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Circular Symmetry in Topologically Massive Gravity

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Abstract

We show that spherical/circular symmetry breaks topologically massive gravity (TMG) down to the separate vanishing of its constituent Einstein and Cotton sectors, effectively reducing it to GR alone, since GR implies vanishing Cotton tensor. Consequences include Birkhoff's theorem (time-independence), verified here for $D = 3$ GR, and absence of novel "Schwarzschild-TMG" metrics: only the GR solutions–conical spaces or their cosmological extensions, are allowed.

1 Introduction

Topologically massive gravity [1] and its cosmological extension [2] (collectively TMG) have always been counter-examples to much of our standard lore due to the synergy between its separately anemic Einstein and Cotton sectors in $D = 3$: GR implies flat or (AS)dS, pure Cotton implies the weaker conformally flat, space. Here, we will study the flip side, the case when the full model necessarily reduces to its separate parts, and its consequences. Specifically, we will see that circular symmetry has precisely the effect of splitting the single TMG equation into its two constituents. This will first be exploited to prove Birkhoff's theorem (B), which states that there is no monopole radiation in theories, in particular Maxwell's and Einstein's, that are devoid of monopole degrees of freedom [3]. There, gauge invariance leaves only massless helicity (± 1,± 2) modes respectively. Their massive extensions, having helicity 0 excitations, do permit radial radiation. Topologically massive vector (TME) and tensor (TMG) $D = 3$ models partake of both camps: they are gauge invariant, but have massive excitations. One would nevertheless expect them to obey B because, although massive, they have (single) helicities (1, 2) but not 0. As a warmup, we first establish B in TME and linearized TMG, then do so for full nonlinear TMG.

As was also previously noted in [4], sector-splitting implies absence of "Schwarzschild-TMG" solutions beyond those of GR. There, it was shown more generally that splitting occurs in the presence of a hypersurface-orthogonal Killing vector X_μ; our circular symmetry is the physically important concrete example, with $X_\mu = g_{\mu\theta}$.

2 Topologically Massive Electrodynamics

We begin with TME; its circularly symmetric vector potentials are necessarily gradients, hence pure gauges: $A_i(r,t) = \partial_i a(r,t)$. From the surviving A_0, only $F_{0i} = -\partial_i A$ (dropping the "0") remains; of its equations

$$V^\nu \equiv \partial_\mu F^{\mu\nu} + m\,\epsilon^{\nu\alpha\beta}\,F_{\alpha\beta} = 0, \tag{1}$$

only the spatial one

$$V^i \equiv \dot{A}_{,i} + 2\,m\,\epsilon^{ij}\,A_{,j} = 0, \tag{2}$$

remains. But since the longitudinal and transverse components of any vector must vanish separately, as is especially clear here, since they are respectively proportional to x^i and $\epsilon^{ij}\,x^j$, we immediately conclude that $A = A(r) + A(t)$, with $A(t)$ pure gauge.

3 Linearized TMG

Linearized TMG goes similarly, just with more indices. We first write the full nonlinear model's field equations, including the cosmological term for completeness; it does not affect the sector separation, but only expands the GR sector:

$$T^\mu_\nu \equiv \sqrt{-g}\,\left(G^\mu_\nu + \Lambda\,\delta^\mu_\nu\right) + m^{-1}\,C^\mu_\nu = 0 \tag{3}$$

where the Cotton (mixed) tensor density is defined by

$$C^\mu_\nu \equiv \epsilon^{\mu\alpha\beta} D_\alpha\,\tilde{R}_{\beta\nu}, \qquad \tilde{R}_{\beta\nu} \equiv R_{\beta\nu} - \frac{1}{4}\,g_{\beta\nu}\,R. \tag{4}$$

[Our conventions are $(-++)$, $R_{\mu\nu} \sim +\partial_\alpha\,\Gamma^\alpha_{\mu\nu} + \ldots$ and $R = R^\mu_\mu$.] The Schouten tensor $\tilde{R}_{\beta\nu}$ ensures the following essential (but non-manifest) properties of the Cotton tensor:

$$D_\mu\,C^\mu_\nu \equiv 0 \quad C_{\mu\nu} \equiv C_{\nu\mu} \quad C^\mu_\mu \equiv 0. \tag{5}$$

Linearization of (4) about flat space (dropping the Λ term for simplicity here) consists of keeping only the linear deviations, $h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$, of the metric in evaluating $R_{\mu\nu}$, using an ordinary rather than covariant derivative, and dropping $\sqrt{-g}$. Gauge invariance permits us to keep only

$$h_{ij} = \delta_{ij}\,f, \qquad h_{0i} = 0, \qquad h_{00} = n, \tag{6}$$

since any 2-metric can always be reduced to its conformal part, and $h_{0i} = \partial_i\,h_0$, is pure gauge; the n merely simplifies notation. We use conformal gauge here for variety, reverting to the more familiar Schwarzschild one for the full theory. The linearized Einstein tensor in terms of the above two independent metric components (f, n) is:

$$G_{00} = -1/2\,\nabla^2\,f, \qquad G_{0i} = -1/2\,\left(x_i/r\right)\,\dot{f}', \qquad G_{ij} = 1/2\,\delta_{ij}\,\ddot{f} - 1/2\,\left(\delta_{ij}\,\nabla^2 - \partial^2_{ij}\right)\,n, \tag{7}$$

where dots and primes denote time and radial derivatives. A brief detour shows that GR obeys B: $G_{0i} = 0$ says $\dot{f}' = 0$, implying as in TME that $f = f(r)$, while from $G_{ij} = 0$ we get $n = n(t)$, a pure gauge. This suffices to fix time-independence of the metric. Next we compute the linearized $C_{\mu\nu}$,

$$C_{00} = 0, \qquad C_{0i} = -1/4\, \epsilon^{ij}\, \partial_j\, \nabla^2\, (n + f)\,, \qquad C_{ij} = 1/2\, \left(\delta_{ij}\, \nabla^2 - \partial^2_{ij}\right)\, \left(\dot{n} + \dot{f}\right). \tag{8}$$

The vanishing of C^μ_ν, being guaranteed by that of G^μ_ν, contains no further information, as is also obvious from (8).

4 Full TMG

Turning to the full nonlinear TMG, we first establish the separation of the GR & Cotton sectors. Vectors and tensors must look like x^i and δ_{ij} or $x^i\, x^j$ respectively, while pseudo-vectors and -tensors (in particular the C^μ_ν components) $\sim$ $(\epsilon^{ij}\, x^j)$ and $\left(\epsilon^{ik}\, x^k\, x^j + i \leftrightarrow j\right)$ respectively; pseudo-scalars like C^0_0 vanish since there is no non-vanishing pseudo-scalar: ϵ^{ij} $(\delta_{ij}$ or $x^i\, x^j) = 0$. Then the FULL field equations ($T^\mu_\nu = 0$) are:

$$\begin{aligned}
T^0_0 &\equiv \sqrt{-g}\, G^0_0 + \Lambda\, \delta^0_0 = 0, && (9)\\
T^i_0 &\equiv a\, x^i + m^{-1}\, b\, \epsilon^{ij}\, x^j = 0, && (10)\\
T^i_j &\equiv c\, \left(r^2\, \delta_{ij} - x^i\, x^j\right) + d\, x^i\, x^j + \Lambda\, \delta^i_j + m^{-1}\, e\, \left(\epsilon^{ik}\, x^k\, x^j + i \leftrightarrow j\right) = 0. && (11)
\end{aligned}$$

From (10) it follows upon contraction with x^i and $\epsilon^{ik}\, x^k$ that $a = 0$ and $b = 0$, while (11) implies (upon projecting with $x^i x^j$ and δ_{ij}) that all remaining functions in T^i_j vanish separately. Hence all G^μ_ν components vanish, irrespective of C^μ_ν. As in the linearized case, there is no new information in the C-sector, which is identically satisfied on GR-shell.

The above results have the further consequence that full TMG has only the same solutions as GR does: there are no "missing" separate "TMG-Schwarzschild" metrics: there are only the well-known conical or constant curvature ones of $D = 3$ GR [5], along with the BTZ solution for negative Λ [6].

As promised, we conclude with the explicit proof of B in pure GR, initially dropping the cosmological term for simplicity. In Schwarzschild (S) gauge, we parameterize the interval as

$$ds^2 = -a\, b^2\, dt^2 + a^{-1}\, dr^2 + r^2\, d\theta^2, \qquad \sqrt{-g} = b\, r, \tag{12}$$

in order to simplify the resulting field equations. Then constraints (as usual in S) are of first derivative order only,

$$\begin{aligned}
G_{00} &\equiv -b^2\, (a^2)'/4\, r = 0, && (13)\\
G_{0r} &\equiv -\left(\log a\right)^{\cdot}/2\, r = 0, && (14)\\
G_{rr} &\equiv \left(\log ab^2\right)'/2\, r = 0, && (15)\\
G_{\theta\theta} &\equiv \frac{r^2}{2\, a^3\, b^3} \left\{-a\, a'\, b' + b\, \left(a\, \ddot{a} - 2\, \dot{a}^2\right) + a^3\, b^3\, a'' + a^3\, b^2\, \left(3\, a'\, b' + 2\, a\, b''\right)\right\} = 0. && (16)
\end{aligned}$$

The two $G_{0\mu} = 0$ equations state that a is constant, $G_{rr} = 0$ then implying $b = b(t)$, which can obviously be made constant by time rescaling in (12); $G_{\theta\theta} = 0$ is then identically satisfied, as is $C^\mu_\nu = 0$. This establishes B (and the conical solution); had we kept the cosmological term, its non-derivative additions to (13) would still imply B, but now yield the (A)dS or BTZ metrics.

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