The Einstein and Møller Energy Complex as Thermodynamic Potentials

I-Ching Yang [1]

Systematic and Theoretical Science Research Group

and Department of Applied Science,

National Taitung University, Taitung, Taiwan 95002, Republic of China

ABSTRACT

In this article, I obtain the Einstein and Møller energy complex in PG coordinates. According to the difference of energy within radius r between Einstein and Møller prescription, I could present the difference of energy within radius r like the fomula of Legendre transformation and propose that the Møller and Einstein energy complex play the role of internal energy and Helmholtz energy in thermodynamics.

PACS No.:04.70.Dy; 04.70.Bw; 04.20.Cv

[1] E-mail:icyang@nttu.edu.tw

1 Introduction

In the theory of general relativity (GR), one of the most important issues which is still unsolved is the localization of energy. According to Noether's theorem, one would define a conserved and localized energy as a consequence of energy-momentum tensor $T^{\mu\nu}$ satisfying the differential conservation law

$$\partial_\nu T^{\mu\nu} = 0. \tag{1}$$

However, in a curved space-time where the gravitational field is presented, the differential conservation law becomes

$$\nabla_\nu T^{\mu\nu} = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\nu} \left(\sqrt{-g} T^{\mu\nu} \right) - \frac{1}{2} g^{\nu\rho} \frac{\partial g^{\nu\rho}}{\partial x^\lambda} T^{\mu\lambda} = 0, \tag{2}$$

and generally does not lead to any conserved quantity. In GR, we shall look for a new quantity $\Theta^{\mu\nu} = \sqrt{-g}\left(T^{\mu\nu} + t^{\mu\nu}\right)$ instead of $T^{\mu\nu}$, which satisfies the differential conservation equation

$$\partial_\nu \Theta^{\mu\nu} = 0, \tag{3}$$

if we want to maintain the localization characteristics of energy. Here, $\Theta^{\mu\nu}$ is an energy-momentum complex of matter plus gravitational fields and $t^{\mu\nu}$ is regarded as the contribution of energy-momentum from the gravitational field. It should be noted that $\Theta^{\mu\nu}$ can be expressed as the divergence of the "superpotential" $U^{\mu[\nu\rho]}$ that is antisymmetric in ν and ρ as

$$\Theta^{\mu\nu} = U^{\mu[\nu\rho]}{}_{,\rho}. \tag{4}$$

Mathematically, it is freedom on the choice of superpotential, because one can add some terms $\psi^{\mu\nu\rho}$, whose divergence or double divergence is zero, to $U^{\mu\nu\rho}$. A large number of definitions for the gravitational energy in GR have been given by many different authors, for example Einstein [1], Møller [2], Landau and Lifshitz [3], Bergmann and Thomson [4], Tolman [5], Weinberg [6], Papapetrou [7], Komar [8], Penrose [9] and Qadir and Sharif [10]. On the other hand, Chang, Nester and Chen [11] showed that every energy-momentum complex is associated with a legitimate Hamiltonian boundary term and actually quasilocal.

One of those problems for using several kinds of energy-momentum complexes is that they may give different results for the same space-time. Especially Virbhadra and his colleagues [12] showed that Einstein, Landau-Lifshitz, Papapetrou, and Weinberg prescriptions (ELLPW) lead to the same results in Kerr-Schild Cartesian coordinates for a specific class of spacetime, i.e. the general nonstatic spherically symmetric space-time of the Kerr-Schild class

$$ds^2 = B(u,r)du^2 - 2dudr - r^2 d\Omega \tag{5}$$

and the most general nonstatic spherically symmetric space-time

$$ds^2 = B(t,r)dt^2 - A(t,r)dr^2 - 2F(t,r)dtdr - D(t,r)r^2 d\Omega, \tag{6}$$

but not in Schwarzschild Cartesian coordinates. Afterward Xulu [13] presented Bergmann-Thomson complex also "coincides" with ELLPW complexes for a more general than the Kerr-Schild class metric. Mirshekari and Abbassi [14] find a unique form for a special general spherically symmetric metric in which the energy of Einstein and Møller prescriptions lead to the same result. In particular, whatever coordinates do not exist the same energy complexes associated with using definitions of Einstein and Møller in some space-time solutions, i.e. Reissner-Nordström (RN) black hole. On the other hand, Yang and Radinschi [15] attemptd to investigate the difference between the energy of Einstein prescription E_{Einstein} and Møller prescription $E_{\text{Møller}}$, and observed the difference $\Delta E = E_{\text{Einstein}} - E_{\text{Møller}}$ can be related to the energy density of the matter fields T_0^0 as

$$\Delta E \sim r^3 \times T_0^0. \tag{7}$$

Matyjasek [16] also presented two analogous relations which are

$$\Delta E = 4\pi r^3 T_0^0 \tag{8}$$

for the simplified stress-energy tensor of the matter field and

$$\Delta E = 4\pi r^3 \langle T_r^r \rangle_{ren}^{(s)} \tag{9}$$

for the approximate renormalized stress-energy tensor of the quantized massive scalar ($s = 0$), spinor ($s = 1/2$) and vector ($s = 1$) field. Later, Vagenas [17] hypothesized that $\alpha_n^{(\text{Einstein})}$ and $\alpha_n^{(\text{Møller})}$ are the expansion coefficients of E_{Einstein} and $E_{\text{Møller}}$ in the inverse powers of r, and found out an interesting relation between these two coefficients

$$\alpha_n^{(\text{Einstein})} = \frac{1}{n+1}\alpha_n^{(\text{Møller})}. \tag{10}$$

Finally, Matyjasek [16] and Yang *et. al.* [18] pointed out the following formula respectively

$$E_{\text{Møller}} = E_{\text{Einstein}} - r\frac{dE_{\text{Einstein}}}{dr}. \tag{11}$$

It should be noted that these relations in Eq. (7)-(11) offer us the mathematical formula between E_{Einstein} and $E_{\text{Møller}}$ only. The remainder of the article is organized as follows. In section 2, I will calculate the energy distribution for generalized Painlevé-Gullstrand (PG) coordinates [19] by using the Einstein and Møller complex. In section 3, the physical explanation of the difference ΔE will be given. I will summarize and conclude finally in section 4. In this article, I use geometrized units in which $c = G = \hbar = 1$ and the metric has signature $(+ - --)$.

2 Using the Einstein and Møller Energy Complex in generalized PG coordinates

The continuation of black holes across the horizon is a well understood problem discussed on GR. The difficulties of the Schwarzschild coordinates (t, r, θ, ϕ) at the horizons of a nonrotating black hole provide a vivid illustration of the fact that the meaning of the coordinates is not independent of the metric tensor $g_{\mu\nu}$ in GR. Several coordinate systems produce a metric that is manifestly regular at horizons, i.e. the Kruskal-Szekeres, Eddington-Finkelstein, and PG coordinates. However, PG coordinates have often been employed to study the physics of black holes. They have been applied to analyse quantum dynamical black holes [20], and used extensively in derivations of Hawking radiation as tunneling following the work of Parikh and Wilczek [21]. In this section, while using PG coordinates, I will find out the energy of static spherically symmetric black hole solutions in Einstein and Møller prescriptions. In four-dimensional theory of gravity, I can write the static spherically symmetric metrics in the form

$$ds^2 = f dt^2 - f^{-1}dr^2 - r^2 d\Omega, \tag{12}$$

where f is a function of r, i.e. $f = f(r)$. Let me transform to generalized PG coordinates [19] and introduce the PG time $dt_p = dt + \beta dr$, thus 4-metric can be written as

$$ds^2 = f dt_p^2 - 2\sqrt{1 - \frac{f}{A^2}} dt_p dr - \frac{1}{A^2} dr^2 - r^2 d\Omega, \tag{13}$$

where $A \equiv \sqrt{f/(1 - f^2\beta^2)}$.

At the outset, the energy component in the Einstein prescription [1] is given by

$$E_{\text{Einstein}} = \frac{1}{16\pi} \int \frac{\partial H_0^{0l}}{\partial x^l} d^3x, \tag{14}$$

where H_0^{0l} is the corresponding von Freud superpotential

$$H_0^{0l} = \frac{g_{0n}}{\sqrt{-g}} \frac{\partial}{\partial x^m} \left[(-g)(g^{0n} g^{lm} - g^{ln} g^{0m}) \right], \tag{15}$$

and the Latin indices take values from 1 to 3. For performing the calculations concerning the energy component of the Einstein energy-momentum complex, I have to transform the spatial parts of above metric (13) into the quasi-Cartesian coordinates (x, y, z)

$$\begin{aligned} ds^2 \;\; &= \;\; A^2 dt_p^2 - 2\sqrt{1 - \frac{f}{A^2}} dt_p (\frac{x}{r} dx + \frac{y}{r} dy + \frac{z}{r} dz) \\ &\quad -(\frac{1}{A^2} - 1)(\frac{x}{r} dx + \frac{y}{r} dy + \frac{z}{r} dz)^2 - (dx^2 + dy^2 + dz^2). \end{aligned} \tag{16}$$

Then, the required nonvanishing components of the Einstein energy-momentum complex H_0^{0l} are

$$\begin{aligned} H_0^{01} &= \frac{2Cx}{r^2}, \\ H_0^{02} &= \frac{2Cy}{r^2}, \\ H_0^{03} &= \frac{2Cz}{r^2}, \end{aligned}$$

and these are easily shown in spherical coordinates to be a vector

$$H_0^{0r} = \frac{2C}{r} \hat{r}, \tag{17}$$

where $C = 1 - f$ and $\hat{r}$ is the outward normal. Applying the Gauss theorem I obtain

$$E_{\text{Einstein}} = \frac{1}{16\pi} \oint H_0^{0r} \cdot \hat{r} r^2 d\Omega, \tag{18}$$

and the integral being taken over a sphere of radius r and the differential solid angle $d\Omega$. The Einstein energy complex within radius r reads

$$E_{\text{Einstein}} = \frac{r}{2}(1-f). \tag{19}$$

Next, the energy component of the Møller energy-momentum complex [2] is described as

$$E_{\text{Møller}} = \frac{1}{8\pi}\int \frac{\partial \chi_0^{0l}}{\partial x^l} d^3x, \tag{20}$$

where χ_0^{0l} is the Møller superpotential

$$\chi_0^{0l} = \sqrt{-g}\left(\frac{\partial g_{0\alpha}}{\partial x^\beta} - \frac{\partial g_{0\beta}}{\partial x^\alpha}\right) g^{0\beta} g^{l\alpha}, \tag{21}$$

and the Greek indices run from 0 to 3. However, the only nonvanishing component of Møller's superpotential is

$$\chi_0^{01} = \frac{df}{dr} r^2 \sin\theta. \tag{22}$$

Applying the Gauss theorem, I evaluate the integral over the surface of a sphere within radius r, and find the energy distribution is

$$E_{\text{Møller}} = \frac{r^2}{2}\frac{df}{dr}. \tag{23}$$

Here, I consider the results of calculation for two cases of the simplest black hole solutions, i.e. Schwarzschild and RN solution. In the first case I have $f = 1 - 2M/r$, therefore the energy complex of Einstein is

$$E_{\text{Einstein}} = M, \tag{24}$$

and of Møller is also

$$E_{\text{Møller}} = M. \tag{25}$$

For the next case it is defined that $f = 1 - 2M/r + Q^2/r^2$, so the energy complex in Einstein prescription is

$$E_{\text{Einstein}} = M - \frac{Q^2}{2r}, \tag{26}$$

and in Møller prescription is

$$E_{\text{Einstein}} = M - \frac{Q^2}{r}. \tag{27}$$

It should be noted that the above results of Einstein energy complex in PG Cartesian coordinates are equivalent to in Schwarzschild Cartesian and Kerr-Schild Cartesian ones [12], but the time coordinate of these three coordinates is different to each other. Using the Schwarzschild black hole as an example, the time coordinate of Schwarzschild Cartesian coordinates t, of Kerr-Schild Cartesian coordinates

$$v = t + r + 2M \ln \left| \frac{r}{2M} - 1 \right|, \tag{28}$$

and of PG Cartesian coordinates

$$t_p = t + 4M \left(\sqrt{\frac{r}{2M}} + \frac{1}{2} \ln \left| \frac{\sqrt{r/2M} - 1}{\sqrt{r/2M} + 1} \right| \right) \tag{29}$$

are not the same. In other words, no matter what the choices of these three kinds of time coordinate the energy complex of Einstein within radius r is $E_{\text{Einstein}} = M$. It need further study if the energy complex of Einstein has universal for any kinds of time coordinate.

3 Legendre transformation between the Einstein's and Møller's Energy Complex

To understand the physical meaning of difference ΔE, let me to begin with examining the static spherically symmetric solutions with two horizons as an example, in which the line element can be written as

$$ds^2 = f(r)dt^2 - f^{-1}(r)dr^2 - r^2 d\Omega, \tag{30}$$

where

$$f(r) = \left(1 - \frac{r_+}{r}\right)\left(1 - \frac{r_-}{r}\right), \tag{31}$$

r_+ is the event horizon and r_- is the inner Cauchy horizon. According to Eq.(19) and Eq.(23), the Einstein energy complex with radius r is

$$E_{\text{Einstein}} = \frac{r_+ + r_-}{2} - \frac{r_+ r_-}{2r}, \tag{32}$$

and the Møller energy complex is

$$E_{\text{Møller}} = \frac{r_+ + r_-}{2} - \frac{r_+ r_-}{r}. \tag{33}$$

Therefore, the difference of energy with radius r between the Einstein and Møller prescription can be obtained as

$$\Delta E = \frac{r_+ r_-}{2r}. \tag{34}$$

In the article of Nester *et. al.* [11], they had stated that *"Consequently, there are various of energy, each corresponding to a different choice of boundary condition; this situation can be compared with thermodynamics with its various energies: internal, enthalpy, Gibbs, and Helmholtz."* Hence, I insert the idea of black hole thermodynamics to compare energy-momentum complex with thermodynamic potential.

Afterward, I would introduce two thermodynamic qualities of black hole, the Hawking temperature [19]

$$T_H = \frac{1}{4\pi} \left.\frac{\partial f}{\partial r}\right|_{r_h} \tag{35}$$

and the Bekenstein-Hawking entropy [22]

$$S_{BH} = \left.\frac{\mathcal{A}}{4}\right|_{r_h} = \pi r_h^2. \tag{36}$$

Because those two qualities are only defined on event horizon, at $r = r_+$, the temperature is given as

$$T_H^+ = \frac{r_+ - r_-}{4\pi r_+^2} \tag{37}$$

and the entropy is also given as

$$S_{BH}^+ = \pi r_+^2, \tag{38}$$

Supposing that we consider the region between those two horizons, shown as $\mathcal{M} = \mathcal{B}^3(r_+) - \mathcal{B}^3(r_-)$, the difference of energy will be obtained in the form

$$\Delta E\,|_{r=r_-}^{r=r_+} = -\frac{r_+ - r_-}{2}. \tag{39}$$

I suggest that the temperature and entropy are definable at inner Cauchy horizon, hence at $r = r_-$ the temperature is

$$T_H^- = -\frac{r_+ - r_-}{4\pi r_+^2} \tag{40}$$

and the entropy is

$$S_{BH}^- = -\pi r_-^2. \tag{41}$$

Because of the normal vector on the boundary of $\mathcal{M}$ at $r = r_-$ is $-\hat{r}$, the area of surface of inner Cauchy horizon will be set $-4\pi r_-^2$ and the entropy at $r = r_-$ is defined to be an negative value. So the difference of energy within radius r between the Einstein prescription and Møller prescription can be written as

$$E_{\text{Møller}}|_{r_-}^{r_+} - E_{\text{Einstein}}|_{r_-}^{r_+} = T_H^+ S_{BH}^+ + T_H^- S_{BH}^-. \tag{42}$$

To compare with the Legendre transformation, I rewrite to the following formula

$$E_{\text{Einstein}}|_{r_-}^{r_+} = E_{\text{Møller}}|_{r_-}^{r_+} - \sum_{\partial\mathcal{M}} T_H S_{BH}. \tag{43}$$

In the region $\mathcal{M}$, I propose that $E_{\text{Møller}}$ and E_{Einstein} play the role of internal energy U and Helmholtz energy F in thermodynamics. But there is a puzzle where $E_{\text{Møller}}$ or E_{Einstein} are not a function of T_H or S_{BH}.

4 Conclusion and Discussion

I attempt to figure out two questions in this article. One is that if the calculation of Einstein energy-momentum complex is acceptable in PG coordinate, and the other is that can those energy-momentum complex be described as thermodynamic potential. Here, the expression for energy of the static spherically symmetric space-time with the PG Cartesian coordinates, like as Eq.(19), is obtained $E_{\text{Einstein}} = (1 - f)r/2$. This is a reasonable and satisfactory result, because Virbhadra [12], using the Kerr-Schild Cartesian ones, and Yang *et. al.* [18], using the Schwarzschild Cartesian ones, also got the same expression. It is an interesting idea to investigate in the future if there is any coincidence between the energy expressions with those three time coordinate.

In addition, I have showed that the relational formula about E_{Einstein} and $E_{\text{Møller}}$ is similar to the Legendre transformation. The $E_{\text{Møller}}$ and E_{Einstein} are regarded as the equivalent of internal energy U and Helmholtz energy F in the region $\mathcal{M}$. Although, the transformation takes us from a function of one pair variables to the other. It means that S_{BH} and T_H must be the variable of $E_{\text{Møller}}$ and E_{Einstein}, but I do not verify that yet. On the other hand, when I set $\mathcal{S}_\perp = \pi r^2$ to be a variable, the second term in the right-hand side of Eq.(11) can be replaced as

$$E_{\text{Møller}} = E_{\text{Einstein}} - 2\mathcal{S}_\perp \frac{dE_{\text{Einstein}}}{d\mathcal{S}_\perp}. \tag{44}$$

To compare with $F = U - TS$, we could obtain

$$T_H = \left.\frac{dE_{\text{Einstein}}}{d\mathcal{S}_\perp}\right|_{r_h}. \tag{45}$$

Here the formula of Eq.(44) presents that E_{Einstein} and $E_{\text{Møller}}$ play the role of U and F, and is opposite to the view of above.

In summary, I have obtained the Einstein and Møller energy complexes of static spherically symmetric black hole with generalized PG coordinates in which has been used in derivations of Hawking radiation as tunneling. Base on the calculation of energy expression in generalized PG coordinates, in Eq.(43) I have combined the difference ΔE with the temperature and entropy of black hole. Although Eq.(43) do not fit in with the Legendre transformation, it is an example to show that the energy-momentum complexes will compare with thermodynamic potential.

ACKNOWLEDGMENTS

I would like to thank Prof. Chopin Soo and Prof. Su-Long Nyeo for useful suggestions and discussions. This work is partially supported by National Center for Theoretical Sciences, Taiwan.

References

[1] A. Trautman, in *Gravitation: an Introduction to Current Research*, edited by L. Witten (Wiley, New York, 1962), pp 169-198; A. Einstein, *Preuss. Akad. Wiss. Berlin* **47**, 778 (1915); *Addendum-ibid.* **47**, 799 (1915).

[2] C. Møller, *Ann. Phys. (NY)* **4**, 347 (1958).

[3] L.D. Landau and E.M. Lifshitz, *The Classiical Theory of Fields* (Addison-Wesley Press, 1987), pp 280.

[4] P.G. Bergmann and R. Thompson, *Phys. Rev.* **89**, 400 (1953).

[5] R.C. Tolman, *Relativity, Thermodynamics and Cosmology* (Oxford Univ. Press, 1934), pp 227.

[6] S. Weinberg, *Gravitation and Cosmology: Principles and Applications of General Theory of Relativity* (John Wiley and Sons, Inc., New York, 1972), pp 168.

[7] A. Papapetrou, *Proc. R. Irish. Acad.* **A52**, 11 (1948).

[8] A. Komar, *Phys. Rev.* **113**, 934 (1959).

[9] R. Penrose, *Proc. Roy. Soc. London* **A381**, 53 (1982).

[10] A. Qadir and M. Sharif, *Phys. Lett.* **A176**, 331 (1992).

[11] C.-C. Chang, J. M. Nester and C.-M. Chen, *Phys. Rev. Lett.* **83**, 1897 (1999).

[12] K.S. Virbhadra, *Phys. Rev.* **D60**, 104041 (1999); J.M. Aguirregabirria, A. Chsmorro and K.S. Virbhadra, *Gen. Rel. Grav.* **28**, 1393 (1996).

[13] S.S. Xulu, *Int. J. Theor. Phys.* **46**, 2915 (2007).

[14] S. Mirshekari and A.M. Abbassi, *Mod. Phys. Lett.* **A24**, 747 (2009).

[15] I.-C. Yang and I. Radinschi, *Chin. J. Phys.* **42**, 40 (2004).

[16] J. Matyjasek, *Mod. Phys. Lett.* **A23**, 591 (2008).

[17] E. Vagenas, *Mod. Phys. Lett.* **A21**, 1947 (2006).

[18] I.-C. Yang, C.-L. Lin, and I. Radinschi, *Int. J. Theor. Phys.* **48**, 248 (2009).

[19] Chun-Yu Lin and Chopin Soo, *Phys. Lett.* **B671**, 493 (2009); Chun-Yu Lin and Chopin Soo, *arXiv:0905.3244*[gr-qc].

[20] V. Husain and O. Winkler, *Phys. Rev.* **D71**, 104001 (2005).

[21] M.K. Parikh and F. Wilczek, *Phys. Rev. Lett.* **85**, 5042 (2000).

[22] S.W. Hawking, *Commun. Math. Phys.* **43**, 199 (1975).