eq. (5). It finally proves the proposition.

The above proposition and its proof contain rich implications in physics. Firstly, because the right hand side of the Eq. (6) is linear with respect to $\partial\rho/\partial t$, the contributions of every term in Eq. (2) to QFI flow correspond to those individuals in the different dissipative channels. We also notice that this is not a simple decomposition since each subflow depends on the whole SLD $L(\theta;t)$. That each term $\mathcal{J}_i$ is not positive means information only streaming from a dissipative individual channel if the corresponding time local parameter $\gamma_i = \gamma_i(t)$ is always positive. Therefore, we can interpret $\mathcal{I}_i = \gamma_i\mathcal{J}_i$ as the flow of the QFI only caused by the dissipative mode described by A_i. The direction of the QFI flow for the A_i mode is determined by the sign of γ_i. For the case that all $\gamma_i(t)$ is positive, Eq. (2) describe a so-called time-dependent Markovian quantum process [3, 11, 15, 16]. In such cases, $\mathcal{I}$ always decreases. The magnitude of the QFI flow is determined by a state-independent factor γ_i and a state-dependent factor $\mathcal{J}_i$. If the total QFI flow $\mathcal{I}$ is negative, it signifies at least one of γ_i is negative. In such cases, the QFI flows back to the open system and the non-Markovian behavior emerges.

Actually, like the trace distance used in Ref. [4], the dynamic return of the QFI is linked to the divisibility property of the dynamical map of quantum processes. If the master equation is of the form (2), the corresponding dynamical map is infinitely divisible provided that all γ_i are positive.[17]. In such cases, for arbitrary time $t > 0$, the dynamical map from time t to $t + dt$ is a completely positive and trace-preserving map. Thus the QFI decreases during this time interval since the QFI is monotonic with respect to a completely positive and trace-preserving map [18].

Secondly, it is seen from the above proof of the proposition that, the free Hamiltonian $H(t)$ does not contribute to the total QFI flow $\mathcal{I}$ directly. Namely, the coherent part of Eq. (2), i.e. $-i\left[H(t), \rho(t)\right]$, does not cause the change of the QFI, or $\mathcal{I}{=}\partial\mathcal{F}/\partial t = 0$. Thus we conclude that the unitary evolutions of a close quantum systems keeps their QFI invariant since the Lindblad terms $\gamma_i\left[A_i\rho A_i^\dagger - \left\{A_i^\dagger A_i, \rho\right\}/2\right]$ vanish for close systems. This observation directly leads to the no-cloning theorem in quantum information, which states that we can not use unitary operations to evolve the states $|\psi(\theta)\rangle \otimes |0\rangle$ into $|\psi(\theta)\rangle \otimes |\psi(\theta)\rangle$ as a quantum copy [19]. This is because the QFI of the target states is twice as the one of the source states, due to the additivity of the QFI for the product states. If such cloning process existed,

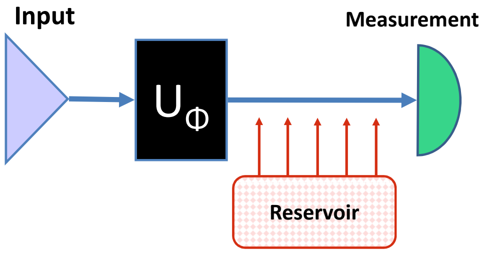

FIG. 1: Estimation of parameter ϕ in an unitary operation. After the phase gate operation, the system interacts with a reservoir. The precision of the estimation is impacted by characteristics of both the reservoir and the interaction.

it would imply that we can increase the QFI by unitary operations, but we have known it is impossible since $\mathcal{I}=\partial\mathcal{F}/\partial t=0$.

Two-level system — Now we use an example of two-level atom to explicitly illustrate our discovery about the intrinsic relation between the QFI flow and the non-Markovianity of the open quantum system. Here, the QFI-based parameter estimation is induced by a single-qubit operation U_ϕ acting on the atom where $\phi=\theta$ is some inference parameter (see Fig. 1). The atom is assumed to be coupled to a reservoir consisting of harmonic oscillators in the vacuum. The total Hamiltonian of this typical model [1, 20, 21] reads

$$H=\omega_0\sigma_+\sigma_- + \sum_k \omega_k b_k^\dagger b_k + (\sigma_+ B + \sigma_- B^\dagger) \qquad (7)$$

with $B=\sum_k g_k b_k$, where ω_0 denotes the transition frequency of the atom with ground and excited states $|g\rangle$ and $|e\rangle$, and $\sigma_\pm$ the raising and lowering operators of atom; $b_k^\dagger$ and b_k are respectively the creating and annihilation operators of the bath mode of frequencies ω_k. g_k denote the coupling constants .

From the total initial state $|\Psi(0)\rangle=|\psi\rangle|0\rangle_r$ where $|\psi\rangle=c_0|g\rangle+c_1|e\rangle$ and $|0\rangle_r$ is the vacuum state of the reservoir, the total system evolves into

$$|\Psi(t)\rangle = c_0|g\rangle|0\rangle_r + c_1(t)|e\rangle|0\rangle_r + \sum_k c_k(t)|g\rangle|1_k\rangle_r, \quad (8)$$

where $|1_k\rangle_r$ is the single particle Fock states of the k-th mode. Here, $c_1(t)$ satisfies

$$\dot{c}_1(t) = -\int_0^t dt_1\, f(t-t_1)c_1(t_1), \qquad (9)$$

where the kernel $f(t-t_1)$ is determined by the spectral density $J(\omega)$ of the reservoir as $f(t-t_1)=\int d\omega J(\omega)\exp\left[i(\omega_0-\omega)(t-t_1)\right]$. This solution corresponds to a time-local master equation of the form (2)

$$\frac{d}{dt}\rho_{\rm S} = F[\rho_{\rm S}] + \gamma(\sigma_-\rho_{\rm S}\sigma_+ - \frac{1}{2}\{\sigma_+\sigma_-,\rho_{\rm S}\}) \qquad (10)$$

with $F[\rho_{\rm S}]=-iS\left[\sigma_+\sigma_-,\rho_{\rm S}\right]/2$ and

$$S(t)=-2\Im\frac{\dot{c}_1(t)}{c_1(t)}, \gamma(t)=-2\Re\frac{\dot{c}_1(t)}{c_1(t)}. \qquad (11)$$

Furthermore, we consider the resonant case with Lorentian spectral density $J(\omega)=\lambda W^2/\{\pi[(\omega_0-\omega)^2+\lambda^2]\}$ where W is the transition strength, and λ defines the spectral width of the coupling, which is related to the reservoir correlation time scale $\tau_{\rm B}$ by $\tau_B=\lambda^{-1}$ [1, 20].

For such given spectral density, the exact solution is given by $c_1(t)=c_1(0)h(t)$:

$$h(t)=\begin{cases} e^{-\lambda t/2}\left[\cosh\left(\frac{dt}{2}\right)+\frac{\lambda}{d}\sinh\left(\frac{dt}{2}\right)\right], & W<\frac{\lambda}{2}, \\ e^{-\lambda t/2}\left[\cos\left(\frac{dt}{2}\right)+\frac{\lambda}{d}\sin\left(\frac{dt}{2}\right)\right], & W>\frac{\lambda}{2}, \end{cases} \qquad (12)$$

where $d=\sqrt{|\lambda^2-4W^2|}$. Then $S(t)=0$ and $\gamma(t)=-2\dot{h}(t)/h(t)$.

Now we consider the estimation of the physical parameter ϕ induced by a single-qubit phase gate $U_\phi=|g\rangle\langle g|+\exp(i\phi)|e\rangle\langle e|$. The optimal input state is chose as $|\psi_{in}\rangle=(|g\rangle+|e\rangle)/\sqrt{2}$ [9]. After the operation U_ϕ, the atom begins to interact with the reservoir described for a time t. Then the reduced density matrix for atomic state is obtained as $\rho_{\rm S}(t)=\frac{1}{2}\left(I+\mathbf{B}\cdot\sigma\right)$. where $\mathbf{B}=(h(t)\cos\phi,-h(t)\sin\phi,h(t)^2-1)$. In order to calculate the QFI flow, we first diagonalize this reduced density matrix as $\rho_{\rm S}(t)=\sum_n\rho_n|\psi_n(t)\rangle\langle\psi_n(t)|$. In this diagonal representation, the SLD with matrix elements $L_{ij}=2\langle\psi_i|\partial_\phi\rho|\psi_j\rangle/(\rho_i+\rho_j)$ is obtained explicitly as

$$L=ih(t)(|\psi_1\rangle\langle\psi_2|-|\psi_2\rangle\langle\psi_1|). \qquad (13)$$

According to Eq. (5), we have $\mathcal{J}\equiv -\mathrm{Tr}(\rho\left[L,\sigma_-\right]^\dagger\left[L,\sigma_-\right])=-h(t)^2$. Then we obtain the exact solution for the QFI flow as follows

$$\mathcal{I}_\phi(t)=\gamma(t)\mathcal{J}(t)=2h(t)\dot{h}(t). \qquad (14)$$

After integrating the above equation with respect to time, we finally obtain $\mathcal{F}_\phi=h(t)^2$.

Therefore, the characteristic of the QFI flow is determined by the function $h(t)$, which has two very different behaviors. The corresponding properties of the QFI flow are shown in Fig. (2). In the weak coupling regime ($W<\lambda/2$), the function $\gamma(t)$ are always positive, thus the QFI are always lost during the time evolution of the open system. In the strong coupling regime ($W>\lambda/2$), the function $\gamma(t)$ takes on negative values within certain intervals of time, see Fig. 2 (d), which displays the non-Markovianity. Obviously in these time intervals, the QFI flow is inward. It is remarkable that although $\gamma(t)$ diverges at certain time, the QFI flow does not, see Figs. 2 (c) and (d). This is because the QFI flow is determined by two factors, $\gamma(t)$ and $\mathcal{J}(t)$.

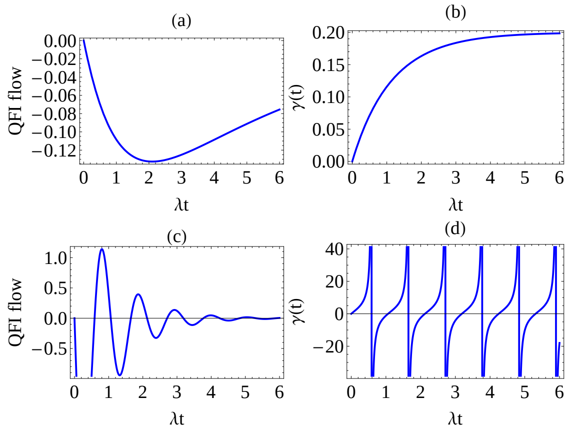

FIG. 2: Damped Jaynes-Cummings model on resonance: (a) QFI flow as a function of rescaled time, plotted in the weak coupling regime ($W = 0.3\lambda$); (b) γ as a function of rescaled time, $W = 0.3\lambda$; (c) QFI flow as a function of rescaled time, plotted in the strong coupling regime ($W = 3\lambda$); (d) γ as a function of rescaled time, $W = 3\lambda$.

Conclusion with remark — In summary, we have proposed an information-theoretical approach for distinguishing between the Markovian and non-Markovian behaviors of open quantum systems. We have suggested a quantitative measure for non-Markovianity using information flow between the system and bath. Using the time local master equation, we showed that the information indeed can change through different dissipative modes. With this contributions split effect, during the time evolution, the return of the QFI, i.e, an inverse QFI flow, is a clear signature for the non-Markovian characters.

The present approach can be associated with the problems in the quantum metrology. This observation concerns on finding an optimal way to make high-resolution and highly sensitive measurement of physical parameters [9]. In the quantum metrology context, the QFI gives a theoretical-achievable limit on the precision when estimating a parameter, according to the Cramér-Rao theorem. The parameter to be estimated is assumed to be induced by unitary operations. To estimate the parameter as precisely as possible, we should optimize input states to maximize the QFI , and then optimize measurements to achieve the Cramér-Rao bound. Due to the interaction with environment, like the photon losses in the optical interferometry or the presence of quantum noise [22], the QFI will change and affect the precision of the parameter estimation. Therefore, it is worthy to study the dynamical evolution of the QFI in the context of quantum metrology even with non-Markovian processes.

This work is supported by NSFC with Grant No., 10874151, 10874091, 10935010; NFRPC with Grant No. 2006CB921205; Program for New Century Excellent Talents in University (NCET), and Science Fundation of Chinese University.

* Electronic address: xgwang@zimp.zju.edu.cn
† Electronic address: suncp@itp.ac.cn

[1] H. P. Breuer and F. Petruccione, *The Theory of Open Quantum Systems* (Oxford University Press, Oxford, 2007).
[2] J. Q. Liao, H. Dong, X. G. Wang, X. F. Liu, C. P. Sun, arXiv:0909.1230 (2009).
[3] M. M. Wolf *et al.*, Phys. Rev. Lett, **101**, 150402 (2008).
[4] H. P. Breuer, E. M. Laine, and J. Piilo, Phys. Rev. Lett. **103**, 210401 (2009).
[5] Á. Rivas, S. F. Huelga and M. B. Plenio, arXiv:0911.4270v1.
[6] R. Alicki and K. Lendi, Quantum Dynamical Semigroups and Applications, Lect. Notes Phys. 717 (Springer, Berlin Heidelberg, 2007).
[7] W. K. Wootters, Phys. Rev. D, **23**, 357 (1981).
[8] S. L. Braunstein and C. M. Caves, Phys. Rev. Lett. **72**, 3439 (1994).
[9] V. Giovannetti, S. Lloyd, and L. Maccone, Phys. Rev. Lett, **96**, 010401 (2006).
[10] V. Gorini *et al.*, J. Math. Phys. **17**, 821 (1976); G. Lindblad, Commun. Math. Phys. **48**, 119 (1976).
[11] H. P. Breuer, Phys. Rev. A **70**, 012106 (2004).
[12] F. Shibata, Y. Takahashi, and N. Hashitsume, J. Stat. Phys. **17**, 171 (1977); S. Chaturvedi and F. Shibata, Z. Phys. B **35**, 297 (1979); F. Shibata and T. Arimitsu, J. Phys. Soc. Jpn. **49**, 891 (1980); A. Royer, Phys. Rev. A **6**, 1741 (1972); A. Royer, Phys. Lett. A **315**, 335 (2003).
[13] R. P. Feynman and F. L. Vernon, Ann. Phys. **24**, 118 (1963); A. O. Caldeira and A. J. Leggett, Physica A **121**, 587 (1983); B. L. Hu *et al.*, Phys. Rev. D **45**, 2843 (1992); J.-H. An *et al.*, Quantum Inf. Comput. **9**, 0317 (2009).
[14] C. W. Helstrom, *Quantum Detection and Estimation Theory* (Academic, New York, 1976), chap. VIII.4; A. S. Holevo, *Probabilistic and Statistical Aspects of Quantum Theory* (North-Holland, Amsterdam, 1982), especially Chaps. III.2 and VI.2.
[15] A. J. van Wonderen and K. Lendi, J. Stat. Phys. **100**, 633 (2000).
[16] S. Maniscalco *et al.*, J. Opt. B: Quantum Semiclass. Opt. **6**, S98 (2004).
[17] M. M. Wolf and J. I. Cirac, Commun. Math. Phys. **279**, 147 (2008).
[18] A. Fujiwara, Phys. Rev. A, **63**, 042304 (2001).
[19] W. K. Wootters and W. H. Zurek, Nature **299,** 802 (1982).
[20] B.M. Garraway, Phys. Rev. A **55**, 2290 (1997).
[21] B. Bellomo,*et al.*, Phys. Rev. Lett, **99**, 160502 (2007).
[22] U. Dorner *et al.*, Phys. Rev. Lett. **102**, 040403 (2009); Y. Watanabe *et al.*, arXiv:0906.4463v2.