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Pseudo-models and propositional Horn inference

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Abstract

A well-known result is that the inference problem for propositional Horn formulae can be solved in linear time. We show that this remains true even in the presence of arbitrary (static) propositional background knowledge. Our main tool is the notion of a cumulated clause, a slight generalization of the usual clauses in Propositional Logic. We show that each propositional theory has a canonical irredundant base of cumulated clauses, and present an algorithm to compute this base.

MSC: 03B05; 03B35; 68T30

Keywords: Horn inference; Horn base; Background knowledge

1. Introduction

Inferring information can be a tedious task, even in the simple setting of Propositional Logic: to decide if a propositional formula follows from a given list of formulae is, in general, an $\mathcal{NP}$-complete problem. If, however, all the formulae under consideration are implications, then the inference problem can easily be solved.

Implications are particularly natural to describe classifications, when objects are grouped with regard to selected attributes. In such a situation, implications encode expressions of the form "each object with attributes $a_1, \ldots, a_n$ also has attributes $b_1, \ldots, b_m$". The simple

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inference mechanism allows to effectively study the system of all implications that hold in a given situation and to construct an irredundant implicational base, see [4, Section 2.3].

Implications are natural to describe classifications because objects are usually grouped according to their common attributes, whereas disjunctions and negations are of lesser importance. Nevertheless, for some purposes implications are insufficient: they cannot express that certain attributes exclude each other or that the absence of an attribute implies the presence of another. In a knowledge acquisition process, such nonimplicational information is often static and has the character of background information. For many-valued attributes (see [4, Section 1.3]), such background knowledge may describe the implicit structure of the attribute values.

An implicational knowledge acquisition method with propositional background knowledge has been described in [3], with emphasis on the methodology rather than on the efficiency. The algorithmic problems have been studied in [7]. The present paper combines and extends results from these. To make it accessible to a wider audience it is formulated in the language of Propositional Logic, in contrast to its predecessors, which use the terminology of Formal Concept Analysis.

We proceed as follows: first we introduce the notion of a *pseudo-model*. This can be done without reference to logic, in terms of elementary set theory. Next, we define a class of propositional formulae which we call *cumulated clauses* (because they are conjunctions of clauses having the same negated part). Cumulated clauses generalize the usual clauses; therefore each propositional theory is generated by its cumulated clauses. We prove an “irredundant basis” result for cumulated clauses and give an algorithm to construct such a base. Unfortunately, unlike in the case of implications, this basis is not necessary of minimal cardinality. Our final result shows that the complexity of implicational inference, modulo background knowledge given in the form of cumulated clauses, depends linearly on the implicational part.

It is due to space limitations that we illustrate our results below only by toy examples. Serious applications exist and will be published elsewhere. The reader is referred to [3] for a first impression.

2. Pseudo-models

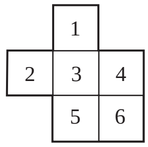
Let M be a finite set and let $\mathscr{F}$ be a set of some subsets of M. For reasons that will become transparent later we call the elements of $\mathscr{F}$ the “models”.

Definition 1. A set $P \subseteq M$ is a *pseudo-model* of $\mathscr{F} \subseteq \mathfrak{P}(M)$ if

(1) $P \notin \mathscr{F}$, and
(2) for each pseudo-model $Q \subseteq P$, $Q \neq P$, there is some $F \in \mathscr{F}$ with $Q \subseteq F \subseteq P$.

This is of course a recursive definition: the set $\emptyset$ is a pseudo-model if and only if $\emptyset \notin \mathscr{F}$, and as soon as the pseudo-models of cardinality less than n are known the definition specifies which sets of cardinality n are pseudo-models.

Example 1. Let $M := \{1, \ldots, 6\}$ and let $\mathscr{F}$ be the set of those subsets which correspond to connected subfigures of the displayed figure, including the empty set. The pseudo-models of $\mathscr{F}$ are precisely the nine disconnected two-element sets $\{1, 2\}, \ldots, \{4, 5\}$.



Example 2. Let $M := \{1, \ldots, 6\}$ and let $\mathscr{F}$ consist of all *additively saturated* subsets of M, that is, of those subsets $S \subseteq M$ which satisfy

$$\text{if } a \in S,\ b \in S, \text{ and } a + b \in M \text{ then } a + b \in S.$$

We have $\mathscr{F} = \{\emptyset, \{6\}, \{5\}, \{5,6\}, \{4\}, \{4,6\}, \{4,5\}, \{4,5,6\}, \{3,6\}, \{3,5,6\}, \{3,4,6\}, \{3,4,5,6\}, \{2,4,6\}, \{2,4,5,6\}, \{2,3,4,5,6\}, \{1,2,3,4,5,6\}\}$. The pseudo-models of $\mathscr{F}$ are $\{1\}$, $\{2\}$, $\{3\}$, and $\{2, 3, 4, 6\}$.

The models, that is, the elements of $\mathscr{F}$, are precisely the subsets $P \subseteq M$ satisfying

(2*) for each pseudo-model $Q \subseteq P$ there is some $F \in \mathscr{F}$ with $Q \subseteq F \subseteq P$.

Note that the sets which satisfy (2) are just those that are either models or pseudo-models.

Proposition 1. *Suppose $F_1 \in \mathscr{F}$, $F_2 \in \mathscr{F}$ implies $F_1 \cap F_2 \in \mathscr{F}$. If Q_1 and Q_2 are models or pseudo-models with $Q_1 \not\subseteq Q_2$ and $Q_2 \not\subseteq Q_1$ then $Q_1 \cap Q_2 \in \mathscr{F}$.*

Proof. We show that if $Q_1 \neq Q_1 \cap Q_2 \neq Q_2$, then $Q_1 \cap Q_2$ satisfies (2*): Let $Q \subseteq Q_1 \cap Q_2$ be a pseudo-model. Since Q_1 and Q_2 satisfy (2), there must be models F_1 and F_2 with $Q \subseteq F_1 \subseteq Q_1$ and $Q \subseteq F_2 \subseteq Q_2$ and therefore $F_1 \cap F_2 \subseteq Q_1 \cap Q_2$. $\square$

Now let M be a finite set of propositional variables and let α be some propositional formula in these variables. We say that a set $X \subseteq M$ is a *model* of α if the truth assignment which maps X to true and $M \setminus X$ to false, makes α true. X is a model of a set $\mathscr{T}$ of formulae if X is a model of each formula in $\mathscr{T}$. $\mathrm{Mod}(\mathscr{T})$ denotes the set of all models of $\mathscr{T}$. By a pseudo-model of $\mathscr{T}$ we mean a pseudo-model of $\mathrm{Mod}(\mathscr{T})$.

For each set $\mathscr{F} \subseteq \mathfrak{P}(M)$ of subsets let $\mathrm{Th}(\mathscr{F})$ denote the *theory* of $\mathscr{F}$, that is, the set of all formulae that have each $F \in \mathscr{F}$ as a model. It is well known that $\mathrm{Mod}(\mathrm{Th}(\mathscr{F})) = \mathscr{F}$.

The theory *generated* by a set $\mathscr{L}$ of formulae is $\mathrm{Th}(\mathrm{Mod}(\mathscr{L}))$. $\mathscr{L}$ is called *nonredundant* if it does not contain a smaller generating set, i.e., if for each $\lambda \in \mathscr{L}$ we have $\mathrm{Mod}(\mathscr{L} \setminus \{\lambda\}) \neq \mathrm{Mod}(\mathscr{L})$. It is well known that every propositional theory is generated by its *clauses*, which are the formulae of the form

$$\bigwedge A \to \bigvee B, \quad A, B \subseteq M.$$

3. Cumulated clauses

Definition 2. A *cumulated clause* is a formula of the form

$$\bigwedge A \to \bigvee_{t\in T} \bigwedge B_t,$$

where T is some finite index set, and where the sets A and B_t, $t \in T$, are all subsets of the fixed finite set M of variables. A is called the *premise* of the cumulated clause.

Cumulated clauses generalize both clauses $(\bigwedge A \to \bigvee B)$ and *implications* $(\bigwedge A \to \bigwedge B)$. A set $X \subseteq M$ is a model of $\bigwedge A \to \bigvee_{t\in T} \bigwedge B_t$ if and only if

$$A \nsubseteq X \quad \text{or} \quad B_t \subseteq X \text{ for some } t \in T.$$

The case $T = \emptyset$ is admitted, but we write $\bigwedge A \to \perp$ instead of $\bigwedge A \to \emptyset$. A set X is a model of $\bigwedge A \to \perp$ if and only if $A \nsubseteq X$. The cumulated clauses which are either implications or of the form $\bigwedge A \to \perp$ are called *cumulated Horn clauses*.

To each family $\mathscr{F} \subseteq \mathfrak{P}(M)$ of subsets and each $X \subseteq M$ we define a cumulated clause $\alpha_X^{\mathscr{F}}$ by

$$\alpha_X^{\mathscr{F}} := \bigwedge X \to \bigvee \left(\bigwedge Y \mid Y \in \mathscr{F} \text{ is minimal w.r.t. } X \subseteq Y\right).$$

Note that the minimality condition is only for simplification; the cumulated clause $\bigwedge X \to \bigvee(\bigwedge Y | Y \in \mathscr{F}, X \subseteq Y)$ is equivalent to (that is, has the same models as) $\alpha_X^{\mathscr{F}}$. Another equivalent version, used in Example 3 below, is

$$\bigwedge X \to \bigvee \left(\bigwedge Y \setminus X \mid Y \in \mathscr{F} \text{is minimal w.r.t. } X \subseteq Y\right).$$

Proposition 2.

(1) *$F \in \mathscr{F} \Rightarrow F$ is a model of $\alpha_X^{\mathscr{F}}$.*
(2) *X is a model of $\alpha_X^{\mathscr{F}} \iff X \in \mathscr{F}$.*
(3) *If P and Q are pseudo-models of $\mathscr{F}$ then P is a model of $\alpha_Q^{\mathscr{F}}$ if and only if $P \neq Q$.*

Theorem 1. *For each $\mathscr{F} \subseteq \mathfrak{P}(M)$ the set*

$$B_{\mathscr{F}} := \{\alpha_P^{\mathscr{F}} | P \textit{ pseudo-model of } \mathscr{F}\}$$

is a nonredundant generating set for $\mathrm{Th}(\mathscr{F})$.

Proof. We show that each model of $B_{\mathscr{F}}$ belongs to $\mathscr{F}$. Let X be a model of $B_{\mathscr{F}}$, and let $P \subseteq X$ be a pseudo-model. Then X is a model of $\alpha_P^{\mathscr{F}} \in B_{\mathscr{F}}$. Therefore there must be an element B_t of $\mathscr{F}$ with $P \subseteq B_t \subseteq X$. Thus X fulfills (2*), and consequently $X \in \mathscr{F}$.

To see that the set $B_{\mathscr{F}}$ is nonredundant, note that, according to Proposition 2, each pseudo-model P is a model of $B_{\mathscr{F}} \setminus \{\alpha_P^{\mathscr{F}}\}$. $\square$

This generalizes a theorem of Gigues and Duquenne [5] (see also [4, Theorem 8]) on implications. We call generating set $B_{\mathscr{F}}$ the *stem base* of $\mathscr{F}$.

Example 3. The stem base of the set $\mathscr{F}$ in Example 1, in slightly simplified notation, consists of the following cumulated clauses:

$$\begin{array}{lll} 1 \wedge 2 \to 3, & 1 \wedge 4 \to 3, & 1 \wedge 5 \to 3, \\ 1 \wedge 6 \to (3 \wedge 4) \vee (3 \wedge 5), & 2 \wedge 4 \to 3, & 2 \wedge 5 \to 3, \\ 2 \wedge 6 \to (3 \wedge 4) \vee (3 \wedge 5), & 3 \wedge 6 \to 4 \vee 5, & 4 \wedge 5 \to 3 \vee 6. \end{array}$$

In the case of cumulated Horn clauses, the stem base is the smallest possible generating set consisting of cumulated clauses, counting the number of generators (Theorem 5 of [10]):

Proposition 3. *A set $\mathscr{L}$ of cumulated clauses satisfying*

$$F_1, F_2 \in \mathrm{Mod}(\mathscr{L}) \Rightarrow F_1 \cap F_2 \in \mathrm{Mod}(\mathscr{L})$$

cannot have more pseudo-models than elements.

Proof. For each pseudo-model P_1 of $\mathscr{L}$ there must be some cumulated clause $\alpha := (\bigwedge A \to \bigvee \wedge B_t)$ in $\mathscr{L}$ that P_1 is not a model of. Then $A \subseteq P_1$. If P_2 is a pseudo-model with $P_1 \subset P_2$ then by Definition 1 $P_1 \subseteq F \subseteq P_2$ for some $F \in \mathrm{Mod}(\mathscr{L})$, and thus P_2 is a model of α. If neither $P_1 \subseteq P_2$ nor $P_2 \subseteq P_1$ then Proposition 1 yields that $P_1 \cap P_2$ is a model. But then P_1 must be a model of α, a contradiction. $\square$

The intersection of any two models of a cumulated Horn clause is a model again. So if $\mathscr{L}$ is a set of cumulated Horn clauses, then $B_{\mathrm{Mod}(\mathscr{L})}$ is of minimal cardinality. However, a small generating set is not necessarily simple. In fact, finding short bases (counting the number of literals) is difficult: Hammer and Kogan [6] (and, in other terms, Maier [8]) have shown that this is $\mathscr{N}\mathscr{P}$-complete already in the case of propositional Horn theories. Even the stem base in Example 3 can be shortened: the fourth and seventh cumulated clause can be replaced by $1 \wedge 6 \to 3$ and $2 \wedge 6 \to 3$, respectively, without changing the theory generated.

4. Finding models of cumulated clauses

Finding the models of a given set of cumulated clauses cannot be an easy task, because it is, in general, an $\mathscr{N}\mathscr{P}$-complete problem to decide if there is a model at all. When the cumulated clauses are of a special form, things may be easier. Implications, for example, admit a linear time algorithm that finds the smallest model containing a given set.

We present an algorithm (similar to that in [2], cf. [4, Section 2.1]) that constructs the models of a given set $\mathscr{L}$ of cumulated clauses in a certain lexicographical order. To define this order, fix some linear order of M. In what follows, we shall w.l.o.g. assume that $M := \{1 < 2 < \cdots < n\}$, but this is only for simplicity (in Section 7 we shall vary this linear order). A subset $A \subseteq M$ is called *lectically smaller* than a subset $B \subseteq M$ if the smallest element of M in which A and B differ belongs to B. Formally,

$$A <_i B :\Longleftrightarrow A \cap \{1, \ldots, i-1\} = B \cap \{1, \ldots, i-1\} \text{ and } i \in B,\ i \notin A,$$
$$A < B :\Longleftrightarrow A <_i B \quad \text{for some } i \in M.$$

Note that $A \subseteq B$ implies $A \leqslant B$. If A is an initial segment in the linear order of M then $A \leqslant B$ implies $A \subseteq B$.

Definition 3. For a set $\mathscr{L}$ of cumulated clauses and a set $X \subseteq M$ let $\mathscr{L}(X)$ be defined as follows: construct a sequence $\mathscr{X}_0, \mathscr{X}_1, \ldots,$ of sets $\mathscr{X}_i \subseteq \mathfrak{P}(M)$, starting with $\mathscr{X}_0 := \{X\}$, by the following rule.

If $\mathscr{X}_i$ is nonempty, then let Y be the lectically smallest element of $\mathscr{X}_i$. If Y is not a model of $\mathscr{L}$ and $\alpha := (\bigwedge A \to \bigvee_{t \in T} \bigwedge B_t) \in \mathscr{L}$ is a cumulated clause which Y is not a model of, then

$$\mathscr{X}_{i+1} := (\mathscr{X}_i \backslash \{Y\}) \cup \{Y \cup B_t \mid t \in T\}.$$

The sequence terminates after a finite number of steps, either because $\mathscr{X}_i$ is empty (in which case we set $\mathscr{L}(X) := \perp$) or because the minimal element Y of $\mathscr{X}_i$ is a model. In the latter case we set $\mathscr{L}(X) := Y$.

It is easy to see that the sequence terminates. In every step, the lectically smallest element is removed. Note that the construction is not fully deterministic since there may be several α that can be used. The next proposition shows that this has no effect on the result.

Proposition 4. *If X is not contained in any model of $\mathscr{L}$, then $\mathscr{L}(X) = \perp$. Otherwise, $\mathscr{L}(X)$ is the lectically smallest model of $\mathscr{L}$ containing X.*

Proof. By definition the sequence $\mathscr{X}_0, \mathscr{X}_1, \ldots,$ either vanishes (with some $\mathscr{X}_i = \emptyset$) or terminates with a model $\mathscr{L}(X)$ containing X. Let Y denote the lectically smallest model containing X. We have to show that $\mathscr{L}(X) = Y$. To do so[1], we prove by induction that each $\mathscr{X}_i$ contains a set Y_i with $Y_i \subseteq Y$. For $i = 0$ we choose $Y_0 := X$. If $Y_i \in \mathscr{X}_i$ has the desired property, then either $Y_i \in \mathscr{X}_{i+1}$ (in which case we let $Y_{i+1} := Y_i$), or Y_i is the lectically smallest element in $\mathscr{X}_i$. If $Y_i \neq Y$ then $\mathscr{X}_{i+1}$ is obtained using some $\alpha := (\bigwedge A \to \bigvee \wedge B_t) \in \mathscr{L}$ which Y_i is not a model of Y is a model and respects α. So there is some $B_t \subseteq Y$. Therefore $Y_{i+1} := Y_i \cup B_t$ is a subset of Y and is contained in $\mathscr{X}_{i+1}$. □

Example 4. Let $M := \{1, \ldots, 6\}$, let $\top$ denote the empty conjunction and $\perp$ denote the empty disjunction. The set $\mathscr{L}$ of cumulated clauses given by $\mathscr{L} := \{\top \to 1 \vee 6, \top \to 2 \vee 5, \top \to 3 \vee 4, 1 \wedge 6 \to \perp, 2 \wedge 5 \to \perp, 3 \wedge 4 \to \perp, 5 \wedge 6 \to \perp\}$ has exactly six models:

$$\mathscr{F} = \{\{2, 4, 6\}, \{2, 3, 6\}, \{1, 4, 5\}, \{1, 3, 5\}, \{1, 2, 4\}, \{1, 2, 3\}\}.$$

[1] Why does this suffice to prove that $\mathscr{L}(X) = Y$? Consider the $\mathscr{X}_i$ for which the algorithm terminates. Since $\mathscr{X}_i$ contains some $Y_i \subseteq Y$, certainly $\mathscr{X}_i$ is not empty. Thus the termination must be caused by the fact that the smallest element, say Z, of $\mathscr{X}_i$ is a model and therefore $\mathscr{L}(X) = Z$. Then $X \subseteq Z \leqslant Y_i \subseteq Y$. But since Y is the smallest model containing X we get $Z = Y$ and thus $\mathscr{L}(X) = Y$.

We demonstrate how the algorithm finds $\mathcal{L}(\{4\})$:

(1) $\mathcal{X}_1 = \{\{4\}\}$. The lectically smallest element of $\mathcal{X}_1$ is $Y := \{4\}$. A cumulated clause in $\mathcal{L}$ which Y is not a model of is $\alpha := (\top \to 1 \vee 6)$. We replace Y by $\{4\} \cup \{1\}$ and $\{4\} \cup \{6\}$ and obtain $\mathcal{X}_2$,
(2) $\mathcal{X}_2 = \{\{4, 6\}, \{1, 4\}\}$, $Y = \{4, 6\}$, $\alpha = (\top \to 2 \vee 5)$,
(3) $\mathcal{X}_3 = \{\{4, 5, 6\}, \{2, 4, 6\}, \{1, 4\}\}$, $Y = \{4, 5, 6\}$, $\alpha = (5 \wedge 6 \to \bot)$. $\{4, 5, 6\}$ is removed and replaced by nothing,
(4) $\mathcal{X}_4 = \{\{2, 4, 6\}, \{1, 4\}\}$, $Y = \{2, 4, 6\}$ is a model. The algorithm terminates with $\mathcal{L}(\{4\}) = \{2, 4, 6\}$.

We can use Proposition 4 to find, for a given set $A \subseteq M$, the lectically "next" model of $\mathcal{L}$, if such a model exists. Define for $A \subseteq M$ and $i \in M \backslash A$

$$A \oplus i := \mathcal{L}((A \cap \{1, \ldots, i-1\}) \cup \{i\}).$$

Theorem 2. *The lectically smallest model of $\mathcal{L}$ which is larger than A, if such a model exists, is $A \oplus i$, where i is the largest element of M satisfying $A <_i A \oplus i$.*

Proof. Let A^+ denote the lectically smallest model larger than A. Then $A < A^+$ implies $A <_i A^+$ for some i, and thus $(A \cap \{1, \ldots, i-1\}) \cup \{i\} \subseteq A^+$. Therefore, $B := \mathcal{L}((A \cap \{1, \ldots, i-1\}) \cup \{i\})$ is a model of $\mathcal{L}$ containing $(A \cap \{1, \ldots, i-1\}) \cup \{i\}$. Since B is the smallest such model, we have $B \leqslant A^+$. This implies that $B \cap \{1, \ldots, i-1\} = A \cap \{1, \ldots, i-1\}$ and thus $A <_i B$. But then B must equal A^+, for A^+ was minimal.

That i is maximal can be seen as follows: from $A <_i A \oplus i$ and $A <_j A \oplus j$ with $j < i$ we can infer $A \oplus i <_j A \oplus j$. $\square$

In practice, the computation can be simplified. Let $X \subseteq M$ be a set with largest element i. For each cumulated clause $\alpha := (\bigwedge R \to \bigvee_{t \in T} \bigwedge S_t)$ we define

$$\alpha_{[X]} := \left[\bigwedge R \to \bigvee \left(\bigwedge S_t \mid t \in T, S_t \cap \{1, \ldots, i-1\} \subseteq X\right)\right].$$

Clearly each model of $\alpha_{[X]}$ is also a model of α (since we have only shortened the disjunction of the conclusion). Conversely, each model Y of α with

$$Y \cap \{1, \ldots, i-1\} \subseteq X$$

is also a model of $\alpha_{[X]}$. Moreover, such a model can never be obtained using the procedure of Definition 3 by applying an $\alpha_{[X]}$ where the premise $\operatorname{premise}(\alpha) := R$ does not satisfy

$$R \cap \{1, \ldots, i-1\} \subseteq X.$$

So if for given $A \subseteq M$ and $i \in M \backslash A$ we set $X := (A \cap \{1, \ldots, i-1\}) \cup \{i\}$ and

$$\mathcal{L}_{A,i} := \{\alpha_{[X]} \mid \alpha \in \mathcal{L}, \quad \operatorname{premise}(\alpha) \cap \{1, \ldots, i-1\} \subseteq X\},$$

then the desired result is given by

$$Y := \mathcal{L}_{A,i}(X).$$

Suppose $Y \neq \perp$, then Y is a model of $\mathscr{L}$ satisfying $X \subseteq Y$, $Y \cap \{1, \ldots, i-1\} \subseteq X$ and thus $A <_i Y$. Actually, Y is the lectically smallest model of $\mathscr{L}$ with these properties, and $Y = \perp$ only if no such model exists. From Theorem 2, we infer

$$A <_i A \oplus i \iff A <_i \mathscr{L}_{A,i}(X) \iff \mathscr{L}_{A,i}(X) \neq \perp$$

and:

Corollary 1. *The lectically smallest model of $\mathscr{L}$ which is larger than A, if such a model exists, is $\mathscr{L}_{A,i}((A \cap \{1, \ldots, i-1\}) \cup \{i\})$, where i is the largest element of $M \backslash A$ for which $\mathscr{L}_{A,i}((A \cap \{1, \ldots, i-1\}) \cup \{i\}) \neq \perp$.*

Both the theorem and the corollary can be used to generate all models of $\mathscr{L}$ in the following manner:

- To check if there is a model of $\mathscr{L}$, compute the sequence $\mathscr{X}_0, \mathscr{X}_1, \ldots,$ as defined above, for $X = \emptyset$. If some $\mathscr{X}_i = \emptyset$ then $\mathscr{L}$ has no model. Otherwise, $\mathscr{L}(\emptyset)$ is the smallest model of $\mathscr{L}$.
- Whenever a model A is found, the lectically next model, if it exists, can be found using the criteria given in the theorem or the corollary. If no "next" model exists, all models were found.

Example 5. In Example 4 we have computed a model $A := \{2, 4, 6\}$. We now demonstrate how to find, using Corollary 1, the lectically next model of the set $\mathscr{L}$ of cumulated clauses from Example 4.

- We first try $i := 5$, because this is the largest element of $M \backslash A$. Then

  $$X := (A \cap \{1, \ldots, 4\}) \cup \{5\} = \{2, 4, 5\}.$$

  But since $2 \wedge 5 \rightarrow \perp$ is in $\mathscr{L}_{A,5}$, we obtain $\mathscr{L}_{A,5}(X) = \perp$. Therefore $i = 5$ does not give a solution.
- The next possible choice is $i := 3$, which leads to

  $$X := (A \cap \{1, 2\}) \cup \{3\} = \{2, 3\}.$$

  We find that $\mathscr{L}_{A,3} = \{\top \rightarrow 6,\ \top \rightarrow 2 \vee 5,\ \top \rightarrow 3 \vee 4,\ 2 \wedge 5 \rightarrow \perp,\ 3 \wedge 4 \rightarrow \perp,\ 5 \wedge 6 \rightarrow \perp\}$.

Using the algorithm of Definition 3 we compute $\mathscr{L}_{A,3}(\{2, 3\})$ to $\{2, 3, 6\}$, which is the lectically next model of $\mathscr{L}$.

5. Finding pseudo-models

The method which we have developed above can also be used to construct the pseudo-models of a given family $\mathscr{F} \subseteq \mathfrak{P}(M)$. As in Theorem 1 let $B_{\mathscr{F}}$ denote the stem base of $\mathscr{F}$.

Proposition 5. *If $\mathscr{L} \subseteq B_{\mathscr{F}}$ and if $P \subseteq M$ is minimal with respect to*

(1) *P is a model of $\mathscr{L}$,*
(2) *$P \notin \mathscr{F}$,*
then P is a pseudo-model of $\mathscr{F}$.

Proof. Let $Q \subseteq P$, $Q \neq P$ be a pseudo-model of $\mathscr{F}$. Q is no model of $\mathscr{L}$, since P was minimal. But by Proposition 2, $\alpha_Q^{\mathscr{F}}$ is the only cumulated clause in $B_{\mathscr{F}}$ not respected by Q. So $\alpha_Q^{\mathscr{F}} \in \mathscr{L}$. Since P respects $\alpha_Q^{\mathscr{F}}$, there must be an element $B_t \in \mathscr{F}$ with $Q \subseteq B_t \subseteq P$. This shows that P satisfies the conditions for being a pseudo-model of $\mathscr{F}$. $\square$

Proposition 5 gives rise to an algorithm that constructs, for a given set $\mathscr{F} \subseteq \mathfrak{P}(M)$, all pseudo-models and thereby the stem base $B_{\mathscr{F}}$. The algorithm generates, step-by-step in lectic order, all models and pseudo-models and a list $\mathscr{L}$ of cumulated clauses that will eventually be equal to $B_{\mathscr{F}}$. We start with $\mathscr{L} := \emptyset$.

The smallest model or pseudo-model is $\emptyset$. Whenever a model or pseudo-model Q is found, we check if it belongs to $\mathscr{F}$. If not, then Q is a pseudo-model and we enlarge $\mathscr{L}$ by $\alpha_Q^{\mathscr{F}}$.

Then the "next" model of $\mathscr{L}$ is constructed with Corollary 1. According to Proposition 4, it is a model or a pseudo-model of $\mathscr{F}$. More formally, we proceed as follows:

Algorithm. for constructing the stem base $B_{\mathscr{F}}$ of a given set $\mathscr{F} \subseteq \mathfrak{P}(M)$. Let M be linearly ordered.

(1) $\mathscr{L} := \emptyset$, $Q := \emptyset$.
(2) If $Q \notin \mathscr{F}$ then $\mathscr{L} := \mathscr{L} \cup \{\alpha_Q^{\mathscr{F}}\}$.
(3) Search for the largest $i \in M$ such that $Q <_i Q \oplus i$.
If such an i exists, then let $Q := Q \oplus i$ and continue with 2,
else let $B_{\mathscr{F}} := \mathscr{L}$, and stop.

6. The complexity of computing $\mathscr{L}(X)$

It cannot be easy to compute $\mathscr{L}(X)$ (the smallest model of $\mathscr{L}$ containing X, cf. Proposition 4), because this solves a well-known hard problem: the question if $\mathscr{L}$ has a model at all is "the" generic $\mathscr{NP}$-complete problem, and it is equivalent to

$$\mathscr{L}(\emptyset) \neq \perp?$$

It is therefore of little interest to analyze the complexity of the algorithm given in Definition 3 in general.

The situation is completely different if $\mathscr{L}$ consists of cumulated Horn clauses only. Then $\mathscr{L}(X)$ can be computed in "linear" time $O(|\mathscr{L}| \cdot |M|^2)$ ([8], see also [1,9]). We aim for a similar result in the case that most of the formulae in $\mathscr{L}$ are Horn. To this end, let us assume that

$$\mathscr{L} = \mathscr{N} \cup \mathscr{H},$$

where $\mathscr{H}$ consists of cumulated Horn clauses only. We denote the $\mathscr{H}$-closure of any set $X \subseteq M$ by $\mathscr{H}(X)$. Thus, $\mathscr{H}(X) = \perp$ if X is not contained in any model of $\mathscr{H}$. Otherwise, $\mathscr{H}(X)$ is the smallest[2] model of $\mathscr{H}$ containing X.

Algorithm. for constructing $\mathscr{L}(X)$, where $X \subseteq M$ and $\mathscr{L} = \mathscr{N} \cup \mathscr{H}$ as above.

- Construct a sequence $\overline{\mathscr{X}}_0, \overline{\mathscr{X}}_1, \ldots,$ of subsets of $\mathfrak{P}(M)$ starting with

$$\overline{\mathscr{X}}_0 := \begin{cases} \{\mathscr{H}(X)\} & \text{if } \mathscr{H}(X) \neq \perp, \\ \emptyset & \text{if } \mathscr{H}(X) = \perp \end{cases}$$

  according to the following rule:
- If $\overline{\mathscr{X}}_i$ is nonempty then let Y denote the lectically smallest element of $\overline{\mathscr{X}}_i$. If Y is not a model of $\mathscr{L}$ and $\alpha := (\bigwedge A \to \bigvee_{t \in T} \bigwedge B_t) \in \mathscr{N}$ is a cumulated clause which Y is not a model of, then

$$\overline{\mathscr{X}}_{i+1} := \overline{\mathscr{X}}_i \backslash \{Y\} \cup \{\mathscr{H}(Y \cup B_t) \mid t \in T, \mathscr{H}(Y \cup B_t) \neq \perp\}.$$

The sequence terminates after a finite number of steps, either because $\overline{\mathscr{X}}_i$ is empty (in which case we set $\mathscr{L}(X) := \perp$) or because the minimal element Y of $\overline{\mathscr{X}}_i$ is a model. In the latter case we set $\mathscr{L}(X) := Y$.

Except for leaving out some intermediate steps, this is actually not a *modification* of the algorithm given in Definition 3, but only a more precise version: if there is a choice, Horn formulae are applied first. It is therefore evident that the result is the same and that Proposition 4 applies.

How many sets are in the $\overline{\mathscr{X}}_i$'s? This depends on the structure of the cumulated clauses in $\mathscr{N}$. Let $N := |\mathscr{N}|$,

$$\mathscr{N} =: \{\bigwedge A_i \to \bigvee_{t \in T_i} \bigwedge B_t \mid i \in \{1, \ldots, N\}\},$$

where the indexing is chosen in such a way that

$$|T_1| \geqslant |T_2| \geqslant \cdots \geqslant |T_N|.$$

Then the total number of sets generated in the above algorithm is at most

$$\sigma(\mathscr{N}) := 1 + |T_1| + |T_1| \cdot |T_2| + \cdots + |T_1| \cdot |T_2| \cdots |T_N|.$$

Another parameter of interest is

$$\tau(\mathscr{N}) := N + |T_1| + \cdots + |T_N|,$$

the total number of sets occurring in $\mathscr{N}$.

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[2] Since the models of Horn clauses are closed under intersection, the lectically smallest model containing X is also the smallest with respect to set inclusion.

Whenever the algorithm generates a new set, we have

- to find a cumulated clause $\bigwedge A_i \rightarrow \bigvee_{t\in T_i} \bigwedge B_t$ in $\mathscr{N}$ of which the lectically smallest set Y in $\overline{\mathscr{X}}_i$ is not a model. This may take $\tau(\mathscr{N})$ set operations,
- to compute $Y \cup B_t$ (a set operation of complexity proportional to $|M|$),
- to compute the $\mathscr{H}$-closure $\mathscr{H}(Y \cup B_t)$ (which is of complexity $\mathrm{O}(|\mathscr{H}| \cdot |M|^2)$), and
- to insert this set in a lectically ordered list (what is again of complexity $\mathrm{O}(|M|^2)$ because there are at most $2^{|M|}$ entries in that list).

So each new set requires an effort of $\mathrm{O}(\tau(\mathscr{N}) + (|\mathscr{H}|+1)\cdot|M|)$ set operations. Since there are at most $\sigma(\mathscr{N})$ such sets, we get the following theorem:

Theorem 3. *Constructing $\mathscr{L}(X)$ with the above algorithm requires at most*

$$\mathrm{O}(\sigma(\mathscr{N}) \cdot (\tau(\mathscr{N}) + (|\mathscr{H}| + 1) \cdot |M|) \cdot |M|)$$

bit operations.

Let us remark that for fixed $\mathscr{N}$ and nonempty $\mathscr{H}$ this is

$$O(|\mathscr{H}| \cdot |M|^2),$$

which with a pinch of salt may be called *linear in the size of $\mathscr{H}$*. If sets are represented by their characteristic vectors, then the input size of $\mathscr{H}$ is $2 \times (|\mathscr{H}| \cdot |M|)$. If set operations (instead of bit operations) are counted as single steps, the complexity bound reduces by a factor of $|M|$. From this viewpoint, we may say that

for fixed $\mathscr{N}$, the algorithm is linear in the input size of $\mathscr{H}$.

7. Implicational inference

Let $\mathscr{L}$ be a set of cumulated clauses. Using the operator $\mathscr{L}(X)$ from Section 4 it is easy to decide if a given indefinite Horn clause $\bigwedge A \rightarrow \bot$ follows from $\mathscr{L}$, because obviously we have

Proposition 6. *An indefinite Horn clause $\bigwedge A \rightarrow \bot$ belongs to* $\mathrm{Th}(\mathrm{Mod}(\mathscr{L}))$ *if and only if $\mathscr{L}(A) = \bot$.*

For the definite case we have that an implication $\bigwedge A \rightarrow \bigwedge B$ is in $\mathrm{Th}(\mathrm{Mod}(\mathscr{L}))$ if and only if every model of $\mathscr{L}$ containing A also contains B. The operator $\mathscr{L}(X)$ computes only the lectically smallest such model, thus in general it is not sufficient to check if $B \subseteq \mathscr{L}(A)$ to decide the problem. But there is a simple trick: the operator $\mathscr{L}(X)$ was introduced in Section 4 with respect to some fixed, but arbitrary linear order of M. The complexity bound for the algorithm does not change if we replace this ordering by a different one.

Proposition 7. *If $\mathscr{L}(A)$ is computed with respect to a linear order of M in which B is an initial segment, then*

$$\bigwedge A \to \bigwedge B \;\in \mathrm{Th}(\mathrm{Mod}(\mathscr{L})) \quad \iff \quad B \subseteq \mathscr{L}(A).$$

Proof. If B is an initial segment then every set that is lectically larger than B contains B as a subset. In this case, B is a subset of every model containing A if and only if B is a subset of the lectically smallest such model, i.e., of $\mathscr{L}(A)$. □

We now can formulate what we mean by *Horn inference with background knowledge*: decide if a given cumulated Horn clause follows from a given set of cumulated Horn clauses and a (small) set of arbitrary propositional "background knowledge".

For a better understanding of our main theorem, let us recall that we consider propositional formulae over a finite variable set M, that cumulated clauses generalize the usual clauses and that a cumulated Horn clause is either indefinite (of the from $\bigwedge A \to \bot$) or an implication (of the form $\bigwedge A \to \bigwedge B$).

Theorem 4. *Let $\mathscr{N}$ be a set of cumulated clauses. The number of set operations necessary to decide if a given cumulated Horn clause α follows from $\mathscr{N}$ and a given set $\mathscr{H}$ of cumulated Horn clauses, i.e., if*

$$\alpha \in \mathrm{Th}(\mathrm{Mod}(\mathscr{N} \cup \mathscr{H})),$$

is (for fixed $\mathscr{N}$) linear in the size $|\mathscr{H}|$ of $\mathscr{H}$.

We expect this result to be of practical value for knowledge acquisition with many-valued attributes, see [3] for details.

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